ACST2002 Assignment

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Note: This report was written in `R`. Code for tables is shown to demonstrate methodology. The methodology can also be understood by viewing cell formulas in the Excel spreadsheet. The source code can be found here after I make the repo public.

```
library(tidyverse)
library(kableExtra)
```

Question (a)

Write down an efficient recurrence relation for $a_{\overline{n}|}$. Explain your thought process in words (e.g., using the timeline approach) or prove the result mathematically from first principles. Also write down the initial value for the sequence.

The cash flows pertaining to $a_{\overline{n}|}$ has one extra cash flow at time n compared to that of $a_{\overline{n-1}|}$. This additional cash flow is discounted n periods in the calculation of $a_{\overline{n}|}$. Hence,

$$a_{\overline{n}|} = a_{\overline{n-1}|} + v^n.$$

Question (b)

Given an effective discrete periodic rate of 4% per period, tabulate the values of $a_{\overline{n}|}$ for $n \in [1, 40] \cap \mathbb{Z}$.

Applying the recursive equation $n-1$ times, noticing that $a_{\overline{1}|} = v$, and applying the sum of terms of a geometric sequence formula,

$$\begin{aligned} a_{\overline{n}|} &= a_{\overline{1}|} + \sum_{t=2}^{n} \left(v^t\right) = \sum_{t=1}^{n} \left(v^t\right) = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{(1+i)-1} \\ a_{\overline{n}|} &= \frac{1-v^n}{i}. \end{aligned} \tag{1}$$

When $i = 4\%$, after applying this formula, the following table gives the values of $a_{\overline{n}|}$ for all $n \in [1, 40] \cap \mathbb{Z}$.

```
n <- 1:40
i <- rep(0.04, times = 40)
a_n <- (1-(1+i)^-n)/i
```

```r
tibble(n. = n, i. = i, a_n. = round(a_n, 2)) %>%
  mutate(i. = paste0(i*100, "\\%")) %>%
  kbl(col.names = c("$n$", "$i$", "$a_{\\angln}$ (2dp)"),
      escape = FALSE, align = "rrr", longtable = TRUE,
      caption = "Table of values for $a_{\\angln}$.") %>%
  kable_styling(latex_options = "HOLD_position") %>%
  column_spec(1, border_left = TRUE) %>%
  column_spec(3, border_right = TRUE)
```

Table 1: Table of values for $a_{\overline{n}|}$.

n	i	$a_{\overline{n}|}$ (2dp)
1	4%	0.96
2	4%	1.89
3	4%	2.78
4	4%	3.63
5	4%	4.45
6	4%	5.24
7	4%	6.00
8	4%	6.73
9	4%	7.44
10	4%	8.11
11	4%	8.76
12	4%	9.39
13	4%	9.99
14	4%	10.56
15	4%	11.12
16	4%	11.65
17	4%	12.17
18	4%	12.66
19	4%	13.13
20	4%	13.59
21	4%	14.03
22	4%	14.45
23	4%	14.86
24	4%	15.25
25	4%	15.62
26	4%	15.98
27	4%	16.33
28	4%	16.66
29	4%	16.98
30	4%	17.29
31	4%	17.59
32	4%	17.87
33	4%	18.15
34	4%	18.41
35	4%	18.66
36	4%	18.91
37	4%	19.14
38	4%	19.37
39	4%	19.58

40	4%	19.79

Question (c)

Repeat parts (a) and (b) for $(Ia)_{\overline{n}|}$ instead of $a_{\overline{n}|}$.

The cash flows pertaining to $(Ia)_{\overline{n}|}$ has one extra cash flow at time n with value n compared to that of $(Ia)_{\overline{n-1}|}$. This additional cash flow is discounted n periods in the calculation of $(Ia)_{\overline{n}|}$. Hence,

$$(Ia)_{\overline{n}|} = (Ia)_{\overline{n-1}|} + nv^n.$$

$(Ia)_{\overline{1}|} = v$. $(Ia)_{\overline{2}|}$ can be found by applying this formula, using the cell entry for $(Ia)_{\overline{1}|}$. This formula can then be applied recursively (over and over again) until one reaches their desired target (in this case $(Ia)_{\overline{40}|}$).

When $i = 4\%$, the following table gives the values of $(Ia)_{\overline{n}|}$ for all $n \in [1, 40] \cap \mathbb{Z}$.

```r
Ia_n <- (1+i[1])^-1
for (t in 2:length(n)) {
  Ia_n[t] <- Ia_n[t-1] + t*(1+i[t])^-t
}

tibble(n. = n, i. = i, Ia_n. = round(Ia_n, 2)) %>%
  mutate(i. = paste0(i*100, "\\%")) %>%
  kbl(col.names = c("$n$", "$i$", "$(Ia)_{\\angln}$ (2dp)"),
      escape = FALSE, align = "rrr", longtable = TRUE,
      caption = "Table of values for $(Ia)_{\\angln}$.") %>%
  kable_styling(latex_options = "HOLD_position") %>%
  column_spec(1, border_left = TRUE) %>%
  column_spec(3, border_right = TRUE)
```

Table 2: Table of values for $(Ia)_{\overline{n}|}$.

n	i	$(Ia)_{\overline{n}|}$ (2dp)
1	4%	0.96
2	4%	2.81
3	4%	5.48
4	4%	8.90
5	4%	13.01
6	4%	17.75
7	4%	23.07
8	4%	28.91
9	4%	35.24
10	4%	41.99
11	4%	49.14
12	4%	56.63
13	4%	64.44
14	4%	72.52
15	4%	80.85
16	4%	89.40
17	4%	98.12

18	4%	107.01
19	4%	116.03
20	4%	125.16
21	4%	134.37
22	4%	143.65
23	4%	152.99
24	4%	162.35
25	4%	171.73
26	4%	181.10
27	4%	190.47
28	4%	199.81
29	4%	209.10
30	4%	218.35
31	4%	227.54
32	4%	236.67
33	4%	245.71
34	4%	254.67
35	4%	263.54
36	4%	272.31
37	4%	280.98
38	4%	289.54
39	4%	297.99
40	4%	306.32

Question (d)

Suppose that we would like to develop the relevant calculations for a new actuarial notation defined as $(Qa)_{\overline{n}|} = \sum_{t=1}^{n}\left(t^2 v^t\right)$. Repeat parts (a) and (b) for $(Qa)_{\overline{n}|}$ instead of $a_{\overline{n}|}$.

The cash flows pertaining to $(Qa)_{\overline{n}|}$ has one extra cash flow at time n with value n^2 compared to that of $(Qa)_{\overline{n-1}|}$. This additional cash flow is discounted n periods in the calculation of $(Qa)_{\overline{n}|}$. Hence,

$$(Qa)_{\overline{n}|} = (Qa)_{\overline{n-1}|} + n^2 v^n.$$

$(Qa)_{\overline{1}|} = v$. $(Qa)_{\overline{2}|}$ can be found by applying this formula, using the cell entry for $(Qa)_{\overline{1}|}$. This formula can then be applied recursively until one reaches their desired target ($(Qa)_{\overline{40}|}$).

When $i = 4\%$, the following table gives the values of $(Qa)_{\overline{n}|}$ for all $n \in [1, 40] \cap \mathbb{Z}$.

```
Qa_n <- (1+i[1])^-1
for (t in 2:length(n)) {
  Qa_n[t] <- Qa_n[t-1] + t^2*(1+i[t])^-t
}

tibble(n. = n, i. = i, Qa_n. = round(Qa_n, 2)) %>%
  mutate(i. = paste0(i*100, "\\%")) %>%
  kbl(col.names = c("$n$", "$i$", "$(Qa)_{\\angln} (2dp)$"),
      escape = FALSE, align = "rrr", longtable = TRUE,
      caption = "Table of values for $(Qa)_{\\angln}$.") %>%
  kable_styling(latex_options = "HOLD_position") %>%
```

```
column_spec(1, border_left = TRUE) %>%
column_spec(3, border_right = TRUE)
```

Table 3: Table of values for $(Qa)_{\overline{n}|}$.

n	i	$(Qa)_{\overline{n}|}(2dp)$
1	4%	0.96
2	4%	4.66
3	4%	12.66
4	4%	26.34
5	4%	46.89
6	4%	75.34
7	4%	112.57
8	4%	159.34
9	4%	216.25
10	4%	283.80
11	4%	362.40
12	4%	452.34
13	4%	553.84
14	4%	667.03
15	4%	791.96
16	4%	928.64
17	4%	1077.01
18	4%	1236.94
19	4%	1408.29
20	4%	1590.84
21	4%	1784.37
22	4%	1988.59
23	4%	2203.22
24	4%	2427.93
25	4%	2662.38
26	4%	2906.21
27	4%	3159.04
28	4%	3420.48
29	4%	3690.15
30	4%	3967.64
31	4%	4252.54
32	4%	4544.44
33	4%	4842.92
34	4%	5147.59
35	4%	5458.02
36	4%	5773.82
37	4%	6094.57
38	4%	6419.88
39	4%	6749.36
40	4%	7082.63

Question (e)

Simplify $(I\ddot{a})_{\overline{n}|} - (Ia)_{\overline{n}|}$ and hence derive a formula for $(Ia)_{\overline{n}|}$.

$$
\begin{aligned}
(I\ddot{a})_{\overline{n}|} - (Ia)_{\overline{n}|} &= \sum_{t=0}^{t=n-1} \left((t+1)v^t\right) - \sum_{t=1}^{n} \left(tv^t\right) \\
&= 1 + \sum_{t=1}^{n-1} \left(v^t\right) - nv^n \\
&= \ddot{a}_{\overline{n}|} - nv^n
\end{aligned}
$$

But $(I\ddot{a})_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|}$, so

$$
\begin{aligned}
(1+i-1)(Ia)_{\overline{n}|} &= \ddot{a}_{\overline{n}|} - nv^n \\
(Ia)_{\overline{n}|} &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}.
\end{aligned} \tag{2}
$$

Question (f)

Suppose that $(Q\ddot{a})_{\overline{n}|} = \sum_{t=1}^{n} \left(t^2 v^{t-1}\right)$, simplify $(Q\ddot{a})_{\overline{n}|} - (Qa)_{\overline{n}|}$ and hence derive a formula for $(Qa)_{\overline{n}|}$.

$$
\begin{aligned}
(Q\ddot{a})_{\overline{n}|} - (Qa)_{\overline{n}|} &= \sum_{t=0}^{t=n-1} \left((t+1)^2 v^t\right) - \sum_{t=1}^{n} \left(t^2 v^t\right) \\
&= 1 + \sum_{t=1}^{n-1} \left((t^2 + 2t + 1 - t^2)v^t\right) - n^2 v^n \\
&= 1 + 2\sum_{t=1}^{n-1} \left(tv^t\right) + \sum_{t=1}^{n-1} \left(v^t\right) - n^2 v^n \\
&= 2(Ia)_{\overline{n-1}|} + \ddot{a}_{\overline{n}|} - n^2 v^n
\end{aligned}
$$

But $(Q\ddot{a})_{\overline{n}|} = (1+i)(Qa)_{\overline{n}|}$, so

$$
\begin{aligned}
(1+i-1)(Qa)_{\overline{n}|} &= 2(Ia)_{\overline{n-1}|} + \ddot{a}_{\overline{n}|} - n^2 v^n \\
(Qa)_{\overline{n}|} &= \frac{2(Ia)_{\overline{n-1}|} + \ddot{a}_{\overline{n}|} - n^2 v^n}{i}.
\end{aligned} \tag{3}
$$

Question (g)

Using the formulae developed in parts (e) and (f), re-calculate the relevant values of annuity functions in parts (c) and (d).

The following table is constructed by applying equations (2) and (3) for all $n \in [1, 40] \cap \mathbb{Z}$.

```r
n0 <- 0:40
i0 <- rep(0.04, times = 41)
Ia_n0 = ((1+i0)*(1-(1+i0)^-n0)/i0 - n0*(1+i0)^-n0) / i0

Ia_n <- Ia_n0[2:length(Ia_n0)]

Ia_n_1 <- Ia_n0[1:(length(Ia_n0)-1)]
Qa_n <- (2*Ia_n_1 + (1+i)*(1-(1+i)^-n)/i - n^2*(1+i)^-n) / i

tibble(n. = n, i. = i, Ia_n. = round(Ia_n, 2), Qa_n. = round(Qa_n, 2)) %>%
  mutate(i. = paste0(i*100, "\\%")) %>%
  kbl(col.names = c("$n$", "$i$", "$(Ia)_{\\angln}$ (2dp)", "$(Qa)_{\\angln}$ (2dp)"),
      escape = FALSE, align = "rrrr", longtable = TRUE) %>%
  kable_styling(latex_options = "HOLD_position") %>%
  column_spec(1, border_left = TRUE) %>%
  column_spec(4, border_right = TRUE)
```

n	i	$(Ia)_{\overline{n}\rceil}$ (2dp)	$(Qa)_{\overline{n}\rceil}$ (2dp)
1	4%	0.96	0.96
2	4%	2.81	4.66
3	4%	5.48	12.66
4	4%	8.90	26.34
5	4%	13.01	46.89
6	4%	17.75	75.34
7	4%	23.07	112.57
8	4%	28.91	159.34
9	4%	35.24	216.25
10	4%	41.99	283.80
11	4%	49.14	362.40
12	4%	56.63	452.34
13	4%	64.44	553.84
14	4%	72.52	667.03
15	4%	80.85	791.96
16	4%	89.40	928.64
17	4%	98.12	1077.01
18	4%	107.01	1236.94
19	4%	116.03	1408.29
20	4%	125.16	1590.84
21	4%	134.37	1784.37
22	4%	143.65	1988.59
23	4%	152.99	2203.22
24	4%	162.35	2427.93
25	4%	171.73	2662.38
26	4%	181.10	2906.21
27	4%	190.47	3159.04
28	4%	199.81	3420.48
29	4%	209.10	3690.15
30	4%	218.35	3967.64
31	4%	227.54	4252.54
32	4%	236.67	4544.44
33	4%	245.71	4842.92

34	4%	254.67	5147.59
35	4%	263.54	5458.02
36	4%	272.31	5773.82
37	4%	280.98	6094.57
38	4%	289.54	6419.88
39	4%	297.99	6749.36
40	4%	306.32	7082.63

Question (h)

Comment on the advantages and disadvantages of the two different approaches (recurrence relation and algebraic formula) used in constructing the relevant tabulated values above.

Both approaches can be split into two steps: writing a formula, and then applying it. The first step is usually easier for the recurrence relation approach, but the second step is usually easier for the algebraic formula approach.

- A recurrence equation is usually easier to find in closed form compared to its corresponding algebraic equation.
  - The algebraic equation may not even be able to be expressed in closed form. A recurrence equation usually will be.
- The recurrence equation needs to be applied many times to find other terms of the sequence before applying it to find the desired term, while the algebratic equation only needs to be applied once.

Question (i)

A loan of \$700,000 is to be fully repaid by 25 level annual repayments made in arrears at an effective annual rate of 4% per annum. Calculate the amount of the level annual repayment.

Let the annual repayment amount be X. Since the loan is fully repaid, the present value of the \$700,000 cash flow must equal the present value of the 25 annual repayment cash flows.

$$\begin{aligned} 700000 &= \sum_{t=1}^{25} \left(X \cdot (1.04)^{-t} \right) \\ &= X a_{\overline{25|}} \\ X &= \frac{700000}{a_{\overline{25|}}} = 44808.37 \text{ (2dp)} \quad \text{from Table 1.} \end{aligned}$$

Question (j)

A loan of \$450,000 is to be fully repaid by level semi-annual repayments made in arrears for the next 8 years. The equivalent constant force of interest for this loan is 7.8441426% per annum. Calculate the amount of the capital repayment between the 3rd and the 6th year.

Let time t be measured in years and let the loan start at $t = 0$. Let t_0 be an arbitrary positive real number measured in years. With a constant force of interest, the effective semi-annual interest rate, $i_{0.5}$, can be calculated as follows.

$$A(t_0, t_0 + 0.5) = (1 + i_{0.5}) = exp\left[\int_{t_0}^{t_0+0.5} \left(7.8441426 \cdot 10^{-2}\right)\, \mathrm{d}t\right]$$
$$i_{0.5} = exp\left(7.8441426 \cdot 10^{-2} \cdot 0.5\right) - 1 = 0.0400 \text{ (4dp)}$$

Let the semi-annual repayment amount be X. Since the loan is fully repaid, the present value of the $450,000 cash flow must equal the present value of the 16 semi-annual repayment cash flows.

$$450000 = \sum_{u=1}^{16} \left(X \cdot (1 + i_{0.5})^{-u}\right) = X a_{\overline{16|}i_{0.5}}$$
$$X = \frac{450000}{a_{\overline{16|}i_{0.5}}} = 38619.00 \text{ (2dp)} \quad \text{from Table 1.}$$

Let L_v represent the outstanding loan amount after the cash flow when $t = v$. The total capital repayment amount between the 3rd and the 6th year is equal to the negative of the change in the outstanding loan amount in that period.

$$\begin{aligned}
-(L_6 - L_2) &= L_2 - L_6 \\
&= \sum_{u=1}^{12} \left(X \cdot (1 + i_{0.5})^{-u}\right) - \sum_{u=1}^{4} \left(X \cdot (1 + i_{0.5})^{-u}\right) \\
&= X \cdot \sum_{u=5}^{12} \left((1 + i_{0.5})^{-u}\right) = (1 + i_{0.5})^{-4} X \cdot \sum_{u=1}^{8} \left((1 + i_{0.5})^{-u}\right) \\
&= (1 + i_{0.5})^{-4} X a_{\overline{8|}i_{0.5}} \\
&= 222259.24 \text{ (2dp)} \quad \text{from Table 1.}
\end{aligned}$$

Question (k)

Calculate the present value at time 0 of a 10-year continuous annuity with a payment rate of $300 per annum under an effective annual rate of 1.9804%.

Let PV be the required present value and let $i_1 = 0.019804$. Notice that $i_2 \equiv (1 + i_1)^2 - 1 = 0.0400$ (4dp).

$$PV = \int_0^{10} \left(300 \cdot (1 + i_1)^{-t}\right)\, \mathrm{d}t$$

Let $t = 2u$.

$$\begin{aligned}
PV &= \int_0^5 \left(300 \cdot (1+i_1)^{-2u}\right) 2\mathrm{d}u \\
&= 600 \int_0^5 \left((1+i_2)^{-u}\right) \mathrm{d}u \\
&= 600 \int_0^5 \left(exp\left[-ln(1+i_2)u\right]\right) \mathrm{d}u \\
&= 600 \left[\frac{-(1+i_2)^{-u}}{ln(1+i_2)}\right]_0^5 \\
&= 600 \cdot \frac{1-(1+i_2)^{-5}}{ln(1+i_2)} \\
&= 600 \cdot \frac{i_2}{ln(1+i_2)} \cdot \frac{1-(1+i_2)^{-5}}{i_2} \\
&= \frac{600\ i_2}{ln(1+i_2)} a_{\overline{5|}i_2} \quad \text{from equation 1} \\
PV &= 2724.17 \text{ (2dp)} \quad \text{from Table 1.}
\end{aligned}$$

Question (l)

Given an effective annual rate of 4%, calculate the present value at time 0 of a 20-year arithmetically increasing annuity immediate whereby the first annual payment is $6,000 and subsequent annual increment is $500.

$$\begin{aligned}
PV &= \sum_{t=1}^{20} \left((5500 + 500t) \cdot 1.04^{-t}\right) \\
&= \sum_{t=1}^{20} \left(5500 \cdot 1.04^{-t}\right) + \sum_{t=1}^{20} \left(500t \cdot 1.04^{-t}\right) \\
&= 5500 a_{\overline{20|}} + 500 (Ia)_{\overline{20|}} \\
&= 137324.30 \text{ (2dp)} \quad \text{from Table 1 and 2.}
\end{aligned}$$

Question (m)

Consider a 25-year annuity immediate with a payment amount of $(26-t)^2$ at the end of year t. For instance, the payment amount at the end of year 6 is $400. Calculate its present value at time 0 given an effective annual rate of 4%.

$$
\begin{aligned}
PV &= \sum_{u=1}^{25} \left((26-u)^2 \cdot 1.04^{-u} \right) \\
&= \sum_{u=1}^{25} \left((26^2 - 52u + u^2) \cdot 1.04^{-u} \right) \\
&= 26^2 \sum_{u=1}^{25} \left(1.04^{-u} \right) - 52 \sum_{u=1}^{25} \left(u \cdot 1.04^{-u} \right) + \sum_{u=1}^{25} \left(u^2 \cdot 1.04^{-u} \right) \\
&= 26^2 a_{\overline{25|}} - 52 (Ia)_{\overline{25|}} + (Qa)_{\overline{25|}} \\
&= 4293.15 \text{ (2dp)} \quad \text{from Table 1, 2, and 3.}
\end{aligned}
$$