MTH630: Graph Theory and Combinatorics

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1 Introduction

Topics to cover: Introduction: who am I, what is this course what is a Proof what is graph theory what are the topics we need to cover what depends on what

Acknowledgments

The author would like to thank ...

2 Combinatorics

This is the section on Combinatorics, still to be completed.

- induction

3 Set Theory

Definition 3.1.

1. *A* ***Set*** *is a collection of distinct objects, none of which is the set itself. If a is an object lying in the set A, we write $ainA$.*
2. *A set containing no elements is called the* ***empty set****, or the* ***null set****, and is written $\varnothing$ or $\{\}$.*
3. *A set A is said to be a* ***subset*** *of the set B, written $A \subseteq B$ if every element of A is also an element of B.*
4. *A set A is said to be a* ***equal to*** *the set B, written $A = B$ if $A \subseteq B$ and $B \subseteq A$.*

If it is possible to enumerate the elements of A, we do so with the notation:

$$A = \{a, \pi, \frac{45}{36}, \text{“}Massachusetts''\}.$$

Add some discussion about what this means?

Theorem 3.2. *There is only one empty set.*

Lemma 3.3. *(transitivity of subset) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Other possibilities: sets are subsets of themselves, sets are not elements of themselves.
cardinality
bijection/one-to-one correspondence

exercises (interject above)

list subsets of $\{1, 2, 3\}$.
how many subsets of a given set? prove!
the natural numbers are in bijection with the even numbers

4 Graphs

This is the section on graphs, still to be completed.

Definition 4.1. *1. A **graph** $G = (V, E)$ is a pair of sets V and E, where V is a non-empty set and E is a (possibly empty) set consisting only of two-element sets of the form $\{a, b\}$, where $a \in V$ and $b \in V$. The set $V = V(G)$ is called the set of **vertices** of G and the set $E = E(G)$ is called the set of **edges** of G.*

2. The number of vertices in a graph is denoted by v and the number of edges in a graph is denoted by e. It is possible that $v = \infty$ or $e = \infty$, meaning that there is no such (finite) number.

*3. If $e = (v_1, v_2) \in E$, then we say that e **connects** v_1 and v_2 and that v_1 and v_2 are **adjacent**.*

4. Graph Diagram finish! what does it mean to "represent"?

*5. Two graphs are **equal** if they have equal vertex and edge sets. Two graph diagrams are equal if they represent equal graphs.*

Another name for vertex is *node*.

Lemma 4.2. *Let G be a graph. Then G has no* loops, i.e. *edges connecting a vertex to itself, and G has no* skeins, i.e. *collections of more than one edge connecting a pair of vertices.*

directed vs. undirected

Example 4.3. *1. null graph*

2. cyclic graph C_v

3. complete graph K_v

Theorem 4.4. *Let K_v denote the complete graph on v vertices. Then $e = |E(K_v)| = \frac{1}{2}v \cdot (v - 1)$.*

Definition 4.5. *compliment and subgraph, isomorphism, supergraph*

equal implies isomorphic
isomorphism is an equivalence relation
isomorphism implies equal numbers of Vs and Es
definition of degree/valence
isomorphism implies the set of degrees is the same and the number of vertices of degree n is the same.
non-isomorphism examples, e.g. `non-iso_graphs.png`

exercises (interject above)

wheel graphs, draw some and prove number of edges.
determine and prove the number of edges in $\overline{G}$
if
determine all numbers v such that $C_v \cong K_v$. prove.
Prove that $C_v \cong \overline{C_v}$ if and only if $v = 5$.
$G \cong \overline{G}$ implies that v or $v - 1$ is divisible by 4.
Maybe theorem: if $G_1 \cong G_2$ and $A_1 \subseteq G_1$ then there exists a subgraph $A_2 \subseteq G_2$ with $A_1 \cong A_2$. Then, reprove the non-iso from 4
? prove the number of isomorphism classes of v=3 is 7. "Classify all graphs with 3 vertices up to isomorphism."
$G_1 \cong G_2$ iff $\overline{G_1} \cong \overline{G_2}$.
define bipartite, prove non-isomorphism of a pair

5 Planar Graphs

This is the section on planar graphs, still to be completed.

- Definition of graph projection
- A Graph is planar if it (is isomorphic to?) a graph with a projection drawn in a plane with no edge-crossings (define)
- examples
- Jordan Curve Theorem: If C is a continuous simple closed curve in a plane and two points x and y of C are joined by a continuous simple arc L such that $L \cap C = \{x, y\}$, then except for its endpoints L is entirely contained in one of the two regions of $\mathbb{R}^2 \backslash C$.
- $K_{3,3}$ is nonplanar (not using Euler)
- K_5 is nonplanar
- Any subgraph of a planar graph is planar
- corollary any supergraph of a nonplanar graph is nonplanar
- If G may be obtained from H by replacing an edge (x, y) of H with another vertex v, and a pair of edges $(x, v), (v, y)$, then G is said to be obtained from H via an **edge expansion**. If G may be obtained from H by a finite sequence of edge expansions, then G is an **expansion** of H.
- (maybe?) If G is obtained from H by a sequence of expansions and passing to supergraphs, then G is said to be an **expanded supergraph** of H (my definition) (NOTE: this is equivalent to being a supergraph of an expansion. Prove!)
- Every expanded supergraph of $K_{3,3}$ or K_5 is nonplanar.
- Kuratowski's Theorem: a graph is nonplanar if and only if it is an expanded supergraph of $K_{3,3}$ or K_5.
- exercise: examples of large graphs, is it planar?
- TODO: add exercises

6 Euler's Formula

- A **walk**, or **path** is a sequence $v_1, v_2, \ldots, v_n$ of not-necessarily-distinct vertices in a graph G such that (v_i, v_{i+1}) is an edge of G for $1 \leqslant i < n$.
- A graph is **connected** if every pair of vertices may be joined by a path. Otherwise, it is disconneted
- Disclaimer: path connected vs connected?
- examples
- Given a planar graph diagram D, a **face** of D is the set of all points in $\mathbb{R}^2 \backslash D$ that may be joined by a continuous arc in $\mathbb{R}^2 \backslash D$. The number of faces of D is denoted as
- prove if G is a planar graph, then the number of faces of *any* planar diagram of G is the same.
- A graph is **polygonal** if it is planar, connected, and has the propery that every edge borders on two different faces
- If G is polygonal then $v - e + f = 2$. (two students, longish)
- If G is planar and connected, then $v - e + f = 2$.

- K_5 and $K_{3,3}$ are nonplanar, revisited.
- If G is planar (and connected? not necessary) then G has a vertex of degree $\leqslant 5$ (Q?)
- exercises from 4
- a graph is **regular** if all its vertices have the same degree, said "regular of degree d".
- examples
- a graph is **platonic** if it is polygonal, regular, and all its faces are bounded by the same number of edges
- (what if we remove the last condition?)
- examples
- Theorem: Apart from K_1 and the cyclic graphs, there are 5 platonic graphs. Prove by breaking into d cases Needs lemata:
  - if G is regular of degree d then $e = dv/2$.
  - If G is platonic of degree d, and n is the number of edges bounding each face, then $f = dv/n$.
- exercises

7 Colorings

- A graph has been (n-)**colored** if each vertex has been assigned a number from $\{1, 2, \ldots, n\}$ such that no edge joins vertices with the same number ("color"). We say that a graph G is n**-colorable** if it may be n-colored.
- examples
- The **chromatic number** of a graph G is the smallest n such that G is n-colorable, denoted $X(G)$.
- examples
- (DNP) Four-color theorem: Every planar graph has $X \leqslant 4$.
- Five-color theorem: Every planar graph has $X \leqslant 5$. (induction)
- Every planar graph having a vertex of degree $\leqslant 4$ has $X \leqslant 4$. This is crazy! Compare to the theorem about every planar graph having a vertex $\leqslant 5$.
- reading about the four-color theorem and its proof. Do you believe it?
- Map colorings! define dual graph
- exercises

8 Eulerian and Hamiltonian Path

- A path is **closed** if $v_1 = v_n$, otherwise it is **open**.
- A path is **simple** if $|\{v_1, v_2, \ldots, v_n\}| = n$ if open or $n - 1$ if closed.
- simple is equivalent to having no interior overlap
- examples
- An Eulerian path uses every edge in the graph exactly once
- examples
- they're simple
- A connected graph has a closed Eulerian path if and only if every vertex is even.
- There is an Eulerian cycle beginning at any vertex in a graph with all even vertices
- A connected graph has an open Eulerian path if and only if every vertex is even except for exactly two.
- the path must begin at one of the two odd vertices
- A Hamiltonian path is one which uses every vertex exactly once (if closed, the first and last vertex is repeated)
- examples
- lemma If the sum of the degrees of every pair of vertices of a graph is at least $v - 1$, then
  - every pair of vertices are either adjacent to each other or to a common third vertex, and
  - G is connected
- if the sum of the degrees of every pair of vertices of G is at least $v - 1$, then G has an open Hamiltonian path
- if the sum of the degrees of every pair of vertices of G is at least v, then G has an closed Hamiltonian path
- a **skein** is an object consisting of two vertices connected by two or more lines (finite?)
- a **multigraph** $M(G)$ is an object consisting of a graph G where some of its edges are replaced by skeins. G is called a **generator** of M
- generators are unique
- examples
- define some terms for multigraphs and prove them?
- a walk in a multigraph is, a euler walk is, a hamilton walk is...
- A connected multigraph has a closed euler walk iff every vertex is even.
- same thing again with the open walk and two odd edges
- Königsberg Bridge Problem
- The sum of the degrees of the vertices of a multigraph is 2e
- Every multigraph has an even number of odd vertices
- applications? aren't there tons?
- exercises