2.1-1 Optimal control of a bilinear system. [Total: 3 pts]

Let the scalar plant

$$x_{k+1} = x_k u_k + 1$$

have performance index

$$J = \frac{1}{2}\sum_{k=0}^{N-1} u_k^2$$

With the final time N = 2. X_0, it is desired to make $x_2 = 0$.

1. Write state and costate equations with u_k eliminated.
2. Assume the final costate λ_2 is known. Solve for λ_0, λ_1 in terms of λ_2 and the state. Use this to express x_2 in terms of λ_2 and x_0. Hence, find a quartic equation for λ_2 in terms of initial state x_0.
3. If $x_0 = 1$, find the optimal state and costate sequences, the optimal control, and the optimal value of the performance index.

Solution for part 1

Recall Table 2.1-1:

System Model:

$$x_{k+1} = f^k(x_k, u_k) = x_k u_k + 1$$

Performance Index:

$$J_i = \phi(N, x_N) + \sum_{k=i}^{N-1} L^k(x_k, u_k) = 0 + \sum_{k=0}^{N-1} \frac{1}{2} u_k^2$$

Hamiltonian:

$$H^k(x_k, u_k) = L^k(x_k, u_k) + \lambda_{k+1}^T f^k(x_k, u_k) = \frac{1}{2} u_k^2 + \lambda_{k+1}^T (x_k u_k + 1)$$

Optimal controller

State equation [.25 pts]

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = f^k(x_k, u_k) = x_k u_k + 1$$

Costate equation [.25 pts]

$$\lambda_k = \frac{\partial H_k}{\partial x_k} = \left(\frac{\partial f^k}{\partial x_k}\right)^T \lambda_{k+1} + \frac{\partial L^k}{\partial x_k} = \lambda_{k+1} u_k$$

Stationary condition

$$0 = \frac{\partial H^k}{\partial u_k} = u_k + \lambda_{k+1} x_k$$

Use stationary condition to eliminate u_k in state and costate. [0.5 pts]

- $u_k = -\lambda_{k+1} x_k$
- $x_{k+1} = -\lambda_{k+1} x_k^2 + 1$
- $\lambda_k = -\lambda_{k+1}^2 x_k$

Solution for part 2

Assume λ_2 is known

$$\lambda_0 = -\lambda_1^2 x_0$$
$$\lambda_1 = -\lambda_2^2 x_1$$

$$x_1 = 1 - \lambda_1 x_0^2$$
$$x_2 = 1 - \lambda_2 x_1^2$$

Solve for λ_0, λ_1 in terms of λ_2 and the states [0.5 pts]

$$\Rightarrow \lambda_0 = -(-\lambda_2^2 x_1)^2 x_0 = -\lambda_2^4 x_1^2 x_0$$
$$\Rightarrow \lambda_1 = -\lambda_2^2 x_1$$

Express x_2 in terms of λ_0 and λ_2

$$\lambda_1 = -\lambda_2^2 x_1$$
$$x_1 = 1 - \lambda_1 x_0^2$$
$$x_2 = 1 - \lambda_2 x_1^2$$

$$\Rightarrow x_1 = 1 - (-\lambda_2^2 x_1) x_0^2 = \frac{1}{1 - \lambda_2^2 x_0^2}$$
$$x_2 = 1 - \lambda_2 \left(\frac{1}{1 - \lambda_2^2 x_0^2} \right)^2 = 1 - \frac{\lambda_2}{(1 - \lambda_2^2 x_0^2)^2}$$

From the problem statement, it is given "it is desired to make x2 = 0". [0.5 pts]

$$\frac{\lambda_2}{(1 - \lambda_2^2 x_0^2)^2} = 1$$

Solution for part 3

If $x_0 = 1$, $\lambda_2 = 1.4902,\ 0.52489, -1.0076 \pm 0.51312j$

$\lambda_2 = 1.4902$	$\lambda_2 = 0.52489$
$x_1 = \frac{1}{1 - \lambda_2^2 x_0^2} = -0.8192$	$x_1 = 1.3803$
$\lambda_1 = -\lambda_2^2 x_1 = 1.8192$	$\lambda_1 = -0.3803$
$u_0 = -\lambda_1 x_0 = -1.8192$	$u_0 = -\lambda_1 x_0 = 0.3803$
$u_1 = -\lambda_2 x_1 = 1.2208$	$u_1 = -\lambda_2 x_1 = -0.7245$
$J = \frac{1}{2} \sum_{k=0}^{N-1} u_k^2 = \frac{1}{2(u_0^2 + u_1^2)}$	$J = \frac{1}{2}((0.3803)^2 + (-0.7245)^2) = 0.3386$
$J = \frac{1}{2}((-1.8192)^2 + (1.2208)^2) = 2.3999$	

The optimal costate is $\lambda_2 = 0.52489$ since $J^* = 0.3386$ is the smallest.

The optimal state sequence, x^* is [0.25 pts]

$$x_0 = 1$$

$$x_1 = 1.3803$$
$$x_2 = 0$$

The optimal costate sequence λ^* is [0.25 pts]

$$\lambda_1 = -0.3803$$
$$\lambda_2 = 0.52489$$

The optimal control sequences u^* is [0.25 pts]

$$u_0 = 0.3803$$
$$u_1 = -0.7245$$

The optimal value of the performance index is $J^* = 0.3386$ [0.25 pts]

2.1-2. Optimal control of a bilinear system. [Total: 3 pts]

Consider the bilinear system

$$x_{k+1} = Ax_k + Dx_k u_k + bu_k$$

Where $x_k \in R^n, u_k \in R$, with quadratic performance index

$$J = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q x_k + r u_k^2)$$

where $S_N \geq 0, Q \geq 0, r > 0$. Show that the optimal control is the bilinear state costate feedback

$$u_k = -(b + Dx_k)^T \frac{\lambda_{k+1}}{r}$$

and that the state and costate equations after eliminating u_k are

$$x_{k+1} = Ax_k - (b + Dx_k)(b + Dx_k)^T \frac{\lambda_{k+1}}{r}$$

$$\lambda_k = Qx_k + A^T \lambda_{k+1} - (b + Dx_k)^T \lambda_{k+1} D^T \frac{\lambda_{k+1}}{r}$$

Solution:

Hamiltonian function

$$H = \frac{1}{2}(x_k^T Q x_k + r u_k^2) + \lambda_{k+1}^T (Ax_k + Dx_k u_k + bu_k)$$

Derive the optimal control [1 pts] with steps.

$$\frac{\partial H^k}{\partial u_k} = 0 = r u_k + (Dx_k + b)^T \lambda_{k+1}$$

$$u_k = -\frac{(Dx_k + b)^T \lambda_{k+1}}{r}$$

Derive the state and costate equations [1 pts each with steps].

$$\begin{aligned} x_{k+1} &= \frac{\partial H^k}{\partial \lambda_{k+1}} = Ax_k + Dx_k u_k + bu_k \\ &= Ax_k + (Dx_k + b)u_k \\ &= Ax_k - (Dx_k + b)\frac{(Dx_k + b)^T \lambda_{k+1}}{r} \end{aligned}$$

$$\lambda_k = \frac{\partial H_k}{\partial x_k} = Qx_k + (A + Du_k)^T \lambda_{k+1}$$

$$= Qx_k + \left(A^T + \left(-\frac{(Dx_k + b)^T \lambda_{k+1}}{r}\right) D^T\right) \lambda_{k+1}$$

$$= Qx_k + A^T \lambda_{k+1} - (b + Dx_k)^T \lambda_{k+1} D^T \frac{\lambda_{k+1}}{r}$$

Detailed derivation for $\frac{\partial H_k}{\partial x_k}$:

$$H = \frac{1}{2}(x_k^T Q x_k + r u_k^2) + \lambda_{k+1}^T (A x_k + D x_k u_k + b u_k)$$

$$\frac{\partial}{\partial x_k}\left(\frac{1}{2} x_k^T Q x_k\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_k}(x_k^T Q x_k) + \frac{\partial}{\partial x_k}(x_k^T Q^T x_k)\right)$$

$$= \frac{1}{2}(Q x_k + Q^T x_k) = Q x_k \ \ Since\ Q\ is\ symmetric$$

$$\begin{aligned}\frac{\partial}{\partial x_k}\left(\lambda_{k+1}^T (A x_k + D x_k u_k + b u_k)\right) &= \frac{\partial}{\partial x_k}(\lambda_{k+1}^T \cdot A x_k) + \frac{\partial}{\partial x_k}(\lambda_{k+1}^T \cdot u_k D x_k)\ since\ u\ is\ scalar \\ &= \frac{\partial}{\partial x_k}(x_k^T A^T \lambda_{k+1}) + \frac{\partial}{\partial x_k}(x_k^T D^T u_k \lambda_{k+1}) \\ &= A^T \lambda_{k+1} + D^T u_k \lambda_{k+1} \\ &= (A^T + u_k D^T)\lambda_{k+1}\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial H_k}{\partial x_k} &= Q x_k \ + (A^T + u_k D^T)\lambda_{k+1} \\ &= Q x_k + \left(A^T + \left(-\frac{(D x_k + b)^T \lambda_{k+1}}{r}\right) D^T\right)\lambda_{k+1} \\ &= Q x_k + A^T \lambda_{k+1} - (b + D x_k)^T \lambda_{k+1} D^T \frac{\lambda_{k+1}}{r}\end{aligned}$$

2.1-3. Optimal control of a generalized state-space system. [Total: 4 pts]

Rederive the equations in Table 2.1-1 to find the optimal controller for the nonlinear generalized statespace (or descriptor) system

$$Ex_{k+1} = f^k(x_k, u_k)$$

where E is singular. These systems often arise in circuit analysis, economics, and similar areas.

Hint: See textbook page 22

Let the NEW system model be (constraint function)

$$g^k(x_k, u_k) = f^k(x_k, u_k) - Ex_{k+1} = 0$$

Let the performance index, given in the general form be

$$J' = \phi(N, x_N) + \sum_{k=i}^{N-1} L^k(x_k, u_k)$$

Using Lagrange-multiplier approach, append the constraint $g^k(x_k, u_k)$ to J' to define an augmented performance index J'

$$J' = \phi(N, x_N) + \sum_{k=i}^{N-1} L^k(x_k, u_k) + \lambda_{k+1}^T\left[f^k(x_k, u_k) - Ex_{k+1}\right]$$

Define the Hamiltonian function as

$$H^k(x_k, u_k) = L^k(x_k, u_k) + \lambda_{k+1}^T f^k(x_k, u_k)$$

We can rewrite J' as

$$J' = \phi(N, x_N) + L^i(x_i, u_i) + \lambda_{i+1}^T f^i(x_i, u_i) - \lambda_N^T Ex_N + \sum_{k=i+1}^{N-1} L^k(x_k, u_k) + \lambda_{k+1}^T\left[f^k(x_k, u_k) - Ex_{k+1}\right]$$

$$= \phi(N, x_N) + H^i(x_i, u_i) - \lambda_N^T Ex_N + \sum_{k=i+1}^{N-1} \left[H^k(x_k, u_k) - \lambda_k^T Ex_k\right]$$

Write the increment dJ'

$$dJ' = \left(\frac{\partial \phi(N, x_N)}{\partial x_N}\right)^T dx_N + \left(\frac{\partial H^{i(x_i, u_i)}}{\partial x_i}\right)^T dx_i + \left(\frac{\partial H^i(x_i, u_i)}{\partial u_i}\right)^T du_i - \left(\frac{\partial}{\partial x_N}(x_N^T E^T \lambda_N)\right)^T dx_N$$
$$+ \sum_{k=i+1}^{N-1}\left[\left(\frac{\partial H^k(x_k, u_k)}{\partial x_k}\right)^T dx_k + \left(\frac{\partial H^k(x_k, u_k)}{\partial u_k}\right)^T du_k - \left(\frac{\partial}{\partial x_k}(x_k^T E^T \lambda_k)\right)^T dx_k\right]$$
$$+ \sum_{k=i+1}^{N} [\left(\frac{\partial H^{k-1}(x_k, u_k)}{\partial \lambda_k}\right)^T d\lambda_k - \left(\frac{\partial}{\partial \lambda_k}(\lambda_k^T E x_k)\right)^T d\lambda_k]$$

$$dJ' = \phi_{x_N}^T dx_N + \left(H_{x_i}^i\right)^T dx_i + \left(H_{u_i}^i\right)^T du_i - (E^T\lambda_N)^T dx_N + \sum_{k=i+1}^{N-1} [(H_{x_k}^k)^T dx_k + \left(H_{u_k}^k\right)^T du_k - (E^T\lambda_k)^T dx_k]$$
$$+ \sum_{k=i+1}^{N} [\left(H_{\lambda_k}^{k-1}\right)^T d\lambda_k - (Ex_k)^T d\lambda_k]$$

$$dJ' = \left(\phi_{x_N} - E^T\lambda_N\right)^T dx_N + \left(H^i_{x_i}\right)^T dx_i + \left(H^i_{u_i}\right)^T du_i + \sum_{k=i+1}^{N-1}\left[\left(H^k_{x_k} - E^T\lambda_k\right)^T dx_k + \left(H^k_{u_k}\right)^T du_k\right] + \sum_{k=i+1}^{N}\left(H^{k-1}_{\lambda_k} - Ex_k\right)^T d\lambda_k$$

At a constrained minimum, this increment dJ' should be zero. Necessary conditions for a constrained minimum are thus given by

State Equation [1 pts]

$$\frac{dJ'}{d\lambda_k} = H^{k-1}_{\lambda_k} - Ex_k = H^k_{\lambda_{k+1}} - Ex_{k+1} = 0, \qquad Ex_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = f^k(x_k, u_k), \qquad k = i, \dots, N-1$$

Costate Equation [1 pts]

$$\frac{dJ'}{dx_k} = H^k_{x_k} - E^T\lambda_k = 0, \qquad E^T\lambda_k = \frac{\partial H^k}{\partial x_k} = \left(\frac{\partial f^k}{\partial x_k}\right)^T \lambda_{k+1} + \frac{\partial L^k}{\partial x_k}, \qquad k = i, \dots, N-1$$

Stationary Condition [1 pts]

$$\frac{dJ'}{du_k} = H^k_{u_k} = \frac{\partial H^k}{\partial u_k} = \left(\frac{\partial f^k}{\partial u_k}\right)^T \lambda_{k+1} + \frac{\partial L^k}{\partial u_k} = 0, \qquad k = i, \dots, N-1$$

Boundary Conditions [1 pts]

$$\left(\phi_{x_N} - E^T\lambda_N\right)^T dx_N = \left(\frac{\partial \phi}{\partial x_N} - E^T\lambda_N\right)^T dx_N = 0$$

$$\left(H^i_{x_i}\right)^T dx_i = \left(\frac{\partial H^i}{\partial x_i}\right)^T dx_i = \left(\frac{\partial L^i}{\partial x_i} + \left(\frac{\partial f^i}{\partial x_i}\right)^T \lambda_{i+1}\right)^T dx_i = 0$$