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# Marino, “Instantons and Large $N$”

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Subject Index

# 1 Definitions

We consider

$$f(z) = \int \mathcal{D}x\, e^{-S[x;z]}, \tag{1.1}$$

where

$$S[x;z] = S_0[x] + zS_I[x]. \tag{1.2}$$

Perturbation series:

$$f_{\mathrm{PT}}(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{1.3}$$

$$a_k = \frac{1}{k!} \int \mathcal{D}x\, e^{-S_0[x]} \left(-S_I[x]\right)^k. \tag{1.4}$$

$$\mathrm{PT}: \int \mathcal{D}x \sum_{k=0}^{\infty} \to \underbrace{\sum_{k=0}^{\infty} \int \mathcal{D}x}_{\text{Asymptotic}} \tag{1.5}$$

## *1.1 Borel resummation*

- Borel transform:

$$B(u) \equiv \sum_{k=0}^{\infty} \frac{a_k}{k!} u^k. \tag{1.6}$$

- Borel resummation:

$$f_{\mathrm{Borel}}(z) \equiv \frac{1}{z} \int_0^{\infty} du\, B(u) e^{-u/z} \tag{1.7}$$

$$= \int_0^{\infty} du\, B(zu) e^{-u} \tag{1.8}$$

Noting that

$$k! = \int_0^{\infty} du\, u^k e^{-u}, \tag{1.9}$$

we have

$$f(z) = \int \mathcal{D}x \sum_{k=0}^{\infty} \int_0^{\infty} du\, e^{-S_0[x]} \frac{1}{(k!)^2} \left(-zS_I[x]\right)^k u^k e^{-u}. \tag{1.10}$$

$$\mathrm{Borel}: \int \mathcal{D}x \sum_{k=0}^{\infty} \int_0^{\infty} du \to \underbrace{\sum_{k=0}^{\infty} \int \mathcal{D}x}_{\text{Asymptotic}} \int_0^{\infty} du \to \int_0^{\infty} du \underbrace{\sum_{k=0}^{\infty} \int \mathcal{D}x}_{u \sim 0} \tag{1.11}$$

### 1.2 't Hooft

Suppose that $S[x;z] \geq 0$ for $z > 0$.

$$f(z) = \int \mathcal{D}x \int_0^\infty dt\, \delta(t - S[x;z]) e^{-S[x;z]} \tag{1.12}$$

$$f_\delta(z) \equiv \int_0^\infty dt\, F_\delta(t;z) e^{-t}, \tag{1.13}$$

$$F_\delta(t;z) \equiv \int \mathcal{D}x\, \delta(t - S[x;z]) \tag{1.14}$$

$$= \sum_{x_i} \left| \frac{\delta S[x;z]}{\delta x} \right|^{-1} \Bigg|_{x\,=\,x_i \text{ s.t. } S[x_i;z]\,=\,t} \tag{1.15}$$

When $x \in \mathbb{R}$, this is a change of variables: $x \mapsto t$.

#### 1.2.1 Saddle points

For $x \sim x_0$ such that $S[x_0;z] = 0$ and $\delta S[x;z]/\delta x|_{x=x_0} = 0$, we have

$$a_0 \equiv \left. \frac{\delta^2 S[x;z]}{\delta x^2} \right|_{x=x_0} > 0 \tag{1.16}$$

$$S[x_0 + \epsilon; z] \sim \frac{1}{2} a_0 \epsilon^2 = t \qquad\qquad \epsilon = \pm\sqrt{2t/a_0} \equiv \epsilon_\pm \tag{1.17}$$

and then

$$\sum_{\epsilon=\epsilon_\pm} \left| \left. \frac{\delta S[x;z]}{\delta x} \right|_{x=x_0+\epsilon} \right|^{-1} \tag{1.18}$$

$$= \sum_{\epsilon=\epsilon_\pm} \left| \epsilon \frac{\delta}{\delta x} \left. \frac{\delta S[x;z]}{\delta x} \right|_{x=x_0} \right|^{-1} = \sqrt{\frac{2}{a_0 t}}. \tag{1.19}$$

$\int_0 dt/\sqrt{t}$ is convergent near $x = x_0$! (We see a branch point at $t = 0$.) Similarly, for $x \sim x_*^\pm$ such that $\delta S[x;z]/\delta x|_{x=x_*^\pm} = 0$, and

$$c_*^\pm \equiv S[x_*^\pm; z] > 0 \qquad\qquad a_*^\pm \equiv \left. \frac{\delta^2 S[x;z]}{\delta x^2} \right|_{x=x_*^\pm} \gtrless 0, \tag{1.20}$$

$\int_{c_*^+} dt/\sqrt{t - c_*^+}$ and $\int^{c_*^-} dt/\sqrt{c_*^- - t}$ are convergent near $x = x_*$.

*Example.*

$$S[x] = (x^2-1)^2 \qquad\qquad f \sim 1.97 \tag{1.21}$$

$$F_\delta(t) = \begin{cases} \frac{1}{2\sqrt{t}}\left(\frac{1}{\sqrt{1+\sqrt{t}}} + \frac{1}{\sqrt{1-\sqrt{t}}}\right) & \text{for } 0<t<1 \\ \frac{1}{2\sqrt{t}}\frac{1}{\sqrt{1+\sqrt{t}}} & \text{for } t>1 \end{cases} \tag{1.22}$$

$$f_\delta \sim 1.97 \tag{1.23}$$

$$\int_{-\sqrt{2}}^{\sqrt{2}} e^{-S} \sim 1.88 \qquad\qquad \int_0^1 dt\, F_\delta(t) \sim 1.88 \tag{1.24}$$

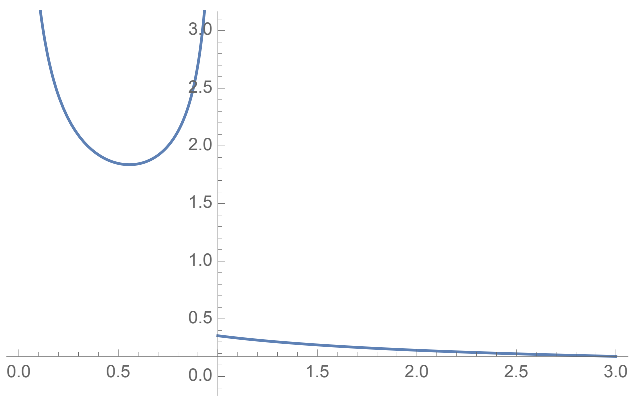


**Fig. 1** Eq. (1.22). $F(t) \overset{x\to 0}{\to} \frac{1}{\sqrt{t}}$, $F(t) \overset{x\to 1-0}{\to} \frac{1}{\sqrt{2(1-t)}} + \frac{1}{2\sqrt{2}}$, and $F(t) \overset{x\to 1+0}{\to} \frac{1}{2\sqrt{2}}$

# 2. 't Hooft ↔ Borel

## *2.1. Inverse Laplace transform*

*"Borel".*

$$f(z) = \int \mathcal{D}x \sum_{k=0}^{\infty} \int_0^\infty \frac{du}{z}\, e^{-S_0[x]} \frac{1}{(k!)^2} (-zS_I[x])^k \left(\frac{u}{z}\right)^k e^{-u/z} \tag{2.1}$$

$$= \frac{1}{z}\int_0^\infty du \int \mathcal{D}x\, e^{-S_0[x]} e^{-u/z} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} (-uS_I[x])^k \tag{2.2}$$

$$= \frac{1}{z}\int_0^\infty du\, e^{-u/z} \underbrace{\int \mathcal{D}x\, e^{-S_0[x]} J_0(2\sqrt{uS_I[x]})}_{\mathcal{B}(u)} \tag{2.3}$$

Not asymptotic expansion!

*'t Hooft.* Suppose that there exsists a mapping,

$$x \mapsto y = y(x), \tag{2.4}$$

such that

$$f(z) = \int \mathcal{D}x\, e^{-S[x;z]} = \sigma(z) \int \mathcal{D}y\, e^{-\frac{1}{z}S[y]}. \tag{2.5}$$

$$f(z) = \sigma(z) \int_0^\infty dt\, e^{-t/z} \underbrace{\int \mathcal{D}y\, \delta(t - S[y])}_{\mathcal{F}_\delta(t)} \tag{2.6}$$

We define the inverse Laplace transform of the overall factor

$$z\sigma(z) = \int_0^\infty dt\, e^{-t/z} \mathcal{G}(t). \tag{2.7}$$

Then,

$$f(z) = \frac{1}{z} \int_0^\infty dt\, e^{-t/z} (\mathcal{F}_\delta * \mathcal{G})(t) \tag{2.8}$$

Here "$*$" denotes the convolution,

$$(f * g)(t) = \int_0^t ds\, f(s) g(t-s) \tag{2.9}$$

*Example.*

$$S[x;z] = \frac{x^2}{2} + z\frac{x^4}{4} \tag{2.10}$$

$$\frac{dS}{dx} = x + zx^3 \tag{2.11}$$

$$t = S[x;z] = \frac{z}{4}\left(\frac{1}{z} + x^2\right)^2 - \frac{1}{4z} \tag{2.12}$$

$$x^2 + \frac{1}{z} = \pm\frac{1}{z}\sqrt{(4tz+1)} \tag{2.13}$$

$$x = \pm\sqrt{-\frac{1}{z} \pm \frac{1}{z}\sqrt{4tz+1}} \tag{2.14}$$

$$\mathcal{F}_\delta(t) = \int dx\, \delta(t - S[y]) = \frac{2}{x_*(1+x_*^2)} = 2\sqrt{\frac{1}{-1+\sqrt{4t+1}}}\frac{1}{\sqrt{4t+1}} \tag{2.15}$$

for $t > 0$, and

$$\mathcal{G}_\delta(t) = \frac{1}{\sqrt{\pi t}} \tag{2.16}$$

$$\mathcal{B}(u) = \int dx\, e^{-x^2/2} J_0(2\sqrt{ux^4/4}) = 2\sqrt{\frac{2}{\pi}}(1+4u)^{-1/4} K(\tfrac{1}{2} - \tfrac{1}{2\sqrt{1+4u}}) \tag{2.17}$$

where $K(z)$ is the complete elliptic integral of the first kind.

In the figure below, we can see $\mathcal{F} * \mathcal{G} = \mathcal{B}$ for $t > 0$. We have $f = f_{\mathcal{B}} = f_\delta \sim 1.94$ $(z = 1)$.

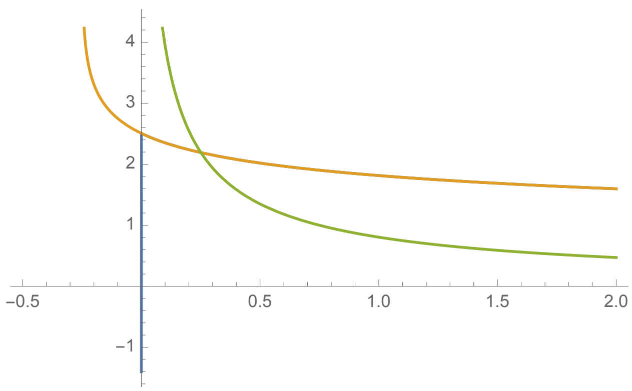


**Fig. 2** $(\mathcal{F} * \mathcal{G})(t)$ (blue) and $\mathcal{B}(t)$ (orange) [and $\mathcal{F}(t)$ (green)].

*2.2. 0D models*

$$S[x; g^2] = \frac{x^2}{2} + \frac{1}{g^2} V[gx], \tag{2.18}$$

$$V[x] = V_3 x^3 + V_4 x^4 + \dots \tag{2.19}$$

and $S[x] \equiv S[x; 1]$.

$$f(z) = \int dx\, e^{-S[x;g^2]} \tag{2.20}$$

$$= \frac{1}{\sqrt{g^2}} \int dx\, e^{-\frac{1}{g^2} S[x]} \tag{2.21}$$

$$= \frac{1}{g^2}\sqrt{g^2} \int_0^\infty dt\, e^{-t/g^2} \underbrace{\int dx\, \delta(t - S[x])}_{F(t)} \tag{2.22}$$

Noting that

$$\sqrt{g^2} = \int_0^\infty dt\, e^{-t/g^2} \underbrace{\frac{1}{\sqrt{\pi t}}}_{G(t)}, \tag{2.23}$$

we have

$$f(z) = \frac{1}{g^2} \int_0^\infty dt\, e^{-t/g^2} (F * G)(t) \tag{2.24}$$

$$\begin{aligned}
(F * G)(t) &= \int_0^t ds\, F(s) G(t-s) && (2.25)\\
&= \int_0^t ds \int dx\, \delta(s - S[x]) \frac{1}{\sqrt{\pi(t-s)}} && (2.26)\\
&= \sum_{x_i} \int_0^t ds \left. \left| \frac{dS[x]}{dx} \right|^{-1} \right|_{x\,=\,x_i \text{ s.t. } S[x]\,=\,s} \frac{1}{\sqrt{\pi(t-s)}} && (2.27)
\end{aligned}$$

- For $x \sim x_0$ ($t \gtrsim 0$),

$$\int_0^t ds\, \frac{1}{\sqrt{s(t-s)}} = \pi \quad \text{(const. and finite)} \tag{2.28}$$

- For $x \sim x_*^+$ ($t \gtrsim c_*^+$)

$$\int_{c_*^+}^t ds\, \frac{1}{\sqrt{(s - c_*^+)(t-s)}} = \pi \quad \text{(const. and finite)} \tag{2.29}$$

- For $x \sim x_*^-$ ($t \lesssim c_*^-$)

$$\begin{aligned}
\int_{(0)}^t ds\, \frac{1}{\sqrt{(c_*^- - s)(t-s)}} &= 2 \operatorname{arcosh}\left( \sqrt{\frac{c_*^-}{c_*^- - t}} \right) && (2.30)\\
&= \ln\left( \frac{4c_*^-}{c_*^- - t} \right) + O(c_*^- - t) && (2.31)
\end{aligned}$$

*Example.*

$$S[x] = (x^2 - 1)^2 \tag{2.32}$$

$$F(t) = \begin{cases} \frac{1}{2\sqrt{t}} \left( \frac{1}{\sqrt{1+\sqrt{t}}} + \frac{1}{\sqrt{1-\sqrt{t}}} \right) & \text{for } 0 < t < 1 \\ \frac{1}{2\sqrt{t}} \frac{1}{\sqrt{1+\sqrt{t}}} & \text{for } t > 1 \end{cases} \tag{2.33}$$

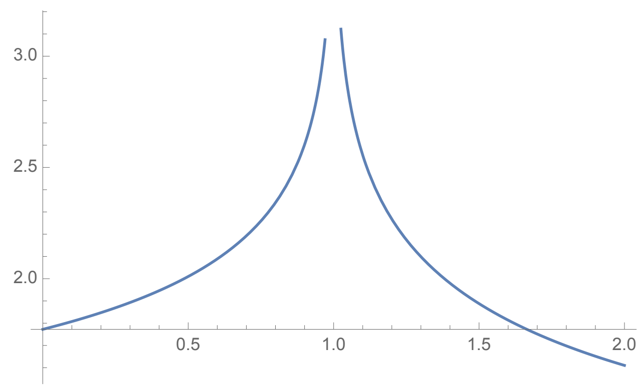


**Fig. 3** $F * G$

*Example.*

$$S[x] = \frac{x^2}{2} - 2\gamma\frac{x^3}{3} + \frac{x^4}{4} \tag{2.34}$$

$$\frac{dS[x]}{dx} = x[1 - \gamma^2 + (x-\gamma)^2] \tag{2.35}$$

$$x_0 = 0, \quad x_*^{\pm} = \gamma \pm \sqrt{\gamma^2 - 1} \tag{2.36}$$

$$S[x_0] = 0 \tag{2.37}$$

$$S[x_*^{\pm}] = -\frac{1}{4} + \gamma^2 - \frac{2}{3}\gamma^4 \mp \frac{2}{3}\gamma(\gamma^2-1)^{3/2} \tag{2.38}$$

$$t = S[x] = x^2\left(\frac{1}{2} - 2\gamma\frac{x}{3} + \frac{x^2}{4}\right) \tag{2.39}$$

$$= x^2\left[\left(\frac{x}{2} - \gamma\frac{2}{3}\right)^2 - \gamma^2\frac{4}{9} + \frac{1}{2}\right] \tag{2.40}$$

$$\sum_{k=0}^{\infty}\int dx\, e^{-x^2/2}\frac{1}{k!}\left(2\gamma g\frac{x^3}{3} - g^2\frac{x^4}{4}\right)^k \tag{2.41}$$

$$= \sum_{k=0}^{\infty}\int dx\, e^{-x^2/2}\frac{1}{k!}\sum_{m=0}^{k} {}_kC_m\left(2\gamma g\frac{x^3}{3}\right)^{k-m}\left(g^2\frac{x^4}{4}\right)^m \tag{2.42}$$

$$= \sum_{k=0}^{\infty}\int dx\, e^{-x^2/2}\frac{1}{k!}\sum_{m=0}^{k} {}_kC_m g^{k+m}x^{3k+m}\left(\frac{\gamma}{3}\right)^{k-m}2^{k-3m} \tag{2.43}$$

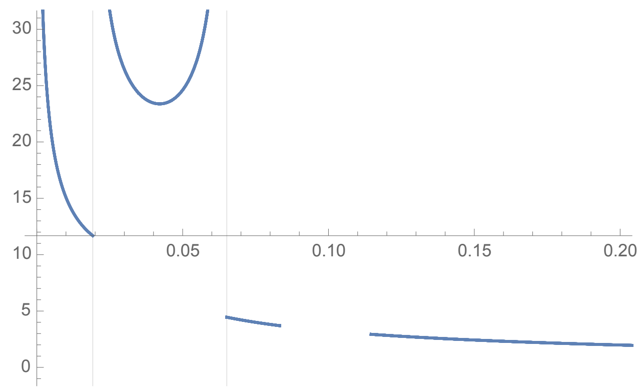


**Fig. 4** $F$ with $\gamma = 1.05$

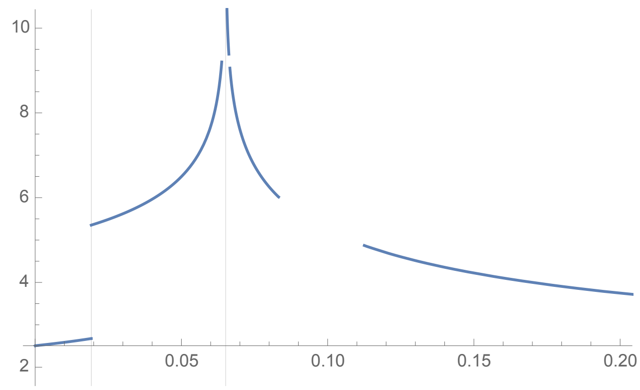


**Fig. 5** $F * G$ with $\gamma = 1.05$

$k' = 2\ell = k + m$

$$= \sum_{k'=0}^{\infty} \left(\frac{2\gamma}{3} g\right)^{k'} \sum_{m=0}^{[k'/2]} \frac{1}{(k'-m)!} {}_{k'-m}C_m \left(\frac{4\gamma}{3}\right)^{-2m} \int dx\, e^{-x^2/2} x^{3k'-2m} \tag{2.44}$$

$$= \sum_{\ell=0}^{\infty} \left(\frac{2\gamma}{3} g\right)^{2\ell} \sum_{m=0}^{\ell} \frac{1}{(2\ell-m)!} {}_{2\ell-m}C_m \left(\frac{4\gamma}{3}\right)^{-2m} 2^{3\ell-m+1/2} \Gamma(3\ell - m + \tfrac{1}{2}) \tag{2.45}$$

$$= \sum_{\ell=0}^{\infty} \left(\frac{2\gamma}{3} g\right)^{2\ell} 2^{3\ell+1/2} \frac{\Gamma(3\ell+\frac{1}{2})}{\Gamma(2\ell+1)} F(\tfrac{1}{2} - \ell, -\ell; \tfrac{1}{2} - 3\ell; -\frac{9}{8\gamma^2}) \tag{2.46}$$