

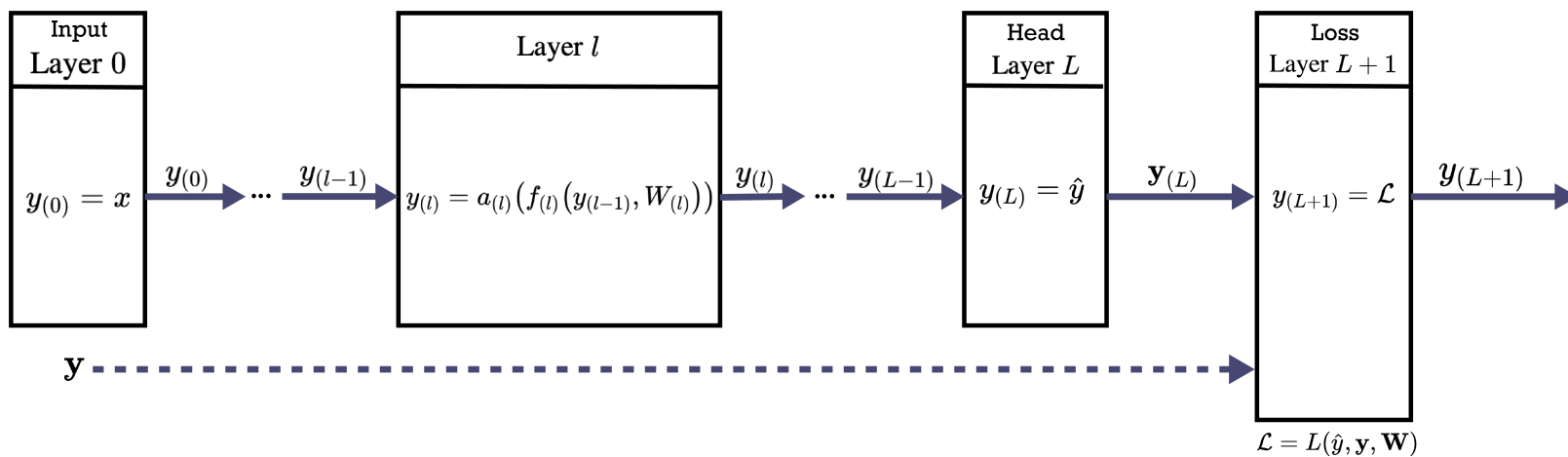
# Inside a layer: Units/Neurons

## Notation 1

Layer  $l$ , for  $1 \leq l \leq L$ :

- Produces output vector  $\mathbf{y}_{(l)}$
- $\mathbf{y}_{(l)}$  is a vector of  $n_{(l)}$  synthetic features
$$n_{(l)} = ||\mathbf{y}_{(l)}||$$
- Takes as input  $\mathbf{y}_{(l-1)}$ , the output of the preceding layer

- Layer  $L$  will typically implement Regression or Classification
- The first  $(L - 1)$  layers create synthetic features of increasing complexity
- We will use layer  $(L + 1)$  to compute a Loss



The input  $\mathbf{x}$

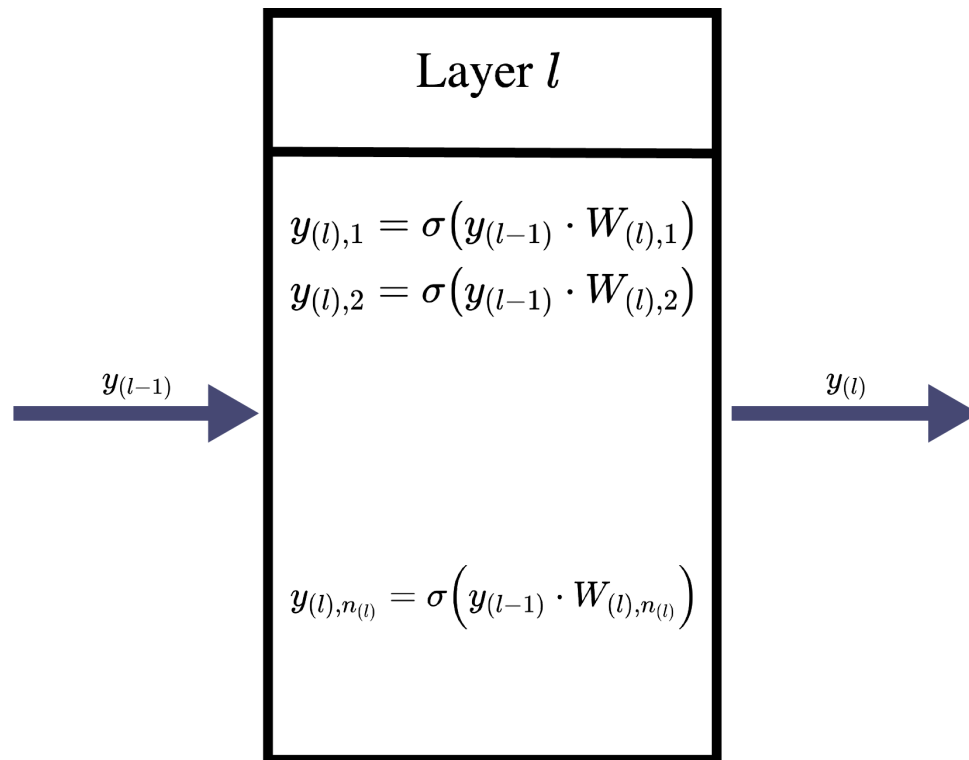
- Is called "layer 0"
- $\mathbf{y}_{(0)} = \mathbf{x}$

The output  $\mathbf{y}_{(L-1)}$  of the penultimate layer ( $L - 1$ )

- Becomes the input of a Classifier/Regression model at layer  $L$

Let's look inside layer  $l$  (of a particular type called *Fully Connected* or *Dense*)

Layer



- Input vector of  $n_{(l-1)}$  features:  $\mathbf{y}_{(l-1)}$
- Produces output vector or  $n_{(l)}$  features  $\mathbf{y}_{(l)}$
- Feature  $j$  defined by the function
$$\mathbf{y}_{(l),j} = \sigma(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),j})$$

Each feature  $\mathbf{y}_{(l),j}$  is produced by a *unit (neuron)*

- There are  $n_{(l)}$  units in layer  $l$
- The units are *homogenous*
  - same input  $\mathbf{y}_{(l-1)}$  to every unit
  - same functional form for every unit
  - units differ only in  $\mathbf{W}_{(l),j}$



*Units* are also sometimes referred to as *Hidden Units*

- They are internal to a layer.
- From the standpoint of the Input/Output behavior of a layer, the units are "hidden"

The functional form

$$\mathbf{y}_{l,j} = \sigma(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),j})$$

is called a *Dense* or *Fully Connected* unit.

It is called Fully connected since

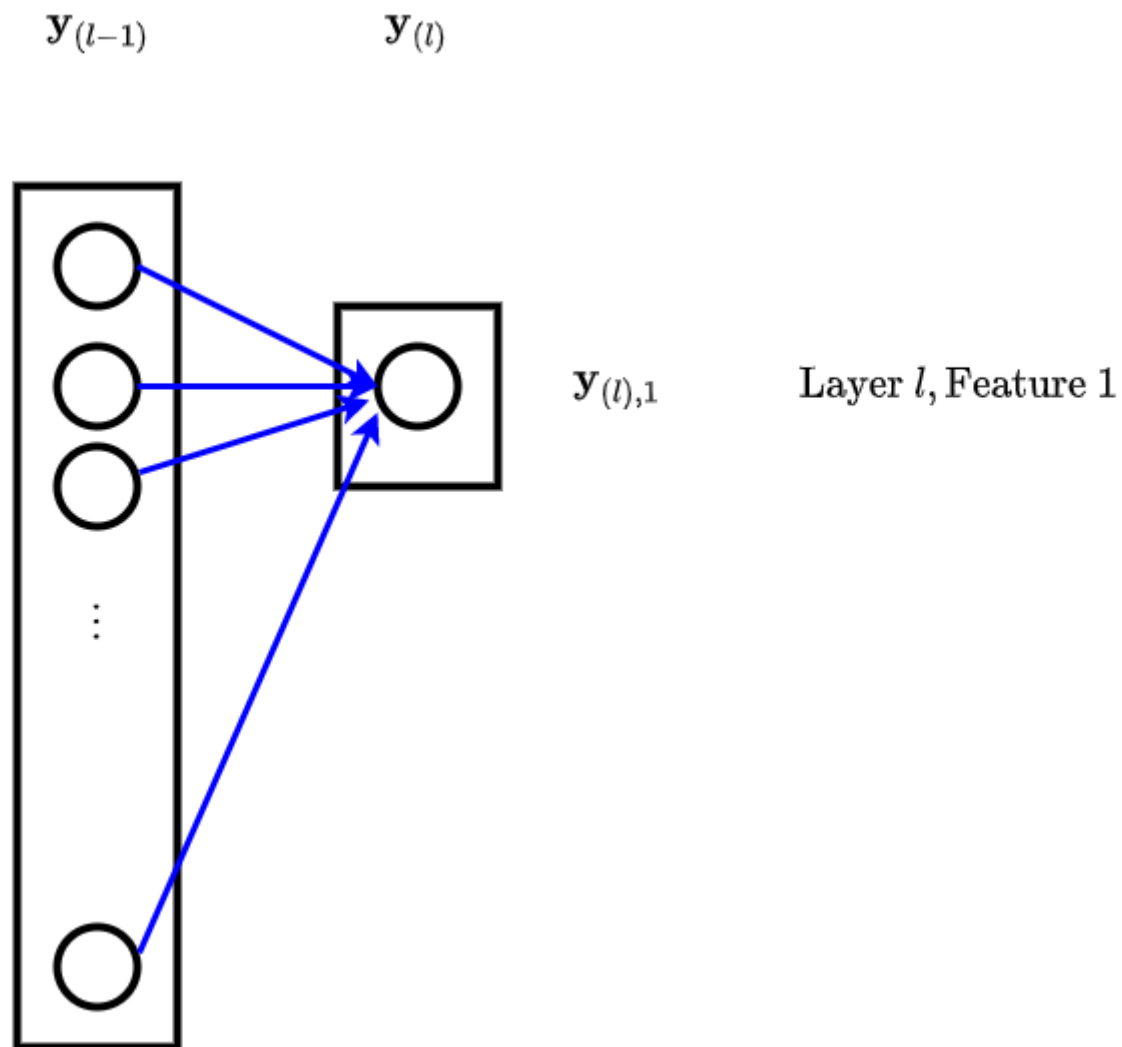
- each unit takes as input  $\mathbf{y}_{(l-1)}$ , **all**  $n_{(l-1)}$  outputs of the preceding layer

The *Fully Connected* part can be better appreciated by looking at a diagram of the connectivity of a *single* unit producing a *single* feature.

A Fully Connected/Dense Layer producing a *single* feature at layer  $l$  computes

$$\mathbf{y}_{(l),1} = a_{(l)}(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),1})$$

## Fully connected unit, single feature

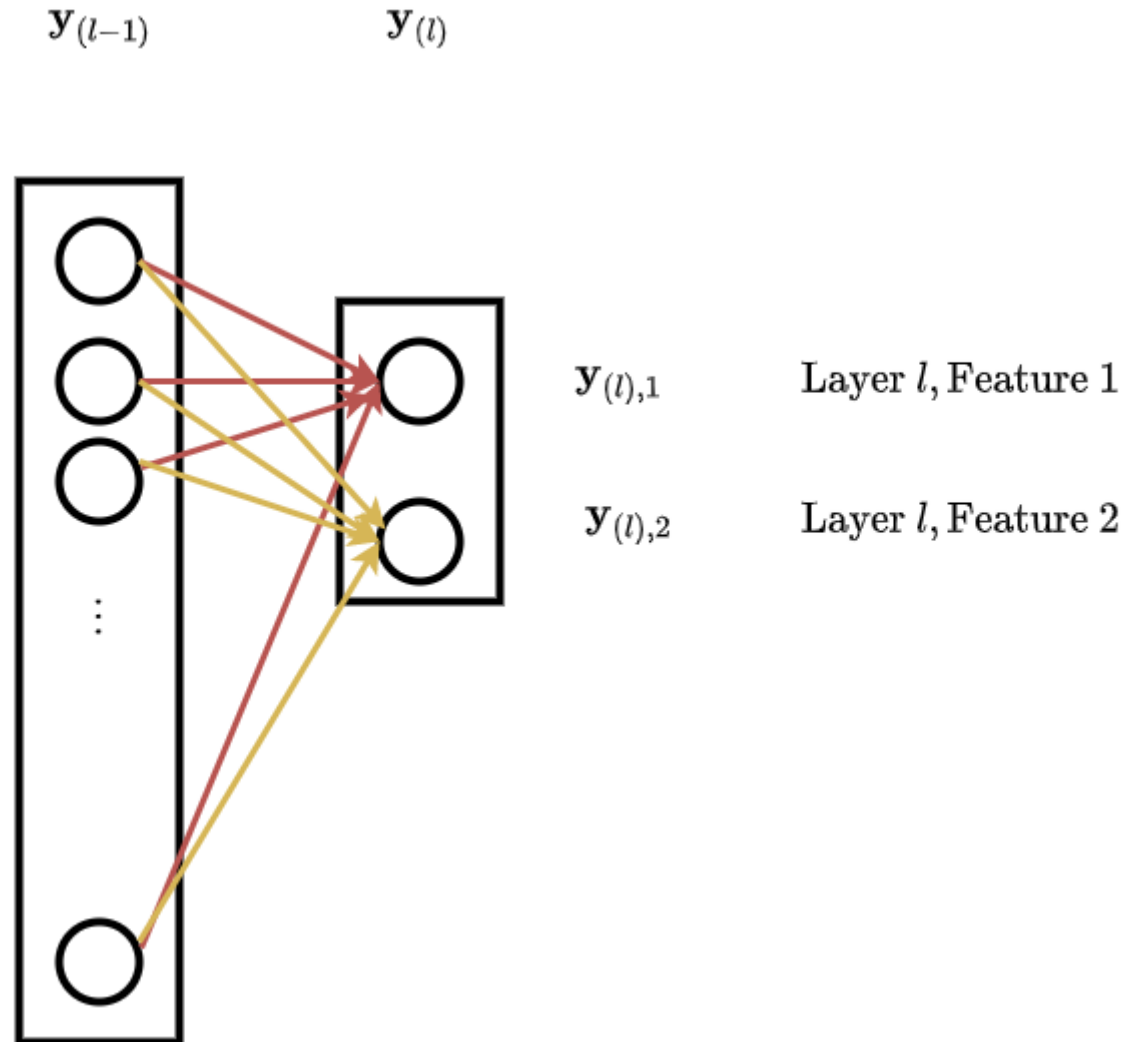


The edges into the single unit of layer  $l$  correspond to  $\mathbf{W}_{(l),1}$ .

A Fully Connected/Dense Layer with multiple units producing *multiple* feature at layer  $l$  computes

$$\mathbf{y}_{(l),j} = a_{(l)}(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),j})$$

## Fully connected, two features



The edges into each unit of layer  $l$  correspond to

- $\mathbf{W}_{(l),1}, \mathbf{W}_{(l),2} \dots$
- Separate colors for each units/row of  $\mathbf{W}$

Each unit  $\mathbf{y}_{(l),j}$  in layer  $l$  creates a new feature using pattern  $\mathbf{W}_{(l),j}$

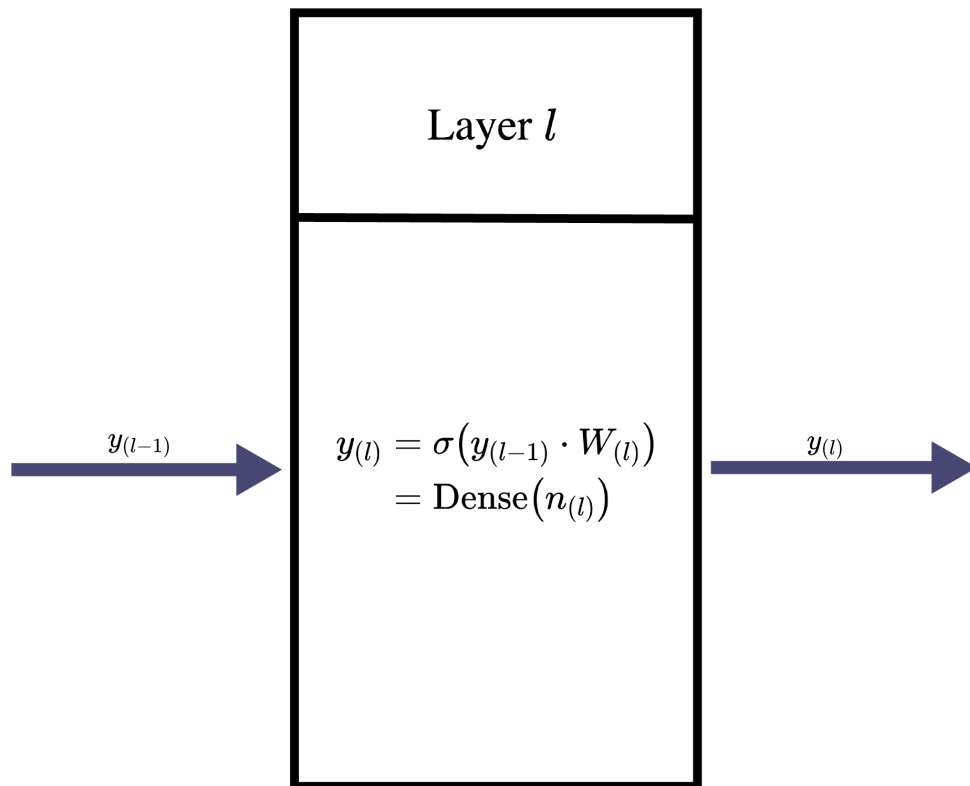


The functional form is of

- A dot product  $\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),j}$ 
  - Which can be thought of matching input  $\mathbf{y}_{(l-1)}$  against pattern  $\mathbf{W}_{(l),j}$
- Fed into  $\sigma$ , the *sigmoid* function we have previously encountered in Logistic Regression.

Because the units are homogeneous, we can depict it as

Layer



where

- $\mathbf{y}_{(l)}$  is a vector of length  $n_{(l)}$
- $\mathbf{W}_{(l)}$  is a matrix
  - $n_{(l)}$  rows
  - $\mathbf{W}_{(l)}^{(j)}$   
 $= \mathbf{W}_{(l),j}$

Written with the shorthand  $\text{Dense}(n_l)$

We will introduce other types of layers.

- Most will be homogeneous
- Not all will be fully Connected
- The dot product will play a similar role

The sigmoid function  $\sigma$  may be the *most significant part* of the functional form

- The dot product is a *linear* operation
- The outputs of sigmoid are *non-linear* in its inputs

So the sigmoid induces a non-linear transformation of the features  $\mathbf{y}_{(l-1)}$

The outer function which applies a non-linear transformation to linear inputs

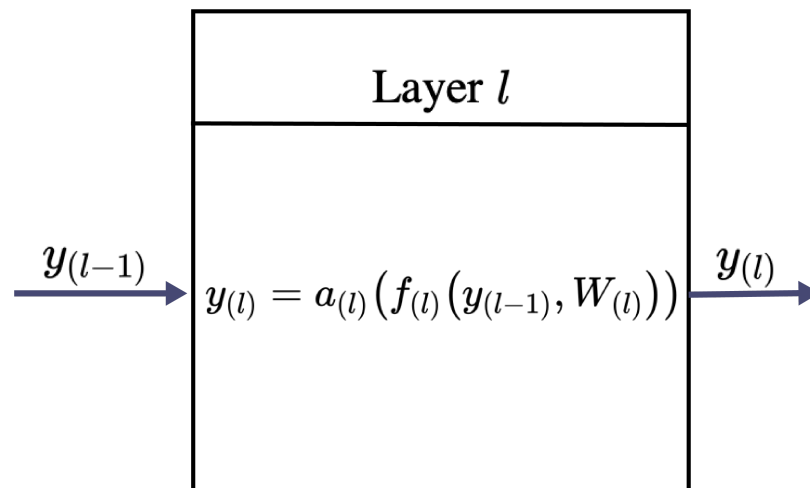
- Is called an *activation function*
- Sigmoid is one of several activation functions we will study

- The operation of a layer does not always need to be a dot production
- The activation function of a layer need not always be the sigmoid

More generically we write a layer as



## Layers



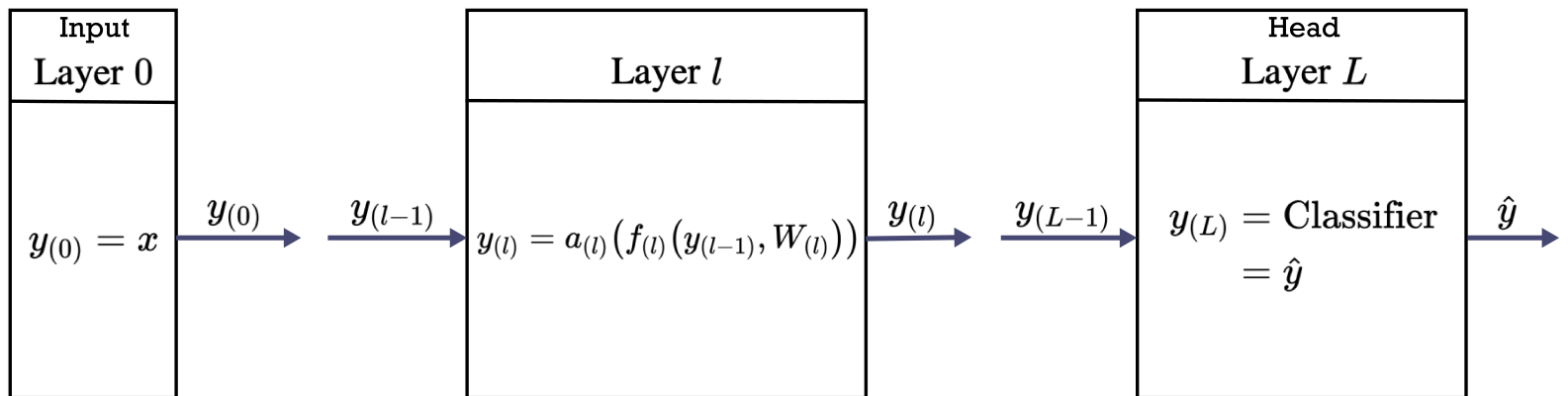
$$\mathbf{y}_{(l)} = \mathbf{a}_{(l)} \left( \mathbf{f}_{(l)} \left( \mathbf{y}_{(l-1)}, \mathbf{W}_{(l)} \right) \right)$$

where

- $\mathbf{f}_{(l)}$  is a function of  $\mathbf{y}_{(l-1)}$  and  $\mathbf{W}_{(l)}$
- $\mathbf{a}_{(l)}$  is an activation function

**So our multi-layer Neural Network (using Dense layers) looks like**

## Layers



In slightly more mathematical terms: Layer  $l$  is computing a function  $\mathbf{y}_{(l)} = \mathbf{F}_{(l)}$

$$\mathbf{F}_{(l)}(\mathbf{y}_{(l-1)}; \mathbf{W}_{(l)}) = \mathbf{y}_{(l)}$$

$$\mathbf{F}_{(l)} : \mathcal{R}^{||\mathbf{y}_{(l-1)}||} \mapsto \mathcal{R}^{||\mathbf{y}_{(l)}||}$$

If we expand  $\mathbf{F}_{(l)}$ , we see that it is the  $l$ -fold composition of functions  $\mathbf{F}_{(1)}, \dots, \mathbf{F}_{(l)}$

$$\begin{aligned} \mathbf{y}_{(l)} &= \mathbf{F}_{(l)}(\mathbf{y}_{(l-1)}; \mathbf{W}_{(l)}) \\ &= \mathbf{F}_{(l)}(\mathbf{F}_{(l-1)}(\mathbf{y}_{(l-2)}; \mathbf{W}_{(l-1)}); \mathbf{W}_{(l)}) \\ &= \mathbf{F}_{(l)}(\mathbf{F}_{(l-1)}(\mathbf{F}_{(l-2)}(\mathbf{y}_{(l-3)}; \mathbf{W}_{(l-2)}); \mathbf{W}_{(l-1)}); \mathbf{W}_{(l)}) \\ &= \vdots \end{aligned}$$

So the layer-wise architecture is nothing more than a way of computing a nested (composed) function.

In [4]: `print("Done")`

**Done**