ACST2002 Assignment

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Question (a)

Write down an efficient recurrence relation for $a_{\overline{n}|}$. Explain your thought process in words (e.g., using the timeline approach) or prove the result mathematically from first principles. Also write down the initial value for the sequence.

The cash flows pertaining to $a_{\overline{n}|}$ has one extra cash flow at time n compared to that of $a_{\overline{n-1}|}$. This additional cash flow is discounted n periods in the calculation of $a_{\overline{n}|}$. Hence,

$$a_{\overline{n}|} = a_{\overline{n-1}|} + v^n.$$

Question (b)

Given an effective discrete periodic rate of 4% per period, tabulate the values of $a_{\overline{n}|}$ for $n \in [1, 40] \cap \mathbb{Z}$.

Applying the recursive equation $n-1$ times, noticing that $a_{\overline{1}|} = v$, and applying the sum of terms of a geometric sequence formula,

$$a_{\overline{n}|} = a_{\overline{1}|} + \sum_{t=2}^{n} \left(v^t\right) = \sum_{t=1}^{n} \left(v^t\right) = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{(1+i)-1} = \frac{1-v^n}{i}.$$

When $i = 4\%$, this formula gives the following values of $a_{\overline{n}|}$ for $n \in [1, 40] \cap \mathbb{Z}$.

n	i	$a_{\overline{n\rceil}}$
1	0.04	0.96
2	0.04	1.89
3	0.04	2.78
4	0.04	3.63
5	0.04	4.45
6	0.04	5.24
7	0.04	6.00
8	0.04	6.73
9	0.04	7.44
10	0.04	8.11
11	0.04	8.76
12	0.04	9.39
13	0.04	9.99
14	0.04	10.56
15	0.04	11.12
16	0.04	11.65
17	0.04	12.17
18	0.04	12.66
19	0.04	13.13
20	0.04	13.59
21	0.04	14.03
22	0.04	14.45
23	0.04	14.86
24	0.04	15.25
25	0.04	15.62
26	0.04	15.98
27	0.04	16.33
28	0.04	16.66
29	0.04	16.98
30	0.04	17.29
31	0.04	17.59
32	0.04	17.87
33	0.04	18.15
34	0.04	18.41
35	0.04	18.66
36	0.04	18.91
37	0.04	19.14
38	0.04	19.37
39	0.04	19.58
40	0.04	19.79

Question (c)

Repeat parts (a) and (b) for $(Ia)_{\overline{n\rceil}}$ instead of $a_{\overline{n\rceil}}$.

The cash flows pertaining to $(Ia)_{\overline{n\rceil}}$ has one extra cash flow at time n with value n compared to that of $(Ia)_{\overline{n-1\rceil}}$. This additional cash flow is discounted n periods in the calculation of $(Ia)_{\overline{n\rceil}}$. Hence,

$$(Ia)_{\overline{n\rceil}} = (Ia)_{\overline{n-1\rceil}} + nv^n.$$

Applying the recursive equation $n-1$ times,

$$\begin{aligned}
(Ia)_{\overline{n}|} &= (Ia)_{\overline{1}|} + \sum_{t=2}^{n}\left(tv^t\right) = \sum_{t=1}^{n}\left(tv^t\right) \\
(1+i)(Ia)_{\overline{n}|} &= \sum_{t=1}^{n}\left(tv^{t-1}\right) = \sum_{t=0}^{n-1}\left((t+1)v^t\right) \\
(1+i-1)(Ia)_{\overline{n}|} &= 1 + \sum_{t=1}^{n-1}\left((t+1-t)v^t\right) - nv^n \\
i(Ia)_{\overline{n}|} &= \sum_{t=0}^{n-1}\left(v^t\right) - nv^n = (1+i)a_{\overline{n}|} - nv^n \\
(Ia)_{\overline{n}|} &= \frac{(1+i)a_{\overline{n}|} - nv^n}{i}.
\end{aligned}$$

When $i = 4\%$, this formula gives the following values of $(Ia)_{\overline{n}|}$ for $n \in [1, 40] \cap \mathbb{Z}$.

n	i	$(Ia)_{\overline{n}\rvert}$
1	0.04	0.96
2	0.04	2.81
3	0.04	5.48
4	0.04	8.90
5	0.04	13.01
6	0.04	17.75
7	0.04	23.07
8	0.04	28.91
9	0.04	35.24
10	0.04	41.99
11	0.04	49.14
12	0.04	56.63
13	0.04	64.44
14	0.04	72.52
15	0.04	80.85
16	0.04	89.40
17	0.04	98.12
18	0.04	107.01
19	0.04	116.03
20	0.04	125.16
21	0.04	134.37
22	0.04	143.65
23	0.04	152.99
24	0.04	162.35
25	0.04	171.73
26	0.04	181.10
27	0.04	190.47
28	0.04	199.81
29	0.04	209.10
30	0.04	218.35
31	0.04	227.54
32	0.04	236.67
33	0.04	245.71
34	0.04	254.67
35	0.04	263.54
36	0.04	272.31
37	0.04	280.98
38	0.04	289.54
39	0.04	297.99
40	0.04	306.32

Question (d)

Suppose that we would like to develop the relevant calculations for a new actuarial notation defined as $(Qa)_{\overline{n}\rvert} = \sum_{t=1}^{n} \left(t^2 v^t\right)$. Repeat parts (a) and (b) for $(Qa)_{\overline{n}\rvert}$ instead of $a_{\overline{n}\rvert}$.

The cash flows pertaining to $(Qa)_{\overline{n}\rvert}$ has one extra cash flow at time n with value n^2 compared to that of $(Qa)_{\overline{n-1}\rvert}$. This additional cash flow is discounted n periods in the calculation of $(Qa)_{\overline{n}\rvert}$. Hence,

$$(Qa)_{\overline{n}\rvert} = (Qa)_{\overline{n-1}\rvert} + n^2 v^n.$$

$(Qa)_{\overline{1}|} = v$. $(Qa)_{\overline{2}|}$ can be found by applying this formula using the cell entry for $(Qa)_{\overline{1}|}$. This formula can then be applied recursively until one reaches their desired target.

n	i	$(Qa)_{\overline{n}|}$
1	0.04	0.96
2	0.04	3.12
3	0.04	6.50
4	0.04	11.18
5	0.04	17.26
6	0.04	24.85
7	0.04	34.07
8	0.04	45.01
9	0.04	57.82
10	0.04	72.63
11	0.04	89.56
12	0.04	108.77
13	0.04	130.42
14	0.04	154.66
15	0.04	181.68
16	0.04	211.64
17	0.04	244.76
18	0.04	281.22
19	0.04	321.25
20	0.04	365.08
21	0.04	412.93
22	0.04	465.07
23	0.04	521.76
24	0.04	583.28
25	0.04	649.92
26	0.04	722.01
27	0.04	799.86
28	0.04	883.82
29	0.04	974.26
30	0.04	1071.56
31	0.04	1176.13
32	0.04	1288.39
33	0.04	1408.78
34	0.04	1537.79
35	0.04	1675.90
36	0.04	1823.65
37	0.04	1981.57
38	0.04	2150.24
39	0.04	2330.28
40	0.04	2522.32

Question (e)

Simplify $(I\ddot{a})_{\overline{n}|} - (Ia)_{\overline{n}|}$ and hence derive a formula for $(Ia)_{\overline{n}|}$.

$$
\begin{aligned}
(I\ddot{a})_{\overline{n}|} - (Ia)_{\overline{n}|} &= \sum_{t=0}^{t=n-1} \left((t+1)v^t\right) - \sum_{t=1}^{n} \left(tv^t\right) \\
&= 1 + \sum_{t=1}^{n-1} \left(v^t\right) - nv^n \\
&= \ddot{a}_{\overline{n}|} - nv^n
\end{aligned}
$$

But $(I\ddot{a})_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|}$, so

$$
\begin{aligned}
(1+i-1)(Ia)_{\overline{n}|} &= \ddot{a}_{\overline{n}|} - nv^n \\
(Ia)_{\overline{n}|} &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}.
\end{aligned}
$$

Question (f)

Suppose that $(Q\ddot{a})_{\overline{n}|} = \sum_{t=1}^{n} \left(t^2 v^{t-1}\right)$, simplify $(Q\ddot{a})_{\overline{n}|} - (Qa)_{\overline{n}|}$ and hence derive a formula for $(Qa)_{\overline{n}|}$.

$$
\begin{aligned}
(Q\ddot{a})_{\overline{n}|} - (Qa)_{\overline{n}|} &= \sum_{t=0}^{t=n-1} \left((t+1)^2 v^t\right) - \sum_{t=1}^{n} \left(t^2 v^t\right) \\
&= 1 + \sum_{t=1}^{n-1} \left((t^2 + 2t + 1 - t^2)v^t\right) - n^2 v^n \\
&= 1 + 2\sum_{t=1}^{n-1} \left(tv^t\right) + \sum_{t=1}^{n-1} \left(v^t\right) - n^2 v^n \\
&= 2(Ia)_{\overline{n-1}|} + \ddot{a}_{\overline{n}|} - n^2 v^n
\end{aligned}
$$

But $(Q\ddot{a})_{\overline{n}|} = (1+i)(Qa)_{\overline{n}|}$, so

$$
\begin{aligned}
(1+i-1)(Qa)_{\overline{n}|} &= 2(Ia)_{\overline{n-1}|} + \ddot{a}_{\overline{n}|} - n^2 v^n \\
(Qa)_{\overline{n}|} &= \frac{2(Ia)_{\overline{n-1}|} + \ddot{a}_{\overline{n}|} - n^2 v^n}{i}.
\end{aligned}
$$

Question (g)