2.3-1. **Digital control of harmonic oscillator.** **[3 pts]** A harmonic oscillator is described by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_n^2 x_1 + u$$

a. Discretize the plant using a sampling period of *T*.
b. With the discretized plant, associate a performance index of

$$J = \frac{1}{2}[s_1(x_N^1)^2 + s_2(x_N^2)^2] + \frac{1}{2}\sum_{k=0}^{N-1}[q_1(x_k^1)^2 + q_2(x_k^2)^2 + ru_k^2],$$

where the state is $x_k = [x_k^1 \; x_k^2]^T$. Write scalar equations for a digital optimal controller.
c. Write a MATLAB subroutine to simulate the plant dynamics, and use the time response program *lsim.m* to obtain zero-input state trajectories.
d. Write a MATLAB subroutine to compute and store the optimal control gains and to update the control u_k given the current state x_k. Write a MATLAB driver program to obtain time response plots for the optimal controller.

Solution to part a:
The continous time-invariant plant is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

the discretized plant using a sampling period of T is

$$x_{k+1} = A^s x_k + B^s u_k$$

Method 1

$$A^s = e^{AT} = \mathcal{L}^{-1}\{(SI - A)^{-1}\}$$

$$(SI - A)^{-1} = \begin{bmatrix} \dfrac{s}{s^2 - (-\omega_n^2)} & \dfrac{1}{s^2 - (-\omega_n^2)} \\ \dfrac{-\omega_n^2}{s^2 - (-\omega_n^2)} & \dfrac{s}{s^2 - (-\omega_n^2)} \end{bmatrix}$$

$$A^s = \begin{bmatrix} \cos(\omega_n T) & \dfrac{\sin(\omega_n T)}{\omega_n} \\ -\omega_n \sin(\omega_n T) & \cos(\omega_n T) \end{bmatrix}$$

$$B^s = \int_0^T e^{A\tau} B \, d\tau = \int_0^T \begin{bmatrix} \dfrac{\sin(\omega_n T)}{\omega_n} \\ \cos(\omega_n T) \end{bmatrix} d\tau = \begin{bmatrix} \dfrac{1 - \cos(\omega_n T)}{\omega_n^2} \\ \dfrac{\sin(\omega_n T)}{\omega_n} \end{bmatrix}$$

Method 2
We let the function of a matrix be, $f(A) = \alpha_1 I + \alpha_2 A + \alpha_2 A^2 + \cdots + \alpha_n A^{n-1}$
Such that

$$A^s = e^{AT} = \alpha_1 I + \alpha_2 A$$

Find the eigenvalues of matrix A

$$\begin{vmatrix} -\lambda & T \\ -\omega_n^2 T & -\lambda \end{vmatrix} = \lambda^2 + \omega_n^2 T^2 = 0$$

$$\lambda_{1,2} = \pm i\omega_n T$$

Also $f(\lambda) = e^{\lambda} = \alpha_1 + \alpha_2 \lambda$

$$f(\lambda_1) = f(i\omega_n T) = e^{i\omega_n T} = \alpha_1 + \alpha_2 i\omega_n T \qquad (1)$$

$$f(\lambda_2) = f(-i\omega_n T) = e^{-i\omega_n T} = \alpha_1 - \alpha_2 i\omega_n T \qquad (2)$$

But

$$e^{i\theta} = \cos(\theta) + isin(\theta)$$

Equation 1 plus 2,

$$e^{i\omega_n T} + e^{-i\omega_n T} = 2\alpha_1$$

$$\alpha_1 = \cos(\omega_n T)$$

Equation 1 minus 2,

$$e^{i\omega_n T} - e^{-i\omega_n T} = 2\alpha_2 i\omega_n T$$

$$\alpha_2 = \frac{\sin(\omega_n T)}{\omega_n T}$$

Therefore,

$$e^{AT} = \cos(\omega_n T)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(\omega_n T)}{\omega_n T}\begin{pmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{pmatrix} T$$

$$\begin{bmatrix} \cos(\omega_n T) & \dfrac{\sin(\omega_n T)}{\omega_n} \\ -\omega_n \sin(\omega_n T) & \cos(\omega_n T) \end{bmatrix} = A^s$$

$$B^s = \int_0^T e^{A\tau} B\, d\tau = \int_0^T \begin{bmatrix} \dfrac{\sin(\omega_n T)}{\omega_n} \\ \cos(\omega_n T) \end{bmatrix} d\tau = \begin{bmatrix} \dfrac{1 - \cos(\omega_n T)}{\omega_n^2} \\ \dfrac{\sin(\omega_n T)}{\omega_n} \end{bmatrix}$$

Solution to part b

We have

$$S_N = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, R = r$$

The control is

$$u_k = -K_k x_k$$

But

$$K_k = [k_1 \quad k_2] = (B^T S_{k+1} A)\delta^{-1}$$

Assume $S = \begin{bmatrix} s_a & s_b \\ s_b & s_c \end{bmatrix}$ a symmetric matrix.

$$\delta = B^T S_{k+1} B + R = \begin{bmatrix} \dfrac{1 - \cos(\omega_n T)}{\omega_n^2} \\ \dfrac{\sin(\omega_n T)}{\omega_n} \end{bmatrix}^T \begin{bmatrix} s_a & s_b \\ s_b & s_c \end{bmatrix} \begin{bmatrix} \dfrac{1 - \cos(\omega_n T)}{\omega_n^2} \\ \dfrac{\sin(\omega_n T)}{\omega_n} \end{bmatrix} + r$$

$$\delta = \frac{s_a(1 - \cos(\omega_n T))^2}{\omega_n^4} + s_c \frac{\sin^2(\omega_n T)}{\omega_n^2} + \frac{2s_b(1 - \cos(\omega_n T))\sin(\omega_n T)}{w_n^3} + r$$

$$K_k = [k_1 \quad k_2] = (B^T S_{k+1} A)\delta^{-1}$$

$$k_1 = \frac{\cos(\omega_n T)\left(s_a \frac{1-\cos(\omega_n T)}{\omega_n^2} + s_b \frac{\sin(\omega_n T)}{\omega_n}\right) - \omega_n \sin(\omega_n T)\left(s_b \frac{1-\cos(\omega_n T)}{\omega_n^2} + s_c \frac{\sin(\omega_n T)}{\omega_n}\right)}{\delta}$$

$$k_2 = \frac{\frac{\sin(\omega_n T)}{\omega_n}\left(s_a \frac{1-\cos(\omega_n T)}{\omega_n^2} + s_b \frac{\sin(\omega_n T)}{\omega_n}\right) + \cos(\omega_n T)\left(s_b \frac{1-\cos(\omega_n T)}{\omega_n^2} + s_c \frac{\sin(\omega_n T)}{\omega_n}\right)}{\delta}$$

Such that $K = [k_1\, k_2]$

Substitute this into the plant to get the closed loop system matrix

$$A_k^{cl} = A - BK_k = \begin{bmatrix} a_{11}^{cl} & a_{12}^{cl} \\ a_{21}^{cl} & a_{22}^{cl} \end{bmatrix} = \begin{bmatrix} \cos(\omega_n T) - k_1 \frac{1-\cos(\omega_n T)}{\omega_n^2} & \frac{\sin(\omega_n T)}{\omega_n} - k_2 \frac{1-\cos(\omega_n T)}{\omega_n^2} \\ -\omega_n \sin(\omega_n T) - k_1 \frac{\sin(\omega_n T)}{\omega_n} & \cos(\omega_n T) - k_2 \frac{\sin(\omega_n T)}{\omega_n} \end{bmatrix}$$

$$S_k = A_{cl}^T S_{k+1} A_k^{cl} + Q = \begin{bmatrix} s_a & s_b \\ s_b & s_c \end{bmatrix}$$

$$\begin{aligned} s_a = q_1 - &\left(s_b \sin(\omega_n T)\left(\omega_n + \frac{k_1}{\omega_n}\right) - s_a\left(\cos(\omega_n T) - \frac{k_1(1-\cos(\omega_n T))}{\omega_n^2}\right)\right)\Bigg(\cos(\omega_n T) \\ &- \frac{k_1(1-\cos(\omega_n T))}{\omega_n^2}\Bigg) \\ &+ \left(s_c \sin(\omega_n T)\left(\omega_n + \frac{k_1}{\omega_n}\right) - s_b\left(\cos(\omega_n T) - \frac{k_1(1-\cos(\omega_n T))}{\omega_n^2}\right)\right)\Bigg(\omega_n \sin(\omega_n T) \\ &+ \frac{k_1 \sin(\omega_n T)}{\omega_n}\Bigg) \end{aligned}$$

$$\begin{aligned} s_b = -&\left(\frac{\sin(\omega_n T)}{\omega_n} - \frac{k_2(1-\cos(\omega_n T))}{\omega_n^2}\right)\left(s_b \sin(\omega_n T)\left(\omega_n + \frac{k_1}{\omega_n}\right) - s_a\left(\cos(\omega_n T) - \frac{k_1(1-\cos(\omega_n T))}{\omega_n^2}\right)\right) \\ &- \left(s_c \sin(\omega_n T)\left(\omega_n + \frac{k_1}{\omega_n}\right) - s_b\left(\cos(\omega_n T) - \frac{k_1(1-\cos(\omega_n T))}{\omega_n^2}\right)\right)\Bigg(\cos(\omega_n T) \\ &- \frac{k_2 \sin(\omega_n T)}{\omega_n^2}\Bigg) \end{aligned}$$

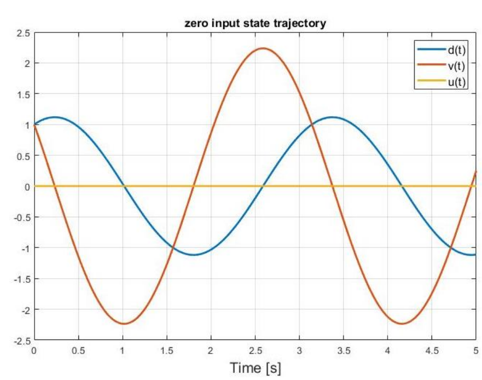
$$\begin{aligned} s_c = q_2 + &\left(\frac{\sin(\omega_n T)}{\omega_n} - \frac{k_2(1-\cos(\omega_n T))}{\omega_n^2}\right)\Bigg(s_b \sin(\omega_n T)\left(\omega_n + \frac{k_1}{\omega_n}\right) \\ &+ s_a\left(\cos(\omega_n T) - \frac{k_1(1-\cos(\omega_n T))}{\omega_n^2}\right)\Bigg) \\ &+ \left(\cos(\omega_n T) - \frac{k_2 \sin(\omega_n T)}{\omega_n}\right)\left(s_c \sin(\omega_n T)\left(\omega_n + \frac{k_1}{\omega_n}\right) \mp \left(\cos(\omega_n T) - \frac{k_1(1-\cos(\omega_n T))}{\omega_n^2}\right)\right) \end{aligned}$$

Solve for S backwards in time and K backwards in time.

$$u_k = -K_k x_k$$

Solution to part c

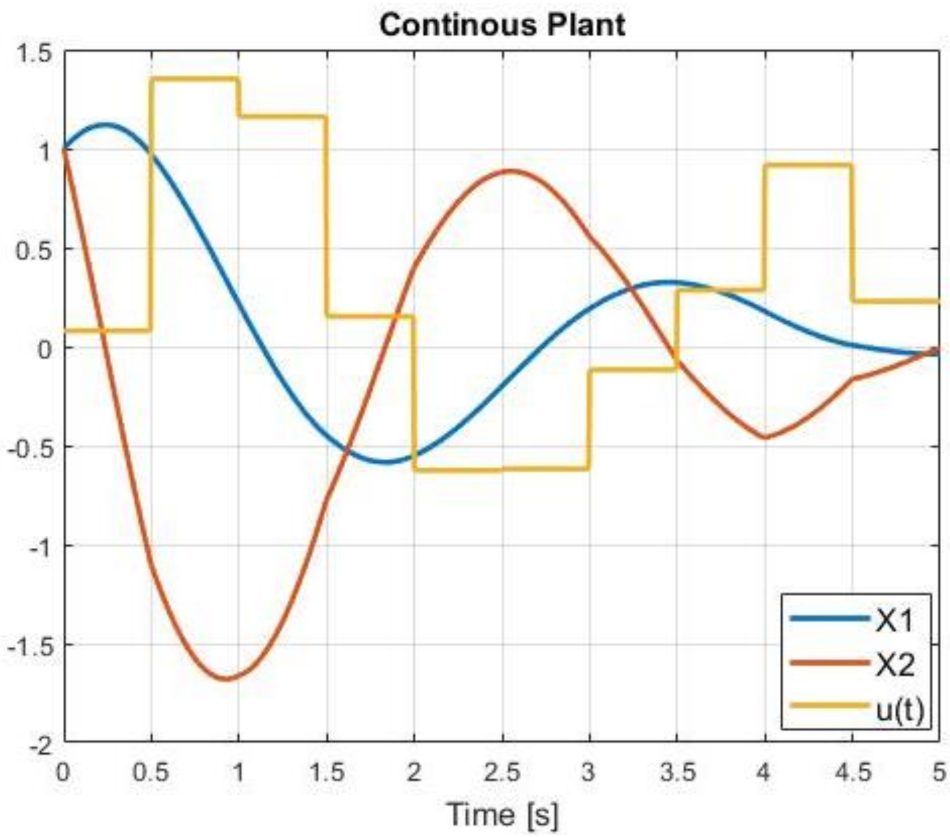
```
clear all
close all
clc

t=0:0.005:5;                    % Time length
U1=zeros(length(t),1);          % Input signal. Zero for homogenous
solution/zero-input

% Let
omega_n=2;          % The natural frequency
d0=1;               % initial position
v0=1;               % initial speed
x0=[d0,v0];

% The plant:
% of the form xdot = Ax + Bu with output y = Cx + Du
A=[0 1;
   -omega_n^2 0];

B=[0;
   1];

C=eye(2);                   % Output the two states x1 and x2
D=zeros(2,1);               % Output is not a function of u

system=ss(A,B,C,D);         % The state space system
[Y,t,X]=lsim(system,U1,t,x0);     % Run simulation

%% Plot
figure(1)
plot(t,Y,'linewidth',2)
hold on
plot(t,U1,'linewidth',2)
xlabel('Time [s]','fontsize',15);
legend('d(t)','v(t)','u(t)','fontsize',12);
title('zero input state trajectory','fontsize',12)
grid on
```

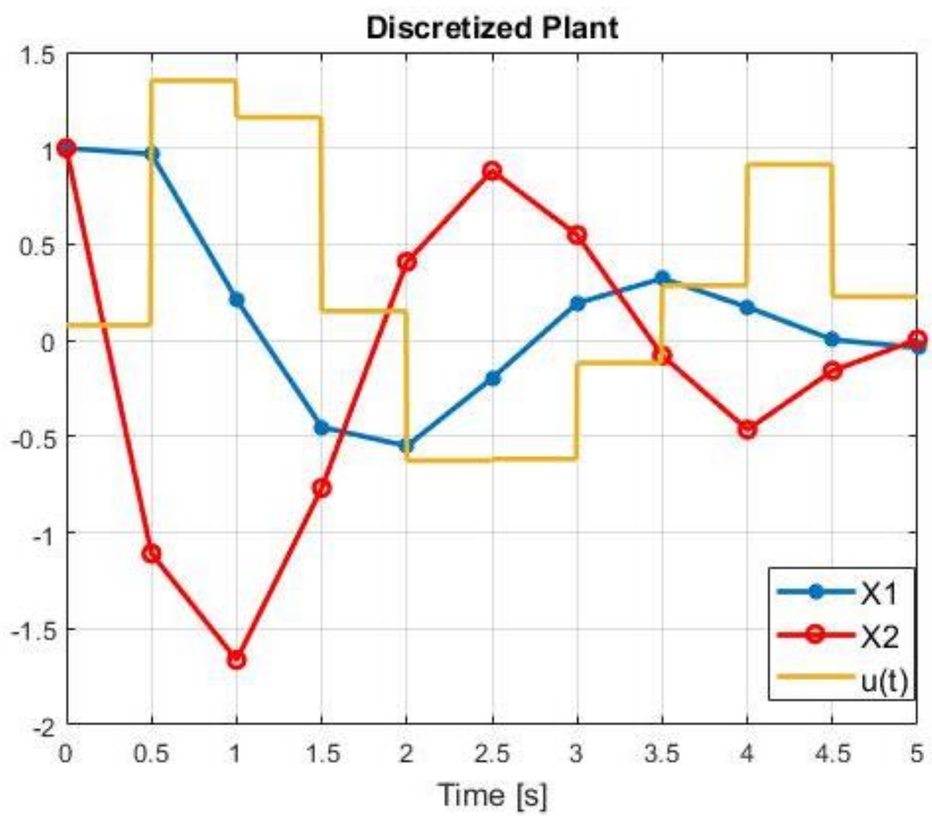


Solution to part d

```
clear all
close all
clc

%Input parameters
t1=0:0.5:5;          % Time length
N=10;                % Number of steps
T=0.5;               % Sampling period

% The Plant variables
a=1; b=1; wn=2;

% Performance weights
% Final value weithing matrix
sd=100; sv=100;
sN=[sd,0;0,sv];

% Control weight
r=1;

% Transient state weight
qd=1; qv=1;
Q=[qd,0;0,qv];


%% Discretized system
As=[cos(wn*T) sin(wn*T)/wn;
    -wn*sin(wn*T) cos(wn*T)];

Bs=[(1-cos(wn*T))/wn^2;
    sin(wn*T)/wn];

% Let
d0=1;      %initial position
v0=1;      %initial speed
x0=[d0,v0];

% calculate Kalman gain Kk and cost kernel Sk in backward
iteration
k=N;
Kk=zeros(N,2);
%Sk=zeros(N+1,2,2);
Sk=sN;

while k>0
    Kk(k,:)=(Bs'*Sk*Bs+r)^(-1)*(Bs'*Sk*As);
    Sk = As'*Sk*(As-Bs*Kk(k,:))+Q;
    k=k-1;
end
```

```
%Find optimal control u(k) (Forward iteration)
x(:,1)=x0;
for k=1:N
    u(k)=-Kk(k,:)*x(:,k);
    x(:,k+1)=As*x(:,k)+Bs*u(k);
end

% Simulate the continous time system with digital optimal
control
ts = .005;
t=0:ts:5;

% Expand the input u to the specified interval T
 U=kron(u,ones(1,ceil(T/ts)));

 U=[U u(end)];

% Continuous plant
A=[0,1;
   -1*wn^2,0];
B=[0;
   1];

C=eye(2);
D=zeros(2,1);

system=ss(A,B,C,D);
[Y,t,X]=lsim(system,U,t,x0);

%%
figure(1)    % Continuous time
plot(t,Y,'linewidth',2)
hold on
plot(t,U,'linewidth',2)
xlabel('Time [s]','fontsize',12);
legend('X1','X2','u(t)','fontsize',12);
grid on
title('Continous Plant','fontsize',12)

%%
figure(3) % Descretized optimal control
plot(t1,x(1,:),'-*','linewidth',2)
hold on
plot(t1,x(2,:),'r-o','linewidth',2);
plot(t,U,'-','linewidth',2)
xlabel('Time [s]','fontsize',12);
legend('X1','X2','u(t)','fontsize',12);
grid on
title('Discretized Plant','fontsize',12)
```


Continous Plant
X1
X2
u(t)
Time [s]


Discretized Plant
X1
X2
u(t)
Time [s]

2.3-2. **Digital control of an unstable system.** **[3 pts]** Repeat the previous problem (2.3-1) for

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = a^2 x_1 + bu$$

a. Discretize the plant using a sampling period of *T*.
b. With the discretized plant, associate a performance index of

$$J = \frac{1}{2}[s_1(x_N^1)^2 + s_2(x_N^2)^2] + \frac{1}{2}\sum_{k=0}^{N-1}[q_1(x_k^1)^2 + q_2(x_k^2)^2 + ru_k^2],$$

where the state is $x_k = [x_k^1 \; x_k^2]^T$. Write scalar equations for a digital optimal controller.
c. Write a MATLAB subroutine to simulate the plant dynamics, and use the time response program *lsim.m* to obtain zero-input state trajectories.
d. Write a MATLAB subroutine to compute and store the optimal control gains and to update the control u_k given the current state x_k. Write a MATLAB driver program to obtain time response plots for the optimal controller.

Solution to part a:
Discretize the plant (using one of the two methods listed in 2.3-1)

$$(SI - A)^{-1} = \begin{bmatrix} \dfrac{s}{s^2 - (a^2)} & \dfrac{1}{s^2 - (a^2)} \\ \dfrac{a^2}{s^2 - (a^2)} & \dfrac{s}{s^2 - (a^2)} \end{bmatrix}$$

$$A^s = \begin{bmatrix} \cosh(aT) & \dfrac{\sinh(aT)}{a} \\ a\sinh(aT) & \cosh(aT) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}(e^{aT} + e^{-aT}) & \dfrac{1}{2a}(e^{aT} - e^{-aT}) \\ \dfrac{a}{2}(e^{aT} - e^{-aT}) & \dfrac{1}{2}(e^{aT} + e^{-aT}) \end{bmatrix}$$

$$B^s = \begin{bmatrix} b\left(\dfrac{-1 + \cosh(aT)}{a^2}\right) \\ \dfrac{b\sinh(aT)}{a} \end{bmatrix} = \begin{bmatrix} \dfrac{b}{a^2}\left(-1 + \dfrac{1}{2}(e^{aT} + e^{-aT})\right) \\ \dfrac{b}{2a}(e^{aT} - e^{-aT}) \end{bmatrix}$$

Solution to part b:
We have

$$S_N = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, R = r$$

The control is

$$u_k = -K_k x_k$$

But

$$K_k = [k_1 \quad k_2] = (B^T S_{k+1} A)\delta^{-1}$$

Assume $S = \begin{bmatrix} s_a & s_b \\ s_b & s_c \end{bmatrix}$ a symmetric matrix.

$$\delta = B^T S_{k+1} B + R = r + \frac{s_a b^2}{a^2}(\cosh(aT) - 1)^2 + \frac{s_c b^2}{a^2}\sinh^2(aT) + \frac{2s_b b^2}{a^3}\sinh(aT)\,(\cosh(aT) - 1)$$

$$K_k = [k_1 \quad k_2] = (B^T S_{k+1} A)\delta^{-1}$$

$$k_1 = \frac{\text{b Sinh[t }\alpha] \left(2 \text{ (sb }\alpha + \text{sa Coth[t }\alpha]) \text{ Sinh}\left[\frac{T\alpha}{2}\right]^2 + \alpha \text{ (sc }\alpha + \text{sb Coth[t }\alpha]) \text{ Sinh[T }\alpha]\right)}{\alpha^2 \delta}$$

$$k_2 = \frac{\text{b Sinh[t }\alpha] \left(2 \text{ (sa} + \text{sb }\alpha \text{ Coth[t }\alpha]) \text{ Sinh}\left[\frac{T\alpha}{2}\right]^2 + \alpha \text{ (sb} + \text{sc }\alpha \text{ Coth[t }\alpha]) \text{ Sinh[T }\alpha]\right)}{\alpha^3 \delta}$$

Such that $K = [k_1\, k_2]$

Substitute this into the plant to get the closed loop system matrix

trixForm=

$$A_k^{cl} = A - BK_k = \begin{bmatrix} a_{11}^{cl} & a_{12}^{cl} \\ a_{21}^{cl} & a_{22}^{cl} \end{bmatrix} = \begin{pmatrix} \text{Cosh[t }\alpha] - \frac{\text{b K1 (}-1+\text{Cosh[T }\alpha])}{\alpha^2} & \frac{\text{b K2} - \text{b K2 Cosh[T }\alpha] + \alpha \text{ Sinh[t }\alpha]}{\alpha^2} \\ \alpha \text{ Sinh[t }\alpha] - \frac{\text{b K1 Sinh[T }\alpha]}{\alpha} & \text{Cosh[t }\alpha] - \frac{\text{b K2 Sinh[T }\alpha]}{\alpha} \end{pmatrix}$$

$$S_k = A^T S_{k+1} A_k^{cl} + Q = \left\{\left\{\text{sa} \to \frac{\text{q1}}{1 - \text{a11 Cosh[t }\alpha]}, \text{ sb} \to 0, \text{ sc} \to \frac{\text{q2}}{1 - \text{a22 Cosh[t }\alpha]}\right\}\right\}$$

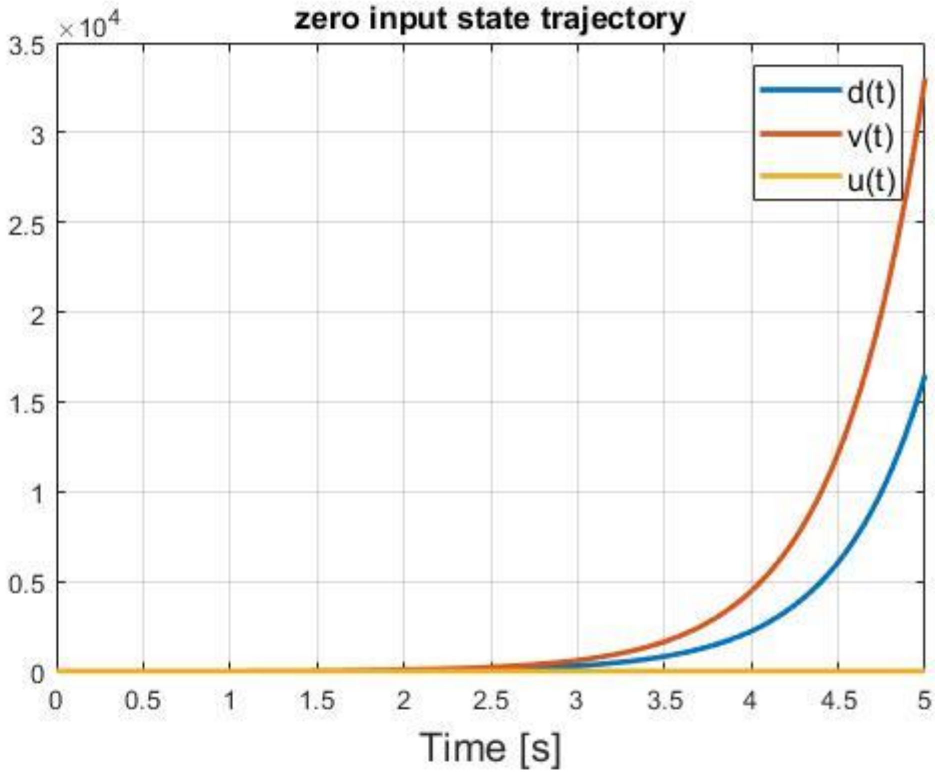
Solve for S backwards in time and S backwards in time.

$$u_k = -K_k x_k$$

Solution to part c:

```
clear all
close all
clc

t=0:0.005:5;                % Time length
U1=zeros(length(t),1);      % Input signal. Zero for homogenous
solution/zero-input

% Let
a = 2; b = 1;
d0 = 1;           % initial position
v0 = 1;           % initial speed
x0 = [d0,v0];

% The plant:
% of the form xdot = Ax + Bu with output y = Cx + Du
A=[0 1;
   a^2 0];

B=[0;
   b];

C=eye(2);                 % Output the two states x1 and x2
D=zeros(2,1);             % Output is not a function of u

system=ss(A,B,C,D);       % The state space system
[Y,t,X]=lsim(system,U1,t,x0);     % Run simulation
```

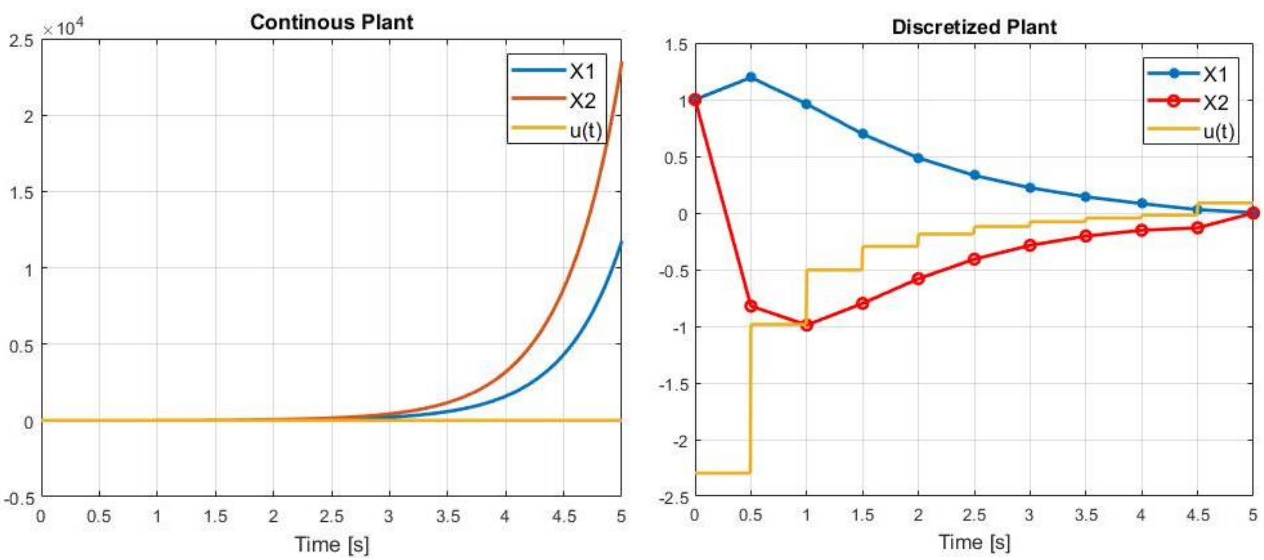
```
%% Plot
figure(1)
plot(t,Y,'linewidth',2)
hold on
plot(t,U1,'linewidth',2)
xlabel('Time [s]','fontsize',15);
legend('d(t)','v(t)','u(t)','fontsize',12);
title('zero input state trajectory','fontsize',12)
grid on
```



Solution to part d:

```
clear all
clc

N=100;
a=2;
b=1;
T=0.05;
s1=100;
s2=100;
sN=[s1,0;0,s2];
r=0.03;
q1=50;
q2=50;
Q=[q1,0;0,q2];
As=[cosh(a*T),sinh(a*T)/a;a*sinh(a*T),cosh(a*T)];
Bs=[(b*cosh(a*T)-b)/a^2,b*sinh(a*T)/a]';
```

```
x0=[1,1];
k=N;
Kk=zeros(N,2);

Sk=sN;
while k>0
    Kk(k,:)=(Bs'*Sk*Bs+r)^(-1)*(Bs'*Sk*As);
    Sk=As'*Sk*(eye(2)+Bs*Bs'*Sk/r)^(-1)*As+Q;
    k=k-1;
end

x(:,1)=x0;
for k=1:N
    u(k)=-Kk(k,:)*x(:,k);
    x(:,k+1)=As*x(:,k)+Bs*u(k);
end
t=0:0.005:5;
%Expand the input u to the specified interval T
U=kron(u,ones(1,10));
U=[U u(length(u))]';

A=[0,1;a^2,0];
B=[0,b]';
C=eye(2);
D=zeros(2,1);
system=ss(A,B,C,D);
[Y,t,X]=lsim(system,U,t,x0);

figure(4)
plot(t,Y)
hold on
plot(t,U)
xlabel('Time [s]');
legend('d(t)','v(t)','u(t)');
grid on
```



2.4-1. **Steady-state behavior. [2 pts]** In this problem we consider a rather unrealistic discrete system because it is simple enough to allow an analytic treatment. Thus, let the plant

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

have performance index of

$$J_0 = \frac{1}{2} x_N^T x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k^T \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix} x_k + r u_k^2)$$

a. Find the optimal steady-state (i.e., $N \to \infty$) Riccati solution s_∞^* and show that it is positive definite. Find the optimal steady-state gain K_∞^* and determine when it is nonzero.
b. Find the optimal steady-state closed-loop plant and demonstrate its stability.
c. Now the suboptimal constant feedback

$$u_k = -K_\infty^* x_k$$

is applied to the plant. Find scalar updates for the components of the suboptimal cost kernel S_k. Find the suboptimal steady-state cost kernel S_∞ and demonstrate that $S_\infty = S_\infty^*$.

Solution to part a

- Find optimal steady state S_∞^*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} q1 & q2 \\ q2 & q1 \end{bmatrix}, \qquad R = r$$

$$S_\infty^* = A^T [S_\infty^* - S_\infty^* B (B^T S_\infty^* B + R)^{-1} B^T S_\infty^*] A + Q$$

$$S_\infty^* = \begin{bmatrix} q_1 & q_2 \\ q_2 & -\dfrac{s_2^2}{r + s_3} + q_1 + s_1 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \\ q_2 & -\dfrac{q_2^2}{r + s_3} + 2q_1 \end{bmatrix}$$

$$S_\infty^* = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 - \dfrac{r}{2} \pm \sqrt{\left(q_1 + \dfrac{r}{2}\right)^2 - q_2^2} \end{bmatrix}$$

- Show that S_∞^* is positive definite

  Start with $Q > 0$,
  $\Rightarrow q_1^2 > q_2^2$ and $q_1 > 0$ since $Q > 0$.
  Next $R > 0$,
  $\Rightarrow r > 0$ since $R > 0$.
  We have $q_1 r > 0$ as a result from above.
  Now check the determinant of S_∞^*

$$\det(S_\infty^*) = q_1^2 - \frac{q_1 r}{2} \pm q_1 \sqrt{\left(q_1 + \frac{r}{2}\right)^2 - q_2^2} - q_2^2$$

  Inside the square root term, $q_1^2 + q_1 r + \frac{r^2}{4} - q_2^2 > 0$ since $q_1^2 > q_2^2$.
  Also $q_1^2 \left(q_1^2 + q_1 r + \frac{r^2}{4}\right) > q_1^2 q_2^2$ since q_1^2 is multiplied on both sides. Note that the square root term is positive.

Then $q_1\sqrt{\left(q_1+\frac{r}{2}\right)^2-q_2^2} > \frac{q_1 r}{2}$ and $q_1^2-\frac{q_1 r}{2}+q_1\sqrt{\left(q_1+\frac{r}{2}\right)^2-q_2^2}-q_2^2>0$
Since $q_1^2 > q_2^2 \Rightarrow |S_\infty^*| > 0$ and $S_\infty^* > 0$

- Find K_∞^* and determine when it is non zero.

$$K_\infty^* = (B^T S_\infty^* B + R)^{-1} B^T S_\infty^* A = \begin{bmatrix} 0 & \dfrac{q_2}{2q_1 + r - \dfrac{q_2^2}{r+s_3}} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{q_2}{r+s_3} \end{bmatrix}$$

K_∞^* is non zero when $q_2 \neq 0$.

Solution to part b:

$$A_{cl} = A - BK_\infty^* = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{2q_2}{2q_1 - r + \sqrt{4q_1^2 - 4q_2^2 + 4q_1 r + r^2}} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{q_2}{r+s_3} \end{bmatrix}$$

Eigenvalues / poles at 0 and $\frac{-2q_2}{2q_1 - r + \sqrt{4q_1^2-4q_2^2+4q_1 r + r^2}}$, both poles inside unit circle ($|z| < 1$) hence stable.

Solution to part c:
For suboptimal S_k
Recall $K_\infty^* = \begin{bmatrix} 0 & \frac{q_2}{s_3} \end{bmatrix}$

$$A_\infty^{cl} = A - BK_\infty^* = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{2q_2}{2q_1 - r + \sqrt{4q_1^2 - 4q_2^2 + 4q_1 r + r^2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{q_2}{r+s_3} \end{bmatrix}$$

$$S_k = {A_\infty^{cl}}^T S_{k+1} A_\infty^{cl} + K_\infty^T R K_\infty + Q$$

With $S_N = I$ the identity matrix.

$$S_k = \begin{bmatrix} s_1^k & s_2^k \\ s_2^k & s_3^k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{q_2}{r+s_3} \end{bmatrix}^T \begin{bmatrix} s_1^{k+1} & s_2^{k+1} \\ s_2^{k+1} & s_3^{k+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{q_2}{r+s_3} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{rq_2^2}{{s_3}^2} \end{bmatrix} + \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}$$
$$= \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 + s_1^{k+1} - \dfrac{q_2 s_2^{k+1}}{r+s_3} + \dfrac{-q_2 s_2^{k+1}(r+s_3) + q_2^2 s_3^{k+1} + q_2^2 r}{(r+s_3)^2} \end{bmatrix}$$

We have

$$s_1^k = s_1^{k+1} = q_1$$
$$s_2^k = s_2^{k+1} = q_2$$

Therefore

$$S_k = \begin{bmatrix} q_1 & q_2 \\ q_2 & 2q_1 - \dfrac{q_2^2}{r+s_3} - \dfrac{q_2^2\left(s_3 - s_3^{k+1}\right)}{(r+s_3)^2} \end{bmatrix}$$

For steady-state S_∞, $s_3^{k+1} = s_3$

$$S_\infty = \begin{bmatrix} q_1 & q_2 \\ q_2 & 2q_1 - \dfrac{q_2^2}{r+s_3} \end{bmatrix} = S_\infty^*$$

2.4-2. **Analytic Riccati solution.** **[2 pts]** Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, S_N = I, Q = I$$

a. Let $r = 0.1$. Find the Hamiltonian matrix H and its eigenvalues and eigenvectors. Find the analytic expression for Riccati solution S_k. Find the steady-state solution S_∞ using (2.4-42). Find the optimal steady-state gain K_∞ using (2.4-63) and also using Ackermann's formula.
b. Let $r = 1$. Find the Hamiltonian matrix and its eigenstructure. Find the steadystate solution S_∞ and gain K_∞. (*Hint*: See the discussion following (2.4-63).)

Solution to part a:

- Find Hamiltonian matrix H

$$H = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^T \\ QA^{-1} & A^T + QA^{-1}BR^{-1}B^T \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & -10 \\ 0 & 1 & 0 & 10 \\ 1 & -1 & 1 & -10 \\ 0 & 1 & 1 & 11 \end{bmatrix}$$

- Find eigenvalue and eigenvector of H

$$\lambda = [10.7802 \quad 2.7654 \quad 0.3616 \quad 0.0928]$$

$$M = \begin{bmatrix} 10.7802 & 0 \\ 0 & 2.7654 \end{bmatrix}, \quad \textit{unstable eigenvalues}$$

$$M^{-1} = \begin{bmatrix} 0.3616 & 0 \\ 0 & 0.0928 \end{bmatrix}, \quad \textit{stable eigenvalues}$$

$$W = \begin{bmatrix} -0.5113 & -0.5081 & 0.4740 & -0.1013 \\ 0.4639 & 0.3243 & 0.8369 & -0.9907 \\ -0.5636 & -0.7959 & -0.2685 & 0.0104 \\ 0.4537 & 0.0573 & -0.0534 & 0.0899 \end{bmatrix}$$

- Find the analytic expression for Riccati solution S_k

$$S_k = (W_{21} + W_{22}T_k)(W_{11} + W_{12}T_k)^{-1}$$

$$T = -(W_{22} - S_N W_{12})^{-1}(W_{21} - S_N W_{11}) = \begin{bmatrix} -0.0787 & -0.4 \\ -0.0554 & -0.0824 \end{bmatrix}$$

$$T_k = M^{-(N-k)} T M^{-(N-k)}$$

$$S_k = (W_{21} + W_{22}T_k)(W_{11} + W_{12}T_k)^{-1}$$

- Find the steady-state solution S_∞

$$S_\infty = (W_{21} W_{11}^{-1}) = \begin{bmatrix} 2.6687 & 1.7266 \\ 1.7266 & 2.8812 \end{bmatrix}$$

- Find the optimal steady state gain K_∞

$$K_\infty = R^{-1}B^T\Lambda \mathrm{M}\mathrm{X}^{-1} = [\,0.5792 \quad 1.5456]$$

$$X = W_{11} = \begin{bmatrix} -0.5113 & -0.5081 \\ 0.4639 & 0.3243 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.0928 & 0 \\ 0 & 0.3616 \end{bmatrix}$$

$$\Lambda = W_{21} = \begin{bmatrix} -0.5636 & -0.7959 \\ 0.4537 & 0.0573 \end{bmatrix}$$

- Find the optimal steady state gain K_∞ using Ackermann's formula

$$K_\infty = e_n^T U_n^{-1} \Delta^d(A) = [\,0.5792 \quad 1.5456]$$

$$e_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U_n = [B\ AB] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Delta^{cl}(\lambda) = (\lambda - 0.3616)(\lambda - 0.0928) = \lambda^2 - 0.4544\lambda + 0.0336$$

$$\Delta^{cl}(A) = A^2 - 0.4544A + 0.0336I = \begin{bmatrix} 0.5792 & 1.5456 \\ 0 & 0.5792 \end{bmatrix}$$

Solution to part b:

Find Hamiltonian matrix and Eigen structure

$$H = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\lambda = [2.122 \pm 1.0528i \quad 0.378 \pm 0.1877i]$$

$$W = \begin{bmatrix} 0.4374 + 0.1320i & 0.4374 - 0.1320i & 0.3330 + 0.3128i & 0.3330 - 0.3128i \\ -0.2472 - 0.1642i & -0.2472 + 0.1642i & 0.7032 & 0.7032 \\ 0.7032 & 0.7032 & -0.0186 - 0.2962i & -0.0186 + 0.2962i \\ -0.1044 - 0.4448i & -0.1044 + 0.4448i & -0.4374 + 0.1320i & -0.4374 - 0.1320i \end{bmatrix}$$

- Find the steady state solution S_∞ and K_∞

$$\begin{bmatrix} X \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0.4374 + 0.1320i \\ -0.2472 - 0.1642i \\ 0.7032 \\ -0.1044 - 0.4448i \end{bmatrix}$$

$$[Re(X_i) \quad Im(X_i)] = \begin{bmatrix} 0.4374 & 0.1320 \\ -0.2472 & -0.1642 \end{bmatrix}$$

$$[Re(\Lambda_i) \quad Im(\Lambda_i)] = \begin{bmatrix} 0.7032 & 0 \\ -0.1044 & -0.4448 \end{bmatrix}$$

$$\begin{bmatrix} Re(\mu_i) & Im(\mu_i) \\ -Im(\mu_i) & Re(\mu_i) \end{bmatrix} = \begin{bmatrix} 0.378 & -0.1877 \\ 0.1877 & 0.378 \end{bmatrix}$$

$$K_\infty = R^{-1}B^T[Re(\Lambda_i) \quad Im(\Lambda_i)] \begin{bmatrix} Re(\mu_i) & Im(\mu_i) \\ -Im(\mu_i) & Re(\mu_i) \end{bmatrix} [Re(X_i) \quad Im(X_i)]^{-1}$$

$$K_\infty = [0.4221 \quad 1.2439]$$

$$S_\infty = [Re(\Lambda_i) \quad Im(\Lambda_i)][Re(X_i) \quad Im(X_i)]^{-1} = \begin{bmatrix} 2.9471 & 2.3692 \\ 2.3692 & 4.6131 \end{bmatrix}$$