MTH630: Graph Theory and Combinatorics

Contents

1 Introduction

Topics to cover: Introduction: who am I, what is this course what is a Proof what is graph theory what are the topics we need to cover what depends on what

Acknowledgments

The author would like to thank ...

2 Combinatorics

This is the section on Combinatorics, still to be completed.

- induction

3 Set Theory

Definition 3.1. *1. A **Set** is a collection of distinct objects, none of which is the set itself. If a is an object belonging to the set A, we write $a \in A$, and say "a is an element of A".*

2. *A set containing no elements is called the **empty set**, or the **null set**, and is written $\varnothing$ or $\{\}$.*

3. *A set A is said to be a **subset** of the set B, written $A \subseteq B$ if every element of A is also an element of B.*

4. *A set A is said to be a **equal to** the set B, written $A = B$ if $A \subseteq B$ and $B \subseteq A$.*

If it is possible to enumerate the elements of A, we do so with the notation:

$$A = \{a, \pi, \frac{45}{36}, \text{“Massachusetts”}\}.$$

Remark 3.2. *You may find the definition of a mathematical set nebulous and confusing. What's a "collection"? What's an "object", and what does it mean for them to be "distinct"? In truth, while it is possible to formally define all of these concepts, it is typically the case that a student has an intuitive understanding of a set, and can begin from that.*

However, this should be the only such definition in the course.

Exercise 3.3. *List all the subsets of $\{1, 2, 3\}$.*

Notation 3.4. *(Set Builder Notation) Let A be a set, and for all $x \in A$, let $p(x)$ be a proposition about x which may be true or false. Then we may build a set by taking all those elements of A for which the proposition is true; such a set may be written down using **set builder notation**:*

$$S = \{x \in A \mid p(x)\},$$

and we read this as "S is (equal to) the set of all x in A such that p of x." One important note is that a set A must exist in order to use set builder notation; as a result of this, we will use the term universe of discourse, often denoted by X, to describe any reasonably conceivable objects that may be placed into a set. You will see this appearing in the definitions ahead (see for example Definition 1).

Exercises 3.5. *Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers.*

1. *Translate the set $\{1, 2, 3, 4, 5\}$ into set builder notation.*

2. *Write down, without the uses of ellipses ("$\ldots$"), notation defining the set of even natural numbers; repeat for the set of odd natural numbers divisible by 5 (one may use "$7 \mid 14$" to say that "7 divides 14").*

Theorem 3.6. *There is only one empty set.*

Theorem 3.7. *(transitivity of subset) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

3.1 Getting new sets from old

Definition 3.8. *Let A and B be sets, and let X denote the universe of discourse.*

1. *The set $A \cup B = \{x \in X \mid x \in A \vee x \in B\}$ is called the **union** of A and B.*

2. *The set $A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$ is called the **intersection** of A and B.*

3. *The set $A \smallsetminus B = \{x \in A \mid x \notin B\}$ is called the **(relative) complement** of A in B.*

Theorem 3.9. *For all sets A and B, if $A \subseteq A \cap B$ then $A \cup B \subseteq B$.*

Theorem 3.10. *For all sets A, B, C, and D, if $A \subseteq C$ and $B \subseteq D$ then $A \cup B \subseteq C \cup D$.*

Theorem 3.11. *For all sets A, B, C, and D, if $A \subseteq C$ and $B \subseteq D$ then $A \cap B \subseteq C \cap D$.*

Theorem 3.12. *Let A, B, and X be sets. If $A \subseteq B$, then $X \smallsetminus B \subseteq X \smallsetminus A$.*

Exercise 3.13. *Write down and prove the inverse of Theorem 3.12. (The inverse of the statement $p(x)$ is $\neg p(x)$.)*

Theorem 3.14. *Let A and B be sets. Then $A \smallsetminus B = \varnothing$ if and only if $A \subseteq B$.*

Theorem 3.15. *For sets A and B, $(A \smallsetminus B) \cup (B \smallsetminus A) = (A \cup B) \smallsetminus (A \cap B)$.*

For our purposes, a *claim* is something that may or may not be true, and we need to determine whether or not it is true.

Claim 3.16. *For all sets A, B, and C, if $A \subseteq B \cup C$ then $A \subseteq B$ or $A \subseteq C$.*

3.2 Bijections and cardinality

Definition 3.17. *Let A and B be sets.*

1. *Let $a \in A$ and $b \in B$. Then the **ordered pair** of a and b, written (a, b), is pairing of the elements a and b into an ordered grouping. Strictly speaking (though this intuitive definition typically suffices), one may define $(a, b) = \{\{a\}, \{a, b\}\}$. We refer to a and b as elements of (a, b), even though strictly speaking they are not.*

2. *A **bijection**, or a **one-to-one correspondence**, between A and B is a set C with all of the following properties.*

   - *Every element of C consists of an ordered pair (a, b) where $a \in A$ and $b \in B$.*
   - *(injective) Every element of a exists as an element of exactly one element of C.*
   - *(sujective) Every element of b exists as an element of exactly one element of C.*

   *We say that A and B are **in bijection** (or sometimes bijective) if there exists a bijection between them; this is sometimes written as $A \cong B$, but it often just written out in words.*

Remark 3.18. *In a traditional set theory course, one uses ordered pairs to first define cartesian products, and then relations, functions, injections, surjections, domain, co-domain, range, etc. before defining bijections. For our purposes, bijections will suffice.*

Theorem 3.19 (Bijectivity is an equivalence relation)**.** *Let A, B, and C be sets.*

1. *(reflexivity) A is in bijection with itself.*

2. *(symmetry) If A is in bijection with B, then B is in bijection with A.*

3. *(transitivity) If A is in bijection with B, and B is in bijection with C, then A is in bijection with C.*

Remark 3.20. *The fact that bijections satisfy the above three properties give it the status of being what's called an **equivalence relation**. We will see equivalence relations again in the future when we discuss graphs. One often considers equivalence relations to be a notion of "sameness": if A is in bijection with B, then they're essentailly the same in my mind.*

Definition 3.21. *If a set A is in bijection with the set $\{1, 2, 3, 4, \ldots, n\}$, then the **cardinality** of A is given by n, written $|A| = n$, and we say that A is **finite**. If a set is in bijection with the natural numbers, then we say that it is **countably infinite**.*

Theorem 3.22. *If $|A| \neq |B|$, then A is not in bijection with B.*

Question 3.23. *Is the converse of Theorem 3.22 true? Prove or disprove. (The converse of a statement $x \implies y$ is $y \implies x$.)*

Theorem 3.24. *Being countably infinite and finite are mutually exclusive set properties.*

3.3 Exercises

1. Let A, B, and C be sets. Prove that if $A \subseteq C$ and $B \subseteq C$ then $AcupB \subseteq C$.

2. Given a set A with $|A| = n$, how many subsets does A have? Prove your answer.

3. Prove that the natural numbers are in bijection with the even numbers.

4. Prove that the natural numbers are in bijection with the integers.

5. Let C be a bijection between the natural numbers ($\mathbb{N}$) and the integers ($\mathbb{Z}$), so that $C \subseteq \{(x, y) \mid x \in \mathbb{N} \wedge \mathbb{Z}\}$. Show that there exist elements (a, b) and (x, y) in C such that $a > x$ and $b < y$.

6. Prove that the natural numbers are in bijection with the set of ordered pairs $\{(n, a) \mid n \in \mathbb{N} \wedge a \in \{1, 2, 3\}\}$.

7. Prove that the natural numbers are in bijection with the set of ordered pairs $\{(n, m) \mid n \in \mathbb{N} \wedge a \in \mathbb{N}\}$.

8. Prove that the set of words in this sentence is not in correspondence with the set of words in the preamble to the U.S. Constitution.

4 Graphs

This is the section on graphs, still to be completed.

Definition 4.1. *1. A **graph** $G = (V, E)$ is a pair of sets V and E, where V is a non-empty set and E is a (possibly empty) set consisting only of two-element sets of the form $\{a, b\}$, where $a \in V$ and $b \in V$. The set $V = V(G)$ is called the set of **vertices** of G and the set $E = E(G)$ is called the set of **edges** of G.*

2. The number of vertices in a graph is denoted by v and the number of edges in a graph is denoted by e. It is possible that $v = \infty$ or $e = \infty$, meaning that there is no (finite) number.

*3. If $e = (v_1, v_2) \in E$, then we say that e **connects** v_1 and v_2 and that v_1 and v_2 are **adjacent**.*

4. Graph Diagram finish! what does it mean to "represent"?

*5. Two graphs are **equal** if they have equal vertex and edge sets. Two graph diagrams are equal if they represent equal graphs.*

Another name for vertex is *node*.

Lemma 4.2. *Let G be a graph. Then G has no* loops, i.e. *edges connecting a vertex to itself, and G has no* skeins, i.e. *collections of more than one edge connecting a pair of vertices.*

directed vs. undirected

Example 4.3. *1. null graph*

2. cyclic graph C_v

3. complete graph K_v

Theorem 4.4. *Let K_v denote the complete graph on v vertices. Then $e = |E(K_v)| = \frac{1}{2}v \cdot (v - 1)$.*

Definition 4.5. *compliment and subgraph, isomorphism, supergraph*

equal implies isomorphic
isomorphism is an equivalence relation
isomorphism implies equal numbers of Vs and Es
definition of degree/valence
isomorphism implies the set of degrees is the same and the number of vertices of degree n is the same.
non-isomorphism examples, e.g. `non-iso_graphs.png`

exercises (interject above)

wheel graphs, draw some and prove number of edges.
determine and prove the number of edges in $\overline{G}$
if
determine all numbers v such that $C_v \cong K_v$. prove.
Prove that $C_v \cong \overline{C_v}$ if and only if $v = 5$.
$G \cong \overline{G}$ implies that v or $v - 1$ is divisible by 4.
Maybe theorem: if $G_1 \cong G_2$ and $A_1 \subseteq G_1$ then there exists a subgraph $A_2 \subseteq G_2$ with $A_1 \cong A_2$. Then, reprove the non-iso from 4
? prove the number of isomorphism classes of v=3 is 7. "Classify all graphs with 3 vertices up to isomorphism."
$G_1 \cong G_2$ iff $\overline{G_1} \cong \overline{G_2}$.
define bipartite, prove non-isomorphism of a pair

5 Planar Graphs

This is the section on planar graphs, still to be completed.

- Definition of graph projection
- A Graph is planar if it (is isomorphic to?) a graph with a projection drawn in a plane with no edge-crossings (define)
- examples
- Jordan Curve Theorem: If C is a continuous simple closed curve in a plane and two points x and y of C are joined by a continuous simple arc L such that $L \cap C = \{x, y\}$, then except for its endpoints L is entirely contained in one of the two regions of $\mathbb{R}^2 \backslash C$.
- $K_{3,3}$ is nonplanar (not using Euler)
- K_5 is nonplanar
- Any subgraph of a planar graph is planar
- corollary any supergraph of a nonplanar graph is nonplanar
- If G may be obtained from H by replacing an edge (x, y) of H with another vertex v, and a pair of edges $(x, v), (v, y)$, then G is said to be obtained from H via an **edge expansion**. If G may be obtained from H by a finite sequence of edge expansions, then G is an **expansion** of H.
- (maybe?) If G is obtained from H by a sequence of expansions and passing to supergraphs, then G is said to be an **expanded supergraph** of H (my definition) (NOTE: this is equivalent to being a supergraph of an expansion. Prove!)
- Every expanded supergraph of $K_{3,3}$ or K_5 is nonplanar.
- Kuratowski's Theorem: a graph is nonplanar if and only if it is an expanded supergraph of $K_{3,3}$ or K_5.
- exercise: examples of large graphs, is it planar?
- TODO: add exercises

6 Euler's Formula

- A **walk**, or **path** is a sequence $v_1, v_2, \ldots, v_n$ of not-necessarily-distinct vertices in a graph G such that (v_i, v_{i+1}) is an edge of G for $1 \leqslant i < n$.
- A graph is **connected** if every pair of vertices may be joined by a path. Otherwise, it is disconneted
- Disclaimer: path connected vs connected?
- examples
- Given a planar graph diagram D, a **face** of D is the set of all points in $\mathbb{R}^2 \backslash D$ that may be joined by a continuous arc in $\mathbb{R}^2 \backslash D$. The number of faces of D is denoted as
- prove if G is a planar graph, then the number of faces of *any* planar diagram of G is the same.
- A graph is **polygonal** if it is planar, connected, and has the propery that every edge borders on two different faces
- If G is polygonal then $v - e + f = 2$. (two students, longish)
- If G is planar and connected, then $v - e + f = 2$.

- K_5 and $K_{3,3}$ are nonplanar, revisited.
- If G is planar (and connected? not necessary) then G has a vertex of degree $\leqslant 5$ (Q?)
- exercises from 4
- a graph is **regular** if all its vertices have the same degree, said "regular of degree d".
- examples
- a graph is **platonic** if it is polygonal, regular, and all its faces are bounded by the same number of edges
- (what if we remove the last condition?)
- examples
- Theorem: Apart from K_1 and the cyclic graphs, there are 5 platonic graphs. Prove by breaking into d cases Needs lemata:
  - if G is regular of degree d then $e = dv/2$.
  - If G is platonic of degree d, and n is the number of edges bounding each face, then $f = dv/n$.
- exercises

7 Colorings

- A graph has been (n-)**colored** if each vertex has been assigned a number from $\{1, 2, \ldots, n\}$ such that no edge joins vertices with the same number ("color"). We say that a graph G is n**-colorable** if it may be n-colored.
- examples
- The **chromatic number** of a graph G is the smallest n such that G is n-colorable, denoted $X(G)$.
- examples
- (DNP) Four-color theorem: Every planar graph has $X \leqslant 4$.
- Five-color theorem: Every planar graph has $X \leqslant 5$. (induction)
- Every planar graph having a vertex of degree $\leqslant 4$ has $X \leqslant 4$. This is crazy! Compare to the theorem about every planar graph having a vertex $\leqslant 5$.
- reading about the four-color theorem and its proof. Do you believe it?
- Map colorings! define dual graph
- exercises

8 Eulerian and Hamiltonian Path

- A path is **closed** if $v_1 = v_n$, otherwise it is **open**.
- A path is **simple** if $|\{v_1, v_2, \ldots, v_n\}| = n$ if open or $n-1$ if closed.
- simple is equivalent to having no interior overlap
- examples
- An Eulerian path uses every edge in the graph exactly once
- examples
- they're simple
- A connected graph has a closed Eulerian path if and only if every vertex is even.
- There is an Eulerian cycle beginning at any vertex in a graph with all even vertices
- A connected graph has an open Eulerian path if and only if every vertex is even except for exactly two.
- the path must begin at one of the two odd vertices
- A Hamiltonian path is one which uses every vertex exactly once (if closed, the first and last vertex is repeated)
- examples
- lemma If the sum of the degrees of every pair of vertices of a graph is at least $v-1$, then
  - every pair of vertices are either adjacent to each other or to a common third vertex, and
  - G is connected
- if the sum of the degrees of every pair of vertices of G is at least $v-1$, then G has an open Hamiltonian path
- if the sum of the degrees of every pair of vertices of G is at least v, then G has an closed Hamiltonian path
- a **skein** is an object consisting of two vertices connected by two or more lines (finite?)
- a **multigraph** $M(G)$ is an object consisting of a graph G where some of its edges are replaced by skeins. G is called a **generator** of M
- generators are unique
- examples
- define some terms for multigraphs and prove them?
- a walk in a multigraph is, a euler walk is, a hamilton walk is…
- A connected multigraph has a closed euler walk iff every vertex is even.
- same thing again with the open walk and two odd edges
- Königsberg Bridge Problem
- The sum of the degrees of the vertices of a multigraph is 2e
- Every multigraph has an even number of odd vertices
- applications? aren't there tons?
- exercises