Convolutional Neural Networks

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Inner Products

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Vector Inner Product

- $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$.

- Inner product:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T \mathbf{b} = \sum_i a_i b_i = 70.$$

Matrix Inner Product

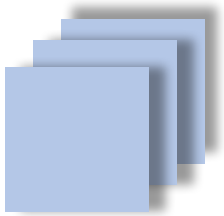
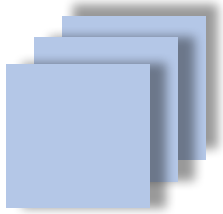
- $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$.

- Inner product:

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_i \sum_j a_{ij} b_{ij} = 70.$$

- Property: $\langle \mathbf{A}, \mathbf{B} \rangle = \langle \mathrm{vec}(\mathbf{A}), \mathrm{vec}(\mathbf{B}) \rangle$.

Tensor Inner Product

- $\mathbf{A} =$  and $\mathbf{B} =$  .

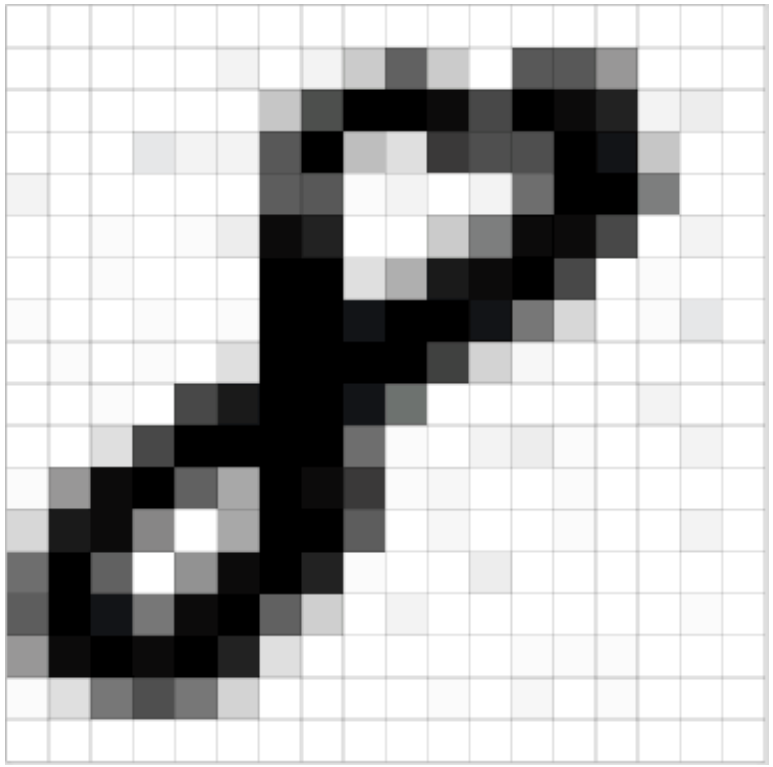
- Inner product:

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_i \sum_j \sum_k a_{ijk} b_{ijk} .$$

- Property: $\langle \mathbf{A}, \mathbf{B} \rangle = \langle \mathrm{vec}(\mathbf{A}), \mathrm{vec}(\mathbf{B}) \rangle$.

Convolution for Images

An Image is A Matrix/Tensor of Pixel Values



The MNIST Dataset

- Each grayscale image, $\mathbf{x}_j$, is a 28×28 matrix.

- (Previously, we reshape it to a 784-dim vector.)

Convolution

Input Image

5×5

1	1	1	0	0
0	1	1	1	0
0	0	1	1	1
0	0	1	1	0
0	1	1	0	0

Filter (Kernel)

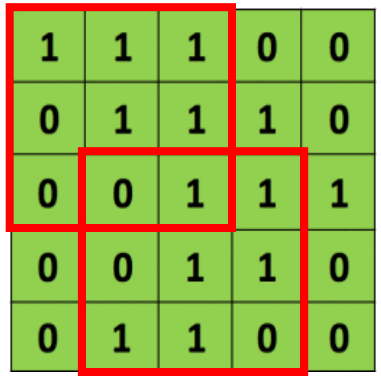
3×3

1	0	1
0	1	0
1	0	1

Convolution

Input Image

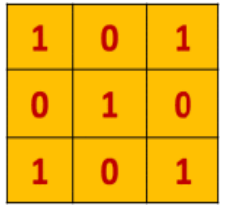
5×5



3×3 patch

Filter (Kernel)

3×3



Question: how many 3×3 patches does a 5×5 image have?

Answer: $(5-3+1)\times(5-3+1)=9.$

Convolution

Input Image

5×5

1	1	1	0	0
0	1	1	1	0
0	0	1	1	1
0	0	1	1	0
0	1	1	0	0

Filter (Kernel)

3×3

1	0	1
0	1	0
1	0	1

Convolution:

4	3	4
2	4	3
2	3	4

Result

3×3

Convolution

Input Image

5×5

1	1	1	0	0
0	1	1	1	0
0	0	1	1	1
0	0	1	1	0
0	1	1	0	0

Filter (Kernel)

3×3

1	0	1
0	1	0
1	0	1

Convolution:

4	3	4
2	4	3
2	3	4

Result

3×3

The value 4 is the inner product of the patch

1	1	1
0	1	1
0	0	1

and the filter

1	0	1
0	1	0
1	0	1

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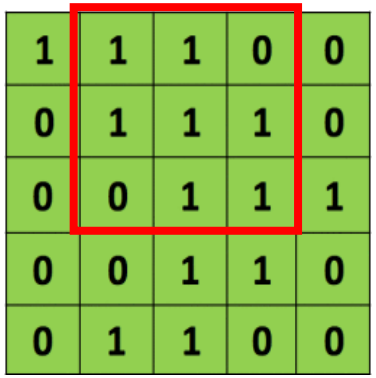
Convolution

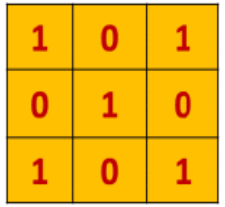
Input Image

5×5

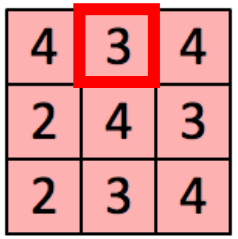


Filter (Kernel)

3×3



Convolution:



Result

3×3

The value 3 is the inner product of the patch

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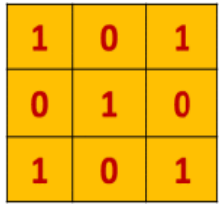
and the filter



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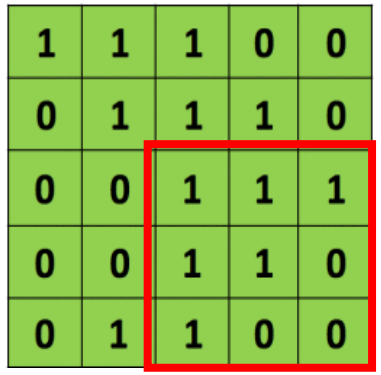
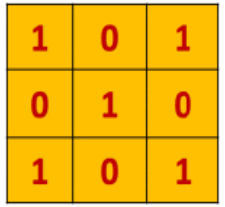
Convolution

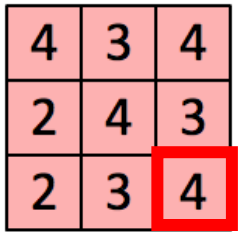
Input Image
5×5



Filter (Kernel)
3×3

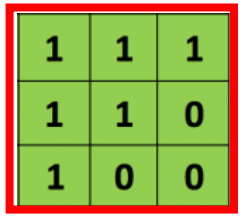


Convolution:

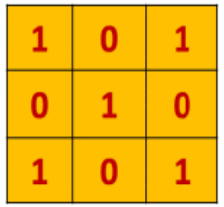


Result
3×3

The value **4** **is the inner product of the patch**



and the filter



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Convolution

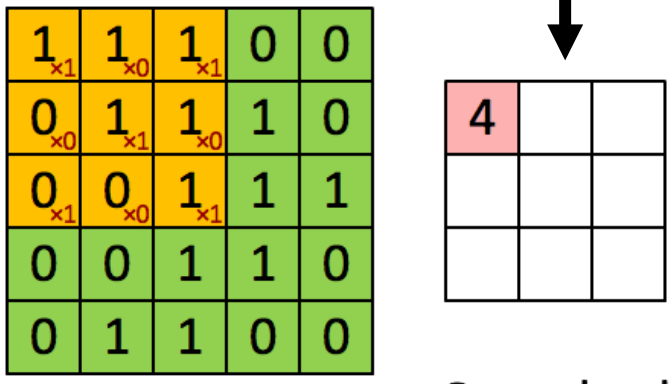
Input Image
5×5

1	1	1	0	0
0	1	1	1	0
0	0	1	1	1
0	0	1	1	0
0	1	1	0	0

Filter (Kernel)
3×3

1	0	1
0	1	0
1	0	1

Convolution:



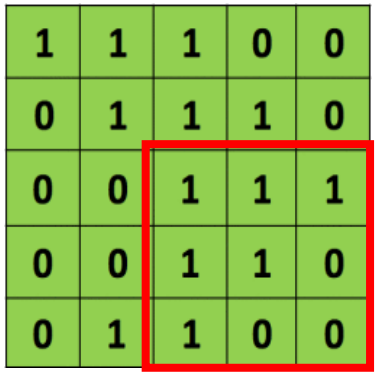
Image

Convolved Feature

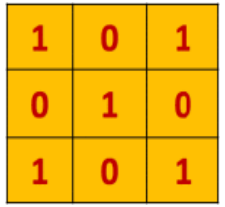
Result
3×3

Convolution

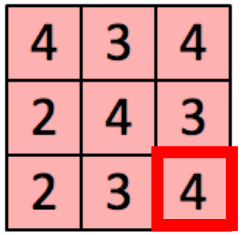
Input Image
5×5



Filter (Kernel)
3×3



Convolution:



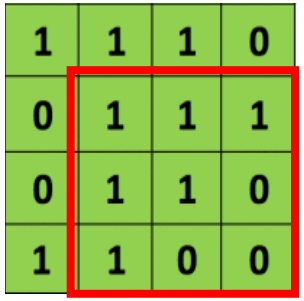
Result
3×3

Question: Why is the result a 3×3 **matrix?**

Convolution

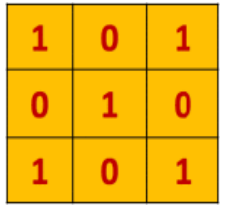
Input Image

4×4

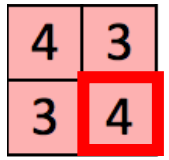


Filter (Kernel)

3×3



Convolution:



Result

2×2

Convolution

Input Image
5×5

1	1	1	0	0
0	1	1	1	0
0	0	1	1	1
0	0	1	1	0
0	1	1	0	0

Filter (Kernel)
3×3

1	0	1
0	1	0
1	0	1

Convolution:

4	3	4
2	4	3
2	3	4

Dimensions:

- Input: $d_1\times d_2$
- Filter: $k_1\times k_2$
- Output: $(d_1-k_1+1)\times(d_2-k_2+1)$

Convolution

Input Image

	Filters	**Outputs**
Identity 等式	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	

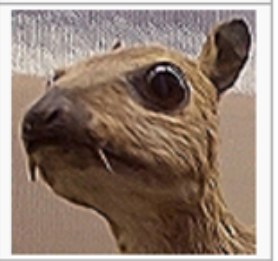
Convolution

Input Image		滤波器 Filters	Outputs
	Edge detection 边缘检测	$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	

- The inner product of $\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ is 0.
- The inner product of $\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ is 0.8.

Convolution

Input Image

	Filters	Outputs
Sharpen 锐化	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	

- The inner product of $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ is 0.1.
- The inner product of $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}$ is 0.6.

Convolution

Input Image

	Filters	**Outputs (or feature map)**
Box blur (normalized)	$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$	
Gaussian blur (approximation)	$\frac{1}{16}\begin{bmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{bmatrix}$	

Convolution

Convolution: Zero Padding

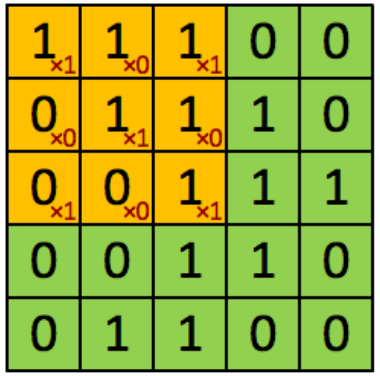
Input Image
5×5

1	1	1	0	0
0	1	1	1	0
0	0	1	1	1
0	0	1	1	0
0	1	1	0	0

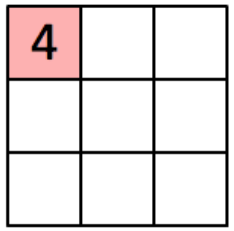
Filter (Kernel)
3×3

1	0	1
0	1	0
1	0	1

Convolution:



Image



Convolved Feature

Dimensions:

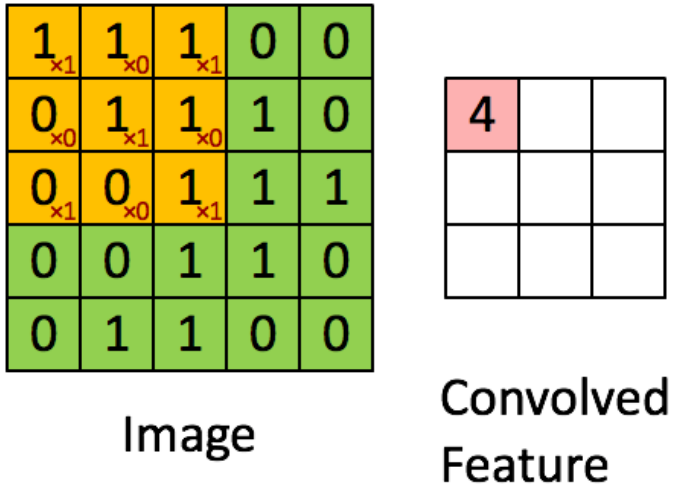
- Input: $d_1\times d_2$
- Filter: 3×3
- Output: $(d_1-2)\times(d_2-2)$

Convolution: Zero Padding

Problem: the output is smaller than the input!

- If the input is 28×28 and the filter is 3×3, you can have at most 13 convolutional layers. (The output of the 13th layer is 2×2.)

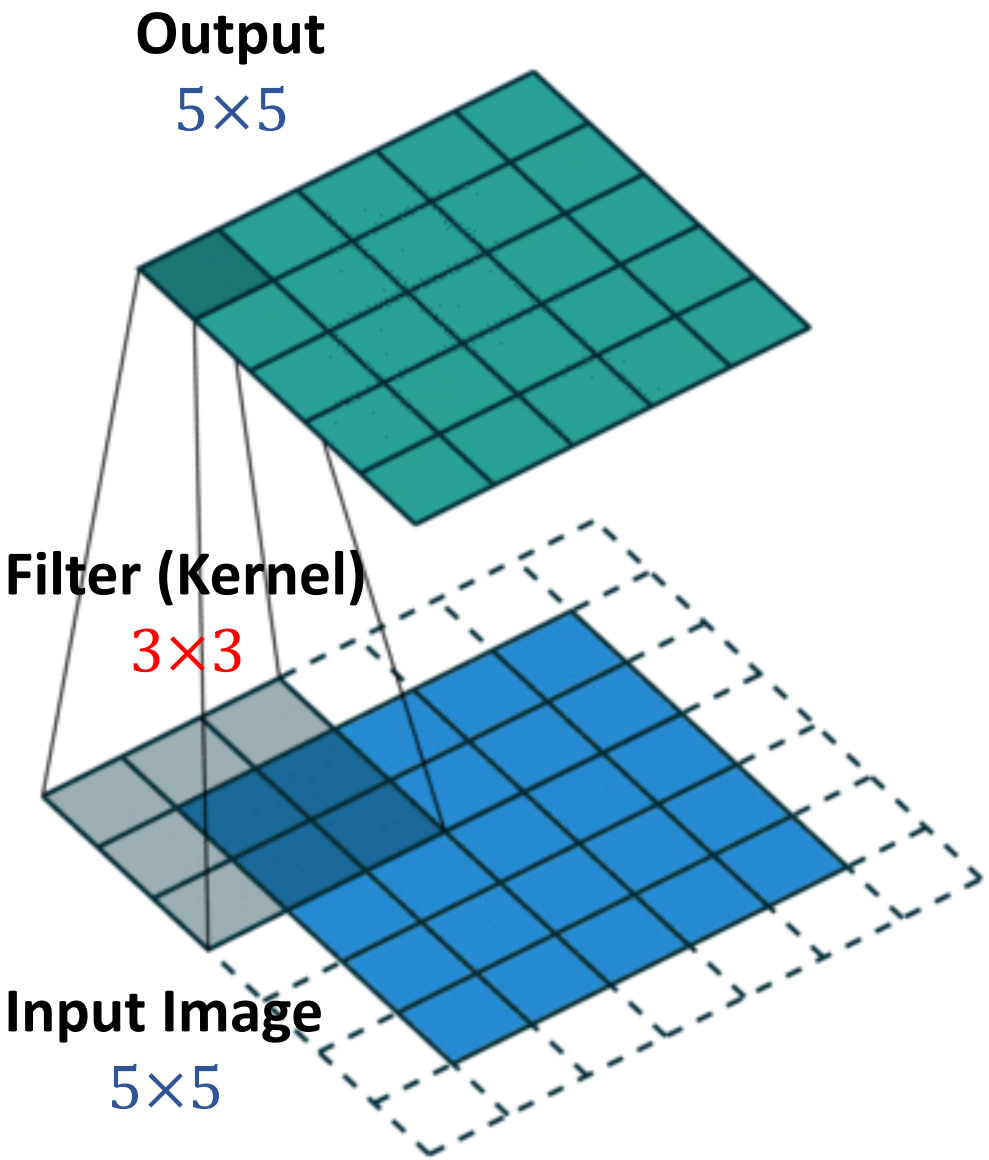
Convolution:



Dimensions:

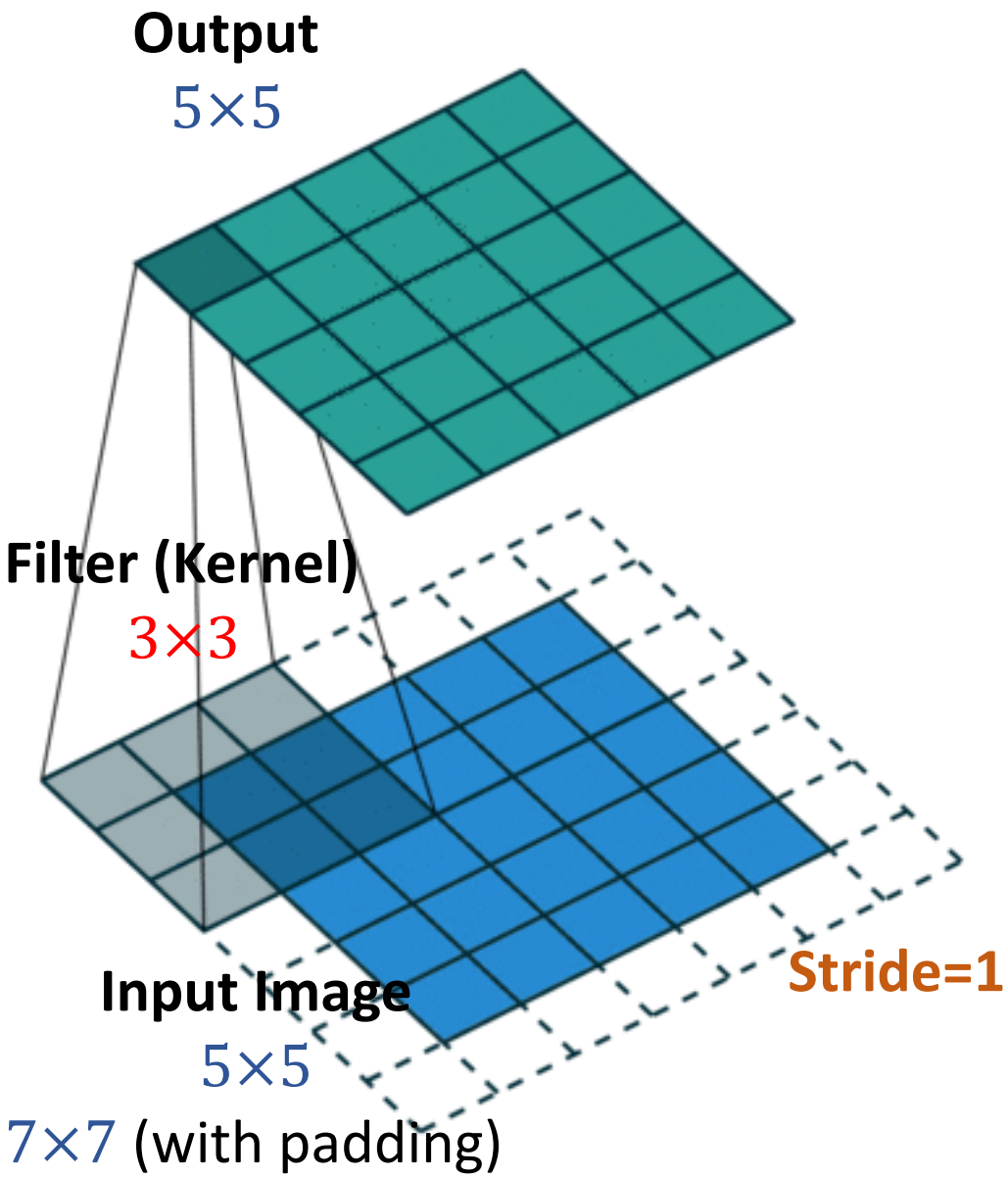
- Input: $d_1\times d_2$
- Filter: 3×3
- Output: $(d_1-2)\times(d_2-2)$

Convolution: Zero Padding



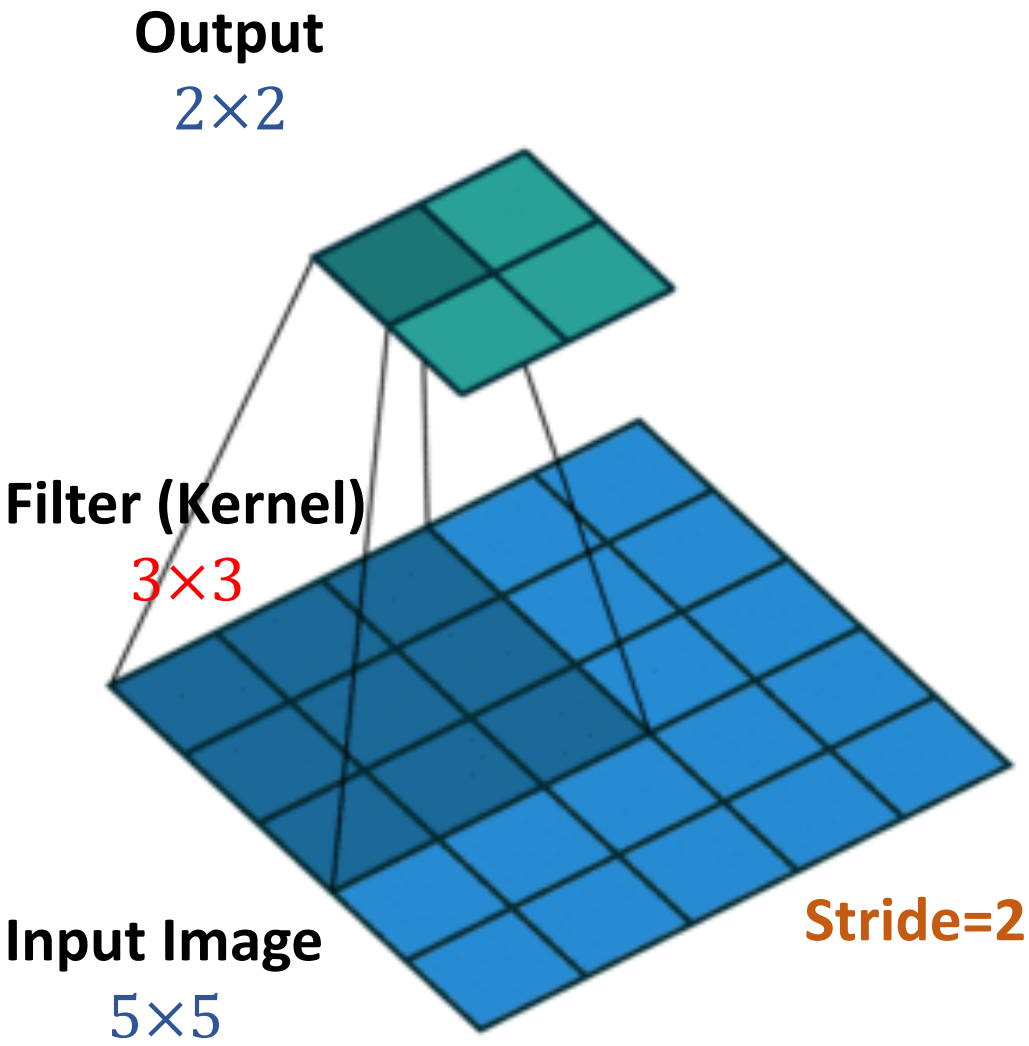
- Add a “boarder” of all-zeros.
- Increase the input shape:
  - From $d_1 \times d_2$ to $(d_1 + 2) \times (d_2 + 2)$.
- If the filter is 3×3, the output is $d_1 \times d_2$.

Convolution: Stride



- In the previous examples, the stride is 1.
  - The filter moves 1 step each time.

Convolution: Stride

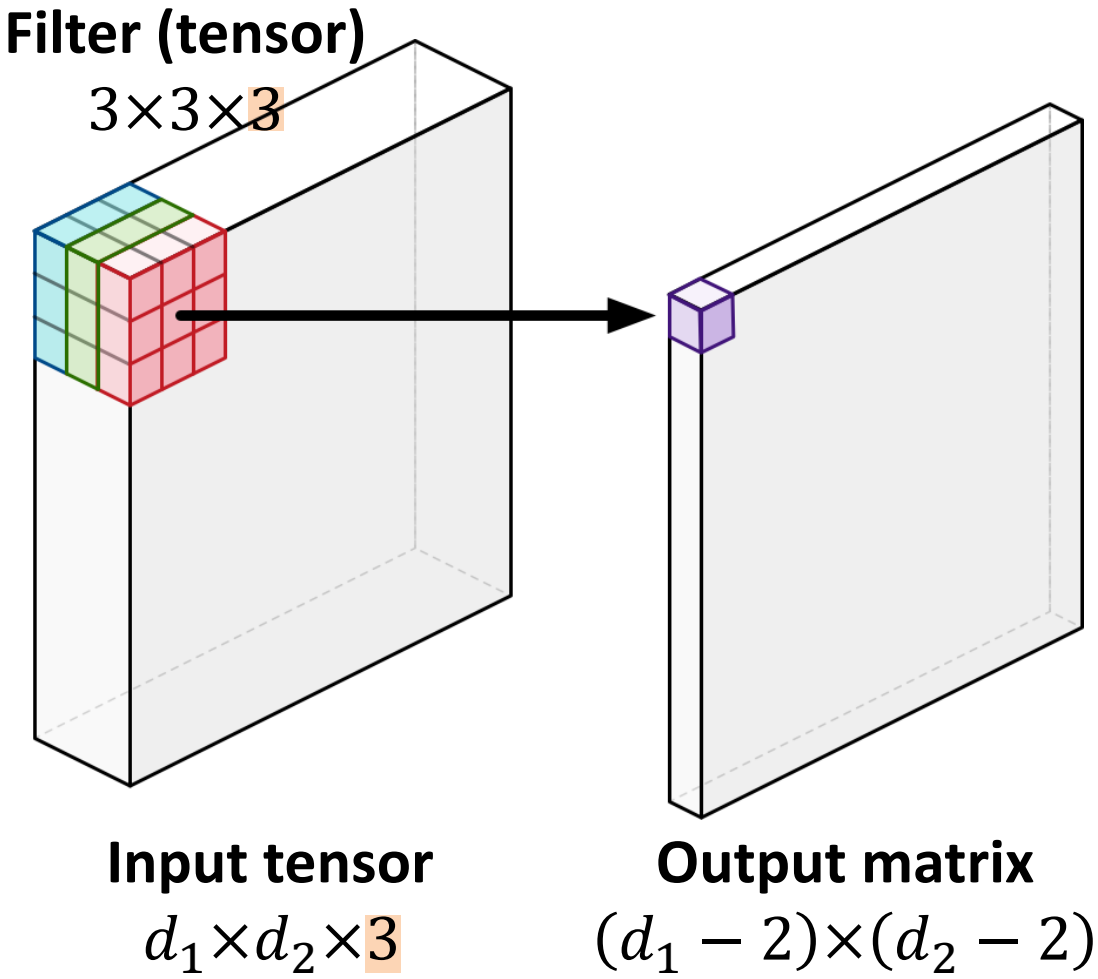


- In the previous examples, the stride is 1.
  - The filter moves 1 step each time.
- The stride can be 2 or even larger.

Dimensions:

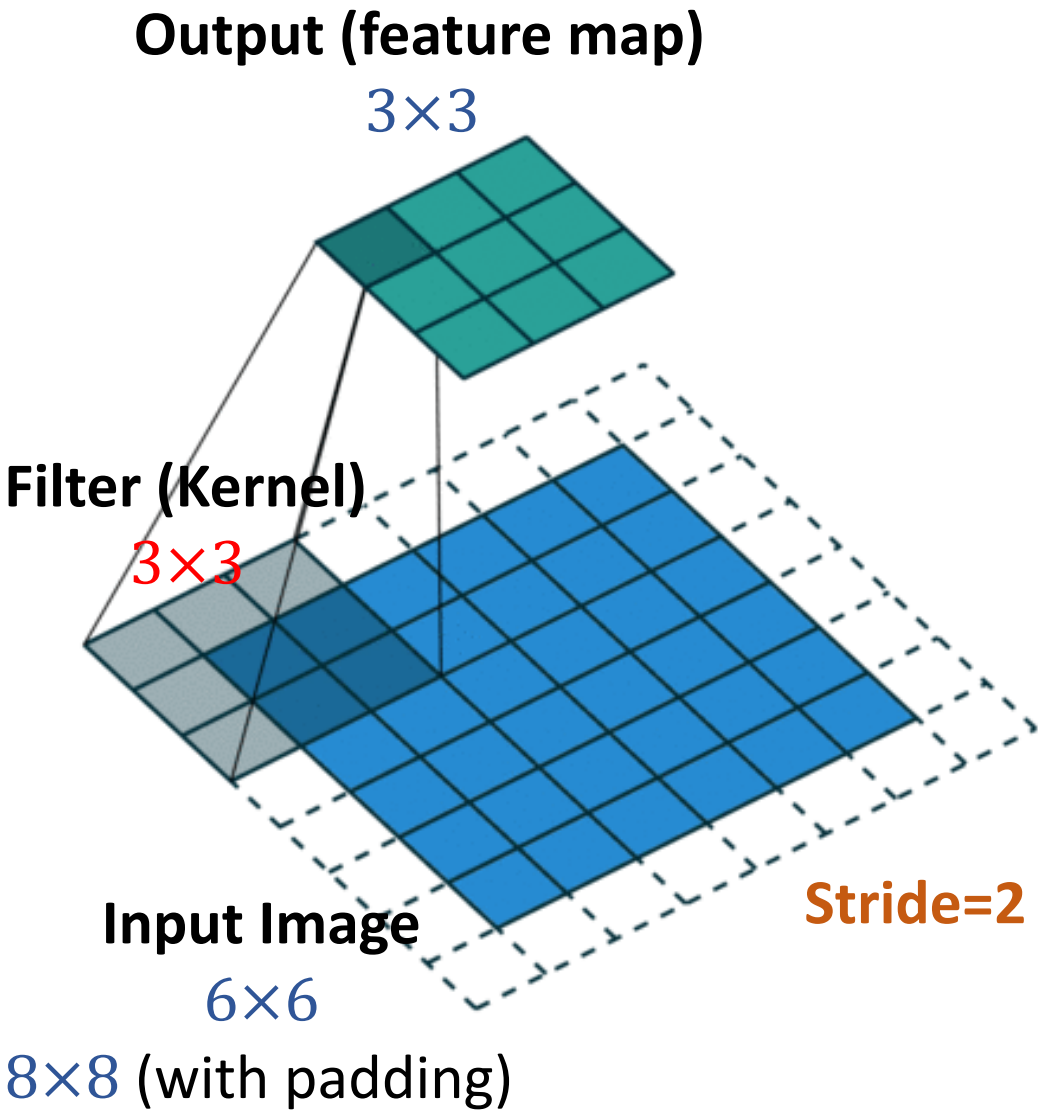
- Input: $d_1 \times d_2$
- Filter: $k_1 \times k_2$
- Stride: s
- Output: $\left(\left\lfloor \frac{d_1 - k_1}{s} \right\rfloor + 1\right) \times \left(\left\lfloor \frac{d_2 - k_2}{s} \right\rfloor + 1\right)$

Convolution for Tensors



- The input is the order-3 tensor of shape $d_1\times d_2\times3$, e.g., a color image.
- The filter is a order-3 tensor of shape $3\times3\times3$.
- The output is a matrix (order-2 tensor) of shape $(d_1-2)\times(d_2-2)$.

Convolution: Key Concepts

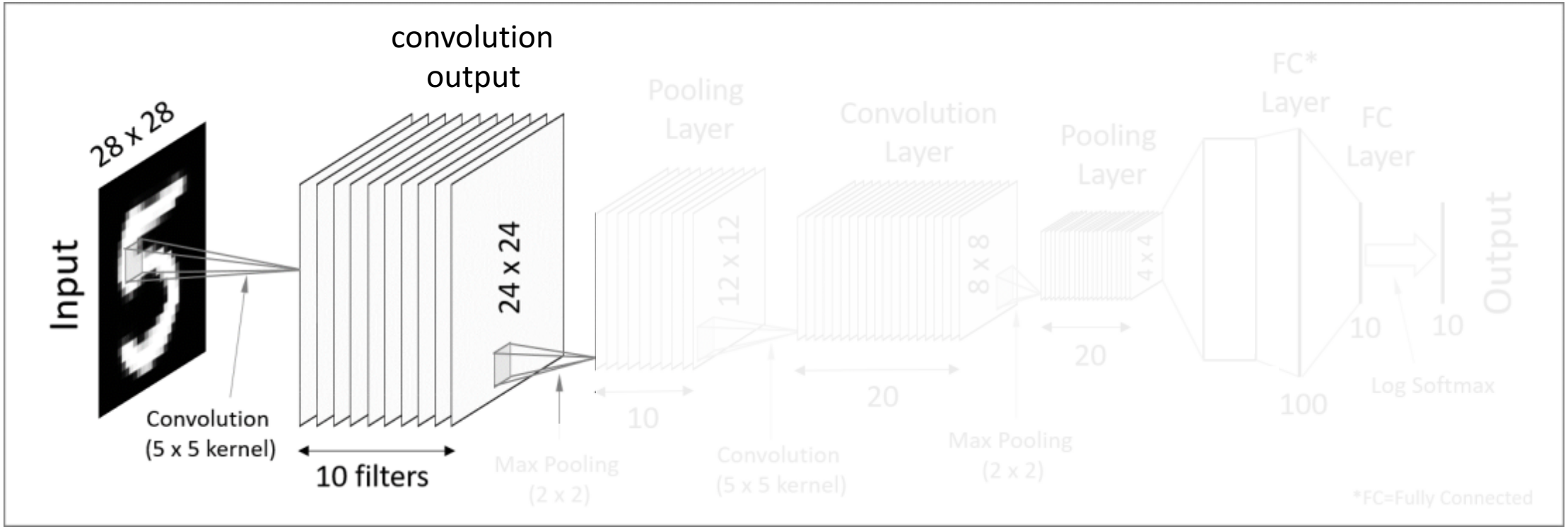


- Input, filter (kernel), and output (feature map).
- Convolution for matrix (order-2 tensor).
- Convolution for order-3 tensor.
- Zero padding.
- Stride.

Convolutional Neural Networks (CNNs)

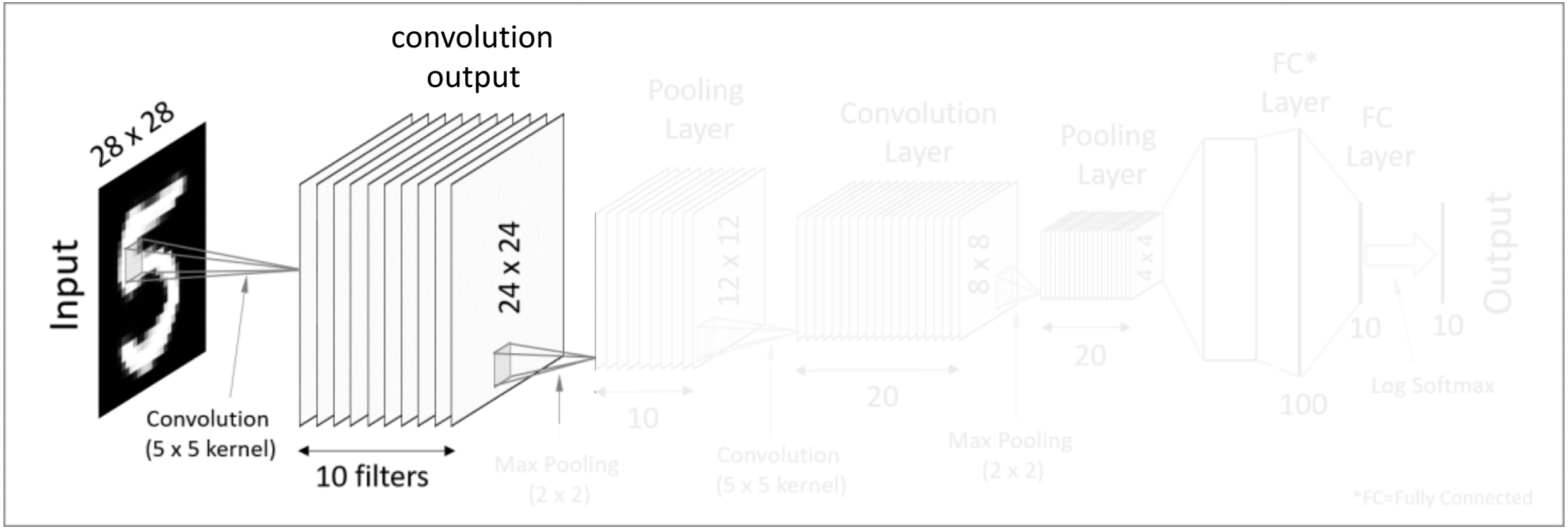
CNN: Convolutional Layer 1

- Use multiple filters (kernels) for the feature mapping. E.g.,
  - using ten 5×5 filters, (merely $10\times5\times5 = 250$ parameters!)
  - the results are ten 24×24 matrices.



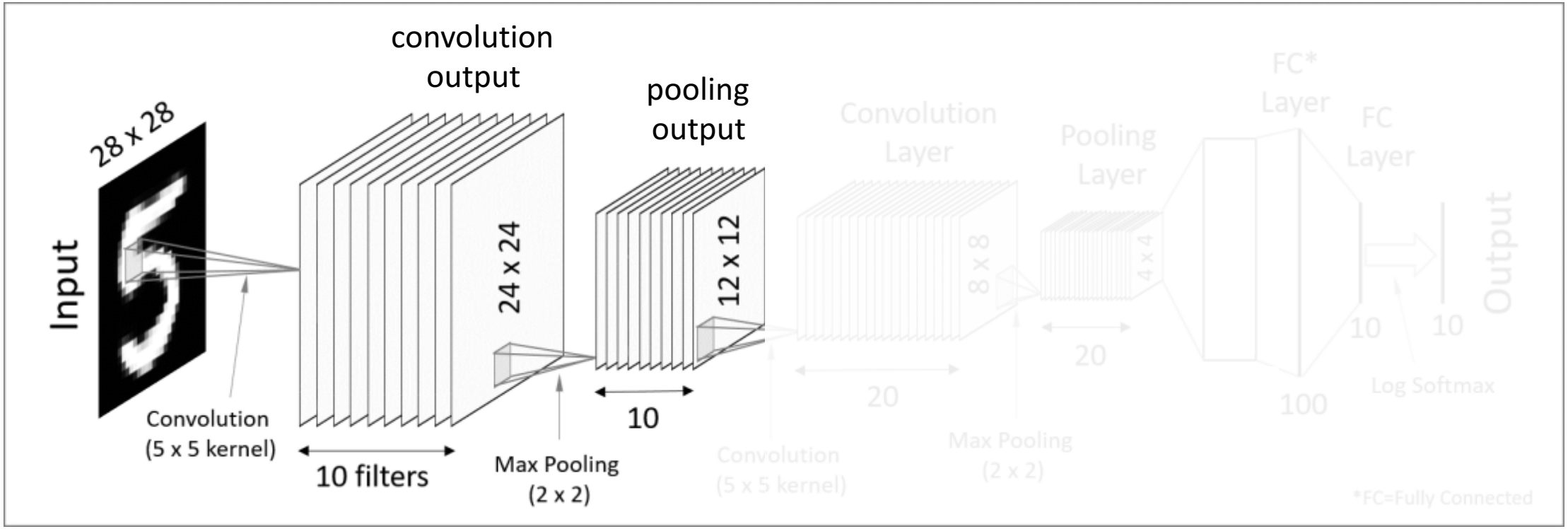
CNN: Convolutional Layer 1

- Use multiple filters (kernels) for the feature mapping.
- Apply an activation function (e.g., ReLU).
- A Convolutional layer = convolution + activation.



CNN: Pooling Layer 1

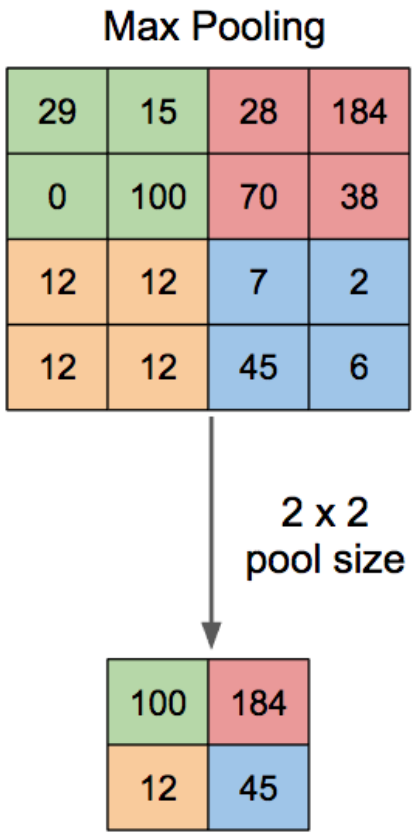
- Pooling reduces the dimensionality of each feature map.
- E.g., Max Pooling, Average Pooling, etc.



Pooling

池化降低了每个特征映射的维数

- Pooling reduces the dimensionality of each feature map.
- E.g., Max Pooling, Average Pooling, etc.

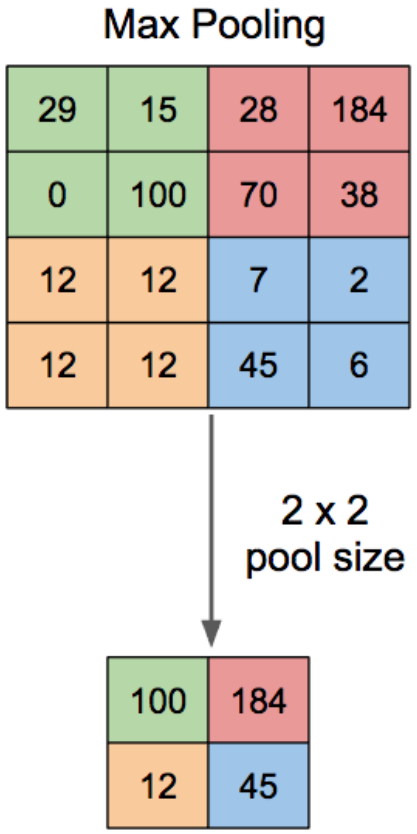


Settings:

- Max Pooling.
- Pool size $= 2\times2$.
- No overlap (stride $= 2\times2$).

Pooling

- Pooling reduces the dimensionality of each feature map.
- E.g., Max Pooling, Average Pooling, etc.



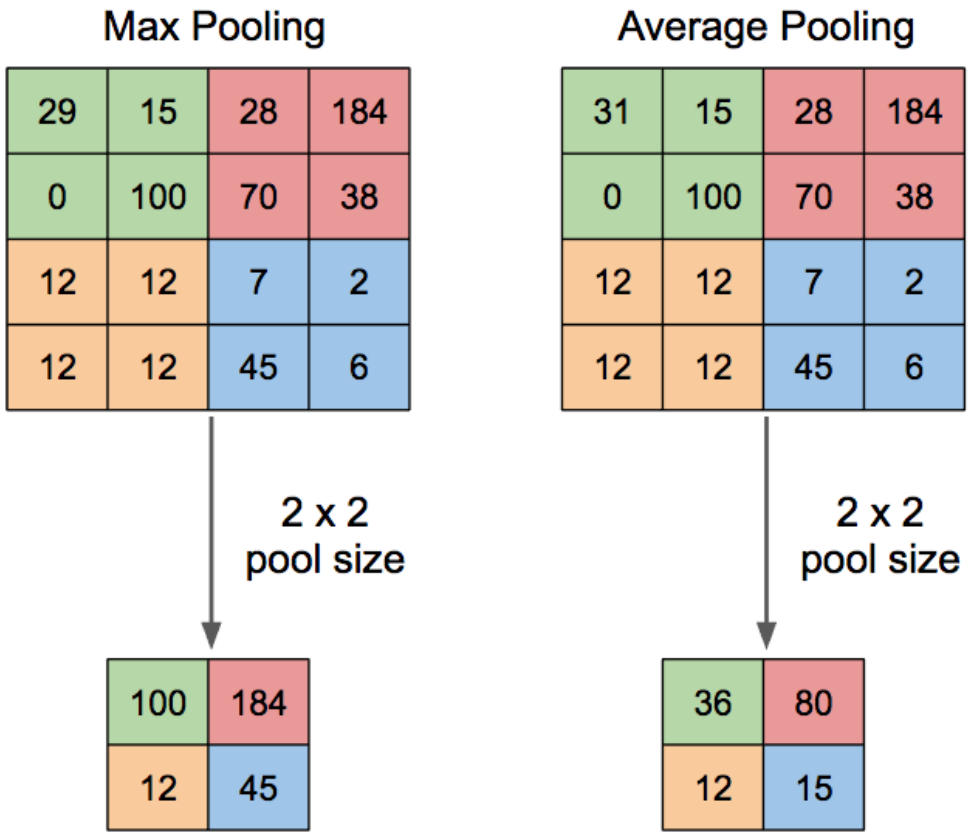
Settings:

- Max Pooling.
- Pool size $= 2\times2$.
- No overlap (stride $= 2\times2$).

Question: What if the stride is 1×1?

Pooling

- Pooling reduces the dimensionality of each feature map.
- E.g., Max Pooling, Average Pooling, etc.



Settings:

- Average Pooling.
- Pool size $= 2 \times 2$.
- No overlap (stride $= 2 \times 2$).

Pooling

- Pooling reduces the dimensionality of each feature map.
- E.g., Max Pooling, Average Pooling, etc.

- **Question:** What is the output shape?
  - Input shape: 12×12.
  - Pool size: 3×3.
  - No overlap (stride: 3×3).

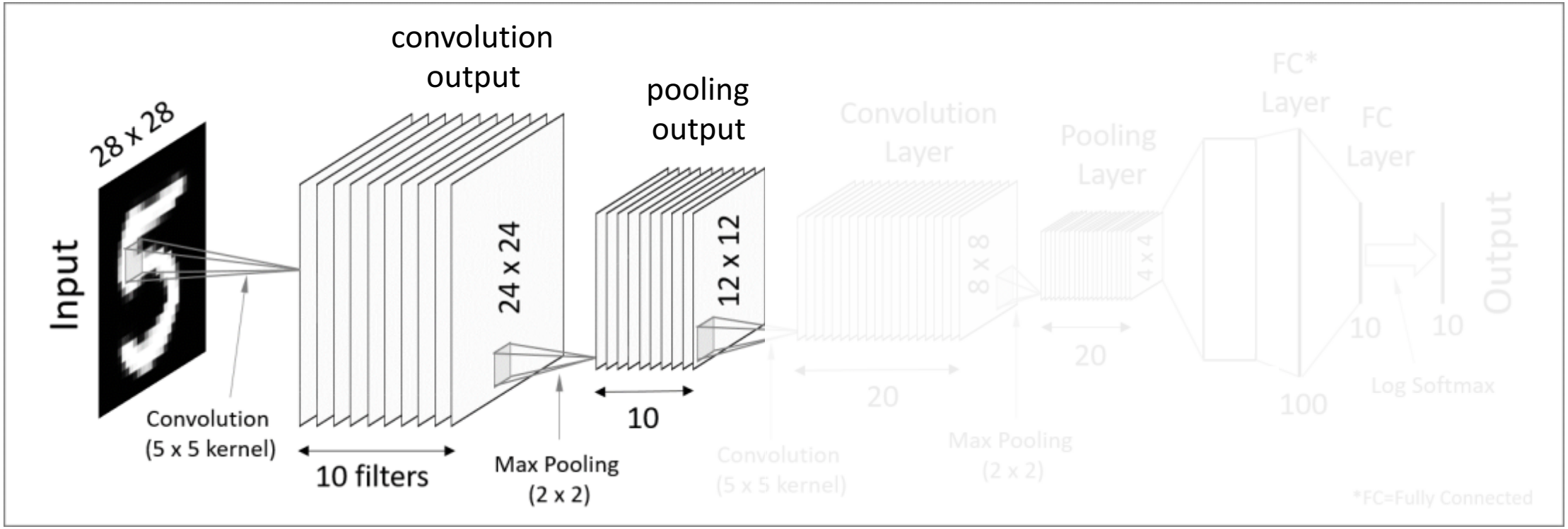
Pooling

- Pooling reduces the dimensionality of each feature map.
- E.g., Max Pooling, Average Pooling, etc.

- **Question:** What is the output shape?
  - Input shape: 12×12.
  - Pool size: 3×3.
  - With overlap (stride: 1×1).

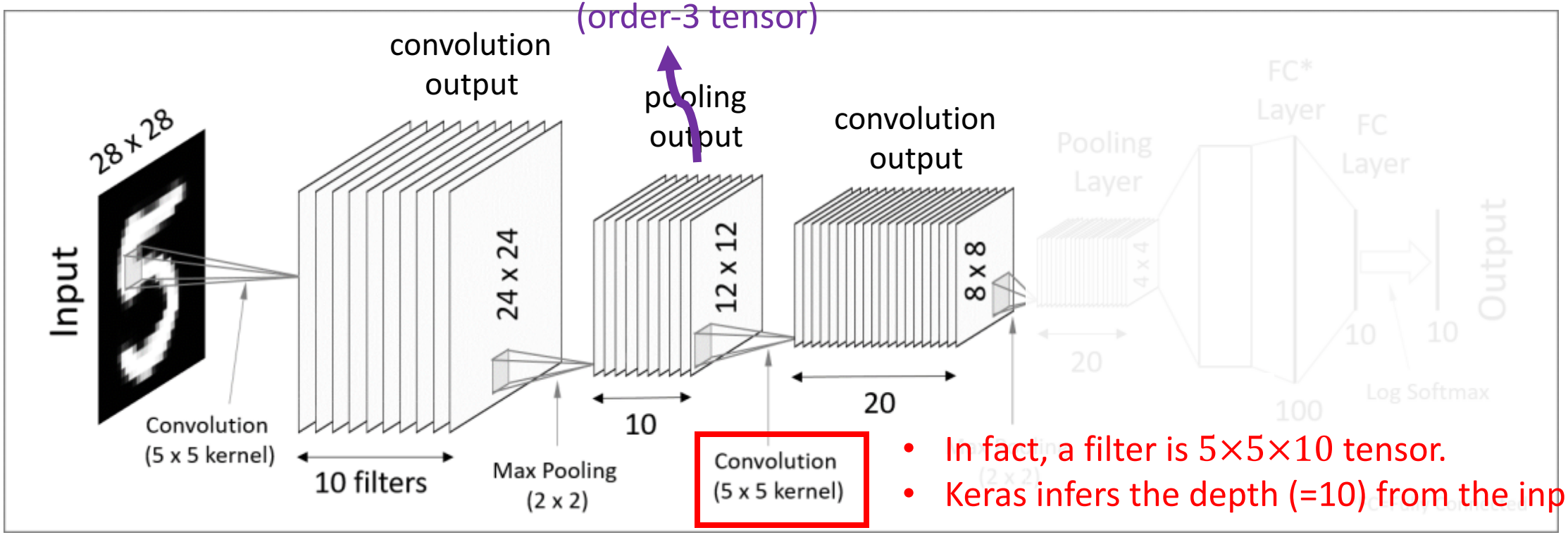
CNN: Pooling Layer 1

- Pooling reduces the dimensionality of each feature map.
- E.g., Max Pooling, Average Pooling, etc.

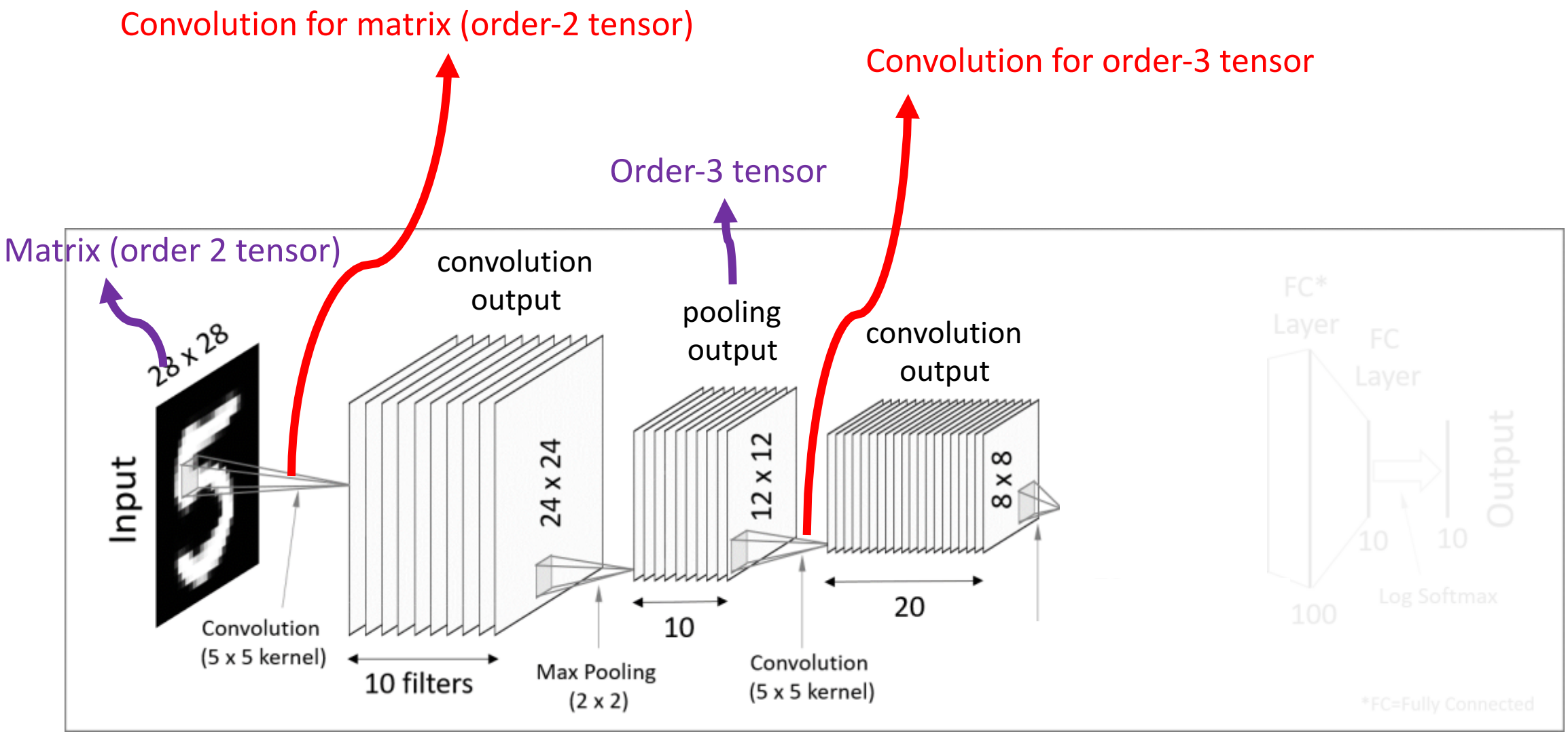


CNN: Convolutional Layer 2

- Perform tensor convolution.
- Apply $5\times5\times10$ tensor filter to the $12\times12\times10$ tensor input.
- Twenty such filters ➔ twenty 8×8 feature maps.

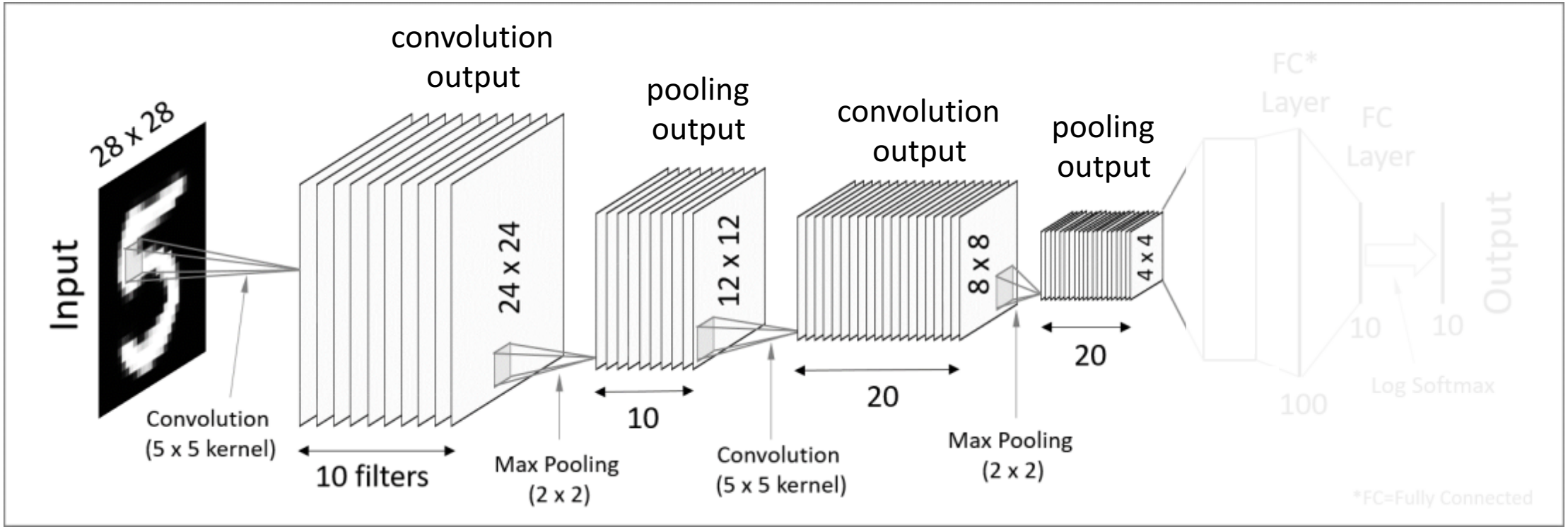


CNN: Convolutional Layers



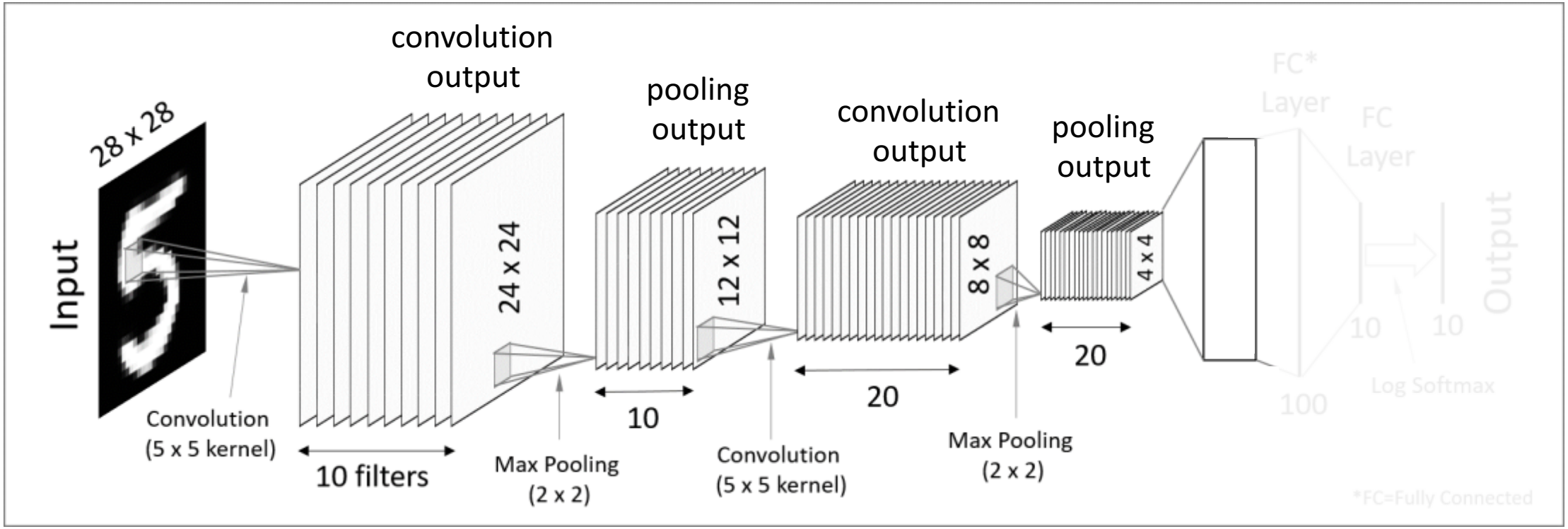
CNN: Pooling Layer 2

- 2×2 Max Pooling applied to an 8×8 matrix ➔ a 4×4 matrix.



CNN: Flatten (Vectorization)

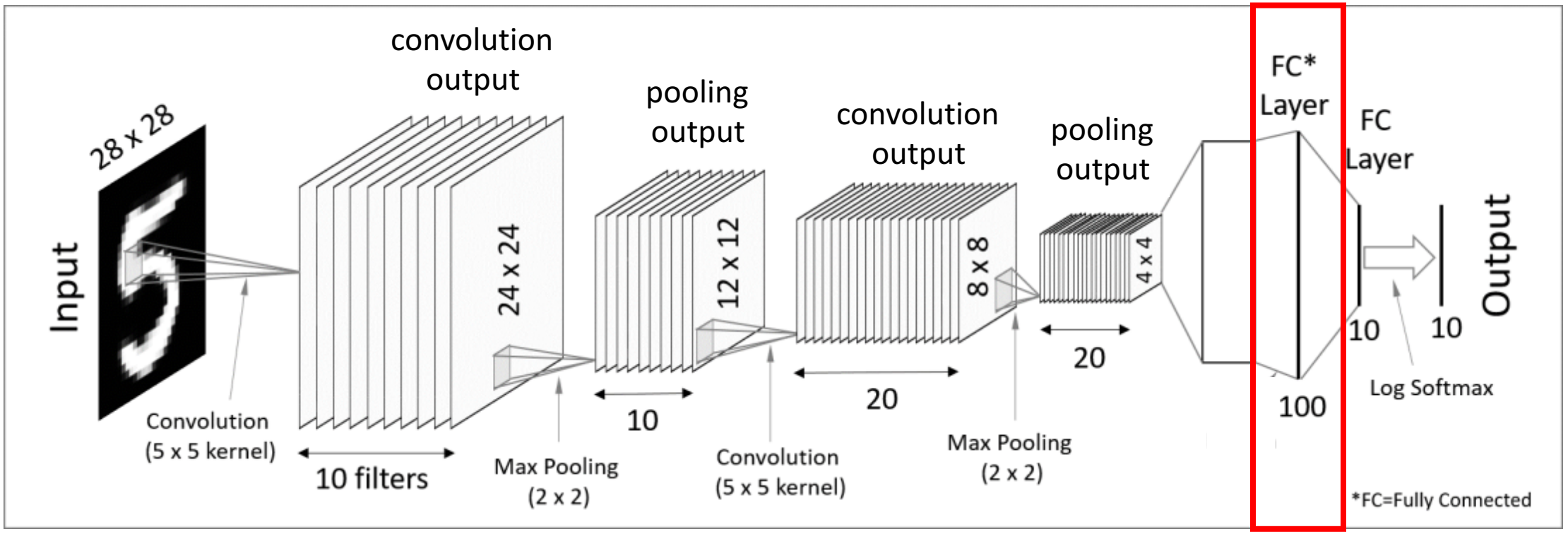
- Flatten the $4\times4\times20$ tensor.
- Get a 320-dim vector.



CNN: Fully-Connected Layers

- Add one or more fully-connected layers

This one is optional.



CNN: How Many Trainable Parameters?

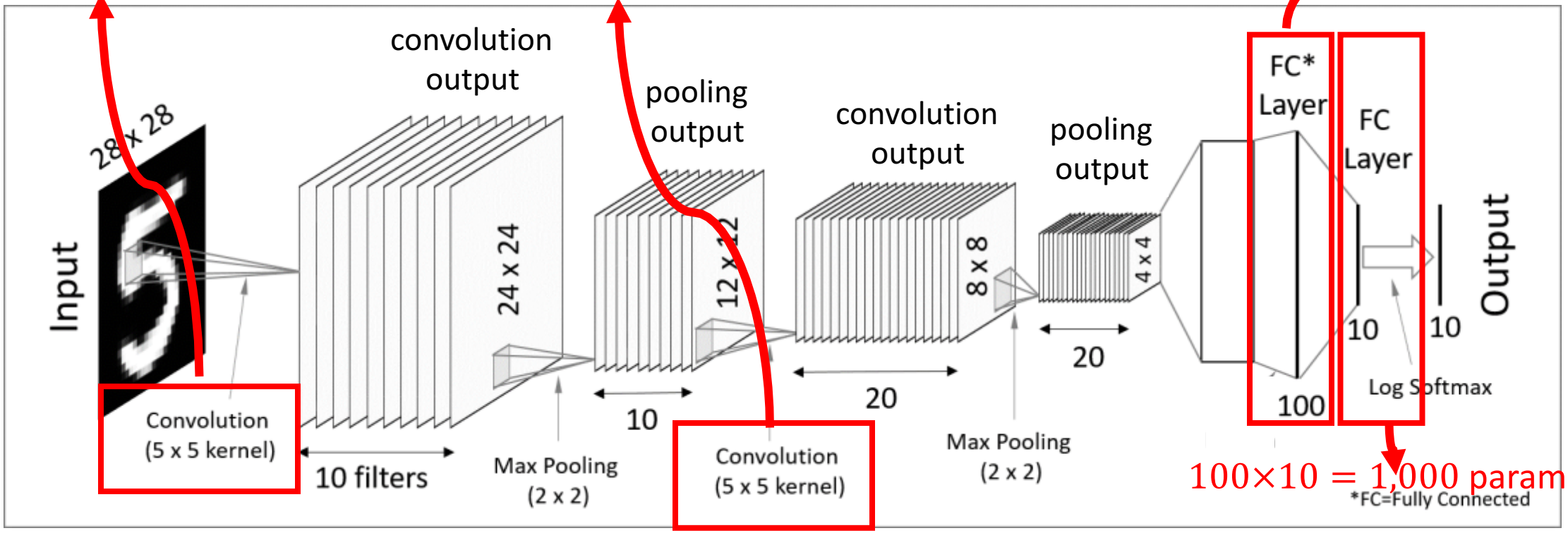
$10 \times 5 \times 5 = 250$ parameters

- 10: number of filters
- 5×5: size of filters

$20 \times 5 \times 5 \times 10 = 5000$ parameters

- 20: number of filters
- $5 \times 5 \times 10$: size of filters

$320 \times 100 = 32{,}000$ parameters



$100 \times 10 = 1{,}000$ parameters

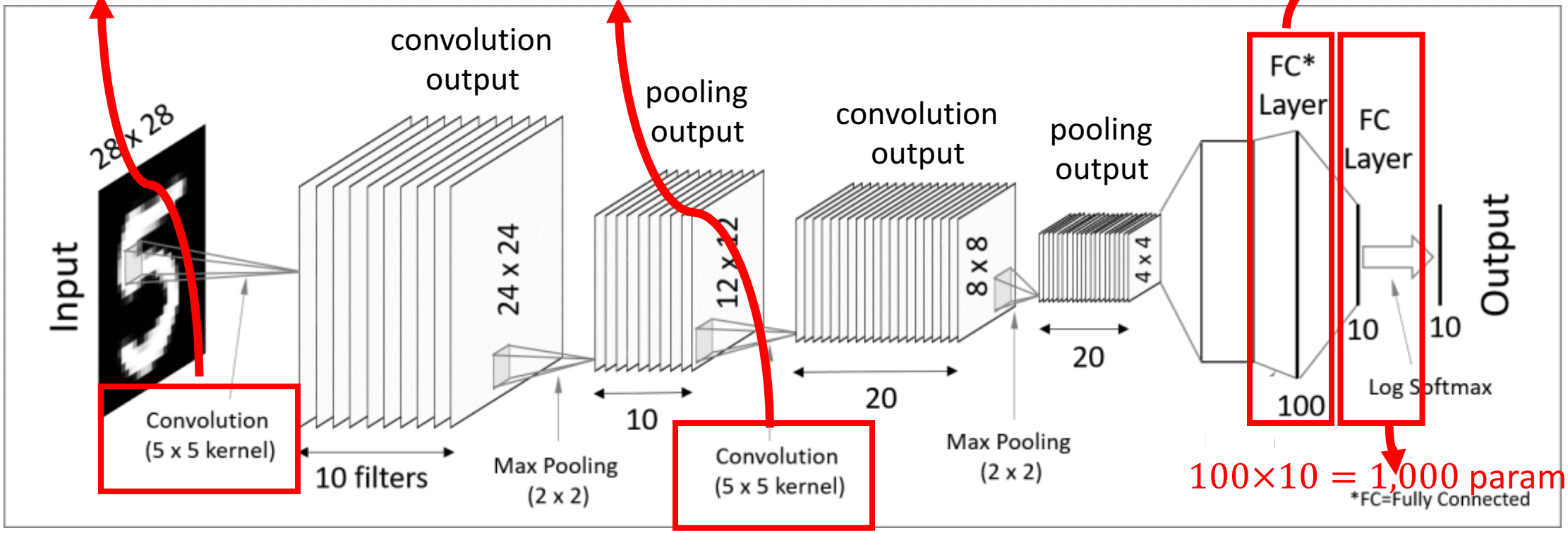
CNN: How Many Trainable Parameters?

Answer: $> 38{,}250$ parameters (not counting the intercepts).

$10 \times 5 \times 5 = 250$ parameters
- 10: number of filters
- 5×5: size of filters

$20 \times 5 \times 5 \times 10 = 5000$ parameters
- 20: number of filters
- 5×5: size of filters
- 10: depth of filters

$320 \times 100 = 32{,}000$ parameters

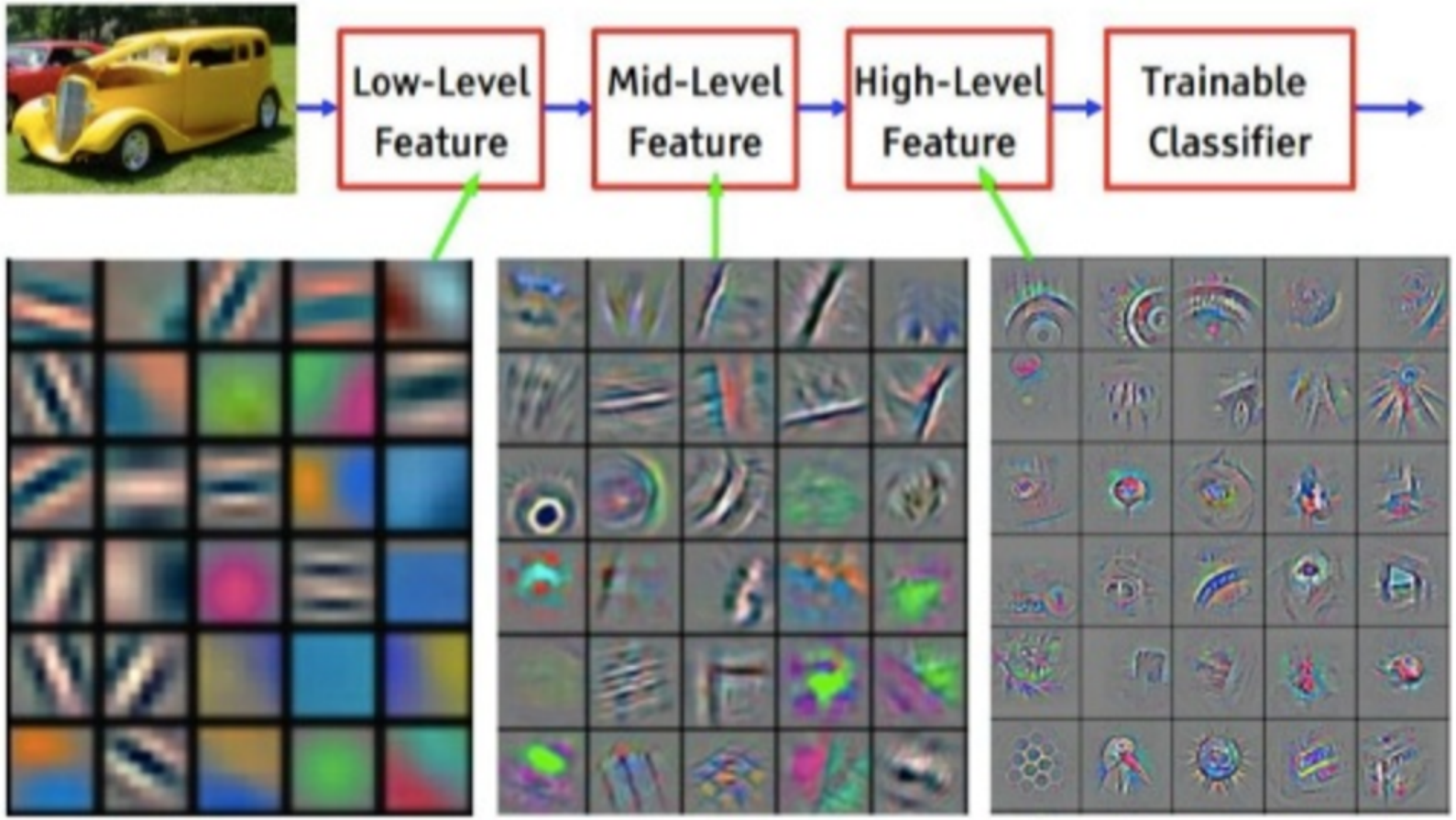


$100 \times 10 = 1{,}000$ parameters

CNN: Re-Cap

Question: Why convolutional layers?

Answer: detect features like edge, shape, pattern, etc.



CNN: Re-Cap

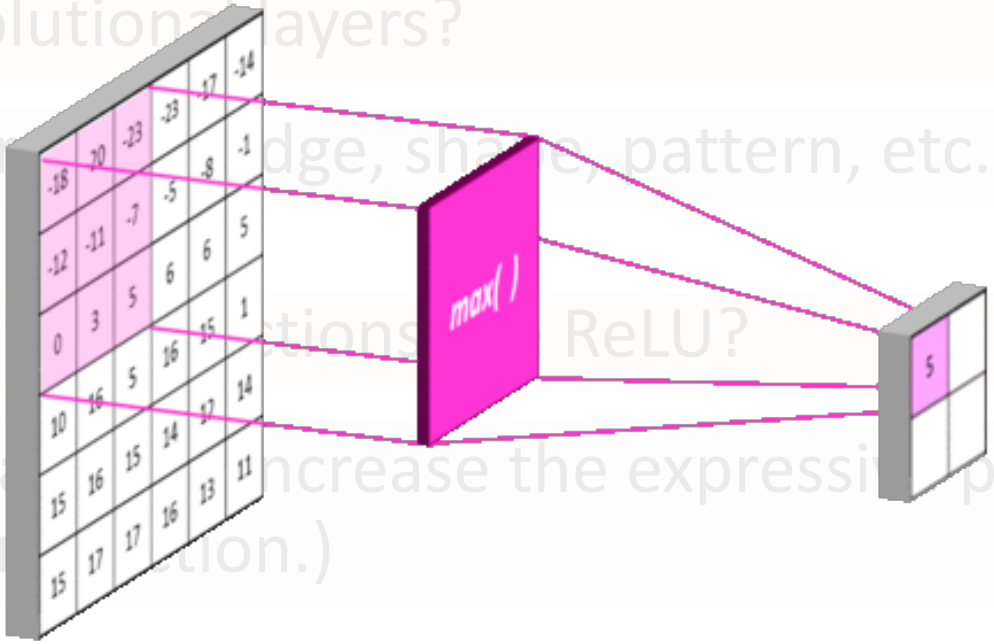
Question: Why convolutional layers?

Answer: detect features like edge, shape, pattern, etc.

Question: Why activation functions like ReLU?

Answer: Add **nonlinearity** to increase the expressive power. (Note that convolution is a **linear** function.) 增加非线性，以增加表达能力。

CNN: Re-Cap



Question: Why pooling?

Answer: Drastically reduce the shape of features.

Hyper-Parameters of CNN

Hyper-Parameters for Network Structure

- Convolutional layers
  - # Filters
  - Filter shape
  - Stride
  - Zero-padding
- Pooling layers
  - MaxPool or AvgPool
  - Pool size
  - Pool stride
- Fully-connected layers
  - Width
- Activation functions
  - ReLU
  - SoftMax
  - Sigmoid

Hyper-Parameters for Training

- Loss function.
  - Cross-entropy for classification
  - L1 or L2 for regression (the labels are continuous)
- Optimization algorithm (and its hyper-parameters, e.g., learning rate).
  - SGD
  - SGD with momentum
  - AdaGrad
  - RMSprop
- Random initialization.
  - Gaussian or uniform?
  - Scaling

Implement a CNN Using Keras

1. Load and Process the MNIST Dataset

Load data

```
from keras.datasets import mnist

(x_train, y_train), (x_test, y_test) = mnist.load_data()

print('Shape of x_train: ' + str(x_train.shape))
print('Shape of x_test: ' + str(x_test.shape))
print('Shape of y_train: ' + str(y_train.shape))
print('Shape of y_test: ' + str(y_test.shape))
```

```
Shape of x_train: (60000, 28, 28)
Shape of x_test: (10000, 28, 28)
Shape of y_train: (60000,)
Shape of y_test: (10000,)
```

1. Load and Process the MNIST Dataset

Reshape: convert the $60000 \times 28 \times 28$ dataset to a $60000 \times 28 \times 28 \times 1$ tensor.

```
x_train_vec = x_train.reshape((60000, 28, 28, 1)) / 255.0
x_test_vec = x_test.reshape((10000, 28, 28, 1)) / 255.0

print('Shape of x_train_vec is ' + str(x_train_vec.shape))
```

```
Shape of x_train_vec is (60000, 28, 28, 1)
```

单通道

1. Load and Process the MNIST Dataset

One-hot encode: convert the labels (scalars in $\{0, 1, \cdots, 9\}$) to 10-dim vectors.

```
def to_one_hot(labels, dimension=10):
    results = np.zeros((len(labels), dimension))
    for i, label in enumerate(labels):
        results[i, label] = 1.
    return results

y_train_vec = to_one_hot(y_train)
y_test_vec = to_one_hot(y_test)
print('Shape of y_train_vec is ' + str(y_train_vec.shape))
```

```
Shape of y_train_vec is (60000, 10)
```

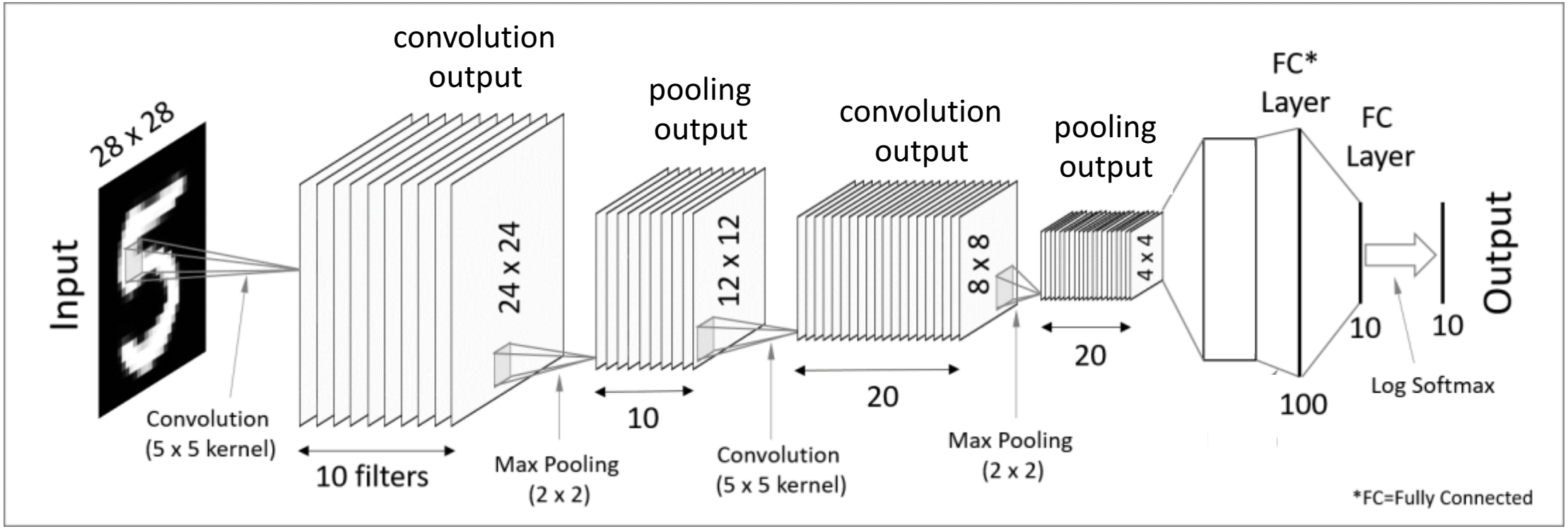
1. Load and Process the MNIST Dataset

Partition the **training** set to **training** and **validation** sets.

```
# Partition to training and validation sets

rand_indices = np.random.permutation(60000)
train_indices = rand_indices[0:50000]
valid_indices = rand_indices[50000:60000]

x_valid_vec = x_train_vec[valid_indices, :, :, :]
y_valid_vec = y_train_vec[valid_indices, :]

x_train_vec = x_train_vec[train_indices, :, :, :]
y_train_vec = y_train_vec[train_indices, :]

print('Shape of x_valid_vec: ' + str(x_valid_vec.shape))
print('Shape of y_valid_vec: ' + str(y_valid_vec.shape))
print('Shape of x_train_vec: ' + str(x_train_vec.shape))
print('Shape of y_train_vec: ' + str(y_train_vec.shape))
```

```
Shape of x_valid_vec: (10000, 28, 28, 1)
Shape of y_valid_vec: (10000, 10)
Shape of x_train_vec: (50000, 28, 28, 1)
Shape of y_train_vec: (50000, 10)
```

2. Build the CNN



2. Build the CNN

```
from keras import models
from keras import layers

model = models.Sequential()
model.add(layers.Conv2D(10, (5, 5), activation='relu', input_shape=(28, 28, 1)))
```

Filter number

Filter shape

Default

- Stride: 1
- Zero-padding: no padding

滤波器： (5 , 5,10)
原图片： (28,28, 1)

2. Build the CNN

```
from keras import models
from keras import layers

model = models.Sequential()
model.add(layers.Conv2D(10, (5, 5), activation='relu', input_shape=(28, 28, 1)))
model.add(layers.MaxPooling2D((2, 2)))
```

pool_size

Default

- Stride: pool_size

2. Build the CNN

```
from keras import models
from keras import layers

model = models.Sequential()
model.add(layers.Conv2D(10, (5, 5), activation='relu', input_shape=(28, 28, 1)))
model.add(layers.MaxPooling2D((2, 2)))  Output: 12×12×10
model.add(layers.Conv2D(20, (5, 5), activation='relu'))  Output: 8×8×20
model.add(layers.MaxPooling2D((2, 2)))  Output: 4×4×20
```

2. Build the CNN

```
from keras import models
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model = models.Sequential()
model.add(layers.Conv2D(10, (5, 5), activation='relu', input_shape=(28, 28, 1)))
model.add(layers.MaxPooling2D((2, 2)))  Output: 12×12×10
model.add(layers.Conv2D(20, (5, 5), activation='relu'))  Output: 8×8×20
model.add(layers.MaxPooling2D((2, 2)))  Output: 4×4×20
model.add(layers.Flatten())             Output: 320
```

2. Build the CNN

```
from keras import models
from keras import layers

model = models.Sequential()
model.add(layers.Conv2D(10, (5, 5), activation='relu', input_shape=(28, 28, 1)))
model.add(layers.MaxPooling2D((2, 2)))
model.add(layers.Conv2D(20, (5, 5), activation='relu'))
model.add(layers.MaxPooling2D((2, 2)))
model.add(layers.Flatten())
model.add(layers.Dense(100, activation='relu'))
model.add(layers.Dense(10, activation='softmax'))
```

Output: 12×12×10

Output: 8×8×20

Output: 4×4×20

Output: 320

Output: 100

Output: 10

2. Build the CNN

```
# print the summary of the model.
model.summary()
```

```
_________________________________________________________________
Layer (type)                 Output Shape              Param #
=================================================================
conv2d_1 (Conv2D)            (None, 24, 24, 10)        260
_________________________________________________________________
max_pooling2d_1 (MaxPooling2 (None, 12, 12, 10)        0
_________________________________________________________________
conv2d_2 (Conv2D)            (None, 8, 8, 20)          5020
_________________________________________________________________
max_pooling2d_2 (MaxPooling2 (None, 4, 4, 20)          0
_________________________________________________________________
flatten_1 (Flatten)          (None, 320)               0
_________________________________________________________________
dense_1 (Dense)              (None, 100)               32100
_________________________________________________________________
dense_2 (Dense)              (None, 10)                1010
=================================================================
Total params: 38,390
Trainable params: 38,390
Non-trainable params: 0
_________________________________________________________________
```

3. Train the CNN

Specify: optimization algorithm, learning rate (LR), loss function, and metric.

```
from keras import optimizers
model.compile(optimizers.RMSprop(lr=0.0001),
              loss='categorical_crossentropy',
              metrics=['accuracy'])
```

3. Train the CNN

Specify: batch size and number of epochs.

```
history = model.fit(x_train_vec, y_train_vec,
                    batch_size=128, epochs=50,
                    validation_data=(x_valid_vec, y_valid_vec))
```

```
Train on 50000 samples, validate on 10000 samples
Epoch 1/50
50000/50000 [==============================] - 12s 239us/step - loss: 1.2455 - acc: 0.7201 - val_loss: 0.4805 - val_acc: 0.8707
Epoch 2/50
50000/50000 [==============================] - 12s 235us/step - loss: 0.3563 - acc: 0.8995 - val_loss: 0.2851 - val_acc: 0.9172
Epoch 3/50
50000/50000 [==============================] - 12s 244us/step - loss: 0.2474 - acc: 0.9271 - val_loss: 0.2230 - val_acc: 0.9365
```

- ●
- ●
- ●

```
Epoch 49/50
50000/50000 [==============================] - 12s 240us/step - loss: 0.0244 - acc: 0.9924 - val_loss: 0.0439 - val_acc: 0.9875
Epoch 50/50
50000/50000 [==============================] - 12s 239us/step - loss: 0.0238 - acc: 0.9927 - val_loss: 0.0446 - val_acc: 0.9881
```

4. Examine the Results

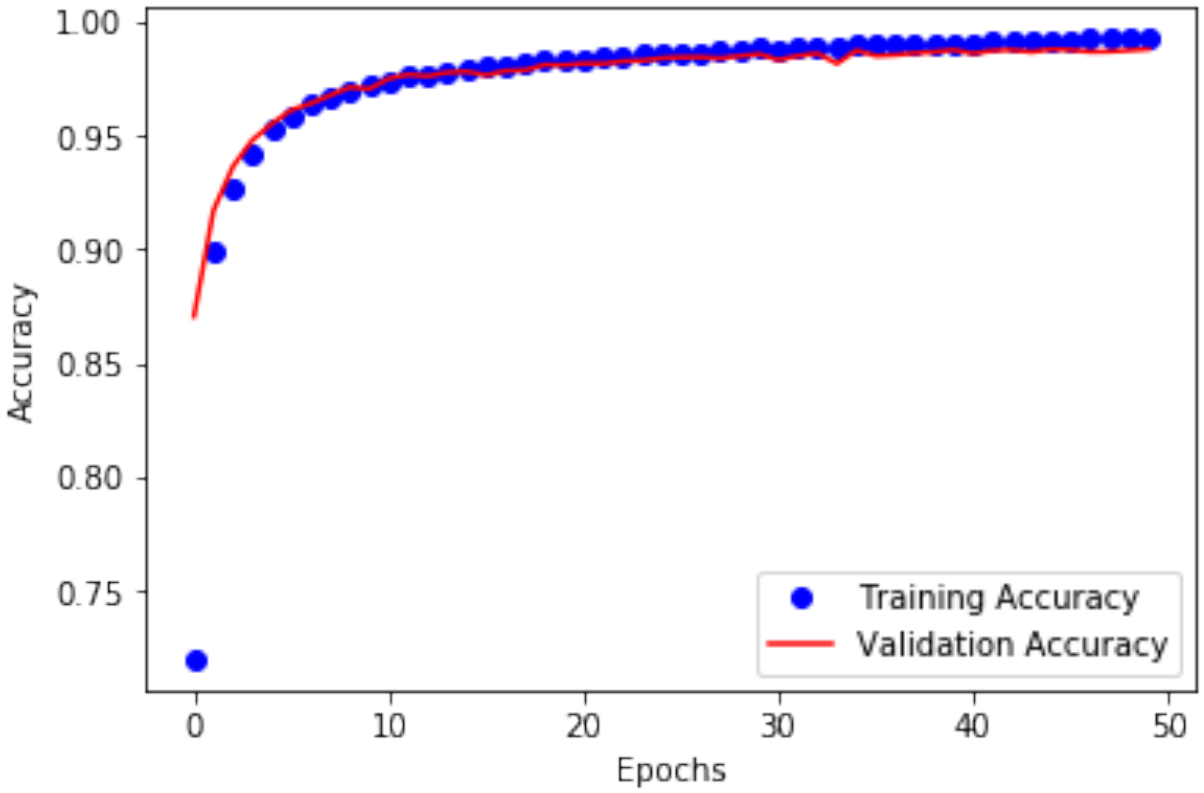
Plot the ***accuracy*** against ***epochs*** (1 epoch = 1 pass over the data).

```
import matplotlib.pyplot as plt
%matplotlib inline

epochs = range(50) # 50 is the number of epochs
train_acc = history.history['acc']
valid_acc = history.history['val_acc']
plt.plot(epochs, train_acc, 'bo', label='Training Accuracy')
plt.plot(epochs, valid_acc, 'r', label='Validation Accuracy')
plt.xlabel('Epochs')
plt.ylabel('Accuracy')
plt.legend()
plt.show()
```

4. Examine the Results

Plot the ***accuracy*** against ***epochs*** (1 epoch = 1 pass over the data).



Why using the validation set?

- Tune the hyper-parameters, e.g., learning rate, filter shape, filter number, etc.
- Check overfitting or underfitting.

4. Examine the Results

Evaluate the model on the ***test*** set. (So far, the model has not seen the ***test*** set.)

```
loss_and_acc = model.evaluate(x_test_vec, y_test_vec)
print('loss = ' + str(loss_and_acc[0]))
print('accuracy = ' + str(loss_and_acc[1]))
```

```
10000/10000 [==============================] - 1s 127us/step
loss = 0.035708001047297 15
accuracy = 0.9883
```

4. Examine the Results

- Training accuracy: 99.3%
- Validation accuracy: 98.8%
- Test accuracy: 98.8%

Summary

1. Convolution

- Matrix convolutional and tensor convolution.
- Concepts:
  - filter (kernel)
  - patch
  - output (feature map)
  - zero-padding
  - stride

2. Convolutional Neural Network

- Convolutional layers.
  - Filter number.
  - Filter shape.
  - Stride (default: one).
  - Zero-padding (default: no padding).

2. Convolutional Neural Network

- Convolutional layers.
  - Filter number.
  - Filter shape.
  - Stride (default: one).
  - Zero-padding (default: no padding).
- Activation functions.
- Pooling layers.
  - MaxPool, AveragePool, etc.
  - Pool size.
  - Pool stride (default: pool size).

3. Build and Train CNN using Keras

- Build a network.
  - Add Conv, Pool, FC layers.
- Compile the model.
  - Specify optimization algorithm, loss function, and metrics.
- Fit the model on training data.