Definitions

Integral and its series expansion

In[∘]:=
```
(1/Sqrt[2 π]) Integrate[Exp[-z^2/2 - g z^4/4], {z, -∞, ∞}]
```

Out[∘]=
$$\frac{e^{\frac{1}{8}/g}\,\mathrm{BesselK}\left[\frac{1}{4}, \frac{1}{8\,g}\right]}{2\sqrt{g}\,\sqrt{\pi}} \quad \text{if } \mathrm{Re}[g] > 0$$

In[∘]:=
```
N[Limit[g^(1/4) %, g → ∞]]
```

Out[∘]= 1.02277

In[24]:=
$$\mathtt{quarticI[g\_] :=} \frac{e^{\frac{1}{8}/g}\,\mathrm{BesselK}\left[\frac{1}{4}, \frac{1}{8\,g}\right]}{2\sqrt{g}\,\sqrt{\pi}}$$

```
(*quarticIseries[g_, nmax_]:= Expand[(1/Sqrt[2π]) Distribute[Integrate[
    Normal[Series[Exp[-z^2/2 - g z^4/4], {g,0,nmax}]], {z,-∞,∞}]]]*)
```

$$\mathtt{quarticIseries[g\_, nmax\_] := Sum}\left[\frac{(-1)^n\,\mathrm{Gamma}\left[\frac{1}{2} + 2\,n\right]}{\sqrt{\pi}\,n!}\,\mathtt{g\^{}n,\ \{n, 0, nmax\}}\right]$$

Pade approximant and Borel-Pade resummation

Optimized perturbation theory

Order dependent mapping

We now consider the order dependent mapping:

$$g = \rho\,\frac{\lambda}{(1-\lambda)^2},$$

where ρ is an adjustable parameter.

Then, from the original integral I(g), we have

In[∘]:=
```
(1/Sqrt[2 π]) (1 - λ)^(1/2)
 Integrate[Exp[-z^2/2 + λ (z^2/2 - ρ z^4/4)], {z, -∞, ∞}]
```

Out[∘]=
$$-\frac{e^{\frac{(-1+\lambda)^2}{8\lambda\rho}}\,\sqrt{1-\lambda}\,(-1+\lambda)\,\mathrm{BesselK}\left[\frac{1}{4}, \frac{(-1+\lambda)^2}{8\lambda\rho}\right]}{\sqrt{2\pi}\,\sqrt{2-2\lambda}\,\sqrt{\lambda\rho}} \quad \text{if } \mathrm{Re}[\lambda] < 1 \,\&\&\, \mathrm{Re}[\lambda\rho] > 0$$

The n-th approximant with $\rho = \rho_n$ is given by

$$\sum_{k=0}^{n} a_k(\rho_n)\,\lambda^k, \quad a_n(\rho_n) = 0.$$

In[28]:=
$$\mathtt{odmcoef[k\_, \rho\_] := If}\left[\mathtt{k == 0, 1,}\ \frac{1}{\sqrt{2\pi}\,\mathtt{Factorial[k]}}\,\mathtt{Integrate}\left[\mathtt{Exp}\left[-s^2/2\right]\left(s^2/2 - \rho\,s^4/4\right)^k, \{s, -\infty, \infty\}\right]\right]$$

For odd nmax only, we may use
 (N)Solve[%==0, ρ, Reals].
For large nmax, (N)Solve for $\rho \in$ Complexes cannot find a “good” solution.
Then, the above assumption, $\rho \in$ Reals, is better.

```
(* For even nmax, ρ ∈ Complexes; Find a maximum value of Abs[ρ] *)
findmaxρ[ρlist_] :=
 Module[{ρabs}, ρabs = Table[Abs[ρ /. %[[i]]], {i, Length[ρlist]}];
  ρlist[[Position[ρabs, Max[ρabs]][[1, 1]]]]]
(* Solver *)
solveρ[coef_, symbolicQ_, complexQ_] := Module[{solve},
  If[symbolicQ, solve = Composition[FullSimplify, Solve], solve = NSolve];
  Flatten[If[complexQ,
    solve[coef == 0 && Re[ρ] > 0 && Im[ρ] ≥ 0, ρ], solve[coef == 0, ρ, Reals]]]]
```

Plots

Borel-Pade resummation

Optimized perturbation theory

Order dependent mapping

In[83]:=
```
kval = 30;
symbolicQ = False; complexQ = True;
odmcoef[kval, ρ];
solveρ[%, symbolicQ, complexQ]
findmaxρ[%]
ρsol = ρ /. %;
N[ρsol kval]
N[1/ρsol]
odm = Sum[odmcoef[k, ρsol] λ^k, {k, 0, kval}];
N[{odmcoef[kval, ρsol], odmcoef[kval + 1, ρsol]}]

odmg = Simplify[(1 - λ)^(1/2) odm /. {λ → (1 + 2 g/ρsol - Sqrt[1 + 4 g/ρsol])/(2 g/ρsol)}];

Plot[{quarticI[g], Re[odmg], Im[odmg]},
 {g, 0, 100}, PlotLegends → {"I(g)", "ODM(re)", "ODM(im)"}]

(* g→∞ *)
1.0227656721131686` - (ρsol λ)^(1/4) odm /. λ → 1
```

$$\lambda \to \frac{1 + 2\,g/\rho sol - \sqrt{1 + 4\,g/\rho sol}}{2\,g/\rho sol}$$

Out[86]= {ρ → 0.00294738 + 0.0178833 i, ρ → 0.0057082 + 0.0184488 i,
ρ → 0.00821163 + 0.0185224 i, ρ → 0.0105591 + 0.0182339 i,
ρ → 0.0127693 + 0.0176401 i, ρ → 0.0148365 + 0.0167784 i,
ρ → 0.016803 + 0.0156152 i, ρ → 0.0182587, ρ → 0.0185227, ρ → 0.0197114,
ρ → 0.019993, ρ → 0.0207318, ρ → 0.0213616, ρ → 0.0213876,
ρ → 0.0214366, ρ → 0.0219547, ρ → 0.0221063, ρ → 0.0229244,
ρ → 0.0253938, ρ → 0.0254101, ρ → 0.0268104, ρ → 0.0269537, ρ → 0.027235}

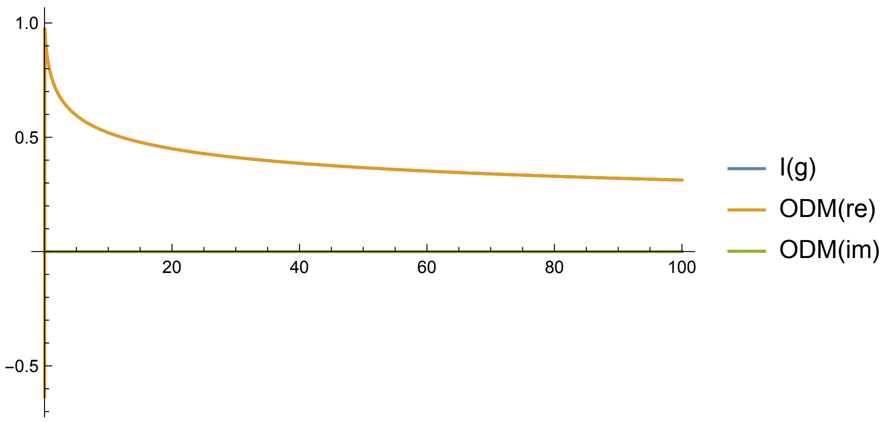
Out[87]= ρ → 0.027235

Out[89]= 0.817049

Out[90]= 36.7175

Out[92]= $\{1.81197 \times 10^{-8}, -2.5253 \times 10^{-8}\}$

Out[94]=



Out[95]= -6.52711×10^{-9}

Various ODMs

We now consider the order dependent mapping:

$$g = \rho \frac{\lambda}{(1-\lambda)^{\alpha}}$$

In[•]:=
```
(1 / Sqrt[2 π]) (1 - λ) ^ (α / 4)
 Integrate[Exp[- (1 - λ) ^ (α / 2) z^2 / 2 - ρ λ z^4 / 4], {z, -∞, ∞}]
```

Out[•]=
$$\boxed{\frac{e^{\frac{(1-\lambda)^{\alpha}}{8\lambda\rho}}\,(1-\lambda)^{\alpha/4}\,\sqrt{(1-\lambda)^{\alpha/2}}\,\text{BesselK}\left[\frac{1}{4},\frac{(1-\lambda)^{\alpha}}{8\lambda\rho}\right]}{2\sqrt{\pi}\,\sqrt{\lambda\rho}} \quad \text{if} \quad \text{Re}\left[(1-\lambda)^{\alpha/2}\right]>0\,\&\&\,\text{Re}[\lambda\rho]>0}$$

In[•]:=
```
Limit[D[Exp[- (1 - λ) ^ (α / 2) z^2 / 2 - ρ λ z^4 / 4], {λ, 1}], λ → 0]
```

Out[•]=
$$e^{-\frac{z^2}{2}}\left(\frac{z^2\,\alpha}{4}-\frac{z^4\,\rho}{4}\right)$$

In[•]:=
```
odmcoef2[k_, ρ_, α_] := If[k == 0, 1, 1/(Sqrt[2 π] Factorial[k]) Integrate[Limit[
      D[Exp[- (1 - λ) ^ (α / 2) z^2 / 2 - ρ λ z^4 / 4], {λ, k}], λ → 0], {z, -∞, ∞}]]
solveρ2[coef_, symbolicQ_, complexQ_] := Module[{solve},
  If[symbolicQ, solve = Composition[FullSimplify, Solve], solve = NSolve];
  Flatten[If[complexQ,
    solve[coef == 0 && Re[ρ] > 0 && Im[ρ] ≥ 0 && α > 0, ρ], solve[coef == 0, ρ, Reals]]]]
```

In[•]:=
```
kval = 2;
odmcoef2[kval, ρ, α]
solveρ2[%, True, True]
Normal[%]
```

Out[•]=
$$\frac{1}{32}\left(\alpha\,(4+\alpha)-30\,\alpha\,\rho+105\,\rho^2\right)$$

Out[•]=
$$\left\{\rho\to\boxed{\frac{\alpha}{7}+\frac{2\,i\,\sqrt{(7-2\,\alpha)\,\alpha}}{7\,\sqrt{15}} \quad \text{if} \quad 0<\alpha<\frac{7}{2}},\right.$$
$$\left.\rho\to\boxed{\frac{\alpha}{7}-\frac{2\,\sqrt{\alpha\,(-7+2\,\alpha)}}{7\,\sqrt{15}} \quad \text{if} \quad \alpha>\frac{7}{2}},\ \rho\to\boxed{\frac{\alpha}{7}+\frac{2\,\sqrt{\alpha\,(-7+2\,\alpha)}}{7\,\sqrt{15}} \quad \text{if} \quad \alpha>\frac{7}{2}}\right\}$$

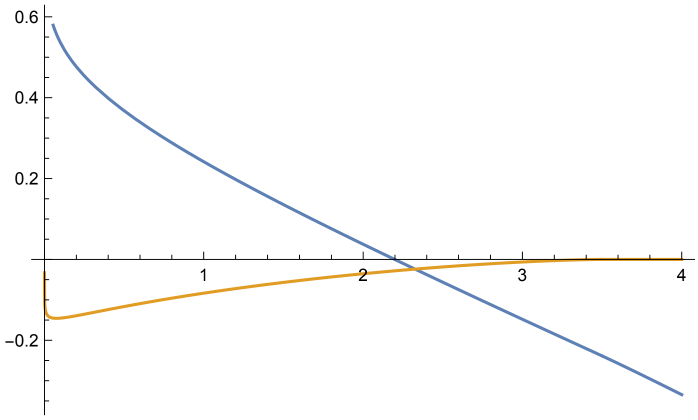
Out[•]=
$$\left\{\rho\to\frac{\alpha}{7}+\frac{2\,i\,\sqrt{(7-2\,\alpha)\,\alpha}}{7\,\sqrt{15}},\ \rho\to\frac{\alpha}{7}-\frac{2\,\sqrt{\alpha\,(-7+2\,\alpha)}}{7\,\sqrt{15}},\ \rho\to\frac{\alpha}{7}+\frac{2\,\sqrt{\alpha\,(-7+2\,\alpha)}}{7\,\sqrt{15}}\right\}$$

In[•]:=
```
kval = 2;
ρsol = ρ /. ρ → α/7 + (2 ⅈ Sqrt[(7 - 2 α) α])/(7 Sqrt[15]);
odm2 = FullSimplify[Sum[odmcoef2[k, ρsol, α] λ^k, {k, 0, kval}]]
(* Error at g→∞ *)
1.0227656721131686` - (ρsol λ)^(1/4) odm2 /. λ → 1
Plot[{Re[%], Im[%]}, {α, 0, 4}]
```

Out[•]= $1+\frac{\alpha\lambda}{7}-\frac{1}{14}\,\mathbb{i}\,\sqrt{\frac{3}{5}}\,\sqrt{(7-2\alpha)\,\alpha}\,\lambda$

Out[•]= $1.02277-\left(1+\frac{\alpha}{7}-\frac{1}{14}\,\mathbb{i}\,\sqrt{\frac{3}{5}}\,\sqrt{(7-2\alpha)\,\alpha}\right)\left(\frac{\alpha}{7}+\frac{2\,\mathbb{i}\,\sqrt{(7-2\alpha)\,\alpha}}{7\,\sqrt{15}}\right)^{1/4}$

Out[•]=



In[∘]:=
```
kval = 3;
ρsol =
  ρ /. ρ → Root[-32 α - 12 α^2 - α^3 + (180 α + 225 α^2) #1 - 2835 α #1^2 + 10395 #1^3 &, 1];
odm2 = FullSimplify[Sum[odmcoef2[k, ρsol, α] λ^k, {k, 0, kval}]]
(* Error at g→∞ *)
1.0227656721131686` - (ρsol λ)^(1/4) odm2 /. λ → 1
Plot[{Re[%], Im[%]}, {α, 0, 4}]
```

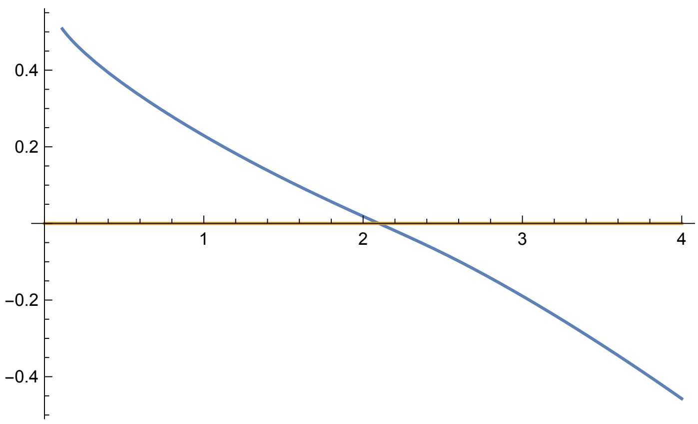
Out[∘]=

$$\frac{1}{32}\left(32+\alpha\lambda\left(8+(4+\alpha)\lambda\right)+3\lambda\,\text{Root}\left[-32\alpha-12\alpha^2-\alpha^3+\left(180\alpha+225\alpha^2\right)\#1-2835\alpha\#1^2+10395\#1^3\&,1\right]\left(-8-10\alpha\lambda+35\lambda\,\text{Root}\left[-32\alpha-12\alpha^2-\alpha^3+\left(180\alpha+225\alpha^2\right)\#1-2835\alpha\#1^2+10395\#1^3\&,1\right]\right)\right)$$

Out[∘]=

$$1.02277-\frac{1}{32}\text{Root}\left[-32\alpha-12\alpha^2-\alpha^3+\left(180\alpha+225\alpha^2\right)\#1-2835\alpha\#1^2+10395\#1^3\&,1\right]^{1/4}\left(32+\alpha(12+\alpha)+3\,\text{Root}\left[-32\alpha-12\alpha^2-\alpha^3+\left(180\alpha+225\alpha^2\right)\#1-2835\alpha\#1^2+10395\#1^3\&,1\right]\left(-8-10\alpha+35\,\text{Root}\left[-32\alpha-12\alpha^2-\alpha^3+\left(180\alpha+225\alpha^2\right)\#1-2835\alpha\#1^2+10395\#1^3\&,1\right]\right)\right)$$

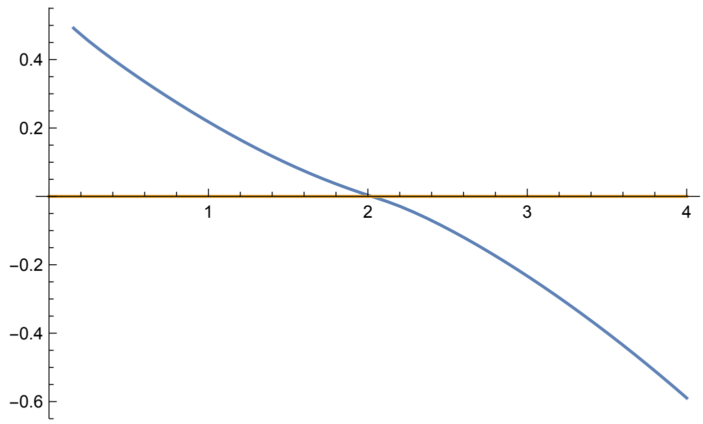
Out[∘]=



```
In[•]:= kval = 5;
        ρsol = Root[odmcoef2[kval, #1, α] &, 1]
        odm2 = FullSimplify[Sum[odmcoef2[k, ρsol, α] λ^k, {k, 0, kval}]];
        (* Error at g→∞ *)
        1.0227656721131686` - (ρsol λ)^(1/4) odm2 /. λ → 1;
        Plot[{Re[%], Im[%]}, {α, 0, 4}]
```

Out[•]= Root$\left[-6144\,\alpha - 3200\,\alpha^2 - 560\,\alpha^3 - 40\,\alpha^4 - \alpha^5 + \left(28\,800\,\alpha + 66\,000\,\alpha^2 + 45\,000\,\alpha^3 + 9375\,\alpha^4\right)\#1 + \left(-302\,400\,\alpha - 1\,020\,600\,\alpha^2 - 765\,450\,\alpha^3\right)\#1^2 + \left(5\,405\,400\,\alpha + 17\,567\,550\,\alpha^2\right)\#1^3 - 172\,297\,125\,\alpha\,\#1^4 + 654\,729\,075\,\#1^5\,\&,\ 1\right]$

Out[•]=




```
In[•]:= realcoupling[ρsol_, αval_, λr_] := Module[{g, θsol, gval, ps},
          g = (ρsol λ / (1 - λ)^α) /. {α → αval, λ → λr e^(i θ)};
          θsol = NSolve[Im[g] == 0 && 0 ≤ θ < 2 π, θ];
          gval = Table[Re[g] /. θsol[[i]], {i, Length[θsol]}];
          ps = Position[gval, Max[gval]][[1, 1]];
          {gval[[ps]], θ /. θsol[[ps]]}]

In[•]:= odmlist[ρsol_, αval_] := Module[{g, θ, λval},
          Table[(
            {g, θ} = realcoupling[ρsol, αval, λr];
            λval = λr e^(i θ);
            {g,
             ((1 - λ)^(α/4) odm2) /. {α → αval, λ → λval}}
           ), {λr, 0.1, 0.9, 0.2}]]
```

In[•]:=
```
odmlist[ρsol, 2]
Show[ListPlot[{Re[%]}], Plot[quarticI[g], {g, 0, 20}]]
{%%[[-1]], quarticI[%%[[-1, 1]]]}
```

NSolve: NSolve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

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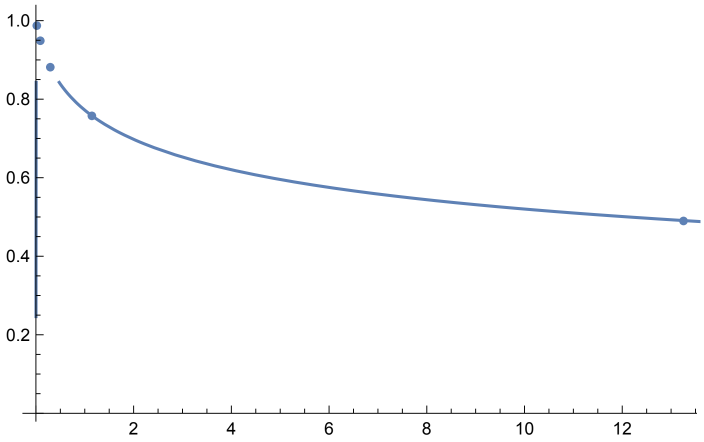
General: Further output of NSolve::ratnz will be suppressed during this calculation.

Out[•]=
```
{{0.0181812, 0.987314}, {0.0901639, 0.948816},
 {0.294535, 0.881456}, {1.14541, 0.757509}, {13.2541, 0.48984}}
```

Out[•]=



Out[•]=
```
{{13.2541, 0.48984}, 0.490888}
```