Kinetic theory of plasma.

Vlasov equation.

([4], p.397-407)

The most complete description of plasma properties is provided by the kinetic equation. Each of the species of particles is described by the distribution function:

$$f = f(\vec{r}, \vec{v}, t)$$

The distribution function is defined in such a way that

$$f(\vec{r}, \vec{v}, t) d^3\vec{v} d^3\vec{r}$$

is the number of particles in a phase-space elementary cell $d^3\vec{v} d^3\vec{r}$. The distribution function is particle density in the **6-dimensional phase space**:

$$\vec{R}_6 = [x, y, z, v_x, v_y, v_z],$$

$$\vec{V}_6 = [\dot{x}, \dot{y}, \dot{z}, \dot{v}_x, \dot{v}_y, \dot{v}_z]$$

The total number of particles is:

$$N = \int_{-\infty}^{\infty} d^3\vec{r} \int_{-\infty}^{\infty} d^3\vec{v} f(\vec{r}, \vec{v}, t)$$

Particle density (number of particles in unit volume) is:

$$n(\vec{r}, t) = \int_{-\infty}^{\infty} d^3\vec{v} f(\vec{r}, \vec{v}, t)$$

The particle flux density (first moment of f):

$$\vec{j} = \int_{-\infty}^{\infty} d^3\vec{v} f(\vec{r}, \vec{v}, t)\vec{v}.$$

Example. Maxwellian distribution:

$$f_M = n \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m\vec{v}^2}{2T}\right)$$

The continuity equation for the distribution function (particle density in the 6-dimensional space):

$$\frac{\partial f}{\partial t} + \mathrm{div}_6(\vec{V}_6 f) = 0$$

where

$$\mathrm{div}_6(\vec{V}_6 f) = \frac{\partial}{\partial x}(\dot{x} f) + \frac{\partial}{\partial y}(\dot{y} f) + \frac{\partial}{\partial z}(\dot{z} f) +$$

$$+\frac{\partial}{\partial v_x}(\dot{v}_x f) + \frac{\partial}{\partial v_y}(\dot{v}_y f) + \frac{\partial}{\partial v_z}(\dot{v}_z f)$$

Using

$$\dot{\vec{r}} = \vec{v}$$

$$\dot{\vec{v}} = \frac{e}{m}\left(\vec{E}(\vec{r},t) + \frac{1}{c}[\vec{v}\times\vec{B}(\vec{r},t)]\right)$$

we proof that

$$\mathrm{div}_6(\vec{V}_6) = 0.$$

Indeed,

$$\mathrm{div}_6(\vec{V}_6) = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{v}_x}{\partial v_x} + \frac{\partial \dot{v}_y}{\partial v_y} + \frac{\partial \dot{v}_z}{\partial v_z}$$

where the first three terms vanish because $\vec{r}$ and $\vec{v} = \dot{\vec{r}}$ are independent variables, and the other three are calculated using the momentum equation. It is easy to check that

$$\frac{\partial}{\partial \vec{v}}[\vec{v}\times\vec{B}] = 0$$

$$[\vec{v}\times\vec{B}] = \vec{i}(v_y B_z - V_z B_y) + \vec{j}(v_z B_x - v_x B_z) + \vec{k}(v_x B_y - v_y B_x)$$

$$\frac{\partial}{\partial v_x}(v_y B_z - v_z B_y) = 0 \quad ...$$

Thus,

$$\mathrm{div}_6(\vec{V}_6) = 0.$$

Then,

$$\mathrm{div}_6(\vec{V}_6 f) = \vec{V}_6 \nabla_6 f = \vec{v} \nabla f + \dot{\vec{v}} \frac{\partial f}{\partial \vec{v}}$$

Finally, the continuity equation

$$\frac{\partial f}{\partial t} + \mathrm{div}_6(\vec{V}_6 f) = 0$$

is written as

$$\frac{\partial f}{\partial t} + \vec{v} \nabla f + \dot{\vec{v}} \frac{\partial f}{\partial \vec{v}} = 0$$

or for particles in electric and magnetic fields:

$$\frac{\partial f}{\partial t} + \vec{v} \nabla f + \frac{e}{m}(\vec{E} + \frac{1}{c}[\vec{v} \times \vec{B}]) \frac{\partial f}{\partial \vec{v}} = 0$$

The electric and magnetic fields include both external fields and fields generated by plasma particles, averaged over a macroscopic volume. These fields can be determined from Maxwell equations:

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{j}$$

$$\nabla \vec{B} = 0 \quad \nabla \vec{E} = 4\pi q,$$

where

$$q = \sum_a q_a \int f_a d^3\vec{v}$$

$$\vec{j} = \sum_a q_a \int \vec{v} f_a d^3\vec{v}$$

The kinetic equation with the self-consistent fields is called **Vlasov equation**.

The physical interpretation of the Vlasov equation.

$$\frac{\partial f}{\partial t} = -\dot{\vec{r}}\frac{\partial f}{\partial \vec{r}} - \dot{\vec{v}}\frac{\partial f}{\partial \vec{v}}$$

means that the distribution function at a point in the phase space can change because particles may come from other locations and because of their velocity change.

Consider the rate of the distribution function in the coordinate system moving with a particle (full derivative):

$$\frac{df}{dt} = [f(\vec{r} + d\vec{r}, \vec{v} + d\vec{v}, d + dt) - f(\vec{r}, \vec{v}, t)]/dt =$$

$$= [dt\frac{\partial f}{\partial t} + d\vec{r}\nabla f + d\vec{v}\frac{\partial f}{\partial \vec{v}}]/dt =$$

$$= \frac{\partial f}{\partial t} + \dot{\vec{r}} \nabla f + \dot{\vec{v}} \frac{\partial f}{\partial \vec{v}} = 0.$$

The condition

$$\frac{df}{dt} = 0$$

means that the particle flux in the phase space is incompressible. This is a direct consequence of $\mathrm{div}_6 \vec{V}_6 = 0$, and has an analogy with the hydrodynamic continuity equation: $\mathrm{div}\vec{u} = 0$ for incompressible fluid.

However, collisions may cause sudden changes of velocity. Hence, we have to add a collision term into the right-hand side:

$$\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{r}} + \dot{\vec{v}} \frac{\partial f}{\partial \vec{v}} = \left(\frac{df}{dt} \right)_{\mathrm{coll}}$$

Illustration of the Vlasov equation: thermal effects of Langmuir waves.

Langmuir (plasma) waves are oscillations of electrons relative to ions. We can consider plasma waves in terms of a small perturbation, f_1 to an equilibrium spatially uniform distribution function,

$f_0(\vec{v})$. In 1-D case, the Vlasov equation for electrons ($q = -e$) is:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{\partial f}{\partial v} = 0$$

We write

$$f(x, v, t) = f_0(v) + f_1(x, v, t)$$

and assume that $f_1 \ll f_0$. We assume that the wave amplitude is small, so they produce only a weak electric field, and write a linearized Vlasov equation:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m}E\frac{\partial f_0}{\partial v} = 0$$

The Maxwell equation for the electric field:

$$\nabla\vec{E} = 4\pi q = -4\pi e\int f_1 d^3\vec{v}$$

in the 1-D case is:

$$\frac{\partial E}{\partial x} = -4\pi e\int f_1 dv$$

Now, we look for a wave-like solution for f_1 and E_1:

$$f_1(x, v, t) = \hat{f}_1(v)\exp\left(-i\omega t + ikx\right)$$

$$E_1(x,v,t) = \hat{E}_1(v)\exp(-i\omega t + ikx)$$

We get:

$$-i(\omega - kv)\hat{f}_1 = \frac{e}{m}\hat{E}\frac{\partial f_0}{\partial v}$$

$$\hat{f}_1 = \frac{ie\hat{E}}{m}\frac{\partial f_0/\partial v}{\omega - kv}$$

$$ik\hat{E} = -4\pi e\int \hat{f}_1 dv = -\frac{4\pi i e^2\hat{E}}{m}\int \frac{\frac{\partial f_0}{\partial v}}{\omega - kv}dv$$

This equation has a non-zero solution only if

$$D(k,\omega) = 1 + \frac{4\pi e^2}{mk}\int_{-\infty}^{\infty}\frac{\frac{\partial f_0}{\partial v}}{\omega - kv}dv = 0$$

This is the dispersion relation for plasma waves.

Consider the case of high-frequency waves: $\omega \gg kv$:

$$\frac{1}{\omega - kv} = \frac{1}{\omega} + \frac{kv}{\omega^2} + \frac{k^2v^2}{\omega^3} + \frac{k^3v^3}{\omega^4} + \ldots$$

For a Maxwellian distribution the integrals can be calculated analytically:

$$\int_{-\infty}^{\infty}\frac{\partial f_0}{\partial v}dv = 0$$

($\partial f_0/\partial v$ is an odd function)

$$\int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} v dv = -n$$

(use integration by parts)

$$\int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} v^2 dv = 0$$

$$\int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} v^3 dv = -3n v_T^2,$$

where $v_T = (T/m)^{1/2}$.

Hence, the dispersion relation is:

$$1 + \frac{4\pi e^2}{mk} \left(-\frac{kn}{\omega^2} - \frac{3n v_T^2 k^3}{\omega^4} \right) = 0$$

$$1 - \frac{4\pi e^2 n}{m\omega^2} \left(1 + \frac{3 v_T^2 k^2}{\omega^2} \right) = 0$$

Note that

$$\frac{4\pi e^2 n}{m} = \omega_p^2$$

is the plasma frequency.

$$\frac{v_T}{\omega_p} = \lambda_D$$

is the Debye length. The dispersion relation:

$$1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3v_T^2 k^2}{\omega^2}\right) = 0$$

or

$$\omega^2 = \omega_p^2 + 3v_T^2 k^2 \equiv \omega_p^2[1 + 3(\lambda_D k)^2]$$

is so-called Bohm-Gross dispersion relation.

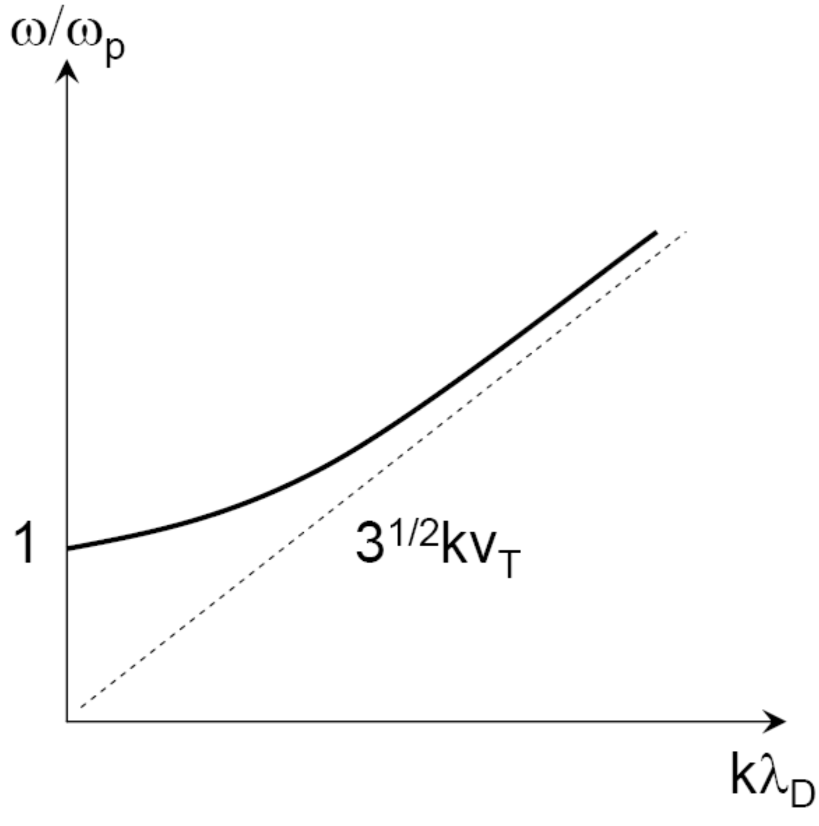

Figure 1: Bohm-Gross dispersion relation for plasma waves.

The group velocity of plasma waves in finite-temperature plasma is non-zero. Thus, the plasma waves can transfer energy. For our previous dispersion relation $\omega = \omega_p$ the group velocity is zero.

Two-steam instability

When initial velocity distribution f_0 departs from Maxwellian plasma waves may become unstable.

We consider two oppositely directed streams of electrons with velocities $\pm v_0$. Assume that temperature is zero. Then the distribution function is a sum of two δ-functions:

$$f_0(v) = \frac{1}{2} n[\delta(v - v_0) + \delta(v + v_0)],$$

where n is the total electron density of the two streams.

calculate the integral of the dispersion relation using integration by parts:

$$\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{\omega - kv} dv =$$

$$= -\int_{-\infty}^{\infty} f_0 \frac{\partial}{\partial v}\left(\frac{1}{\omega - kv}\right) dv + \left[\frac{f_0}{\omega - kv}\right]_{-\infty}^{\infty} =$$

$$= -k \int_{-\infty}^{\infty} \frac{f_0}{(\omega - kv)^2} dv =$$

$$= -\frac{kn}{2}\left(\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right)$$

Hence,

$$D(k,\omega) = 1 - \frac{1}{2}\left(\frac{\omega_p^2}{(\omega - kv_0)^2} + \frac{\omega_p^2}{(\omega + kv_0)^2}\right) = 0$$

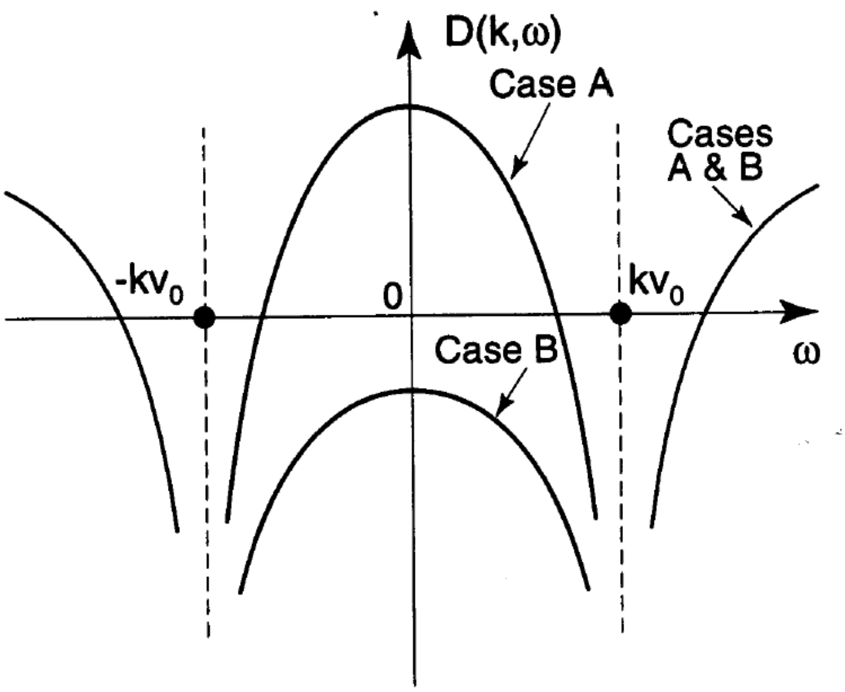

Figure 2: The dispersion relation for the two-steam instability.

This is a quadratic equation for ω^2. It has four roots for ω, real or complex. The complex roots are in case B, when $D(k,0) < 0$.

$$D(k,0) = 1 - \frac{\omega_p^2}{k^2 v_0^2} < 0.$$

Hence, the instability condition is:

$$v_0 > \omega_p / k$$

(stream velocity is higher than the phase speed of plasma waves).

This is the **two-stream instability**. It prevents two oppositely directed beams of electrons from passing each other, even if the electrons are neutralized by a uniform background of ions. The instability produces spatial inhomogeneities, in which electrons are bunched together, and their energy is transformed into the energy of plasma waves.

Ion-acoustic waves

When both electrons and ions are allowed to oscillate a new type of waves (**ion-acoustic waves**) appears. The dispersion relation has two contributions for electrons and ions:

$$D(k,\omega) = 1 + \sum_{j=e,i} \frac{4\pi q_j^2}{m_j k} \int \frac{\partial f_{0,j}/\partial v}{\omega - kv} dv = 0$$

Now, we consider the case when the wave frequency is relatively small for electrons (this is opposite to the previous case of Langmuir waves), but it is still high for ions:

$$kv_{T_i} \ll \omega \ll kv_{T_e}$$

Then for ions, the calculations are similar to the Langmuir waves:

$$\frac{1}{\omega - kv} = \frac{1}{\omega} + \frac{kv}{\omega^2} + \frac{k^2v^2}{\omega^3} + \frac{k^3v^3}{\omega^4} + \ldots$$

$$\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{\omega - kv} dv = -\frac{nk}{\omega^2} - \frac{3nk^3 v_{T_i}^2}{\omega^4}$$

For electrons, we get:

$$\frac{1}{\omega - kv} \approx -\frac{1}{kv}$$

$$\frac{\partial f_0}{\partial v} = -\frac{v f_0}{v_{T_e}^2}$$

(for a Maxwellian distribution)

$$\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{\omega - kv} dv \approx \frac{n}{k v_{T_e}^2}.$$

Hence, the dispersion relation is:

$$1 + \frac{4\pi e^2 n}{m k^2 v_{T_e}^2} - \frac{4\pi e^2 n}{M\omega^2}\left(1 + \frac{3k^2 v_{T_i}^2}{\omega^2}\right)$$

or

$$1 + \frac{\omega_p^2}{k^2 v_{T_e}^2} - \frac{\Omega_p^2}{\omega^2}\left(1 + \frac{3k^2 v_{T_i}^2}{\omega^2}\right) = 0,$$

where $\Omega_p = 4\pi e^2 n/M$ is the ion plasma frequency.

If the wavelength is much longer than the Debye wavelength:

$$k\lambda_D = \frac{k v_{T_e}}{\omega_p} \ll 1$$

then we can neglect the first term (1) and solve the dispersion equation for $\omega/k \equiv x$:

$$x^2 - \frac{\Omega_p}{\omega_p} v_{T_e}^2 \left(1 + \frac{3v_{T_i}^2}{x^2}\right) = 0$$

or

$$x^4 - c_S^2 x^2 - 3v_{T_i}^2 c_S^2 = 0$$

where

$$c_S = \frac{\Omega_p}{\omega_p} v_{T_e} = \sqrt{\frac{T_e}{M}}$$

For our case, $kv_{T_i} \ll \omega$, and hence, $x \ll v_{T_i}$:

$$x^2 \approx c_S^2 \left(1 + \frac{3v_{T_i}^2}{c_S^2}\right)$$

or

$$\left(\frac{\omega}{k}\right)^2 = \frac{T_e + 3T_i}{M} \equiv C_S^2$$

$$C_S = \sqrt{\frac{T_e + 3T_i}{M}}$$

is called **ion-acoustic speed**.

Thus, in the long-wavelength limit $(1/k \gg \lambda_D)$ the ion-acoustic waves are very similar to normal sound waves. In this case, both electrons and ions contribute to pressure, but ions provide all the inertia. However, often in plasma $T_e \gg T_i$ because of the difference in the collision times, and then electrons contribute most pressure. The phase and group velocities are constant.

In the short-wavelength limit (shorter than the Debye wavelength), the waves become constant frequency ion plasma waves: $\omega = \Omega_p$.