MECE6388 Optimal Control Theory, 2025 Fall
Dr. Zheng Chen
Chapter 2: Optimal Control of Discrete Time System

2.1 Solution of the General Discrete-Time Optimization problem

Let the plant be

$$\boxed{x_{k+1} = f^k(x_k, u_k)}$$

with initial condition x_0. The superscript on the function f indicates that, in general, the system and thus its model can have time-varying dynamics. Let the state x_k, be a vector of size n and the control input u_k be a vector of size m.

Let an associated scalar performance index

$$\boxed{J_i = \phi(N, x_N) + \sum_{k=i}^{N-1} L^k(x_k, u_k)}$$

where $[i, N]$ is the time interval, on a discrete time scale with a fixed sample step, $\phi(N, x_N)$ is a function of the final time N and the state at the final time, and $L^k(x_k, u_k)$ is a general time-varying function of the state and control input.

The optimal control problem is to find the control u^* on the interval $[i, N]$ that drives the system along a trajectory x_k^* such that the value of the performance index is minimized (optimized).

Example 2:1-1

A. Minimum-time Problems

Suppose we want to find the control u_k to drive the system from the given initial state x_0 to a desired final state $x \in \mathbb{R}^n$ in minimum time, then we could select the performance index

$$J = N = \sum_{k=0}^{N-1} 1$$

and specify the boundary condition

$$x_N = x$$

In this case, one can consider either $\phi = N$ and $L = 0$. Or equivalently $\phi = 0$ and $L = 1$.

B. Minimum- fuel Problems

To find the scalar control u_k to drive the system from x_0 to a desired final state x at a fixed time N using minimum fuel, we could use

$$J = \sum_{k=0}^{N-1} |u_k|$$

Since the fuel burned is proportional to the magnitude of final state and all intermediate state, we define a new cost

$$J = \frac{1}{2} s x_N^T x_N + \frac{1}{2} \sum_{k=0}^{N-1} (q x_k^T x_k + r u_k^T u_k)$$

where q, r, and s are scalar weighting factors.

For more generality, we could select weighting matrices Q, R, and S instead of scalars.

$$J = \frac{1}{2} x_N^T S x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k)$$

Problem Solution:

Let $\lambda \in \mathbb{R}^n$ and append the constraints to the performance index

$$J' = \phi(N, x_N) + \sum_{k=i}^{N-1} [L^k(x_k, u_k) + \lambda_{k+1}^T (f^k(x_k, u_k) - x_{k+1})]$$

Note that we have associated with f^k the multiplier λ_{k+1} not λ.

Define the Hamiltonian function as

$$H^k(x_k, u_k) = L^k(x_k, u_k) + \lambda_{k+1}^T f^k(x_k, u_k)$$

We can write

$$J' = \phi(N, x_N) - \lambda_N^T x_N + H^i(x_i, u_i) + \sum_{k=i+1}^{N-1} [H^k(x_k, u_k,) - \lambda_k^T x_k]$$

where some minor manipulations with indies have been performed.

Now, we want to examine the increment J' due to increment in all the variables x_k, λ_k and u_k.

$$\begin{aligned} dJ' = {} & (\phi_{x_N} - \lambda_N)^T dx_N + (H_{x_i}^i)^T dx_i + (H_{u_i}^i)^T du_i \\ & + \sum_{k=i+1}^{N-1} \left[(H_{x_k}^k - \lambda_k)^T dx_k + (H_{u_k}^k)^T du_k\right] \\ & + \sum_{k=i+1}^{N} (H_{\lambda_k}^{k-1} - x_k)^T d\lambda_k \end{aligned}$$

where

$$H_{x_k}^k \triangleq \frac{\partial H^k}{\partial x_k}$$

Necessary conditions for a constrained minimum are thus given by

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}}, \qquad k = i, \dots, N-1\,, \qquad (1)$$

$$\lambda_k = \frac{\partial H^k}{\partial x_k}, \qquad k = i+1, \dots, N-1\,, \qquad (2)$$

$$0 = \frac{\partial H^k}{\partial u_k}, \qquad k = i, \dots, N-1\,, \qquad (3)$$

which arise from the terms inside the summations and the coefficient of du_i and

$$\left(\frac{\partial \phi}{\partial x_N} - \lambda_N\right)^T dx_N = 0, \qquad (4)$$

$$\left(\frac{\partial H^i}{\partial x_i}\right)^T dx_i = 0, \qquad (5)$$

The stationary condition allows the optimal control u_k to be expressed in terms of the costate λ_k.

If the initial state x_i is fixed, then $dx_i = 0$. Then (5) holds regardless of the value of $H_{x_i}^i$.

In the case of free initial state, dx_i is not zero, then from (5)

$$\frac{\partial H^i}{\partial x_i} = 0$$

In our applications, the system starts at a known initial state x_i. Thus, the case (5) holds and there are no constraints on the value of $H_{x_i}^i$.

In the case of a fixed final state, we use the desired value of x_N as the terminal condition. Since x_N is not free to be varied, $dx_N = 0$ and (4) can always be satisfied.

In the case of free-final-state situation, dx_N is not zero. Then (4) gives us:

$$\frac{\partial \phi}{\partial x_N} - \lambda_N = 0 \Rightarrow \lambda_N = \frac{\partial \phi}{\partial x_N}$$

In a summary, the discrete Nonlinear Optimal Controller can be found based on the following equations:

System Model:

$$x_{k+1} = f^k(x_k, u_k), \qquad k > i$$

Performance Index:

$$J_i = \phi(N, x_N) + \sum_{k=i}^{N-1} L^k(x_k, u_k)$$

Hamiltonian:

$$H^k = L^k + \lambda_{k+1}^T f^k(x_k, u_k)$$

Optimal controller

State equation:

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = f^k(x_k, u_k)$$

Costate equation:

$$\lambda_k = \frac{\partial H_k}{\partial x_k} = \left(\frac{\partial f^k}{\partial x_k}\right)^T \lambda_{k+1} + \frac{\partial L^k}{\partial x_k}$$

Stationary condition:

$$0 = \frac{\partial H^k}{\partial u_k} = \left(\frac{\partial f^k}{\partial u_k}\right)^T \lambda_{k+1} + \frac{\partial L^k}{\partial u_k}$$

Boundary conditions:

$$\left(\frac{\partial L^i}{\partial x_i} + \left(\frac{\partial f^i}{\partial x_i}\right)^T \lambda_{i+1}\right)^T dx_i = 0$$

$$\left(\frac{\partial \phi}{\partial x_N} - \lambda_N\right)^T dx_N = 0$$

Example

Consider a simple linear dynamical system:

$$x_{k+1} = ax_k + bu_k$$

where a, b are scalar constant values.

Given an initial condition x_0 and the interval $[0, N]$, we need to find a control to minimize the control energy

$$J_0 = \frac{r}{2}\sum_{k=0}^{N-1} u_k^2$$

for some scalar weighting factor r.

Case A: Fixed final state

$$x_N = r_N$$

To find the optimal control sequence $u_0, u_1, \dots, u_{N-1}$ that drives x from the given initial point x_0 to the desired final state $x_N = r_N$ while minimizing J_0.

Step 1: Formulate the Hamiltonian

$$H^k = L^k + \lambda_{k+1}^T f^k = \frac{r}{2}u_k^2 + \lambda_{k+1}(ax_k + bu_k)$$

And the optimal conditions are

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = ax_k + bu_k, \qquad (1)$$

$$\lambda_k = \frac{\partial H_k}{\partial x_k} = a\lambda_{k+1}, \qquad (2)$$

$$0 = \frac{\partial H_k}{\partial u_k} = ru_k + b\lambda_{k+1}, \qquad (3)$$

Solving the stationary condition (3) yields

$$u_k = -\frac{b}{r}\lambda_{k+1}, \qquad (4)$$

So, if we find the optimal λ_k, then we obtain the optimal control u_k. To find the optimal λ_k, we look at equation (2)

$$\lambda_k = a\lambda_{k+1} \Longrightarrow \lambda_k = a^{N-k}\lambda_N, \qquad (5)$$

If we know λ_N, we solve the problem. However, λ_N is still unknown. To find it, we look at eq (1).

$$x_{k+1} = ax_k + bu_k$$

From (4)

$$x_{k+1} = ax_k - \frac{b^2}{r}\lambda_{k+1}$$

From (5)

$$x_{k+1} = ax_k - \frac{b^2}{r}a^{N-k-1}\lambda_N$$

Define $\gamma \triangleq \frac{b^2}{r}$, we get

$$x_{k+1} = ax_k - \gamma a^{N-k-1}\lambda_N$$

With initial state x_0

$$x_k = a^k x_0 - \sum_{i=0}^{k-1} a^{k-i-1}\left(\gamma\lambda_N a^{N-i-1}\right)$$

$$= a^k x_0 - \gamma\lambda_N a^{N+k-2}\sum_{i=0}^{k-1} a^{-2i}$$

Using the formula for the sum of a geometric series

$$x_k = a^k x_0 - \gamma\lambda_N a^{N+k-2}\frac{1-a^{-2k}}{1-a^{-2}}$$
$$= a^k x_0 - \gamma\lambda_N a^{N-k}\ \frac{1-a^{2k}}{1-a^2}$$

To find λ_N, since the final state is fixed ($x_N = r_N$) and

$$x_N = a^N x_0 - \gamma\lambda_N a^{N-N}\frac{1-a^{2N}}{1-a^2}$$
$$= a^N x_0 - \frac{(1-a^{2N})\gamma}{1-a^2}\lambda_N$$

So

$$\lambda_N = -\frac{1}{\Lambda}(\mathrm{r_N} - a^N x_0)$$

Where

$$\Lambda \triangleq \frac{\gamma(1-\mathrm{a}^{2\mathrm{N}})}{1-\mathrm{a}^2}$$

Since $u_k = -\frac{b}{r}\lambda_{k+1}$ and $\lambda_{k+1} = a^{N-k-1}\lambda_N$

$$u_k^* = \frac{b}{r\Lambda}(r_N - a^N x_0)a^{N-k-1} = \frac{(1-a^2)}{b(1-a^{2N})}(r_N - a^N x_0)a^{N-k-1}$$

Note: u_k^* does not depend on r.

To find out x_k^* and J_0^* with the optimal control u^*, we put u^* into the dynamical equation

$$x_{k+1}^* = a x_k^* + \frac{1-a^2}{1-a^{2N}}(r_N - a^N x_0)a^{N-k-1}$$

The solution of this dynamical system with forcing function given by the second term is

$$x_k^* = a^k x_0 + \frac{1-a^2}{1-a^{2N}}(r_N - a^N x_0)\sum_{i=0}^{k-1} a^{k-i-1}a^{N-i-1}$$

Rewrite it as

$$x_k^* = \frac{(1 - a^{2(N-k)})a^k x_0 + (1 - a^{2k})a^{N-k} r_N}{1 - a^{2N}}$$

where $k = 0$, $x_0^* = x_0$ and when $k = N$, $x_N^* = x_N$.

The optimal performance index is

$$\begin{aligned}
J_0^* &= \frac{r}{2} \sum_{k=0}^{N-1} {u_k^*}^2 \\
&= \frac{r}{2} \sum_{k=0}^{N-1} \left[\frac{1 - a^2}{b(1 - a^{2N})} (r_N - a^N x_0) a^{N-k-1} \right]^2 \\
&= \frac{r}{2} \frac{(1 - a^2)^2}{b^2(1 - a^{2N})^2} (r_N - a^N x_0)^2 \sum_{k=0}^{N-1} a^{2(N-k-1)} \\
&= \frac{1}{2\Lambda} (r_N - a^N x_0)^2
\end{aligned}$$

The optimal performance index is only related to the final state and the initial state.

Case B: Free Final State

We desire the system state variable $x_N \to r_N$ at times N but not exactly $x_N = r_N$. Therefore, we make the difference $x_N - r_N$ small by including it in the performance index

$$J_0 = \frac{1}{2}(x_N - r_N)^2 + \frac{r}{2} \sum_{k=0}^{N-1} u_k^2$$

Now the optimal control will attempt to make $|x_N - r_N|$ small while also using low control energy. In this case, the function ϕ

$$\phi = \frac{1}{2}(x_N - r_N)^2$$

But f_k and L_k are not changed. The Hamiltonian is still the same as we discussed in Case A. The only change is the boundary condition. Since $dx_N \neq 0$, we must have

$$\lambda_N = \frac{\partial \phi}{\partial x_N} = x_N - r_N$$

Since

$$\lambda_k = a^{N-k}\lambda_N \Rightarrow \lambda_k = a^{N-k}(x_N - r_N)$$

And

$$u_k = -\frac{b}{\gamma}\lambda_{k+1} \Rightarrow u_k = -\frac{b}{r}a^{N-k-1}(x_N - r_N)$$

Since

$$\begin{aligned} x_N &= a^N x_0 - \Lambda\lambda_N \\ &= a^N x_0 - \Lambda(x_N - r_N) \end{aligned}$$

Solving for x_N gives

$$x_N = \frac{\Lambda r_N + a^N x_0}{1+\Lambda}$$

With x_N, we get the costate

$$\lambda_k = a^{N-k}(x_N - r_N) = -\frac{r_N - a^N x_0}{1+\Lambda}a^{N-k}$$

And the optimal control

$$\begin{aligned} u_k^* &= -\frac{b}{r}\lambda_{k+1} \\ &= \frac{b}{r(1+\Lambda)}(r_N - a^N x_0)a^{N-k-1} \end{aligned}$$

Note: the optimal control for free final state does depend on r,

As $r \to 0$, we are concerned less and less about the control energy since u_k^2 is weighted less and less heavily in J_0.
When we look at the final state

$$x_N = \frac{r_N + \frac{a^N}{\Lambda}x_0}{1+\frac{1}{\Lambda}}$$

As $r \to 0$, $\Lambda = \frac{b^2}{r}\frac{(1-a^{2N})}{(1-a^2)} \to \infty$, $x_N \to \frac{r_N+0}{1+0} = r_N$

In summary, if we are not concerned about energy, the final state will reach its desired value. There is a trade-off between the energy consumed and final state desired. The value of r effects the tradeoff.

To find the optimal value of the performance index, use optimal control u_k^* in (2) and simplify to obtain

$$J_0^* = \frac{\Lambda}{2(1+\Lambda)^2}(r_N - a^N x_0)^2$$

As $r \to 0$, $\Lambda \to \infty$, $J_0^* \to \frac{1}{2\Lambda}(r_N - a^N x_0)^2$, which is the fixed final state cost.