Name: ______________________________

(한글/漢字 and Romanization)

Student ID: ______________________________

Quiz 7 GEDB003 Linear Algebra

Spring 2020

Tuesday, 28 April 2020

1. An invertible square matrix A satisfies $A^3+3A^2-25A+21\,I=O$, where I and O are the identity and zero matrices, respectively. Show that the inverse of A^2 satisfies

$$-\frac{25}{21^2}A^2-\frac{32}{147}A-\frac{562}{441}I.$$

2. A parallelepiped with one vertex at the origin and adjacent vertices at the points $(x-7,\ 6,\ x-2)$, $(-8,\ x-5,\ x+4)$, and $(x+9,\ x-3,\ 1)$, where $x\in\mathbb{R}$, has volume of 11. Find the possible exact values for $x\in\mathbb{R}$.

Solution:

1. Since A is invertible, multiplying the matrix equation either from the left or right with A^{-1} yields (we multiply from the right in the following)

$$\begin{aligned} A^3A^{-1}+3A^2A^{-1}-25AA^{-1}+21\,IA^{-1} &= OA^{-1} \\ A^2+3A-25\,I+21A^{-1} &= O \\ A^{-1} &= \frac{1}{21}\left(-A^2-3A+25\,I\right). \end{aligned}$$

Multipyling again the quadratic equation in A with A^{-1} either from the left or right yields (we multiply from the right in what follows)

$$\begin{aligned} A^2A^{-1}+3AA^{-1}-25\,IA^{-1}+21A^{-1}A^{-1} &= OA^{-1} \\ A+3\,I-25A^{-1}+21\left(A^{-1}\right)^2 &= 0 \\ 21\left(A^{-1}\right)^2 &= -A-3\,I+25A^{-1} \\ \left(A^{-1}\right)^2 &= \frac{1}{21}\left(-A-3\,I+25A^{-1}\right) \end{aligned}$$

Substituting A^{-1} from the previous expression and using the fact that the inverse of a square matrix is the square of inverse matrix, we obtain

$$\begin{aligned} \left(A^{-1}\right)^2 &= \frac{1}{21}(-A-3\,I)+\frac{25}{21}\left(-A^2-3A+25\,I\right) \\ \left(A^2\right)^{-1} &= -\frac{25}{21^2}A^2-\frac{1}{21}\left(1+\frac{25}{7}\right)A+\frac{1}{21}\left(\frac{25^2}{21}-3\right)I \\ &= -\frac{25}{21^2}A^2-\frac{32}{147}A-\frac{562}{441}I. \qquad \square \end{aligned}$$

2. Note to the graders: It is fine if the students could not finish the first case, but we are looking the values for $x\in\mathbb{R}$ that can be expressed exactly in a relatively simple form, which is manageable albeit a little bit hard for the second case.

For a 3×3 matrix M, the volume of parallelepiped determined by the columns of M is $|\det(M)|$. We construct the matrix M as follows:

$$M = \begin{bmatrix} x-7 & -8 & x+9 \\ 6 & x-5 & x-3 \\ x-2 & x+4 & 1 \end{bmatrix}.$$

The determinant of M is given by

$$\begin{aligned} \det(M) &= (x-7)(x-5)(1) + (-8)(x-3)(x-2) + (x+9)(6)(x+4) \\ &\quad - [(x-2)(x-5)(x+9) + (x+4)(x-3)(x-7) + 1(6)(-8)] \\ &= -x^2 + 106x + 203 - (2x^3 - 4x^2 - 72x + 126) = -2x^3 + 3x^2 + 178x + 77 \\ |\det(M)| &= \left|-2x^3 + 3x^2 + 178x + 77\right| = 11 \\ &- 2x^3 + 3x^2 + 178x + 77 = \pm 11. \end{aligned}$$

For the first case, we have the following cubic equation:

$$\begin{aligned} -2x^3 + 3x^2 + 178x + 77 &= 11 \\ 2x^3 - 3x^2 - 178x - 66 &= 0. \end{aligned}$$

It turns out that even though this cubic equation has three real roots, it cannot be factorized easily. You may leave it here and continue to the next case. Just in case you are interested in the numerical value of the roots, they are $x_1 \approx -8.50652$, $x_2 \approx -0.37373$, and $x_3 \approx 10.38024$. For the second case, we have the following cubic equation:

$$\begin{aligned} -2x^3 + 3x^2 + 178x + 77 &= -11 \\ 2x^3 - 3x^2 - 178x - 88 &= 0 \\ (2x+1)(x^2 - 2x - 88) &= 0 \\ x = -\frac{1}{2}, \qquad\qquad x &= \frac{2 \pm \sqrt{4 - 4(-88)}}{4} = 1 \pm \sqrt{89}. \end{aligned}$$

Thus, the possible exact values of $x \in \mathbb{R}$ are $-1/2$, $1 + \sqrt{89}$, and $1 - \sqrt{89}$.

#		Grading elements	points
Quiz7	#1	$A^{-1}=\frac{1}{21}(-A^2-3A+25I)$	1
		$(A^{-1})^2=\frac{1}{21}(-A-3I+25A^{-1})$	1
		$(A^2)^{-1}=-\frac{25}{21^2}A^2-\frac{32}{147}A+\frac{562}{441}I$	3
		※ If there are no errors in the process even if solved in many different ways, a corresponding score is given. ※ If you find an error in the problem and write corrected answer, you'll get all the points.	
	#2	Expressing matrix by $M=\begin{bmatrix} x-7 & -8 & x+9 \\ 6 & x-5 & x-3 \\ x-2 & x+4 & 1 \end{bmatrix}$	1
		$\lvert\det(M)\rvert=\lvert-2x^3+3x^2+178x+77\rvert=11$	1
		1st case ; $x\approx-8.50652,\ -0.37373,\ 10.38024$ 2nd case ; $x=\frac{1}{2},1\pm\sqrt{89}$	3
		$\{x \mid -2x^3+3x^2+178x+77=11\}$, $x=\frac{1}{2},1\pm\sqrt{89}$	
		The possible exact values are $x=\frac{1}{2},1\pm\sqrt{89}$	