Chapter 2: Optimal Control of Discrete Time System

2.2 Discrete-time Linear Quadratic Regulator

- The state and costate equations:

Let the plant to be controlled be described by

$$x_{k+1} = A_k x_k + B_k u_k$$

with $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$. The associated performance index is the quadratic function:

$$J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k)$$

defined over the time interval of interest $[i, N]$. The initial plant state is given as x_i. We assume the Q_k, R_k and S_N are symmetric positive semi definite matrices and that $|R_k| \neq 0$ for all k.

The objective is to find the control sequences $u_i, u_{i+1}, \ldots, u_{N-1}$ that minimizes J_i.

The Hamiltonian function:

$$H^k = \frac{1}{2} (x_k^T Q_k x_k + u_k^T R_k u_k) + \lambda_{k+1}^T (A_k x_k + B_k u_k)$$

The state and costate equations:

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = A_k x_k + B_k u_k, \qquad (1)$$

$$\lambda_k = \frac{\partial H^k}{\partial x_k} = Q_k x_k + A_k^T \lambda_{k+1}, \qquad (2)$$

And the stationary condition:

$$0 = \frac{\partial H^k}{\partial u_k}, \qquad k = i, \ldots, N-1, \qquad (3)$$

Based on the stationary condition and $|R_k| \neq 0$

$$u_k = -R_k^{-1} B_k^T \lambda_{k+1}, \qquad (4)$$

If we can find the costate sequence λ_k, then we find the optimal control.

Using (4), equation (1) can be written as

$$x_{k+1} = A_k x_k - B_k R_k^{-1} B_k^T \lambda_{k+1}, \qquad (5)$$

From (2) and (5), we find that the state and costate are coupled. We can form a single unforced system

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ Q & A^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix}$$

Where $A_k = A$, $B_k = B$, $R_k = R$, $Q_k = Q$ for time invariant system.

The system described by (6) is difficult to solve since part of it develops forward and part of it develops backwards in time.

If $|A| \neq 0$, we can rewrite (1) by

$$x_k = A^{-1} x_{k+1} + A^{-1} B R^{-1} B^T \lambda_{k+1}$$

Then the system of (6) can be written by

$$\begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^T \\ QA^{-1} & A^T + QA^{-1}BR^{-1}B^T \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix}$$

This equation develops purely backwards in time. If we know x_N and λ_N. Then we can find x_k and λ_k and hence the optimal control. However, we only are given x_0 not x_N and λ_N.

If $u_k = 0$ and $Q_k = 0$, then

$$x_{k+1} = Ax_k$$
$$\lambda_k = A^T \lambda_{k+1}$$

The solutions are

$$x_k = A^k x_0$$
$$\lambda_k = (A^T)^{N-k} \lambda_N$$

Therefore

$$\lambda_k^T x_k = \lambda_0^T x_0 = \lambda_N^T x_N = constant\ for\ all\ k$$

Three special cases:

Case A: Zero input cost and the Lyapunov Equation

Let the input $u_k = 0$, we want to determine the value of J_i in this uncontrolled situation, we solve J_i backwards to find $J_N, J_{N-1}, J_{N-2}, \ldots$

When $i = N$

$$J_N = \frac{1}{2} x_N^T S_N x_N$$

When $i = N-1$ and $u_k = 0$

$$J_{N-1} = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} x_k^T Q x_k$$

$$= \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} x_{N-1}^T Q x_{N-1}$$

Using the plant dynamics $x_N = A x_{N-1}$

$$J_{N-1} = \frac{1}{2} x_{N-1}^T (A^T S_N A + Q) x_{N-1}$$

We define anew intermediate variable c an $n \times n$ matrix.

$$S_{N-1} = A^T S_N A + Q$$

Then

$$J_{N-1} = \frac{1}{2} x_{N-1}^T S_{N-1} x_{N-1}$$

If we repeat this procedure, for time-invarying system.

$$S_k = A^T S_{k+1} A + Q, \qquad k < N, \qquad (1)$$

For time-varying system:

$$S_k = A_k^T S_{k+1} A_k + Q_k, \qquad k < N\,, \qquad (2)$$

With boundary condition S_N given as the final state weighting matrix. This is a discrete Lyapunov equation for S_k , also known as the observability Lyapunov equation. The zero input performance index over interval $[k, N]$ is

$$J_k = \frac{1}{2} x_k^T S_k x_k$$

We call S_k the performance index kernel sequence. S_k can be calculated offline without knowing the trajectory x_k . If we know the initial condition x_k, then the performance index with zero-input over the interval $[k, N]$ can be computed.

The solution of (1) can be written as

$$S_k = (A^T)^{N-k} S_N A^{N-k} + \sum_{i=k}^{N-1} (A^T)^{N-i-1} Q A^{N-i-1}$$

By the lyapunov stability theory, as $(N-k) \to \infty$, this converges to

$$S_\infty = \sum_{i=0}^{\infty} (A^T)^i Q A^i$$

if the plant is asymptotically stable.

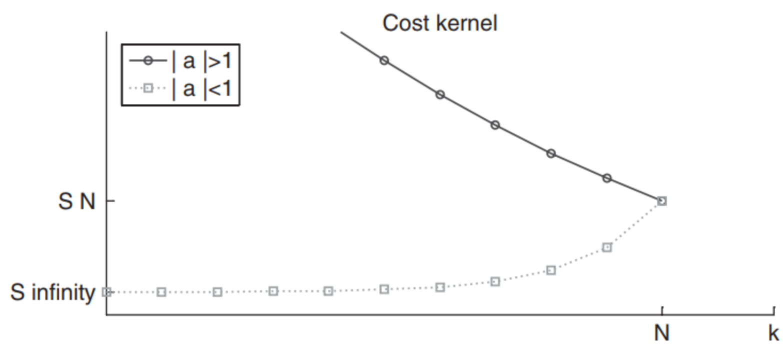

FIGURE 2.2-2 Zero-input cost kernel for a stable and an unstable plant.

If the plant is stable, the cost over the interval $[-\infty, N]$ or equivalently $[0, N]$ is given by the steady-state cost.

$$J_\infty = \frac{1}{2} x_0^T S_\infty x_0$$

If A is not stable, the zero input steady state cost can be infinite (depending on Q) since $||x_k||$ is unbounded.
In the steady state case, $S \triangleq S_k = S_{k+1}$ when $k \to \infty$
becomes

$$S = A^T S A + Q$$

which is called algebraic Lyapunov equation.

By the Lyapunov theory, this equation has a positive semidefinite solution S_∞ if A is stable.

If $\sqrt{Q}$ is defined by

$$Q = \sqrt{Q}^T \cdot \sqrt{Q}$$

then S_∞ is the unique positive definite solution if A is stable and $(A, \sqrt{Q})$ is observable.

Case B: Fixed final state and open loop control

Let the initial time be $i = 0$. The initial state x_0 is given. The terminal objective is to make x_N match exactly the desired final reference

$$x_N = r_N$$

Since $dx_N = 0$,

$$\left(\frac{\partial \phi}{\partial x_N} - \lambda_N\right)^T dx_N = 0$$

Since $x_N = r_N$, the final state contribution to J_0 is a fixed value $\frac{1}{2} r_N^T S_N r_N$. It is therefore redundant to include a final state weighting term in J_0, so we set

$$S_N = 0$$

Let the cost function be

$$J_0 = \frac{1}{2} \sum_{k=0}^{N-1} u_k^t R u_k$$

and we need to find a control that drives x_0 exactly to $x_N = r_N$ using minimum control energy.

The state and costate equation are

$$x_{k+1} = Ax_k - BR^{-1}B^T\lambda_{k+1}$$
$$\lambda_k = A^T\lambda_{k+1}$$

Since $Q = 0$, the costate equation is decoupled from the state equation

$$\lambda_k = (A^T)^{N-k}\lambda_N$$

Use this to eliminate λ_{k+1} in (1)

$$x_{k+1} = Ax_k - BR^{-1}B^T(A^T)^{N-k-1}\lambda_N$$

Considering this as a first order difference equation with the second term as the input

$$x_k = A^k x_0 - \sum_{k=0}^{k-1} A^{k-i-1}BR^{-1}B^T(A^T)^{N-i-1}\lambda_N$$

when $k = N$

$$x_N = A^N x_0 - \sum_{i=0}^{N-1} A^{N-i-1}BR^{-1}B^T(A^T)^{N-i-1}\lambda_N$$

Since $x_N = r_N$, solve (2) for λ_N we get

$$\lambda_N = -G_{0,N}^{-1}(r_N - A^N x_0)$$

where

$$G_{o,N} = \sum_{i=0}^{N-1} A^{N-i-1}BR^{-1}B^T(A^T)^{N-i-1}$$

Then the costate is

$$\lambda_k = (A^T)^{N-k}\lambda_N = -(A^T)^{N-k}G_{0,N}^{-1}(r_N - A^N x_0)$$

and the optimal control

$$u_k^* = R^{-1}B^T(A^T)^{N-k-1}G_{0,N}^{-1}(r_N - A^N x_0)$$

This is the minimum control energy solution to the fixed final state LQR problem. To verify the control drives x_0 to $x_N = r_N$, we get

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1}Bu_i$$

where $k = N$ and $u_i = u_i^*$

$$x_N = A^N x_0 + \sum_{i=0}^{N-1} A^{N-i-1}BR^{-1}B^T(A^T)^{N-i-1}\, G_{0,N}^{-1}(r_N - A^N x_0)$$

$$= A^N x_0 + G_{0,N} \cdot G_{0,N}^{-1}(r_N - A^N x_0) = r_N$$

$G_{0,N}$ is the weighted reachability gramian of the system.

The system reachability matrix is $U_k = [B \quad AB \quad \ldots \quad A^{k-1}B]$
Then

$$G_{0,N} = U_N \begin{bmatrix} R^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R^{-1} \end{bmatrix} U_N^T$$

If $R = I$, then $G_{0,N} = U_N \cdot U_N^T$. The optimal control exist if and only if $|G_{0,N} \neq 0|$. Since we assumed $|R| \neq 0$, this is equivalent to U_N having full rank n, where n is the state dimension. Therefore, we can drive any given x_0 to any desired $x_N = r_N$ for some N steps if and only if the system is reachable (controllable)

To compute the reachability gramian, we can form the Lyapunov equation

$$P_{k+1} = AP_kA^T + BR^{-1}B^T, \qquad k > 0$$

The solution is

$$P_k = A^k P_0 (A^k)^T + \sum_{i=0}^{k-1} A^{k-i-1} BR^{-1}B^T (A^T)^{k-i-1}$$

with $P_0 = 0$, we get $G_{0,k} = P_k$ for any $k > 0$.
When $k = N$, $G_{0,N} = P_N$

<u>Case C:</u> Free final state and closed loop control
The state and costate equations:

$$x_{k+1} = A_k x_k - B_k R_k^{-1} B_k^T \lambda_{k+1}$$
$$\lambda_k = Q_k x_k + A_k^T \lambda_{k+1}$$

The control is given as

$$u_k = -R_k^{-1} B^T \lambda_{k+1}$$

The initial state x_i is given and final state x_N is free.

Since $dx_N \neq 0$, then

$$\lambda_N = S_N x_N$$

Assume a linear relation

$$\lambda_k = S_k x_k, \qquad (1)$$

For some intermediate sequence of $n \times n$ matrices S_k

$$x_{k+1} = A_k x_k - B_k R_k^{-1} B_k^T S_{k+1} x_{k+1}$$

$$x_{k+1} = (I + B_k R_k^{-1} B_k^T S_{k+1})^{-1} A_k x_k, \qquad (2)$$

Now substitute (1) into the costate equation

$$\lambda_k = S_k x_k = Q_k x_k + A_k^T S_{k+1} x_{k+1}$$

$$S_k x_k = Q_k x_k + A_k^T S_{k+1} (I + B_k R^{-1} B_k^T S_{k+1})^{-1} A_k x_k$$

Since x_k is generally nonzero,

$$S_k = A_k^T S_{k+1} (I + B_k R_k^{-1} B_k^T S_{k+1})^{-1} A_k + Q_k$$

Using matrix inversion lemma

$$S_k = A_k^T [S_{k+1} - S_{k+1} B_k (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1}] A_k + Q_k \text{ (3)}$$

This is a backward recursion for the postulatedS_k. The boundary S_N is known. The matrices S_k can be precomputed offline. Equation (3) is called a Riccati equation.

If $|S_k| \neq 0$ for all k, the Riccati equation can be rewritten by

$$S_k = A_k^T (S_{k+1}^{-1} + B_k R_k^{-1} B_k^T)^{-1} A_k + Q_k$$

The optimal control

$$u_k^* = -R_k^{-1} B_k^T \lambda_{k+1} = -R_k^{-1} B_k^T S_{k+1} x_{k+1}$$

with $x_{k+1} = A_k x_k + B_k u_k$

$$u_k^* = -(B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k x_k$$

Define the Kalman gain sequence

$$K_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k$$

the control takes the form

$$u_k^* = -K_k x_k$$

The Kalman gain is given in terms of the Riccati equation solution S_k and the system and weighting matrices.

In the free final state linear quadratic (LQ) regulator, the optimal control is given by a close loop control law.

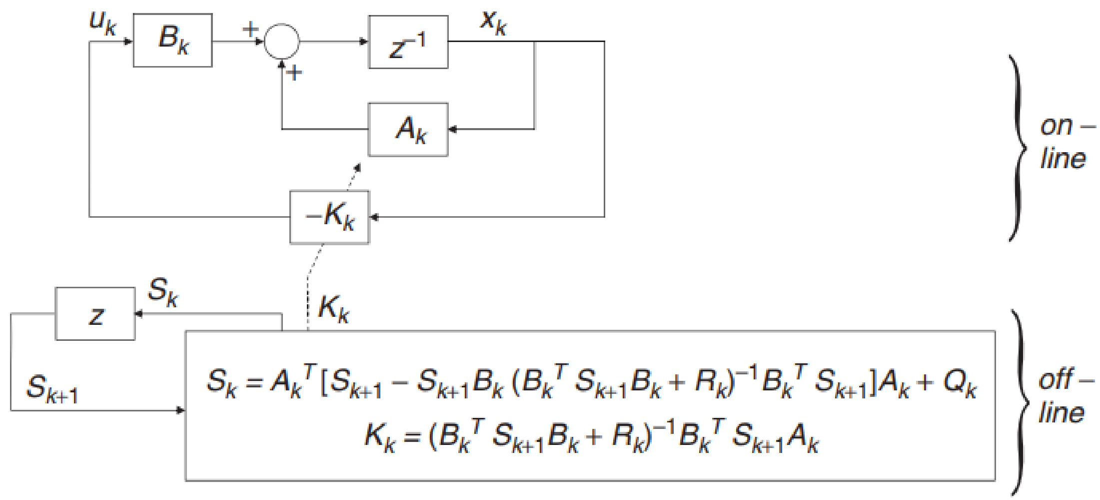

FIGURE 2.2-3 Free-final-state LQ regulator optimal control scheme.

A more efficient way to compute K_k and S_k by using the recursion

$$K_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k S_{k+1} A_k$$
$$S_k = A_k^T S_{k+1}(A_k - B_k K_k) + Q_k$$

Table: Discrete Linear Quadratic Regulator (Final state free)
System model: $x_{k+1} = A_k x_k + B_k u_k, \qquad k > i$
Performance index: $J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k)$
Assumption: $S_n \geq 0, \qquad Q_k \geq 0, \qquad R_k \geq 0$ and all three are symmetric
Optimal feedback control: $S_k = A_k^T[S_{k+1} - S_{k+1}B_k(B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1}]A_k + Q_k$ Or $S_k = A_k^T S_{k+1}(A_k - B_k K_k) + Q_k,\ K < N, S_N$ given. $K_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k, \qquad k < N$

$$\boxed{\begin{gathered} u_k = -K_k x_k, \qquad k < N \\ J_i^* = \frac{1}{2} x_i^T S_i x_i \end{gathered}}$$

Example: Optimal Feedback Control of a scalar system

The plant is described by

$$x_{k+1} = a x_k + b u_k$$

with performance index

$$J_i = \frac{1}{2} S_N x_N^2 + \frac{1}{2} \sum_{k=i}^{N-1} (q x_k^2 + r u_k^2)$$

The Riccati equation is

$$\begin{aligned} S_k &= a^2 S_{k+1} - \frac{a^2 b^2 S_{k+1}^2}{b^2 S_{k+1} + r} + q \\ &= \frac{a^2 r S_{k+1}}{b^2 S_{k+1} + r} + q \end{aligned}$$

The Kalman gain is

$$K_k = \frac{a b S_{k+1}}{b^2 S_{k+1} + r} = \frac{\frac{a}{b}}{1 + \frac{r}{b^2} S_{k+1}}$$

And the optimal control is

$$u_k = -K_k x_k$$

The optimal value of the performance index is

$$J_k = \frac{1}{2} S_k x_k^2$$

$S_k$is hard to find analytically but for a particular value of a, b, q, r, S_N, it is very easy to compute the optimal control sequence u_k^*.

<u>Case A: No Control Weighting</u>

Let $r = 0$, meaning that we do not care how much control is used. Then

$$S_k = \frac{a^2 r S_{k+1}}{b^2 S_{k+1} + r} + q = q$$

And

$$K_k = \frac{\frac{a}{b}}{1 + \frac{r}{b^2} S_{k+1}} = -\frac{a}{b}, \qquad u_k = -\frac{a}{b} x_k$$

Under the influence of this control

$$J_k = \frac{1}{2} q x_k^2$$

And closed loop system becomes

$$x_{k+1} = a x_k + b u_k = 0$$

If we have a given value of x_k for the state at time K, then to minimize the magnitude of the state vector, we solve the state equation for u_k so that $x_{k+1} = 0$

Case B: Very large control weighting

When $r \rightarrow \infty$, then

$$S_k = \frac{a^2 r S_{k+1}}{b^2 S_{k+1} + r} + q \xrightarrow{r \to \infty} a^2 S_{k+1} + q$$

The solution of this equation is

$$S_k = S_N a^{2(N-k)} + \sum_{i=k}^{N-1} q a^{2(N-i-1)}$$

$$= S_N a^{2(N-k)} + \left(\frac{1 - a^{2(N-k)}}{1 - a^2}\right) q$$

The Kalman gain

$$K_k = \frac{\frac{a}{b}}{1 + \frac{r}{b^2} S_{k+1}} \xrightarrow{r \to \infty} 0$$

and $u_k = 0$

The closed loop system becomes

$$x_{k+1} = ax_k$$

If we are very concerned about using too much control, the best policy is to use none at all.

Case C: No intermediate state weighting

Let $q = 0$, then we are concerned about making X_N^2 small without using too much control energy.

$$S_k = \frac{a^2 r S_{k+1}}{b^2 S_{k+1} + r} + q$$

$$S_k = \frac{a^2 r S_{k+1}}{b^2 S_{k+1} + r} \Rightarrow S_k^{-1} = \frac{b^2 + r S_{k+1}^{-1}}{a^2 r}$$

$$\Rightarrow S_k^{-1} = \frac{S_{k+1}^{-1}}{a^2} + \frac{b^2}{a^2 r}$$

The solution is

$$S_k^{-1} = a^{-2(N-k)} S_N^{-1} + \sum_{i=k}^{N-1} \frac{b^2}{a^2 r} a^{-2(N-i-1)}$$

$$= S_N^{-1} a^{-2(N-k)} + \frac{b^2}{r a^{2(N-k)}} \frac{1 - a^{2(N-k)}}{1 - a^2}$$

$$\Rightarrow S_k = \frac{S_N a^{2(N-k)}}{1 + S_N \left(\frac{b^2}{r}\right) \left[\frac{1 - a^{2(N-k)}}{1 - a^2}\right]}$$