2.4 Steady-state Closed-loop Control and Suboptimal Feedback

Motivation: The solution to LQR control problem is a state feedback control

$$u_k = -K_k x_k$$

With gain sequence K_k given in terms of the solution S_k to the Riccati equation as

$$S_k = A^T[S_{k+1} - S_{k+1}B(B^T S_{k+1} B + R)^{-1} B^T S_{k+1})]A + Q$$
$$K_k = (B^T S_{k+1} B + R)^{-1} B^T S_{k+1} A$$

The closed-loop system

$$x_{k+1} = (A - BK_k)x_k$$

Is a time varying system since K_k is time varying.

The time-varying feedback is not always convenient to implement. It requires the storage for K_k ($m \times n$ matrices). We might be more interested in using instead a suboptimal feedback gain that does not actually minimize the performance index but is a constant

$$u_k = -Kx_k$$

Suboptimal Feedback Gain

The time invariant plant

$$x_{k+1} = Ax_k + Bu_k$$

The performance index

$$J_i = \frac{1}{2}x_N^T S_N x_N + \frac{1}{2}\sum_{k=i}^{N-1}(x_k^T Q x_k + u_k^T R u_k)$$

Since

$$\frac{1}{2}\sum_{k=i}^{N-1}(x_{k+1}^T S_{k+1} x_{k+1} - x_k^T S_k x_k) = \frac{1}{2}x_N^T S_N x_N - \frac{1}{2}x_i^T S_i x_i$$

The performance index can be rewritten by

$$J_i = \frac{1}{2}x_i^T S_i x_i + \frac{1}{2}\sum_{k=i}^{N-1}[x_{k+1}^T S_{k+1} x_{k+1} + x_k^T(Q - S_k + K_k^T R K_k)x_k]$$

The S_k satisfies the Riccati equation, the summation part in J_i is zero. Then the resulting cost on $[k, N]$ is given for each time k by

$$J_k = \frac{1}{2}x_k^T S_k x_k$$

Where the kernel is the solution to

$$S_k = (A - BK_k)^T S_{k+1}(A - BK_k) + K_k^T R K_k + Q \quad (1)$$

Joseph stabilized version of the Riccati equation.

It becomes Joseph-Riccati equation only if the optimal gain K_k sequence is used. If K_k is an arbitrary given gain then the equation (1) is simply a Lyapunov equation in terms of the known closed-loop plant matrix $A' = (A - BK_k)$. If K_k is not the optimal gain, then $J_k = \frac{1}{2}x_k^T S_k x_k$ is greater than J_k^*.

Example 2.4-1 Suboptimal Feedback Control of a Scalar System

Let us reconsider the system

$$x_{k+1} = ax_k + bu_k$$

With performance index

$$J_0 = \frac{1}{2}S_N x_N^2 + \frac{1}{2}\sum_{k=0}^{N-1}(qx_k^2 + ru_k^2)$$

The optimal control is a time-varying state feedback

$$u_k = -K_k x_k$$

With gain determined by the Riccati equation as

$$S_k = \frac{a^2 r S_{k+1}}{b^2 S_{k+1} + r} + q$$

$$K_k = \frac{abS_{k+1}}{b^2 S_{k+1} + r}$$

For parameters of $a = 1.05$, $b = 0.01$, $r = q = S_N = 5$ with final time $N = 100$, a simulation was run to obtain the Kalman gain sequence. A steady-state value $K_\infty \triangleq K_0 = 9.808$.

Now let us suppose we want a simpler feedback control

$$u_k = -K_\infty x_k = -9.808x_k$$

Then the cost is given by

$$J_k = \frac{1}{2}S_k x_k^2$$

Where S_k is the solution to the Lyapunov equation

$$S_k = S_{k+1}(a - bK_\infty)^2 + rK_\infty^2 + q$$

With boundary condition $S_N = 5$.

$$J_k^* = \frac{1}{2}S_k^* x_k^{*2} \leq \frac{1}{2}S_k x_k^2 = J_k$$

Where J_k^*, S_k^* and x_k^* are the optimal performance index, optimal S_k and optimal trajectory. Simulation code is posted on BlackBoard.

The Algebraic Riccati Equation

The Riccati equation is solved backward in time beginning at time N. As $k \to \infty$, the sequence S_k can converge to a steady-state matrix S_∞, which may be zero, positive semi-definite or positive definite. But it can also fail to converge to a finite matrix.

If S_k does converge, then for large negative k, evidently $S \triangleq S_k = S_{k+1}$. Riccati equation becomes the algebraic Riccati equation:

$$S = A^T[S - SB(B^T SB + R)^{-1}B^T S]A + Q$$

If the limiting solution exists and is denoted by S_∞ then the corresponding steady state Kalman gain is

$$K_\infty = (B^T S_\infty B + R)^{-1}B^T S_\infty A$$

This is a constant feedback gain

$$u_k = -K_\infty x_k$$

Limiting Behavior of the Riccati Equation Solution

Question:

1. When does there exist a bounded limiting solution S_∞ to the Riccati equation for all choices of S_N
2. In general, the limiting solution S_∞ depends on boundary condition S_N when is S_∞ the same for all choices of S_N?
3. When is the closed loop plant asymptotically stable?

Theorem 2.4-1

Let $< A, B >$ be stabilizable. Then for every choice of S_N there is a bounded limiting solution S_∞. Furthermore, S_∞ is a positive semidefinite solution to the algebraic Riccati equation.

Proof: Since (A, B) is stabilizable, there exists a constant feedback L so that $u_k = -Lx_k$ and $x_{k+1} = (A - BL)x_k$ is asymptotically stable. Thus, x_k is bounded and goes to zero as $k \to \infty$. Therefore, the associated cost

$$J_i = \frac{1}{2}x_N^T S_N x_N + \frac{1}{2}\sum_{k=i}^{N-1}(x_k^T Q x_k + u_k^T R u_k)$$

Is finite as $N - i \to \infty$. S_i satisfies the Lyapunov equation

$$S_i = (A - BL)^T S_{i+1}(A - BL) + L^T RL + Q$$

The optimal cost

$$J_i^* = \frac{1}{2}x_i^T S_i^* x_i^*$$

Where S_i^* is the condition to Riccati equation with S_N as boundary condition. Since $J_i^* \leq J_i$ for any initial state x_i so S_i provides an upper bound for S_i^*. Hence the solution to Riccati equation is bounded by a finite sequence so S_i^* converges to a constant limit S_∞.

Since the Riccati equation is symmetric, then for all i, S_i is symmetric if S_N is symmetric. The structure of the equation and the assumptions on Q, R also imply the positive semi definiteness of S_∞.

Theorem 2.4-2

Let C be a square root of the intermediate state weighting matrix, so that $Q = C^T C \geq 0$ and suppose $R > 0$. Suppose that (A, C) is observable, then (A, B) is stabilizable if and only if

a. There is a unique positive definite limiting solution S_∞ to the Riccati equation. Furthermore, S_∞ is the unique positive definite solution to the algebraic Riccati equation.
b. The closed loop plant

$$x_{k+1} = (A - BK_\infty)x_k$$

Is asymptotically stable, where K_∞ is given by

$$K_\infty = (B^T S_\infty B + R)^{-1} B^T S_\infty A$$

Proof,

Necessity:

Define D by $R = D^T D$. Since $|D| \neq 0$, then $|D| \neq 0$. So that there is $M = BD^{-1}$ or $B = MD$,

$$rank\begin{bmatrix} zI - A \\ C \\ DK \end{bmatrix} = rank\begin{bmatrix} I & 0 & M \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} zI - A \\ C \\ DK \end{bmatrix} = rank\begin{bmatrix} zI - (A - BK) \\ C \\ DK \end{bmatrix}$$

If (A, C) is observable, then D by PBH (Popov-Belevitch-Hautus) rank test

$$rank\begin{bmatrix} zI - A \\ C \end{bmatrix} = n, \qquad for\ every\ z$$

So

$$rank\begin{bmatrix} zI - (A - BK) \\ C \\ DK \end{bmatrix} = rank\begin{bmatrix} zI - A \\ C \\ DK \end{bmatrix} = rank\begin{bmatrix} zI - A \\ C \end{bmatrix} = n$$

We get $\left((A - BK), \begin{bmatrix} C \\ DK \end{bmatrix}\right)$ is observable for any K.

The cost kernel function with $K = K_\infty$ and $R = D^T D$, $Q = C^T C$

$$\begin{aligned} S_k &= (A - BK_\infty)^T S_{k+1}(A - BK_\infty) + K_\infty^T R K_\infty + Q \\ &= (A - BK_\infty)^T S_{k+1}(A - BK_\infty) + K_\infty^T D^T D K_\infty + C^T C \\ &= (A - BK_\infty)^T S_{k+1}(A - BK_\infty) + \begin{bmatrix} C \\ DK_\infty \end{bmatrix}^T \begin{bmatrix} C \\ DK_\infty \end{bmatrix} \end{aligned}$$

The limiting solution to this cost kernel function satisfies

$$S = (A - BK_\infty)^T S (A - BK_\infty) + \begin{bmatrix} C \\ DK_\infty \end{bmatrix}^T \begin{bmatrix} C \\ DK_\infty \end{bmatrix}$$

This is the Lyapunov equation with

$$\left((A - BK_\infty), \begin{bmatrix} C \\ DK_\infty \end{bmatrix}\right)$$

Observable and$(A - BK_\infty)$ stable. There is a unique positive definite solutionS^* to the algebraic Riccati equation.

$$S = A^T[S - SB(B^T S B + R)^{-1} B^T S]A + Q$$

Which is the limiting solution to the Riccati equation

$$S_k = A^T[S_{k+1} - S_{k+1}B(B^T S_{k+1} B + R)^{-1} B^T S_{k+1}]A + Q$$

The stability implies: There exist a feedback control $u_k = -Lx_k$ so that $x_{k+1} = (A - BL)x_k$ is asymptotically stable. The cost function of such control on $[i, \infty]$ is

$$J_i = \frac{1}{2} x_i^T S x_i$$

With S the limiting solution to

$$S_k = (A - BL)^T S_{k+1}(A - BL) + L^T R L + Q$$

The optimal cost function on $[i, \infty]$

$$J_i^* = \frac{1}{2}\sum_{k=i}^{\infty}\left({x_k^*}^T Q x_k^* + {u_k^*}^T R u_k^*\right) = \frac{1}{2}\, x_i^T S^* x_i \le J_i$$

With S^* the limiting solution to

$$S_k = (A - BK_k)^T S_{k+1}(A - BK_k) + K_k^T R K_k + Q$$
$$K_k = (B^T S_{k+1} B + R)^{-1} B^T S_{k+1} A$$

Therefore ${x_k^*}^T Q x_k^* + {u_k^*}^T R u_k^* \to 0$ as $k \to \infty$.
Since $R > 0$, $Q \ge 0$, ${x_k^*}^T Q x_k^* \to 0$ and ${u_k^*}^T R u_k^* \to 0$ as $k \to \infty$
Since $|R| \ne 0$, $u_k^* \to 0$ as $k \to \infty$
Since $Q = C^T C$

$$\begin{aligned} {x_k^*}^T Q x_k^* \to 0 \ &\Rightarrow {x_k^*}^T C^T C x_k^* \to 0 \\ &\Rightarrow (Cx_k^*)^T C x_k^* \to 0\ as\ k \to \infty \\ &\Rightarrow C x_k^* \to 0\ as\ k \to \infty \end{aligned}$$

Select an N so that Cx_k^* and u_k^* are negligible for $k > N$.

$$0 = \begin{bmatrix} Cx_k^* \\ Cx_{k+1}^* \\ \vdots \\ Cx_{k+n-1}^* \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_k^*$$

And the observability matrix $Q_c = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank, $x_k^* = 0$ as $k > N \Rightarrow x_k^* \to 0$ as $k \to \infty$.

Hence, the optimal closed loop system is asymptotical stable.

Sufficiency:

If $x_{k+1} = (A - BK_\infty)x_k$ is asymptotically stable, then (A, B) is certainly stabalizabe.

Infinite horizon Optimal Control

$$u_k = -K_\infty x_k$$

Minimize the cost over infinite time $[0, \infty]$

$$J_0 = \frac{1}{2}\sum_{k=0}^{\infty}(x_k^T Q x_k + u_k^T R u_k)$$

Example: Steady-state control of a scalar system:

Let the plant be

$$x_{k+1} = a x_k + b u_k$$

With performance index

$$J_0 = \frac{1}{2}\sum_{k=0}^{\infty}(q x_k^2 + r u_k^2)$$

The optimal control minimizing J_0 is the constant feedback

$$u_k = -K_\infty x_k$$

Where the gain

$$K_\infty = (B^T S_\infty B + R)^{-1} S_\infty A$$
$$= \frac{abS_\infty}{b^2 S_\infty + r}$$

With $A = a, B = b, R = r$.

The steady-state kernel is the unique positive definite root of the algebraic Riccati equation (ARE)

$$S = A^T[S - SB(B^T SB + R)^{-1} B^T S]A + Q$$

With $A = a, B = b, R = r, Q = q$

$$s = a^2 s - \frac{a^2 b^2 s^2}{b^2 s + r} + q$$

The ARE can be written as

$$b^2 s^2 + [(1 - a^2)r - b^2 q]s - qr = 0$$

Define

$$\Lambda = \frac{b^2 q}{(1 - a^2)r}$$

This becomes

$$\frac{\Lambda}{q} s^2 + (1 - \Lambda)s - \frac{\Lambda r}{b^2} = 0$$

Which has two solutions

$$s = \frac{q}{2\Lambda}\left[\pm\sqrt{(1 - \Lambda)^2 + \frac{4\Lambda}{1 - a^2}} - (1 - \Lambda)\right]$$

We need to pick up a non-negative solution, we must consider two cases

a. *Original system is stable:* $|a| < 1$

If $|a| < 1$ then $(1 - a^2) > 0$ and $\Lambda > 0$. In this case the unique non-negative solution is

$$s_\infty = \frac{q}{2\Lambda}\left[\sqrt{(1 - \Lambda)^2 + \frac{4\Lambda}{1 - a^2}} - (1 - \Lambda)\right]$$

And the steady state feedback is $u_k = -\frac{abs_\infty}{b^2 s_\infty + r}\, x_k$

b. *Original system is unstable:* $|a| > 1$

If $|a| > 1$ then $(1 - a^2) < 0$ and $\Lambda < 0$. Then the unique non-negative solution to the ARE is

$$s_\infty = -\frac{q}{2\Lambda}\left[\sqrt{(1 - \Lambda)^2 + \frac{4\Lambda}{1 - a^2}} - (1 - \Lambda)\right]$$

With the steady-state feedback gain, the system becomes

$$x_{k+1} = (a - bK_\infty)x_k$$

Define

$$a^{cl} = a - bK_\infty = \frac{a}{1 + \left(\frac{b^2}{r}\right) S_\infty}$$

If $|a^{cl}| < 1$, then the system is stable.

An analytic solution to the Riccati Equation

$$\begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} = H \begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix}$$

With

$$H \triangleq \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^T \\ QA^{-1} & A^T + QA^{-1}BR^{-1}B^T \end{bmatrix}$$

Assuming A is non-singular
The final condition is

$$\lambda_N = S_N x_N$$

And the initial condition is x_0 if the final state is free.
We assume $\lambda_k = S_k x_k$
Define $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$
Since H is symplectic , we have

$$H^T J H = J$$
$$H^T J = J H^{-1}$$
$$J^{-1} H^T J = H^{-1}$$

Since $J^{-1} = -J$

$$H^{-1} = -J H^T J$$

Performing these multiplications, we get

$$H^{-1} = \begin{bmatrix} A + BR^{-1}B^T A^{-T} Q & -BR^{-1}B^T A^{-T} \\ -A^{-T} Q & A^{-T} \end{bmatrix}$$

If μ is an eigenvalue of H and $\begin{bmatrix} f \\ g \end{bmatrix}$ is an eigenvector associated with it, then

$$\begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^T \\ QA^{-1} & A^T + QA^{-1}BR^{-1}B^T \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \mu \begin{bmatrix} f \\ g \end{bmatrix}$$

Or

$$\begin{bmatrix} A^T + QA^{-1}BR^{-1}B^T & -QA^{-1} \\ -A^{-1}BR^{-1}B^T & A^{-1} \end{bmatrix} \begin{bmatrix} g \\ -f \end{bmatrix} = \mu \begin{bmatrix} g \\ -f \end{bmatrix}$$

So

$$H^{-T} \begin{bmatrix} g \\ -f \end{bmatrix} = \mu \begin{bmatrix} g \\ -f \end{bmatrix}$$

μ is also an eigenvalue of H^{-T} . So $\frac{1}{\mu}$ is an eigenvalue of H^T. Since

$$|\lambda I - A| = |(\lambda I - A)^T| = |\lambda I - A^T|,$$

for any square matrix A, A and A^T have same eigenvalues so $\frac{1}{\mu}$ is also an eigenvalue of H
What this means is that the $2n$ eigenvalues of H can be arranged in a matrix

$$D = \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix}$$

where M is a diagonal matrix containing n eigenvalues outside the unity circle. Hence, M^{-1} is stable.

There is a similarity matrix W whose columns are the eigenvectors of H.

$$W^{-1}HW = D$$

Define the new state variables $\begin{bmatrix} w_k \\ z_k \end{bmatrix} = W^{-1}\begin{bmatrix} x_k \\ \lambda_k \end{bmatrix}$ so

$$\begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} = W\begin{bmatrix} w_k \\ z_k \end{bmatrix}$$

$$= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}\begin{bmatrix} w_k \\ z_k \end{bmatrix}$$

The Hamiltonian system takes on its Jordan normal form

$$\begin{bmatrix} w_k \\ z_k \end{bmatrix} = D\begin{bmatrix} w_{k+1} \\ z_{k+1} \end{bmatrix} \Rightarrow \begin{bmatrix} w_k \\ z_k \end{bmatrix} = \begin{bmatrix} M^{N-k} & 0 \\ 0 & M^{-(N-k)} \end{bmatrix}\begin{bmatrix} w_N \\ z_N \end{bmatrix}$$

The problem with this solution is that M^{N-k} does not go to zero as $N-k \to \infty$

We rewrite the solution

$$\begin{bmatrix} w_N \\ z_k \end{bmatrix} = \begin{bmatrix} M^{-(N-k)} & 0 \\ 0 & M^{-(N-k)} \end{bmatrix}\begin{bmatrix} w_k \\ z_N \end{bmatrix}$$

Since

$$\begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}\begin{bmatrix} w_k \\ z_k \end{bmatrix}$$

And

$$\lambda_N = S_N \cdot x_N$$

$$\begin{cases} \lambda_N = W_{21}w_N + W_{22}z_N = S_N x_N \\ x_N = W_{11}w_N + W_{12}z_N \end{cases}$$

$$\Rightarrow W_{11}w_N + W_{22}z_N = S_N(W_{11}w_N + W_{12}z_N)$$

Solving for z_N in terms of w_N

$$z_N = Tw_N$$

Where

$$T = -(W_{22} - S_N W_{12})^{-1}(W_{21} - S_N W_{11})$$

Since

$$z_k = M^{-(N-k)}z_N = M^{-(N-k)} \cdot Tw_N$$

And

$$w_N = M^{-(N-k)} \cdot w_k$$

$$z_k = M^{-(N-k)}TM^{-(N-k)}w_k$$

At each value of k, we have

$$z_k = T_k w_k$$

Where $T_k = M^{-(N-k)}TM^{-(N-k)}$

To relate S_k to T_k

$$\lambda_k = W_{21}w_k + W_{22}z_k = S_k x_k = S_k(W_{11}w_k + W_{12}z_k)$$

$$(W_{21} + W_{22}T_k)w_k = S_k(W_{11} + W_{12}T_k)w_k$$

Which implies

$$S_k = (W_{21} + W_{22}T_k)(W_{11} + W_{12}T_k)^{-1}$$

Since as $N - k \to \infty$, $M^{-(N-k)}$ goes to zero. So

$$T_k = M^{-(N-k)}TM^{-(N-k)} \to 0$$

So the steady state limit

$$S_\infty = W_{21}W_{11}^{-1}$$

Thus S_∞ can be constructed from the unstable eigenvector of H. Hence the optimal steady state feedback K_∞ can be found without solving the Riccati equation.

$$K_\infty = (B^T S_\infty B + R)^{-1} S_\infty A$$

Example: Let the plant and the cost function be

$$x_{k+1} = x_k + u_k$$

$$J_i = \frac{10}{2}x_N^2 + \frac{1}{2}\sum_{k=1}^{N-1}(x_k^2 + u_k^2)$$

Then

$$H = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

has eigenvalues: 0.382, 2.618, so $M = 2.618$ and $M^{-1} = 0.382$

The matrix of eigenvectors (the unstable one first)

$$W = \begin{bmatrix} 1 & 1 \\ 1.618 & -0.618 \end{bmatrix}$$

Since $S_N = 10$

$$T = -(W_{22} - S_N W_{12})^{-1}(W_{21} - S_N W_{11})$$
$$T = -(-0.618 - 10 * 1)^{-1}(1.618 - 10 * 1)$$
$$T = -(-10.618)^{-1}(-8.382)$$

$$T = -0.789$$

And

$$T_k = M^{-(N-k)}TM^{-(N-k)}$$
$$T_k = -0.789(0.382)^{2(N-k)}$$

Therefore

$$S_k = (W_{21} + W_{22}T_k)(W_{11} + W_{12}T_k)^{-1}$$
$$S_k = \frac{1.618 + 0.488(0.382)^{2(N-k)}}{1 - 0.789(0.382)^{2(N-k)}}$$

As $N - k \to \infty$

$$S_\infty = \frac{1.618}{1} = 1.618$$
$$K_\infty = (B^T S_\infty B + R)^{-1} B^T S_\infty A$$
$$= \frac{1.618}{2.618} = 0.618$$

With $B = 1$, $R = 1$, $A = 1$

The control law

$$u_k = -0.618\, x_k$$

Results in a stable closed loop system

$$x_{k+1} = (A - BK_\infty)x_k = 0.382\, x_k$$

Design of steady state Regulators by Eigenstructure Assignment

Assume (A, B) is stabilizable and $\left(A, \sqrt{Q}\right)$ is detectable and A is nonsingular.

The Hamiltonian system as the forward recursion

$$\begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} A + BR^{-1}B^TA^{-T}Q & -BR^{-T}B^TA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_k \end{bmatrix}$$

Where the coefficient matrix is H^{-1}

Let μ be an eigenvalue of H^{-1}. Then the eigenvectors of H^{-1} corresponding to μ are the eigenvectors of H corresponding to $\frac{1}{\mu}$.

The steady state closed loop system with the optimal control $u_k = -K_\infty x_k$ is

$$x_{k+1} = (A - BK_\infty)x_k$$

Suppose that u_i is an eigenvalue of the closed loop system. If only the mode corresponding to μ_i is excited then the state control and costate are

$$x_k = X_i \mu_i^k$$
$$u_k = U_i \mu_i^k$$
$$\lambda_k = \Lambda_i \mu_i^k$$

For some vectors X_i, U_i, Λ_i

Since

$$x_{k+1} = Ax_k + Bu_k$$

or

$$X_i \mu_i^{k+1} = AX_i \mu_i^k + BU_i \mu_i^k$$

So that

$$(\mu_i I - A)X_i = BU_i$$

The optimal control $u_k = -K_\infty x_k$

$$U_i = -K_\infty X_i$$

And

$$(u_i I - A + BK_\infty)X_i = 0$$

Thus, X_i is an eigenvector of the closed loop plant for eigenvalue μ_i

Now focus on the Hamiltonian system

$$\mu_i \begin{bmatrix} X_i \\ \Lambda_i \end{bmatrix} = H^{-1} \begin{bmatrix} X_i \\ \Lambda_i \end{bmatrix}$$

Hence $\begin{bmatrix} X_i \\ \Lambda_i \end{bmatrix}$ is an eigenvector of H^{-1} for eigenvalue μ_i

Since $(A - BK_\infty)$ is stable, so that $|\mu_i| < 1$. Hence the eigenvalues of the closed loop system with $\mu_k = -K_\infty x_k$ are the stable eigenvalues of H^{-1}. We can use pole assignment approach to find the feedback gain K_∞ instead of solving ARE or solving eigenvectors of H^{-1}.

For example if the plant is single input, we can use Ackerman's formula to find the required feedback K_∞ with the given desired closed loop eigenvalues.

According to this formula, the state feedback K is required to assign a desired close loop characteristic polynomial $\Delta^d(s)$ is

$$K = e_n^T U_n^{-1} \Delta^d(A)$$

Where e_n is the last column of the $n \times n$ identity matrix

$$e_n = [0 \quad \dots \quad 0 \quad 1]^T$$

And U_n is the reachability matrix

$$U_n = [B \quad AB \quad \dots \quad A^{n-1}B]$$

$\Delta^d(A)$ is the desired characteristic polynomial evaluation at A. If the desired eigenvalues (or poles) are $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$\begin{aligned} \Delta^d(s) &= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)\dots(s-\lambda_n) \\ &= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 \end{aligned}$$

And

$$\Delta^d(A) = A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I$$

In the multiple input case the desired closed loop eigenvalues are not sufficient to determine the required feedback gain. The eigenvectors are also required.

In general, we can compute K_∞ from the eigenstructure of H^{-1} as follows:
Suppose the closed loop eigenvalues are district. The optimal control is

$$u_k = -R^{-1}B^T \lambda_{k+1}$$

So that

$$U_i = -R^{-1}B^T \mu_i \Lambda_i$$

Since

$$u_k = -K_\infty x_k \Rightarrow U_i = -K_\infty X_i$$

So that

$$K_\infty X_i = R^{-1}B^T \mu_i \Lambda_i$$

Let X be a matrix whose columns are X_i and Λ be a matrix whose columns are Λ_i, where $\begin{bmatrix} \mathrm{X_i} \\ \Lambda_i \end{bmatrix}$ is the eigenvector of the stable μ_i of H^{-1}. Let $M = diag[\mu_1 \quad \cdots \quad \mu_i]$. So

$$K_\infty X = R^{-1}B^T \Lambda M$$

$$K_\infty = R^{-1}B^T \Lambda M X^{-1}$$

If μ_i is complex then so are X_i and Λ_i. In this event there is a block in $\Lambda \mathrm{M} \mathrm{X}^{-1}$ of the form

$$[\Lambda_\mathrm{i} \quad \Lambda_\mathrm{i}^*]\begin{bmatrix}\mu_i & 0\\ 0 & \mu_i^*\end{bmatrix}[X_i \quad X_i^*]^{-1}$$

By pre multiplying and post multiplying $diag[u_i, \quad \mu_i^*]$ by

$$I = \frac{1}{2}\begin{bmatrix}1 & -j\\ 1 & j\end{bmatrix}\begin{bmatrix}1 & 1\\ j & -j\end{bmatrix}$$

This block becomes

$$[Re(\Lambda_\mathrm{i}) \quad Im(\Lambda_\mathrm{i})]\begin{bmatrix}Re(\mu_i) & Im(\mu_i)\\ -Im(\mu_i) & Re(\mu_i)\end{bmatrix}[Re(X_i) \quad Im(X_i)]^{-1}$$

Which results in a real feedback gain K_∞.

If the μ_i are not distinct, then the generalized eigenvector must be used to construct X, and there is Jordan block

$$J = \begin{bmatrix}\mu_i & 1 & 0\\ 0 & \mu_i & 1\\ 0 & 0 & \mu_i\end{bmatrix}$$

If μ_i is a triple repeated eigenvalue.

Example: Eigenstructure design of steady state regulator for harmonic oscillator

Suppose out plane is the harmonic oscillator with natural frequency $\omega_n = \sqrt{2}$ and damping ratio $\delta = -\frac{1}{\sqrt{2}}$ so

$$\dot{x} = \begin{bmatrix}0 & 1\\ -2 & 2\end{bmatrix}x + \begin{bmatrix}0\\ 10\end{bmatrix}u$$

The plants is unstable with poles at $1 \pm j$.
Discretizing with $T = 25\ msec$ ($T = 0.025$)

$$\begin{aligned} x_{k+1} &= \begin{bmatrix}0.999 & 0.026\\ -0.051 & 1.051\end{bmatrix}x_k + \begin{bmatrix}0.003\\ 0.256\end{bmatrix}\mu_k \\ &\triangleq Ax_k + Bu_k \end{aligned}$$

The open loop poles are at

$$z = 1.025 \pm j0.026$$

Let us associate the system with the infinite horizon performance index

$$J_0 = \frac{1}{2}\sum_{k=0}^{\infty}(qx_k^T x_k + u_k^2), \qquad (Q = qI)$$

We are seeking the optimal steady state control

a. **Locus of Optimal closed loop poles versus *q***

$$H^{-1} = \begin{bmatrix}A + BB^T A^{-T} q & -BB^T A^{-T}\\ -A^T q & A^{-T}\end{bmatrix}$$

The poles of the optimal closed loop plant

$$x_{k+1} = (A - BK_\infty)x_k \triangleq A_\infty^{cl} x_k$$

Are given by the stable poles of H^{-1}. For $q = 0$.

$$z = 0.975 \pm j0.024$$

Which are the original plant reflected inside the unit circle (their reciprocals). As $q \to \infty$, the optimal closed loop poles tend to

$$z = 0, 0.975$$

For the case of $q = 0.07$, we find three different ways to calculate K_∞.

b. Solution of the Algebraic Riccati Equation (ARE)

To solve the ARE, we use a final condition of $S = I$. Any final condition will do since (A, B) is reachable (stabilizable) and $(A, \sqrt{Q})$ is observable.

After 200 iteration of

$$S = A^T[S - SB(B^T SB + R)^{-1} B^T S]A + Q$$

The solution converges

$$S_\infty = \begin{bmatrix} 6.535 & 0.528 \\ 0.528 & 2.314 \end{bmatrix}$$

Then

$$\begin{aligned} K_\infty &= (B^T S_\infty B + R)^{-1} B^T S_\infty A \\ &= [0.109 \quad 0.545] \end{aligned}$$

The resulting closed loop plant is

$$A_\infty^{cl} = (A - BK_\infty) = \begin{bmatrix} 0.909 & 0.024 \\ -0.079 & 0.911 \end{bmatrix}$$

Which has stable poles of

$$z = 0.962, 0.948$$

We can find these poles are the stable poles of H^{-1}.

c. Ackermann's Formula

We can avoid solving the ARE as follows. If $q = 0.07$, we can find the stable poles of H^{-1} which are

$$z_1 = 0.962, \qquad z_2 = 0.948$$

Then the desired closed-loop characteristics polynomial is

$$\Delta^d(z) = (z - 0.962)(z - 0.948) = z^2 - 1.910z + 0.912$$

The reachability matrix is

$$U_2 = [B \quad AB] = \begin{bmatrix} 0.003 & 0.01 \\ 0.256 & 0.269 \end{bmatrix}$$

The

$$e_2^T = [0 \quad 1]$$

And $\Delta^d(A) = A^2 - 1.910A + 0.912I$

$$\begin{aligned} K_\infty &= [0 \quad 1]U_2^{-1}\Delta^d(A) \\ &= [0.109 \quad 0.545] \end{aligned}$$

d. Eigenstructure Assignment

The diagonal matrix of stable eigenvalues of H^{-1} for $q = 0.07$ is

$$M = \begin{bmatrix} 0.962 & 0 \\ 0 & 0.948 \end{bmatrix}$$

And the associated eigenvectors are the columns in

$$\begin{bmatrix} 0.148 & 0.764 \\ -0.229 & -1.640 \\ 0.849 & 4.130 \\ -0.452 & -3.392 \end{bmatrix} \triangleq \begin{bmatrix} X \\ \Lambda \end{bmatrix}$$

Then the optimal feedback gain

$$\begin{aligned} K_\infty &= R^{-1}B^T \Lambda M X^{-1} \\ &= [0.109 \quad 0.545] \end{aligned}$$

We can also check our analytic ARE equation

$$S_\infty = W_{21}W_{11}^{-1} = \Lambda \mathrm{X}^{-1} = \begin{bmatrix} 6.535 & 0.528 \\ 0.528 & 2.314 \end{bmatrix}$$

Which is the numerical solution of ARE.