Convolutional Neural Networks

Our introduction was of a very limited Convolutional Layer

- Recognizing a single feature
- One dimensional

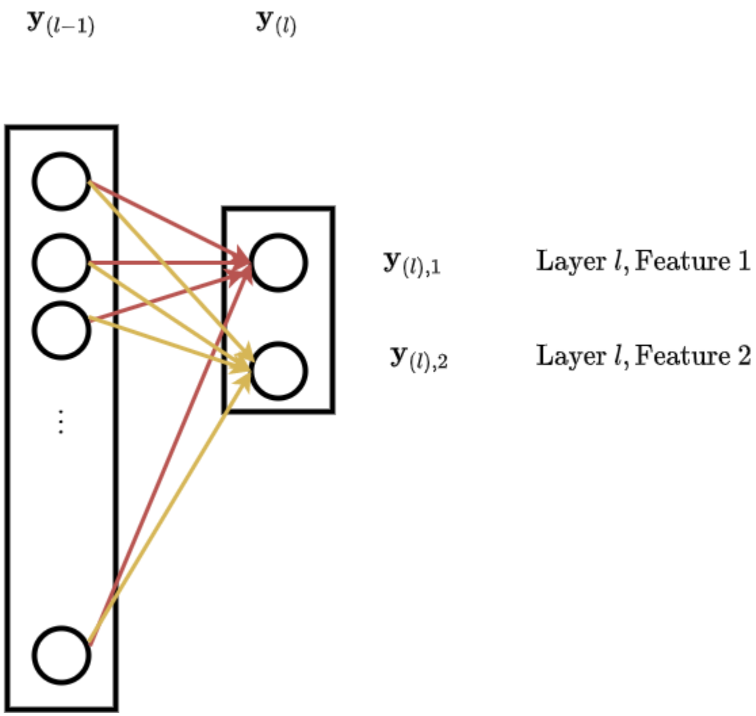
We will relax each restriction in turn.

Multiple features

Recall that a Fully Connected layer may have multiple units, so as to compute *multiple* features.

A Fully Connected/Dense Layer producing multiple features at layer l computes
$$\mathbf{y}_{(l),j} = a_{(l)}(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),j})$$
using separate weights to recognize each feature

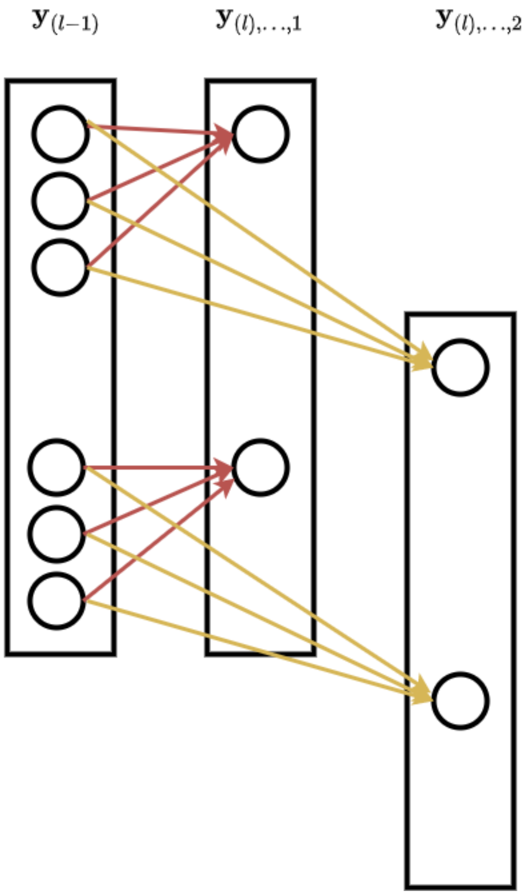
Fully connected, two features



Similary, a Convolutional layer may compute *multiple* features:

- Using separate kernels to recognize each output feature map
- Indicated via separate colors

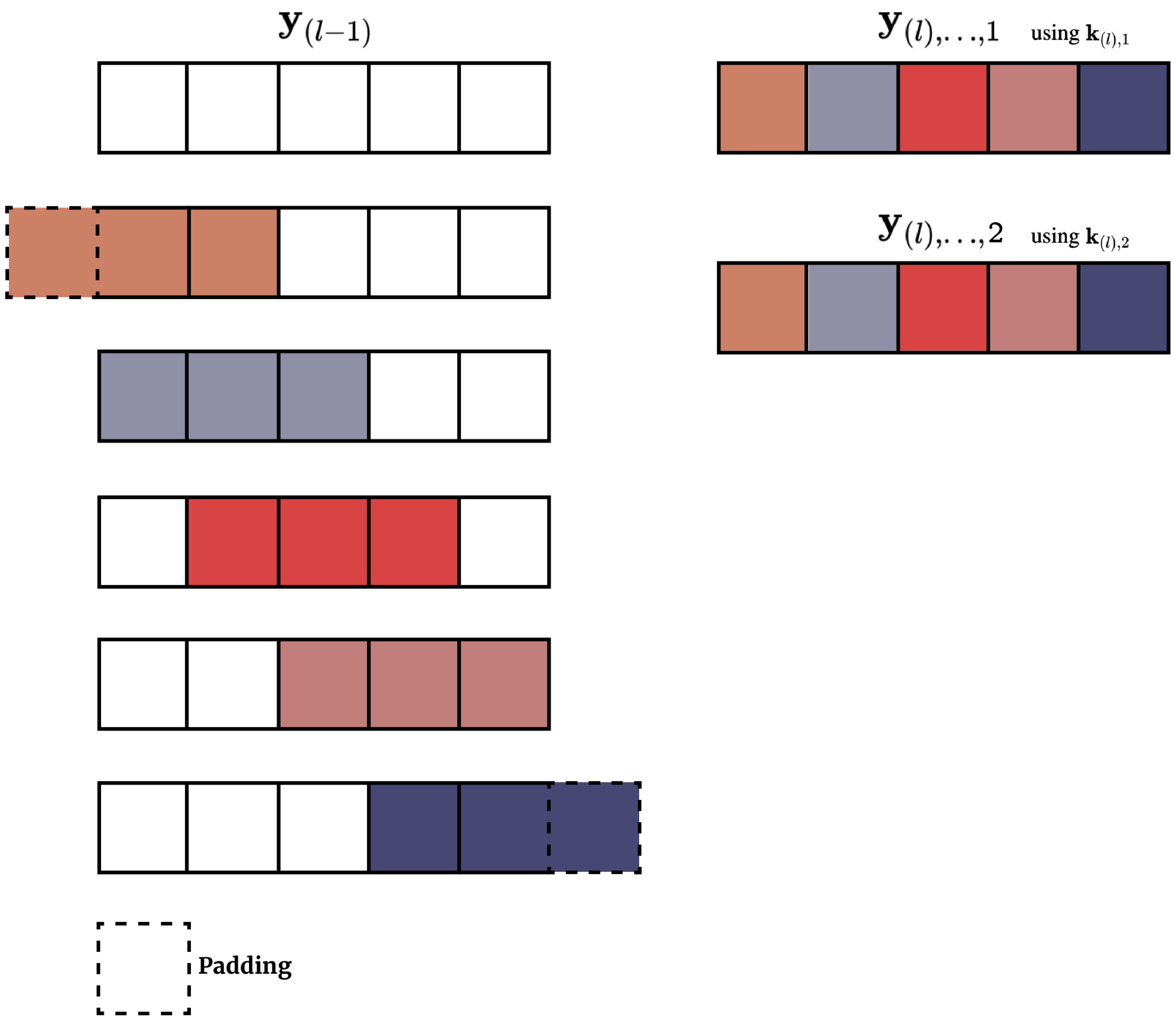
CNN layer, multiple features



Each output feature, of the same shape as the spatial dimension of the input, is called a *feature map*

- Different feature maps $\mathbf{y}_{(l),j}$ use *different* kernels
    - e.g., $\mathbf{k}_{(l),1}, \mathbf{k}_{(l),2}, \ldots$
- But are applied over the *same* input locations
- Recognizing *different* features at the same location
- e.g., $\mathbf{y}_{(l),1}, \mathbf{y}_{(l),2}, \ldots$

Conv 1D, single input, multiple output features



Notation

Input dimensions: Spatial, channel

Our examples thus far have input layers that are one dimensional (having a single feature).

This will not always be the case:

- When Convolutional Layer l creates *multiple* features, as above
- Layer l output is 2 dimensional

We will soon deal with even higher dimensional inputs (e.g, 3 dimensional).

First, some common terminology.

Suppose the input $\mathbf{y}_{(l-1)}$ is $(N+1)$ dimensional of shape

$$||\mathbf{y}_{(l-1)}|| = (d_{(l-1),1} \times d_{(l-1),2} \times \ldots d_{(l-1),N} \quad \times n_{(l-1)})$$

(Thus far: $N = 1$ and $n_{(l-1)} = 1$ but that will soon change)

The first N dimensions $(d_{(l-1),1} \times d_{(l-1),2} \times \ldots d_{(l-1),N})$

- Are called the *spatial* dimensions

The last dimension (of size $n_{(l-1)}$)

- Indexes the features i.e., varies over the number of features
- Called the *feature* or *channel* dimension

Notation

- N denotes the *number* of spatial dimensions
- $n_{(l)}$ denotes the *number of features* in layer l
- Thus far: $N = n_{(l)} = 1$

Rather than treating the single feature input as a special case

- The shape of $\mathbf{y}_{(l-1)}$ would be better written with an extra dimension of length 1: $$||\mathbf{y}_{(l-1)}|| = (d_{(l-1),1} \times d_{(l-1),2} \times \ldots d_{(l-1),N} \times \mathbf{1})$$
- More clearly indicating that layer $l-1$ has just one feature

With this terminology we can say that a Convolution

- Uses a different kernel $\mathbf{k}_{(l),j}$ for each output feature/channel $1 \le j \le n_{(l)}$
- Applies this kernel to *each* element in the *spatial* dimensions
- Feature map for feature number $1 \le j \le n_{(l)}$
    - Is of same shape as the spatial dimension
    - Recognizing a single feature at each location within the spatial dimension

Channel Last/First

As we have seem: we are dealing with objects of $(N+1)$ dimensions

- Have identified the first N dimensions as "spatial"
- The last ($(N+1)^{th}$) as the feature/channel dimension

This is known as *channel last* because the feature/channel dimension is the last.

Some toolkits

- Identify the *first* dimension as the feature/channel dimension
- The remaining N dimensions as the spatial dimensions

This is called *channel first* because the feature/channel dimension is first.

You may arrange the data in Keras according to *either* convention, but it defaults to channel last so we will use that as well.

That's why we write the output of layer l at feature j as

$$\mathbf{y}_{(l),\ldots,j}$$

where the dots $(\ldots)$ indicate the (variable number of) spatial dimensions

Conv1d when input layer has multiple features: $n_{(l-1)} > 1$

Our examples thus far have input layer $(l-1)$ with a single feature

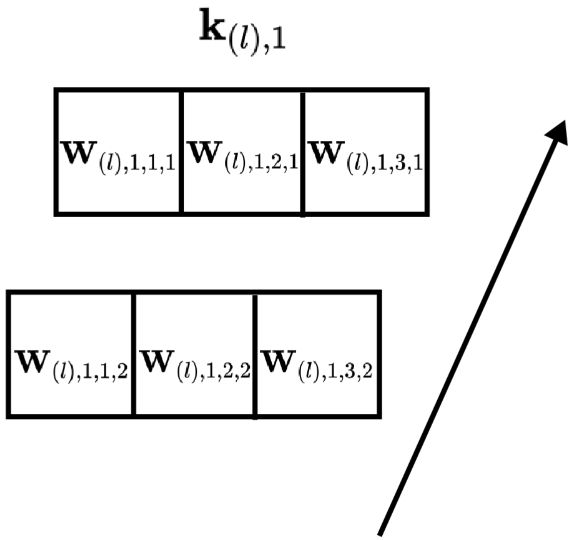
How does a convolution work when the input layer has *more than one* feature ?

- As would be the case of layer l which is the *result* of applying a Convolutional Layer to layer $l-1$

The answer is that we again slide a kernel over each location in the spatial dimension

- **but** each spatial location is now a *vector* of all $n_{(l-1)}$ input features
- Hence the kernel has an extra dimension of length $n_{(l-1)}$
    - That is, of shape $(f_{(l)} \times n_{(l-1)})$

Conv 1D: 2 input features: kernel 1



Note: Weights notation

- $\mathbf{W}_{(l),k,j,f}$
    - layer l
    - output feature k
    - spatial location j
    - input feature f

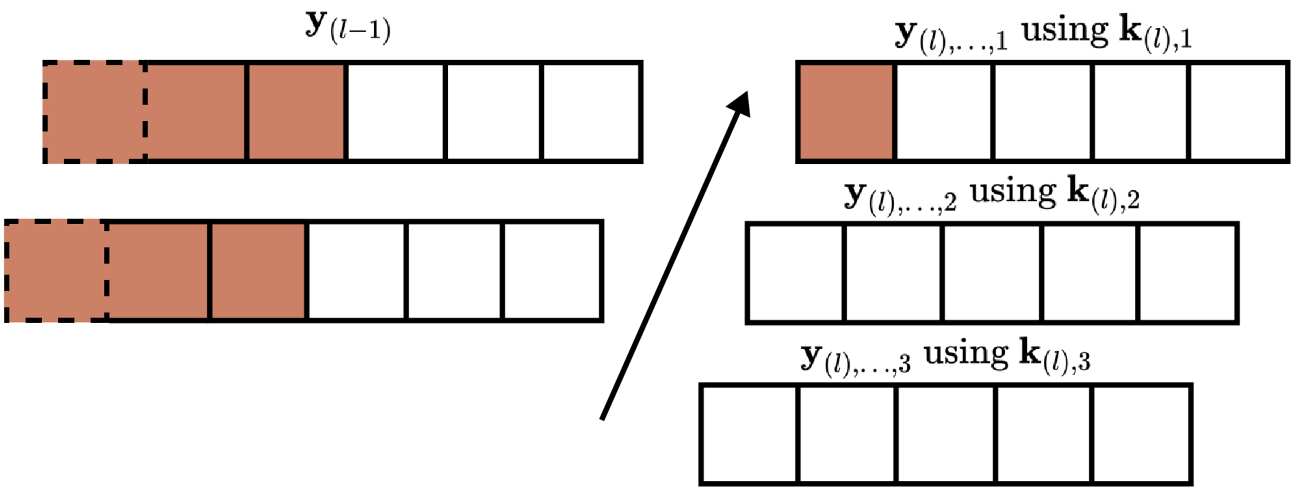
Note

- Dot product is only defined over one dimensional vectors
- When we use "dot product" on two higher dimensional objects of the same shape:
    - Element-wise product
    - Reduced to a scalar by summing the products
- Consider it to be the dot product of the flattened versions of the two objects

Let's illustrate how this works.

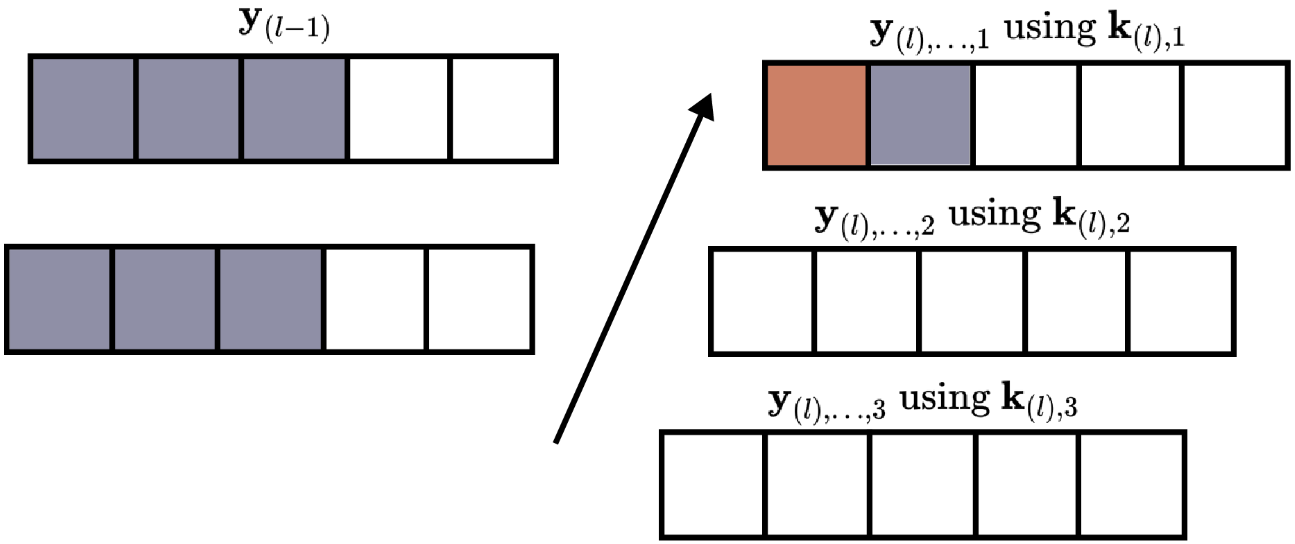
- Output feature 1
- Spatial location 1

Conv 2D: 2 features to 3 features: kernel 1



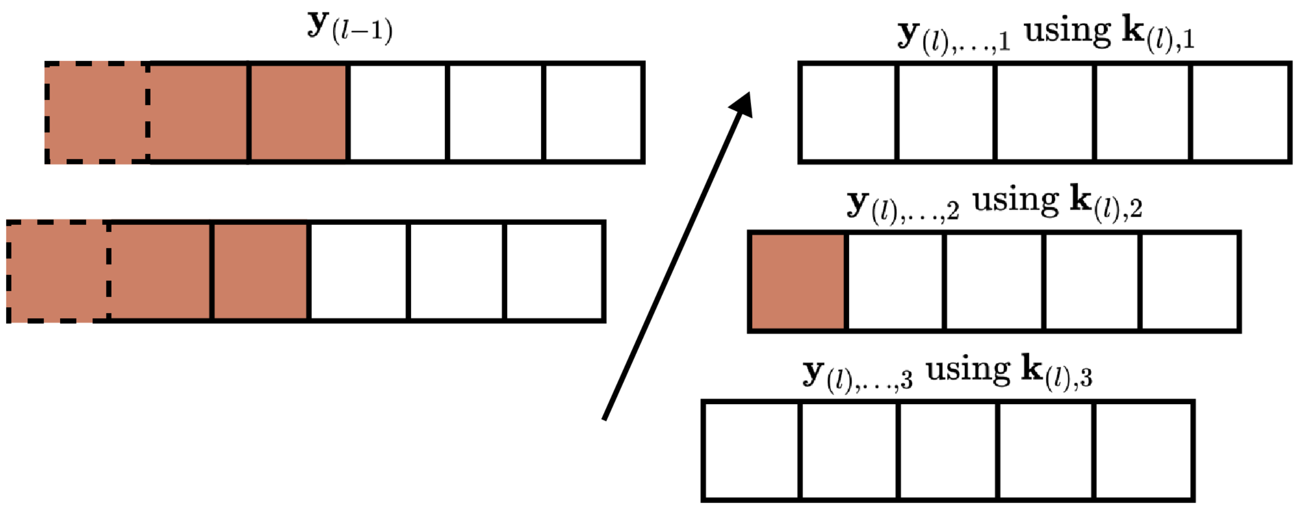
- Output feature 1
- Spatial location 2

Conv 2D: 2 features to 3 features: kernel 1



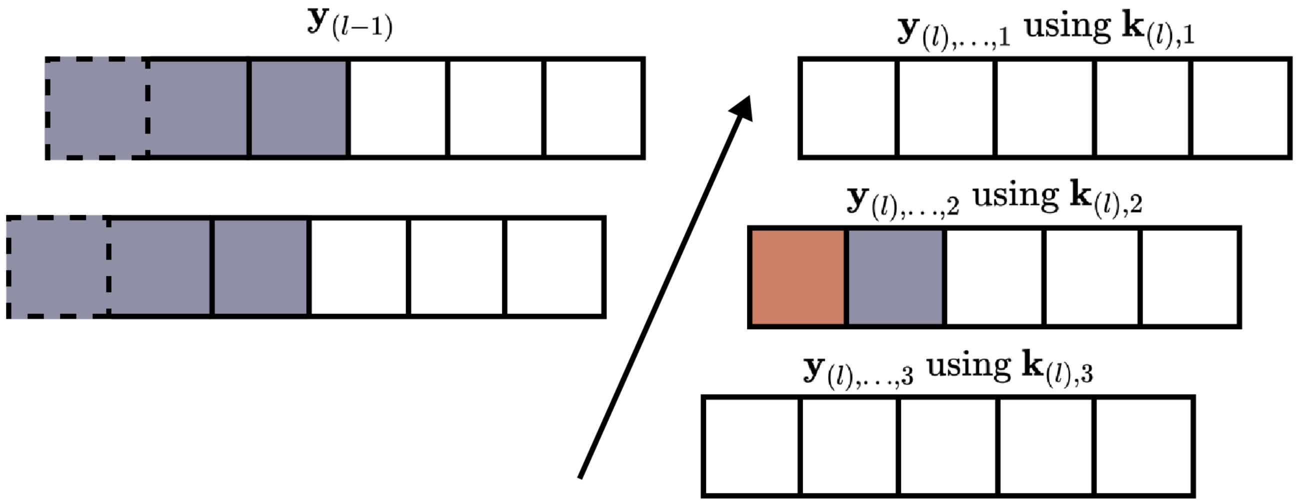
- Output feature 2
- Spatial location 1

Conv 2D: 2 features to 3 features: kernel 2



- Output feature 2
- Spatial location 2

Conv 2D: 2 features to 3 features: kernel 2



With an input layer having N spatial dimensions, a Convolutional Layer l producing $n_{(l)}$ features

- Preserves the "spatial" dimensions of the input
- Replaces the channel/feature dimensions

That is\

$$\begin{aligned}
||\mathbf{y}_{(l-1)}|| &= (n_{(l-1),1} \times n_{(l-1),2} \times \ldots n_{(l-1),N}, \quad \mathbf{n_{(l-1)}}) \\
||\mathbf{y}_{(l)}|| &= (n_{(l-1),1} \times n_{(l-1),2} \times \ldots n_{(l-1),N}, \quad \mathbf{n_{(l)}})
\end{aligned}$$

Conv2d: Two dimensional convolution ($\mathbf{N = 2}$)

Thus far, the spatial dimension has been of length $N = 1$.

Generalizing to $N = 2$ is straightforward.

For example, here is a two dimensional convolution with a single input and output feature ($n_{(l-1)} = n_{(l)} = 1$)

- Kernel
    - Two spatial dimensions of size $f_{(l)}$ each
    - A single input feature dimension of size $n_{(l-1)} = 1$
    - Dimension $\left(f_{(l)} \times f_{(l)} \times n_{(l-1)} \right)$
- Is "slid" over 2 dimensional segments of the input
- The "dot product" of the kernel and a two dimensional region of $\mathbf{y}_{(l-1)}$ is performed
- There are $n_{(l)} = 1$ kernels and output features

Conv 2D: single input feature: kernel 1

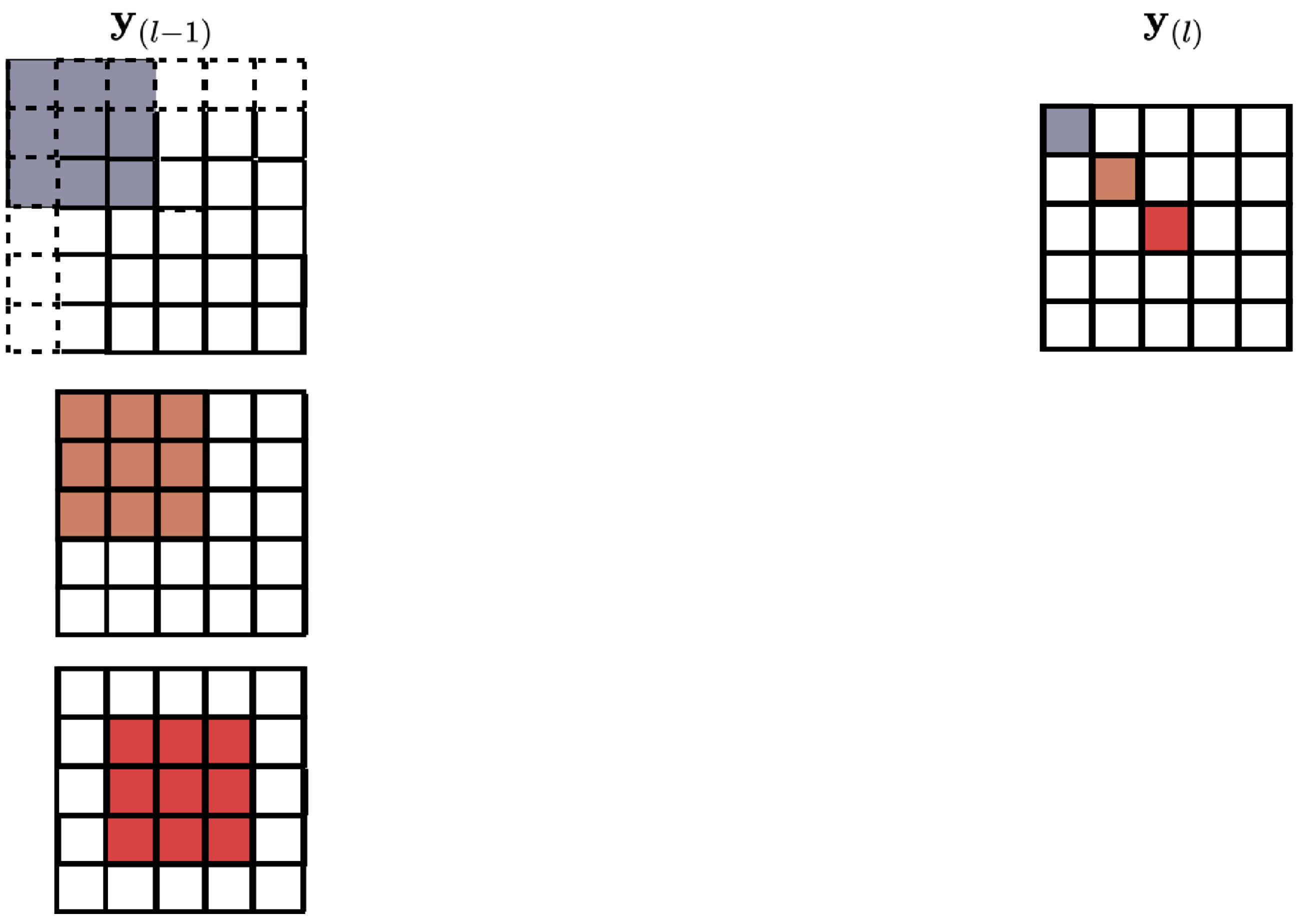
$\mathbf{k}_{(l),1,1}$

$\mathbf{W}_{(l),1,1,1}$	$\mathbf{W}_{(l),1,2,1}$	$\mathbf{W}_{(l),1,3,1}$
$\mathbf{W}_{(l),2,1,1}$	$\mathbf{W}_{(l),2,2,1}$	$\mathbf{W}_{(l),2,3,1}$
$\mathbf{W}_{(l),3,1,1}$	$\mathbf{W}_{(l),3,2,1}$	$\mathbf{W}_{(l),3,3,1}$

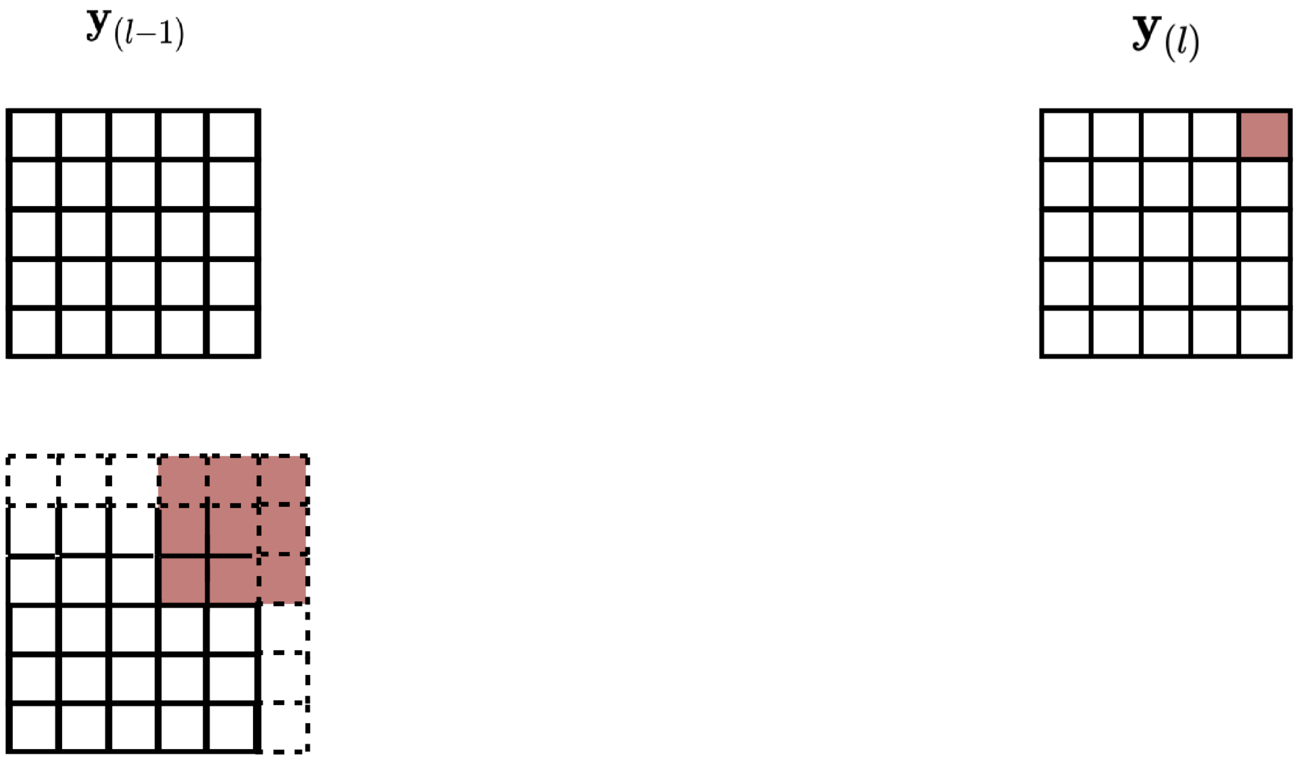
$\mathbf{k}_{(l),j,j'}$

- layer l
- output feature j
- input feature j'

Conv 2D, single input, single output feature: padding at border



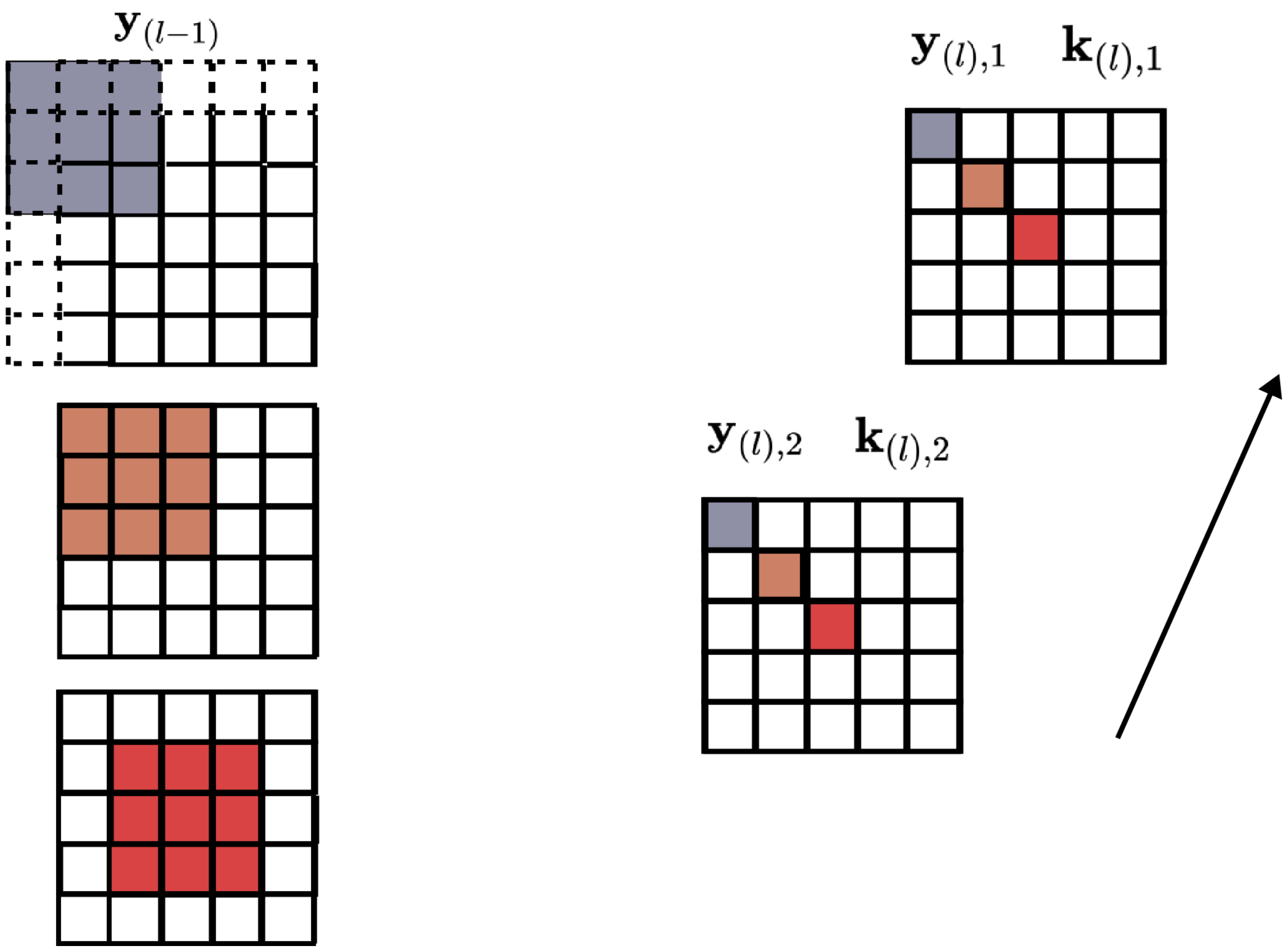
Conv 2D, single input, single output feature: padding at borderpadding at border



The above example was for a single feature.

Of course, we can (and it's common) to recognize multiple features ($n_{(l)} > 1$)

Conv 2D, single input, multiple output feature: padding at border



Dealing with multiple input features works similarly as for $N = 1$:

- The dot product
- Is over a spatial region that now has a "depth" $n_{(l-1)}$ equal to the number of input features
- Which means the kernel has a depth $n_{(l-1)}$

Conv 2D: multiple input features: kernel 1

$\mathbf{k}_{(l),1,1}$

$\mathbf{W}_{(l),1,1,1}$	$\mathbf{W}_{(l),1,2,1}$	$\mathbf{W}_{(l),1,3,1}$
$\mathbf{W}_{(l),2,1,1}$	$\mathbf{W}_{(l),2,2,1}$	$\mathbf{W}_{(l),2,3,1}$
$\mathbf{W}_{(l),3,1,1}$	$\mathbf{W}_{(l),3,2,1}$	$\mathbf{W}_{(l),3,3,1}$

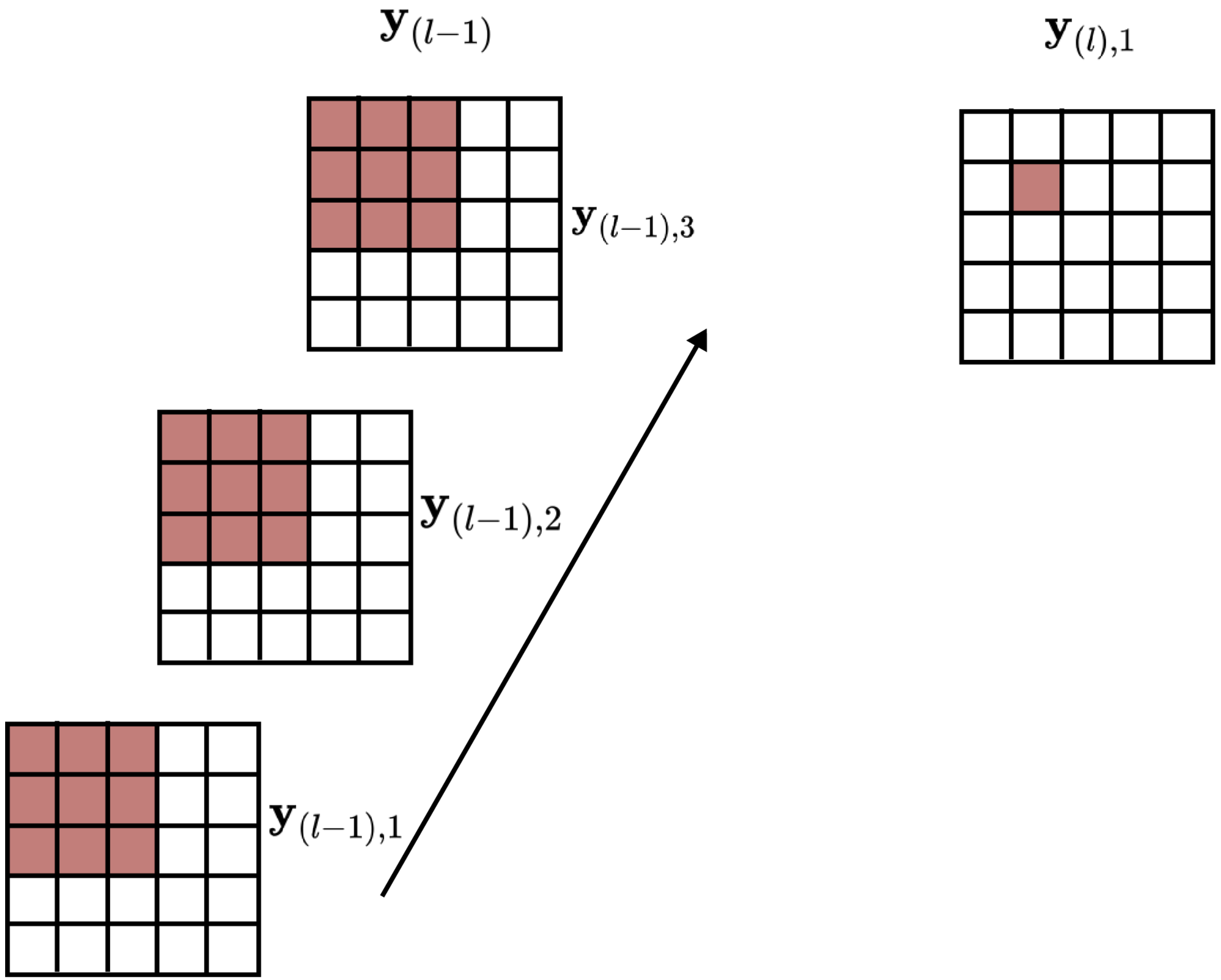
$\mathbf{k}_{(l),1,2}$

$\mathbf{W}_{(l),1,1,2}$	$\mathbf{W}_{(l),1,2,2}$	$\mathbf{W}_{(l),1,3,2}$
$\mathbf{W}_{(l),2,1,2}$	$\mathbf{W}_{(l),2,2,2}$	$\mathbf{W}_{(l),2,3,2}$
$\mathbf{W}_{(l),3,1,2}$	$\mathbf{W}_{(l),3,2,3}$	$\mathbf{W}_{(l),3,3,2}$

$\mathbf{k}_{(l),j,j'}$

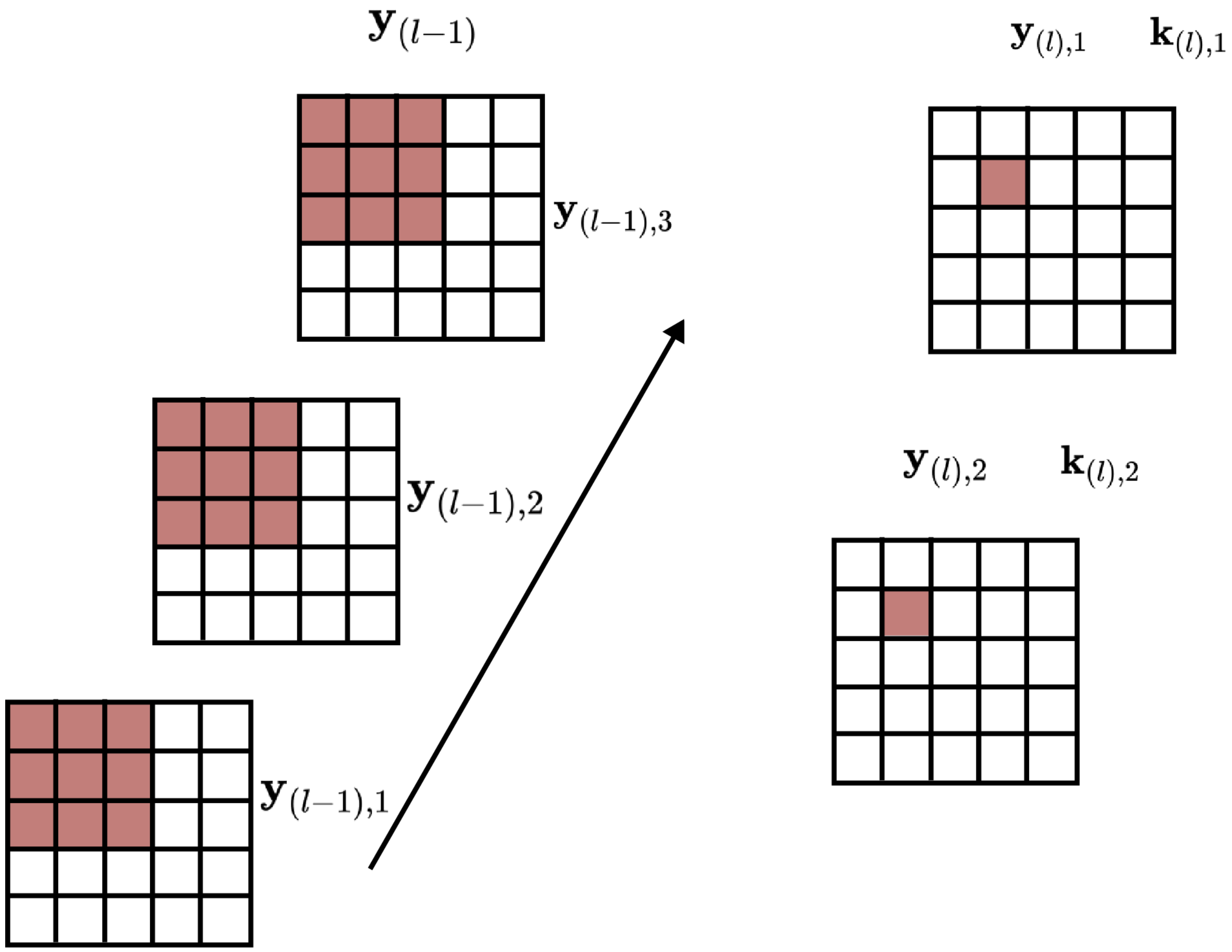
- layer l
- output feature j
- input feature j'

Conv 2D, multiple input, single output feature: padding at border



When we compute *multiple* feature maps, we get

Conv 2D, multiple input, multiple output features



Conv2d in action

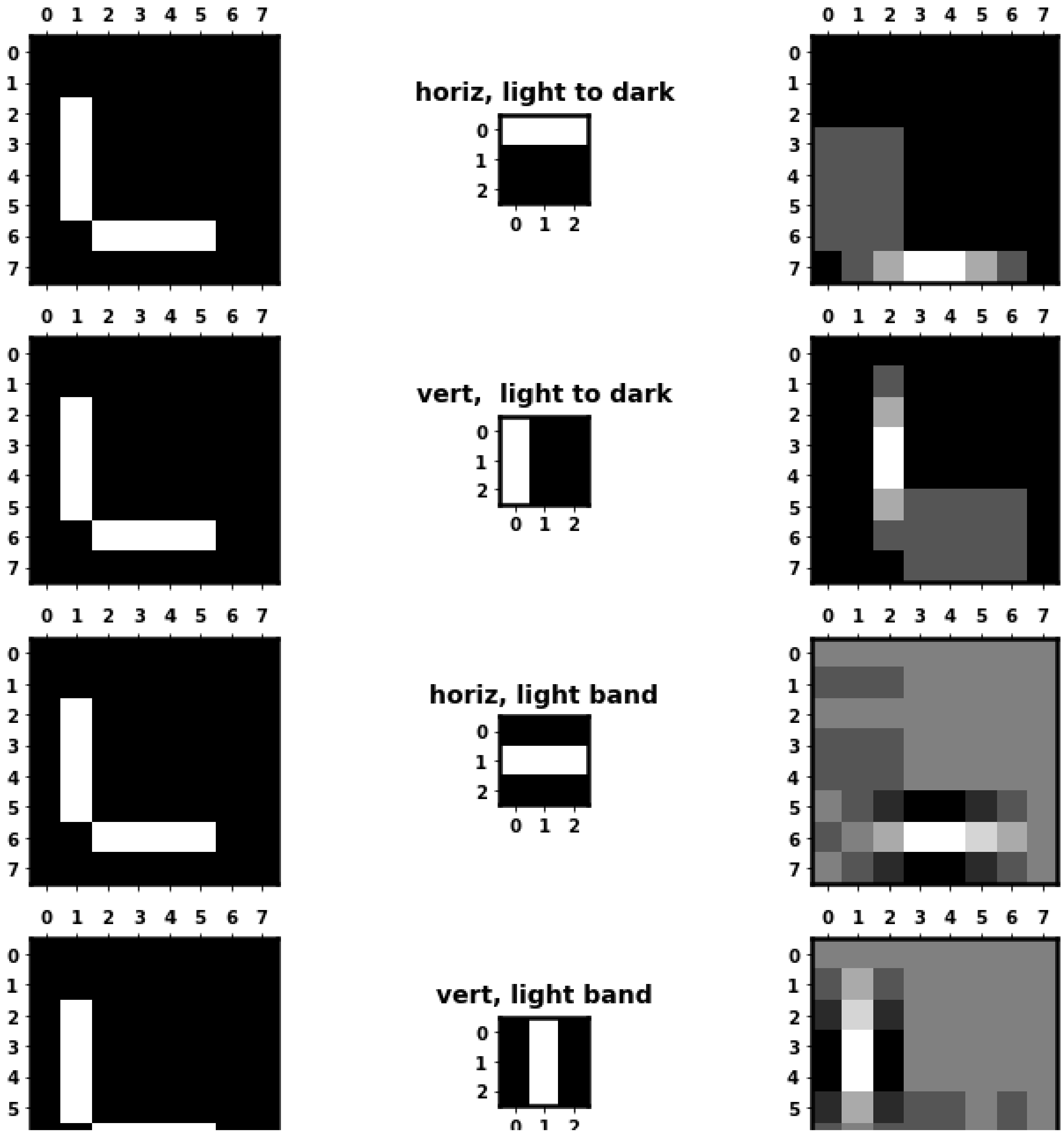
Pre-Deep Learning: manually specified filters have a rich history for image recognition.

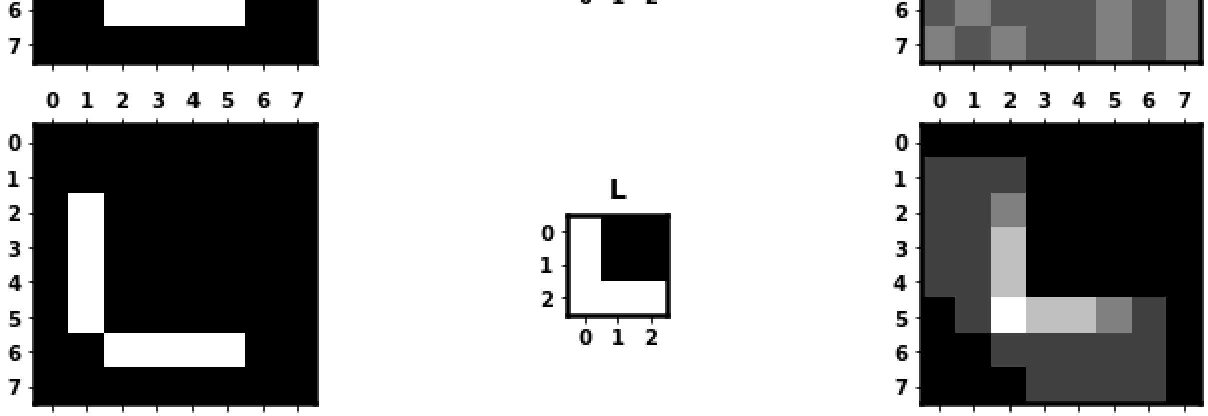
Here is a list of manually constructed kernels (templates) that have proven useful

- list of filter matrices (https://en.wikipedia.org/wiki/Kernel_(image_processing))

Let's see some in action to get a better intuition.

In [5]:
```
_= cnnh.plot_convs()
```




L

- A bright element in the output indicates a high, positive dot product
- A dark element in the output indicates a low (or highly negative) dot product

In our example

- $N = 2$: Two spatial dimensions
- One input feature: $n_{(l-1)} = 1$
- One output feature $n_{(l)} = 1$
- $f_{(l)} = 3$
    - Kernel is $(3 \times 3 \times 1)$.

The template match will be maximized when

- high values in the input correspond to high values in the matching location of the template
- low values in the input correspond to low values in the matching locations of the template

Training a CNN

Hopefully you understand how kernels are "feature recognizers".

But you may be wondering: how do we determine the weights in each kernel ?

Answer: a Convolutional Layer is "just another" layer in a multi-layer network

- The kernels are just weights (like the weights in Fully Connected layers)
- We solve for all the weights $\mathbf{W}$ in the multi-layer network in the same way

The answer is: exactly as we did in Classical Machine Learning

- Define a loss function that is parameterized by $\mathbf{W}$:
$$\mathcal{L} = L(\hat{\mathbf{y}}, \mathbf{y}; \mathbf{W})$$

- The kernel weights are just part of $\mathbf{W}$

- Our goal is to find $\mathbf{W}^*$ the "best" set of weights
$$\mathbf{W}^* = \underset{W}{\operatorname{argmin}} \, L(\hat{\mathbf{y}}, \mathbf{y}; \mathbf{W})$$
- Using Gradient Descent !

In other words: their is nothing special about finding the "best" kernels.

In [6]:

```
print("Done")
```

Done