2.3 Digital Control of Continuous-time System

Design of Digital Controls

The continuous time-invariant plant is given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

The discretized plant with a sampling period of T is

$$x_{k+1} = A^s x_k + B^s u_k$$

The sampled plant and control are

$$A^s = e^{AT}$$

$$B^s = \int_0^T e^{At}\, B dt$$

The discretization process assumes that the control input $u(t)$ to the continuous plant is switched only at times kT and it is held constant for $1 < T \leq t \leq (k+1)T$.

Simulation of Digital Control

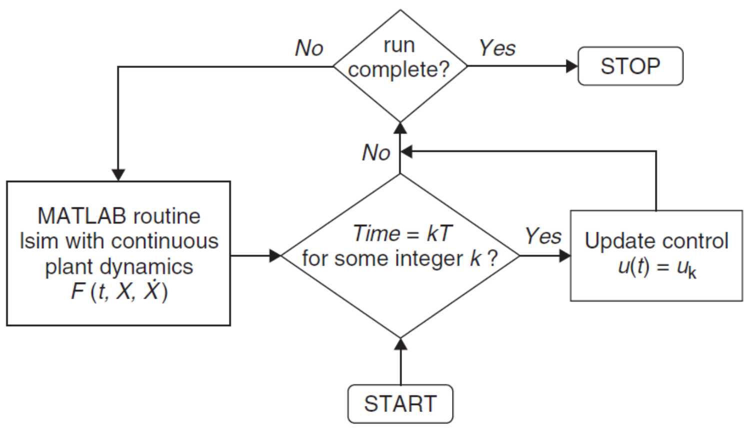

FIGURE 2.3-1 Digital control simulation scheme.

Example 2.3.1 Digital Control of an RC Circuit

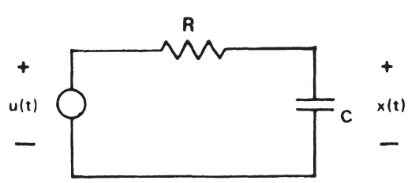

FIGURE 2.3-2 RC circuit.

$$\dot{x} = \frac{1}{\tau}x + \frac{1}{\tau}u$$

with time constant $\tau = \frac{1}{RC}$. Let $\tau = 5$ so that $\dot{x} = -0.2x + 0.2u$.

a) <u>Design a Digital Control Law for Free Final State</u>

Let the sampling period $T = 0.5$ sec

$$
\begin{aligned}
x_{k+1} &= e^{-\frac{T}{\tau}} x_k + \int_0^T e^{-\frac{\lambda}{\tau}} \frac{1}{\tau} d\lambda \, u_k \\
&= e^{-\frac{T}{\tau}} x_k + \left(1 - e^{-\frac{T}{\tau}}\right) u_k
\end{aligned}
$$

or

$$x_{k+1} = a x_k + b u_k$$

with $a = e^{-\frac{T}{\tau}} = 0.9048$ and $b = \left(1 - e^{-\frac{T}{\tau}}\right) = 0.0952$

Suppose that we want the control and state samples u_k and x_k to be small over a 5 sec interval for any initial voltage $x(0)$. Then $N = \frac{5}{T} = 10$. Select the performance index

$$J = \frac{1}{2} S_N x_N^2 + \frac{1}{2} \sum_{k=0}^{N-1} (q x_k^2 + r u_k^2)$$

The optimal control is given by Riccati equation

$$K_k = \frac{ab s_{k+1}}{b^2 S_{k+1} + r}$$

$$S_k = \frac{a^2 r S_{k+1}}{b^2 S_{k+1} + r} + q$$

$$u_k = -K_k x_k$$

By selecting S_N large, we can force $x_N = X(5)$ small.

b) Simulation of Digital control law for Free Final state

Let $a_c = -0.2$, $b_c = 0.2$ and $a_d = 0.9048$, b_d=0.0952, $q = r = 1$, $S_N = 100$, $N = 100$, and $x_0 = 10$.

c) Design of control law for Fixed Final State

To achieve a fixed final value of $x(5) = 0$, we can use a very large value of S_n. To achieve a nonzero value of r, we must use the open loop control for the fixed final value problem. Thus, suppose we want to drive the capacitor voltage from $x(0) = x_0 = 10\ V$ exactly to $x(5) = r_N = 20V$, while minimizing the energy.

$$J_9 = \frac{r}{2} \sum_{k=0}^{N-1} u_k^2$$

The optimal control sequence is

$$u_k = \frac{1 - a^2}{b(1 - a^{2N})} (r_N - a^N x_0) \cdot a^{N-k-1}, \qquad k = 0, \dots, N$$

Example 2.3-2 Digital Control of System obeying Newton's Law

Newton laws $m\ddot{d} = F$ can be expressed in state-variable form. If we define $x_1 = d$, $x_2 = \dot{d} = v$, $u(t) = \ddot{d}$ which is acceleration, $x \triangleq [x_1 \quad x_2]^T$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \Rightarrow A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a) The Rendezvous Problem

A target aircraft A_t is moving in the y_1 direction with a constant velocity of v_t. Its initial y_1 coordinate is Y_t. Our aircraft A is moving in y_1 direction with a constant velocity of $v > v_t$. Out initial y_1 coordinate is 0. Thus, out velocity relative to A_t is $(v - v_t)$. Then at time

$$t_f = \frac{Y_t}{v - v_t}$$

The two aircraft A and A_t will be abreast of each other (have the same coordinate in y_1)

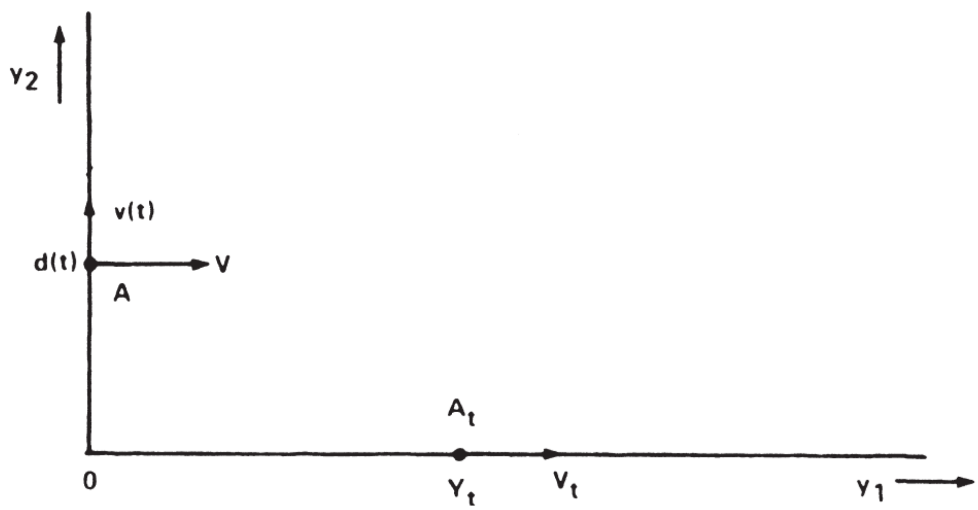

FIGURE 2.3-8 Rendezvous problem geometry.

The optimal control problem is as follow. The y_2 position and velocity of A relative to A_t are $d(t)$ and $v(t)$ and the y_2 dynamics of A are captured by Newtons Laws. It is required to find the control $u(t)$ needed in y_2 direction so that A will rendezvous with target A_t at time t_f, which means $d(t_f) = 0$ and $v(t_f) = 0$.

b) Design of Digital Control Law

The dynamics in continuous time can be discretized by

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u_k$$

Note that $e^{AT} = I + AT$ since $A^2 = 0$ and $\int_0^T e^{A\tau}\, d\tau\, B = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$

T is the sampling period. Suppose that $t_f = 5\ sec$ then $T = 0.5\ sec$ is reasonable.

The performance index

$$J_0 = \frac{1}{2} x_N^T \begin{bmatrix} s_d & 0 \\ 0 & s_v \end{bmatrix} x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T \begin{bmatrix} q_d & 0 \\ 0 & q_v \end{bmatrix} x_k + r u_k^2)$$

Select control weighting $r = 1$, position weighting $q_d = 1$ and velocity weighting $q_v = 1$. To make sure that the final y_2 position d_N and v_N are very small, select the final state component weights as $s_d = 100$, $s_v = 100$. The number of iterations

$$N = \frac{t_f}{T} = 10$$

The Kalman gain

$$K_k = \left(B^{s^T} S_{k+1} B^s + r\right)^{-1} (B^T S_{k+1} A^s)$$

And

$$S_k = A^{s^T} S_{k+1} (A^s - B^s K_k) + Q$$

Where

$$A^s = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \qquad B^s = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$$

The optimal control sequence

$$u^*(k) = -K_k x_k$$

Simulation code is posted on BlackBoard.