

STAT2372 Assignment

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```
library(tidyverse)
```

The original source code for this assignment can be found [here](#) after I make the repo public.

Question 1

- (a) Let X be a random variable with mean 0 and finite variance σ^2 . By applying Markov's inequality to the random variable $W = (X + t)^2$, where $t > 0$, show that

$$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2} \quad \text{for any } a > 0.$$

$$\begin{aligned} P(X \geq a) &= P(X + t > a + t) \\ &\leq P((X + t)^2 > (a + t)^2) \\ &\leq \frac{E((X + t)^2)}{(a + t)^2} \quad \text{from applying Markov's inequality} \\ &= \frac{E(X^2) + 2tE(X) + t^2}{(a + t)^2} \\ &= \frac{\sigma^2 + t^2}{(a + t)^2} \quad \text{since } E(X) = 0. \end{aligned}$$

Let $f(t) = \frac{\sigma^2 + t^2}{(a + t)^2}$. Let the stationary point of $f(t)$ occur when $t = t_m$.

$$\begin{aligned} 0 &= f'(t_m) \\ &= \frac{(a + t_m)^2 \cdot 2t_m - (\sigma^2 + t_m^2)(2(a + t_m))}{(a + t_m)^4} \\ 0 &= (a + t_m)^2 \cdot 2t_m - (\sigma^2 + t_m^2)(2(a + t_m)) \\ &= (a + t_m)(2at_m + 2t_m^2 - 2\sigma^2 - 2t_m^2) \\ &= 2(a + t_m)(at_m - \sigma^2) \\ t_m &= \frac{\sigma^2}{a} \quad \text{since } t > 0. \end{aligned}$$

Substituting $t = t_m$ into the inequality,

$$P(X \geq a) \leq \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{(a + \frac{\sigma^2}{a})^2}$$

$$\begin{aligned}
&= \frac{\frac{\sigma^2}{a} \left(a + \frac{\sigma^2}{a}\right)}{\left(a + \frac{\sigma^2}{a}\right)^2} \\
&= \frac{\frac{\sigma^2}{a}}{a + \frac{\sigma^2}{a}} \\
P(X \geq a) &\leq \frac{\sigma^2}{\sigma^2 + a^2}.
\end{aligned}$$

(b) Hence show that, for any $a > 0$,

$$\begin{aligned}
P(Y \geq \mu + a) &\leq \frac{\sigma^2}{\sigma^2 + a^2} \\
P(Y \leq \mu - a) &\leq \frac{\sigma^2}{\sigma^2 + a^2}
\end{aligned}$$

where $E(Y) = \mu$, $Var(Y) = \sigma^2$.

$$\begin{aligned}
E(Y - \mu) &= E(Y) - \mu = 0. \\
Var(Y - \mu) &= Var(Y) = \sigma^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
E(\mu - Y) &= 0. \\
Var(\mu - Y) &= \sigma^2.
\end{aligned}$$

Applying the result in part (a),

$$\begin{aligned}
P(Y \geq \mu + a) &= P(Y - \mu \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}. \\
P(Y \leq \mu - a) &= P(\mu - Y \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.
\end{aligned}$$

Question 2

(a) Let X have a Poisson distribution with parameter λ .

(i) Determine $K_X(t)$, the cumulant generating function. Hence find the third and the fourth central moments of X .

$$\begin{aligned}
K_X(t) &= \log(E(e^{tX})) \\
&= \log\left(\sum_{k=0}^{\infty} \left(e^{tk} \frac{e^{-\lambda} \lambda^k}{k!}\right)\right) \\
&= \log\left(e^{-\lambda} \sum_{k=0}^{\infty} \left(\frac{(\lambda e^t)^k}{k!}\right)\right) \\
&= \log\left(e^{-\lambda} e^{\lambda e^t}\right) \quad (\text{series expansion of an exponential function})
\end{aligned}$$

$$= \lambda(e^t - 1).$$

$$\begin{aligned}\mu_3 &= K_X^{(3)}(0) \\ &= \lambda e^0\end{aligned}$$

$$\mu_3 = \lambda.$$

$$\begin{aligned}\sigma^2 &= K_X^{(2)}(0) \\ &= \lambda\end{aligned}$$

$$\begin{aligned}\mu_4 - 3\sigma^4 &= K_X^{(4)}(0) \\ \mu_4 &= \lambda + 3\lambda^2.\end{aligned}$$

(ii) Show that the moment-generating function of $Y = (X - \lambda)/\sqrt{\lambda}$ is given by

$$M_Y(t) = \exp\left(\lambda e^{t/\sqrt{\lambda}} - \sqrt{\lambda}t - \lambda\right).$$

$$\begin{aligned}M_Y(t) &= E\left(\exp\left(\frac{X - \lambda}{\sqrt{\lambda}}t\right)\right) \\ &= E\left(\exp\left(\frac{Xt}{\sqrt{\lambda}}\right) \cdot \exp(-\sqrt{\lambda}t)\right) \\ &= e^{-\sqrt{\lambda}t} \sum_{k=0}^{\infty} \left(e^{\frac{kt}{\sqrt{\lambda}}} \frac{e^{-\lambda} \lambda^k}{k!}\right) \\ &= e^{-\sqrt{\lambda}t - \lambda} \sum_{k=0}^{\infty} \left(\frac{\left(e^{\frac{t}{\sqrt{\lambda}}} \lambda\right)^k}{k!}\right) \\ &= e^{-\sqrt{\lambda}t - \lambda} \exp\left(e^{\frac{t}{\sqrt{\lambda}}} \lambda\right) \\ &= \exp\left(\lambda e^{t/\sqrt{\lambda}} - \sqrt{\lambda}t - \lambda\right).\end{aligned}$$

(iii) Use the expansion

$$e^{t/\sqrt{\lambda}} = \sum_{i=0}^{\infty} \left(\frac{(t/\sqrt{\lambda})^i}{i!}\right)$$

to show that

$$\lim_{\lambda \rightarrow \infty} (M_Y(t)) = e^{t^2/2}$$

and hence show that the distribution function of Y converges to a standard normal distribution function as $\lambda \rightarrow \infty$.

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} (M_Y(t)) &= \lim_{\lambda \rightarrow \infty} \left(\exp\left[\lambda e^{t/\sqrt{\lambda}} - \sqrt{\lambda}t - \lambda\right]\right) \\ &= \lim_{\lambda \rightarrow \infty} \left(\exp\left[\lambda \sum_{i=0}^{\infty} \left(\frac{(t/\sqrt{\lambda})^i}{i!}\right) - \sqrt{\lambda}t - \lambda\right]\right) \\ &= \lim_{\lambda \rightarrow \infty} \left(\exp\left[\lambda \sum_{i=2}^{\infty} \left(\frac{(t/\sqrt{\lambda})^i}{i!}\right)\right]\right) \\ &= \lim_{\lambda \rightarrow \infty} \left(\exp\left[t^2 \sum_{i=2}^{\infty} \left(\frac{(t/\sqrt{\lambda})^{i-2}}{i!}\right)\right]\right)\end{aligned}$$

$$\begin{aligned}
&= \exp \left[t^2 \lim_{\lambda \rightarrow \infty} \left(\sum_{i=2}^{\infty} \left(\frac{(t/\sqrt{\lambda})^{i-2}}{i!} \right) \right) \right] \\
&= \exp \left[t^2 \left(1/2 + \lim_{\lambda \rightarrow \infty} \left[\sum_{i=3}^{\infty} \left(\frac{(t/\sqrt{\lambda})^{i-2}}{i!} \right) \right] \right) \right] \\
&= e^{t^2/2}.
\end{aligned}$$

This is the mgf of a standard normal distribution. That is, $\lim_{\lambda \rightarrow \infty} (M_Y(t)) = M_Z(t)$. The random variable Y converges in distribution to a standard normal random variable as $\lambda \rightarrow \infty$.

- (b) X , the number of accidents per year at a given intersection, is assumed to have a Poisson distribution. Over the past few years, an average of 36 accidents per year have occurred at this intersection. If the number of accidents per year is at least 45, an intersection can qualify to be redesigned under an emergency program set up by the state. Approximate the probability that this intersection will come under the emergency program at the end of the next year.

Let $X \sim \text{Pois}(36)$.

$$\begin{aligned}
P(X \geq 45) &= 1 - P(X \leq 44) \\
&= 0.0819.
\end{aligned}$$

Hence, the probability that this intersection will come under the emergency program at the end of the next year is 0.0819.

The inline code used is shown below.

```
(1 - ppois(44, 36)) %>% round(4) %>% format(nsmall = 4)
```

```
## [1] "0.0819"
```

- (c) Suppose V_i , $i \in [1, n] \cap \mathbb{N}$, are independent exponential random variables with rate 1. Denote

$$X = \max \left(\left\{ n \mid \sum_{i=1}^n (V_i) \leq \lambda \right\} \right),$$

so X can be thought of as being the maximum number of exponentials having rate 1 that can be summed and still be less than or equal to λ .

- (i) Using properties of a Poisson process with rate 1, explain why X has a Poisson distribution with parameter λ .

The exponential distribution with parameter 1 can be defined as the distribution for the random variable that represents the time between two successive events in the Poisson process with parameter 1. Let $\{Y(t) : t \geq 0\}$ be this particular Poisson process. Thus, the random variable X represents the number of times that the time between two successive events has been observed within a time period of λ in $\{Y(t) : t \geq 0\}$. Let $(t_0, t_0 + \lambda)$ represent this particular time interval, where $t_0 \geq 0$. Since this is the same as the number of events that have occurred within $(t_0, t_0 + \lambda)$, $X = Y(t_0 + \lambda) - Y(t_0) \sim \text{Pois}(\lambda)$.

(ii) Let $V_i = -\log(U_i)$, $U_i \sim \text{Uniform}(0, 1)$, for all $i \in [1, n] \cap \mathbb{N}$. Show that

$$X = \max \left(\left\{ n \left| \prod_{i=1}^n (U_i) \geq e^{-\lambda} \right. \right\} \right), \quad (1)$$

where $\prod_{i=1}^0 (U_i) = 1$.

$$\begin{aligned} \lambda &\geq \sum_{i=1}^n (V_i) \\ &= \sum_{i=1}^n (-\log(U_i)) \\ &= -\log \left(\prod_{i=1}^n (U_i) \right) \\ \log \left(\prod_{i=1}^n (U_i) \right) &\geq -\lambda \\ \prod_{i=1}^n (U_i) &\geq e^{-\lambda}. \end{aligned}$$

Hence,

$$\begin{aligned} X &= \max \left(\left\{ n \left| \sum_{i=1}^n (V_i) \leq \lambda \right. \right\} \right) \\ &= \max \left(\left\{ n \left| \prod_{i=1}^n (U_i) \geq e^{-\lambda} \right. \right\} \right). \end{aligned}$$

(iii) It can be shown that (1) is equivalent to

$$X = \min \left(\left\{ n \left| \prod_{i=1}^n (U_i) < e^{-\lambda} \right. \right\} \right) - 1.$$

This result may be used to simulate a Poisson random variable with parameter λ . If we continue generating $\text{Uniform}(0, 1)$ random variables U_i until their product falls below $e^{-\lambda}$, then the number required, minus 1, is Poisson with parameter λ . Implement this procedure in R to generate 1000 realizations of a Poisson random variable with parameter $\lambda = 5$. Produce the histogram of the generated values $\{x_i : i \in [1, 1000] \cap \mathbb{N}\}$. Find the mean of these values, and compare with its theoretical counterpart.

```
set.seed(46375058)
elam <- exp(-5) # e to the power of negative lambda
x_i <- rep(1 - 1, times = 1000) # all x_i start at 0
prod <- runif(1000, 0, 1) # starting test values
logic_test <- prod >= elam # check which x_i needs adjustment

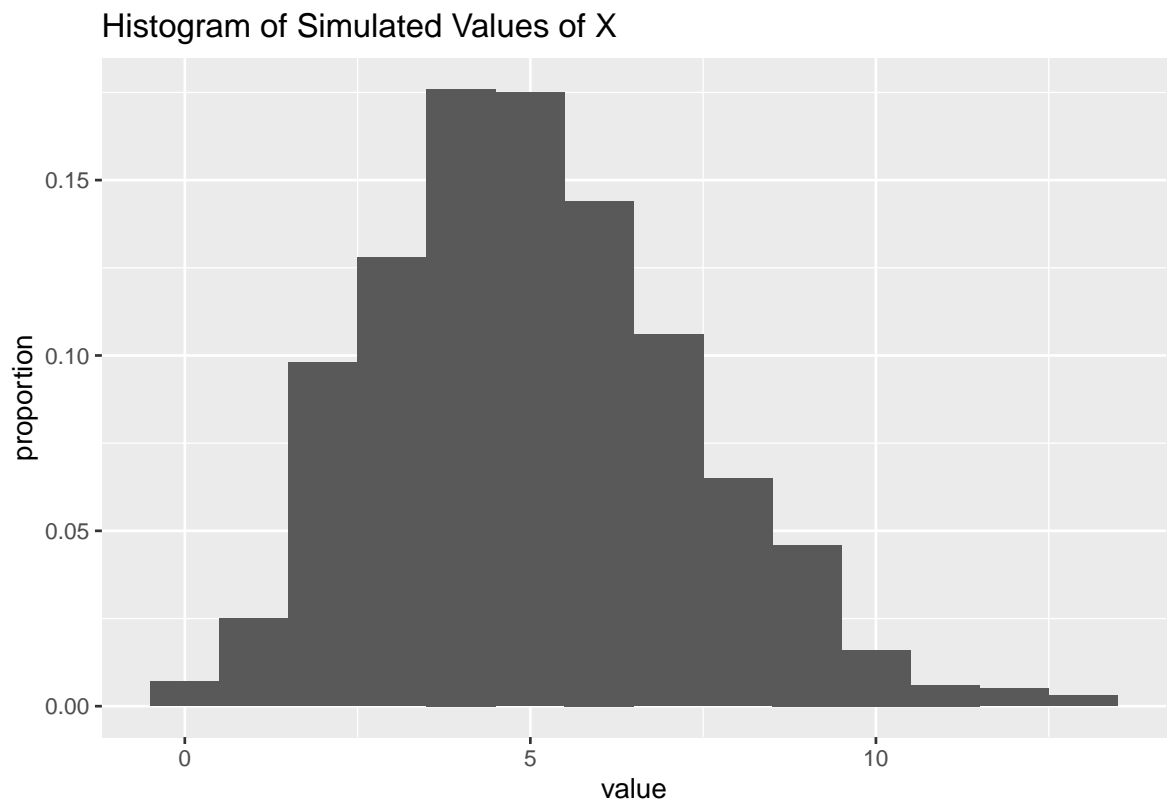
while (FALSE == all(!logic_test)) {
  t <- sum(logic_test) # number of x_i that will increase by 1
  x_i[logic_test] <- x_i[logic_test] + rep(1, times = t) # increase required x_i by 1
```

```

prod[logic_test] <- prod[logic_test] * runif(t, 0, 1) # change required test values
logic_test <- prod >= elam # check which x_i needs adjustment
}

ggplot(tibble(x_i = x_i)) +
  geom_histogram(aes(x = x_i, y = ..count../sum(..count..)), binwidth = 1) +
  labs(title = "Histogram of Simulated Values of X",
       x = "value", y = "proportion")

```



```
(mu <- mean(x_i))
```

```
## [1] 5.049
```

The mean of the simulation data is 5.0490, while the theoretical mean is 5. The relative error is 0.0098. A low absolute relative error and a histogram shaped like a Poisson distribution provides empirical evidence that X is distributed as a Poisson distribution.

Question 3

Let $\{X_n, n \in \mathbb{N} \setminus \{0\}\}$ be a sequence of i.i.d. Bernoulli random variables with parameter $\frac{1}{2}$. Let X be a Bernoulli random variable taking the values 0 and 1 with probability $\frac{1}{2}$ each and let $Y = 1 - X$.

(a) Explain why $X_n \xrightarrow{d} X$ and $X_n \xrightarrow{d} Y$.

$$p_X(x) = \begin{cases} 0.5 & \text{if } x = 1 \\ 0.5 & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases}$$

$$= p_Y(x).$$

$$\lim_{n \rightarrow \infty} (p_{X_n}(x)) = \lim_{n \rightarrow \infty} \left(\begin{cases} 0.5 & \text{if } x = 1 \\ 0.5 & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases} \right)$$

$$= \begin{cases} 0.5 & \text{if } x = 1 \\ 0.5 & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases}$$

$$= p_X(x) = p_Y(x).$$

Hence, $X_n \xrightarrow{d} X$ and $X_n \xrightarrow{d} Y$.

(b) Show that $X_n \not\xrightarrow{P} Y$, that is, X_n does not converge to Y in probability.

Assuming that X_n and Y are independent to each other,

$$X_n - Y = x \quad \text{with probability} \quad \sum_{a \in \text{supp}(X)} (p_{X_n}(a)p_Y(x-a))$$

$$= \begin{cases} -1 & \text{with probability } 0.25 \\ 0 & \text{with probability } 0.5 \\ 1 & \text{with probability } 0.25 \\ k & \text{with probability } 0, \text{ where } k \text{ represents any other value} \end{cases}.$$

$$|X_n - Y| = \begin{cases} 0 & \text{with probability } 0.5 \\ 1 & \text{with probability } 0.5 \\ k & \text{with probability } 0, \text{ where } k \text{ represents any other value} \end{cases}.$$

Let $\varepsilon > 0$.

$$P(|X_n - Y| \geq \varepsilon) = \begin{cases} 0.5 & \text{if } \varepsilon \leq 1 \\ 0 & \text{if } \varepsilon > 1 \end{cases}.$$

Hence, $P(|X_n - Y| \geq \varepsilon) \neq 0$ for all $\varepsilon > 0$, and $X_n \not\xrightarrow{P} Y$.

Question 4

Consider the function

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 \exp(-x_1^2) & \text{if } x_1 > 0 \text{ and } x_2 > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Check whether it is a valid joint probability density function.

$$\begin{aligned}
 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (f(x_1, x_2)) \, dx_1 \right) dx_2 &= \int_0^{\infty} \left(\int_0^{\infty} (4x_1 x_2 \exp(-x_1^2)) \, dx_1 \right) dx_2 \\
 &= \int_0^{\infty} \left(\int_0^{\infty} \left(-2 \frac{d(-x_1^2)}{dx_1} x_2 \exp(-x_1^2) \right) dx_1 \right) dx_2 \\
 &= \int_0^{\infty} \left([-2x_2 \exp(-x_1^2)]_{x_1=0}^{x_1=\infty} \right) dx_2 \\
 &= \int_0^{\infty} (2x_2) \, dx_2 \\
 &\neq 1.
 \end{aligned}$$

Hence, $f(x_1, x_2)$ is not a valid probability density function.

Question 5

Let X be a discrete random variable with probability function $f_X(x)$, and suppose that $a \leq X \leq b$. Define the *tail generating function*

$$T_X(z) = \sum_{x=a}^{b-1} (z^x P(X > x)).$$

- (a) Show that $(1-z)T_X(z) = z^a - G_X(z)$, where $G_X(z)$ is the probability generating function of X . In particular, if X is a non-negative discrete random variable, show that $(1-z)T_X(z) = 1 - G_X(z)$.

$$\begin{aligned}
 (1-z)T_X(z) &= \sum_{x=a}^{b-1} (z^x P(X > x)) - \sum_{x=a}^{b-1} (z^{x+1} P(X > x)) \\
 &= \sum_{x=a}^{b-1} (z^x P(X > x)) - \sum_{x=a+1}^b (z^x P(X > x-1)) \\
 &= \sum_{x=a+1}^b (z^x P(X > x)) + z^a P(X > a) - \sum_{x=a+1}^b (z^x P(X > x-1)) \quad \text{since } P(X > b) = 0 \\
 &= z^a P(X > a) + \sum_{x=a+1}^b (z^x [P(X > x) - P(X > x-1)]) \\
 &= z^a (1 - P(X \leq a)) + \sum_{x=a+1}^b (z^x [P(X > x) - P(X > x) - P(X = x)]) \\
 &= z^a (1 - P(X = a)) - \sum_{x=a+1}^b (z^x P(X = x)) \\
 &= z^a - \sum_{x=a}^b (z^x P(X = x)) \\
 &= z^a - G_X(z).
 \end{aligned}$$

If X is non-negative and discrete, then $0 \leq X$, and substituting $a = 0$,

$$\begin{aligned}(1-z)T_X(z) &= z^0 - G_X(z) \\ &= 1 - G_X(z).\end{aligned}$$

(b) Using the result from (a) for a non-negative discrete random variable X , show that $E(X) = T_X(1)$ and $Var(X) = 2T'_X(1) + T_X(1) - (T_X(1))^2$.

$$\begin{aligned}E(X) &= \left[\frac{d}{dz} (G_X(z)) \right] (z=1) \\ &= \left[\frac{d}{dz} (1 - (1-z)T_X(z)) \right] (z=1) \quad \text{from part (a)} \\ &= [T_X(z) - (1-z)T'_X(z)] (z=1) \\ &= T_X(1).\end{aligned}$$

$$\begin{aligned}E(X(X-1)) &= \left[\frac{d^2}{(dz)^2} (G_X(z)) \right] (z=1) \\ &= \left[\frac{d}{dz} (T_X(z) - (1-z)T'_X(z)) \right] (z=1) \\ &= [T'_X(z) + T'_X(z) - (1-z)T''_X(z)] (z=1) \\ &= 2T'_X(1) \\ E(X^2) &= 2T'_X(1) + E(X) \\ &= 2T'_X(1) + T_X(1).\end{aligned}$$

$$\begin{aligned}Var(X) &= E(X^2) - (E(X))^2 \\ &= 2T'_X(1) + T_X(1) - (T_X(1))^2.\end{aligned}$$

(c) Let random variables $\{Y_i, i \geq 1\}$ be independently and uniformly distributed on $[1, n] \cap \mathbb{N}$. Let $S_k = \sum_{i=1}^k (Y_i)$, and define $\tau_n = \min(\{k | S_k > n\})$. Thus, τ_n is the smallest number of the Y_i required to achieve a sum exceeding n . Show that

$$P(S_j \leq n) = \frac{1}{n^j} \binom{n}{j}.$$

The total number of ways that $1 \leq S_1 < S_2 < \dots < S_k \leq n$ can occur is $\binom{n}{k}$. The total number of ways that $S_j \in \mathbb{N}$ can occur is n^j . Hence, $P(S_j \leq n) = \frac{1}{n^j} \binom{n}{j}$.

(d) Show that $\tau_n \geq j+1$ if and only if $S_j \leq n$.

$$\begin{aligned}S_j \leq n &\iff \max(\{k | S_k \leq n\}) \geq j \\ &\iff \min(\{k | S_k > n\}) - 1 \geq j \\ &\iff \tau_n \geq j+1.\end{aligned}$$

(e) Find the tail generating function $T_{\tau_n}(z) = \sum_{j=0}^n (z^j P(\tau_n > j))$.

$$\begin{aligned}
 T_{\tau_n}(z) &= \sum_{j=0}^n (z^j P(\tau_n > j)) \\
 &= \sum_{j=0}^n (z^j P(\tau_n \geq j+1)) \\
 &= \sum_{j=0}^n (z^j P(S_j \leq n)) \quad \text{from part (d)} \\
 &= \sum_{j=0}^n \left(z^j \frac{1}{n^j} \binom{n}{j} \right) \quad \text{from part (c)} \\
 &= \sum_{j=0}^n \left(\binom{n}{j} \left(\frac{z}{n} \right)^j 1^{n-j} \right) \\
 &= \left(1 + \frac{z}{n} \right)^n \quad \text{from applying the binomial formula.}
 \end{aligned}$$

(f) Using the results from (b) and (e), calculate $E(\tau_n)$ and $Var(\tau_n)$.

$$\begin{aligned}
 E(\tau_n) &= T_{\tau_n}(1) \\
 &= \left(1 + \frac{1}{n} \right)^n. \\
 Var(\tau_n) &= 2T'_{\tau_n}(1) + T_{\tau_n}(1) - (T_{\tau_n}(1))^2 \\
 &= 2 \left[n \left(1 + \frac{z}{n} \right)^{n-1} n^{-1} \right] (z=1) + \left(1 + \frac{1}{n} \right)^n - \left(1 + \frac{1}{n} \right)^{2n} \\
 &= 2 \left(1 + \frac{1}{n} \right)^{n-1} + \left(1 + \frac{1}{n} \right)^n - \left(1 + \frac{1}{n} \right)^{2n}.
 \end{aligned}$$

(g) Find the probability generating function $G_{\tau_n}(z)$.

$$\begin{aligned}
 G_{\tau_n}(z) &= 1 - (1-z)T_{\tau_n}(z) \\
 &= 1 - (1-z) \left(1 + \frac{z}{n} \right)^n.
 \end{aligned}$$

(h) Find the probability function of τ_n .

$$p_{\tau_n}(k) = \begin{cases} \frac{1}{k!} \left[\frac{d^k}{(dz)^k} (G_{\tau_n}(z)) \right] (z=0) & \text{if } k \in [2, n] \cap \mathbb{N} \\ 0 & \text{if otherwise} \end{cases}.$$

$$\left[\frac{d}{dz} (G_{\tau_n}(z)) \right] (z=0) = \left[\frac{d}{dz} \left(1 - (1-z) \left(1 + \frac{z}{n} \right)^n \right) \right] (z=0)$$

$$\begin{aligned}
&= \left[\left(1 + \frac{z}{n}\right)^n - (1-z) \left(1 + \frac{z}{n}\right)^{n-1} \right] (z=0) \\
&= 0. \\
\left[\frac{d^2}{(dz)^2} (G_{\tau_n}(z)) \right] (z=0) &= \left[\left(1 + \frac{z}{n}\right)^{n-1} + \left(1 + \frac{z}{n}\right)^{n-1} - \frac{n-1}{n} (1-z) \left(1 + \frac{z}{n}\right)^{n-2} \right] (z=0) \\
&= \left[2 \left(1 + \frac{z}{n}\right)^{n-1} - \frac{n-1}{n} (1-z) \left(1 + \frac{z}{n}\right)^{n-2} \right] (z=0) \\
&= 2 - \frac{n-1}{n}. \\
\left[\frac{d^3}{(dz)^3} (G_{\tau_n}(z)) \right] (z=0) &= \left[2 \frac{n-1}{n} \left(1 + \frac{z}{n}\right)^{n-2} + \frac{n-1}{n} \left(1 + \frac{z}{n}\right)^{n-2} - \frac{(n-1)(n-2)}{n^2} (1-z) \left(1 + \frac{z}{n}\right)^{n-3} \right] (z=0) \\
&= \left[3 \frac{n-1}{n} \left(1 + \frac{z}{n}\right)^{n-2} - \frac{(n-1)(n-2)}{n^2} (1-z) \left(1 + \frac{z}{n}\right)^{n-3} \right] (z=0). \\
\left[\frac{d^4}{(dz)^4} (G_{\tau_n}(z)) \right] (z=0) &= \left[4 \frac{(n-1)(n-2)}{n^2} \left(1 + \frac{z}{n}\right)^{n-3} - \frac{(n-1)(n-2)(n-3)}{n^3} (1-z) \left(1 + \frac{z}{n}\right)^{n-4} \right] (z=0). \\
\\
\left[\frac{d^k}{(dz)^k} (G_{\tau_n}(z)) \right] (z=0) &= \left[k \frac{P_{k-2}^{n-1}}{n^{k-2}} \left(1 + \frac{z}{n}\right)^{n-(k-1)} - \frac{P_{k-1}^{n-1}}{n^{k-1}} (1-z) \left(1 + \frac{z}{n}\right)^{n-k} \right] (z=0) \\
&= k \frac{P_{k-2}^{n-1}}{n^{k-2}} - \frac{P_{k-1}^{n-1}}{n^{k-1}} \\
&= k \frac{\frac{(n-1)!}{((n-1)-(k-2))!}}{n^{k-2}} - \frac{\frac{(n-1)!}{((n-1)-(k-1))!}}{n^{k-1}} \\
&= \frac{(n-1)!}{n^{k-2}} \left(\frac{k}{(n-k+1)!} - \frac{1}{n(n-k)!} \right) \\
&= \frac{(n-1)!}{n^{k-2}} \left(\frac{kn}{n(n-k+1)!} - \frac{n-k+1}{n(n-k+1)!} \right) \\
&= \frac{(n-1)!}{n^{k-2}} \left(\frac{kn - n + k - 1}{n(n-k+1)!} \right) \\
&= \frac{(n-1)!}{n^{k-2}} \cdot \frac{(n+1)(k-1)}{n(n-k+1)!} \\
&= \frac{(k-1) \cdot (n+1)!}{n^k(n-k+1)!}. \\
\frac{1}{k!} \left[\frac{d^k}{(dz)^k} (G_{\tau_n}(z)) \right] (z=0) &= \frac{(k-1) \cdot (n+1)!}{n^k \cdot k! (n+1-k)!} \\
&= \frac{(k-1) \binom{n+1}{k}}{n^k}.
\end{aligned}$$

Hence,

$$p_{\tau_n}(k) = \begin{cases} \frac{(k-1) \binom{n+1}{k}}{n^k} & \text{if } k \in [2, n] \cap \mathbb{N} \\ 0 & \text{if otherwise} \end{cases}.$$

(i) What is the limiting probability function of τ_n as $n \rightarrow \infty$?

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\frac{(k-1) \binom{n+1}{k}}{n^k} \right) &= \lim_{n \rightarrow \infty} \left(\frac{(k-1) \cdot \frac{1}{k!} \cdot \prod_{j=0}^{k-1} (n+1-j)}{n^k} \right) \\
&= \lim_{n \rightarrow \infty} \left((k-1) \cdot \frac{1}{k!} \cdot \prod_{j=0}^{k-1} \left(1 + \frac{1-j}{n} \right) \right) \\
&= \frac{k-1}{k!}.
\end{aligned}$$

$$\text{So } \lim_{n \rightarrow \infty} (p_{\tau_n}(k)) = \begin{cases} \frac{k-1}{k!} & \text{if } k \in \mathbb{N} \setminus \{0, 1\} \\ 0 & \text{if otherwise} \end{cases}.$$