Lec 2

Steven Strogatz - uses graphical way to explain dynamical systems
Relevant for robots, learning theory
Long-term behavior of complicated robotics system, gonna need dynamics

Way people optimize performance of gradient descent algorithms is very similar to actuation optimization tools

Simple pendulum (all robots are connected pendulums)

walking systems (double pendulum is almost like one of our simplest walking models)

Kinetic energy

$T = \frac{1}{2} m l^2 \dot{\theta}^2 \quad (\frac{1}{2} m v^2)$

Potential

$U = -mgl\cos\theta$

lagrangian mechanics

$ml^2\ddot{\theta} + mgl\sin\theta = Q$ ← generalized force (torque around joint in this case)

$Q =$ simplest friction model around joint (linear damping) + additional torque $\quad (Q = -b\dot{\theta} + u)$

adds non-linearity

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$$

given $\dot{\theta}, \theta, u \to$ get $\ddot{\theta}$

$\dot{\theta} = f(\theta, \dot{\theta}, u)$

Given: $\theta(0), \dot{\theta}(0)$ (I.C.)

You tell me: $\theta(t)$ with diff eq

But can't solve nonlinear diff eq (no close form expression)

→ no damping still get elliptic integrals

w/ damping, get nothing

(can get numerical approximation w/ simulator?

with numerical soln

Another Approach

(control theory hard bc long term consequence of action)

can't answer "where at time t". There are other analytically precise questions to ask

Where at time ∞

→ easier questions

If it starts at some place will it visit other place

time bad, other variables can be determined precisely

Long-term behavior, stability (lim $t \to \infty$, easier than details to get there)

What is $\lim_{t\to\infty} \theta(t)$?

Will robot fall down?

Lec 2

Summarize robot by examining some characteristic behaviors of system and get very simple concept out
Simplest dynamics ideas can be more powerful than representation learning, machine learning, and other tools

Graphical Analysis (Steve Strogatz)

$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

↑ damping

Theoretical physicists transformed robots beautiful things to say about how systems walked

heavily damped regime, simplifies 2nd to 1st order system $(\underset{kg\frac{m^2}{s}}{b\dot{\theta}} >> \underset{kg m^2 \frac{1}{s^2}}{ml^2\ddot{\theta}})$ units N·m, $kg\frac{m^2}{s^2}$

$\sqrt{\frac{g}{l}}\ [\frac{1}{s}]$ $b >> \cancel{m}l^2$

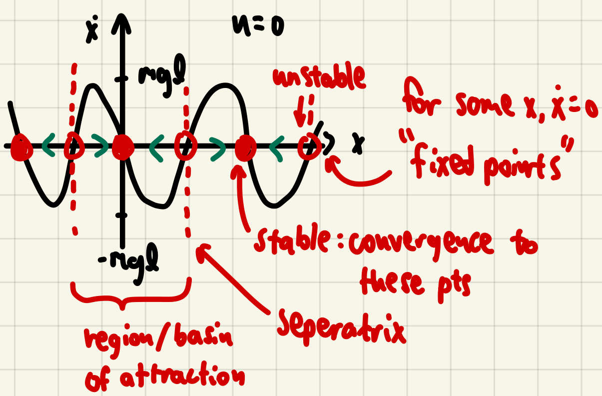
$b\sqrt{\frac{l}{g}} >> ml^2$ heavily damped regime

Tack on natural frequency term to reach dimensional parity

1st order

$b\dot{\theta} \approx u - mgl\sin\theta$

$b\dot{x} = u - mgl\sin x$ ← $x \in \mathbb{R}$ (x no wrapping restriction of angles)



linear: stable at origin no matter what
non-linear: wierd behavior
glancing contact w/ origin at pts like sin wave

Defn of Stability

global stability: all I.C.s converge to a pt

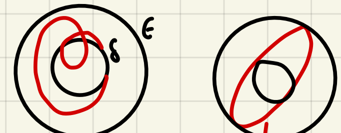
ϵ, δ are small pos constants

"local" stability

- In the sense of Lyapunov (i.s.L.): start at region, won't get too far

  for every ϵ, $\exists\delta$ (there exists) s.t. $\underbrace{\|x(0) - x^*\| < \delta}_{\text{within } \delta \text{ ball of fixed pt in state space}} \Rightarrow \forall t\ \|x(t) - x^*\| < \epsilon$ (x^*: f.p. (fixed pt); $\forall t$: for all t)

  if true for all t, for it to be true at x(0) then $\delta < \epsilon$



local invariant set not contained in euclidean δ norm non-circular trajectories necessitates additional analytical machinery of ε & δ

- locally attractive: will converge to region
- Asymptotically stable: attractive and isL $\lim_{t\to\infty} x(t) \Rightarrow x^*$
- Exponentially stable: get to stable pt faster than linear system w/ particular const

  $\forall t,\ \|x(t) - x^*\| < Ce^{-at}$ where $C, a > 0$

  implies others

  also, linear systems exponentially stable if stable

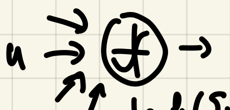
Craziness: # of fixed pts change w/ param change
fixed pts come together/explode apart
limit cycles, manifold stability

Lec 2

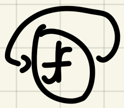
Simple Recurrent Neural Network

growing dendritic processes
simple illustrations of short-term memory

Short-term unit

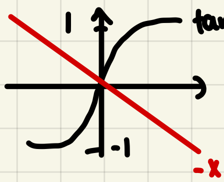


$\tanh(\sum w_i u)$

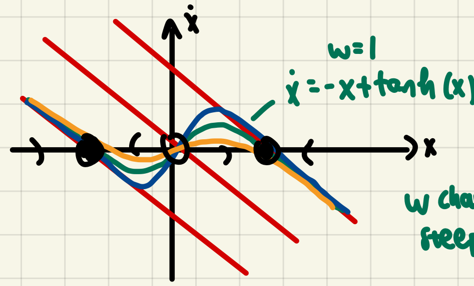


$\dot{x} = -x + \tanh(wx)$

Autapse (not known if it exists in brain but is found in a dish when cultured neurons are lonely & connect to themselves)



$\Rightarrow$

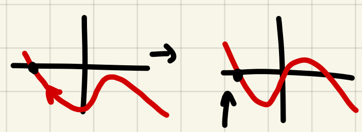


w changes gain of fn
steep or less steep

SIMPLEST analogy of short-term memory

"latching" mechanism with 2 fixed pts
bistable in off config

Can turn on to latch on one side
or turn off to latch to other side



stays here when reset to off

Will remember it was on that side

Long short-term memory (LSTM) - standard unit in RNN until transformers killed it
works on this principle

JANET (autapse with forgetting gate and more I/O)

Neuron = circuits.
Bistability = transistors
Direct analogs to analog electronics

Transformers also have a good dynamical systems interpretation if you use causal version of transformer
Dynamical systems theory and neural network crossover

- Rates of convergence, convergence (inequalities)
- Convergence not just to a fixed point. Optimization in neural network for instance, all minima are global minima. A bunch of fixed points are equally good

Hopfield network (hopfield's model of associate learning)- a recurrent neural network can store memories. Have multiple fixed points. Each one associated with different memory. Simple recipe to program a recurrent network to have fixed networks to be exactly where you want

DYNAMICS (time step in neural nets)

Set weights of neural network in simple case
Images are fixed points
Region of attraction: if close in some pixel space, will converge back to fixed points (images)

Function approximator paradigm being used to approach neural net, don't think about dynamics of learning enough. Memory happens in some flow. Time matters in way brain processes info. LLMs are dynamical system neural networks. Words build off previous words, sequence to sequence. Not flow beautifully like dynamical system would, but has the sense that the current state of the current thing has something about the state of the whole sentence

Recurrent networks didn't scale well. Longer the sentence, hit memory cap. Transformers take over. More neurons doesn't provide arbitrarily large amount of memory capacity

Lec 2

2nd order

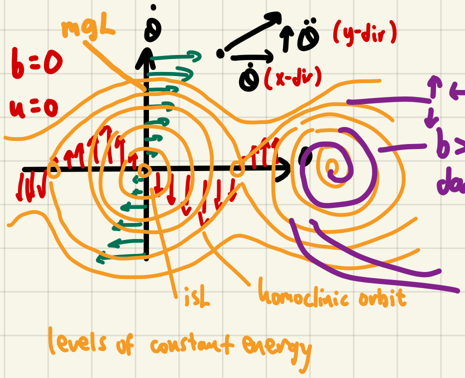
$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$$

$$\dot{x} = f(x,u) \quad x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$$

$$\dot{x} = \begin{bmatrix}\dot{\theta} \\ \frac{1}{ml^2}[u - b\dot{\theta} - mgl\sin\theta]\end{bmatrix} \begin{array}{l}\leftarrow x \\ \leftarrow y\end{array}$$

convert 2nd order to 1st order
write 2 eqns

Phase Portrait - Undamped pendulum



torque limited
impact local output vector magnitude
at time step

b>0
damping

underactuate may mean no
ability to change vector of a given
time

bands of I.C.s that
land at fixed pt

Fully actuated:
impose vector field
wiping field from natural
dynamics

Plot on top of this results of solving for value functions. Physics of problems will be revealed by optimal control problem

Control: Change the Vector Field (if u nonzero / a function)

Game we want to play: what's the minimal change in vector field that shapes the dynamics, reflows the dynamics to the place we want