Black Scholes Option Pricing Model

1 Introduction

The Black-Scholes formula is often used in the finance sector to estimate option prices. To understand Black-Scholes formula and its derivation, we need to introduce some relevant concepts in finance.

An **option** is a security that gives the right to buy or sell an asset within a specified period of time. A **call option** is the kind of option that gives the right, but not the obligation to **buy** a single share of common stock. A **put option** is the kind of option that gives the right, but not the obligation to **sell** a single share of common stock. The **striking price** is the price that is paid for the asset when the option is exercised, i.e it is the price at which the stock is bought when the call option is exercised, or it is the price at which the stock is sold when the put option is exercised.

The price of a stock option is a function of the underlying stock's price and time. More generally, we can say that the price of any option is a function of the stochastic variables underlying the derivative and time.

2 Ito's Lemma and the Black Scholes equation

Stock prices follow a **Wiener process**, also called **Geometric Brownian Motion**. A process $\{X(t)\}_{t=0}^{\infty}$ is a Wiener process with mean μt (called drift) and variance $\sigma^2 t$ and has the following properties-
1) $\{X(t)\}$ has independent incremements.
2)Every increment $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$.

It is a continuous time continuous state Markov process. **Standard Brownian Motion** has drift parameter $\mu = 0$ and $\sigma^2 = t$, it is denoted as $B(t)$ where $B(0) = 0$. Standard Brownian Motion follows normal distribution with mean 0 and variance t. Moreover, any Wiener process $X(t)$ can be written as

$$X(t) = \mu t + X(0) + \sigma B(t)$$

where B(t) is standard Brownian motion.

Geometric Brownian motion is used to model stock prices in the Black–Scholes model and is the most widely used model of stock price behavior. It is used because of several reasons-
1)The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.
2) A GBM process only assumes positive values, just like real stock prices.
3) A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.

Calculations with GBM processes are relatively easy. However, GBM is not a completely realistic model, in particular it falls short of reality in the following ways-
1) In real stock prices, volatility changes over time (possibly stochastically), but in GBM, volatility is assumed constant.
2) In real life, stock prices often show jumps caused by unpredictable events or news, but in GBM, the path is continuous (no discontinuity).

An important result required to derive the Black-Scholes equation for option pricing is **Ito's Lemma**. An **Ito process** is a Wiener process where the parameters μ and σ depend on the underlying variable x and time t. Thus, an Ito process can be written as

$$X(t) = \mu(x,t)t + \sigma(x,t)B(t)$$

We can also write the differential form of the above equation-

$$dX(t) = \mu(x,t)dt + \sigma(x,t)dB(t)$$

An alternative form of Ito's Lemma is stated and used to derive the differential equation which gives us the Black-Scholes formula.

Ito's Lemma: *If X(t) is an Ito process satisfying the stochastic differential equation*

$$dX(t) = \mu(x,t)dt + \sigma(x,t)dB(t)$$

If $B(t)$ is a standard Brownian motion and f is a C^2 function, then $f(t, X(t))$ is also an Ito process with its differential given by

$$df(t,X(t)) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X(t)}\mu(x,t) + 1/2\frac{\partial^2 f}{\partial^2 X(t)}\sigma^2(x,t)\right]dt + \frac{\partial f}{\partial X(t)}\sigma(x,t)dB(t)$$

Now, the Black Scholes equation can be derived if we consider ϕ units of stock and ψ units of cash in a particular portfolio. If the amount of shares at time t is denoted as ϕ_t and the amount of cash as ψ_t , then the value of the portfolio at time t is-

$$V_t = \phi_t S_t + \psi_t r P dt$$

here r is the risk-free interest rate and P is the profit. Now, Ito's lemma can be applied. The partial derivatives of V_t are-

$$\frac{\partial V_t}{\partial t} = \psi_t r P dt$$

$$\frac{\partial V_t}{\partial S_t} = \phi_t$$

$$\frac{\partial^2 V_t}{\partial^2 S_t} = 0$$

Comparing the co-efficients with the terms of Ito's lemma and putting $V_t = f$ (considering it to be an Ito process) , we obtain

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} r S_t + 1/2\sigma^2 \frac{\partial^2 f}{\partial^2 S_t} = rf$$

The solution of the above differential equation gives us the Black-Scholes equations-

$$C_o = S_o F(d_1) - X e^{-rT} F(d_2)$$
$$d_1 = \left(ln(S_o/X) + (r + \sigma^2/2)T \right) \Big/ \sigma\sqrt{T}$$
$$d_2 = \left(ln(S_o/X) + (r - \sigma^2/2)T \right) \Big/ \sigma\sqrt{T}$$

Here, $F(x)$ is the cumulative function of the normally distributed random variables d_1 and d_2. C_o is the price of the option as calculated by the formula. S_o is the spot price, X is the striking price, T is the time left until expiry of the option, r is the risk-free interest rate and σ^2 is the yearly volatility.

3 Limitations and Applications

The Black-Scholes equation is obtained as a solution of the Black-Scholes differential equation, with the boundary condition that $f = max\{S_t - X, 0\}$ which is the return on the stock. The equation also has the following assumptions-

a) The short-term interest rate is known and is constant through time.

b) The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any fortnite interval is log-normal. The variance rate of the return on the stock is constant.

c) The stock pays no dividends or other distributions.

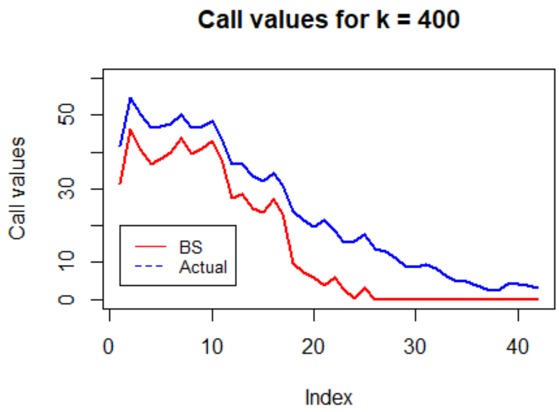
d) The option is "European," that is, it can only be exercised at maturity.

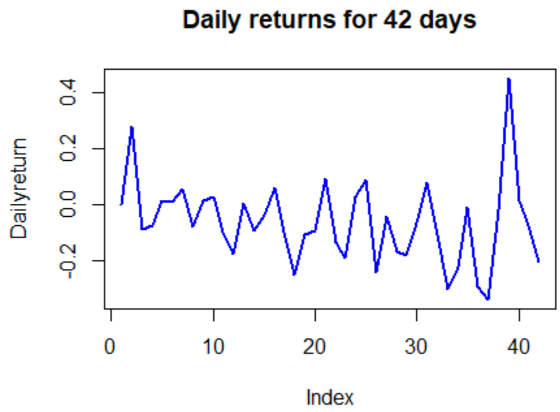
e) There are no transaction costs in buying or selling the stock or the option.

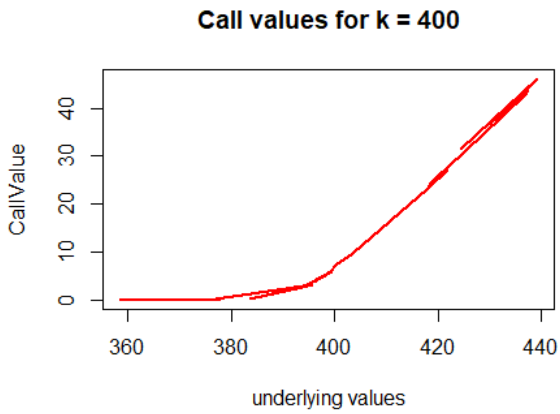
f) It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.

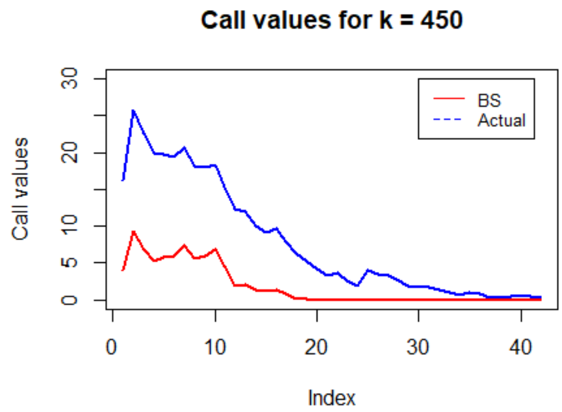
g) There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

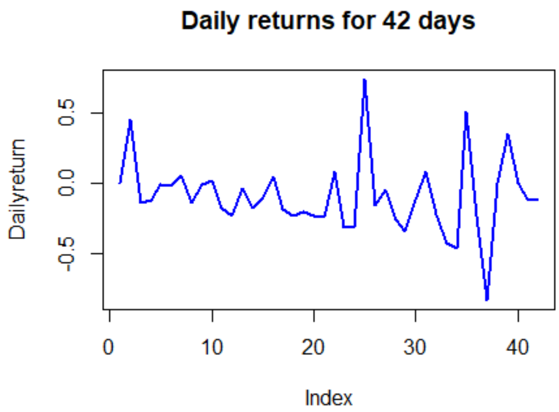
The R code which utilised the Black-Scholes equation for calculating call option prices for the Tata Motors stock prices gave the outputs as depicted in the following figures. The data is for the period of 1 Jan- 1 March 2018, with options expiring on 28 March 2018. The National Stock Exchange (NSE) website was used to obtain the historical data. It was calculated for three different values of striking price (k).

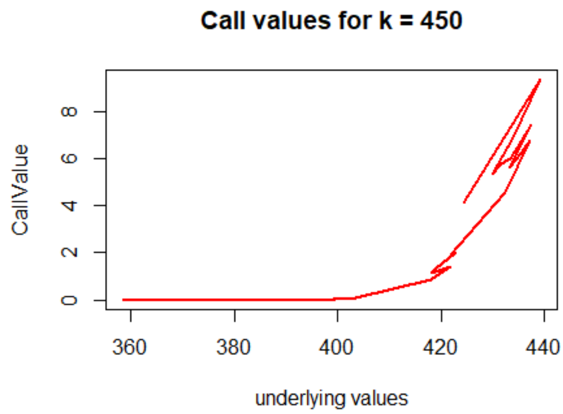

Call values for k = 400
Call values
Index
BS
Actual

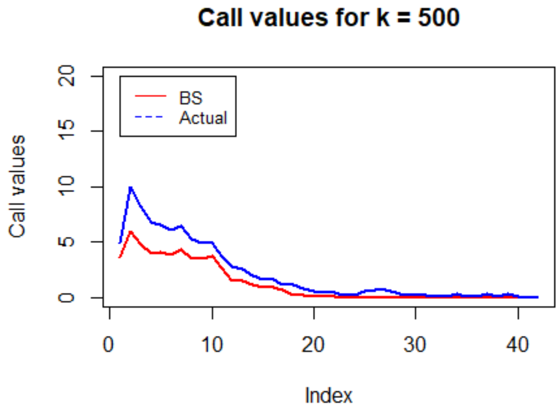

Daily returns for 42 days
Dailyreturn
Index

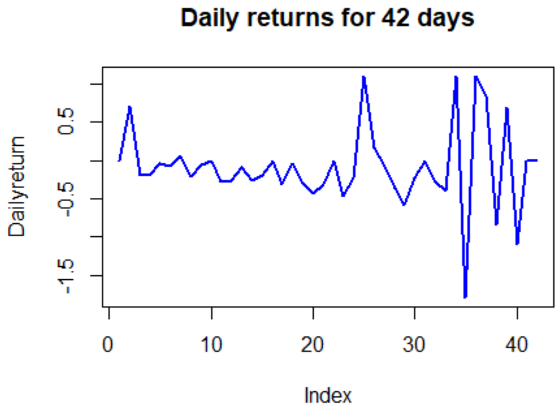

Call values for k = 400
CallValue
underlying values

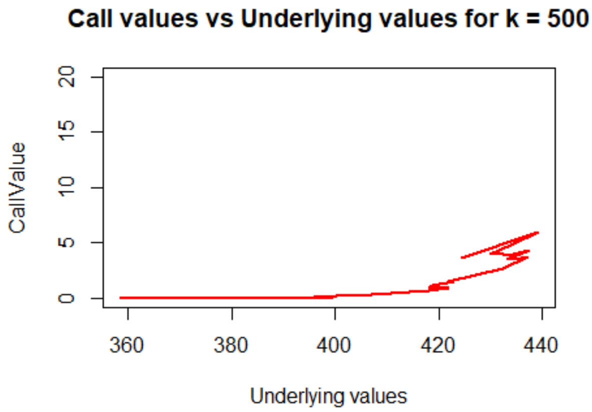

Call values for k = 450
Call values
Index
BS
Actual


Daily returns for 42 days
Dailyreturn
Index


Call values for k = 450
CallValue
underlying values







4 References

1) Hull, John C. *"Options, Futures and Other Derivatives"* , Seventh Edition.
2) Yoo, Younggeun *"Stochastic Calculus and Black-Scholes Model"*