Linear Regression

Shusen Wang

Warm-up: Vector and Matrix

Vector and Matrix

Vector (n-dim)

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Matrix ($n \times d$)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nd} \end{bmatrix}$$

矩阵的向量化表示

Row and columns

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{:1} & \mathbf{a}_{:2} & \cdots & \mathbf{a}_{:d} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{1:} \\ \mathbf{a}_{2:} \\ \vdots \\ \mathbf{a}_{n:} \end{bmatrix}$$

Vector Norms

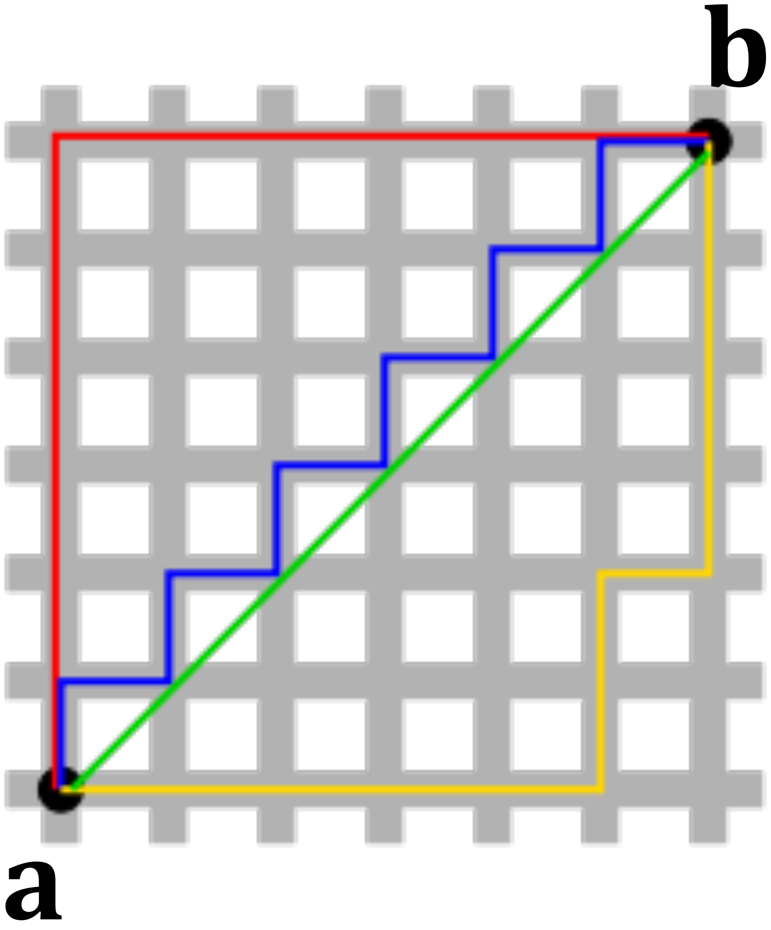
向量的范数

- The ℓ_p norm: $\|\mathbf{x}\|_p := \left(\sum_i |x_i|^p\right)^{1/p}$.

欧几里得范数

- The ℓ_2 norm: $\|\mathbf{x}\|_2 = \left(\sum_i x_i^2\right)^{1/2}$ (the Euclidean norm).
- The ℓ_1 norm $\|\mathbf{x}\|_1 = \sum_i |x_i|$.
- The ℓ_∞ norm is defined by $\|\mathbf{x}\|_\infty = \max_i |x_i|$.

Vector Norms



- The ℓ_2-distance (Euclidean distance): 欧氏距离
  $||\mathbf{a} - \mathbf{b}||_2$ (green line)

- The ℓ_1-distance (Manhattan distance): 曼哈顿距离
  $||\mathbf{a} - \mathbf{b}||_1$ (red, blue, yellow lines)

Transpose and Rank

转置 秩

Transpose:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 4 & -9 \\ 24 & 8 \end{bmatrix}$$

Square matrix: a matrix with the same number of rows and columns.

Symmetric: a square matrix $\mathbf{A}$ is symmetric if $\mathbf{A}^T = \mathbf{A}$.

Rank: the number of linearly independent rows (or columns).

方阵 才有秩 **Full rank**: a square matrix is full rank if the rank equals to #columns.

Eigenvalue Decomposition

特征值分解

- Let $\mathbf{A}$ be any $n \times n$ symmetric matrix.
- Eigenvalue decomposition: $\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T$.
- Eigenvalues satisfy $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.
- Eigenvectors satisfy $\mathbf{v}_i^T \mathbf{v}_j = 0$ for all $i \neq j$.

- $\mathbf{A}$ is full rank $\Longleftrightarrow$ all the eigenvalues are nonzero.

A 是满秩的话 → 所有特征值都是非零的

Warm-up: Optimization

Optimization: Basics

Optimization problem: $\min_{\mathbf{w}} f(\mathbf{w})$; s. t. $\mathbf{w} \in \mathcal{C}$.

- $\mathbf{w} = [w_1, \cdots, w_d]$: optimization variables
- $f : \mathbb{R}^d \mapsto \mathbb{R}$: objective function
- $\mathcal{C}$ (a subset of $\mathbb{R}^d$) : feasible set

Optimization: Basics

Optimization problem: $\min_{\mathbf{w}} f(\mathbf{w})$; $\boxed{\text{s.t. } \mathbf{w} \in \mathcal{C} \,.}$

Constraint

- $\mathbf{w} = [w_1, \cdots, w_d]$: optimization variables
- $f : \mathbb{R}^d \mapsto \mathbb{R}$: objective function
- $\mathcal{C}$ (a subset of $\mathbb{R}^d$) : feasible set 可行集合

Optimization: Basics

Optimization problem: $$\min_{\mathbf{w}} f(\mathbf{w}); \quad \text{s.t. } \mathbf{w} \in \mathcal{C}.$$

Optimal solution: $$\mathbf{w}^\star = \operatorname*{argmin}_{\mathbf{w}\in\mathcal{C}} f(\mathbf{w}).$$

- $f(\mathbf{w}^\star) \leq f(\mathbf{w})$ for all the vectors $\mathbf{w}$ in the set $\mathcal{C}$.
- $\mathbf{w}^\star$ may not exist, e.g., $\mathcal{C}$ is the empty set.
- If $\mathbf{w}^\star$ exists, it may not be unique.

Least Squares Regression

最小二乘回归

Linear Regression

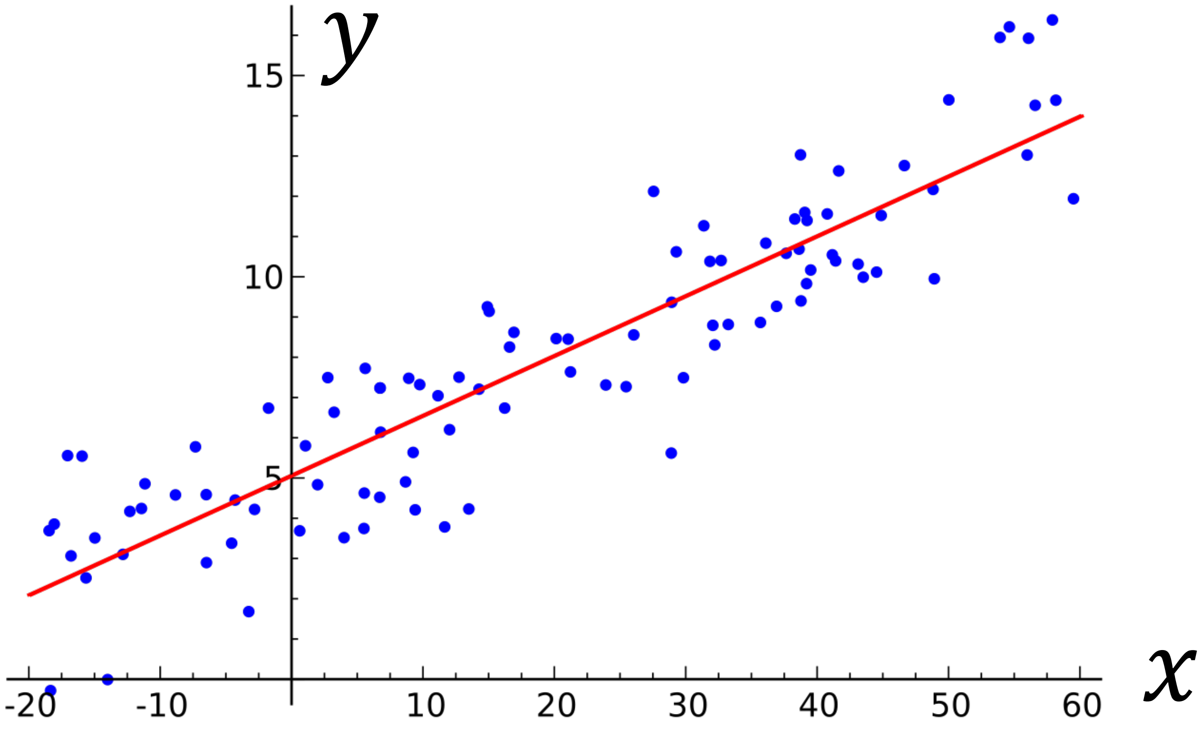
Input: vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n \in \mathbb{R}^d$ and labels $y_1, \cdots, y_n \in \mathbb{R}$

Output: a vector $\mathbf{w} \in \mathbb{R}^d$ and scalar $b \in \mathbb{R}$ such that $\mathbf{x}_i^T \mathbf{w} + b \approx y_i$.

1-dim ($d = 1$) example:

Solution:

$$y_i \approx 0.15\, x_i + 5.0$$

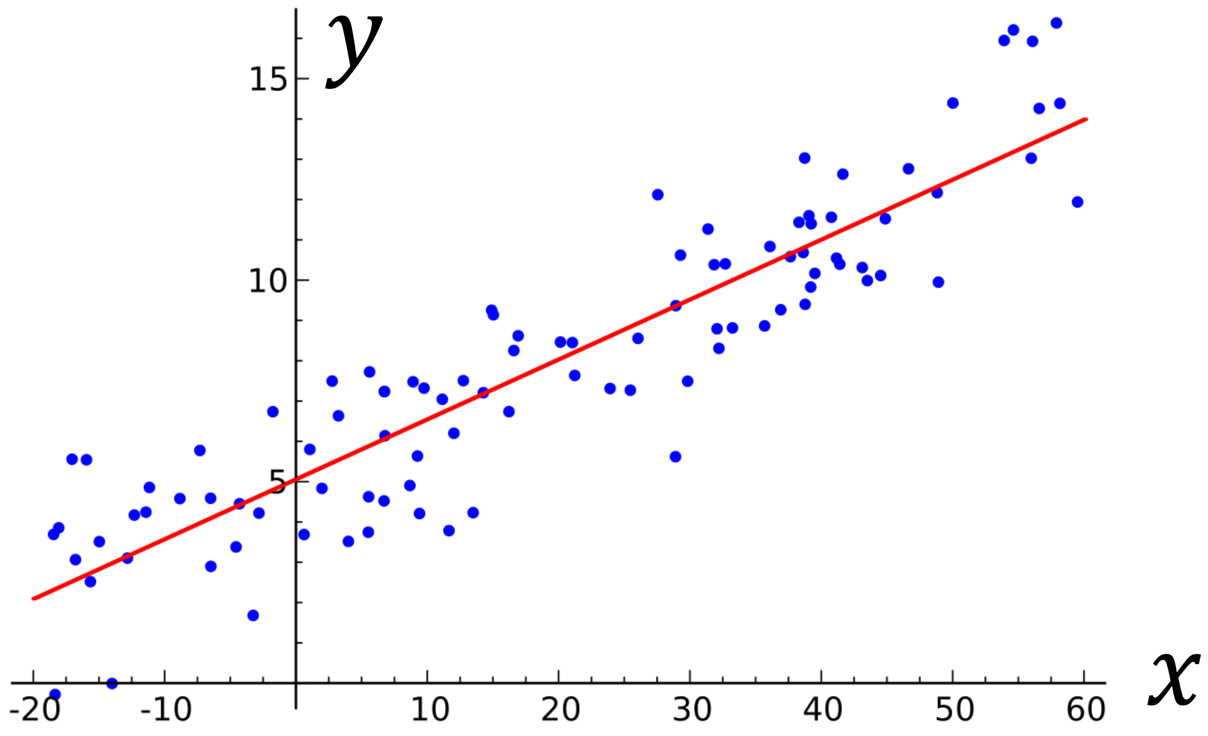


Linear Regression

Input: vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n \in \mathbb{R}^d$ and labels $y_1, \cdots, y_n \in \mathbb{R}$

Output: a vector $\mathbf{w} \in \mathbb{R}^d$ and scalar $b \in \mathbb{R}$ such that $\mathbf{x}_i^T \mathbf{w} + b \approx y_i$.

Question (regard training): how to compute $\mathbf{w}$ and b?



Least Squares Regression

Input: vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n \in \mathbb{R}^d$ and labels $y_1, \cdots, y_n \in \mathbb{R}$

Output: a vector $\mathbf{w} \in \mathbb{R}^d$ and scalar $b \in \mathbb{R}$ such that $\mathbf{x}_i^T \mathbf{w} + b \approx y_i$.

Method: least squares regression.

- The optimization model:

$$\min_{\mathbf{w}, b} \sum_{i=1}^{n} \left(\mathbf{x}_i^T \mathbf{w} + b - y_i \right)^2$$

Least Squares Regression

- The optimization model:

$$\min_{\mathbf{w},b} \sum_{i=1}^{n} \left(\mathbf{x}_i^T \mathbf{w} + \boxed{b} - y_i\right)^{\mathbf{2}}$$

Intercept (or bias)

Least Squares Regression

- The optimization model:

$$\min_{\mathbf{w},b} \sum_{i=1}^{n}\left(\mathbf{x}_i^T\mathbf{w} + b - y_i\right)^2$$

$$\bar{\mathbf{x}}_i = \begin{bmatrix} \mathbf{x}_i \\ 1 \end{bmatrix}$$

Least Squares Regression

- The optimization model:

$$\min_{\mathbf{w},b} \sum_{i=1}^{n} \left(\mathbf{x}_i^T \mathbf{w} + b - y_i\right)^2$$

$$\bar{\mathbf{x}}_i = \begin{bmatrix} \mathbf{x}_i \\ 1 \end{bmatrix} \qquad \bar{\mathbf{w}} = \begin{bmatrix} \mathbf{w} \\ b \end{bmatrix}$$

Least Squares Regression

- The optimization model:

$$\min_{\mathbf{w}, b} \sum_{i=1}^{n} \left(\mathbf{x}_i^T \mathbf{w} + b - y_i \right)^2$$

$$\bar{\mathbf{x}}_i = \begin{bmatrix} \mathbf{x}_i \\ 1 \end{bmatrix} \qquad \bar{\mathbf{w}} = \begin{bmatrix} \mathbf{w} \\ b \end{bmatrix}$$

$$\bar{\mathbf{x}}_i^T \bar{\mathbf{w}} = \mathbf{x}_i^T \mathbf{w} + b$$

Least Squares Regression

- The optimization model:

$$\min_{\mathbf{w},b} \sum_{i=1}^{n} \big(\underbrace{\mathbf{x}_i^T \mathbf{w} + b}_{= \bar{\mathbf{x}}_i^T \bar{\mathbf{w}}} - y_i\big)^2$$

$$\bar{\mathbf{x}}_i = \begin{bmatrix} \mathbf{x}_i \\ 1 \end{bmatrix} \qquad \bar{\mathbf{w}} = \begin{bmatrix} \mathbf{w} \\ b \end{bmatrix}$$

$$\bar{\mathbf{x}}_i^T \bar{\mathbf{w}} = \mathbf{x}_i^T \mathbf{w} + b$$

Least Squares Regression

- The optimization model:

$$\min_{\mathbf{w},b} \sum_{i=1}^{n} \big(\underbrace{\mathbf{x}_i^T \mathbf{w} + b}_{= \bar{\mathbf{x}}_i^T \bar{\mathbf{w}}} - y_i\big)^{\mathbf{2}}$$

$$\Updownarrow$$

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \sum_{i=1}^{n} \left(\bar{\mathbf{x}}_i^T \bar{\mathbf{w}} - y_i\right)^{\mathbf{2}}$$

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \sum_{i=1}^{n} \left(\bar{\mathbf{x}}_i^T \bar{\mathbf{w}} - y_i \right)^2$$

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \sum_{i=1}^{n} \left(\bar{\mathbf{x}}_i^T \bar{\mathbf{w}} - y_i \right)^2$$

$$\bar{\mathbf{X}} = \begin{bmatrix} \mathbf{x}_1^T & 1 \\ \mathbf{x}_2^T & 1 \\ \mathbf{x}_3^T & 1 \\ \vdots & \vdots \\ \mathbf{x}_n^T & 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \sum_{i=1}^{n} \left(\bar{\mathbf{x}}_i^T \bar{\mathbf{w}} - y_i \right)^2$$

$$\bar{\mathbf{X}} \bar{\mathbf{w}} = \begin{bmatrix} \bar{\mathbf{x}}_1^T \bar{\mathbf{w}} \\ \bar{\mathbf{x}}_2^T \bar{\mathbf{w}} \\ \bar{\mathbf{x}}_3^T \bar{\mathbf{w}} \\ \vdots \\ \bar{\mathbf{x}}_n^T \bar{\mathbf{w}} \end{bmatrix}$$

Least Squares Regression

- The optimization model:

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$$\bar{\mathbf{X}} \bar{\mathbf{w}} = \begin{bmatrix} \bar{\mathbf{x}}_1^T \bar{\mathbf{w}} \\ \bar{\mathbf{x}}_2^T \bar{\mathbf{w}} \\ \bar{\mathbf{x}}_3^T \bar{\mathbf{w}} \\ \vdots \\ \bar{\mathbf{x}}_n^T \bar{\mathbf{w}} \end{bmatrix} \qquad \bar{\mathbf{X}} \bar{\mathbf{w}} - \mathbf{y} = \begin{bmatrix} \bar{\mathbf{x}}_1^T \bar{\mathbf{w}} - y_1 \\ \bar{\mathbf{x}}_2^T \bar{\mathbf{w}} - y_2 \\ \bar{\mathbf{x}}_3^T \bar{\mathbf{w}} - y_3 \\ \vdots \\ \bar{\mathbf{x}}_n^T \bar{\mathbf{w}} - y_n \end{bmatrix}$$

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \sum_{i=1}^{n} \left(\bar{\mathbf{x}}_i^T \bar{\mathbf{w}} - y_i \right)^2$$

$$||\bar{\mathbf{X}}\bar{\mathbf{w}} - \mathbf{y}||_2^2 \quad = \left\| \begin{bmatrix} \bar{\mathbf{x}}_1^T \bar{\mathbf{w}} - y_1 \\ \bar{\mathbf{x}}_2^T \bar{\mathbf{w}} - y_2 \\ \bar{\mathbf{x}}_3^T \bar{\mathbf{w}} - y_3 \\ \vdots \\ \bar{\mathbf{x}}_n^T \bar{\mathbf{w}} - y_n \end{bmatrix} \right\|_2^2$$

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}}\in\mathbb{R}^{d+1}} \quad \sum_{i=1}^{n}\left(\bar{\mathbf{x}}_i^T\bar{\mathbf{w}} - y_i\right)^2$$

$$||\bar{\mathbf{X}}\bar{\mathbf{w}} - \mathbf{y}||_2^2 = \left\|\begin{bmatrix} \bar{\mathbf{x}}_1^T\bar{\mathbf{w}} - y_1 \\ \bar{\mathbf{x}}_2^T\bar{\mathbf{w}} - y_2 \\ \bar{\mathbf{x}}_3^T\bar{\mathbf{w}} - y_3 \\ \vdots \\ \bar{\mathbf{x}}_n^T\bar{\mathbf{w}} - y_n \end{bmatrix}\right\|_2^2 = \sum_{i=1}^{n}\left(\bar{\mathbf{x}}_i^T\bar{\mathbf{w}} - y_i\right)^2.$$

L2 范数

L2 范数

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \sum_{i=1}^{n} \left(\bar{\mathbf{x}}_i^T \bar{\mathbf{w}} - y_i \right)^2$$

$$\Updownarrow$$

重要！！！！

Matrix form: $$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad ||\bar{\mathbf{X}}\, \bar{\mathbf{w}} - \mathbf{y}||_2^2$$

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \left|\left|\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}\right|\right|_2^2$$

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad ||\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}||_2^2$$

Tasks	Methods	Algorithms
Linear Regression	Least Squares Regression	?
	LASSO	
	Least Absolute Deviations	

Least Squares Regression

- The optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad ||\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}||_2^2$$

Tasks	Methods	Algorithms
	Least Squares Regression	Analytical Solution
Linear Regression	LASSO	Gradient Descent (GD)
	Least Absolute Deviations	Conjugate Gradient (CG)

Least Squares Regression

- Solve the optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad ||\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}||_2^2$$

Gradient: $$\frac{\partial \; ||\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}||_2^2}{\partial \; \bar{\mathbf{w}}} = 2(\bar{\mathbf{X}}^T \bar{\mathbf{X}}\,\bar{\mathbf{w}} \; - \bar{\mathbf{X}}^T \mathbf{y})$$

Algorithms

Analytical Solution

Gradient Descent (GD)

Conjugate Gradient (CG)

求导重要

Least Squares Regression

- Solve the optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad ||\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}||_2^2$$

Gradient: $$\frac{\partial\ ||\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}||_2^2}{\partial\ \bar{\mathbf{w}}} = 2(\bar{\mathbf{X}}^T\bar{\mathbf{X}}\,\bar{\mathbf{w}}\ - \bar{\mathbf{X}}^T\mathbf{y})\boxed{= \mathbf{0}}$$

1st-order optimality condition

Algorithms

Analytical Solution

Gradient Descent (GD)

Conjugate Gradient (CG)

Least Squares Regression

- Solve the optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \left|\left|\bar{\mathbf{X}}\, \bar{\mathbf{w}} - \mathbf{y}\right|\right|_2^2$$

Gradient: $\dfrac{\partial \; \left|\left|\bar{\mathbf{X}}\, \bar{\mathbf{w}} - \mathbf{y}\right|\right|_2^2}{\partial \; \bar{\mathbf{w}}} = 2(\bar{\mathbf{X}}^T \bar{\mathbf{X}}\, \bar{\mathbf{w}} \; - \bar{\mathbf{X}}^T \mathbf{y}) = \mathbf{0}$

Normal equation: $\bar{\mathbf{X}}^T \bar{\mathbf{X}}\, \bar{\mathbf{w}}^\star \; = \; \bar{\mathbf{X}}^T \mathbf{y}$

Assume $\bar{\mathbf{X}}^T \bar{\mathbf{X}}$ is full rank.

Analytical solution: $\bar{\mathbf{w}}^\star \; = (\bar{\mathbf{X}}^T \bar{\mathbf{X}})^{-1}\, \bar{\mathbf{X}}^T \mathbf{y}$

Algorithms

Analytical Solution

Gradient Descent (GD)

Conjugate Gradient (CG)

Least Squares Regression

- Solve the optimization model:

$$\min_{\bar{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \left|\left|\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}\right|\right|_2^2$$

Gradient: $\frac{\partial\ \left|\left|\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \mathbf{y}\right|\right|_2^2}{\partial\ \bar{\mathbf{w}}} = 2(\bar{\mathbf{X}}^T\bar{\mathbf{X}}\,\bar{\mathbf{w}} - \bar{\mathbf{X}}^T\mathbf{y}) = \mathbf{0}$

Gradient descent repeats:

1. Compute gradient: $\mathbf{g}_t = \bar{\mathbf{X}}^T\bar{\mathbf{X}}\,\bar{\mathbf{w}}_t - \bar{\mathbf{X}}^T\mathbf{y}$
2. Update: $\bar{\mathbf{w}}_{t+1} = \bar{\mathbf{w}}_t - \alpha_t\,\mathbf{g}_t$

Algorithms

Analytical Solution

Gradient Descent (GD)

Conjugate Gradient (CG)

Least Squares Regression

- Solve the optimization model:

$$\min_{\overline{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \left|\left|\overline{\mathbf{X}}\, \overline{\mathbf{w}} - \mathbf{y}\right|\right|_2^2$$

Convergence: after $O\left(\kappa \log \frac{1}{\epsilon}\right)$ iterations, 迭代

$$\left|\left|\overline{\mathbf{X}}\left(\overline{\mathbf{w}}_t - \overline{\mathbf{w}}^{\star}\right)\right|\right|_2 \leq \epsilon \left|\left|\overline{\mathbf{X}}\left(\overline{\mathbf{w}}_0 - \overline{\mathbf{w}}^{\star}\right)\right|\right|_2.$$

$\kappa = \frac{\lambda_{\max}(\overline{\mathbf{X}}^T \overline{\mathbf{X}})}{\lambda_{\min}(\overline{\mathbf{X}}^T \overline{\mathbf{X}})}$ is the condition number.

Algorithms

- Analytical Solution
- Gradient Descent (GD)
- Conjugate Gradient (CG)

Least Squares Regression

- Solve the optimization model:

$$\min_{\overline{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad \left|\left|\overline{\mathbf{X}}\, \overline{\mathbf{w}} - \mathbf{y}\right|\right|_2^2$$

Convergence: after $O\left(\sqrt{\kappa} \log \frac{1}{\epsilon}\right)$ iterations,

$$\left|\left|\overline{\mathbf{X}}\, (\overline{\mathbf{w}}_t - \overline{\mathbf{w}}^\star)\right|\right|_2 \leq \epsilon \left|\left|\overline{\mathbf{X}}\, (\overline{\mathbf{w}}_0 - \overline{\mathbf{w}}^\star)\right|\right|_2.$$

$\kappa = \frac{\lambda_{\max}(\overline{\mathbf{X}}^T \overline{\mathbf{X}})}{\lambda_{\min}(\overline{\mathbf{X}}^T \overline{\mathbf{X}})}$ is the condition number.

The pseudo-code of CG is available at the Wikipedia.

Algorithms

- Analytical Solution
- Gradient Descent (GD)
- Conjugate Gradient (CG)

Least Squares Regression

- Solve the optimization model:

$$\min_{\overline{\mathbf{w}} \in \mathbb{R}^{d+1}} \quad ||\overline{\mathbf{X}}\, \overline{\mathbf{w}} - \mathbf{y}||_2^2$$

Tasks	Methods	Algorithms
	Least Squares Regression	Analytical Solution
Linear Regression	LASSO	Gradient Descent (GD)
	Least Absolute Deviations	Conjugate Gradient (CG)

Solve Least Squares in Python

1. Load Data

```python
from keras.datasets import boston_housing

(x_train, y_train), (x_test, y_test) = boston_housing.load_data()

print('shape of x_train: ' + str(x_train.shape))
print('shape of x_test: ' + str(x_test.shape))
print('shape of y_train: ' + str(y_train.shape))
print('shape of y_test: ' + str(y_test.shape))
```

```
shape of x_train: (404, 13)
shape of x_test: (102, 13)
shape of y_train: (404,)
shape of y_test: (102,)
```

2. Add A Feature

```python
import numpy

n, d = x_train.shape
xbar_train = numpy.concatenate((x_train, numpy.ones((n, 1))),
                               axis=1)

print('shape of x_train: ' + str(x_train.shape))
print('shape of xbar_train: ' + str(xbar_train.shape))
```

```
shape of x_train: (404, 13)
shape of xbar_train: (404, 14)
```

3. Solve the Least Squares

Analytical solution: $\bar{\mathbf{w}} = (\boxed{\bar{\mathbf{X}}^T \bar{\mathbf{X}}})^{-1} \bar{\mathbf{X}}^T \mathbf{y}$

```
xx = numpy.dot(xbar_train.T, xbar_train)
xx_inv = numpy.linalg.pinv(xx) 算 -1
xy = numpy.dot(xbar_train.T, y_train) 点乘
w = numpy.dot(xx_inv, xy)
```

3. Solve the Least Squares

Analytical solution: $\bar{\mathbf{w}} = \boxed{(\bar{\mathbf{X}}^T \bar{\mathbf{X}})^{-1}} \bar{\mathbf{X}}^T \mathbf{y}$

```
xx = numpy.dot(xbar_train.T, xbar_train)
xx_inv = numpy.linalg.pinv(xx)
xy = numpy.dot(xbar_train.T, y_train)
w = numpy.dot(xx_inv, xy)
```

3. Solve the Least Squares

Analytical solution: $\bar{\mathbf{w}} = \boxed{(\bar{\mathbf{X}}^T \bar{\mathbf{X}})^{-1}} \boxed{\bar{\mathbf{X}}^T \mathbf{y}}$

```
xx = numpy.dot(xbar_train.T, xbar_train)
xx_inv = numpy.linalg.pinv(xx)
xy = numpy.dot(xbar_train.T, y_train)
w = numpy.dot(xx_inv, xy)
```

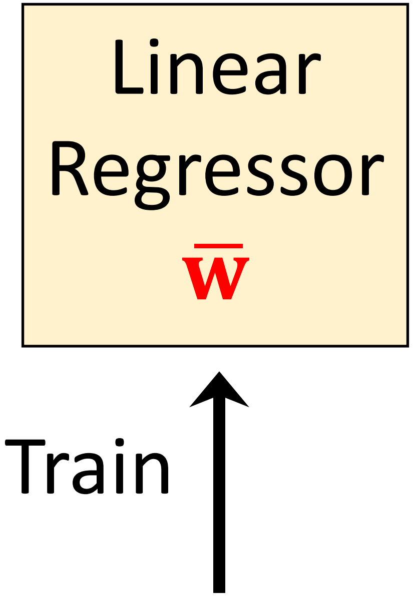
3. Solve the Least Squares

Training Mean Squared Error (MSE): $\frac{1}{n} \left|\left|\mathbf{y} - \overline{\mathbf{X}}\overline{\mathbf{w}}\right|\right|_2^2$

```
y_lsr = numpy.dot(xbar_train, w)
diff = y_lsr - y_train
mse = numpy.mean(diff * diff)
print('Train MSE: ' + str(mse))
```

```
Train MSE: 22.004800838348148
```

Linear Regression for Housing Price



label, y_i

features, $\mathbf{x}_i$

Price	Selling price of the property ($1,000)
Beds	Number of bedrooms in the house
Baths	Number of bathrooms in the house
Square Feet	Size of the house in square feet
Miles to Resort	Miles from the property to the downtown resort area
Miles to Base	Miles from the property to the base of the ski resort's mountain
Acres	Lot size in number of acres
Cars	Number of cars that will fit into the garage
Years Old	Age of the house, in years, at the time it was listed
DoM	Number of days the house was on the market before it sold

Linear Regression for Housing Price

Features of a House, $\mathbf{x}'$

➔ Extend it to $\bar{\mathbf{x}}'$

Linear Regressor $\bar{\mathbf{w}}$

Predict

Price: $\bar{\mathbf{w}}^T\bar{\mathbf{x}}' = \$500K$

4. Make Prediction for Test Samples

- Add a feature to the test feature matrix: $\mathbf{X}_{\text{test}} \rightarrow \overline{\mathbf{X}}_{\text{test}}$.
- Make prediction by: $\mathbf{y}_{\text{pred}} = \overline{\mathbf{X}}_{\text{test}} \overline{\mathbf{w}}$.

```
n_test, _ = x_test.shape
xbar_test = numpy.concatenate((x_test, numpy.ones((n_test, 1))), axis=1)
y_pred = numpy.dot(xbar_test, w)
```

4. Make Prediction for Test Samples

- Add a feature to the test feature matrix: $\mathbf{X}_{\text{test}} \rightarrow \overline{\mathbf{X}}_{\text{test}}$.
- Make prediction by: $\mathbf{y}_{\text{pred}} = \overline{\mathbf{X}}_{\text{test}} \overline{\mathbf{w}}$.
- MSE (test): $\frac{1}{n_{\text{test}}} \left|\left| \mathbf{y}_{\text{pred}} - \mathbf{y}_{\text{test}} \right|\right|_2^2$

```
# mean squared error (testing)

diff = y_pred - y_test
mse = numpy.mean(diff * diff)
print('Test MSE: ' + str(mse))
```

```
Test MSE: 23.195599256409857
```

Training MSE is **22.0**

4. Make Prediction for Test Samples

Test MSE is **23.2**

Also known as out-of-sample error.

样本外误差

The gap is an indicator of generalization.

Training MSE is **22.0**

Also known as in-sample error.

样本内误差

5. Compare with Baseline

Trivial baseline:

- whatever the features are, the prediction is $\mathrm{mean}(\mathbf{y})$.

```
y_mean = numpy.mean(y_train)

diff = y_pred - y_mean
mse = numpy.mean(diff * diff)
print('Test MSE: ' + str(mse))
```

```
Test MSE: 57.38297638530044
```

Test MSE of least squares is **23.19**

Summary

- Linear regression problem.
- Least squares model.
- 3 algorithms for solving the model.
- Make predictions for never-seen-before test data.
- Evaluation of the model (training MSE and test MSE).
- Compare with baselines.