In[∘]:= Integrate[x^n / (a + x)^(k + 1), {x, δ, ∞}]

Out[∘]= ConditionalExpression[$(-a)^{-k+n}$ Beta[$-\frac{a}{\delta}$, k − n, −k],
Re[δ] > 0 && δ == Re[δ] && Re[a] > 0 && Re[k] > Re[n]]

(1) Bound (rough)

(2) Bound~$O((a+\delta)^{-k})$

$$\int_\delta^\infty \frac{x^n}{(x+a)^{k+1}}\,dx$$

$$= a^{-k+n}\int_0^{a/\delta} t^{k-n-1}(1+t)^{-k-1}\,dt$$

Noting that

$$\frac{d}{dt}t^{k-n-1}(1+t)^{-k-1} \propto (-n-2)\left(t-\frac{k-n-1}{n+2}\right),$$

the integrand is maximized at $t=\frac{a}{\delta}$ when $\frac{k-n-1}{n+2}\geq\frac{a}{\delta}$, and at $t=\frac{k-n-1}{n+2}$ when $0<\frac{k-n-1}{n+2}\leq\frac{a}{\delta}$.

We have the following upper bounds:

$$< \delta^{n+1}(a+\delta)^{-k-1}$$

for

$$\frac{k-n-1}{n+2}\geq\frac{a}{\delta},$$

and

$$< a^{-k+n+1}\delta^{-1}\left(\frac{k-n-1}{n+2}\right)^{k-n-1}\left(\frac{k+1}{n+2}\right)^{-k-1} \leq \delta^{-k+n}\left(\frac{k+1}{n+2}\right)^{-k-1}$$

for

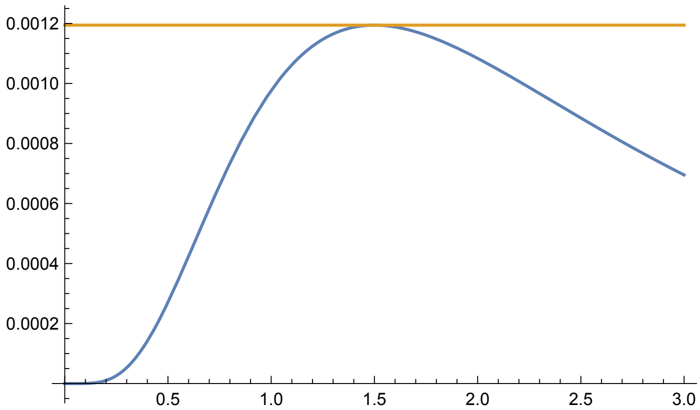
$$0\leq\frac{k-n-1}{n+2}\leq\frac{a}{\delta}.$$

In[•]:=
```
(* t^a(1+t)^-b *)
vals = {a → 6, b → 10}
t0 = - a/(a - b) /. vals
t^a (1 + t) ^ (-b);
{%, % /. t → t0} /. vals
Plot[%, {t, 0, 2 t0}]
```

Out[•]= $\{a \to 6,\ b \to 10\}$

Out[•]= $\frac{3}{2}$

Out[•]= $\left\{\frac{t^6}{(1+t)^{10}},\ \frac{11\,664}{9\,765\,625}\right\}$

Out[•]=


In[•]:=
```
(* (k-n-1)/(n+2) ≥ a/δ *)
bound1 = δ ^ (n + 1)  (a + δ) ^ (-k - 1)
kmin1  =  (a / δ)  (n + 2)  + n + 1
```

Out[•]= $\delta^{1+n}\ (a+\delta)^{-1-k}$

Out[•]= $1 + n + \frac{a\ (2+n)}{\delta}$

In[•]:=
```
(* 0 ≤ (k-n-1)/(n+2) ≤ a/δ *)
bound2 =
 a ^ (-k + n + 1) δ ^ (-1) ((k - n - 1) / (n + 2)) ^ (k - n - 1) ((k + 1) / (n + 2)) ^ (-k - 1)
(*bound2 = δ^(-k+n)  ((k+1)/(n+2))^(-k-1)*)
kmin2 = n + 1
kmax2 =  (a / δ)  (n + 2)  + n + 1
```

Out[•]= $\frac{a^{1-k+n}\ \left(\frac{1+k}{2+n}\right)^{-1-k}\ \left(\frac{-1+k-n}{2+n}\right)^{-1+k-n}}{\delta}$

Out[•]= $1 + n$

Out[•]= $1 + n + \frac{a\ (2+n)}{\delta}$

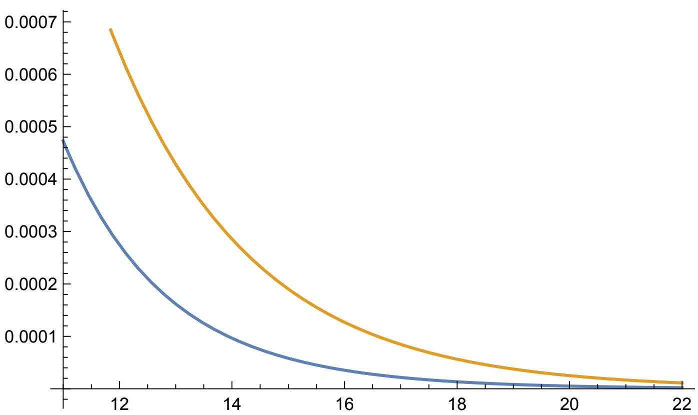
$a+\delta > 1$

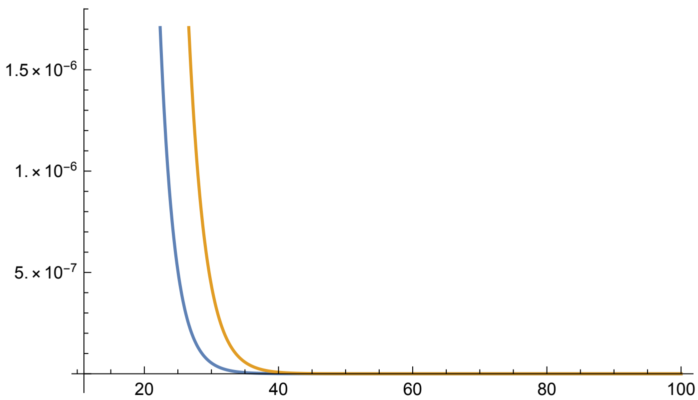
In[∘]:=
```
vals = {δ → 1/2, a → 1, n → 2};

Print[Style["(k-n-1)/(n+2) ≥ a/δ", Red]]
Print["k ≥ ", (kmin1 /. vals)]
{(-a)^(-k+n) Beta[-a/δ, k - n, -k], bound1} /. vals
plt1 = Plot[%, {k, kmin1 /. vals, 2 kmin1 /. vals}]
Plot[%%, {k, kmin1 /. vals, 100}]

Print[Style["0 ≤ (k-n-1)/(n+2) ≤ a/δ", Red]]
Print[(kmin2 /. vals), " ≤ k ≤ ", (kmax2 /. vals)]
{Normal[(-a)^(-k+n) Beta[-a/δ, k - n, -k]], bound2} /. vals
plt2 = Plot[%, {k, kmin2 /. vals, kmax2 /. vals}]

Print[Style["0 ≤ (k-n-1)/(n+2)", Red]]
Print["k ≥ ", (kmin2 /. vals)]
Show[plt1, plt2, PlotRange → Full]
```

(k-n-1)/(n+2) ≥ a/δ

k ≥ 11

Out[∘]= $\{(-1)^{2-k}\,\mathrm{Beta}[-2, -2+k, -k],\ 2^{-2+k} \times 3^{-1-k}\}$

Out[∘]=


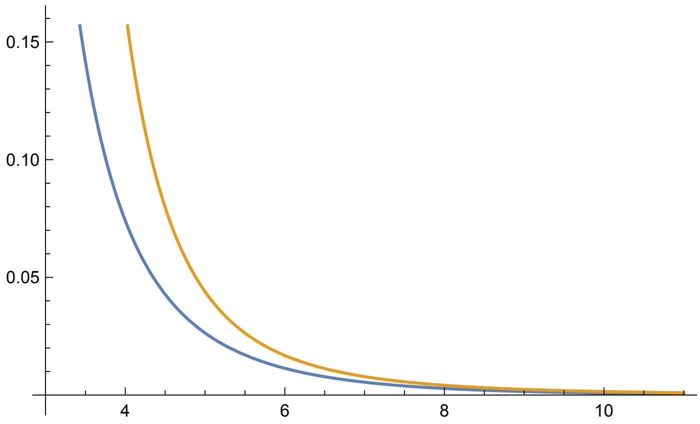
Out[∘]=


0 ≤ (k-n-1)/(n+2) ≤ a/δ

3 ≤ k ≤ 11

Out[•]= $\left\{(-1)^{2-k}\,\mathrm{Beta}[-2,\,-2+k,\,-k],\ 512\,(-3+k)^{-3+k}\,(1+k)^{-1-k}\right\}$

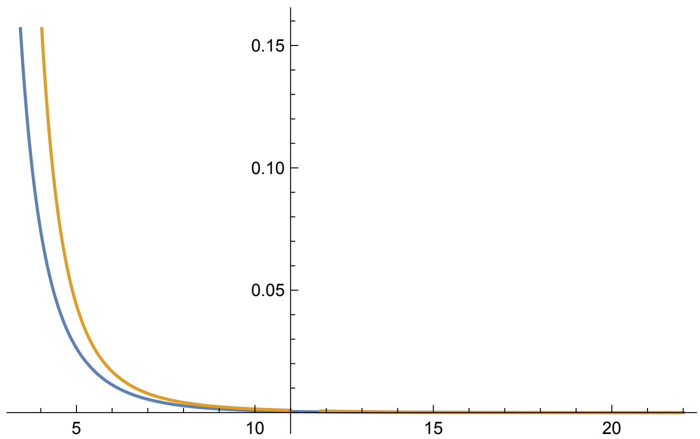
Out[•]=



0 ≤ (k-n-1)/(n+2)

k ≥ 3

Out[•]=



$a+\delta<1$

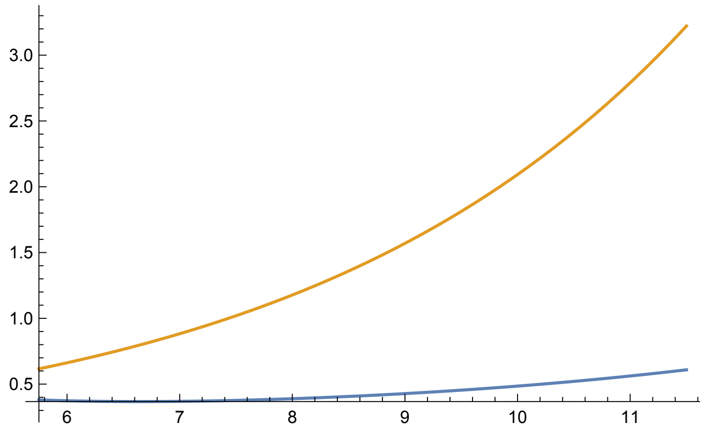
In[∘]:=
```
vals = {δ → 1/2, a → 1/4, n → 5/2};

Print[Style["(k-n-1)/(n+2) ≥ a/δ", Red]]
Print["k ≥ ", (kmin1 /. vals)]
{(-a)^(-k+n) Beta[-a/δ, k - n, -k], bound1} /. vals
plt1 = Plot[%, {k, kmin1 /. vals, 2 kmin1 /. vals}]
Plot[%%, {k, kmin1 /. vals, 100}]

Print[Style["0 ≤ (k-n-1)/(n+2) ≤ a/δ", Red]]
Print[(kmin2 /. vals), " ≤ k ≤ ", (kmax2 /. vals)]
{Normal[(-a)^(-k+n) Beta[-a/δ, k - n, -k]], bound2} /. vals
plt2 = Plot[%, {k, kmin2 /. vals, kmax2 /. vals}]

Print[Style["0 ≤ (k-n-1)/(n+2)", Red]]
Print["k ≥ ", (kmin2 /. vals)]
Show[plt1, plt2, PlotRange → Full]
```

(k-n-1)/(n+2) ≥ a/δ

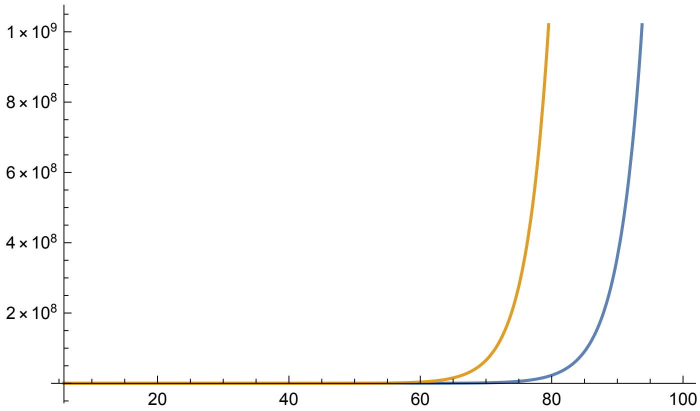
k ≥ $\frac{23}{4}$

Out[∘]= $\left\{\left(-\frac{1}{4}\right)^{\frac{5}{2}-k} \text{Beta}\left[-\frac{1}{2}, -\frac{5}{2}+k, -k\right], 2^{-\frac{3}{2}+2k} \times 3^{-1-k}\right\}$

Out[∘]=



Out[•]=

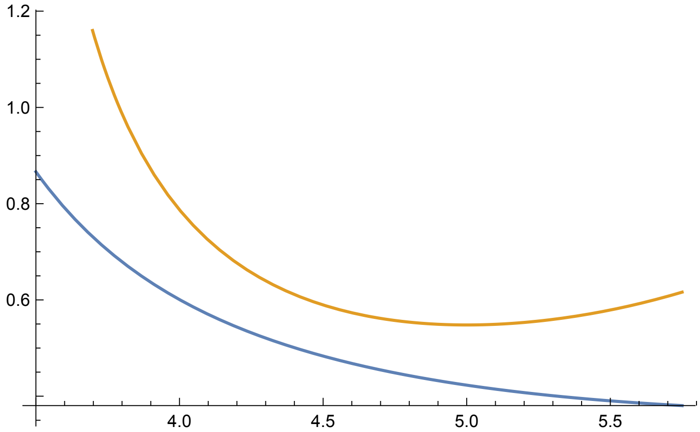


$0 \le (k-n-1)/(n+2) \le a/\delta$

$$\frac{7}{2} \le k \le \frac{23}{4}$$

Out[•]= $\left\{\left(-\frac{1}{4}\right)^{\frac{5}{2}-k} \mathrm{Beta}\left[-\frac{1}{2}, -\frac{5}{2}+k, -k\right], 19\,683 \times 2^{-\frac{21}{2}+2k} \left(-\frac{7}{2}+k\right)^{-\frac{7}{2}+k} (1+k)^{-1-k}\right\}$

Out[•]=



$0 \le (k-n-1)/(n+2)$

$$k \ge \frac{7}{2}$$

Out[•]=