MTH630: Graph Theory and Combinatorics

Contents

1 Set Theory

Definition 1.1.

1. A **Set** is a collection of distinct objects, none of which is the set itself. If a is an object belonging to the set A, we write $a \in A$, and say "a is an element of A".

2. A set containing no elements is called the **empty set**, or the **null set**, and is written $\varnothing$ or $\{\}$.

3. A set A is said to be a **subset** of the set B, written $A \subseteq B$, if every element of A is also an element of B.

4. A set A is said to be a **equal to** the set B, written $A = B$ if $A \subseteq B$ and $B \subseteq A$.

If it is possible to enumerate the elements of A, we do so with the following notation:

$$A = \left\{a, \pi, \frac{45}{36}, \text{"Massachusetts"}\right\}.$$

Remark 1.2. You may find the definition of a mathematical set nebulous and confusing. What's a "collection"? What's an "object", and what does it mean for them to be "distinct"? In truth, while it is possible to formally define all of these concepts, it is typically the case that a student has an intuitive understanding of a set, and can begin from that.

Exercise 1.3. List all the subsets of $\{1, 2, 3\}$.

Notation 1.4. (Set Builder Notation) Let A be a set, and for all $x \in A$, let $p(x)$ be a proposition about x which may be true or false. Then we may build a set S by taking all those elements of A for which the proposition is true; such a set may be written down using **set builder notation**:

$$S = \{x \in A \mid p(x)\},$$

and we read this as "S is (equal to) the set of all x in A such that p of x." One important note is that a set A must exist in order to use set builder notation; as a result of this, we will use the term *universe of discourse*, often denoted by X, to describe any reasonably conceivable objects that may be placed into a set. You will see this appearing in the definitions ahead (see for example Definition 1.8).

Exercises 1.5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers.

1. Translate the set $S = \{1, 2, 3, 4, 5\}$ into set builder notation, where $A = \mathbb{N}$.

2. Write down, without the uses of ellipses ("$\ldots$"), notation defining the set of even natural numbers

3. Repeat the above exercise for the set of odd natural numbers divisible by 5 (one may use "$7 \mid 14$" to say that "7 divides 14").

Theorem 1.6. There is only one empty set.

Theorem 1.7 (transitivity of subset)**.** If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

1.1 Getting new sets from old

Definition 1.8. Let A and B be sets, and let X denote the universe of discourse.

1. The set $A \cup B = \{x \in X \mid x \in A \vee x \in B\}$ is called the **union** of A and B.

2. The set $A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$ is called the **intersection** of A and B.

3. The set $A \smallsetminus B = \{x \in A \mid x \notin B\}$ is called the **(relative) complement** of A in B.

Theorem 1.9. For all sets A and B, if $A \subseteq A \cap B$ then $A \cup B \subseteq B$.

Theorem 1.10. For all sets A, B, C, and D, if $A \subseteq C$ and $B \subseteq D$ then $A \cup B \subseteq C \cup D$.

Theorem 1.11. For all sets A, B, C, and D, if $A \subseteq C$ and $B \subseteq D$ then $A \cap B \subseteq C \cap D$.

Theorem 1.12. Let A, B, and X be sets. If $A \subseteq B$, then $X \setminus B \subseteq X \setminus A$.

Exercise 1.13. Write down the *converse* of Theorem 1.12. It isn't quite true (why not?) unless you also add the assumption that $A \subseteq X$ and $B \subseteq X$. Prove the converse with that additionally assumption. (The *converse* of a statement $p \to q$ is $q \to p$.)

Theorem 1.14. Let A and B be sets. Then $A \setminus B = \varnothing$ if and only if $A \subseteq B$.

Theorem 1.15. For sets A and B, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

For our purposes, a *claim* is something that may or may not be true, and we need to determine whether or not it is true and justify it with a proof or counterexample.

Claim 1.16. For all sets A, B, and C, if $A \subseteq B \cup C$ then $A \subseteq B$ or $A \subseteq C$.

1.2 Bijections and cardinality

Definition 1.17. Let A and B be sets.

1. Let $a \in A$ and $b \in B$. Then the **ordered pair** of a and b, written (a, b), is pairing of the elements a and b into an ordered grouping. Strictly speaking (though this intuitive definition typically suffices), one may define $(a, b) = \{\{a\}, \{a, b\}\}$. We refer to a and b as *elements* of (a, b), even though strictly speaking they are not.
2. A **bijection**, or a **one-to-one correspondence**, between A and B is a set C with all of the following properties.
   - Every element of C consists of an ordered pair (a, b) where $a \in A$ and $b \in B$.
   - (injective) Every element of A exists as the first element of exactly one element of C.
   - (sujective) Every element of B exists as the second element of exactly one element of C.

   We say that A and B are **in bijection** (or sometimes *bijective*) if there exists a bijection between them; this is sometimes written as $A \cong B$, but it often just written out in words.

Remark 1.18. In a traditional set theory course, one uses ordered pairs to first define cartesian products, and then relations, functions, injections, surjections, domain, co-domain, range, *etc.* before defining bijections. For our purposes, bijections as defined above will suffice.

Theorem 1.19 (Bijectivity is an equivalence relation)**.** Let A, B, and C be sets.

1. (reflexivity) A is in bijection with itself.
2. (symmetry) If A is in bijection with B, then B is in bijection with A.
3. (transitivity) If A is in bijection with B, and B is in bijection with C, then A is in bijection with C.

Remark 1.20. The fact that bijections satisfy the above three properties give it the status of being what's called an **equivalence relation**. We will see equivalence relations again in the future when we discuss graphs. One often considers equivalence relations to be a notion of "sameness": if A is in bijection with B, then they're essentially the same object.

Definition 1.21. If a set A is in bijection with the set $\{1, 2, 3, 4, \ldots, n\}$, then the **cardinality** of A is given by n, written $|A| = n$, and we say that A is **finite**. If a set is in bijection with the natural numbers, then we say that it is **countably infinite**.

Theorem 1.22. If $|A| \neq |B|$, then A is not in bijection with B.

Question 1.23. Is the *converse* of Theorem 1.22 true? Prove or disprove.

Theorem 1.24. Being countably infinite and finite are mutually exclusive set properties.

1.3 Exercises

1. Let A, B, and C be sets. Prove that if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.
2. Given a set A with $|A| = n$, how many subsets does A have? Prove your answer.
3. Prove that the natural numbers are in bijection with the even natural numbers.
4. Prove that the natural numbers are in bijection with the integers.
5. Let C be a bijection between the natural numbers ($\mathbb{N}$) and the integers ($\mathbb{Z}$), so that $C \subseteq \{(x, y) \mid x \in \mathbb{N} \wedge y \in \mathbb{Z}\}$. Show that there exist elements (a, b) and (x, y) in C such that $a > x$ and $b < y$.
6. Prove that the natural numbers are in bijection with the set of ordered pairs $\{(n, a) \mid n \in \mathbb{N} \wedge a \in \{1, 2, 3\}\}$.
7. Prove that the natural numbers are in bijection with the set of ordered pairs $\{(n, m) \mid n \in \mathbb{N} \wedge a \in \mathbb{N}\}$.
8. Prove that the set of words in this sentence is not in correspondence with the set of words in the preamble to the U.S. Constitution.

1.4 Addendum: Bijective Functions

This section originally appeared a single sheet, with a theorem labeled "Theorem 1".

Definition 1.25. Let A and B be sets.

1. A set C of the form $C = \{x \in X \mid \exists a \in A\ \exists b \in B\ x = (a, b)\}$ is called a **relation** between A and B.
2. A **function** from A to B, written $f : A \to B$, is a relation $C = C(f)$ that has the additional properties:
   (a) For all $a \in A$, there exists an element $b \in B$ such that $(a, b) \in C$, and
   (b) For all $a \in A$ and $b_1, b_2 \in B$, if $(a, b_1) \in C$ and $(a, b_2) \in C$ then $b_1 = b_2$.
3. A function is said to be **injective**, or **one-to-one**, if it has the property: for all $a_1, a_2 \in A$ and $b \in B$, if $(a_1, b) \in C$ and $(a_2, b) \in C$ then $a_1 = a_2$.
4. A function is said to be **surjective**, or **onto**, if for all $b \in B$, there is an $a \in A$ such that $(a, b) \in C$.
5. A **bijective function**, also called a **one-to-one correspondence**, is a function which is injective and surjective.

We typically use the notation $f(a) = b$ to denote that $(a, b) \in C(f)$. So then the two pieces of the definition of a function f become what you're likely used to:

a. For every $a \in A$, there exists an element $b \in B$ such that $f(a) = b$, and

b. For all $a \in A$ and $b_1, b_2 \in B$, if $f(a) = b_1$ and $f(a) = b_2$ then $b_1 = b_2$.

These are colloquially "the domain is all of A" and "f passes the vertical line test," respectively. The definition of injective and surjective become:

- The function $f : A \to B$ is injective if for all $a_1, a_2 \in A$ and $b \in B$, if $f(a_1) = b$ and $f(a_2) = b$ then $a_1 = a_2$.
- The function $f : A \to B$ is surjective if for all $b \in B$, there is an $a \in A$ such that $f(a) = b$.

After you do this change of notation, you never need to discuss C again, so it is dropped from the discussion. What we'd like to do is equate the two definitions of "bijective", so that one has access to whichever definition makes more sense to them.

Theorem 1.26. Let A and B be sets.

a. If C is a bijection between A and B (in the sense of Definition 1.17), then there is a bijective function $f : A \to B$ such that for all $a \in A$, the (unique) element $b \in B$ such that $(a, b) \in C$ satisfies $f(a) = b$.

b. If $f : a \to b$ is a bijective function (in the sense of Definition 1.25), then the set $C = \{x \in X \mid \exists a \in A\ x = (a, f(a))\}$ is a bijection.

In other words, for any sets A and B, there is a bijection between the set of all bijections between A and B in the sense of Definition 1.17 and the set of all bijective functions between A and B in the sense of Definition 1.25.

2 Combinatorics

2.1 The Pigeonhole Principle

Theorem 2.1 (Pigeonhole Principle)**.** Let n be a natural number. If $n+1$ objects are to be placed into n boxes, then at least one of the boxes must contain at least two objects.

Examples 2.2.

a. Show that there is some day of the week on which over 1 billion currently-living people have been born.

b. Show that given m integers $A = \{a_1, a_2, \dots, a_m\}$, there exists a consecutive subset $\{a_k, a_{k+1}, \dots, a_l\} \subseteq A$ whose sum is divisible by m, for k and l natural numbers with $1 \leqslant k \leqslant l \leqslant m$.

c. Show that given $2n$ integers, from any subset of $n+1$ of them there is a pair where one element of which is divisible by the other.

Theorem 2.3 (Chinese Remainder Theorem)**.** Let m and n be relatively prime positive integers, and let a and b be integers with $0 \leqslant a \leqslant m-1$ and $0 \leqslant b \leqslant n-1$. Then there is a positive integer x such that the remainder when x is divided by m is a, and the remainder when x is divided by n is b.

Theorem 2.4 (Strong Pigeonhole Principle)**.** Let $q_1, q_2, \dots, q_n$ be positive integers. If $q_1 + q_2 + \dots + q_n - n + 1$ objects are put into n boxes, then for at least one i, the i^{th} box contains at least q_i objects.

Examples 2.5.

a. If $n+1$ numbers are chosen from the set $\{1, \dots, 2n\}$, then there is a pair that differ by 1.

b. If $n+1$ numbers are chosen from the set $\{1, \dots, 3n\}$, then there is a pair that differ by 2.

Exercise 2.6. Generalize the examples above and prove.

Example 2.7. Let n be a natural number. Determine a natural number m_n such that if m_n points are chosen from an equilateral triangle of side length 1, then there are two whose distance is less than or equal to $\frac{1}{n}$.

2.2 Permutations and Combinations

Definition 2.8. Let A be a set. The set $X = \{A_1, A_2, \dots, A_n\}$ is said to be a **partition** of A if:

a. $A_i \subseteq A$ for all i, $1 \leqslant i \leqslant n$,

b. $A_i \cap A_j = \varnothing$ for all i and j, $1 \leqslant i < j \leqslant n$, and

c. $A = A_1 \cup A_2 \cup \dots \cup A_n$.

This definition is primarily useful for the language in proofs surrounding the following four "Rules of (arithmetic)".

Theorem 2.9 (Rule of Addition)**.** If a set S can be written as $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \varnothing$, then $|S| = |S_1| + |S_2|$.

Remark 2.10. This should be viewed, along with all the other "Rules of (arithmetic)" in this section, as breaking down or otherwise simplifying a counting problem.

Example 2.11. A student at Phillips Academy wants to take one Mathematics elective or one English elective, but cannot take both. If there are 2 Mathematics electives and 3 English electives on offer, how many options are available to the student this term?

Theorem 2.12 (Rule of Multiplication)**.** Let S be the set of ordered pairs (a, b) of objects, where the a lies in a set of size n and b lies in a set of size m. Then $|S| = m \cdot n$.

Example 2.13. Determine the number of positive integers that divide $32189975201412589275 = 3^4 \cdot 5^2 \cdot 11^7 \cdot 13^8$.

Question 2.14. Note that the Rule of Multiplication is not:

Let A and B be sets, and let S be the set of ordered pairs (a, b) of objects, where the $a \in A$ and $b \in B$. Then $|S| = |A| \cdot |B|$.

Why not? In order to successfully answer this question, consider the following problem: How many 2-digit numbers have distinct and nonzero digits?

Theorem 2.15 (Rule of Subtraction)**.** Let A and U be subsets such that $A \subseteq U$. Then $|A| = |U| - |U \backslash A|$.

Example 2.16. A set of length-8 computer passwords are taken from the characters 0-9 and *a*-*z*. How many have repeated symbols?

Theorem 2.17 (Rule of Division)**.** Let S be a finite set that is partitioned into k parts of equal size, say n. Then

$$k = \frac{|S|}{n}.$$

Examples 2.18.

a. How many odd numbers between 1000 and 9999 have distinct digits?

b. How many different 5-digit numbers can be constructed from the digits $1, 1, 1, 3, 5$?

Definition 2.19. An r**-permutation** of a set S of n elements is an ordered arrangement of elements of S of size r . The number of such r-permutations is $P(n, r) = {}_nP_r$.

Theorem 2.20. Let S be a set of size n and $1 \leqslant r \leqslant n$. Then $P(n, r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$.

Corollary 2.21. Let S be a set of size n and $1 \leqslant r \leqslant n$. Then $P(n, r) = \dfrac{n!}{(n-r)!}$.

Exercises 2.22.

a. Find a closed-form expression (*i.e.* no "...") for the number of possible positions of the "15-puzzle": a game consisting of a 4×4 grid of 15 numbers and one blank space, where one may swap the positions of the blank space with one of the adjacent (up, down, left, or right) numbered squares.

b. How many 5-digit numbers with unique, non-zero digits are there such that a 5 is never followed by a 6, and vice versa?

Definition 2.23. An r**-combination** of a set S of n elements is a subset of size r. The number of such may be written as $\binom{n}{r} = C(n, r) = {}_nC_r$.

Theorem 2.24. For integers r and n with $0 \leqslant r \leqslant n$, $P(n, r) = r!\binom{n}{r}$, hence

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Corollary 2.25. $\binom{n}{r} = \binom{n}{n-r}$.

Example 2.26. 25 points are drawn on a piece of paper such that no 3 are colinear. How many line segments pass through a pair of them?

Theorem 2.27. $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$.

Exercises 2.28.

a. How many straight-flushes are there in a deck of cards? (A straight-flush consists of 5 consecutive cards of the same suit.)

b. How many integers larger than 5400 have all distinct digits, none of which are 2 or 6?

c. In how many ways can six students and six faculty members be seated at a table if the members of the two groups must alternate?

d. In how many ways can 8 indistinguishable rooks be placed on a chess board so that they cannot attack one another?

e. A woman works in a building 9 blocks east and 8 blocks north of her home. When commuting, she never backtracks or takes any route that is longer than the shortest possible path. Suppose the grid of roads between her home and her work is full of possible paths, except for the road that travels one block east from the block which lies 4 blocks east and 3 blocks north from her home. How many viable paths to work does she have?

f. A group of mn players are to be arranged into m teams each with n players. Determine the number of ways this can be arranged if the teams

   (a) have names, and

   (b) are indistinguishable.

Theorem 2.29 (Pascal's Formula)**.** Let n and r be integers with $0 \leqslant r \leqslant n$. Then $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

Exercise 2.30. What does the above have to do with the so-called Pascal's triangle?

Theorem 2.31 (Binomial Theorem)**.** $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + y^n$.

Exercise 2.32. Prove that

a. $3^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k$, and

b. $2^n = \displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$.

2.3 The Inclusion-Exclusion Principle

Exercise 2.33. Find the number of integers between 1 and 600 which are not divisible by 6. Do so in two ways: first, by directly counting it, then by combining the processes of counting those numbers divisible by 2 and those numbers divisible by 3.

Theorem 2.34 (Inclusion-Exclusion Principle)**.** Let S be a finite set and let $P_1, P_2, \ldots, P_n$ be a collection of properties that some elements of S satisfy. Let A_i denote the subset of S satisfying property P_i for $1 \leqslant i \leqslant n$. Let n_k denote the number of elements in any k-fold intersection of the sets $\{A_i\}$, that is,

$$n_k = \sum_{i_1, i_2, \ldots, i_k} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|,$$

where $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$. Then

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_i}| = |S| - n_1 + n_2 - n_3 + \ldots + (-1)^n\, n_n.$$

Examples 2.35.

a. How many permutations of the letters `C, A, T, D, O, G, M, A, T, H` have none of the words `CAT`, `DOG`, or `MATH` in them?

b. Find the number of integers between 1 and 10000 (inclusive) that are *not* divisible by 4, 5, or 6.

c. Find the number of integers between 1 and 10000 (inclusive) that are not perfect squares, cubes, or fourth powers (sometimes called perfect *tesseractic*s).

Remark 2.36. Oftentimes when learning about the Inclusion-Exclusion Principle one studies questions of the form "find the number of nonnegative solutions to the equation $x_1 + x_2 + x_3 + x_4 = 14$ subject to the conditions $x_1 \leqslant 4$, $x_2 \leqslant 7$, etc.", which proves useful in many contexts. However, this would require the formulation of a notion of a *multiset*, namely a set that may have repeats. This is not a particularly difficult task, but requires reproving many theorems. The reader is encouraged to explore creating her or his own definition of a *multiset* and to formalize how the Inclusion-Exclusion Principle applies in this setting.

3 Graphs

Definition 3.1.

a. A **graph** $G = (V, E)$ is a pair of sets $V = V(G)$ and $E = E(G)$, where V is a non-empty set and E is a (possibly empty) set consisting only of two-element sets of the form $\{a, b\}$, where $a \in V$ and $b \in V$. The set $V(G)$ is called the set of **vertices** of G and the set $E(G)$ is called the set of **edges** of G.

b. The number of vertices in a graph is denoted by $v = v(G) = |V(G)|$ and the number of edges in a graph is denoted by $e = e(G) = |E(G)|$. It is possible that $v = \infty$ or $e = \infty$, meaning that there is no such (finite) number.

c. If $\gamma = (v_1, v_2) \in E$, then we say that γ **connects** v_1 and v_2 and that v_1 and v_2 are **adjacent**.

d. Let D be a subset of a space (typically the Euclidean Plane, $\mathbb{R}^2$) consisting of points and arcs connecting those points, such that the arcs only meet the points in their boundaries. Given a graph G, D is said to be a **graph diagram** for G if:

   (a) the vertices of G are in one-to-one correspondance with the points of D, and

   (b) the edges of G are in one-to-one correspondance with the arcs of D. Note that a graph diagram D is sometimes referred to as an **embedding**, particularly if the space is not $\mathbb{R}^2$. We will often not distinguish between a graph and its projection unless it is important to do so. (For example, we may say "draw a graph that..." which clearly means "draw a projection of a graph in $\mathbb{R}^2$ such that...".)

e. Two graphs are **equal** if they have equal vertex and edge sets. Two graph diagrams are equal if they represent equal graphs.

Another name for vertex is *node*, and another name for an edge is a *link*.

Lemma 3.2. Let G be a graph. Then G has no *loops, i.e.* edges connecting a vertex to itself, and G has no *skeins, i.e.* collections of more than one edge connecting a pair of vertices.

Remark 3.3. In other formulations of Graph Theory, using multisets one can define graphs that have loops and skeins, and then one would create a definition of a graph without these, typically called *simple graphs.* In this other formulation, Lemma 3.2 shows that we will restrict our discussion to simple graphs.

Often, one may consider graphs whose edges have a direction to them, *i.e.* whose edges are ordered pairs, rather than sets, which has huge ramifications when formulating a notion of paths (see Chapter 6). We will not study these types of graphs either, called *directed graphs*, but they are nonetheless a very important concept in Graph Theory.

Examples 3.4. Let v be a natural number. Draw several examples of each of the following graphs.

a. The **null graph** is a graph with $E = \varnothing$.

b. The **cyclic graph** C_v consists of v vertices $v_1, v_2, \ldots, v_v$ and edges of the form $\{v_i, v_{i+1}\}$ for all i with $1 \leqslant i \leqslant v$ along with $\{v_v, v_1\}$.

c. The **complete graph** on v vertices, K_v, is the graph consisting of v vertices and all possible edges between them.

Theorem 3.5. Let K_v denote the complete graph on v vertices. Then $e = |E(K_v)| = \frac{1}{2}v \cdot (v-1)$.

Definition 3.6. Suppose G and H are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Then H is a **subgraph** of G, and G is a **supergraph** of H.

Example 3.7. Given any graph G and any subset X of $V(G)$, the **induced subgraph** of G by X is the subgraph whose vertex set is X and whose edge set is all the edges of G whose vertices both lie in X. Draw an example of a graph with 7 vertices and 10 edges, and draw three different induced subgraphs.

Lemma 3.8. Every graph G is the subgraph of a complete graph, denoted $K_{v(G)}$.

Definition 3.9. Let G and H be graphs.

a. The **complement** of G, denoted $\overline{G}$, is the graph given by:

- $V(\overline{G}) = V(G)$, and
- $E(\overline{G}) = K_{v(G)} \backslash E(G)$. That is, the edges of $\overline{G}$ are precisely the edges "missing" from G to make up $K_{v(G)}$.

b. Suppose there are a pair of bijections C_V and C_E between the vertex and edges sets of G and H, respectively, which *respect one another* in the following way: if the edge $\{v_1, v_2\} \subseteq E(G)$ is paired under C_E to an edge $\{w_1, w_2\} \subseteq E(H)$, then without loss of generality the vertices v_1 and w_1 are paired and the vertices v_2 and w_2 are paired under C_V. Then the pair (C_V, C_E) is called an **isomorphism** between G and H, and G and H are said to be **isomorphic**, written $G \cong H$.

Theorem 3.10 (Isomorphism is an equivalence relation)**.** Let G, H, and K be graphs.

a. (Reflexivity) G is isomorphic to G.

b. (Symmetry) If G is isomorphic to H, then H is isomorphic to G.

c. (Transitivity) If G is isomorphic to H and H is isomorphic to K, then G is isomorphic to K.

Corollary 3.11. If G and H are isomorphic graphs, then $v(G) = v(H)$ and $e(G) = e(H)$.

Definition 3.12. Let G be a graph and $a \in V$ be a vertex.

a. The **degree** of a, $\deg(a)$, is the number of edges containing a as one of its vertices. (This is sometimes called the **valence**). Vertices may be called **even** or **odd** if they have even or odd degreee, respectively.

b. If G is a graph such that every vertex has the same degree, say r, then G is said to be **regular of degree** r.

c. The **degree sequence** of G is the ordered v-tuple of nonincreasing degrees of the vertices of G. That is, suppose that the vertices of G are $v_1, v_2, \ldots, v_v$, and the ordering of them has been choosen such that $\deg(v_i) \geqslant \deg(v_{i+1})$ for all i with $1 \leqslant i \leqslant v$. Then the degree sequence of G is an ordered list $(\deg(v_1), \deg(v_2), \ldots, \deg(v_v))$.

Remark 3.13. As in the definition of ordered pair (Definition 1.17), an ordered n-tuple, for n some natural number, is an ordered list where repetition is possible. The word "tuple" is a generalization of the word "pair". Similarly to an ordered pair, one may form a definition of an ordered pair using sets for absolute clarity, such as for a 4-tuple:

$$(1, 2, 3, 3) = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}.$$

Two ordered n-tuples are the same if they contain the same items in the same order.

Theorem 3.14. If G and H are isomorphic graphs, then their degree sequences are equal.

Example 3.15. Show that the two graphs drawn in Figure 3 are not isomorphic.

Exercises 3.16. a. Let v be a natural number. The **wheel graph** on v vertices, denoted W_v, is the graph obtained from C_{v-1} by adding a new "complete" vertex, *i.e.* a vertex that is adjacent to every other vertex. Draw some examples of wheel graphs, then determine and prove the correctness of a formula for the number of edges in W_v.

b. Let G be a graph. Determine and prove correctness of a formula for the number of edges in $\overline{G}$ given only $v(G)$ and $e(G)$.

c. Determine all numbers v such that $C_v \cong K_v$. Prove your claim.

d. The degree sequence is what is called an **isomorphism invariant** meaning that if $G \cong H$, then the degree sequence of G equals the degree sequence of H (by Theorem 3.14). State the contrapositive of this statement, and find a pair of graphs with the same number over vertices (at least 5), edges (at least 7), yet a different degree sequence. What can you conclude about these graphs? Find a pair of non-isomorphic graphs with at least 5 vertices, none of degree 2, for which their degree sequences do not prove their non-isomorphic status. This shows that the degree sequence is not a *complete* invariant.

e. Prove that $C_v \cong \overline{C_v}$ if and only if $v = 5$.

f. Prove that if $G \cong \overline{G}$, then v or $v - 1$ is divisible by 4.

g. Prove that if $G_1 \cong G_2$ and $A_1 \subseteq G_1$ then there exists a subgraph $A_2 \subseteq G_2$ with $A_1 \cong A_2$. Use this to reprove the non-isomorphism of Example 3.15.

h. Classify all graphs with 3 vertices up to isomorphism. That is, find a list of pairwise non-isomorphic graphs such that any graph with 3 vertices is isomorphic to exactly one of the graphs in your list.

i. Prove that $G_1 \cong G_2$ if and only if $\overline{G_1} \cong \overline{G_2}$.

j. A *partition* of a set is a decomposition into non-intersecting sets. That is, $\{1, 2, 3\} = \{1, 2\} \cup \{3\}$ is a partition, but $\{1, 2, 3\} = \{1, 2\} \cup \{2, 3\}$, which is a true statement, is not a partition.

   Let G be a graph such that $V(G)$ can be partitioned into two sets A and B with the property that no edges of G have both vertices in A or both vertices in B. Then G is called a **bipartite graph**.

   One family of examples of bipartite graphs are the **complete bipartite graphs**, $K_{m,n}$: given natural numbers m and n, the complete bipartite graph on m and n vertices is a bipartite graph with partition $A \cup B$, where $|A| = m$ and $|B| = n$, and all possible edges. Draw some examples of complete bipartite graphs.

   Let m, n, a, and b be natural numbers. Prove that $K_{m,n} \cong K_{a,b}$ if and only if $m = a$ and $n = b$.

k. Create formulae (and prove their correctness) for the number of edges of each of the following families of graphs, based on the number of vertices (v):

   - K_v
   - C_v
   - $\overline{C}_v$
   - W_v (wheel graph)
   - $K_{m,n}$.

l. Prove that a graph with $v \geqslant 2$ has two vertices with the same degree.

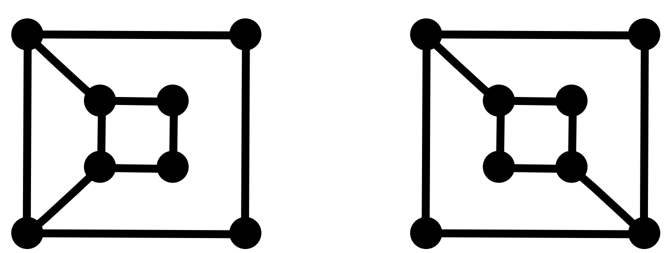

Figure 1: Two non-isomorphic graphs

4 Planar Graphs

Definition 4.1.

a. In a graph diagram D in the space X (such as the plane, R^2), an **edge-crossing** is a point $x \in X$ such that x lies in more than one edge/arc of D.

b. A graph G is **planar** if it has a graph diagram drawn in the plane with no edge-crossings.

Examples 4.2.

a. Draw a graph diagram K_4 which has an edge-crossing, and one without. Is K_4 planar?

b. Draw a planar graph diagram for $K_{m,n}$ for $m < 3$ and $n < 4$.

Theorem 4.3 (Jordan Curve Thoerem, Do Not Prove)**.** If C is a continuous simple closed curve in a plane and two points x and y of C are joined by a continuous simple arc L such that $L \cap C = \{x, y\}$, then except for its endpoints L is entirely contained in one of the two regions of $\mathbb{R}^2 \backslash C$.

Theorem 4.4.

a. $K_{3,3}$ is not planar.

b. K_5 is not planar.

(Do not use Theorem **??**).

Theorem 4.5. Any subgraph of a planar graph is planar.

Corollary 4.6. Any supergraph of a nonplanar graph is nonplanar.

Definition 4.7. Let G and H be graphs.

a. If G may be obtained from H by replacing an edge (x, y) of H with another vertex v, and a pair of edges $(x, v), (v, y)$, then G is said to be obtained from H via an **edge expansion**.

b. If G may be obtained from H by a finite sequence of edge expansions, then G is an **expansion** of H.

c. If G is obtained from H by a finite sequence of expansions and passing to supergraphs, then G is said to be an **expanded supergraph** of H.

Theorem 4.8. Let G and H be graphs. If G is an expanded supergraph of H, then G is a supergraph of an expansion of H.

Theorem 4.9. Every expanded supergraph of $K_{3,3}$ or K_5 is nonplanar.

Theorem 4.10 (Kuratowski's Theorem, Do Not Prove)**.** A graph G is nonplanar if and only if G is an expanded supergraph of $K_{3,3}$ or K_5.

Example 4.11. Show that the following graph is nonplanar.

4.1 Euler's Formula

Definition 4.12. Let G be a graph.

a. A **walk**, or **path** on G is a sequence $v_1, v_2, \ldots, v_n$ of not-necessarily-distinct vertices of G such that (v_i, v_{i+1}) is an edge of G for $1 \leqslant i < n$. A path with n vertices is said to have **length** n.

b. G is **connected** if every pair of vertices of G may be joined by a path. Otherwise, it is **disconneted**; the maximal connected subgraphs of a disconnected graph are called the **components** of the graph.

Remark 4.13. Those with a keen understanding of topology might notice that what has actually been defined here is the concept of *path-connectedness*, and that *connectedness* is a more general concept. However, for graphs, we tend to equate these two definitions.

Examples 4.14.

a. Draw examples of paths of length n on a graph with v vertices, for $v = 5, 6$, and 7 and $n = 3, 5, 8$.

b. A *cycle* or *circuit* is a path whose beginning and ending vertices are equal. Show that K_4 cannot be decomposed into a single cycle. That is, show that no cycle of K_4 that uses each edge at most once uses every edge.

Definition 4.15. Given a planar graph diagram D of a graph G, a **face** of D is the set of all points in $\mathbb{R}^2 \backslash D$ that may be joined by a continuous arc in $\mathbb{R}^2 \backslash D$. The number of faces of D is denoted as $f(D)$ (or just f or $f(G)$ if D is clear from context.)

Theorem 4.16. If G is a planar graph with planar diagrams D_1 and D_2, then $f(D_1) = f(D_2)$.

Definition 4.17. A graph diagram D is **polygonal** if it is planar, connected, and has the propery that every edge of D borders on two distinct faces.

Theorem 4.18 (Euler)**.** If G is polygonal then $v - e + f = 2$.

Corollary 4.19. If G is planar and connected, then $v - e + f = 2$.

Corollary 4.20. K_5 and $K_{3,3}$ are nonplanar.

Theorem 4.21. If G is planar then G has a vertex of degree $\leqslant 5$

Definition 4.22. Let $d \in \mathbb{N}$. A graph is **regular of degree** d if all its vertices have degree d.

Examples 4.23. Construct an example of a connected graph with v vertices which is regular of degree d for $v = 5, 6$, and 7 and $d = 2, 3$, and 4, if possible. If it isn't possible, explain why.

Definition 4.24. A graph is **platonic** if it is polygonal, regular, and all its faces are bounded by the same number of edges.

Examples 4.25.

a. Construct an example of a platonic graph which is regular of degree 4.

b. Construct an example of a polygonal, regular graph which is not platonic.

We now aim to classify the platonic graphs. In order to do so, we'll need the following lemmas.

Lemma 4.26. If G is regular of degree d then $e = \frac{dv}{2}$.

Lemma 4.27. If G is platonic of degree d, and n is the number of edges bounding each face, then $f = dv/n$.

Theorem 4.28. (Euclid) Apart from K_1 and the cyclic graphs, there are 5 platonic graphs.

5 Colorings

Definition 5.1. A graph G is said to have been (n-)**colored** if each vertex has been assigned a (non-unique) number from $\{1, 2, \ldots, n\}$ such that no edge joins vertices with the same number ("color"). We say that a graph G is n**-colorable** if it may be n-colored.

Examples 5.2.

a. Show that

Definition 5.3. The **chromatic number** of a graph G is the smallest n such that G is n-colorable, denoted $X(G)$.

Examples 5.4.

a. Determine $X(K_4)$, $X(K_5)$, $X(W_5)$.

b. Determine $X(K_{m,n})$ for all natural numbers m and n.

Theorem 5.5 (Four-color Theorem, Do Not Prove)**.** Every planar graph has $X \leqslant 4$.

Theorem 5.6 (Five-color Theorem)**.** Every planar graph has $X \leqslant 5$.

Claim 5.7. Every planar graph having a vertex of degree $\leqslant 4$ has $X \leqslant 4$.

Theorem 5.8. It is sufficient to prove the four-color theorem for trivalent graphs, or graphs which are regular of degree three.

Definition 5.9. Map colorings! define dual graph and why no bridges

Definition 5.10 (Brooks)**.** Let G be a connected graph. If G is not complete and its largest vertex degree is n, then G is n-colorable.

Examples 5.11. Construct examples of graphs where Brooks theorem is useful and informative, and graphs for which Brooks theorem is less informative.

Theorem 5.12. A map is 2-colorable if and only if G (the dual graph?) is Eulerian

Theorem 5.13. Let G be a graph. Then $1 \leqslant X(G) \leqslant v$; $X(G) = v$ if and only if G is a complete graph.

Corollary 5.14. If G has a subgraph isomorphic to the complete graph K_p, then $p \leqslant X(G)$.

Theorem 5.15. Let G be a graph with at least one edge. Then $X(G) = 2$ if and only if G is a bipartite graph.

6 Eulerian and Hamiltonian Paths

Definition 6.1.

a. A **cycle**, **circuit**, or **closed path** is a path $(v_1, \ldots, v_n)$ with $v_1 = v_n$. A path which is not closed is **open**.

b. A path $\gamma = (v_1, \ldots, v_n)$ is **simple** if $|\{v_1, v_2, \ldots, v_n\}| = n$ if γ is open or $n-1$ if γ is closed.

Exercise 6.2. Prove that any two connected graphs with the same number of vertices and degree sequences $(2, 2, \ldots, 2)$ are isomorphic.

Theorem 6.3. Let G be a graph with

$$\frac{(n-1)(n-2)}{2} + 1$$

edges. Then G is connected. This bound is "sharp", *i.e.* that this bound cannot be improved for graphs in general.

Theorem 6.4. A path $\gamma = (v_1, \ldots, v_n)$ is simple if and only if for all pairs of distinct i, j with $1 \leqslant i < j \leqslant n$, $v_i \neq v_j$ unless perhaps $i = 1$ and $j = n$.

Examples 6.5. Do it!

Definition 6.6. A path **traverses** an edge e if it contains the pair v_i, v_{i+1} and $e = (v_i, v_{i+1})$. An **Eulerian path** is a path in a graph G traverses every edge in G exactly once. An **Eulerian cycle** is a cycle which is Eulerian.

Examples 6.7. Do it!

Theorem 6.8. A connected graph G has an Eulerian cycle if and only if every vertex of G is even (*i.e.* has even degree).

Corollary 6.9. There is an Eulerian cycle beginning at any vertex in a graph with all even vertices.

Corollary 6.10. A connected graph has an open Eulerian path if and only if every vertex is even except for exactly two. Such a path must begin at one of the two odd vertices.

Definition 6.11. An open **Hamiltonian path** is one which visits every vertex exactly once. A **Hamiltonian cycle** γ is a cycle in which every vertex G is visited exactly once, apart from the first and last vertex of γ.

Examples 6.12. Do it!

Lemma 6.13. If the sum of the degrees of every pair of vertices of a graph G is at least $v - 1$, then

- every pair of vertices are either adjacent to each other or to a common third vertex, and
- G is connected.

Theorem 6.14.

a. If the sum of the degrees of every pair of vertices of G is at least $v - 1$, then G has an open Hamiltonian path

b. If the sum of the degrees of every pair of vertices of G is at least v, then G has an closed Hamiltonian path

Remark 6.15. It is an important open problem in Mathematics to classify Hamiltonian cycles. Do a little research on the current status of the problem!

Theorem 6.16. If G is a graph such that every pair u and w of nonadjacent vertices, $\deg(u) + \deg(w) \geqslant v$, then G is Hamiltonian.

Theorem 6.17 (Dirac)**.** If every vertex v of G has $\deg(v) \geqslant \frac{1}{2}v$, then G is Hamiltonian.

Remark 6.18. Research the Königsberg Bridge Problem. We haven't discussed it here because it requires the concept of multigraphs (*i.e.* graphs with multiple edges between pairs of vertices). State the problem, and justify the answer to the problem in terms of multigraphs.

7 Trees

Definition 7.1. A **tree** is a connected graph with no circuits.

Theorem 7.2. Let T be a graph. The following are equivalent:

a. T is a tree,

b. T contains no circuits/cycles and has $v-1$ edges,

c. T is connected and has $v-1$ edges,

d. T is connected and every edge of T is a **bridge**, *i.e.* it lies in a disconnecting set of a size one,

e. Any two vertices of T have a unique path conntecting them,

f. T contains no circuits/cycles but the addition of any new edge produces exactly one circuit/cycle.

Theorem 7.3. Let T be a tree. If T has a vertex of degree p, then it has at least p *leaves*, *i.e.* vertices of degree 1.

Exercise 7.4. Determine the number of non-isomorphic trees of size:

a. 5

b. 6

c. 7

Definition 7.5. Let G be a connected graph. A **spanning tree** for G is a subgraph T of G that has the properties:

a. T is a tree, and

b. every vertex of G is a vertex of T.

Exercise 7.6. Construct an algorithm to "grow" a spanning tree, *i.e.* given an edge e of a connected graph G, describe an algorithm which will result in a spanning tree containing e. Prove that your algorithm always works.

Exercise 7.7. What can be said (and proven) about those edges that are contained in every spanning tree? In none?

Theorem 7.8. Let G be a graph with v vertices. Then the length of the shortest circuit that includes each edge of G at least once is $2(v-1)$.

Exercise 7.9. The following theorem is due to Kirchhoff, and is quite challenging. Do not attempt to prove it! Look up what all the terms mean, be able to describe it to a classmate, compute a couple of examples and perhaps even look up a proof of the theorem!

Theorem 7.10 (Kirchhoff's Theorem)**.** Let G be a connected graph. Construct the following matrix $L = L(G)$, called the *Laplacian matrix* of G. Order the vertices of G as $v_1, v_2, \ldots, v_v$, and then $L_{i,j}$, the entry of L located in entry i, j, is given by:

$$L_{i,j} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are adjacent}, \textit{or} \\ 0 & \text{otherwise.} \end{cases}$$

Then L is a singular matrix, and so has determinant 0. However, any submatrix of rank $n-1$, *i.e.* any matrix obtained from L by deleting the i^{th} row and column, for some $i = 1, \ldots, v$, is not singular, and the absolute value of its determinant is equal to the number of distinct spanning trees of G.

Corollary 7.11 (Cayley's Theorem)**.** For a complete graph K_n, the number of spanning trees of K_n is n^{n-2}.