MECE6388 Optimal Control Theory
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2.2-2 Solutions to the algebraic Lyapunov equation.

a. Find all possible solutions to (2.2-26) if

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 \end{bmatrix}, \qquad Q = C^T C = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

(Hint: Let

$$P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix},$$

Substitute into (2.2-26), and solve for the scalars p_i. Alternatively, the results of Problem 2.2-1 can be used.)

b. Find the symmetric solution

Solution to part a:

From equation (2.2-26)

$$S = P = A^T P A + Q$$

$$\Rightarrow \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}p_1 + 4 & \frac{1}{2}p_1 - \frac{1}{4}p_2 \\ \frac{1}{2}p_1 - \frac{1}{4}p_3 & p_1 - \frac{1}{2}p_2 - \frac{1}{2}p_3 + \frac{1}{4}p_4 \end{bmatrix}$$

Therefore

$$\begin{aligned} p_1 &= \frac{1}{4}p_1 + 4, & p_1 &= \frac{16}{3} \\ p_2 &= \frac{1}{2}p_1 - \frac{1}{4}p_2, & p_2 &= \frac{32}{15} \\ p_3 &= \frac{1}{2}p_1 - \frac{1}{4}p_3, & p_3 &= p_2 \\ p_4 &= p_1 - \frac{1}{2}p_2 - \frac{1}{2}p_3 + \frac{1}{4}p_4, & p_4 &= \frac{64}{15} \end{aligned} \qquad \rightarrow P = \begin{bmatrix} \frac{16}{3} & \frac{32}{15} \\ \frac{32}{15} & \frac{64}{15} \end{bmatrix}$$

Solution to part b:

Assume $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}p_1 + 4 & \frac{1}{2}p_1 - \frac{1}{4}p_2 \\ \frac{1}{2}p_1 - \frac{1}{4}p_2 & p_1 - \frac{1}{2}p_2 - \frac{1}{2}p_2 + \frac{1}{4}p_3 \end{bmatrix}$$

Therefore

$$\begin{aligned} p_1 &= \frac{1}{4}p_1 + 4, & p_1 &= \frac{16}{3} \\ p_2 &= \frac{1}{2}p_1 - \frac{1}{4}p_2, & p_2 &= \frac{32}{15} \\ p_3 &= p_1 - \frac{1}{2}p_2 - \frac{1}{2}p_2 + \frac{1}{4}p_3, & p_3 &= \frac{64}{15} \end{aligned} \qquad \rightarrow p = \begin{bmatrix} \frac{16}{3} & \frac{32}{15} \\ \frac{32}{15} & \frac{64}{15} \end{bmatrix}$$

2.2-4 Control of a scalar system.

Let

$$x_{k+1} = 2x_k + u_k$$

a. Find the homogeneous solution x_k for $k = [0,5]$ if $x_0 = 3$
b. Find the minimum energy control sequence u_k required to drive $x_0 = 3$ to $x_5 = 0$. Check your answer by finding the resulting state trajectory.
c. Find the optimal feedback gain sequence K_k to minimize the performance index

$$J_0 = 5x_5^2 + \frac{1}{2}\sum_{k=0}^{4}(x_k^2 + u_k^2)$$

Find the resulting state trajectory and the cost to go J_k^* for $k = [0,5]$.

Solution to part a:

Homogeneous solution implies no control $u_k = 0$

$$x_{k+1} = 2x_k \;\rightarrow x_k = 2^k x_0$$
$$x_k = [3, 6, 12, 24, 48, 96]$$

Solution to part b:

Method 1.

This is a fixed final state open loop problem. $x_N = r_N = x_5 = 0$

Recall from class notes, $dx_N = 0$, the boundary condition for dx_N is satisfied.

$$\left(\frac{\partial \phi}{\partial x_N} - \lambda_N\right)^T dx_N = 0$$

Set $S_N = 0$ since final state contribution to J_0 is always fixed.

Let the cost function be

$$J_0 = \frac{1}{2}\sum_{k=0}^{N-1} u_k^T R u_k$$

State and costate equations are

$$x_{k+1} = Ax_k - BR^{-1}B^T\lambda_{k+1}$$
$$\lambda_k = A^T\lambda_{k+1}$$
$$\lambda_k = (A^T)^{N-k}\lambda_N$$

Substitute in state equation

$$x_{k+1} = Ax_k - BR^{-1}B^T(A^T)^{N-k-1}\lambda_N$$

This as a first order difference equation with the second term as the input. Therefore

$$x_k = A^k x_0 - \sum_{i=0}^{k-1} A^{k-i-1}BR^{-1}B^T(A^T)^{N-i-1}\lambda_N$$

when $k = N$

$$x_N = A^N x_0 - \sum_{i=0}^{N-1} A^{N-i-1}BR^{-1}B^T(A^T)^{N-i-1}\lambda_N$$

Since $x_N = r_N$,

$$\lambda_N = -G_{0,N}^{-1}(r_N - A^N x_0)$$

where

$$G_{o,N} = \sum_{i=0}^{N-1} A^{N-i-1}BR^{-1}B^T(A^T)^{N-i-1}$$

Then the costate is

$$\lambda_k = (A^T)^{N-k}\lambda_N = -(A^T)^{N-k}G_{0,N}^{-1}(r_N - A^N x_0)$$

and the optimal control

$$u_k^* = R^{-1}B^T\lambda_{k+1} = R^{-1}B^T(A^T)^{N-k-1}G_{0,N}^{-1}(r_N - A^N x_0)$$

Using this equation and $R = 1$, $B = 1$, $r_N = 5$

$$u_k^* = 2^{5-(k+1)} G_{0,5}^{-1}(0 - 2^5 x_0) = -2^{4-k} G_{0,5}^{-1} \cdot 96 = -2^{4-k}(341)^{-1} \cdot 96$$

$$G_{0,N} = \sum_{i=0}^{N-1} A^{N-i-1} B R^{-1} B^T (A^T)^{N-i-1} = \sum_{i=0}^{4} 2^{4-i}(2^T)^{4-i} = 341$$

$$u_k^* = [-4.5044, \quad -2.2522, \quad -1.1261, \quad -0.5630, \quad -0.2815]$$

The resulting state trajectory:

$$x_k^* = [3, \quad 1.4956, \quad 0.7390, \quad 0.3519, \quad 0.1408, \quad 0]$$

Method 2

From Example 2.2-1 of the textbook

$$u_k^* = \frac{1 - a^2}{b(1 - a^{2N})}(r_N - a^N x_0) a^{N-k-1}$$

Where $a = 2,\ b = 1,\ r_N = 0,\ N = 5$

$$u_k^* = [-4.5044, \quad -2.2522, \quad -1.1261, \quad -0.5630, \quad -0.2815]$$

The resulting state trajectory:

$$x_k^* = [3, \quad 1.4956, \quad 0.7390, \quad 0.3519, \quad 0.1408, \quad 0]$$

Solution to part c:

We know since $S_N \neq 0$, this is a free final state problem.

Given $S_N = 10$, $Q_k = 1$, $R_k = 1$, $A_k = 2$, $B_k = 1$

The performance index kernel is

$$S_k = A_k^T \left[S_{k+1} - S_{k+1} B_k \left(B_k^T S_{k+1} B_k + R_k \right)^{-1} B_k^T S_{k+1} \right] A_k + Q_k$$

$$S_k = 4\left(S_{k+1} - \frac{S_{k+1}^2}{S_{k+1} + 1} \right) + 1 = \frac{4 S_{k+1}}{S_{k+1} + 1} + 1$$

The Kalman gain is

$$K_k = \left(B_k^T S_{k+1} B_k + R_k \right)^{-1} B_k^T S_{k+1} A_k$$

$$K_k = \frac{2 S_{k+1}}{1 + S_{k+1}}$$

The cost to go is

$$J_k^* = \frac{1}{2} x_k^T S_k x_k$$

When $N = 5$, starting from $S_5 = 5$

	S_i	K_i	u_i	x_i	J_k^*
0	4.2362	1.6181	-4.8544	3.0000	19.0631
1	4.2372	1.6186	-1.8544	1.1456	2.7807
2	4.2439	1.6220	-0.7087	0.4369	0.4051
3	4.2903	1.6452	-0.2718	0.1652	0.0585
4	4.6364	1.8182	-0.1066	0.0586	0.0080
5	10.0000			0.0107	0.0006

2.2-7 Cubic performance index.

Let

$$x_{k+1} = ax_k + bu_k$$

Where x_k and u_k are scalars, and

$$J = \frac{1}{3}s_N x_N^3 + \frac{1}{3}\sum_{k=0}^{N-1}(qx_k^3 + ru_k^3)$$

a. Write state and costate and stationarity condition

b. When can we solve for u_k? Under this condition, eliminate u_k from the state equation.

c. Solve the open-loop control problem (i.e., x_N fixed, $s_N = 0$, $q = 0$).

Solution to part a:

Hint: See textbook Table 2.1-1

Let the NEW system model be (constraint function)

$$g^k(x_k, u_k) = f^k(x_k, u_k) - x_{k+1} = ax_k + bu_k - x_{k+1} = 0$$

Let the performance index, given in the general form be

$$J' = \phi(N, x_N) + \sum_{k=i}^{N-1} L^k(x_k, u_k) = \frac{1}{3}s_N x_N^3 + \frac{1}{3}\sum_{k=0}^{N-1}(qx_k^3 + ru_k^3)$$

Define the Hamiltonian

$$H^k(x_k, u_k) = L^k(x_k, u_k) + \lambda_{k+1}^T f^k(x_k, u_k) = \frac{1}{3}(qx_k^3 + ru_k^3) + \lambda_{k+1}^T(ax_k + bu_k)$$

State Equation

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = f^k(x_k, u_k) = ax_k + bu_k$$

Costate Equation

$$\lambda_k = \frac{\partial H^k}{\partial x_k} = \left(\frac{\partial f^k}{\partial x_k}\right)^T \lambda_{k+1} + \frac{\partial L^k}{\partial x_k} = a\lambda_{k+1} + qx_k^2$$

Stationary Condition

$$0 = \frac{\partial H^k}{\partial u_k} = \left(\frac{\partial f^k}{\partial u_k}\right)^T \lambda_{k+1} + \frac{\partial L^k}{\partial u_k} = ru_k^2 + \lambda_{k+1} b$$

Solution to part b: From stationary condition

$$u_k = \pm\sqrt{-\frac{b}{r}\lambda_{k+1}}$$

We can solve for u_k only when $\frac{b}{r}\lambda_{k+1} < 0$.

Substitute u_k in the state equation

$$x_{k+1} = ax_k + b\sqrt{-\frac{b}{r}\lambda_{k+1}}$$

Solution to part c:

Recall x_N fixed, $s_N = 0$, $q = 0$

From the costate equation,

$$\lambda_k = a\lambda_{k+1} + qx_k^2 = a\lambda_{k+1} = a^{N-k}\lambda_N$$

$$\lambda_{k+1} = a^{N-k-1}\lambda_N$$

This gives

$$u_k = \pm\sqrt{-\frac{b}{r}\lambda_{k+1}} = \pm\sqrt{-\frac{b}{r}a^{N-k-1}\lambda_N}$$

Substituting in state equation

$$x_{k+1} = ax_k + bu_k = ax_k + b\sqrt{-\frac{b}{r}\lambda_{k+1}} = ax_k + b\sqrt{-\frac{b}{r}a^{N-k-1}\lambda_N}$$

This is a difference equation with a forcing function. It can be written as,

$$x_k = a^k x_0 + \frac{b^{\frac{3}{2}}}{r^{\frac{1}{2}}}\sqrt{-\lambda_N}\sum_{i=0}^{k-1} a^{\frac{N-3+2k-3i}{2}}$$

Let $k = N$,

$$x_N = a^N x_0 + b\sqrt{-\frac{b}{r}\lambda_N}\cdot\sum_{i=0}^{N-1} a^{\frac{3N-3-3i}{2}}$$

The gramian is

$$G_{0,N} = b\sqrt{\frac{b}{r}}\cdot\sum_{i=0}^{N-1} a^{\frac{3N-3-3i}{2}} = a^{\frac{3N-3}{2}}\left(\frac{1-a^{-\frac{3N}{2}}}{1-a^{-\frac{3}{2}}}\right)\cdot b\sqrt{\frac{b}{r}}$$

Therefore

$$x_N = a^N x_0 + G_{0,N}\sqrt{-\lambda_N}$$

$$\lambda_N = \frac{-r\left(1-a^{-\frac{3}{2}}\right)^2 (x_N - a^N x_0)^2}{b^3\left(1-a^{-\frac{3N}{2}}\right)^2 a^{3N-3}}$$

$$\therefore \lambda_k = \frac{-r\left(1-a^{-\frac{3}{2}}\right)^2 (x_N - a^N x_0)^2}{b^3\left(1-a^{-\frac{3N}{2}}\right)^2 a^{3N-3}}\cdot a^{N-k}$$

$$\Longrightarrow u_k = \sqrt{\frac{\left(1-a^{-\frac{3}{2}}\right)^2 (x_N - a^N x_0)^2}{b^2\left(1-a^{-\frac{3N}{2}}\right)^2 a^{3N-3}}\cdot a^{N-k-1}}$$

$$u_k = \frac{1}{b}a^{\frac{-2N-k+2}{2}}\left(\frac{1-a^{-\frac{3}{2}}}{1-a^{-\frac{3N}{2}}}\right)(x_N - a^N x_0)$$