1.1-2 [Total: 3 pt.]

$$L(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 + 3x_1$$

- Find the minimum value. [1 pt.]
- Find the curvature matrix at the minimum. [1 pt.]
- Sketch the contours, showing the gradient at several points. [1 pt.]

<u>Method 1</u>

Write $L(x_1, x_2) = L(x) = \frac{1}{2} x^T \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} x + [s_1 \quad s_2] x$

$$Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad S = [s_1 \quad s_2]^T = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Then

$$x^* = -Q^{-1} S = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \qquad L_{xx} = Q = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0$$

<u>Method 2</u>

$$L_{x_1} = \frac{\partial L}{\partial x_1} = 2x_1 - x_2 + 3 = 0, \qquad L_{x_2} = \frac{\partial L}{\partial x_2} = -x_1 + 2x_2 = 0$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$L_{xx} = \begin{bmatrix} L_{x_1 x_1} & L_{x_1 x_2} \\ L_{x_2 x_1} & L_{x_2 x_2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0$$

L_{xx} is positive definite, hence the critical point is <u>minimum</u>.

$$L(x^*) = \frac{1}{2} x^{*T} Q x^* + S^T x^* = -3$$

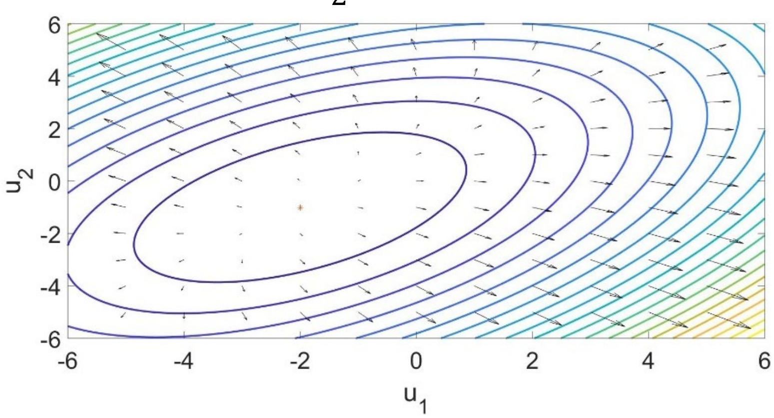


Figure: Contours and the gradient vector.

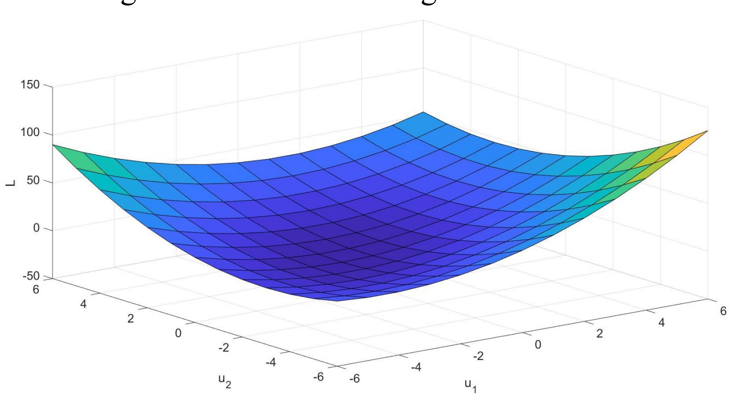


Figure: Plot of Performance index

1.2-5: **Rectangles with maximum area, minimum perimeter** [Total: 3 pt.]

a. Find the rectangle of maximum area with perimeter *p*. [1 pt.]

$$L(x,y) = xy$$
$$f(x,y) = 2x + 2y - p = 0$$

$$H = L + \lambda^T f = xy + \lambda(2x + 2y - p)$$
$$H_\lambda = 2x - p + 2y = 0$$
$$H_x = 2\lambda + y = 0$$
$$H_y = 2\lambda + x = 0$$

$$\lambda = -\frac{p}{8}, \qquad x = \frac{p}{4}, \qquad y = \frac{p}{4}$$

The stationary point

$$(x^*, y^*) = \left(\frac{p}{4}, \frac{p}{4}\right)$$

To verify that the stationary point is a maximum or minimum, check the curvature matrix

$$L_{yy}^f = H_{yy} - f_y^T f_x^{-T} H_{xy} - H_{yx} f_x^{-1} f_y + f_y^T f_x^{-T} H_{xx} f_x^{-1} f_y$$

where

$$H_{yy} = \frac{\partial^2 H}{\partial y^2} = 0, \qquad f_y = \frac{\partial f}{\partial y} = 2, \qquad f_x = \frac{\partial f}{\partial x} = 2$$
$$H_{xy} = \frac{\partial^2 H}{\partial x \partial y} = 1, \qquad H_{yx} = \frac{\partial^2 H}{\partial y \partial x} = 1, \qquad H_{xx} = \frac{\partial^2 H}{\partial x^2} = 0$$
$$L_{yy}^f = 0 - 2 \times \frac{1}{2} \times 1 - 1 \times \frac{1}{2} \times 2 + 0 = -2 < 0$$

Therefore, (x^*, y^*) is a maximum.

b. Find the rectangle of minimum perimeter with area a^2. [1 pt.]

$$L(x,y) = 2x + 2y$$
$$f(x,y) = xy - a^2$$

$$H = 2x + 2y + \lambda(xy - a^2)$$
$$\frac{\partial H}{\partial x} = 2 + \lambda y = 0$$
$$\frac{\partial H}{\partial y} = 2 + \lambda x = 0$$
$$\frac{\partial H}{\partial \lambda} = xy - a^2 = 0$$
$$x = -\frac{2}{\lambda}, \qquad y = -\frac{2}{\lambda}, \qquad \lambda = \pm\frac{2}{a}$$

There are two stationary points $(x_1^*, y_1^*) = (a, a)$ when $\lambda_1^* = -\frac{2}{a}$ and $(x_2^*, y_2^*) = (-a, -a)$ when $\lambda_2^* = \frac{2}{a}$

To verify the curvature matrix with constraint

$$L_{yy}^f = H_{yy} - f_y^T f_x^{-T} H_{xy} - H_{yx} f_x^{-1} f_y + f_y^T f_x^{-T} H_{xx} f_x^{-1} f_y$$

where

$$H_{yy} = \frac{\partial^2 H}{\partial y^2} = 0, f_y = \frac{\partial f}{\partial y} = x, f_x = \frac{\partial f}{\partial x} = y$$

$$H_{xy} = \frac{\partial^2 H}{\partial x \partial y} = \lambda, H_{yx} = \frac{\partial^2 H}{\partial y \partial x} = \lambda \, , H_{xx} = \frac{\partial^2 H}{\partial x^2} = 0$$

$$L_{yy}^f = 0 - x \times \frac{1}{y} \times \lambda - \lambda \times \frac{x}{y} + 0$$

$$= -\frac{2\lambda x}{y}$$

For $(x_1^*, y_1^*), L_{yy}^f(x_1^*, y_1^*) = -\frac{2\lambda_1^* x_1^*}{y_1^*} = \frac{4}{a}$

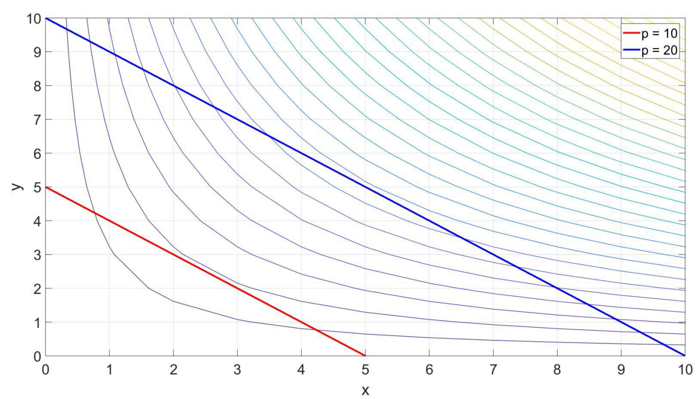
For $(x_2^*, y_2^*), L_{yy}^f(x_2^*, y_2^*) = -\frac{2\lambda_2^* x_2^*}{y_2^*} = -\frac{4}{a}$

If $a < 0$, $L_{yy}^f(x_1^*, y_1^*) < 0$ and $L_{yy}^f(x_2^*, y_2^*) > 0$, then (x_1^*, y_1^*) is a maximum, and (x_2^*, y_2^*) is a minimum. (This has no physical meaning since area a cannot be negative).

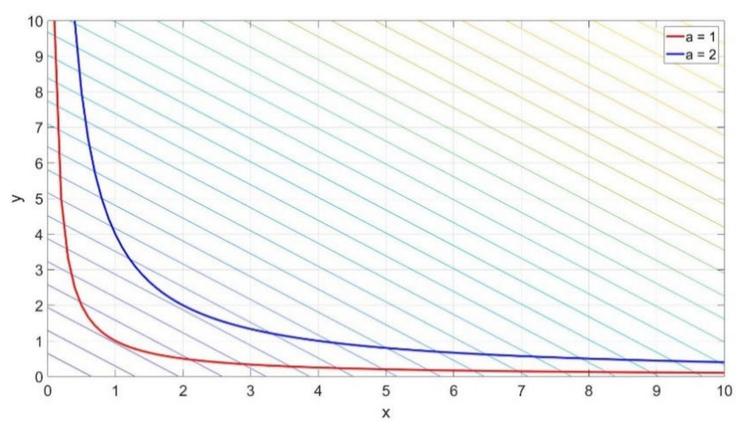
If $a > 0$, $L_{yy}^f(x_1^*, y_1^*) > 0$ and $L_{yy}^f(x_2^*, y_2^*) < 0$, then (x_1^*, y_1^*) is a minimum, and (x_2^*, y_2^*) is a maximum. (As is expected).

c. In each case, sketch the contours of $L(x, y)$ and the constraint. Optimization problems related like these two are said to be *dual*.

- Case a: [0.5 pt.]



- Case b: [0.5 pt.]



1.2-6 **Linear quadratic case**. Minimize [Total: 4 pt]

$$L = \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \frac{1}{2}u^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u$$
$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u$$

$$L(x,u) = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, R = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$f(x,u) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -Ix + Bu + C$$

where

$$B = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- [1 pt.]

$$H = L + \lambda^T f = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + \lambda^T (Ax + Bu + C)$$

- [1 pt.]

$$H_\lambda = -Ix + Bu + C = 0$$
$$H_x = Qx - I\lambda = 0$$
$$H_u = Ru + B^T \lambda = 0$$

- [1 pt.]

From second equation

$$\lambda = Qx$$

Third equation becomes:

$$Ru + B^T Q x = 0$$
$$Ru = -B^T Q x$$
$$u = -R^{-1} B^T Q x$$

Using this expression in first equation:

$$-Ix - BR^{-1}B^T Q x + C = 0$$
$$(I + BR^{-1}B^T Q)x = C$$
$$x^* = (I + BR^{-1}B^T Q)^{-1} C = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$$
$$\lambda^* = Qx = \begin{bmatrix} 0.2 \\ 2 \end{bmatrix}$$
$$u^* = -R^{-1}B^T Q x = \begin{bmatrix} -2 \\ 1.6 \end{bmatrix}$$

- L^* [1 pt.]

$$L^* = \frac{1}{2}x^{*T} Q x^* + \frac{1}{2}u^{*T} R u^* = 3.1$$