HW 4: 2.3-1/2, 2.4-1/2

2.3-1 Harmonic oscillator

Continuous model: $\dot{x}_1 = x_2$

$\dot{x}_2 = -\omega_n^2 x_1 + u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a.) Discretize w/ sampling period T

$$x_{k+1} = A^s x_k + B^s u_k \Rightarrow \text{Need } A^s, B^s$$

$$A^s = e^{A \cdot T} = \sum_{k=0}^{\infty} \frac{(A \cdot T)^k}{k!} = \boxed{\begin{bmatrix} \cos(\omega_n T) & \frac{1}{\omega_n}\sin(\omega_n T) \\ \omega_n \sin(\omega_n T) & \cos(\omega_n T) \end{bmatrix}}$$

$$B^s = \int_0^T e^{At} \cdot B\,dt = \int_0^T \begin{bmatrix} \frac{1}{\omega_n}\sin(\omega_n t) \\ \cos(\omega_n t) \end{bmatrix} dt = \begin{bmatrix} -\frac{1}{\omega_n^2} \cos(\omega_n t) \\ \frac{1}{\omega_n} \sin(\omega_n t) \end{bmatrix} \Bigg|_0^T = \boxed{\begin{bmatrix} \frac{-1}{\omega_n^2}(\cos(\omega_n T) - 1) \\ \frac{1}{\omega_n} \sin \omega_n T \end{bmatrix}}$$

b.) $J = \frac{1}{2}(s_1 (x_N^1)^2 + s_2 (x_N^2)^2) + \frac{1}{2} \sum_{k=0}^{N-1} (q_1 (x_k^1)^2 + q_2 (x_k^2)^2 + r u_k^2)$

$$J = \frac{1}{2} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} x_N + \frac{1}{2} \sum_{k=0}^{N-1} \left(x_k^T \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} x_k + r u_k^2 \right)$$

Optimal controller found by solving the Riccati equation backwards in time for S_k with $S_N = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$. With this S_k, we can get Kalman gains at each time k.

$$K_k = \left[(B^s)^T S_{k+1} B^s + r\right]^{-1} \left[B^T S_{k+1} A^s\right]$$

$$S_k = (A^s)^T S_{k+1} (A^s - B^s K_k) + Q$$

Note, since r is a scalar, $\left[(B^s)^T S_{k+1} B^s + r\right]$ must be a scalar.

Let $\delta = (B^s)^T S_{k+1} B^s + r$

$$\Rightarrow K_k = B^T S_{k+1} A^s / \delta = [K_1 \quad K_2]$$

We know S_{k+1} is symmetric $\Rightarrow S_k = \begin{bmatrix} s_{1k} & s_{2k} \\ s_{2k} & s_{3k} \end{bmatrix}$

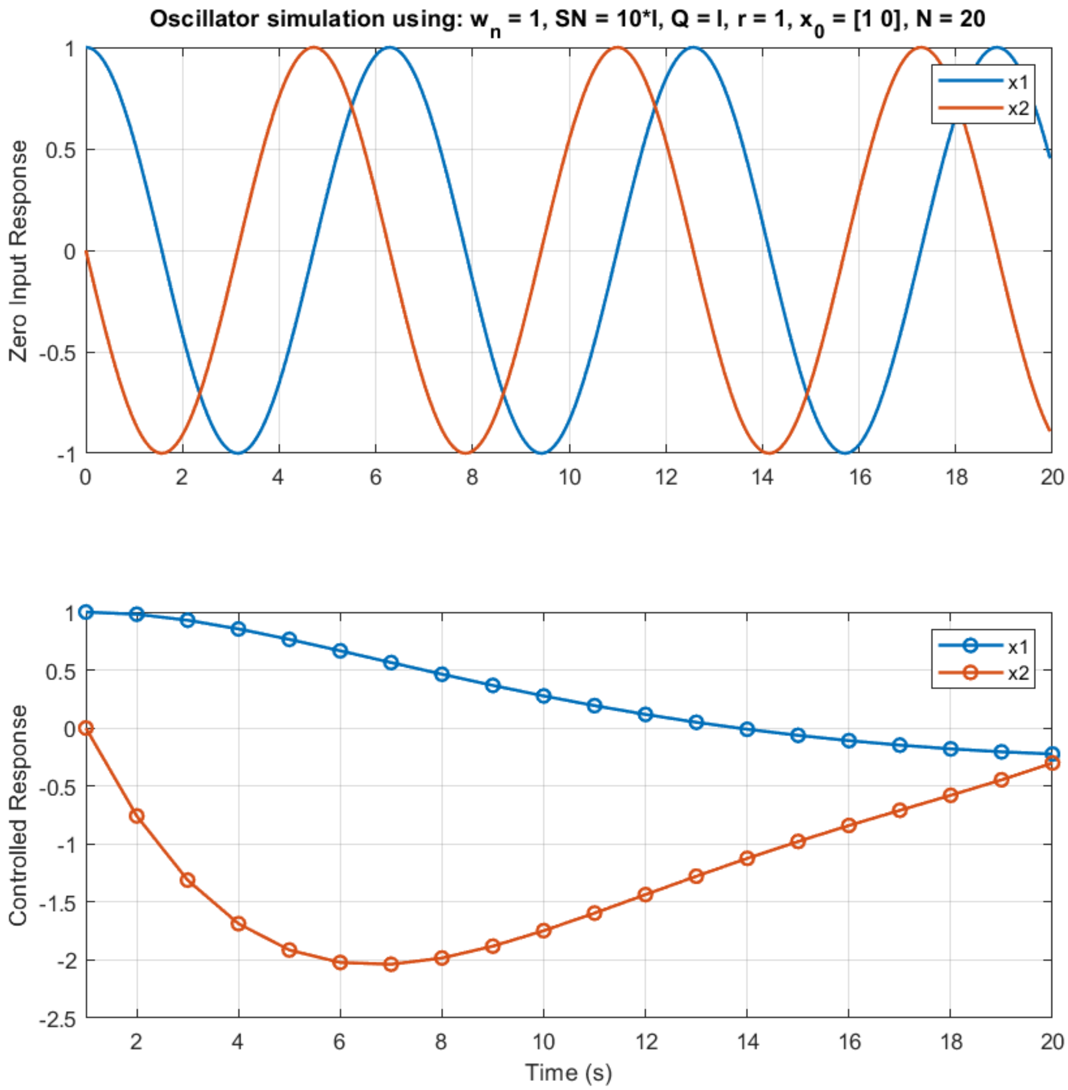
$$\Rightarrow [K_1 \; K_2] = [0 \quad 1] \begin{bmatrix} s_{1k} & s_{2k} \\ s_{2k} & s_{3k} \end{bmatrix} \begin{bmatrix} \cos w_n T & \frac{1}{w_n} \sin w_n T \\ -w_n \sin w_n T & \cos w_n T \end{bmatrix} / \delta$$

$$= [0 \quad 1] \begin{bmatrix} s_{1k} \cos w_n T - s_{2k} w_n \sin w_n T & \frac{s_{1k}}{w_n} \sin w_n T + s_{2k} \cos w_n T \\ s_{2k} \cos w_n T - s_{3k} w_n \sin w_n T & \frac{s_{2k}}{w_n} \sin w_n T + s_{3k} \cos w_n T \end{bmatrix}$$

$$[K_1 \quad K_2] = \left[s_{2k} \cos w_n T - s_{3k} w_n \sin w_n T \qquad \frac{s_{2k}}{w_n} \sin w_n T + s_{3k} \cos w_n T \right] \cdot \frac{1}{\delta}$$

$$\boxed{\begin{aligned} K_1 &= (s_{2k} \cos w_n T - s_{3k} w_n \sin w_n T) \frac{1}{\delta} \\ K_2 &= \left(\frac{s_{2k}}{w_n} \sin w_n T + s_{3k} \cos w_n T\right) \frac{1}{\delta} \\ \Rightarrow u_k &= -K_k x_k = -K_1 x_k^1 - K_2 x_k^2 \end{aligned}}$$

2.3-1
Parts b and c:



2.3-2 Same as 2.3-1 but with:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$A = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Discretization:

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -a^2 & \lambda \end{vmatrix} = \lambda^2 - a^2 = 0 \Rightarrow \lambda_1 = a, \ \lambda_2 = -a$$

$$A^s = e^{AT} = \sum_{k=0}^{\infty} \frac{(AT)^k}{k!} = \sum_{1}^{2} e^{\lambda_k T} \cdot P_k(A) = e^{\lambda_1 T} P_1(A) + e^{\lambda_2 T} P_2(A)$$

$$\text{where } P_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \frac{1}{2a}(A + aI)$$

$$P_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \frac{1}{-2a}(A - aI) = \frac{1}{2a}(aI + A)$$

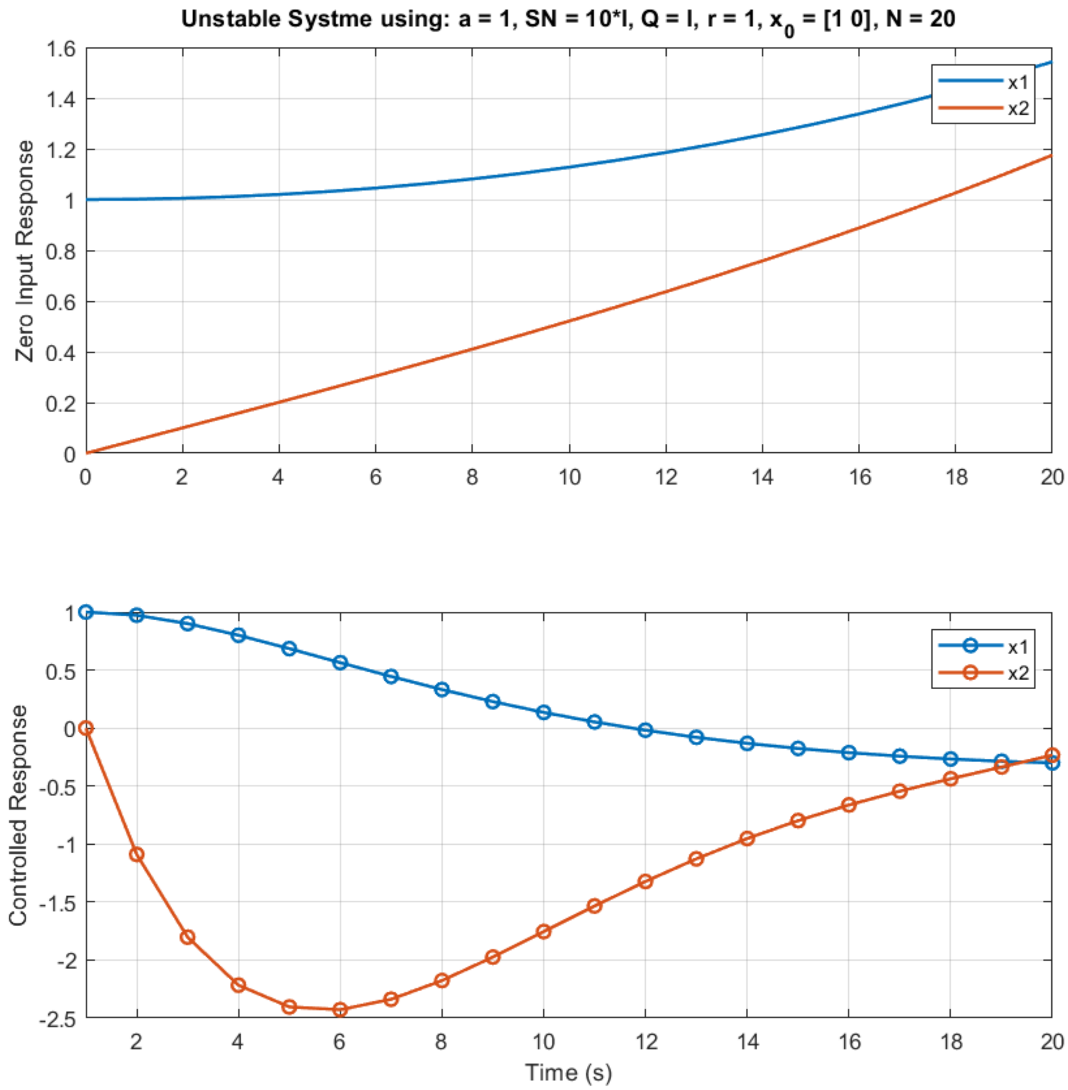
$$\Rightarrow A^s = \frac{e^{aT}(A + aI) + e^{-aT}(aI - A)}{2a} = \frac{aI\overbrace{(e^{aT} + e^{-aT})}^{\frac{1}{2}\cosh(aT}+ A\overbrace{(e^{aT} - e^{-aT})}^{\frac{1}{2}\sinh(aT)}}{2a} = \frac{aI\cosh(aT) + A\sinh(aT)}{a}$$

$$\Rightarrow A^s = \begin{bmatrix} \cosh(aT) & 0 \\ 0 & \cosh(aT) \end{bmatrix} + \begin{bmatrix} 0 & \frac{\sinh(aT)}{a} \\ a\sinh(aT) & 0 \end{bmatrix} = \begin{bmatrix} \cosh(aT) & \frac{1}{a}\sinh(aT) \\ a\sinh(aT) & \cosh(aT) \end{bmatrix}$$

$$B^s = \int_0^T A^s B \, dt = \int_0^T \begin{bmatrix} \frac{1}{a}\sinh(at) \\ \cosh(at) \end{bmatrix} dt = \begin{bmatrix} \frac{1}{a^2}(\cosh(aT) - 1) \\ \frac{1}{a}\sinh(aT) \end{bmatrix}$$

From here, the steps are the same, just use these corresponding matrices

2.3-2



Unstable Systme using: a = 1, SN = 10*I, Q = I, r = 1, x_0 = [1 0], N = 20
Zero Input Response
x1
x2
Controlled Response
x1
x2
Time (s)

24-1

$$X_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$J_0 = \frac{1}{2} X_N^T X_N + \frac{1}{2} \sum_{k=0}^{N-1} \left(X_k^T \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix} X_k + r u_k^2 \right)$$

a.) As $N \to \infty$ $S_k = S_{k+1} = S_\infty$

The Riccati equation is:

$$S = A^T [S - SB(B^T SB + R)^{-1} B^T S] A + Q$$

① $B^T SB + r = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + r = \begin{bmatrix} s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + r = s_3 + r = \delta$

② $SB = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s_2 \\ s_3 \end{bmatrix}$

③ $B^T S = \begin{bmatrix} 0 & 1 \end{bmatrix} S = \begin{bmatrix} s_2 & s_3 \end{bmatrix}$

④ $SBB^T S = \begin{bmatrix} s_2 \\ s_3 \end{bmatrix} \begin{bmatrix} s_2 & s_3 \end{bmatrix} = \begin{bmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{bmatrix}$

$$\Rightarrow S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \left[\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} - \begin{bmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{bmatrix} \frac{1}{\delta} \right] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ s_1 - \frac{s_2^2}{\delta} & s_2 - \frac{s_2 s_3}{\delta} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & s_1 - \frac{s_2^2}{\delta} \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}$$

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \\ q_2 & s_1 - \frac{s_2^2}{\delta} + q_1 \end{bmatrix} \Rightarrow \begin{cases} s_1 = q_1 \\ s_2 = q_2 \\ s_3 = s_1 - \frac{s_2^2}{\delta} + q_1 \end{cases}$$

$s_1 = q_1$

$s_2 = q_2$

$$s_3 = s_1 - \frac{s_2^2}{s} + q_1 = q_1 - \frac{q_2^2}{s_3 + r} + q_1 = 2q_1 - \frac{q_2^2}{s_3+r}$$

Solving for s_3:

$$s_3^2 + s_3 \cdot r = 2q_1 s_3 + 2q_1 r - q_2^2$$

$$s_3^2 + (r - 2q_1)s_3 + (q_2^2 - 2q_1 r) = 0$$

$$\Rightarrow \quad s_3 = \frac{1}{2}\left[(2q_1 - r) \pm \sqrt{(r-2q_1)^2 - 4(q_2^2 - 2q_1 r)}\right]$$

$$= \frac{1}{2}\left[(2q_1 - r) \pm \sqrt{r^2 - \cancel{4q_1 r} + 4q_1^2 + \overset{4q_1 r}{\cancel{8q_1 r}} - 4q_2^2}\right]$$

$$= \frac{1}{2}\left[(2q_1 - r) \pm \sqrt{(r+2q_1)^2 - 4q_2^2}\right] \qquad \sqrt{\left(\frac{r}{2}+q_1\right)^2 - q_2^2} > \frac{r}{2} + q_1 - q_2$$

$$= \left(q_1 - \frac{r}{2}\right) \pm \sqrt{\left(\frac{r}{2} + q_1\right)^2 - q_2^2} \qquad \Rightarrow$$

s_3 must be positive $\Rightarrow$ $s_3 = \left(q_1 - \frac{r}{2}\right) - \sqrt{\left(\frac{r}{2} + q_1\right)^2 - q_2^2}$

Gain:

$$K_\infty = (B^T S_\infty B + r)^{-1} B^T S_\infty A$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Big/ (s_3 + r)$$

$$= \begin{bmatrix} s_2 & s_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Big/ (s_3 + r)$$

$$\boxed{K_\infty = \begin{bmatrix} 0 & s_2 \end{bmatrix} \cdot \frac{1}{s_3 + r}, \quad \text{NON zero if } s_2 = q_2 \neq 0}$$

b.) $A^{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{s_2}{s_3+r} \end{bmatrix} = \begin{bmatrix} 0 & 1-0 \\ 0 & -\frac{s_2}{s_3+r} \end{bmatrix}$

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{s_2}{s_3+r} \end{bmatrix} x_k$$

$$\sqrt{\left(\frac{r}{2}+q_{b_1}\right)^2-q_{b_2}^2} > \frac{r}{2}+q_{b_1}-q_{b_2}$$

$$\Rightarrow \quad -1\cdot \quad < \quad -1\cdot$$

$$= \left(q_{b_1}-\frac{r}{2}\right) \pm \sqrt{\left(\frac{r}{2}+q_{b_1}\right)^2-q_{b_2}^2}$$

$$q_{b_1}-\frac{r}{2}+\sqrt{\left(\frac{r}{2}+q_{b_1}\right)^2-q_{b_2}^2} > q_{b_1}\cancel{-\frac{r}{2}}\cancel{+\frac{r}{2}}+q_{b_1}-q_{b_2} = 2q_{b_1}-q_{b_2}$$

$$q_{b_1}-\frac{r}{2}-\sqrt{\left(\frac{r}{2}+q_{b_1}\right)^2-q_{b_2}^2} < q_{b_1}-\frac{r}{2}-\left[\frac{r}{2}+q_{b_1}-q_{b_2}\right] = -r+q_{b_2}$$

$$q_{b_1}-\frac{r}{2} > \sqrt{\left(\frac{r}{2}+q_{b_1}\right)^2-q_{b_2}^2} > \frac{r}{2}+q_{b_1}-q_{b_2}$$

$$\cancel{q_{b_1}}-\frac{r}{2} > \cancel{q_{b_1}}+\frac{r}{2}-q_{b_2}$$

$$-\frac{r}{2} > \frac{r}{2}-q_{b_2}$$

$$\Rightarrow q_{b_2} > r$$

$$|\lambda I - A^{cl}| = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda + \frac{s_2}{s_3+r} \end{vmatrix} = \lambda^2 + \lambda \frac{s_2}{s_3+r} = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = -\frac{s_2}{s_3+r} = -\frac{q_2}{\left(q_1 - \frac{r}{2}\right) + \sqrt{\left(\frac{r}{2}+q_1\right)^2 - q_2^2}}$$

For stability, we must have : $q_2 < s_3 + r$

Using triangle inequality :

$$s_3 + r = \left(q_1 - \frac{r}{2}\right) - \sqrt{\left(\frac{r}{2}+q_1\right)^2 - q_2^2} + r > q_1 + \frac{r}{2} - \left(\frac{r}{2}+q_1\right) + q_2 = q_2$$

$$\Rightarrow s_3 + r > q_2$$

$$\boxed{\text{Thus } \left|-\frac{s_2}{s_3+r}\right| < 1 \quad \text{so} \quad A^{cl} \text{ is stable.}}$$

$c^2 = a^2 + b^2$

$c < a + b$

$\sqrt{a^2+b^2} < a + b$

$c^2 - a^2 = b^2$

$c - a < b = \sqrt{c^2 - a^2}$

c.) $u_k = -K_\infty^* x_k$

$$S_\infty = (A^{cl})^T S_\infty (A^{cl}) + K_\infty^T r K_\infty + Q$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & -\frac{s_2}{s_3+r} \end{bmatrix} \begin{bmatrix} \bar{s}_1 & \bar{s}_2 \\ \bar{s}_2 & \bar{s}_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\frac{s_2}{s_3+r} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{s_2}{s_3+r} \end{bmatrix} \begin{bmatrix} 0 & \frac{s_2}{s_3+r} \end{bmatrix} \cdot r + Q$$

$$\begin{bmatrix} \bar{s}_1 & \bar{s}_2 \\ \bar{s}_2 & \bar{s}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \bar{s}_1 - \frac{s_2 \bar{s}_2}{s_3+r} & \bar{s}_2 - \frac{s_2 \bar{s}_3}{s_3+r} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\frac{s_2}{s_3+r} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{s_2}{s_3+r}\right)^2 \end{bmatrix} \cdot r + Q$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \bar{s}_1 - \frac{s_2 \bar{s}_2}{s_3+r} - \left(\bar{s}_2 - \frac{s_2 \bar{s}_3}{s_3+r}\right)\left(\frac{s_2}{s_3+r}\right) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{s_2}{s_3+r}\right)^2 \end{bmatrix} \cdot r + Q$$

$$\Rightarrow \bar{s}_1 = q_1 = s_1 \qquad \bar{s}_2 = q_2$$

$$\bar{s}_3 = \bar{s}_1 - \frac{s_2 \bar{s}_2}{s_3+r} - \left(\bar{s}_2 - \frac{s_2 \bar{s}_3}{s_3+r}\right)\left(\frac{s_2}{s_3+r}\right) + \left(\frac{s_2}{s_3+r}\right)^2 r + q_3$$

$$= q_1 - 2\left(\frac{s_2 \bar{s}_2}{s_3+r}\right) + \frac{s_2^2 \bar{s}_3}{(s_3+r)^2} + \left(\frac{s_2}{s_3+r}\right)^2 \cdot r + q_3$$

$$\left[\bar{s}_3\left(1+\left(\frac{q_2}{s_3+r}\right)^2\right) = q_1 - 2\frac{q_2^2}{s_3+r} + \frac{q_2^2}{(s_3+r)^2} \cdot r + q_3\right](s_3+r)^2$$

$$\bar{s}_3(1+q_2^2) \cdot (s_3+r)^2 = q_1(s_3+r) - 2q_2^2(s_3+r) + q_2^2 r + q_3(s_3+r)^2$$

$$\bar{s}_3 = \left[q_1(s_3+r) - 2q_2^2(s_3+r) + q_2^2 r + q_3(s_3+r)^2\right]\frac{1}{(1+q_2^2)(s_3+r)^2}$$

$$= \left[q_1(s_3+r) - q_2^2 r - 2q_2^2 s_3 + q_3(s_3+r)^2\right]\frac{1}{(1+q_2^2)(s_3+r)^2}$$

$$= \left[q_1(s_3+r) - q_2^2\right.$$

2.42 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $S_N = I$ $Q = I$

a.) $r = 0.1$

Hamiltonian:

$$H = \begin{bmatrix} A^{-1} & A^{-1}BB^T/r \\ QA^{-1} & A^T + QA^{-1}BB^T/r \end{bmatrix} = \begin{bmatrix} A^{-1} & A^{-1}BB^T/r \\ A^{-1} & A^T + A^{-1}BB^T/r \end{bmatrix} \quad (Q = I)$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad B \cdot B^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} \cdot BB^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad H = \begin{bmatrix} 1 & -1 & 0 & -10 \\ 0 & 1 & 0 & 10 \\ 1 & -1 & 1 & -10 \\ 0 & 1 & 1 & 11 \end{bmatrix}$$

eigenvalues & eigenvectors:

$$v_1 = \begin{bmatrix} -0.10 \\ -0.99 \\ 0.01 \\ 0.09 \end{bmatrix} \quad \lambda_1 = 10.78 \qquad v_3 = \begin{bmatrix} -0.51 \\ 0.32 \\ -0.80 \\ 0.06 \end{bmatrix} \quad \lambda_3 = 0.36$$

$$v_2 = \begin{bmatrix} -0.47 \\ -0.84 \\ 0.27 \\ 0.05 \end{bmatrix} \quad \lambda_2 = 2.77 \qquad v_4 = \begin{bmatrix} -0.51 \\ 0.46 \\ -0.56 \\ 0.45 \end{bmatrix} \quad \lambda_4 = 0.093$$

Unstable eigenvalues.

$$M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \Rightarrow \quad M^{-1} = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix}$$

$$W = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} W_{11} & W_{12} \end{array} \\ \left[\begin{array}{cc|cc} -0.10 & -0.47 & -0.51 & -0.51 \\ -0.99 & -0.84 & 0.32 & 0.46 \\ \hline 0.01 & 0.27 & -0.80 & -0.56 \\ 0.09 & 0.05 & 0.06 & 0.45 \end{array}\right] \\ \begin{array}{cc} W_{21} & W_{22} \end{array} \end{array}$$

Analytic Riccati Equation:

$$S_k = (W_{21} + W_{22} T_k)(W_{11} + W_{12} T_k)^{-1}$$

where

$$T = -(W_{22} - S_N W_{12})^{-1}(W_{21} - S_N W_{11}) = -(W_{22} - W_{12})^{-1}(W_{21} - W_{11}) \quad S_N = I$$

$$T_k = (M^{-1})^{N-k}\, T \cdot (M^{-1})^{N-k}$$

Since M^{-1} is stable $(M^{-1})^{N-k} \to 0$ as $N-k \to \infty$

Thus $T_k = 0$

$$\Rightarrow S_\infty = W_{21} W_{11}^{-1} = \begin{bmatrix} -0.67 & 0.06 \\ 0.06 & -0.10 \end{bmatrix} \qquad (2.4-42)$$

Using 2-4-63 to get K_∞:

$$K_\infty = R^{-1} B^T \Lambda M_{stable} X^{-1} = \frac{1}{r} \cdot B^T \cdot W_{21} \cdot M^{-1} W_{11}^{-1}$$

$\Lambda = W_{21}$
$M_{stable} = M^{-1}$
$X = W_{11}$

$$\Rightarrow \boxed{K_\infty = \begin{bmatrix} 0.0579 & -0.0387 \end{bmatrix}}$$

Using Ackermann's Formula

$$H^{-1} = \begin{bmatrix} A + BR^{-1}B^T A^{-T} Q & -BR^{-T} B^T A^{-T} \\ -A^{-T} Q & A^{-T} \end{bmatrix}$$

$$H^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -10 & 11 & 10 & -10 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

The stable eigenvalues of H^{-1} are the optimal closed loop poles, which are the unstable eigenvalues of H ($\frac{1}{\mu}$ stable for H^{-1}, μ unstable for H)

$$\Rightarrow \quad \mu_1 = 0.3616 \qquad \mu_2 = 0.0928$$

$$\Delta^{cl}(z) = (z - 0.3616)(z - 0.0928) = z^2 - .4544z + 0.0335$$

$$\Rightarrow \Delta^{cl}(A) = A^2 - 0.4544A + 0.0335I$$

$$= \begin{bmatrix} 0.5792 & 1.5456 \\ 0 & 0.5792 \end{bmatrix}$$

Also need the reachability matrix:

$$\mathcal{U}_2 = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathcal{U}_2^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

Using Ackermann's Formula:

$$K_\infty = [0 \quad 1]\,\mathcal{U}_2^{-1}\,\Delta^{cl}(A)$$

$$= [0 \quad 1]\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.5792 & 1.5456 \\ 0 & 0.5792 \end{bmatrix}$$

$$\boxed{K_\infty = [0.5792 \quad 1.5456]}$$ (? different)

2.4-2

b.) $r=1 \Rightarrow H = \begin{bmatrix} A^{-1} & A^{-1}BB^T \\ A^{-1} & A^T + A^{-1}BB^T \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

$\text{eig}(H) \Rightarrow \lambda_{1,2} = 2.12 \pm 1.05i$

$\lambda_{3,4} = 0.38 \pm 0.19i \longrightarrow \text{stable } (|\lambda| < 1)$

$$[\Lambda_1 \ \Lambda_1^*] \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} [X_1 \ X_1^*]^{-1} \Longleftrightarrow [Re(\Lambda_1) \ Im(\Lambda)] \begin{bmatrix} Re(\lambda_3) & Im(\lambda_3) \\ -Im(\lambda_3) & Re(\lambda_3) \end{bmatrix} [Re(X_1) \ Im(X_1)]^{-1}$$

$$[\Lambda_1 \ \Lambda_1^*] = W_{21} = \begin{bmatrix} \underbrace{\begin{matrix} 0.019 - 0.30i \\ 0.437 + 0.132i \end{matrix}}_{\Lambda_i} & \begin{matrix} 0.019 + 0.30i \\ 0.437 - 0.132i \end{matrix} \end{bmatrix}$$

$$[X_1 \ X_1^*] = W_{11} = \begin{bmatrix} \underbrace{\begin{matrix} -0.33 + 0.31i \\ -0.703 + 0i \end{matrix}}_{X_i} & \begin{matrix} -0.33 - 0.31i \\ -0.703 + 0i \end{matrix} \end{bmatrix}$$

$$Re(\Lambda_1) = \begin{bmatrix} 0.019 \\ 0.437 \end{bmatrix} \qquad Im(\Lambda_1) = \begin{bmatrix} -0.3 \\ 0.132 \end{bmatrix}$$

$$Re(X_1) = \begin{bmatrix} -0.33 \\ -0.703 \end{bmatrix} \qquad Im(X_1) = \begin{bmatrix} 0.31 \\ 0 \end{bmatrix}$$

$$[Re(\Lambda_1) \ Im(\Lambda)] \begin{bmatrix} Re(\lambda_3) & Im(\lambda_3) \\ -Im(\lambda_3) & Re(\lambda_3) \end{bmatrix} [Re(X_1) \ Im(X_1)]^{-1} = \begin{bmatrix} -0.35 & 0.075 \\ 0.42 & -0.40 \end{bmatrix} = \xi$$

$$\Rightarrow \boxed{K_\infty = \cancel{R}^{-1} B^T \cdot \xi = [0.4221 \quad -0.3998]}$$

$$\boxed{S_\infty = W_{21} \cdot W_{11}^{-1} = \begin{bmatrix} -0.947 & 0.4221 \\ 0.4221 & -0.8281 \end{bmatrix}}$$