MTH630: Graph Theory and Combinatorics

Contents

1 Introduction

Topics to cover: Introduction: who am I, what is this course what is a Proof what is graph theory what are the topics we need to cover what depends on what

Acknowledgments

The author would like to thank ...

2 Set Theory

Definition 2.1.

1. A **Set** is a collection of distinct objects, none of which is the set itself. If a is an object belonging to the set A, we write $a \in A$, and say "a is an element of A".
2. A set containing no elements is called the **empty set**, or the **null set**, and is written $\varnothing$ or $\{\}$.
3. A set A is said to be a **subset** of the set B, written $A \subseteq B$ if every element of A is also an element of B.
4. A set A is said to be a **equal to** the set B, written $A = B$ if $A \subseteq B$ and $B \subseteq A$.

If it is possible to enumerate the elements of A, we do so with the notation:

$$A = \{a, \pi, \frac{45}{36}, \text{“Massachusetts”}\}.$$

Remark 2.2. You may find the definition of a mathematical set nebulous and confusing. What's a "collection"? What's an "object", and what does it mean for them to be "distinct"? In truth, while it is possible to formally define all of these concepts, it is typically the case that a student has an intuitive understanding of a set, and can begin from that.

However, this should be the only such definition in the course.

Exercise 2.3. List all the subsets of $\{1, 2, 3\}$.

Notation 2.4. (Set Builder Notation) Let A be a set, and for all $x \in A$, let $p(x)$ be a proposition about x which may be true or false. Then we may build a set by taking all those elements of A for which the proposition is true; such a set may be written down using **set builder notation**:

$$S = \{x \in A \mid p(x)\},$$

and we read this as "S is (equal to) the set of all x in A such that p of x." One important note is that a set A must exist in order to use set builder notation; as a result of this, we will use the term *universe of discourse*, often denoted by X, to describe any reasonably conceivable objects that may be placed into a set. You will see this appearing in the definitions ahead (see for example Definition 1).

Exercises 2.5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers.

1. Translate the set $\{1, 2, 3, 4, 5\}$ into set builder notation.
2. Write down, without the uses of ellipses ("$\ldots$"), notation defining the set of even natural numbers; repeat for the set of odd natural numbers divisible by 5 (one may use "$7 \mid 14$" to say that "7 divides 14").

Theorem 2.6. There is only one empty set.

Theorem 2.7. (transitivity of subset) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

2.1 Getting new sets from old

Definition 2.8. Let A and B be sets, and let X denote the universe of discourse.

1. The set $A \cup B = \{x \in X \mid x \in A \vee x \in B\}$ is called the **union** of A and B.
2. The set $A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$ is called the **intersection** of A and B.
3. The set $A \smallsetminus B = \{x \in A \mid x \notin B\}$ is called the **(relative) complement** of A in B.

Theorem 2.9. For all sets A and B, if $A \subseteq A \cap B$ then $A \cup B \subseteq B$.

Theorem 2.10. For all sets A, B, C, and D, if $A \subseteq C$ and $B \subseteq D$ then $A \cup B \subseteq C \cup D$.

Theorem 2.11. For all sets A, B, C, and D, if $A \subseteq C$ and $B \subseteq D$ then $A \cap B \subseteq C \cap D$.

Theorem 2.12. Let A, B, and X be sets. If $A \subseteq B$, then $X \setminus B \subseteq X \setminus A$.

Exercise 2.13. Write down and prove the *inverse* of Theorem 2.12. (The inverse of the statement $p(x)$ is $\neg p(x)$.)

Theorem 2.14. Let A and B be sets. Then $A \setminus B = \varnothing$ if and only if $A \subseteq B$.

Theorem 2.15. For sets A and B, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

For our purposes, a *claim* is something that may or may not be true, and we need to determine whether or not it is true.

Claim 2.16. For all sets A, B, and C, if $A \subseteq B \cup C$ then $A \subseteq B$ or $A \subseteq C$.

2.2 Bijections and cardinality

Definition 2.17. Let A and B be sets.

1. Let $a \in A$ and $b \in B$. Then the **ordered pair** of a and b, written (a, b), is pairing of the elements a and b into an ordered grouping. Strictly speaking (though this intuitive definition typically suffices), one may define $(a, b) = \{\{a\}, \{a, b\}\}$. We refer to a and b as *elements* of (a, b), even though strictly speaking they are not.

2. A **bijection**, or a **one-to-one correspondence**, between A and B is a set C with all of the following properties.

   - Every element of C consists of an ordered pair (a, b) where $a \in A$ and $b \in B$.
   - (injective) Every element of a exists as an element of exactly one element of C.
   - (sujective) Every element of b exists as an element of exactly one element of C.

   We say that A and B are **in bijection** (or sometimes *bijective*) if there exists a bijection between them; this is sometimes written as $A \cong B$, but it often just written out in words.

Remark 2.18. In a traditional set theory course, one uses ordered pairs to first define cartesian products, and then relations, functions, injections, surjections, domain, co-domain, range, *etc.* before defining bijections. For our purposes, bijections will suffice.

Theorem 2.19 (Bijectivity is an equivalence relation)**.** Let A, B, and C be sets.

1. (reflexivity) A is in bijection with itself.

2. (symmetry) If A is in bijection with B, then B is in bijection with A.

3. (transitivity) If A is in bijection with B, and B is in bijection with C, then A is in bijection with C.

Remark 2.20. The fact that bijections satisfy the above three properties give it the status of being what's called an **equivalence relation**. We will see equivalence relations again in the future when we discuss graphs. One often considers equivalence relations to be a notion of "sameness": if A is in bijection with B, then they're essentailly the same in my mind.

Definition 2.21. If a set A is in bijection with the set $\{1, 2, 3, 4, \ldots, n\}$, then the **cardinality** of A is given by n, written $|A| = n$, and we say that A is **finite**. If a set is in bijection with the natural numbers, then we say that it is **countably infinite**.

Theorem 2.22. If $|A| \neq |B|$, then A is not in bijection with B.

Question 2.23. Is the *converse* of Theorem 2.22 true? Prove or disprove. (The *converse* of a statement $x \implies y$ is $y \implies x$.)

Theorem 2.24. Being countably infinite and finite are mutually exclusive set properties.

2.3 Exercises

1. Let A, B, and C be sets. Prove that if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.
2. Given a set A with $|A| = n$, how many subsets does A have? Prove your answer.
3. Prove that the natural numbers are in bijection with the even numbers.
4. Prove that the natural numbers are in bijection with the integers.
5. Let C be a bijection between the natural numbers ($\mathbb{N}$) and the integers ($\mathbb{Z}$), so that $C \subseteq \{(x, y) \mid x \in \mathbb{N} \wedge \mathbb{Z}\}$. Show that there exist elements (a, b) and (x, y) in C such that $a > x$ and $b < y$.
6. Prove that the natural numbers are in bijection with the set of ordered pairs $\{(n, a) \mid n \in \mathbb{N} \wedge a \in \{1, 2, 3\}\}$.
7. Prove that the natural numbers are in bijection with the set of ordered pairs $\{(n, m) \mid n \in \mathbb{N} \wedge a \in \mathbb{N}\}$.
8. Prove that the set of words in this sentence is not in correspondence with the set of words in the preamble to the U.S. Constitution.

3 Combinatorics

3.1 The Pigeonhole Principle

Theorem 3.1 (Pigeonhole Principle)**.** Let n be a natural number. If $n+1$ objects are to be placed into n boxes, then at least one of the boxes must contain at least two objects.

Examples 3.2.

1. Show that there is some day of the week on which over 1 billion people have been born.
2. Show that given m integers $A = \{a_1, a_2, \ldots, a_m\}$, there exists a consecutive subset $\{a_k, a_{k+1}, \ldots, a_l\} \subseteq A$ whose sum is divisible by m.
3. Show that given $2n$ integers, from any subset of $n+1$ of them there is a pair where one element of which is divisible by the other.

Theorem 3.3 (Chinese Remainder Theorem)**.** Let m and n be relatively prime positive integers, and let $0 \leqslant a \leqslant m-1$ and $0 \leqslant b \leqslant n-1$. Then there is a positive integer x such that the remainder when x is divided by m is a, and the remainder when x is divided by n is b.

Theorem 3.4 (Strong Pigeonhole Principle)**.** Let $q_1, q_2, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \ldots + q_n - n + 1$ objects are put into n boxes, then for at least one i, the i^{th} box contains at least q_i objects.

Examples 3.5.

1. If $n+1$ numbers are chosen from the set $\{1, \ldots, 2n\}$, then there is a pair that differ by 1.
2. If $n+1$ numbers are chosen from the set $\{1, \ldots, 3n\}$, then there is a pair that differ by 2.

Exercise 3.6. Generalize the two examples above and prove.

Example 3.7. Determine an integer m_n such that if m_n points are chosen from an equilateral triangle of side length 1, then there are two whose distance is equal to $\frac{1}{n}$.

3.2 Permutations and Combinations

Theorem 3.8 (Rule of Addition)**.** If a set S can be written as $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \varnothing$, then $|S| = |S_1| + |S_2|$.

Remark 3.9. Note that this should be viewed, along with all the other "Rules of (arithmetic)" in this section, as breaking down or otherwise simplifying a counting problem.

Example 3.10. A student at Phillips Academy wants to take one Mathematics elective or one English elective, but cannot take both. If there are 2 Mathematics electives and 3 English electives on offer, how many options are available to the student this term?

Theorem 3.11 (Rule of Multiplication)**.** Let S be the set of ordered pairs (a, b) of objects, where the a lies in a set of size n and b lies in a set of size m. Then $|S| = m \cdot n$.

Example 3.12. Determine the number of positive integers that divide $32189975201412589275 = 3^4 \cdot 5^2 \cdot 11^7 \cdot 13^8$.

Remark 3.13. Note that the Rule of Multiplication is not:

Let A and B be sets, and let S be the set of ordered pairs (a, b) of objects, where the $a \in A$ and $b \in B$. Then $|S| = |A| \cdot |B|$.

Why not? In order to successfully answer this question, consider the following question: How many 2-digit numbers have distinct and nonzero digits?

Theorem 3.14 (Rule of Subtraction)**.** Let $A \subseteq U$ be sets. Then $|A| = |U| - |\overline{A}|$.

Example 3.15. A set of computer passwords are taken from the characters 0-9 and a-z. How many have repeated symbols? (Hint: use the Rule of Subtraction.)

Theorem 3.16 (Rule of Division)**.** Let S be a finite set that is partitioned into k parts of equal size, say n. Then

$$k = \frac{|S|}{n}.$$

Examples 3.17. 1. How many odd numbers between 1000 and 9999 have distinct digits?

2. How many different 5-digit numbers can be constructed from the digits $1, 1, 1, 3, 5$?

Definition 3.18. An r**-permutation** of a set S of n elements in an ordered arrangement of elements of S of size r . The number of such r-permutations is $P(n, r) = {}_nP_r$.

Theorem 3.19. Let S be a set of size n and $1 \leqslant r \leqslant n$. Then $P(n, r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$.

Corollary 3.20. Let S be a set of size n and $1 \leqslant r \leqslant n$. Then $P(n, r) = \dfrac{n!}{(n-r)!}$.

Exercises 3.21. 1. Find a closed-form expression (*i.e.* no ...) for the number of possible positions of the "15-puzzle": a game consisting of a 4×4 grid of 15 numbers and one blank space, where one may swap positions of the blank space with one of the adjacent (up, down, left, or right) numbered squares.

2. How many 5-digit numbers with unique, non-zero digits are there such that a 5 is never followed by a 6, and vice versa?

Definition 3.22. An r**-combination** of a set S of n elements is a subset of size r. The number of such may be written as $C(n, r) = {}_nC_r = \binom{n}{r}$.

Theorem 3.23. For integers r and n with $0 \leqslant r \leqslant n$, $P(n, r) = r!\binom{n}{r}$, hence

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Corollary 3.24. $\binom{n}{r} = \binom{n}{n-r}$.

Example 3.25. 25 points are drawn on a piece of paper such that no 3 are colinear. How many line segments pass through a pair of them?

Theorem 3.26. $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$.

Exercises 3.27. 1. How many straight-flushes are there in a deck of cards? (5 consecutive cards of the same suit)

2. How many integers larger than 5400 have all distinct digits which are not 2 or 6?

3. In how many ways can six students and six faculty members be seated at a table if the two groups must alternate?

4. In how many ways can 8 indistinguishable rooks be placed of a chess board so that they cannot attack one another?

5. A woman works in a building 9 blocks east and 8 blocks north of her home. When commuting, she never backtracks or takes any route that is longer than the shortest possible path. Suppose the grid of roads between her home and her work is full of possible paths, except for the road that travels one block east from the block which begins 4 blocks east and 3 blocks north from her home. How many viable paths to work does she have?

6. A group of mn players are to be arranged into m teams each with n players. Determine the number of ways this can be arranged if the teams

   (a) have names, and

   (b) are indistinguishable.

Theorem 3.28 (Pascal's Formula)**.** Let n and r be integers with $0 \leqslant r \leqslant n$. Then $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

Exercise 3.29. What does the above have to do with the so-called Pascal's triangle?

Theorem 3.30 (Binomial Theorem)**.** $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + y^n$.

Exercise 3.31. Prove that

1. $3^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k$, and

2. $2^n = \displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$.

3.3 The Inclusion-Exclusion Principle

Exercise 3.32. Find the number of integers between 1 and 600 which are not divisible by 6.

Theorem 3.33 (Inclusion-Exclusion Principle)**.** Let S be a finite set and let $P_1, P_2, \ldots, P_n$ be a collection of properties that some elements of S satisfy. Let A_i denote the subset of S satisfying property P_i for $1 \leqslant i \leqslant n$. Let n_k denote the number of elements in any k-fold intersection of the sets $\{A_i\}$, that is,

$$n_k = \sum_{i_1, i_2, \ldots, i_k} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|,$$

where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$. Then

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_i} = |S| - n_1 + n_2 - n_3 + \ldots + (-1)^n\ n_n.$$

Examples 3.34.

1. How many permutations of the letters `C, A, T, D, O, G, M, A, T, H` have none of the words `CAT`, `DOG`, or `MATH` in them?

2. Find the number of integers between 1 and 10000 (inclusive) that are *not* divisible by 4, 5, or 6.

3. Find the number of integers between 1 and 10000 (inclusive) that are not perfect squares, cubes, or fourth powers (sometimes called perfect *tesseractic* numbers).

Remark 3.35. Oftentimes when learning about the Inclusion-Exclusion Principle one studies questions of the form "find the number of nonnegative solutions to the equation $x_1 + x_2 + x_3 + x_4 = 14$ subject to the conditions $x_1 \leqslant 4$, $x_2 \leqslant 7$, etc.", which proves useful in many contexts. However, this would require the formulation of a notion of a *multiset*, namely a set that may have repeats. This is not a particularly difficult task, but requires reproving many theorems. The reader is encouraged to explore creating her or his own definition of a *multiset* and to formalize how the inclusion-exclusion principle applies in this setting.

4 Graphs

This is the section on graphs, still to be completed.

Definition 4.1. 1. A **graph** $G = (V, E)$ is a pair of sets V and E, where V is a non-empty set and E is a (possibly empty) set consisting only of two-element sets of the form $\{a, b\}$, where $a \in V$ and $b \in V$. The set $V = V(G)$ is called the set of **vertices** of G and the set $E = E(G)$ is called the set of **edges** of G.

2. The number of vertices in a graph is denoted by v and the number of edges in a graph is denoted by e. It is possible that $v = \infty$ or $e = \infty$, meaning that there is no such (finite) number.

3. If $e = (v_1, v_2) \in E$, then we say that e **connects** v_1 and v_2 and that v_1 and v_2 are **adjacent**.

4. Let D be a subset of a space (typically the Euclidean Plane, $\mathbb{R}^2$) consisting of points and arcs connecting those points, such that the arcs only meet the points in their boundaries. Given a graph G, D is said to be a **graph diagram** for G if:

   (a) the vertices of G are in one-to-one correspondance with the points of D, and

   (b) the edges of G are in one-to-one correspondance with the arcs of D. Note that a graph diagram D is sometimes referred to as an **embedding**, particularly if the space is not $\mathbb{R}^2$.

5. Two graphs are **equal** if they have equal vertex and edge sets. Two graph diagrams are equal if they represent equal graphs.

Another name for vertex is *node*.

Lemma 4.2. Let G be a graph. Then G has no loops, i.e. edges connecting a vertex to itself, and G has no skeins, i.e. collections of more than one edge connecting a pair of vertices.

directed vs. undirected

Example 4.3. 1. null graph

2. cyclic graph C_v

3. complete graph K_v

Theorem 4.4. Let K_v denote the complete graph on v vertices. Then $e = |E(K_v)| = \frac{1}{2}v \cdot (v - 1)$.

Definition 4.5. compliment and subgraph, isomorphism, supergraph

equal implies isomorphic
isomorphism is an equivalence relation
isomorphism implies equal numbers of Vs and Es
definition of degree/valence
isomorphism implies the set of degrees is the same and the number of vertices of degree n is the same.
non-isomorphism examples, e.g. `non-iso_graphs.png`

exercises (interject above)

1. wheel graphs, draw some and prove number of edges.

2. determine and prove the number of edges in $\overline{G}$ given data about G.

3. determine all numbers v such that $C_v \cong K_v$. prove.

4. Prove that $C_v \cong \overline{C_v}$ if and only if $v = 5$.

5. $G \cong \overline{G}$ implies that v or $v - 1$ is divisible by 4.

6. Maybe theorem: if $G_1 \cong G_2$ and $A_1 \subseteq G_1$ then there exists a subgraph $A_2 \subseteq G_2$ with $A_1 \cong A_2$. Then, reprove the non-iso from 4
7. ? prove the number of isomorphism classes of v=3 is 7. "Classify all graphs with 3 vertices up to isomorphism."
8. $G_1 \cong G_2$ iff $\overline{G_1} \cong \overline{G_2}$.
9. define bipartite, prove non-isomorphism of a pair
10. Let X and Y be a partition of the vertices....
11. Create formulae for the number of edges of each of the following families of graphs, based on the number of vertices (v):
    - K_v
    - C_v
    - $\overline{C}_v$
    - W_v (wheel graph)
    - $K_{m,n}$.
12. degree sequence e.g. (5,4,3,3,3,3,3,2,2) as an isom invt, find a pair with at least 4 verts for which the invt fails
13. draw a connected graph with degree sequence equal to that above.
14. Prove that a graph with $v \geqslant 2$ has two vertices with the same degree.
15. Prove that any two connected graphs with the same number of vertices and degree sequences $(2, 2, \ldots, 2)$ are isomorphic.
16. Prove that a graph with at least
$$\frac{(n-1)(n-2)}{2} + 1$$
edges must be connected. Find an example to prove that this bound is "sharp", *i.e.* that this bound cannot be improved in general.
17.

5 Planar Graphs

This is the section on planar graphs, still to be completed.

- Definition of graph projection
- A Graph is planar if it (is isomorphic to?) a graph with a projection drawn in a plane with no edge-crossings (define)
- examples
- Jordan Curve Theorem: If C is a continuous simple closed curve in a plane and two points x and y of C are joined by a continuous simple arc L such that $L \cap C = \{x, y\}$, then except for its endpoints L is entirely contained in one of the two regions of $\mathbb{R}^2 \backslash C$.
- $K_{3,3}$ is nonplanar (not using Euler)
- K_5 is nonplanar
- Any subgraph of a planar graph is planar
- corollary any supergraph of a nonplanar graph is nonplanar
- If G may be obtained from H by replacing an edge (x, y) of H with another vertex v, and a pair of edges $(x, v), (v, y)$, then G is said to be obtained from H via an **edge expansion**. If G may be obtained from H by a finite sequence of edge expansions, then G is an **expansion** of H.
- (maybe?) If G is obtained from H by a sequence of expansions and passing to supergraphs, then G is said to be an **expanded supergraph** of H (my definition) (NOTE: this is equivalent to being a supergraph of an expansion. Prove!)
- Every expanded supergraph of $K_{3,3}$ or K_5 is nonplanar.
- Kuratowski's Theorem: a graph is nonplanar if and only if it is an expanded supergraph of $K_{3,3}$ or K_5.
- exercise: examples of large graphs, is it planar?
- TODO: add exercises

6 Euler's Formula

- A **walk**, or **path** is a sequence $v_1, v_2, \ldots, v_n$ of not-necessarily-distinct vertices in a graph G such that (v_i, v_{i+1}) is an edge of G for $1 \leqslant i < n$.
- A graph is **connected** if every pair of vertices may be joined by a path. Otherwise, it is **disconneted**; the maximal connected subgraphs of a disconnected graph are called the **components** of the graph.
- Disclaimer: path connected vs connected?
- examples
- Given a planar graph diagram D, a **face** of D is the set of all points in $\mathbb{R}^2 \backslash D$ that may be joined by a continuous arc in $\mathbb{R}^2 \backslash D$. The number of faces of D is denoted as
- prove if G is a planar graph, then the number of faces of *any* planar diagram of G is the same.
- A graph is **polygonal** if it is planar, connected, and has the propery that every edge borders on two different faces
- If G is polygonal then $v - e + f = 2$. (two students, longish)

- If G is planar and connected, then $v - e + f = 2$.
- K_5 and $K_{3,3}$ are nonplanar, revisited.
- If G is planar (and connected? not necessary) then G has a vertex of degree $\leqslant 5$ (Q?)
- exercises from 4
- a graph is **regular** if all its vertices have the same degree, said "regular of degree d".
- examples
- a graph is **platonic** if it is polygonal, regular, and all its faces are bounded by the same number of edges
- (what if we remove the last condition?)
- examples
- Theorem: Apart from K_1 and the cyclic graphs, there are 5 platonic graphs. Prove by breaking into d cases
  Needs lemata:
  - if G is regular of degree d then $e = dv/2$.
  - If G is platonic of degree d, and n is the number of edges bounding each face, then $f = dv/n$.
- exercises

7 Colorings

- A graph has been (n-)**colored** if each vertex has been assigned a number from $\{1, 2, \ldots, n\}$ such that no edge joins vertices with the same number ("color"). We say that a graph G is n**-colorable** if it may be n-colored.
- examples
- The **chromatic number** of a graph G is the smallest n such that G is n-colorable, denoted $X(G)$.
- examples
- (DNP) Four-color theorem: Every planar graph has $X \leqslant 4$.
- Five-color theorem: Every planar graph has $X \leqslant 5$. (induction)
- Every planar graph having a vertex of degree $\leqslant 4$ has $X \leqslant 4$. This is crazy! Compare to the theorem about every planar graph having a vertex $\leqslant 5$.
- It is sufficient to prove the four-color theorem for cubic maps (trivalent graphs).
- reading about the four-color theorem and its proof. Do you believe it?
- Map colorings! define dual graph and why no bridges
- exercises
- (Brooks) Let G be a connected (simple) graph. If G is not complete and its largest vertex degree is n, then G is n-colorable.
- discuss examples where Brooks theorem is useful vs. not
- A map is 2-colorable if and only if G (the dual graph?) is Eulerian
- Let G be a graph. Then $1 \leqslant X(G) \leqslant v$; $X(G) = v$ if and only if G is a complete graph.
- cor if G has a subgraph isomorphic to the complete graph K_p, then $p \leqslant X(G)$.
- Let G be a graph with at least one edge. Then $X(G) = 2$ if and only if G is a bipartite graph.
-

8 Eulerian and Hamiltonian Paths

- A path is **closed** if $v_1 = v_n$, otherwise it is **open**.
- A path is **simple** if $|\{v_1, v_2, \ldots, v_n\}| = n$ if open or $n-1$ if closed.
- simple is equivalent to having no interior overlap
- examples
- An Eulerian path uses every edge in the graph exactly once
- examples
- they're simple
- A connected graph has a closed Eulerian path if and only if every vertex is even.
- There is an Eulerian cycle beginning at any vertex in a graph with all even vertices
- A connected graph has an open Eulerian path if and only if every vertex is even except for exactly two.
- the path must begin at one of the two odd vertices
- A Hamiltonian path is one which uses every vertex exactly once (if closed, the first and last vertex is repeated)
- examples
- lemma If the sum of the degrees of every pair of vertices of a graph is at least $v-1$, then
  - every pair of vertices are either adjacent to each other or to a common third vertex, and
  - G is connected
- if the sum of the degrees of every pair of vertices of G is at least $v-1$, then G has an open Hamiltonian path
- if the sum of the degrees of every pair of vertices of G is at least v, then G has an closed Hamiltonian path
- Open Problem: classify Hamiltonian cycles
- If G is a graph such that every pair u and w of nonadjacent vertices, $\deg(u) + \deg(w) \geqslant v$, then G is Hamiltonian.
- (Dirac) If every vertex has $\deg(v) \geqslant \frac{1}{2}v$, then G is Hamiltonian.
- a **skein** is an object consisting of two vertices connected by two or more lines (finite?)
- a **multigraph** $M(G)$ is an object consisting of a graph G where some of its edges are replaced by skeins. G is called a **generator** of M
- generators are unique
- examples
- define some terms for multigraphs and prove them?
- a walk in a multigraph is, a euler walk is, a hamilton walk is...
- A connected multigraph has a closed euler walk iff every vertex is even.
- same thing again with the open walk and two odd edges
- Königsberg Bridge Problem

- The sum of the degrees of the vertices of a multigraph is 2e
- Every multigraph has an even number of odd vertices
- applications? aren't there tons?
- exercises

9 Trees

This is the section on Trees, still to be completed.

- Definition: A tree is a connected graph with no circuits/cycles.
- TFAE:
  1. T is a tree,
  2. T contains no circuits/cycles and has $v - 1$ edges,
  3. T is connected and has $v - 1$ edges,
  4. T is connected and every edge of T is a **bridge**, *i.e.* it lies in a disconnecting set of a size one,
  5. Any two vertices of T have a unique path conntecting them,
  6. T contains no circuits/cycles but the addition of any new edge produces exactly one circuit/cycle.
- Def spanning trees
- prove that if a tree has a vertex of degree p, then it has at least p *leaves*, *i.e.* vertices of degree 1.
- Let n be a positive integer and let $2 \leqslant k \leqslant n - 1$. Construct a family of trees $T_{n,k}$ with n vertices and k leaves. Can a tree have any other combination of n and k?
- Let T be a tree with v vertices. Prove that the length of the shortest cycle (or closed path) that includes each edge of G at least once is $2(n - 1)$.