The Hamiltonian is

$$H = L + \lambda^{\mathrm{T}} f = \frac{1}{2}x^2 + xu + u^2 + u + \lambda(x - 3), \tag{3}$$

where λ is a scalar. The conditions for a stationary point are (1.2-25), or

$$H_\lambda = x - 3 = 0, \tag{4}$$

$$H_x = x + u + \lambda = 0, \tag{5}$$

$$H_u = x + 2u + 1 = 0. \tag{6}$$

Solving in the order (4), (6), (5) yields $x = 3$, $u = -2$, and $\lambda = -1$. The stationary point is therefore

$$(x, u)^* = (3, -2). \tag{7}$$

To verify that (7) is a minimum, find the constrained curvature matrix (1.2-31):

$$L_{uu}^f = 2. \tag{8}$$

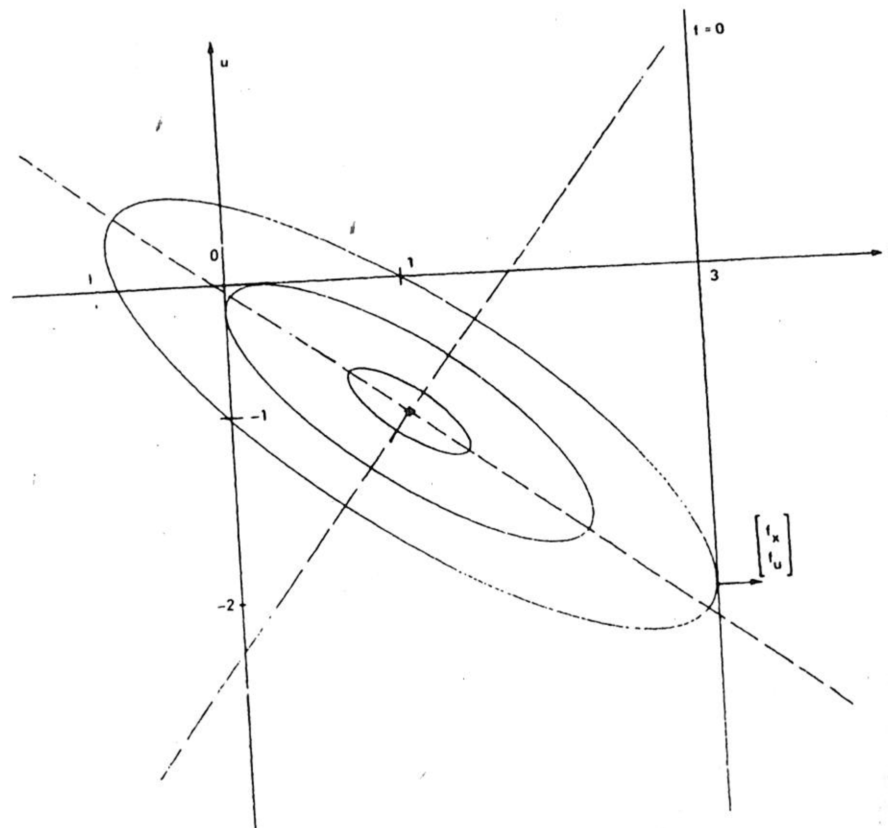


FIGURE 1.2-1 Contours of $L(x, u)$, and the constraint $f(x, u)$.

then u^* is a saddle point. If $|Q| = 0$, then u^* is a *singular point* and in this case L_{uu} does not provide sufficient information for characterizing the nature of the critical point.

By substituting (4) into (2) we find the extremal value of the performance index to be

$$L^* \triangleq L(u^*) = \frac{1}{2}S^T Q^{-1} Q Q^{-1} S - S^T Q^{-1} S$$
$$= -\frac{1}{2} S^T Q^{-1} S. \tag{6}$$

Let

$$L = \frac{1}{2}u^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} u + [0 \quad 1] u. \tag{7}$$

Then

$$u^* = -\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8}$$

a minimum, since $L_{uu} > 0$. Using (6), we see that the minimum value of L is $L^* = -\frac{1}{2}$.

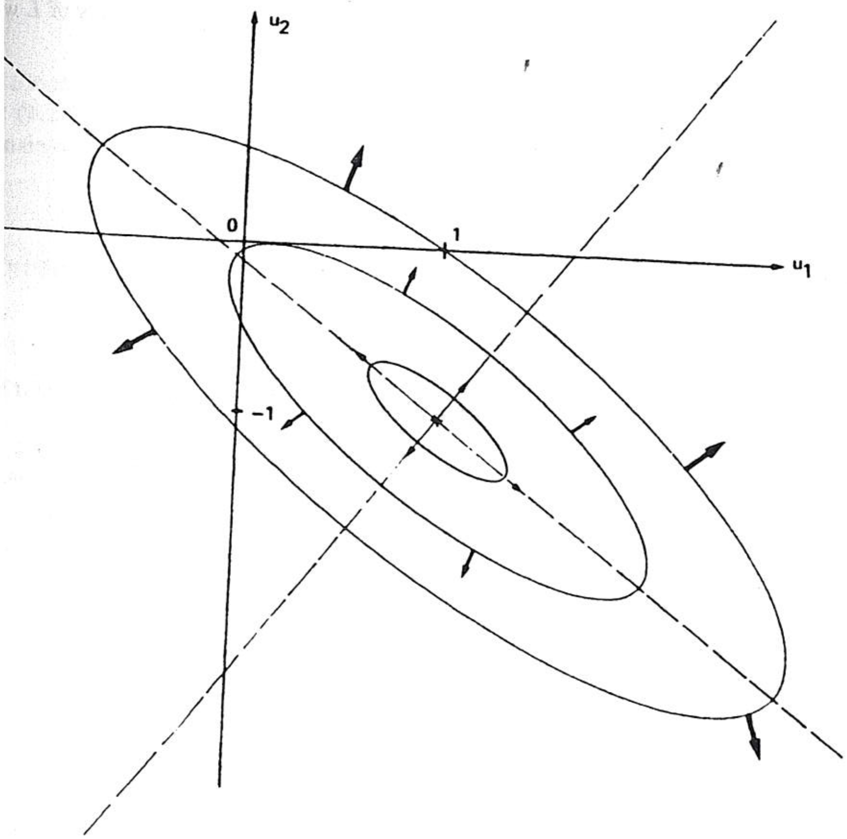
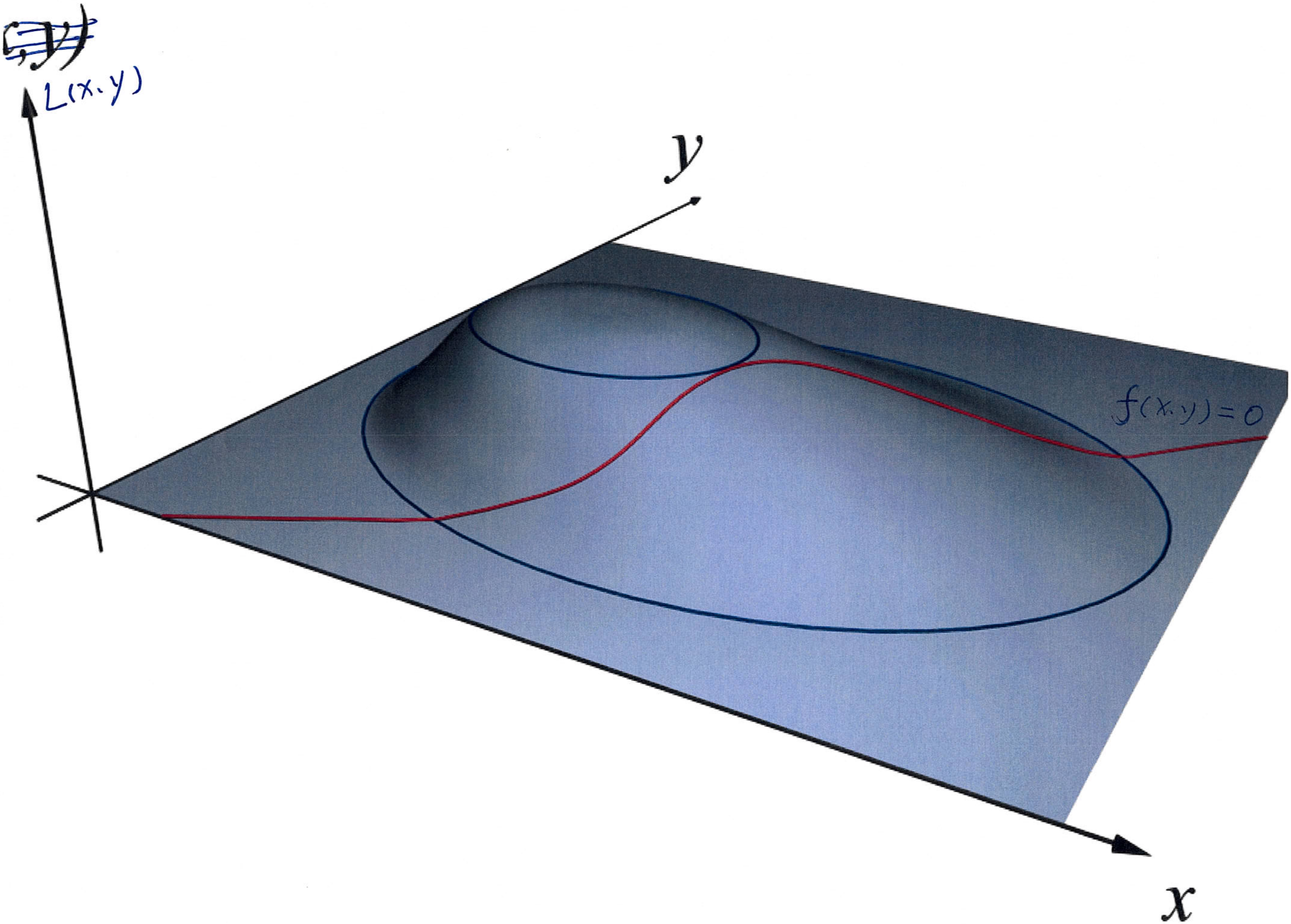
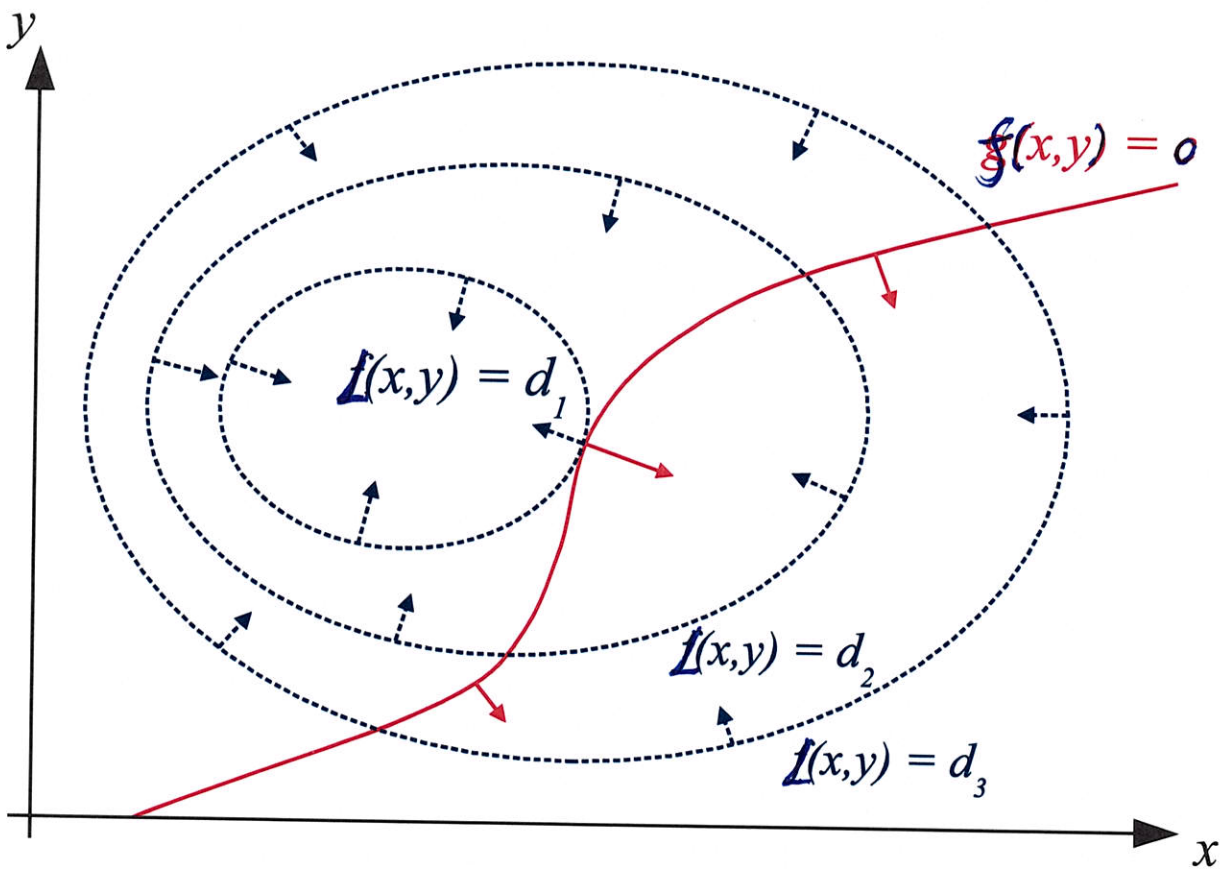


FIGURE 1.1-1 Contours and the gradient vector.


L(x,y)
y
f(x,y)=0
x



y
g(x,y) = 0
f(x,y) = d_1
f(x,y) = d_2
f(x,y) = d_3
x