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HW3: 2.2-2, 4, 7

2.2-2 Find all possible solutions to (2.2-26):

(2.2-26) $$S = A^T S A + Q$$

Where $A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$ $C = \begin{bmatrix} 2 & 0 \end{bmatrix}$ $Q = C^T C$

Let $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$, thus (2.2-26) becomes:

$$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 0 & -1/2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} = \begin{bmatrix} \frac{P_1}{2} & \frac{P_2}{2} \\ P_1 - \frac{P_3}{2} & P_2 - \frac{P_4}{2} \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 0 & -1/2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} = \begin{bmatrix} \frac{P_1}{4} + 4 & \frac{P_1}{2} - \frac{P_2}{4} \\ \frac{P_1}{2} - \frac{P_3}{4} & \left(P_1 - \frac{P_3}{2}\right) - \left(\frac{P_2}{2} - \frac{P_4}{4}\right) \end{bmatrix}$$

Solving simultaneously:

$$P_1 = \frac{P_1}{4} + 4$$

$$\frac{3P_1}{4} = 4$$

$$\boxed{P_1 = \frac{16}{3}}$$

$$P_2 = \frac{P_1}{2} - \frac{P_2}{4} = \frac{8}{3} - \frac{P_2}{4}$$

$$\frac{5P_2}{4} = \frac{8}{3}$$

$$\boxed{P_2 = \frac{32}{15}}$$

$$P_3 = \frac{P_1}{2} - \frac{P_3}{4} = \frac{8}{3} - \frac{P_3}{4}$$

$$\boxed{P_3 = P_2 = \frac{32}{15}}$$ ~ symmetric

$$P_4 = P_1 - \underbrace{\frac{P_3}{2}}_{= P_2} - \frac{P_2}{2} + \frac{P_4}{4} = \frac{16}{3} - \frac{32}{15} + \frac{P_4}{4} = \frac{80-32}{15} + \frac{P_4}{4}$$

$$\frac{3P_4}{4} = \frac{48}{15} = \frac{16}{5}$$

$$P_4 = \frac{64}{15}$$

2.2-2 CONTD a. Thus $P = \begin{bmatrix} \frac{16}{3} & \frac{32}{15} \\ \frac{32}{15} & \frac{64}{15} \end{bmatrix}$

b. Since $|\text{eig}(A)| < 1$, A is stable, and since the observability matrix is,

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix},$$

which has full rank, the solution to the Riccati equation has the unique solution found in part a.

2.2-4 Let $x_{k+1} = 2x_k + u_k$

a. find the homogeneous solution x_k for $k = 0, 5$ if $x_0 = 3$

homogeneous solution $\Rightarrow u_k = 0 \ \forall k$

$\Rightarrow x_{k+1} = 2x_k$

k	x_k	x_{k+1}
0	3	6
1	6	12
2	12	24
3	24	48
4	48	96
5	96	192

b. Minimum-energy control to drive state from $x_0 = 3$ to $x_5 = 0$

cost function: $J_0 = \frac{1}{2} \sum_{k=0}^{5} r u_k^2$

Thus the Hamiltonian is:

$$H^k = \frac{1}{2} r u_k^2 + \lambda_{k+1}(2x_k + u_k)$$

From this we get the state & costate equations:

state: $x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = 2x_k + u_k$

costate: $\lambda_k = \frac{\partial H^k}{\partial x_k} = 2\lambda_{k+1}$

Also have the stationarity condition:

$$0 = \frac{\partial H^k}{\partial u_k} = r u_k + \lambda_{k+1} \Rightarrow u_k = -\frac{\lambda_{k+1}}{r}$$

Substituting this u_k into the state equation:

$$x_{k+1} = 2x_k - \frac{\lambda_{k+1}}{r}$$

2.2-4 CONTD

b)

Next we write λ_K in terms of λ_N:

$$\lambda_{N-1} = 2\cdot\lambda_N$$

$$\lambda_{N-2} = 2^2\lambda_N$$

$$\vdots$$

$$\lambda_K = 2^{N-K}\lambda_N = 2^{5-K}\lambda_5$$

Thus $\lambda_{K+1} = 2^{N-(K+1)}\lambda_N = 2^{N-K-1}\lambda_N = 2^{4-K}\lambda_5$

Substituting into state equation:

$$x_{K+1} = 2x_K - \frac{2^{4-K}\lambda_5}{r} \quad (1^{st}\text{ order difference equation})$$

$$\Rightarrow \quad x_K = 2^K x_0 - \sum_{i=0}^{K-1} 2^{K-i-1}\cdot\frac{1}{r}\cdot 2^{N-i-1}\cdot\lambda_N$$

$$x_K = 2^K x_0 - \frac{\lambda_N}{r}\sum_{i=0}^{K-1} 2^{N+K-2i-2}$$

Setting $K=N=5$, we can solve for $\lambda_N = \lambda_5$

$$x_5 = 2^5(3) - \frac{\lambda_5}{r}\sum_{i=0}^{4} 2^{5+5-2i-2} = 2^5\cdot 3 - \frac{\lambda_5}{r}\sum_{i=0}^{4} 2^{8-2i} = 96 - \frac{\lambda_5}{r}\cdot 341 = 0$$

$$\Rightarrow \lambda_5 = \frac{96}{341}\cdot r$$

Thus, $\lambda_{K+1} = 2^{4-K}\cdot\frac{96}{34}r$

$$\Rightarrow u_K = -\frac{\lambda_{K+1}}{r} = -\frac{2^{4-K}\cdot\frac{96}{341}\cdot r}{r} = -2^{4-K}\frac{96}{341}$$

K	u_K	x_K
0	$-2^4\cdot\frac{96}{341} = 4.5044$	3
1	$-2^3\frac{96}{341} = 2.522$	1.4956
2	1.1261	0.7390
3	0.5630	0.3519
4	0.2815	0.1408
5	—	$-9.38\times10^{-15}\sim 0$ → rounding error from MATLAB

2.2-4 CONT'D C. optimal K_k to minimize $J_0 = 5x_5^2 + \frac{1}{2}\sum_{k=0}^{4}(x_k^2 + u_k^2)$

The key is finding the relation: $\frac{1}{2}S_N x_N^2$

$$\lambda_N = S_N x_N \,, \quad S_N = 10$$

To do this we let $\lambda_k = S_k x_k \quad \forall\, k = 0, \dots, N$

To find the sequence of S_k, we solve the Ricatti equation backwards:

$$S_k = \frac{a^2 r S_{k+1}}{b^2 S_{k+1} + r} + q$$

This can then be used to find the optimal gains K_k:

$$K_k = \frac{a b S_{k+1}}{b^2 S_{k+1} + r}$$

For our system, $a=2$, $r=1$, $q=1$, $b=1$ =>

K	S_k	K_k
5	10	—
4	$\frac{4\cdot1\cdot10}{1\cdot10+1}+1=4.6364$	$\frac{2\cdot10}{10+1}=1.818$
3	4.2903	1.6452
2	4.2439	1.6220
1	4.2372	1.6186
0	4.2362	1.6181

With these gains, we can get the trajectory and cost:

$$u_k^* = -K_k x_k$$

$$x_{k+1} = 2x_k - K_k x_k = (2-K_k)x_k \,, \quad J_k^* = \frac{1}{2}S_k x_k^2$$

K	x_k	J_k^*
0	3	19.0631
1	0.5455	4.74
2	0.1935	1.16
3	0.0732	0.27
4	0.0279	0.046
5	0.0107	$4.4\times10^{-28} \sim 0$

2.2-7 Let $x_{k+1} = \overbrace{a x_k + b u_k}^{f^k}$ (scalar system)

$$J = \underbrace{\frac{1}{3} S_N x_N^3}_{\phi} + \frac{1}{3}\sum_{k=0}^{N-1}(\underbrace{q x_k^3 + r u_k^3}_{L^k})$$

a.) Defining the Hamiltonian:

$$H^k = L^k + \lambda_{k+1} f^k = \frac{1}{3}(q x_k^3 + r u_k^3) + \lambda_{k+1}(a x_k + b u_k)$$

$$\boxed{\begin{aligned} &\text{State equation:} && x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = a x_k + b u_k \\ &\text{Costate equation:} && \lambda_k = \frac{\partial H^k}{\partial x^k} = x_k^2 q + a\lambda_{k+1} \\ &\text{Stationarity Condition:} && 0 = \frac{\partial H^k}{\partial u_k} = r u_k^2 + b\lambda_{k+1} \end{aligned}}$$

b. Solving the stationarity condition for u_k:

$$r u_k^2 = -b\lambda_{k+1}$$

$$u_k = \sqrt{-\frac{b}{r}\lambda_{k+1}}$$

$\Rightarrow$ Since we assume $r > 0$, $\lambda_k > 0$, b must be negative to solve for u_k

Substituting u_k into state (assuming $b < 0$)

$$\boxed{x_{k+1} = a x_k + b\sqrt{-\frac{b}{r}\lambda_{k+1}}}$$

2.2-7 cont'd C. Solve open-loop problem ($dX_N = 0$, $S_N = 0$, $q = 0$)

The performance index becomes:

$$J = \frac{1}{3} \sum_{k=0}^{N-1} r u_k^3$$

and the Hamiltonian:

$$H^k = \frac{1}{3} r u_k^3 + \lambda_{k+1} (a x_k + b u_k)$$

$$\Rightarrow \quad x_{k+1} = a x_k + b u_k \quad \text{— state}$$

$$\lambda_k = \frac{\partial H^k}{\partial x_k} = a \lambda_{k+1} \quad \text{— costate}$$

$$0 = \frac{\partial H^k}{\partial u_k} = r u_k^2 + b \lambda_{k+1} \quad \text{— stationarity}$$

From part b.: $x_{k+1} = a x_k + b \sqrt{-\frac{b}{r} \lambda_{k+1}}$

Since the costate is a difference equation, we can write

$$\lambda_k = a^{N-k} \lambda_N$$

$$\Rightarrow \quad x_{k+1} = a x_k + b \underbrace{\sqrt{\left(\frac{-b}{r}\right) a^{N-k-1} \lambda_N}}_{f}$$

This new state equation is also a difference equation, with forcing function f.

$$\Rightarrow \quad x_k = a^k x_0 + \sum_{i=0}^{k-1} a^{k-i-1} \left(b \sqrt{\left(\frac{-b}{r}\right)\left(a^{N-k-i} \cdot \lambda_N\right)} \right)$$

$$= a^k x_0 + b \cdot \sqrt{\frac{-b}{r}} \cdot \sqrt{\lambda_N} \cdot \sum_{i=0}^{k-1} \underbrace{a^{k-i-1} \cdot a^{(N-k-i)/2}}$$

$$a^{\frac{N+k-3i-2}{2}} = a^{\frac{N+k-2}{2}} \cdot a^{\frac{-3i}{2}}$$

$$\Rightarrow \quad x_k = a^k x_0 + b \sqrt{-\frac{b}{r} \cdot \lambda_N \cdot a^{N+k-2}} \cdot \sum_{i=0}^{k-1} \left(\frac{1}{a^{3/2}}\right)^i \quad \text{geometric series}$$

$$= a^k x_0 + b \sqrt{\frac{-b}{r} \lambda_N a^{N+k-2}} \cdot \frac{1 - a^{-\frac{3}{2}k}}{1 - a^{-3/2}} \quad \longrightarrow \text{only unknown is } \lambda_N$$

c)

2.2-7 cont'd Let $k=N$

$$x_N = a^N x_0 + b\sqrt{\frac{-b}{r}\lambda_N a^{N+N-2}} \cdot \frac{1-a^{-\frac{3}{2}N}}{1-a^{-3/2}}$$

$$\frac{x_N - a^N x_0}{b} = \left(-\frac{b}{r}\lambda_N\right)^{1/2} \underbrace{\left(a^{\cancel{2N-2}\,N-1}\right)^{\cancel{1/2}} \frac{1-a^{-\frac{3N}{2}}}{1-a^{-\frac{3}{2}}}}_{\frac{a^{2N/2}}{a^{2/2}} \cdot \frac{1-a^{-3N/2}}{1-a^{-3/2}} = \frac{a^N - a^{-N/2}}{a-a^{-1/2}}}$$

$$\Rightarrow \left(-\frac{b}{r}\lambda_N\right)^{1/2} = \left[\frac{x_N - a^N x_0}{b}\right]\left[\frac{a-a^{-1/2}}{a^N - a^{-N/2}}\right]$$

$$\lambda_N = -\frac{r}{b}\left[\frac{x_N - a^N x_0}{b}\right]^2 \left[\frac{a-a^{1/2}}{a^N - a^{-N/2}}\right]^2$$

Thus the costate for arbitrary $k \leq N$

$$\lambda_k = -a^{N-k} \cdot \frac{r}{b} \cdot \left[\frac{x_N - a^N x_0}{b}\right]^2 \cdot \left[\frac{a-a^{1/2}}{a^N - a^{-N/2}}\right]^2$$

Then the optimal control sequence is:

$$u_k^* = \sqrt{-\frac{b}{r}\lambda_{k+1}} = \left\{+\cancel{\frac{b}{r}} \cdot \cancel{\frac{r}{b}}\, a^{N-k-1} \cdot \left[\frac{x_N - a^N x_0}{b}\right]^2 \left[\frac{a-a^{1/2}}{a^N - a^{-N/2}}\right]\right\}^{\frac{1}{2}}$$

$$\boxed{u_k^* = a^{\frac{N-k-1}{2}} \cdot \frac{x_N - a^N x_0}{b} \cdot \frac{a-a^{1/2}}{a^N - a^{-N/2}}}$$