From Math to Program

Neural Networks have the flavor of a Functional Program

- A Sequential Model computes the composition of per-layer functions
- Layer l is computing a function $\mathbf{y}_{(l)} = F_{(l)}$

$$F_{(l)}(\mathbf{y}_{(l-1)};\mathbf{W}_{(l)}) = \mathbf{y}_{(l)}$$

$$F_{(l)}: \mathcal{R}^{||\mathbf{y}_{(l-1)}||} \mapsto \mathcal{R}^{||\mathbf{y}_{(l)}||}$$

If we expand $F_{(l)}$, we see that it is the l-fold composition of functions $F_{(1)},\dots,F_{(l)}$

$$egin{array}{lll} \mathbf{y}_{(l)} &=& F_{(l)}(\mathbf{y}_{(l-1)}; \mathbf{W}_{(l)}) \ &=& F_{(l)}(\ F_{(l-1)}(\mathbf{y}_{(l-2)}; \ \mathbf{W}_{(l-1)}); \ \mathbf{W}_{(l)}\) \ &=& F_{(l)}(\ F_{(l-1)}(\ F_{(l-2)}(\mathbf{y}_{(l-3)}; \ \mathbf{W}_{(l-2)}); \ \mathbf{W}_{(l-1)}\); \mathbf{W}_{(l)}\) \ &=& dots \ &dots \ \end{array}$$

It turns out that it is not too difficult to endow a Neural Network with familiar *imperative* programming constructs

- if statement
- switch/case statement

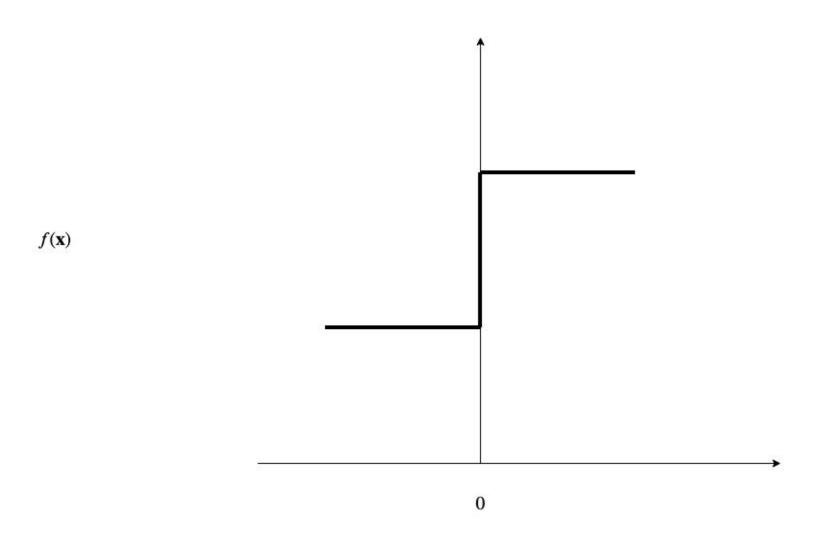
This is sometimes called *Neural Programming*.

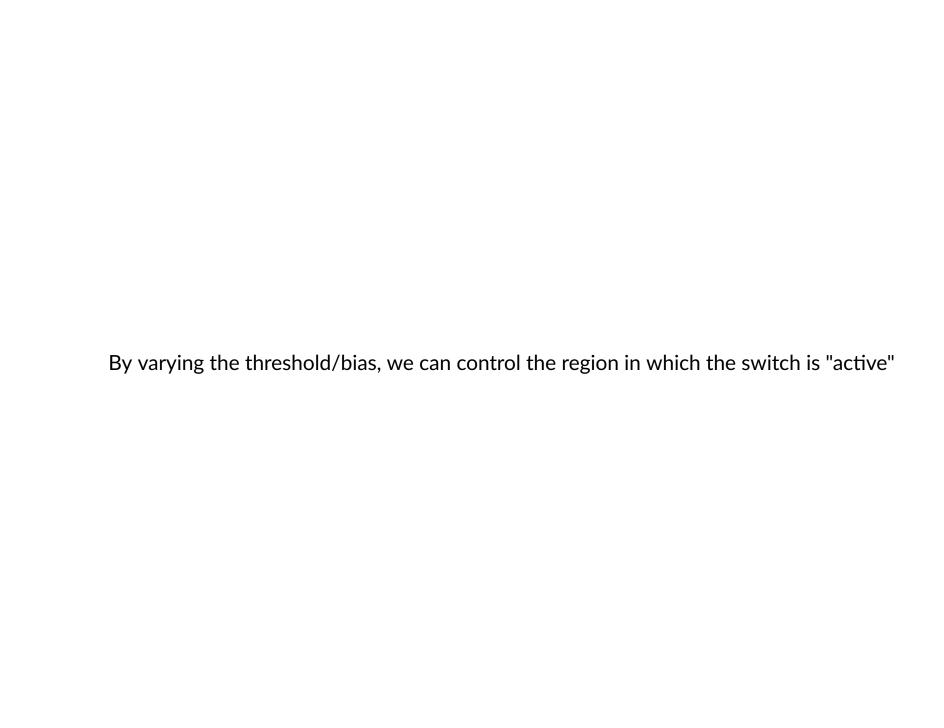
Although interesting in its own right, we introduce this topic as an introduction to more advanced recurrent layer types.

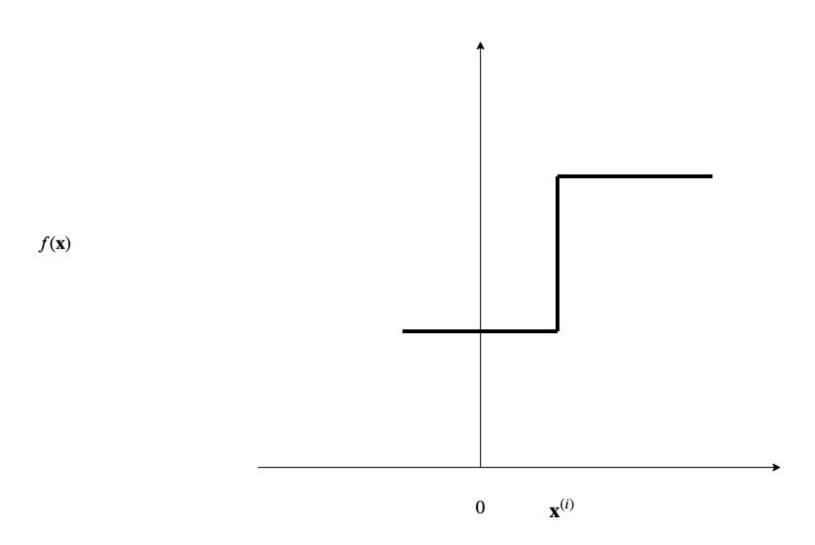
Binary switches

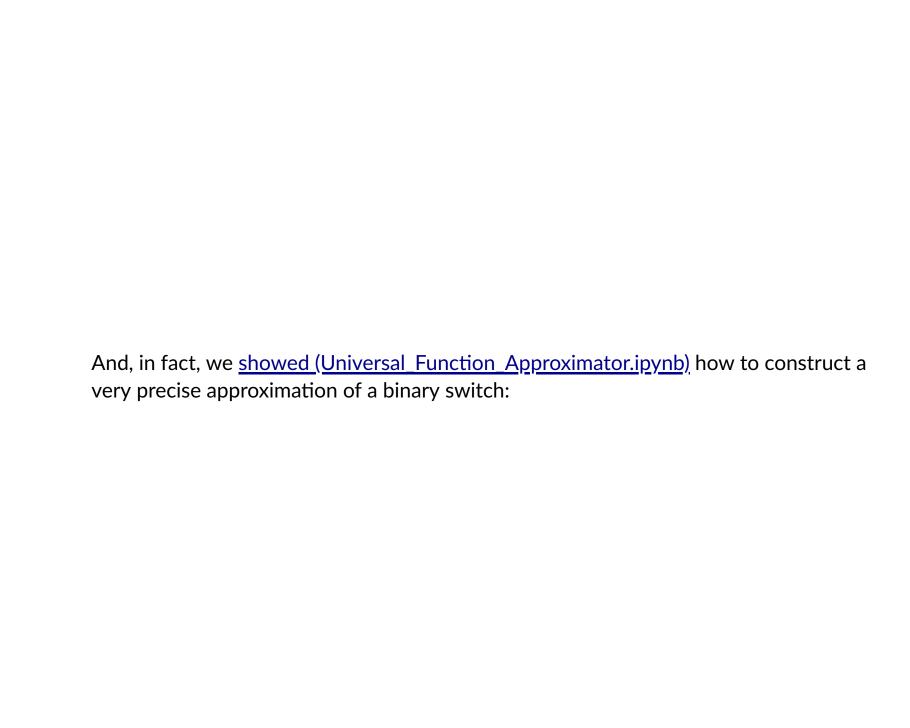
When we introduced Neural Networks, we argued that their power derived from the ability of Activation Functions

- To act like binary "switches"
- Converting the scalar value computed by the dot product
- Into a True/False answer
- To the question: "Is a particular feature present"?

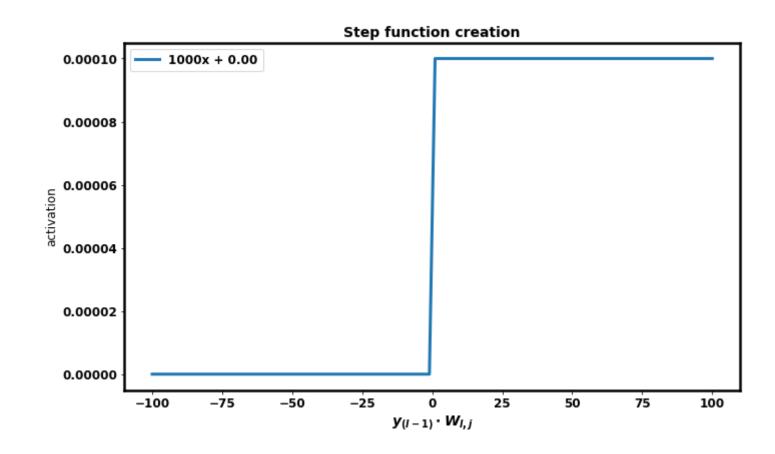








In [5]: fig, ax = nnh.step_fn_plot()



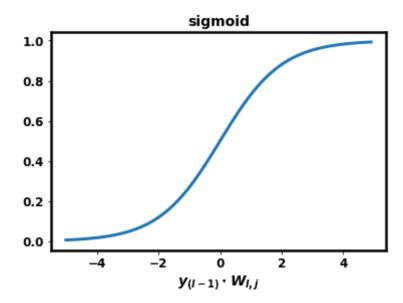
Neurons as statements

With the ability to implement a binary switch

- We can construct Neural Networks
- With elements that look like primitive statements of a programming language

Rather than building a true step function	
$ullet$ We will settle for the approximation offered by the Sigmoid function σ	

```
In [6]: _= nnh.sigmoid_fn_plot()
```



This is more than laziness or convenience

- The step function is **not** differentiable
- The sigmoid function is differentiable

Recall that Gradient Descent is the tool we use to train Neural Networks

• Hence it is important that our functions be differentiable!

"If" statements - Gates

Suppose we want a Neural Network to

- Compute a (vector) output ${f y}$
- $\bullet\,$ That takes on vector value T if some condition g is True
- ullet And F otherwise.

This would be trivial in any programming language having an if statement:

```
if (g):
    y = T
else:
    y = F
```

Let's show how to construct the if statement with just a little arithmetic.

Suppose scalar $g \in \{0,1\}$ was the value output by a switch.

Then

$$\mathbf{y} = (g * \mathbf{T}) + (1 - g) * \mathbf{F}$$

does the trick.

In general, we tend to compute vectors rather than scalars.

Let

- $oldsymbol{\cdot}$ $oldsymbol{g}, oldsymbol{y}$ be vectors of equal length
- f T, F be vectors of equal length (not necessarily the same as f g, y)
 - ullet So elements of $oldsymbol{y}$ have length $||oldsymbol{T}||=||oldsymbol{F}||$

We will construct a "vector" if statement

 \bullet Making a conditional choice for each element of y, independently.

$$\mathbf{y}_j = (\mathbf{g}_j * \mathbf{T}) + (1 - \mathbf{g}_j) * \tilde{\mathbf{F}}$$

Letting

- \otimes denote element-wise vector multiplication (*Hadamard product*)
- ullet $\sigma(\ldots)$ be a sigmoid approximation of a binary switch

The following product (almost) does the trick

$$egin{aligned} \mathbf{g} &= \sigma(\ldots) \ \mathbf{y} &= \mathbf{g} \otimes \mathbf{T} + (1-\mathbf{g}) \otimes \mathbf{F} \end{aligned}$$

It is only "almost"

- ullet Because the sigmoid only takes a value in the range [0,1]
- $\bullet\,$ Rather than exactly either 0 or $1\,$

What we have is

- A continuous (soft) decision **g**.
- That creates a vector if
- ullet Whose elements are mixtures of ${f T}$ and ${f F}$

This is the price we pay for having ${f g}$ be differentiable!

Note that the individual elements of vector \mathbf{y} are independent

- ullet \mathbf{y}_j is influenced only by \mathbf{g}_j
- \bullet The synthetic features represented by ${\bf y}$ are not dependent on one another.
- Most importantly: the derivatives of each feature are independent

"Switch/Case" statements

We can easily generalize from a two-case if to a switch/case statement with $||\mathbf{C}||$ cases.

Suppose we need to set ${f y}$ to one value from among multiple choices in ${f C}$

$$\mathbf{g} = \operatorname{softmax}(\ldots)$$

 $\mathbf{y} = \mathbf{g} \otimes \mathbf{C}$

The *softmax* function

- Was introduced in Multinomial Classification
- ullet Computes a vector (of length ||C||) values
- \bullet With each element being in the range [0,1]
- $\bullet \ \ \text{And summing to} \ 1$

We refer to \mathbf{g} as a mask for \mathbf{C} .

The if statement is a special case of the switch/case statement where

$$\mathbf{C} = egin{bmatrix} \mathbf{T} \ \mathbf{F} \end{bmatrix}$$

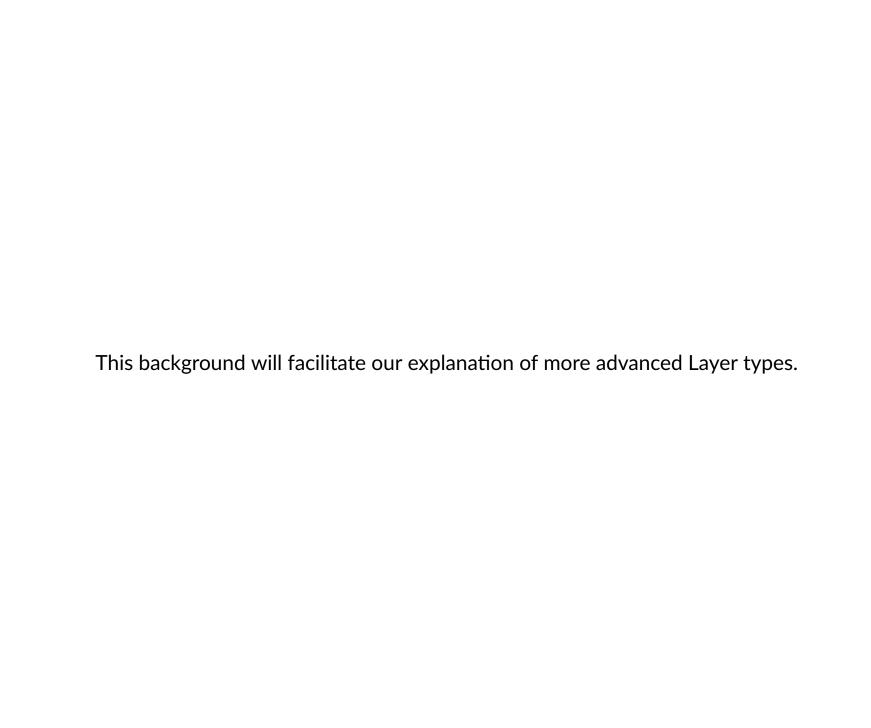
Conclusion

We wanted to show that, in concept

- We could create the logic of a simple imperative program
- Using the machinery of Neural Networks

The only catch was

- We cannot use true binary logic (hard decisions)
- All choices are soft
- In order to preserve differentiability
- Which is necessary for training with Gradient Descent



```
In [7]: print("Done")
```

Done