o-o

2020-04-28

Contents

R bookdown

Chapter 1

{Gaussian elimination}

$$A\mathbf{x} = \mathbf{b}$$

$A \in R^{m\times n}, x \in R^n, b \in R^m,$ A

1.1

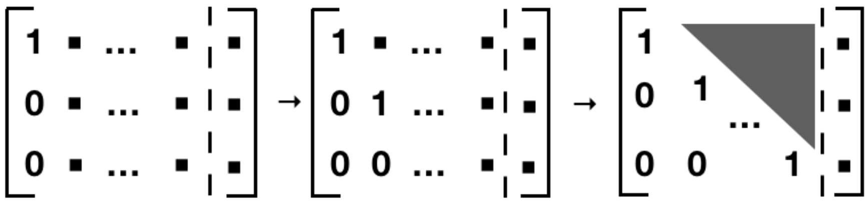
$$a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n = b_1 a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n = b_2 \cdots a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

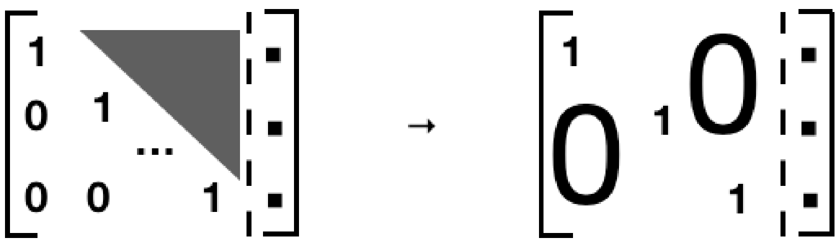
- row_1/a_{11}
- $row_2 - k * row_1, k = \frac{a_{21}}{a_{11}}$
- ...

$$\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$$

- $row_2/(current)a_{22}$
- $\cdots$



A x_m , $x_1 \cdots x_m$



- x_m get
- $row_{m-1} - k * row$
- $\cdots$

- $row * k$
- $row_j + row_i * k$

- LU decomposition: A = LU ;
- Forward substitution: solve Ly = b ;
- Backward substitution: solve U x = y .

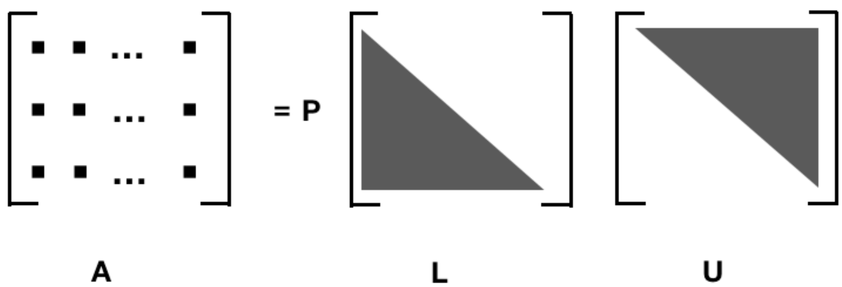
- a_{11} 0 0
- $a_{11} \ll 1$ overflow

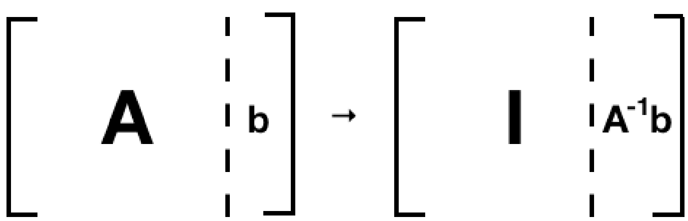
pivot $A\mathbf{x} = \mathbf{b}$ Gaussian elimination with partial pivoting (GEPP).

https://web.mit.edu/10.001/Web/Course_Notes/GaussElimPivoting.html

partial pivoting total pivoting a_{11},

$$A = PLU$$





A^{-1}

1.2

`scipy.linalg.solve` , A^{-1} $A^{-1}b$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_1 - 3x_2 - 3x_3 = -1 \\ 4x_1 + 2x_2 + 2x_3 = 3 \end{cases}$$

```
import numpy as np
from scipy import linalg

a = np.array([[2 ,4 ,-2],
              [1, -3, -3],
              [4, 2, 2,]])
b = np.array([2, -1, 3])

x = linalg.solve(a, b)
np.dot(a, x) # array([ 2., -1.,  3.])
x # array([0.5       , 0.33333333, 0.16666667])
```

```
array([0.5       , 0.33333333, 0.16666667])
```

$$\begin{cases} x_1 = 1/2 \\ x_2 = 1/3 \\ x_3 = 1/6 \end{cases}$$

lu scipy.linalg.lu

Chapter 2

Cholesky {Cholesky decomposition}

Cholesky o( )o ‘ ’

2.1 $A^T A$

$$A\mathbf{x} = \mathbf{b}$$

$x = A^{-1}\mathbf{b}$ A A^{-1}.

$$||A\mathbf{x} - \mathbf{b}||_2$$

$$\begin{aligned}
||A\mathbf{x} - \mathbf{b}||^2 &= (A\mathbf{x} - \mathbf{b}) \cdot (A\mathbf{x} - \mathbf{b}) \\
&= (A\mathbf{x} - \mathbf{b})^T \cdot (A\mathbf{x} - \mathbf{b}) \\
&= (\mathbf{x}^T A^T - \mathbf{b}^T) \cdot (A\mathbf{x} - \mathbf{b}) \\
&= (\mathbf{x}^T A^T A\mathbf{x} - 2\mathbf{b}^T A\mathbf{x} + \mathbf{b}^T\mathbf{b})
\end{aligned}$$

Error $\mathbf{x}$

$$\frac{\partial E}{\partial \mathbf{x}} = 2A^TA\mathbf{x} - 2A^T\mathbf{b} = 0$$

$$A^TA\mathbf{x} = A^T\mathbf{b} \tag{1}$$

(1) least_squares_SP

(1)

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} \tag{2}$$

(1) (2) $(A^TA)^{-1}$, A A^TA Tikhonov regularization

2.2

A^TA $(AB)^T = B^TA^T$

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

2.3

$$M \iff x^{\mathsf{T}}Mx > 0 \text{ for all } x \in \mathbb{R}^n \quad \mathbf{0}$$

$$M \iff x^{\mathsf{T}}Mx \geq 0 \text{ for all } x \in \mathbb{R}^n$$

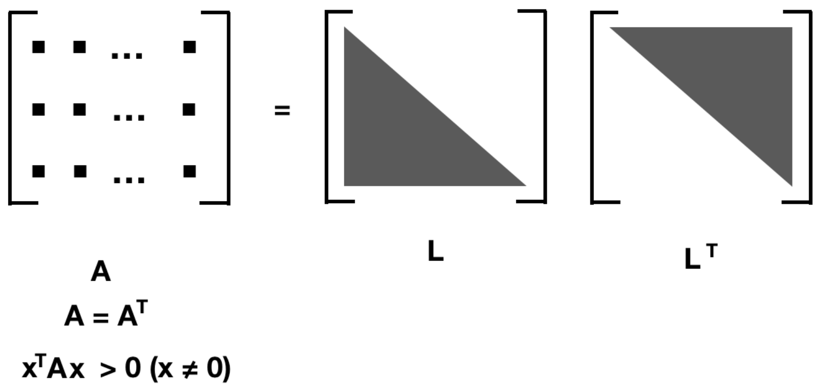
A A A^TA :

$$x^TA^TAx = (Ax)^T \cdot Ax = ||Ax||^2$$

Cholesky

2.4 Cholesky

A SPD (real Symmetric positive definite matrix A A lower triangle L upper triangle L^T.



How to prove the existence and uniqueness of Cholesky decomposition?

$$A = PLU$$

A

$$A = LL^T$$

Cholesky LU

2.5

$$\begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix}$$

`scipy.linalg.cholesky`

```
import numpy as np
from scipy import linalg
```

```
a = np.array([[4, 12, -16],
              [12, 37, -43],
              [-16, -43, 98]])

L = linalg.cholesky(a, lower=True) #    upper     lower = True

# array([[ 2.,  0.,  0.],
#        [ 6.,  1.,  0.],
#        [-8.,  5.,  3.]])

np.allclose(np.dot(L, L.T) , a) #
```

Chapter 3

QR

QR

$$A = QR$$

Q R upper triangle .

Q

- $QQ^T = I$
- det(Q) = ±1, det(Q) = 1 SO(n)
-

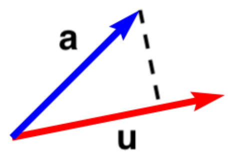
3.1 Gram-Schmidt

Gram–Schmidt Q:

projection

$$\mathrm{proj}_{\mathbf{u}}\,\mathbf{a} = \frac{\langle \mathbf{u}, \mathbf{a}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}$$

a **u**



$A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$,

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{a}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{a}_2 - \mathrm{proj}_{\mathbf{u}_1}\, \mathbf{a}_2, & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_1}\, \mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_2}\, \mathbf{a}_3, & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{a}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}\, \mathbf{a}_k, & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}
\end{aligned}$$

A

$$\begin{aligned}
\mathbf{a}_1 &= \langle \mathbf{e}_1, \mathbf{a}_1 \rangle \mathbf{e}_1 \\
\mathbf{a}_2 &= \langle \mathbf{e}_1, \mathbf{a}_2 \rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_2 \rangle \mathbf{e}_2 \\
\mathbf{a}_3 &= \langle \mathbf{e}_1, \mathbf{a}_3 \rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_3 \rangle \mathbf{e}_2 + \langle \mathbf{e}_3, \mathbf{a}_3 \rangle \mathbf{e}_3 \\
&\vdots \\
\mathbf{a}_k &= \sum_{j=1}^{k} \langle \mathbf{e}_j, \mathbf{a}_k \rangle \mathbf{e}_j
\end{aligned}$$

$$Q = [\mathbf{e}_1, \dots, \mathbf{e}_n]$$

$$R = \begin{pmatrix} \langle \mathbf{e}_1, \mathbf{a}_1 \rangle & \langle \mathbf{e}_1, \mathbf{a}_2 \rangle & \langle \mathbf{e}_1, \mathbf{a}_3 \rangle & \dots \\ 0 & \langle \mathbf{e}_2, \mathbf{a}_2 \rangle & \langle \mathbf{e}_2, \mathbf{a}_3 \rangle & \dots \\ 0 & 0 & \langle \mathbf{e}_3, \mathbf{a}_3 \rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

R $\mathbf{a}_1$ $\mathbf{e}_1$ $\mathbf{a}_2$ $\mathbf{e}_1, \mathbf{e}_2$ $\mathbf{a}_k$ $\mathbf{e}_1 \cdots \mathbf{e}_k$

Gram-Schmidt $\begin{pmatrix} 1 & 1+\varepsilon \\ 1 & 1 \end{pmatrix}, \varepsilon \ll 1,$

3.2 Householder

projection

$$\operatorname{proj}_{\mathbf{u}} \mathbf{a} = \frac{\langle \mathbf{u}, \mathbf{a} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

$$\operatorname{proj}_{\mathbf{u}} \mathbf{a} = \frac{\mathbf{u}^T \cdot \mathbf{a}}{\mathbf{u}^T \cdot \mathbf{u}} \cdot \mathbf{u} = \frac{(\mathbf{u}^T \cdot \mathbf{a}) \cdot \mathbf{u}}{\mathbf{u}^T \cdot \mathbf{u}}$$

$$\operatorname{proj}_{\mathbf{u}} \mathbf{a} = \frac{\mathbf{u} \cdot \mathbf{u}^T \cdot \mathbf{a}}{\mathbf{u}^T \cdot \mathbf{u}}$$

$\mathbf{u} \cdot \mathbf{u}^T$ 3 x 3

$$P = \frac{\mathbf{u} \cdot \mathbf{u}^T}{\mathbf{u}^T \cdot \mathbf{u}}$$

$P^2 = P$....

$$\begin{aligned} \mathbf{b} - 2 \operatorname{proj}_{\mathbf{u}} \mathbf{b} &= \mathbf{b} - 2 \frac{\langle \mathbf{u}, \mathbf{b} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \\ &= \mathbf{b} - 2 \frac{\mathbf{u} \cdot \mathbf{u}^T}{\mathbf{u}^T \cdot \mathbf{u}} \mathbf{b} \\ &= (I - 2 \frac{\mathbf{u} \cdot \mathbf{u}^T}{\mathbf{u}^T \cdot \mathbf{u}}) \mathbf{b} \\ &= H_u \mathbf{b} \end{aligned}$$

$$H_u = I - 2 \frac{\mathbf{u} \cdot \mathbf{u}^T}{\mathbf{u}^T \cdot \mathbf{u}}$$

wikipedia

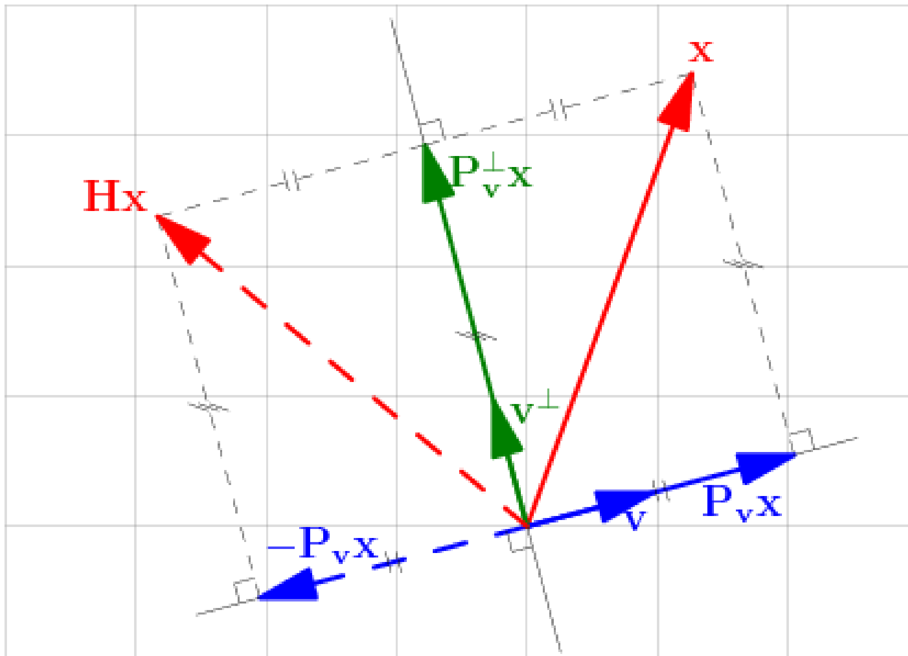


Figure 3.1: from wikipedia

$$\mathbf{x} = [x_1, \dots, x_n]^T \qquad \mathbf{e} = [1, 0, \dots, 0]^T$$

$\mathbf{H} = \mathbf{I} - \frac{2}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}\mathbf{v}^H$

$\mathbf{v}$

$\mathbf{v} = \mathbf{x} + \mathrm{sign}(\mathrm{x}_1)\|\mathrm{x}\|_2 \mathbf{e}_1.$

- $H^T = H$
- $H^{-1} = H$
- $H^2 = I$

$\mathbf{a}$ A H_u :

$$c\mathbf{e}_1 = H_u \mathbf{a}$$

$$\begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} \to \begin{pmatrix} 1 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix}$$

$$R = H_{u_n} \cdots H_{u_1} A Q = H_{u_1}^T \cdots H_{u_n}^T$$

Gram-Schmidt Householder $A^{m \times n}$

- Gram-Schmidt : $Q \in R^{m \times n}, R \in R^{n \times n}$
- Householder: $Q \in R^{m \times m}, R \in R^{m \times n}$

Householder Gram-Schmidt.

3.3

$$A = \begin{pmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.8571 & -0.3943 & 0.3314 \\ 0.4286 & 0.9029 & -0.0343 \\ -0.2857 & 0.1714 & 0.9429 \end{pmatrix}$$

$$R = \begin{pmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & -35 \end{pmatrix}$$

QR `scipy.linalg.qr`:

```
import numpy as np
from scipy import linalg

a = np.array([[12, -51, 4],
              [6, 167, -68],
              [-4, 24, -41]])

q, r = linalg.qr(a)

q
# array([[-0.85714286,  0.39428571,  0.33142857],
#        [-0.42857143, -0.90285714, -0.03428571],
#        [ 0.28571429, -0.17142857,  0.94285714]])
```

```
r
# array([[ -14.,  -21.,   14.],
#        [   0., -175.,   70.],
#        [   0.,    0.,  -35.]])

np.allclose(np.dot(q, r), a) # True
```

$\mathbf{e}_1 \cdots \mathbf{e}_n$.

A = QR A QR

Chapter 4

{Eigen decomposition}

4.1

4.1.1

eigen $\mathbf{x}_i$,

$\mathbf{v}_i$

$$\text{minimize} \sum_i ||\mathbf{x}_i - \text{proj}_{\mathbf{v}}\mathbf{x}_i||^2 ||\mathbf{v}|| = 1$$

$||\mathbf{v}|| = 1$ $\mathbf{v}$

$$\begin{aligned}\sum ||\mathbf{x}_i - \text{proj}_{\mathbf{v}}\mathbf{x}_i||^2 &= ||\mathbf{x}_i - (\mathbf{x}_i \cdot \mathbf{v})\mathbf{v})||^2 \\ &= ||\mathbf{x}_i||^2 - 2(\mathbf{x}_i \cdot \mathbf{v})^2 + (\mathbf{x}_i \cdot \mathbf{v})^2 \\ &= ||\mathbf{x}_i||^2 - (\mathbf{x}_i \cdot \mathbf{v})^2\end{aligned}$$

$\mathbf{x}_i$ $\mathbf{v}$

$$\sum_i (\mathbf{x}_i \cdot \mathbf{v})^2$$

$$||X^T\mathbf{v}||^2 ||\mathbf{v}|| = 1$$

$\mathbf{x}_i$ $\mathbf{v}$ $X^T\mathbf{v}$ $||\mathbf{v}|| = 1$,
$\mathbf{v}$

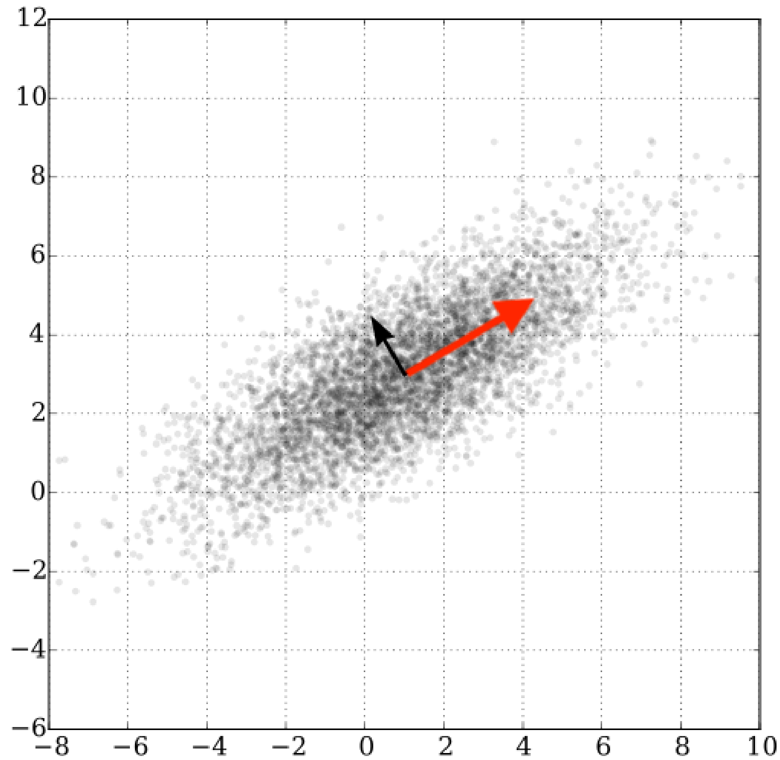


Figure 4.1: wikipedia

$||X^T\mathbf{v}||^2$ $\mathbf{v}^T XX^T\mathbf{v}$

$$XX^T\mathbf{v} = \lambda\mathbf{v}$$

XX^T eigenvalue eigenvector.

4.1.2

eigevalue eigenvector

$$F = m\frac{d^2\mathbf{x}}{dt} = -k\mathbf{x}$$

D:

$$D^2 : f[\mathbf{x}] \to f[\mathbf{x}]\mathbf{x} \mapsto D^2\mathbf{x} = \lambda\mathbf{x}$$

4.1.3 Quadratic Energy

setup

Have:

- n items in a dataset
- $w_{ij} \geq 0$ similarity of items i and j
- $w_{ij} = w_{ji}$

Want: - x_i embedding on R

energy

$$E(\mathbf{x}) = \sum_{ij} w_{ij}(x_i - x_j)^2$$

$E(\mathbf{x})$

$$||\mathbf{x}||^2 = 1\mathbf{1} \cdot \mathbf{x} = 0$$

0.

$$\Lambda = \sum_{ij} w_{ij}(x_i - x_j)^2 - \lambda(\mathbf{x}^T \cdot \mathbf{x} - 1) - \mu(\mathbf{1} \cdot \mathbf{x})$$

$$E(\mathbf{x}) = \mathbf{x}^T(2A - 2W)\mathbf{x}$$

2A − 2W 0 **0**.

- Spectral Graph Partitioning and the Laplacian with Matlab
- The Smallest Eigenvalues of a Graph Laplacian

4.2

4.2.1

$$A\mathbf{x} = \lambda\mathbf{x}\lambda \in \mathbb{R}, A \in \mathbb{R}^{n\times n}$$

$$A\mathbf{x} = \lambda\mathbf{x}\lambda \in \mathbb{C}, \mathbf{x} \in \mathbb{C}^n$$

λ $\mathbf{x}$

4.2.2

A n × n (A) $\rho(A) = \max\{|\lambda_1|, \cdots, |\lambda_n|\}$

A A

4.2.3

- $A \in \mathbb{R}^{n\times n}$
-

A n rank n n column vector

4.2.4

$z = a + ib \in \mathbb{C}$ $\bar{z} = a - ib$

A (m x n)

$$\left(A^{\mathrm{H}}\right)_{ij} = \overline{A_{ji}}$$

$$A^{\mathrm{H}} = \left(\overline{A}\right)^{\mathsf{T}} = \overline{A^{\mathsf{T}}}$$

Hermitian

$$A \text{ Hermitian} \quad \Longleftrightarrow \quad A = A^H$$

A

$$\begin{bmatrix} 3 & 2+i \\ 2-i & 1 \end{bmatrix}$$

4.2.5

$A \in \mathbb{C}^{n\times n}$ $A \in \mathbf{R}^{n\times n}$, A n $\mathbf{x}_1, \cdots, \mathbf{x}_n$ ( ) $\lambda_1, \cdots, \lambda_n$.

span $\mathbb{R}^n$

I :

I n eigenvalue 1 I span $\mathbb{R}^n$

Gram–Schmidt eigenvalue eigenvector span plane o( )o

If a real symmetric matrix has repeated eigenvalues, why does it still have n linearly independent eigenvectors?

- $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots, \}$
- $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

4.2.6

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$p(\lambda) = det(A - \lambda I) = 0$$

Fundamental theorem of algebra $p(\lambda)$ N “ ” Spectrum

: n n

$$p\left(\lambda\right)=\left(\lambda-\lambda_1\right)^{n_1}\left(\lambda-\lambda_2\right)^{n_2}\cdots\left(\lambda-\lambda_k\right)^{n_k}=0$$

$$\sum_{i=1}^{k}n_i=N$$

λ_i

$$\left(\mathbf{A}-\lambda_i\mathbf{I}\right)\mathbf{v}=0$$

$m_i(1\leq m_i\leq n_i)$ m_i λ_i m_i λ_i geometric multiplicity n_i algebraic multiplicity $m_i=n_i=1$

“ A” n x n A n

4.3

$$A^k\mathbf{x}=\lambda^k\mathbf{x}$$

A

$$A^{-1}\mathbf{x}=\frac{1}{\lambda}\mathbf{x}$$

setup

$A\in\mathbb{R}^{n\times n}\mathbf{x}_1,\cdots,\mathbf{x}_n\in\mathbb{R}^n$ eigenvector$|\lambda_1|\geq|\lambda_2|\geq\cdots\geq|\lambda_n|$ eigenvalues

$\mathbf{v}\in\mathbb{R}^n$

$$\mathbf{v}=c_1\mathbf{x}_1+\cdots+c_n\mathbf{x}_n$$

$$A^k\mathbf{v} = \lambda_1^k\left(c_1\mathbf{x}_1 + (\frac{\lambda_2}{\lambda_1})^k c_2\mathbf{x}_2 + \cdots + (\frac{\lambda_n}{\lambda_1})^k c_n\mathbf{x}_n\right)$$

4.3.1 eigenvalue eigenvector

$|\lambda_2| \leq |\lambda_1|$

$$A^k\mathbf{v} = \lambda_1^k\mathbf{x}$$

$$\mathbf{v}_k = A\mathbf{v}_{k-1}$$

eigenvalue **eigenvector** $|\lambda_1| \geq 1$ normalize

$$\mathbf{w}_k = A\mathbf{v}_{k-1}\mathbf{v}_k = \frac{\mathbf{w}_k}{|\mathbf{w}_k|}$$

norm

4.3.2 eigenvalue eigenvector

eigenvalue **eigenvector**

$$\mathbf{w}_k = A^{-1}\mathbf{v}_{k-1}\mathbf{v}_k = \frac{\mathbf{w}_k}{|\mathbf{w}_k|}$$

A^{-1}

$$|\frac{1}{\lambda_1}| < |\frac{1}{\lambda_2}| < \cdots < |\frac{1}{\lambda_n}|$$

4.3.3 σ eigenvalue eigenvector

σ **eigenvalue** **eigenvector** :

$$\mathbf{v}_{k+1} = \frac{(A-\sigma I)^{-1}\mathbf{v}_k}{||(A-\sigma I)^{-1}\mathbf{v}_k||}$$

$(A - \sigma I)$ eigenvector eigenvalue 0 make sense.

4.3.4 eigenvector eigenvalue

v eigenvector eigenvalue

$$A\mathbf{v} \approx \lambda \mathbf{v}$$

λ

$$\arg\min_{\lambda} ||A\mathbf{v} - \lambda \mathbf{v}||^2$$

λ,

$$\lambda = \frac{\mathbf{v}^T A \mathbf{v}}{||\mathbf{v}||^2}$$

Rayleigh quotient)

Rayleigh quotient iteration

1. $\mathbf{v} \in R^n$
2.
   - $\sigma_k = \frac{\mathbf{v}^T A \mathbf{v}}{||\mathbf{v}||^2}$
   - $\mathbf{w}_k = (A - \sigma I)^{-1} \mathbf{v}_{k-1}$
   - $\mathbf{v}_k = \frac{\mathbf{w}_k}{|\mathbf{w}_k|}$

wikipedia Rayleigh quotient iteration

eigenvalue eigenvalue σ

4.3.5 All eigenevalue

eigenvalue

$$\mathbf{v} = c_1 \mathbf{x}_1 + \cdots + c_n \mathbf{x}_n$$

$\mathbf{x}$ $\mathbf{v}_0$

$$\mathbf{v}_0 \cdot \mathbf{x}_1 = 0$$

$\mathbf{v}$ $\mathbf{x}_1$ $\mathbf{v}_1$

$$\mathbf{v}_1 = c_2 \mathbf{x}_2 + \cdots + c_n \mathbf{x}_n$$

$$A\mathbf{v_1} = \lambda_2\left(c_2\mathbf{x}_2 + \cdots + (\frac{\lambda_n}{\lambda_2})c_n\mathbf{x}_n\right)A^k\mathbf{v_1} = \lambda_2^k\left(c_2\mathbf{x}_2 + \cdots + (\frac{\lambda_n}{\lambda_2})^k c_n\mathbf{x}_n\right)$$

$\mathbf{x}_2$

eigenvalue.

A

$\mathbf{x}_1, \cdots, \mathbf{x}_n$

4.3.6 Householder

A Householder H

$$H\mathbf{x}_1 = \mathbf{e}_1$$

$$\begin{aligned} HAH^T\mathbf{e}_1 &= HAH\mathbf{e}_1 \qquad H = H^T \\ &= HAHH\mathbf{x}_1 \qquad H^2 = I \\ &= HA\mathbf{x}_1 \\ &= \lambda_1 H\mathbf{x}_1 \\ &= \lambda_1\mathbf{e}_1 \end{aligned}$$

:

$$HAH^T = \begin{bmatrix} \lambda_1 & b \\ 0 & B \end{bmatrix}$$

H λ_1,

4.3.7 QR

$$A = QRQ^{-1} = Q^TQ^{-1}AQ = Q^TAQ = Q^TQRQ = RQ$$

$$A_1 = AA_k = Q_kR_kA_{k+1} = R_kQ_k$$

$$A_{k+1} = R_k Q_k = Q_k^{-1} Q_k R_k Q_k = Q_k^{-1} A_k Q_k = Q_k^{\top} A_k Q_k$$

A_k eigenvalue

QR algorithm

`scipy.linalg`

```
>>> import numpy as np
>>> from scipy import linalg
>>> a = np.array([[1, 0], [1, 3]])
>>> linalg.eigvals(a)
array([3.+0.j, 1.+0.j])
```

Chapter 5

{SVD decomposition}

5.1

SVD $A\vec{x}$ $\vec{x}$

$$R(\vec{x}) = \frac{\| A\vec{x} \|_2}{\| \vec{x} \|_2}$$

$$R(\alpha\vec{x}) = \frac{\| A\alpha\vec{x} \|_2}{\| \alpha\vec{x} \|_2} = \frac{\| \alpha \| \cdot \| A\vec{x} \|_2}{\| \alpha \| \cdot \| \vec{x} \|_2} = \frac{\| A\vec{x} \|_2}{\| \vec{x} \|_2}$$

- $R(\alpha\vec{x}) = R(\vec{x})$ $\| \vec{x} \|_2 = 1$
- $R(\vec{x}) \geq 0$ $R^2(\vec{x})$

,

$$L(\vec{x}) = (A\vec{x})^2 - \lambda(\vec{x}^2 - 1)$$

$$(A^T A)\vec{x}_i = \lambda_i \vec{x}_i \tag{1}$$

$A\vec{x}$ $\vec{x}$ $A^T A$ o( )o

- $\lambda_i \geq 0 \forall i,$ $A^T A$
-

A A^TA $\vec{x}_i$, $\vec{y}_i = A\vec{x}_i$

$\vec{y}_i$ $\vec{0}$ AA^T

A^TA, $\vec{y}$ AA^T, $A \in \mathbb{R}^{m\times n}$, $AA^T \in \mathbb{R}^{m\times m}$, $A^TA \in \mathbb{R}^{n\times n}$.

$A \in \mathbb{R}^{n\times n}$ AA^T A^TA :

```
>>> import numpy as np
>>> a = np.random.rand(3,3)
>>> a
array([[0.73741709, 0.2207241 , 0.60793118],
       [0.00490906, 0.18066958, 0.44795408],
       [0.70657397, 0.5650763 , 0.29043162]])
>>> aat = np.dot(a, a.T)
>>> aat
array([[0.96208341, 0.31582341, 0.82232812],
       [0.31582341, 0.23332846, 0.23566075],
       [0.82232812, 0.23566075, 0.90290852]])
>>> ata = np.dot(a.T, a)
>>> ata
array([[1.04305484, 0.56292085, 0.6557093 ],
       [0.56292085, 0.40067186, 0.37923277],
       [0.6557093 , 0.37923277, 0.6545937 ]])
>>> np.allclose(aat, ata)
False
>>> from scipy import linalg
>>> linalg.eigvals(aat)
array([1.84996505+0.j, 0.0737421 +0.j, 0.17461325+0.j])
>>> linalg.eigvals(ata)
array([1.84996505+0.j, 0.0737421 +0.j, 0.17461325+0.j])
```

AA^T A^TA

$A, B \in \mathbb{R}^{n\times n}, AB$ BA .

$$AB\vec{x} = \lambda\vec{x}$$

$\vec{y} = B\vec{x}$, $\lambda \neq 0, \vec{x} \neq 0)$

$$BA\vec{y} = BAB\vec{x} = B\lambda\vec{x} = \lambda B\vec{x} = \lambda\vec{y}$$

$\vec{y}_i = A\vec{x}_i$ $\vec{y}_i$ $\vec{0}$ AA^T

$$\begin{aligned}\lambda_i\vec{y}_i &= \lambda_i A\vec{x}_i\\ &= A(\lambda_i\vec{x}_i)\\ &= A(A^TA\vec{x}_i) \quad \text{from (1)}\\ &= (AA^T)\vec{y}_i \qquad\qquad (2)\end{aligned}$$

$\vec{y}_i$ AA^T

$$\begin{aligned}\parallel \vec{y}_i \parallel &= \parallel A\vec{x}_i \parallel\\ &= \sqrt{\parallel \lambda_i A\vec{x}_i \parallel^2}\\ &= \sqrt{\vec{x}_i^T A^T A\vec{x}_i}\\ &= \sqrt{\vec{x}_i^T A^T A\vec{x}_i}\\ &= \sqrt{\vec{x}_i^T \lambda_i\vec{x}_i} \qquad \text{from (1)}\\ &= \sqrt{\lambda_i\vec{x}_i^T\vec{x}_i}\\ &= \sqrt{\lambda_i} \parallel \vec{x}_i \parallel\end{aligned}$$

$\lambda_i = 0, \vec{y}_i = \vec{0}$, $\vec{y}_i$ $\vec{0}$ AA^T $\parallel \vec{y}_i \parallel = \sqrt{\lambda_i} \parallel \vec{x}_i \parallel$.
k A^TA 0 $\lambda_1, \cdots, \lambda_k$, $\vec{x}_1, \cdots, \vec{x}_k \in \mathbb{R}^n$, AA^T A^TA

$$k = \text{number of} \lambda_i > 0 A^TA\vec{x}_i = \lambda_i\vec{x}_i AA^T\vec{y}_i = \lambda_i\vec{y}_i$$

$\parallel \vec{x}_i \parallel = 1$,

$$\vec{y}_i = \frac{1}{\sqrt{\lambda_i}} A\vec{x}_i \tag{3}$$

$$\parallel \vec{y}_i \parallel = \frac{1}{\sqrt{\lambda}} \parallel A\vec{x}_i \parallel = \frac{1}{\sqrt{\lambda}}\sqrt{\lambda} \parallel \vec{x}_i \parallel = 1$$

$\vec{x}_i$ $\vec{y}_i$

$$\bar{V} = \begin{pmatrix}\vec{x}_1 & \cdots & \vec{x}_k\end{pmatrix} \in \mathbb{R}^{n\times k} \bar{U} = \begin{pmatrix}\vec{y}_1 & \cdots & \vec{y}_k\end{pmatrix} \in \mathbb{R}^{m\times k}$$

$\vec{e}_1$ i

$$
\begin{aligned}
\bar{U}^T A \bar{V} \vec{e}_1 &= \bar{U}^T A \vec{x}_i && \bar{V} \text{ defination} \\
&= \frac{1}{\lambda_i} \bar{U}^T A (\lambda_i \vec{x}_i) \\
&= \frac{1}{\lambda_i} \bar{U}^T A (A^T A \vec{x}_i) && \text{from (1)} \\
&= \frac{1}{\lambda_i} \bar{U}^T (A A^T) A \vec{x}_i \\
&= \frac{1}{\sqrt{\lambda_i}} \bar{U}^T (A A^T) \vec{y}_i && \text{from (3)} \\
&= \frac{1}{\sqrt{\lambda_i}} \bar{U}^T \lambda_i \vec{y}_i && \text{from (2)} \\
&= \sqrt{\lambda_i} \bar{U}^T \vec{y}_i \\
&= \sqrt{\lambda_i} \vec{e}_i
\end{aligned}
$$

$\Sigma = diag(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_k})$,

$$\bar{U}^T A \bar{V} = \Sigma$$

$$\bar{U} \in \mathbb{R}^{m\times k}, \bar{V} \in \mathbb{R}^{n\times k}, A \in \mathbb{R}^{m\times n}, \Sigma \in \mathbb{R}^{k\times k}$$

$\bar{U}, \bar{V}$ $\quad A^T A \vec{x}_i = \vec{0} \quad A A^T \vec{y}_i = \vec{0}, \quad \bar{U}, \bar{V}$

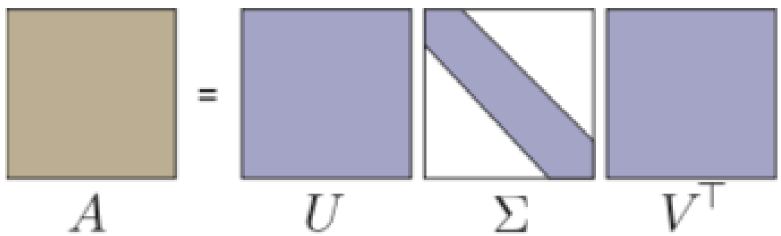
$$\bar{U} \in \mathbb{R}^{m\times k}, \bar{V} \in \mathbb{R}^{n\times k} \mapsto U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$$

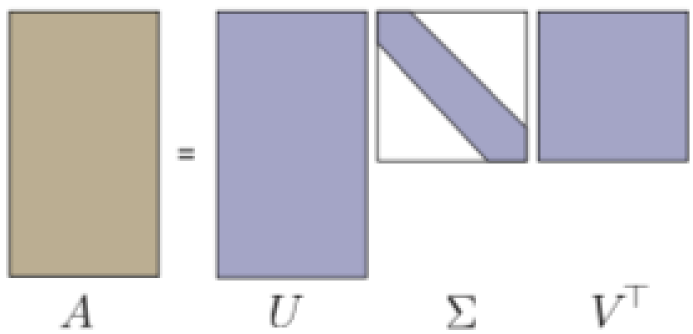
Σ

$$\Sigma_{ij} = \begin{cases} \sqrt{\lambda_i} & i = j, i \le k \\ 0 & \text{otherwise} \end{cases}$$

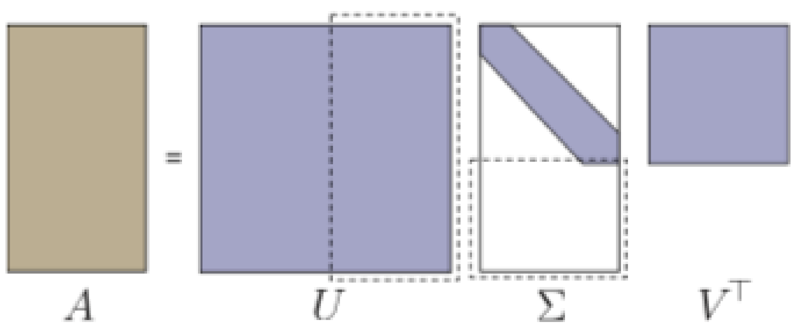
$$A = U \Sigma V^T$$

$A = U \Sigma V^T$ **A**





0 U V



5.2

$$A = U\Sigma V^T$$

- left singular vector) : U span col A
- right singular vector : V span row A (V V^T

- (singular value): Σ $\sigma_1 \geq \sigma_2 \cdots \geq 0$

SVD = x x A A

singular value decompostion, non-invertable singular invertible nonsingular) singular single special /

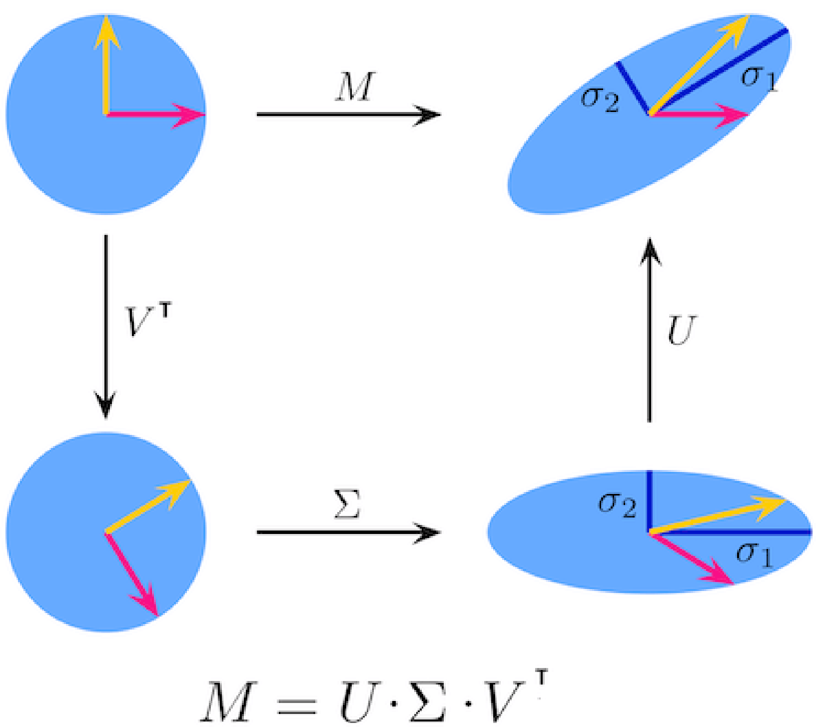
n x n R singular R^{-1}, singular value decompostion o( )o

SVD

SVD

- V^T :
- Σ :
- U :

\ + +



5.3

1. V $A^T A$
2. $AV = U\Sigma$ 0 $\vec{u}_i$ AV
3. $AA^T\vec{u}_i = 0$

5.4

5.4.1

A SVD

$$A = U\Sigma V^T$$

A^{-1} A nonsingular):

$$\begin{aligned} A^{-1} &= (U\Sigma V^T)^{-1} \\ &= (V^T)^{-1}\Sigma^{-1}U^{-1} \\ &= V \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}^{-1} U^{-1} \\ &= V \begin{pmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{pmatrix} U^T \end{aligned}$$

$A\vec{x} = \vec{b}$

$$\begin{aligned} A\vec{x} &= \vec{b} \\ U\Sigma V^T\vec{x} &= \vec{b} \\ \vec{x} &= V\Sigma^{-1}U^T\vec{b} \end{aligned}$$

5.4.2

setup

$$\text{minimize } \parallel \vec{x} \parallel_2^2 \text{ such that } A^T A\vec{x} = A^T\vec{b}$$

$A^T A$:

$$\begin{aligned} A^T A &= (U\Sigma V^T)^T(U\Sigma V^T) \\ &= V\Sigma U^T U\Sigma V^T \\ &= V\Sigma^2 V^T \end{aligned}$$

$A^T A\vec{x} = A^T\vec{b}$

$$\begin{aligned} A^T A\vec{x} = A^T\vec{b} \iff V\Sigma^2 V^T\vec{x} &= (U\Sigma V^T)^T\vec{b} \\ V\Sigma^2 V^T\vec{x} &= V\Sigma U^T\vec{b} \\ \Sigma V^T\vec{x} &= U^T\vec{b} \end{aligned}$$

$$\begin{aligned} A^T A\vec{x} = A^T\vec{b} \iff \Sigma\vec{y} &= \vec{d} \\ \vec{y} &= V^T\vec{x} \\ \vec{d} &= U^T\vec{b} \end{aligned}$$

setup

$$\text{minimize } \parallel \vec{y} \parallel_2^2 \text{ such that } \Sigma\vec{y} = \vec{d}$$

$\vec{y} = V^T\vec{x}$

setup $\vec{x}$ $\vec{y}$, Σ

$$\begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_k & \\ & & & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ \\ y_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ \\ d_m \end{pmatrix}$$

:

$$\Sigma_{ij}^+ = \begin{cases} \frac{1}{\sigma}i & i = j, \sigma_i \neq 0, i \leq k \\ 0 & \text{otherwise} \end{cases} \implies \vec{y} = \Sigma_{ij}^+\vec{d} \implies \vec{x} = V\Sigma_{ij}^+ U^T\vec{b}$$

$V\Sigma_{ij}^+ U^T$ Pseudoinverse ? ,

- A square and invertible $A^+ = A^{-1}$
- A overdetermined $A^+\vec{b}$ gives least-squares solution to $A\vec{x} = \vec{b}$
- A underdetermined $A^+\vec{b}$ gives least-squares solution to $A\vec{x} = \vec{b}$ with least (Euclidean) norm

5.4.3

A

$$A = U\Sigma V^T \implies A = \sum_{i=1}^{l} \sigma_i \vec{u}_i \vec{v}_i^T l = \min\{m, n\}$$

/ A x

$\vec{u}\vec{v}^T$:

$$\vec{u} \otimes \vec{v} = \vec{u}\vec{v}^T$$

$A\vec{x}$:

$$A\vec{x} = \sum_i \sigma_i (\vec{v}_i \cdot \vec{x}) \vec{u}_i$$

$A\vec{x}$ σ_i

$A^+\vec{x}$

$$A^+ = \sum_{\sigma_i \neq 0} \frac{\vec{v}_i \cdot \vec{u}^T}{\sigma_i}$$

A^+ σ_i

> **Eckart-Yound Theorem** :
>
> Suppose $\widetilde{A}$ is obtained from $A = U\Sigma V^T$ by truncating all but the k largest singular values σ_i of A to zero. Then,$\widetilde{A}$ minimizes both $\| A - \widetilde{A} \|_{Fbo}$ and $\| A - \widetilde{A} \|_2$ subject to the constraint that the column space of $\tilde{A}$ has at most dimension k.

A rank r $\widetilde{A}$ A Frobenius norm 2-norm, $\widetilde{A}$ SVD A $A = U\Sigma V^T$ Σ k 0

Frobenius norm

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\operatorname{trace}(A^* A)} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2(A)}$$

2-norm

$$\| A \|_2 = \max_{\vec{v} \neq \vec{0}} \frac{\| A\vec{v} \|_2}{\| \vec{v} \|_2} = \max\{\sigma_i\}$$

$A = U\Sigma V^T$ Σ $\sigma_i \geq 0$ (The singular values are non-negative real numbers.) $\Sigma = diag(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_k})$

$A^TA $ $A^T A$

5.5

M:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

M $U\Sigma V^*$:

$$U = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \Sigma = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} V^* = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ -\sqrt{0.2} & 0 & 0 & 0 & -\sqrt{0.8} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

```
>>> from scipy import linalg
>>> import numpy as np
>>>
>>> a = np.array([[1, 0, 0, 0, 2],
...               [0, 0, 3, 0, 0],
...               [0, 0, 0, 0, 0],
...               [0, 2, 0, 0, 0]])
>>>
>>> u, s, vh = linalg.svd(a)
>>>
>>> u
array([[ 0.,  1.,  0.,  0.],
       [ 1.,  0.,  0.,  0.],
       [ 0.,  0.,  0., -1.],
       [ 0.,  0.,  1.,  0.]])
>>> s
array([3.        , 2.23606798, 2.        , 0.        ])
>>> vh
```

```
array([[-0.        ,  0.        ,  1.        , -0.        ,  0.        ],
       [ 0.4472136 ,  0.        ,  0.        ,  0.        ,  0.89442719],
       [-0.        ,  1.        ,  0.        , -0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  1.        ,  0.        ],
       [-0.89442719,  0.        ,  0.        ,  0.        ,  0.4472136 ]])
```

scipy u vh o( )o

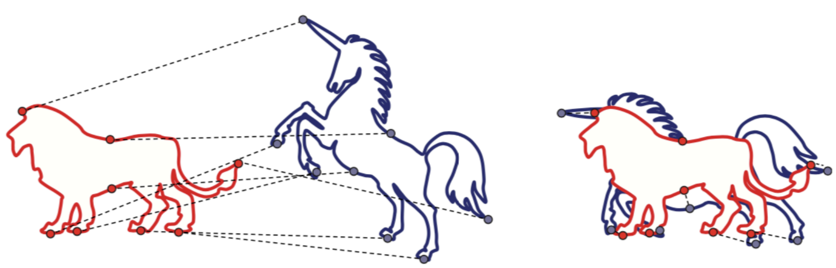
Chapter 6

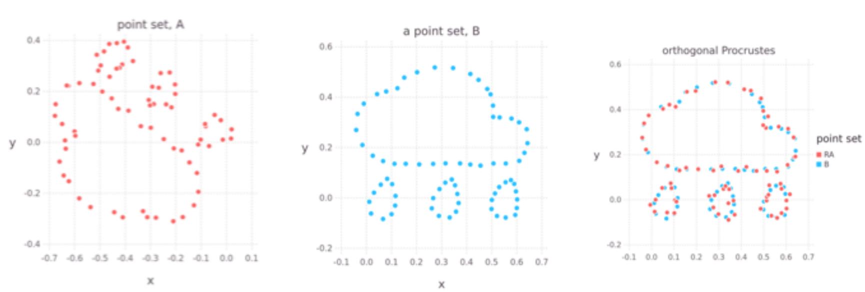
{SVD application}

SVD

6.1 Rigid Alignment / Procrustes Problem

,





3D 3D point clouds 3D mesh Rigid Alignment.

set up ,

$$E = \sum_{i=1}^{n} \parallel p_i - (Rq_i + t) \parallel^2 \; p_i \in P q_i \in Q$$

,

$$p = \frac{1}{n}\sum_{i=1}^{n} p_i = \frac{1}{n}\sum_{i=1}^{n}(Rq_i + t) = R\frac{1}{n}\sum_{i=1}^{n} q_i + t = Rq + tt = p - Rq$$

$$\begin{aligned}
\sum_{i=1}^{n} \parallel p_i - (Rq_i + t) \parallel^2 &= \sum_{i=1}^{n} \parallel p_i - Rq_i - (p - Rq) \parallel^2 \\
&= \sum_{i=1}^{n} \parallel (p_i - p) - R(q_i - q) \parallel^2
\end{aligned}$$

$p_i - p$ $q_i - q$ p q $x_i = p_i - p, y_i = q_i - q,$

$$\begin{aligned}
\parallel x_i - Ry_i \parallel^2 &= (x_i - Ry_i)^T(x_i - Ry_i) \\
&= (x_i^T - y_i^T R^T)(x_i - Ry_i) \\
&= (x_i^T x_i - x_i^T Ry_i - y_i^T R^T x_i + y_i^T R^T Ry_i) \quad (R^T R = I) \\
&= (x_i^T x_i + y_i^T y_i - x_i^T Ry_i - y_i^T R^T x_i)
\end{aligned}$$

$x_i^T Ry_i$ x_i^T 1xd, R dxd, y_i dx1 $a^T = a,$

$$x_i^T Ry_i = (x_i^T Ry_i)^T = y_i^T Rx_i$$

$$\sum_{i=1}^{n} \parallel x_i - Ry_i \parallel^2 = \sum_{i=1}^{n} x_i^T x_i + y_i^T y_i - 2y_i^T Rx_i$$

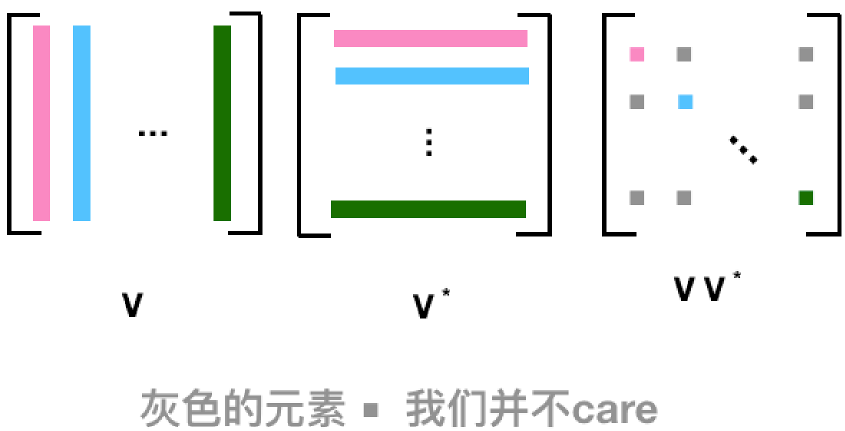
$x_i^T x_i$ $y_i^T y_i$ $\sum_{i=1}^{n} y_i^T Rx_i$

$$\sum_{i=1}^{n} \parallel x_i - Ry_i \parallel^2 = \parallel X - RY \parallel_{Fro}^2 = const - 2tr(Y^T RX)$$

- $ tr(A + B) = tr(A) + tr(B)$
- $ tr(A^T) = tr(A)$
- $\parallel A \parallel_{Fro}^2 = \sum_{i,j} |a_{ij}|^2$
- $tr(AB) = tr(BA)$

$tr(A+B) = tr(A) + tr(B)$ $ tr(A^T) = tr(A)$

Frobenius norm AA* , AA^*
care



$tr(AB) = tr(BA)$ $Tr(AB) = Tr(BA) = \sum a_{ij} b_{ji}$

- $\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}\left((\mathbf{ABC})^{\mathsf{T}}\right) = \mathrm{tr}(\mathbf{CBA}) = \mathrm{tr}(\mathbf{ACB})$,
- $\mathrm{tr}(\mathbf{ABCD}) = \mathrm{tr}(\mathbf{BCDA}) = \mathrm{tr}(\mathbf{CDAB}) = \mathrm{tr}(\mathbf{DABC})$, ABCD

$-2tr(Y^T RX)$ $tr(Y^T RX)$

$$\begin{aligned}
tr(Y^T RX) &= tr(RXY^T) & tr(AB) &= tr(BA) \\
&= tr(RU\Sigma V^T) & XY^T &= U\Sigma V^T \\
&= tr(\Sigma V^T RU) & tr(AB) &= tr(BA) \\
&= tr(\Sigma M) & M &= V^T RU, \text{also orthogonal} \\
&= \sum_i \sigma_i m_{ii} & & \Sigma \text{ is diagonal}
\end{aligned}$$

$M = V^T RU$

$$AA^T = A^T A = IBB^T = B^T B = I(AB)^T(AB) = B^T A^T AB = B^T B = I$$

$M = V^T RU = V^T (RU),$

$$tr(\Sigma M) = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & ... & m_{1d} \\ m_{21} & m_{22} & ... & m_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ m_{d1} & m_{d2} & ... & m_{dd} \end{bmatrix} = \sum_{i=1}^{d} \sigma_i m_{ii} \leq \sum_{i=1}^{d} \sigma_i$$

M

$$1 = m_j^T m_j = \sum_{i=1}^{d} m_{ij}^2 \implies m_{ij}^2 \leq 1 \implies |m_{ij}| \leq 1$$

M $m_i i \leq 1,$ $\sigma_i \geq 0,$ M M = I

$$I = M = V^T RU \implies V = RU \implies R = VU^T$$

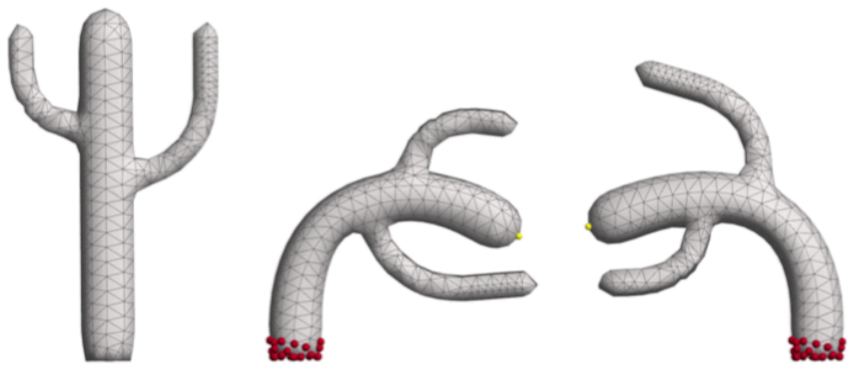
Procrustes Problem

Π Δ

$\| X - RY \|^2$ **R** $R = VU^T$, $XY^T = U\Sigma V^T$

1. R t E
2. t R E R $R^T R = I$
3.

6.2 APAR



paper As-Rigid-As-Possible Surface Modeling, key idea SVD

6.3 PCA

PCA

- : $m = \frac{1}{n}\sum_{i=1}^{n} x_i$
- : $y_i = x_i - m$
- / : $S = YY^T$ Y y_i
- : $S = V\Lambda V^T$
- : $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$
- : $v_1, \cdots, v_d$
-

6.4

Eckart-Yound Theorem):

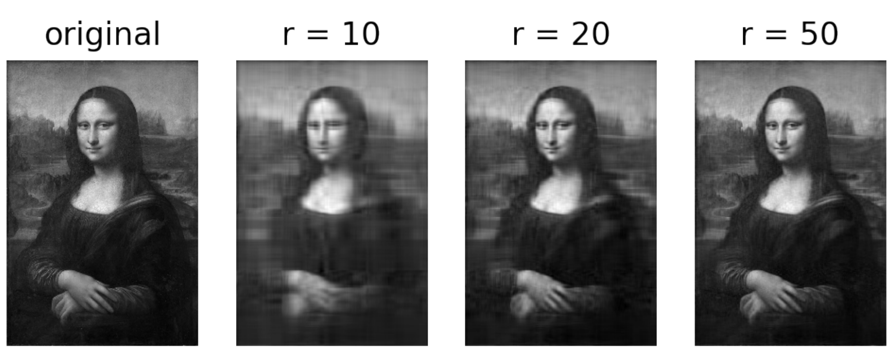
rank r $\tilde{A}$ A A SVD , $A = U\Sigma V^T$, r
$\tilde{A} = U\Sigma_r V^T$, $\| \tilde{A} - A \|_{Fro}$

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

RGBA m x n x 4, m x n, m x n x 4)

SVD 10 20 50

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.image as mpimg

def rgb2gray(rgb):
    return np.dot(rgb[...,:3], [0.299, 0.587, 0.144])

img = mpimg.imread('Mona_Lisa.png')
gray = rgb2gray(img)
plt.imshow(gray, cmap = plt.get_cmap('gray'))

U, s, Vh = np.linalg.svd(gray)

def composite(U, s, Vh, n):
    return np.dot(U[:, :n], np.dot(np.diag(s[:n]), Vh[:n,:]))

for i in [10, 20, 50]:
    new_img = composite(U, s, Vh, i)
    plt.imshow(new_img, cmap='gray')
    title = "new_img_%s" % i
    plt.title(title)
    plt.savefig(title + '.png')
    plt.show()
```



/

$(x, y, z)^T$ xy $(x, y)^T$

$$f(x) = a_n x^n + a_{n-1} x^{n-a} + \cdots + a_1 x + a_0$$

$a_n \cdots, a_r$ $f(x)$ o( )o

Chapter 7

{matirx application}

7.1

7.1.1 Conjugate transpose

A

$$A = \begin{bmatrix} 1 & -2-i & 5 \\ 1+i & i & 4-2i \end{bmatrix}$$

A^T

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & 1+i \\ -2-i & i \\ 5 & 4-2i \end{bmatrix}$$

$\overline{A^T}$:

$$\overline{A^{\mathrm{T}}} = \begin{bmatrix} 1 & 1-i \\ -2+i & -i \\ 5 & 4+2i \end{bmatrix}$$

A^*, A^H $Hermitian$ $, \overline{A^T}$

7.1.2 Hermitian

Hermitian matrix i j j i

$z = a + ib \in \mathbb{C}, \quad \bar{z} = a - ib$

$$A \text{ Hermitian} \iff a_{ij} = \overline{a_{ji}} A \text{ Hermitian} \iff A = A^{\mathsf{H}}$$

$A \in \mathbb{R}^{n\times n}$ Hermitian $a_{ij} = a_{ji}$ Hermitian

A

$$A = \begin{bmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{bmatrix}$$

Hermitian

7.1.3 positive definite

n × n M z $z^T M z > 0$ z^T z

$$M \text{ positive definite} \iff x^{\mathsf{T}} M x > 0 \text{ for all } x \in \mathbb{R}^n \quad \mathbf{0}$$

M M = -I :

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -2 < 0$$

n×n M z $z^* M z > 0$ z* z M z $z^* M z$ 0

$$M \text{ positive definite} \iff x^* M x > 0 \text{ for all } x \in \mathbb{C}^n \quad \mathbf{0}$$

Hermitian

- M λ_i
- ...

7.1.4 orthogonal matrix

$$Q^T = Q^{-1} \Leftrightarrow Q^T Q = Q Q^T = I.$$

$$1 = det(I) = det(Q^T Q) = det(Q^T) det(Q) = (det(Q))^2 \Rightarrow det(Q) = \pm 1$$

-
- +1 (special orthogonal group)
- -1
- n × n O(n)
- SO(n)

7.1.5 unitary matrix

/

$$U^*U = UU^* = I_n$$

U U^*

yǒu 17:00 19:00

unitary unit take

- $U^{-1} = U^*$,
- $|\lambda_n| = 1$, U λ_n 1
- $|\det(U)| = 1$, U 1
- $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$, U $\vec{x}$ $\vec{y}$
- ...

7.1.6 normal matrix

normal matrix A A

$$\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$$

A^* A

A $A^* = A^T$ $AA^T = A^TA.$

A $A = U\Lambda U^*$ $\Lambda = diag(\lambda_1, \lambda_2, \dots)$ U

/ /

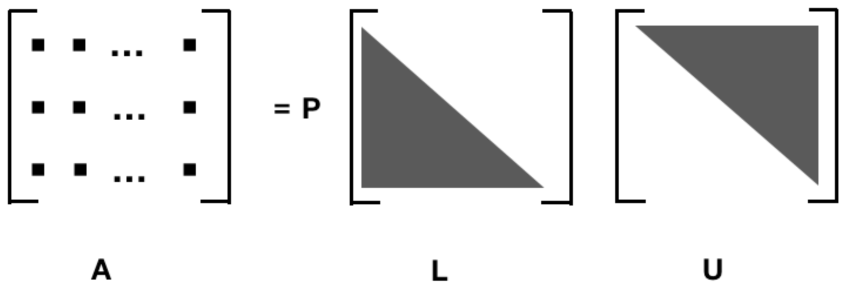
7.1.7

-
-
- 1
-
-

7.2

7.2.1 A = PLU

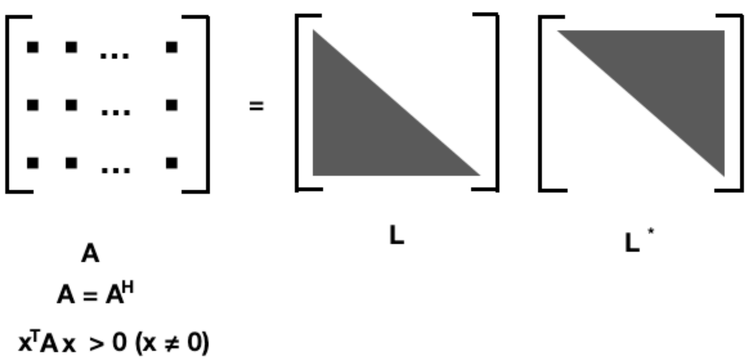
-
- A = PLU, L U P permutation A LU. PLU



PLU

7.2.2 Cholesky

- hermitian positive definite
- $A = LL^*$

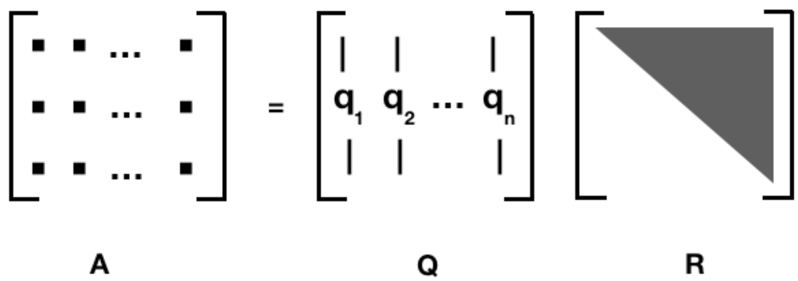


A Hermitian L L^* L .

7.2.3 QR

- m x n, m n
- A = QR, Q m x m unitary matrix , R

QR



A 0 QR

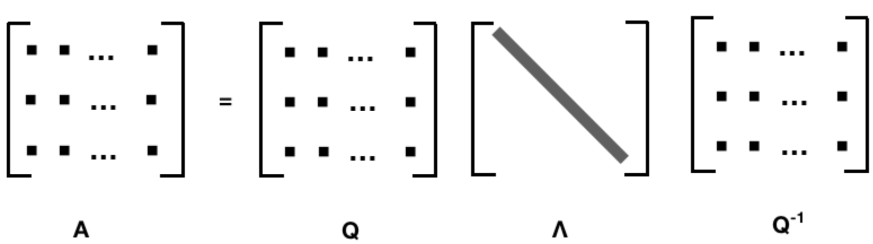
$$A = QR = Q\begin{bmatrix} R_1 \\ 0 \end{bmatrix} = [Q_1, Q_2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1$$

where R1 is an n×n upper triangular matrix, 0 is an (m − n)×n zero matrix, Q1 is m×n, Q2 is m×(m − n), and Q1 and Q2 both have orthogonal columns.

QR Gram–Schmidt Householder reflections.

7.2.4 / Eigendecomposition / spectral decomposition

- A
- $\mathbf{A} = \mathbf{Q}\ \mathbf{Q}^{-1}$



Q n x n i A $\vec{q}_i$, Λ i $\Lambda_{ii} = \lambda_i$ $\vec{q}_i$ λ_i.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\vec{q}_i, (i = 1, \cdots, N)$ $\vec{q}_i, (i = 1, \cdots, N)$ Q Q Q^{-1}

Q

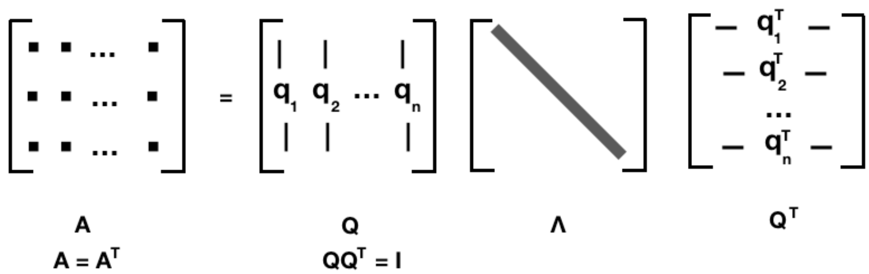
$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$
$$\mathbf{A}\mathbf{Q} = \mathbf{Q}$$
$$\mathbf{A} = \mathbf{Q}\ \mathbf{Q}^{-1}.$$

-

n x n n 1

$$\mathbf{A} = \mathbf{Q}\ \mathbf{Q}^{T}$$

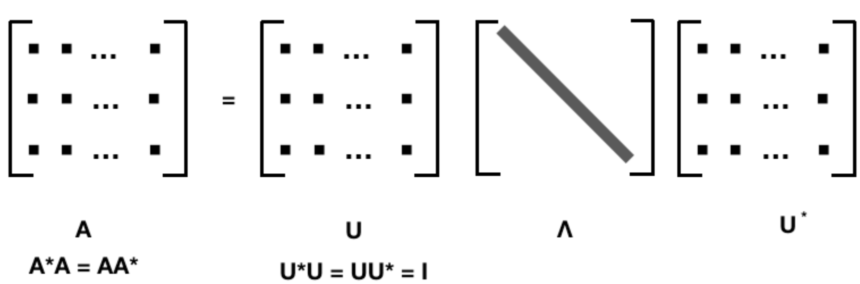
Q Λ



-

$$\mathbf{A} = \mathbf{U}\ \mathbf{U}^{*}$$

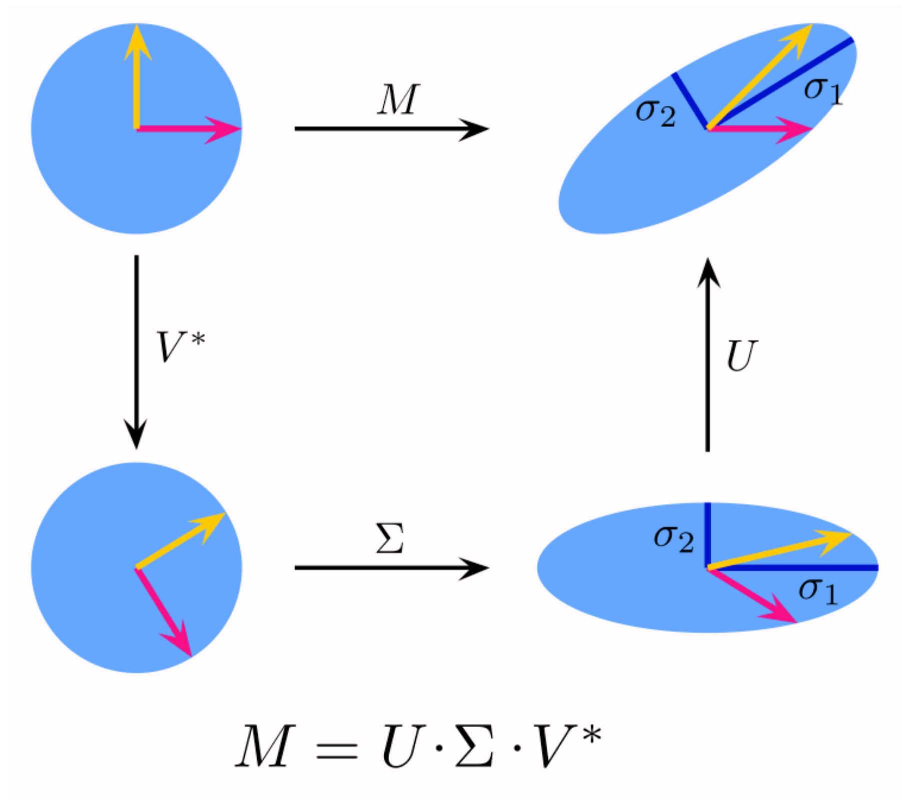
U



$x_{t+1} = Ax_t$ $x_0 = c$ $x_t = A^t c$
$x_{t}=VD^{t}V^{-1}c$ V D A D D t D^{t}
t A t A

7.2.5

- m x n A
- $A = U\Sigma V^{*}$, U V / $U^{*}U = V^{*}V = I$, Σ A ,
- U V



/

Matlab/numpy

wikipedia

Chapter 8

{nonlinear equation}

set up

$$f : \mathbb{R}^n \to \mathbb{R}^m \vec{x}^* : f(\vec{x}^*) = \vec{0}$$

$f : \mathbb{R} \to \mathbb{R}$

- $x \to y, f(x) \to f(y)$
- Lipschitz $|f(x) - f(y)| \leq C|x - y|$
- $\forall x, \exists f'(x)$
- C^k : k C^∞

8.1 Bisection method

$f : [a, b] \mapsto \mathbb{R}$ f(a) < f(b) u f(a) < u < f(b) c, a < c < b f(c)=u f(a) > f(b)

[a,b]

f(x) = 0 (x) :

1. [a, b] f(a) f(b)
2. $m = \frac{a+b}{2}$ f(m)
3. f(m) = 0 m
4. f(m) f(a) [m, b] , [a, m].
5. 2 3

work

8.2 fixed point

fixed point

$$g(x) = x$$

$$x_0 x_{k+1} = g(x_k)$$

$$g(x) = f(x) + x$$

g(x) f(x)

8.3 Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

f'(x) o( )o

8.4 Secant method

f'(x)

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

f'(x) $x_0, x_1,$ $\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$

8.5

$$\cos(x) - x^3 = 0$$

numpy

```
import numpy as np
from scipy.optimize import fsolve

def func(x):
    return np.cos(x) - x**3

result = fsolve(func, 1)
print(result)
# 0.86547403
print( func(result) )
#2.22044605e-16
```

x = 0.86547 ok

Chapter 9

{nonlinear equations}

$$f : \mathbb{R}^n \to \mathbb{R}^m \vec{x}^* : f(\vec{x}^*) = \vec{0}$$

n m. $f(\vec{x}) = A\vec{x} - \vec{b}$

9.1

9.1.1 Jacobian matrix

$$(Df)_{ij} = \frac{\partial f_i}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$f : \mathbb{R}^n \to \mathbb{R}^m$ x x
f x

$$f(\vec{x}) \approx f(\vec{x}_k) + Df(\vec{x}_k)(\vec{x} - \vec{x}_k)$$

$$\vec{x}_{k+1} = \vec{x}_k - [Df(\vec{x}_k)]^{-1} f(\vec{x}_k)$$

$$[Df(\vec{x}_k)]^{-1} f(\vec{x}_k) = \vec{y}_k$$

$$[Df(\vec{x}_k)]\vec{y}_k = f(\vec{x}_k)$$

$[Df(\vec{x}_k)]^{-1}$, $\vec{y}_k$.

1

-
- $Df(\vec{x}_k)$

9.2 Broyden's method

Broyden's method f'(x):

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$J \cdot (\vec{x}_k - \vec{x}_{k-1}) \approx f(\vec{x}_k) - f(\vec{x}_{k-1}) J \approx Df(\vec{x}_k)$$

- $\vec{x}_k$ J_k
- $\vec{x}_k$
- J_k

$$\text{minimize}_{J_k} \parallel J_k - J_{k-1} \parallel^2_{Fro} \text{ such that } J \cdot (\vec{x}_k - \vec{x}_{k-1}) \approx f(\vec{x}_k) - f(\vec{x}_{k-1})$$

$$J_k = J_{k-1} + \frac{(f(\vec{x}_k) - f(\vec{x}_{k-1}) - J_{k-1}\Delta\vec{x})}{\|\vec{x}_k - \vec{x}_{k-1}\|^2}(\Delta\vec{x}^T)\vec{x}_{k+1} = \vec{x}_k - J_k^{-1} f(\vec{x}_k)$$

$J_0 = I$, J_k^{-1},

$$J_k = J_{k-1} + \frac{(f(\vec{x}_k) - f(\vec{x}_{k-1}) - J_{k-1}\Delta\vec{x})}{\|\vec{x}_k - \vec{x}_{k-1}\|^2}(\Delta\vec{x}^T)$$

$$J_k = J_{k-1} + \vec{u}_k \vec{v}_k^T$$

Sherman-Morrison Formula:

$$\left(A + uv^{\mathsf{T}}\right)^{-1} = A^{-1} - \frac{A^{-1}uv^{\mathsf{T}}A^{-1}}{1 + v^{\mathsf{T}}A^{-1}u}.$$

$$J_k^{-1} = J_{k-1}^{-1} - \frac{J_{k-1}^{-1}\vec{u}_k\vec{v}_k^T J_{k-1}^{-1}}{1 + \vec{v}_k^T J_{k-1}^{-1}\vec{u}_k}$$

J_k^{-1}.

9.3

scipy fsolve

$$\begin{cases} x_0 + x_1^2 = 4 \\ e^{x_0} + x_0 x_1 = 3 \end{cases}$$

```
from scipy.optimize import fsolve
import math

def equations(p):
    x0, x1 = p
    return ( x0 + x1**2 - 4, math.exp(x0) + x0 * x1 -3 )

x0, x1 = fsolve(equations, (1, 1))

print(x0, x1)
# 0.6203445234801195 1.8383839306750887
print(equations((x0, x1)))
# (4.4508396968012676e-11, -1.0512035686360832e-11)
```

Chapter 10

{Points Concepts}

10.1 critial point

- $f : \mathbb{R} \to \mathbb{R}$ 0
- $f : \mathbb{C} \to \mathbb{C}$: 0
- $f : \mathbb{R}^n \to \mathbb{R}$: 0
- $f : \mathbb{R}^m \to \mathbb{R}^n$: Jacobian

10.2 stationary point

(stationary point) $f : \mathbb{R} \to \mathbb{R}$

$$\left.\frac{dy}{dx}\right|_p = 0$$

(stationary point) 0 x y stationary point

10.3 inflection point

inflect

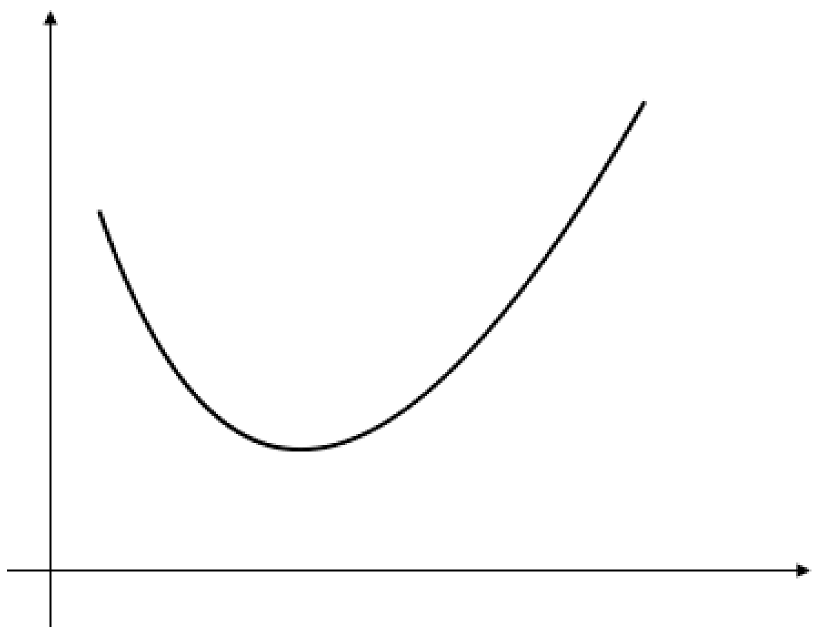
' ' inflection flex

Inflection point

convex concave

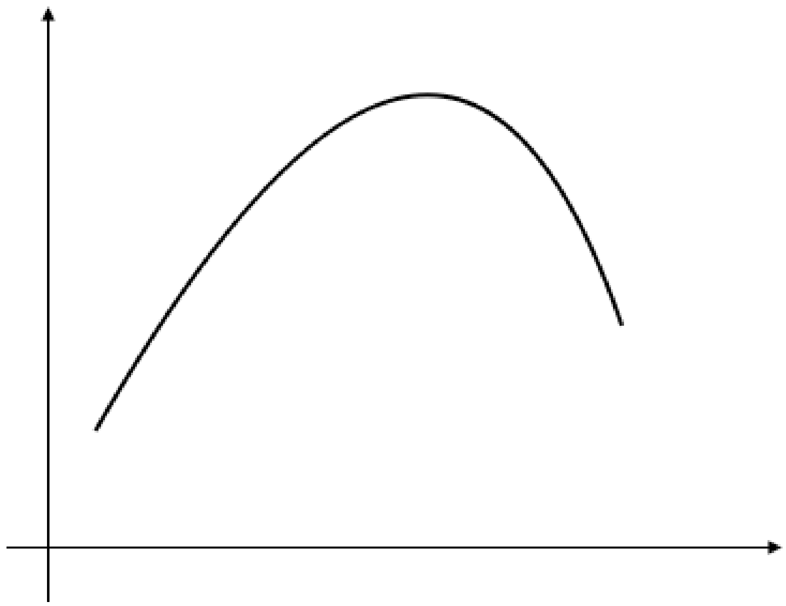
10.3.1 convex

$$f((1-\alpha)x+\alpha y) \le (1-\alpha)f(x)+\alpha f(y)$$

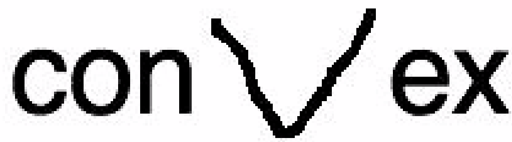


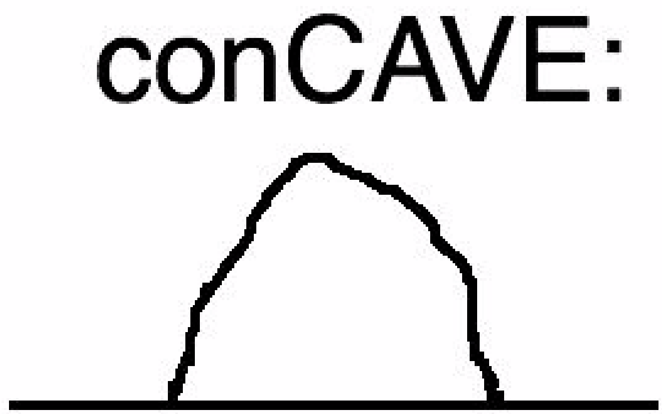
10.3.2 concave

$$f((1-\alpha)x+\alpha y) \ge (1-\alpha)f(x)+\alpha f(y)$$



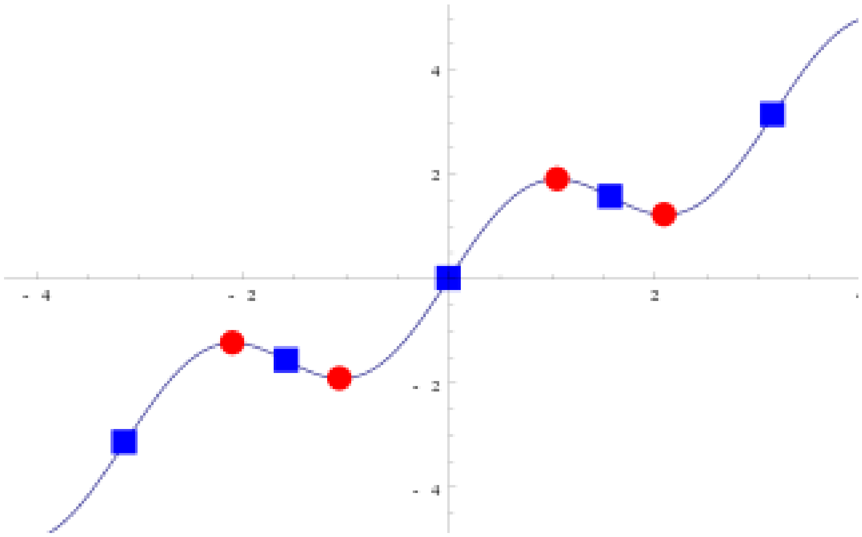
convex v , concave/cave



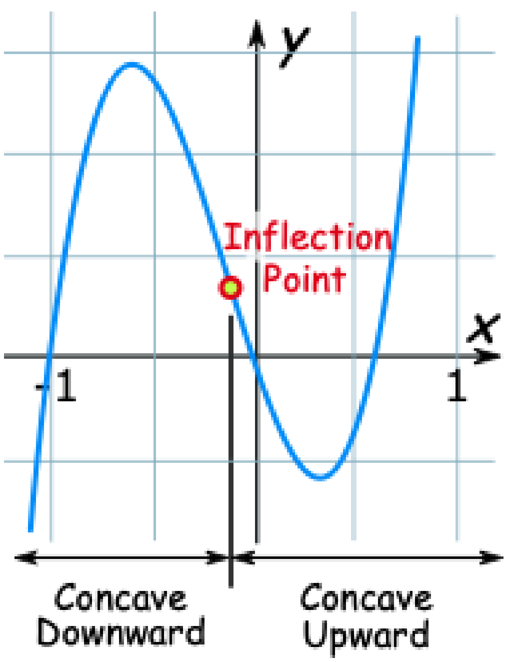


convex concave

/ stationary points/critial points, inflection points.



inflection point

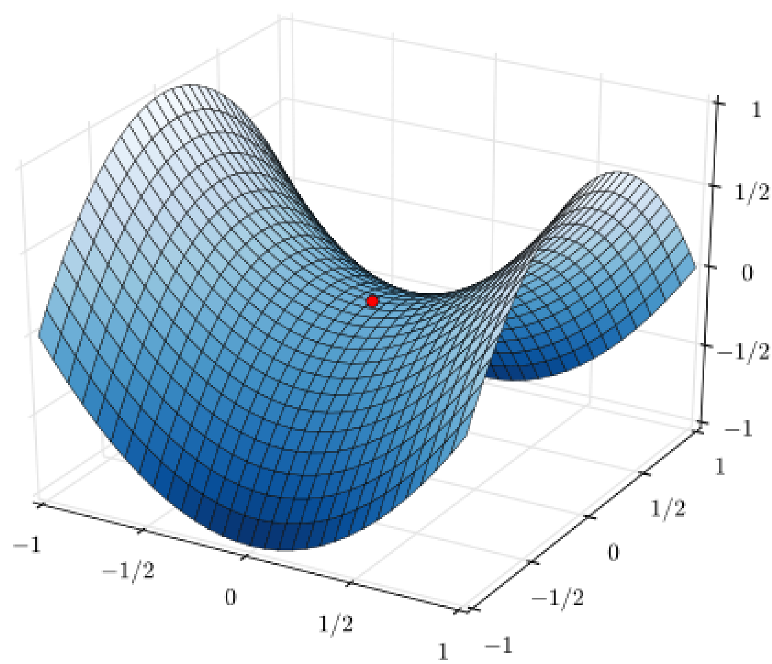


C^0 C^∞ convex/concave

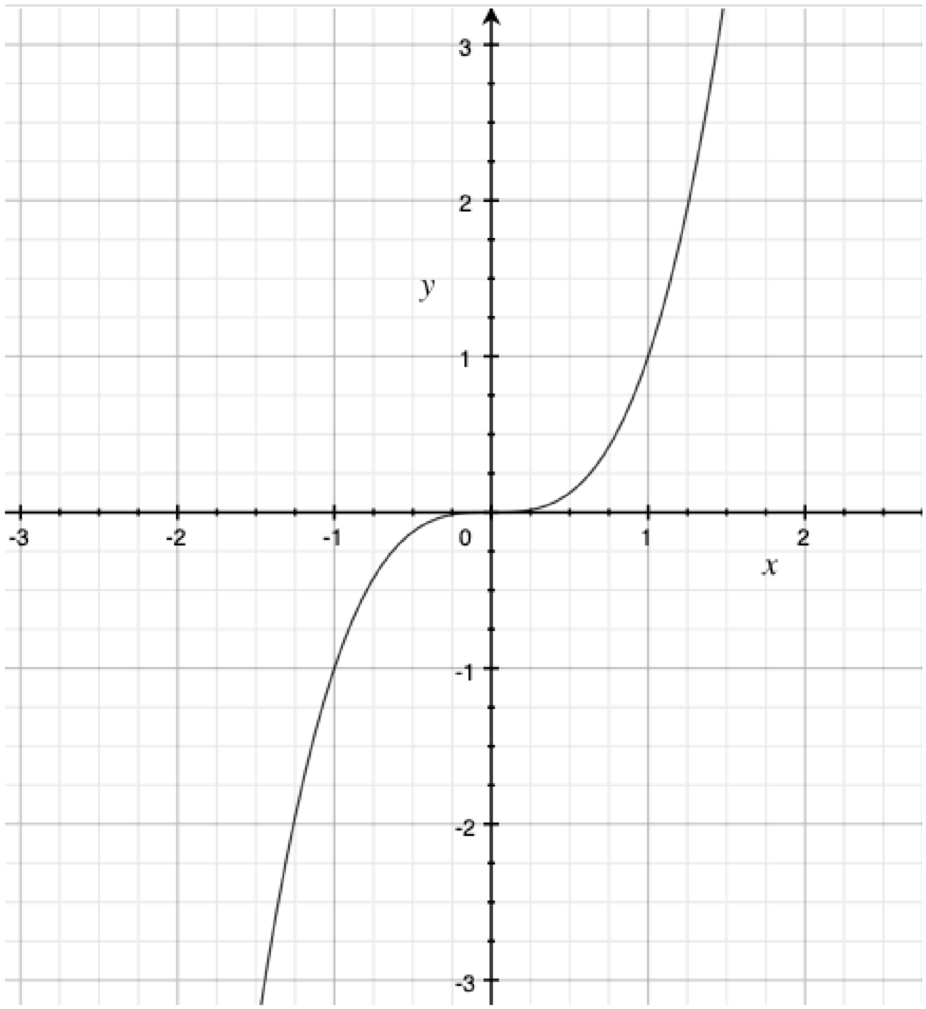
10.4 saddle point

saddle point minmax point.

$f(x, y) = x^2 - y^2$, (0, 0)



.



\$y = x^3 \$ (0, 0

10.5 vertex (curve)

$y = ax^2 + bx + c$

$$k(x) = \frac{2a}{(1 + (2ax + b)^2)^{\frac{3}{2}}}.$$

$x = -b/2a$ 0

Chapter 11

Jacobian Hessian {Gradient Related Concepts}

:

- $f : \mathbb{R} \to \mathbb{R}$
- $f : \mathbb{R}^n \to \mathbb{R}$
- $f : \mathbb{R}^n \to \mathbb{R}^m$

f

11.1

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

11.2

$f : \mathbb{R}^n \to \mathbb{R}$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

n x 1 $\vec{x}$ ∇_x:

$$\nabla_x \overset{\text{def}}{=} \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right]^T = \frac{\partial}{\partial x}$$

$$f(\vec{x}) \approx f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$

11.3 Jacobian

Jacobian

$f : \mathbb{R}^n \to \mathbb{R}^m$

$f : \mathbb{R}^n \to \mathbb{R}^m$ x x
f x

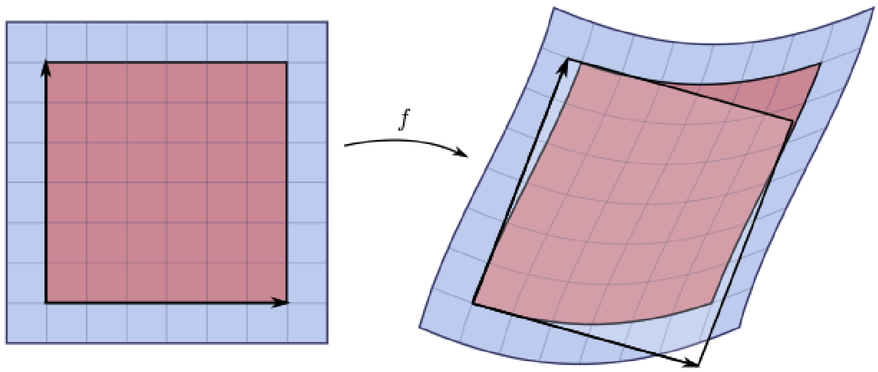
$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial x_j}.$$

Df D$\mathbf{f}$ $\mathbf{J}_{\mathbf{f}}(x_1, \ldots, x_n)$ $\dfrac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}$.

$$f(\vec{x}) \approx f(\vec{x}_k) + J(\vec{x}_k)(\vec{x} - \vec{x}_k)$$

m = n Jacobian



Jacobian Matrix

11.4 Hessian

$f : \mathbb{R}^n \to \mathbb{R}$

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

n x n

$$\mathbf{H}_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

- $f : \mathbb{R} \to \mathbb{R}$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

- $f : \mathbb{R}^n \to \mathbb{R}$:

$$f(x_1, x_2) = f(x_{10}, x_{20}) + f_{x_1}(x_{10}, x_{20})\Delta x_1 + f_{x_2}(x_{10}, x_{20})\Delta x_2 + \frac{1}{2}[f_{x_1 x_1}(x_{10}, x_{20})\Delta x_1^2 + 2 f_{x_1 x_2}(x_{10}, x_{20})\Delta x_1 \Delta x_2 + f_{x_2 x_2}($$

$$\Delta x_1 = x_1 - x_{10} \ \ \Delta x_2 = x_2 - x_{20} \ \ f_{x_1} = \frac{\partial f}{\partial x_1} \ \ f_{x_2} = \frac{\partial f}{\partial x_2} \ \ f_{x_1 x_1} = \frac{\partial^2 f}{\partial x_1^2} \ \ f_{x_2 x_2} = \frac{\partial^2 f}{\partial x_2^2} \ \ f_{x_1 x_2} = \frac{\partial^2 f}{\partial x_1 \partial x_2} =$$

$$f(\vec{x}) \approx f(\vec{x_0}) + \nabla f(\vec{x_0}) \cdot (\vec{x} - \vec{x_0}) + \frac{1}{2}(\vec{x} - \vec{x_0})^T H(\vec{x_0})(\vec{x} - \vec{x_0})$$

$$H(x_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \, \partial x_2} \\ \\ \frac{\partial^2 f}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}_{x_0}$$

Hessian

$f : \mathbb{R} \to \mathbb{R}$, $ x=x_0$ $f'(x_0) = 0, f''(x_0) \neq 0,$

- $f''(x) < 0,$
- $f''(x) > 0,$
- $f''(x) = 0,$
- $f''(x)$

$f : \mathbb{R}^n \to \mathbb{R}$ $\vec{x}_0$ $\vec{0}$ $H(\vec{x_0})$

- H ,
- H ,
- H
- H

- Cholesky
-

Chapter 12

{Optimization without constraintss}

12.1

	$E(\vec{x}) = \| A\vec{x} - \vec{b} \|^2$	
$\vec{b}$ $\vec{a}$	$E(c) = \| c\vec{a} - \vec{b} \|^2$	
	$E(\vec{x}) = \vec{x}^T A\vec{x}$	$\| \vec{x} \| = 1$
Pseudoinverse	$E(\vec{x}) = \| \vec{x} \|^2$	$A^T A\vec{x} = A^T\vec{b}$
	$E(C) = \| X - CC^T X \|_{Fro}$	$C^T C = I_{d\times d}$
Broyden step	$E(J_k) = \| J_k - J_{k-1} \|_{Fro}^2$	$J_k \cdot (\vec{x}_k - \vec{x}_{k-1}) = f(\vec{x}_k) - f(\vec{x}_{k-1})$

set up

$$min_{\vec{x}} f(\vec{x})$$

-

$$E(a,c) = \sum_i (y_i - ce^{ax_i})^2$$

-

$$g(h;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(h-\mu)^2/2\sigma^2}$$

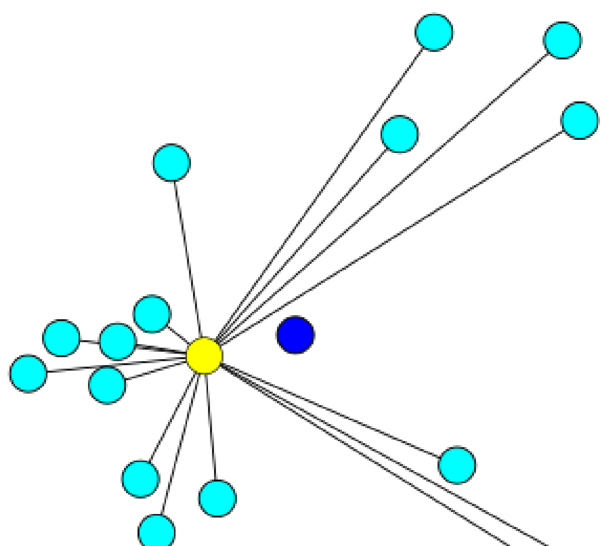
$h_1,\cdots,h_n$

$$P(h_1,\cdots,h_n;\mu,\sigma)=\prod_i g(h_i,\mu,\sigma)$$

μ,σ /NLP

-

geometric median



$$E(\vec{x}) = \sum_i \parallel \vec{x} - \vec{x}_i \parallel_2$$

l2 norm,

12.2

$$\vec{x}^* \in \mathbb{R}^n f : \mathbb{R}^n \to \mathbb{R} \forall \vec{x} \in \mathbb{R}^n f(\vec{x})^* \leq f(\vec{x})$$

$$\vec{x}^* \in \mathbb{R}^n f : \mathbb{R}^n \to \mathbb{R} \forall \parallel \vec{x} - \vec{x}^* \parallel < \varepsilon, f(\vec{x})^* \leq f(\vec{x})$$

12.3

12.3.1

$f : \mathbb{R} \to \mathbb{R}$ $f'(x) = 0$ $f(x) = 0$ $f'(x) = 0$

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}.$$

12.3.2 Golden-section search

Unimodal function unimodular

$f : [a, b] \to \mathbb{R}$ $x^* \in [a, b]$ f $x \in [a, x^*]$ $x \in [x^*, b]$

$a < x_0 < x_1 < b$,

- $f(x_0) \leq f(x_1)$, $x \in [x_1, b]$, $f(x) \geq f(x_0)$, $x^* \in [a, x_1]$, $[x_1, b]$
- $f(x_0) \geq f(x_1)$, $x \in [a, x_0]$, $f(x) \geq f(x_1)$, $x^* \in [x_0, b]$, $[a, x_0]$

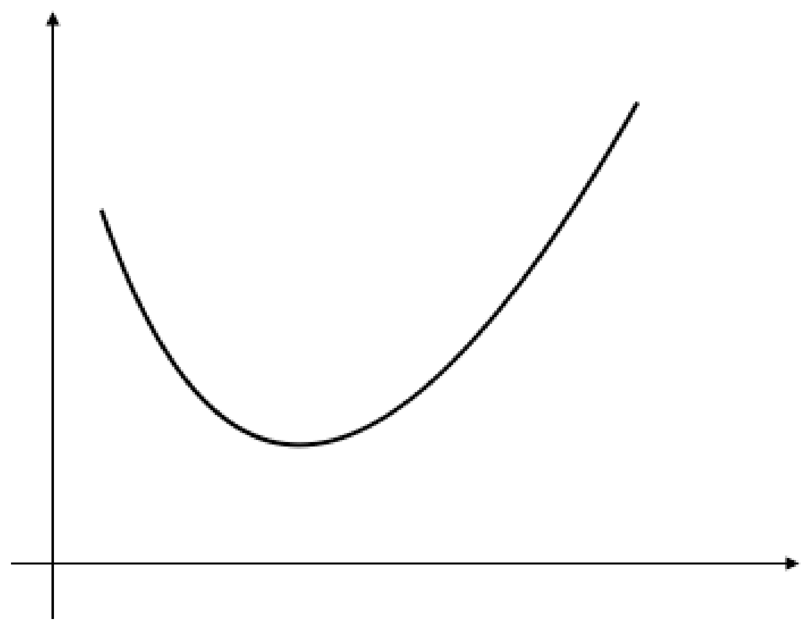


Figure 12.1: unimodular.png

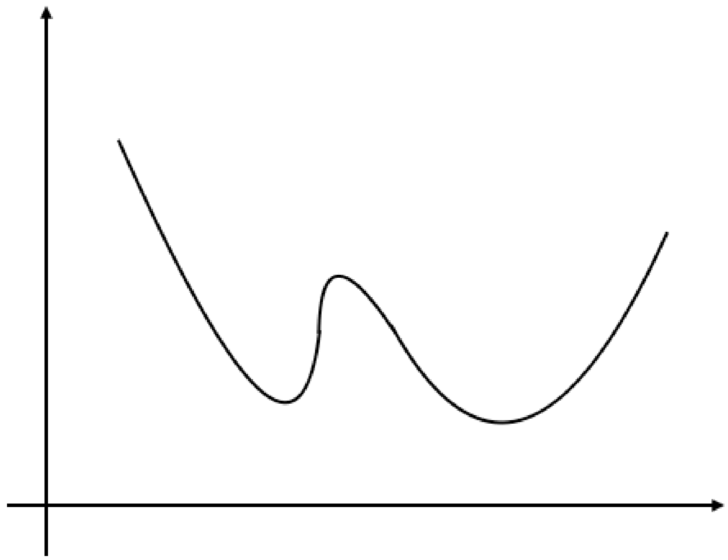


Figure 12.2: Bimodal.png

$f(x_0)$ $f(x_1)$, a = 0, b = 1,

$$x_0 = \alpha, x_1 = 1 - \alpha, \alpha \in (0, \frac{1}{2})$$

- $f(x_0) \leq f(x_1)$

$$[a, x_1] = [0, 1 - \alpha]$$

$$\alpha(1 - \alpha), (1 - \alpha)^2$$

$$x_0 = \alpha = (1 - \alpha)^2$$

$$\alpha^2 - 3\alpha + 1 = 0 \alpha = \frac{1}{2}(3 - \sqrt{5}) 1 - \alpha = \frac{1}{2}(\sqrt{5} - 1)$$

$1 - \alpha = \tau$

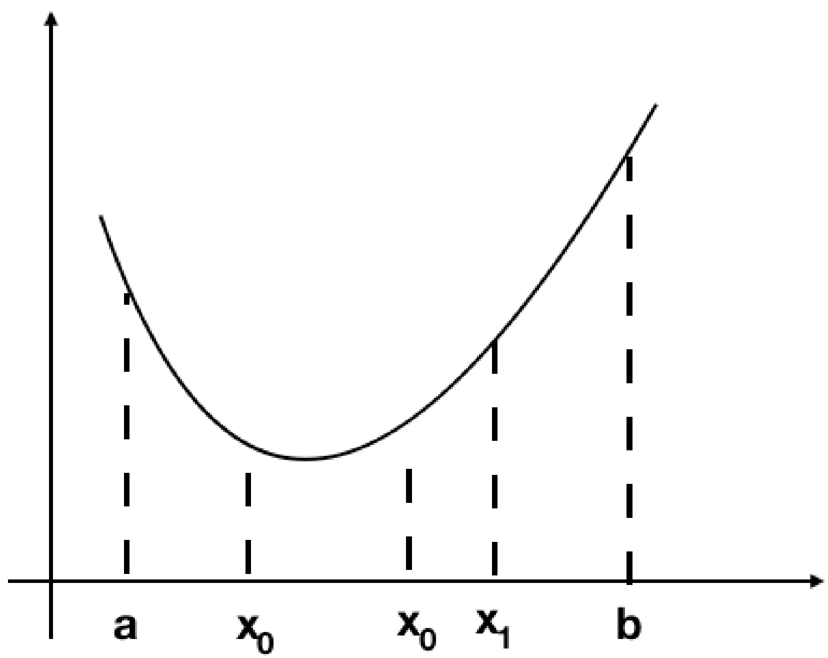


Figure 12.3: unimodal_01.png

- $f(x_0) \geq f(x_1)$

$$[x_0, 1] = [\alpha, 1]$$

$$\alpha + \alpha(1-\alpha), \alpha + (1-\alpha)^2$$

$$x_0 = 1 - \alpha = \alpha + \alpha(1-\alpha)$$

1. a, b f [a, b] unimodular
2. $x_0 = a + (1-\tau)(b-a), x_1 = a + \tau(b-a), f_0 = f(x_0), f_1 = f(x_1)$
3. b - a
   - $f_0 \geq f_1,$ $[a, x_0]$
     - $a \leftarrow x_0$
     - $x_0 \leftarrow x_1, f_0 \leftarrow f_1$
     - $x_1 \leftarrow a + \tau(b-a), f_1 \leftarrow f(x_1)$
   - $f_1 > f_0,$ $[x_1, b]$
     - $b \leftarrow x_1$
     - $x_1 \leftarrow x_0, f_1 \leftarrow f_0$
     - $x_0 \leftarrow a + (1-\tau)(b-a), f_0 \leftarrow f(x_0)$

12.4

$f : \mathbb{R}^n \to \mathbb{R},$

12.4.1

$F(\vec{x})$ $\vec{a}$ $F(\vec{x})$ $\vec{a}$ $-\nabla F(\vec{a})$

$\vec{b} = \vec{a} - \gamma \nabla F(\vec{a})$ $ >0$ $F(\vec{a}) \geq F(\vec{b})$

- x_0
- $g_k(t) = f(\vec{x}_k - t\nabla f(\vec{x}_k))$
- $t^* \geq 0$ g_k
- $\vec{x}_{k+1} = \vec{x}_k - t^* \nabla f(\vec{x}_k)$

12.4.2

$$\vec{x}_{k+1} = \vec{x}_k - [H_f(\vec{x}_k)]^{-1}\nabla f(\vec{x}_k)$$

$\nabla f(\vec{x})$ Hessian +n Quasi-Newton method

12.4.3 BFGS

BFGS Broyden–Fletcher–Goldfarb–Shanno algorithm Shanno Shanno Broyden's method Hessian :

$$\vec{x}_{k+1} = \vec{x}_k - \alpha_k B_k^{-1}\nabla f(\vec{x}_k) B_k \approx H_f(\vec{x}_k)$$

$$B_{k+1}(\vec{x}_{k+1} - \vec{x}_k) = \nabla f(\vec{x}_{k+1}) - \nabla f(\vec{x}_k)$$

B

-
-

$$min_{B_{k+1}} \parallel B_{k+1} - B_k \parallel s.t. B_{k+1}^T = B_{k+1} B_{k+1}(\vec{x}_{k+1} - \vec{x}_k) = \nabla f(\vec{x}_{k+1}) - \nabla f(\vec{x}_k)$$

$ B_{k+1} - B_k $ $ B_{k+1}^{-1} - B_k^{-1} $

$$min_{H_{k+1}} \parallel H_{k+1} - H_k \parallel s.t. H_{k+1}^T = H_{k+1} \vec{x}_{k+1} - \vec{x}_k = H_{k+1}(\nabla f(\vec{x}_{k+1}) - \nabla f(\vec{x}_k))$$

Broyden–Fletcher–Goldfarb–Shanno algorithm

Chapter 13

KKT {Lagrange multiplier to KKT condition}

/ level set/contour

$$\text{min/max } f(x, y) \text{subject to } g(x, y) = c$$

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \cdot (g(x, y) - c) \mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \cdot (g(x, y) - c)$$

$$\text{min/max } f(x, y) \text{subject to } g(x, y) = 0$$

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y) \mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \cdot g(x, y)$$

$g(\mathbf{x}) = \mathbf{0}$

13.1

-

$$f(x,y) = x + y \text{s.t. } x^2 + y^2 = 1$$

f(x,y) g(x,y), f(x,y) g(x,y) :

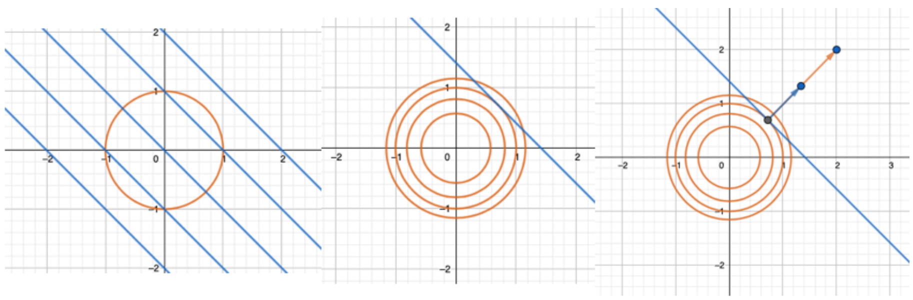


Figure 13.1: level_set_0.png

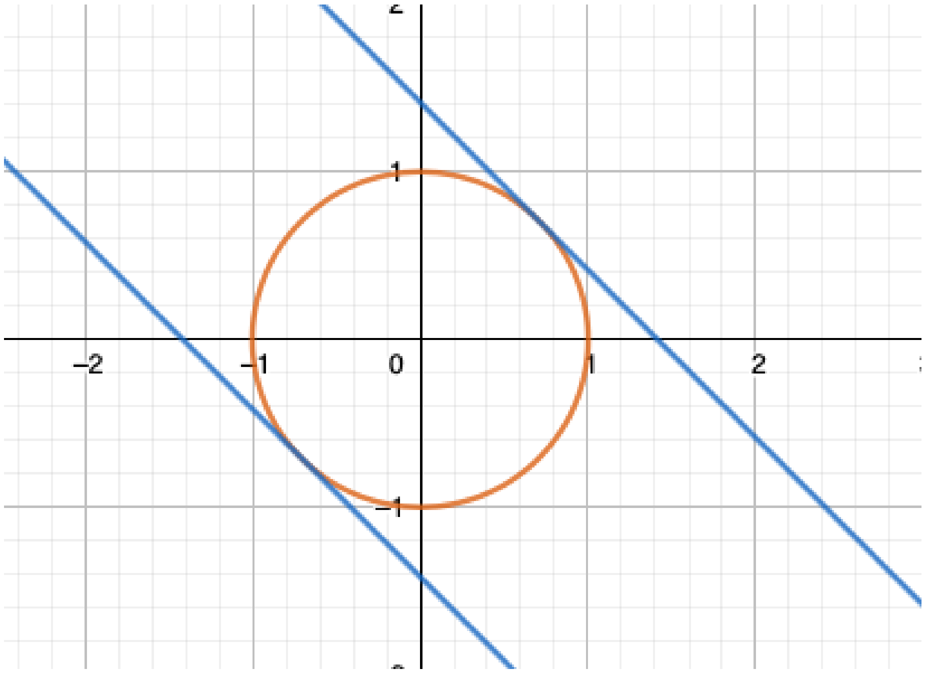
$$\nabla f = \lambda \nabla g$$

pack :

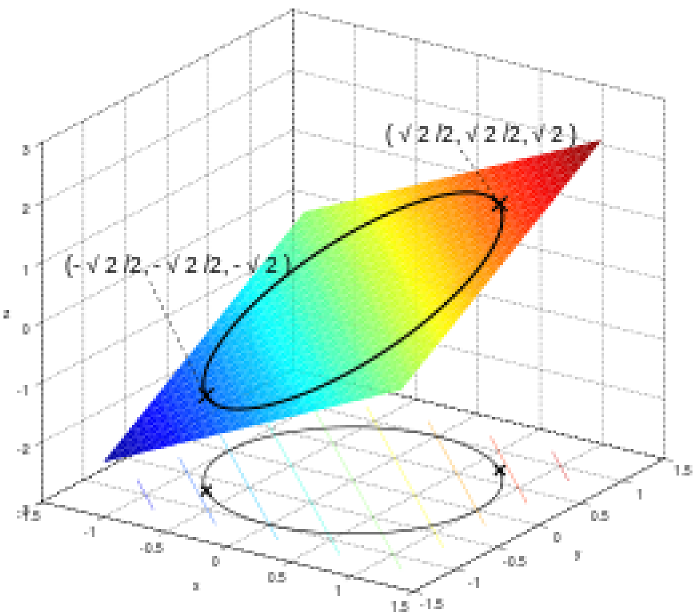
$$\begin{aligned} \mathcal{L}(x,y,\lambda) &= f(x,y) - \lambda \cdot g(x,y) \\ &= x + y - \lambda(x^2 + y^2 - 1). \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x = 0 \frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda y = 0 \frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 1 = 0$$

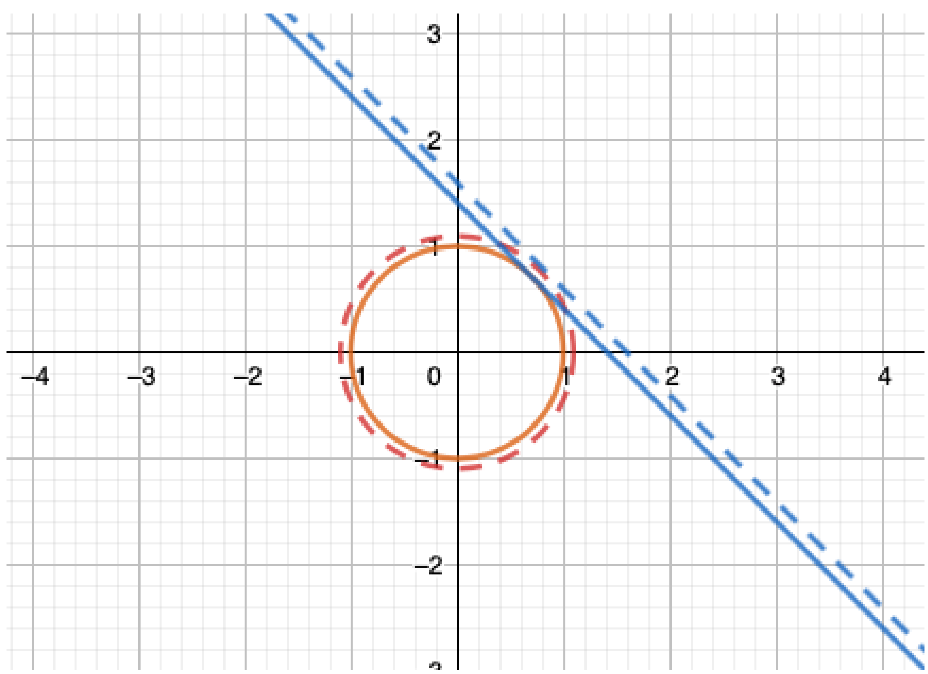
$\lambda = \pm \frac{1}{\sqrt{2}}$



$$f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \sqrt{2} f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\sqrt{2}$$

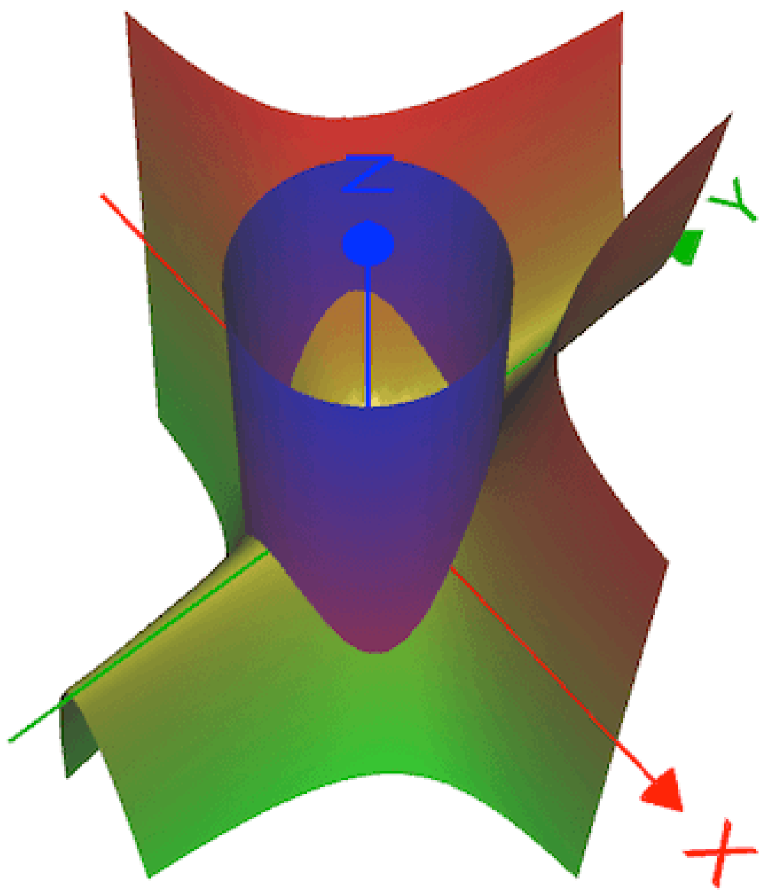


λ $\lambda = \frac{1}{\sqrt{2}}$ f(x,y) .



-

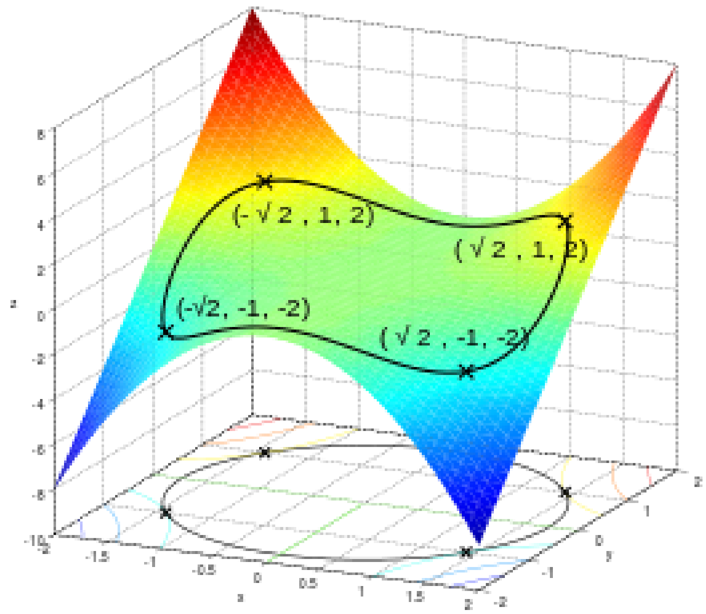
$$f(x, y) = x^2 y \text{s.t. } x^2 + y^2 = 3$$



$$\begin{aligned}\mathcal{L}(x, y, \lambda) &= f(x, y) - \lambda \cdot g(x, y) \\ &= x^2 y - \lambda(x^2 + y^2 - 3).\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2xy - 2\lambda x = 0 \frac{\partial \mathcal{L}}{\partial x} = x^2 - 2\lambda y = 0 \frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 3 = 0$$

6



- $\{p_1, p_2, \dots, p_n\}$, :

$$f(p_1, p_2, \dots, p_n) = -\sum_{j=1}^{n} p_j \log_2 p_j \text{s.t. } g(p_1, p_2, \dots, p_n) = \sum_{j=1}^{n} p_j = 1$$

$$\begin{aligned}\mathcal{L}(x, y, \lambda) &= f(x, y) - \lambda \cdot g(x, y) \\ &= -\sum_{j=1}^{n} p_j \log_2 p_j - \lambda(\sum_{j=1}^{n} p_j - 1)\end{aligned}$$

p_k

$$-\left(\frac{1}{\ln 2} + \log_2 p_k^*\right) - \lambda = 0.$$

$$p_k^* = \frac{1}{n}.$$

13.2

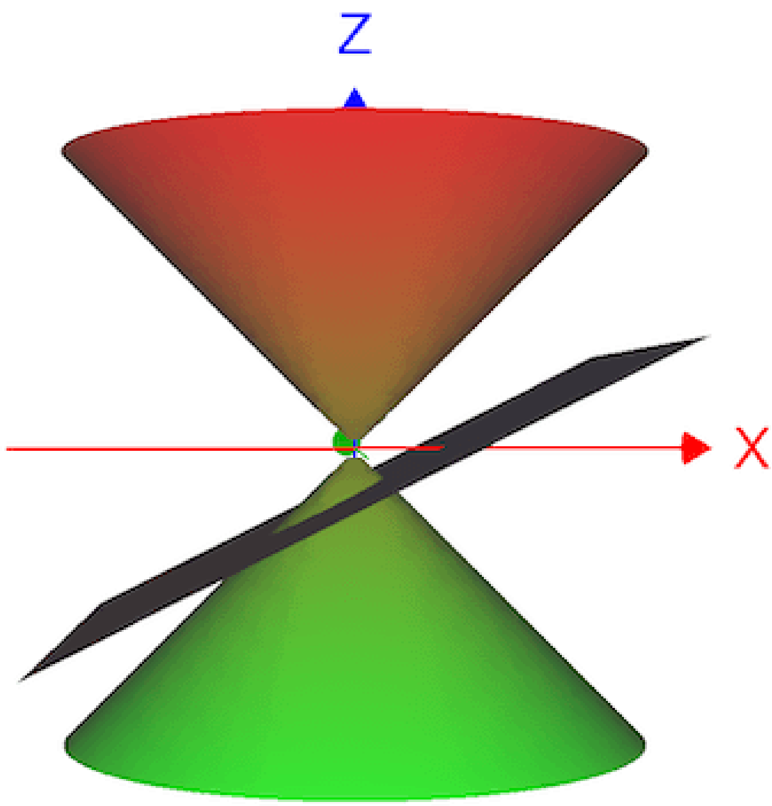
$$f(x,y)\text{s.t. } g_i(x)=0, i=1,\cdots,M$$

$$\mathcal{L}\left(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_M\right)=f\left(x_1,\ldots,x_n\right)-\sum_{k=1}^{M}\lambda_k g_k\left(x_1,\ldots,x_n\right)$$

$$\nabla_{x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_M}\mathcal{L}(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_M)=0 \iff \begin{cases}\nabla f(\mathbf{x})-\sum_{k=1}^{M}\lambda_k\nabla g_k(\mathbf{x})=0\\ g_1(\mathbf{x})=\cdots=g_M(\mathbf{x})=0\end{cases}$$

13.2.1

$z^2=x^2+y^2$ $x-2z=3$,



$$d = \sqrt{x^2 + y^2 + z^2}, \qquad f(x, y, z) = d^2$$

$$f(x, y, z) = x^2 + y^2 + z^2 x^2 + y^2 = z^2 x - 2z = 3$$

$$\begin{aligned}\mathcal{L}(x, y, z, \lambda, \mu) &= f(x, y, z) - \lambda \cdot g(x, y, z) - \mu \cdot h(x, y, z) \\ &= x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - z^2) - \mu(x - 2z - 3)\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 2\lambda x - \mu = 0 \frac{\partial \mathcal{L}}{\partial y} = 2y - 2\lambda y = 0 \frac{\partial \mathcal{L}}{\partial y} = 2z + 2\lambda z + 2\mu = 0 \frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - z^2 = 0 \frac{\partial \mathcal{L}}{\partial \mu} = x - 2z - 3 = 0$$

:

$$(-3, 0, 3) \rightarrow d_{max} = 3\sqrt{2}(1, 0, -1) \rightarrow d_{min} = \sqrt{2}$$

13.3 KKT

13.3.1

KKT Karush–Kuhn–Tucker conditions

$$f(x, y) \text{s.t } g(\mathbf{x}) \leq 0$$

primal feasibility , feasible region

$$K = \{\mathbf{x} \in \mathbb{R}^n | g(\mathbf{x}) \leq 0\}$$

- $g(\mathbf{x}) = 0$, (boundary solution) (active)
- $g(\mathbf{x}) < 0$, (interior solution) (inactive)

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda \cdot g(\mathbf{x})$$

-

$$\nabla f = \mathbf{0}\lambda = 0$$

-

$$g(\mathbf{x}) = 0\nabla f = -\lambda\nabla g$$

f ∇f f $\mathbf{x}$ ∇g K $g(\mathbf{x}) > 0$

$$\lambda \geq 0$$

$\lambda \geq 0$ dual feasibility.

$$\lambda g(\mathbf{x}) = 0$$

$\lambda g(\mathbf{x}) = 0$ complementary slackness.

13.4 KKT

Optimize

$$f(\mathbf{x})$$

subject to

$$g_i(\mathbf{x}) \leq 0, h_i(\mathbf{x}) = 0.$$

g_i $(i = 1, \dots, m)$ h_i $(i = 1, \dots, \ell)$

- Stationarity

For maximizing $\nabla f(x^*) - \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) - \sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*) = 0,$

For minimizing : $\nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) + \sum_{j=1}^{\ell} \lambda_j \nabla h_j(x^*) = 0,$

- Primal feasibility

$$g_i(x^*) \le 0,\ \text{for } i = 1, \dots, m h_j(x^*) = 0,\ \text{for } j = 1, \dots, \ell$$

- Dual feasibility

$\mu_i \ge 0,\ \text{for } i = 1, \dots, m$

- Complementary slackness

$\mu_i g_i(x^*) = 0,\ \text{for }\ i = 1, \dots, m.$

13.5

$$\text{minimize } x_1^2 + x_2^2 - 4x_1 - 4x_2 \text{s.t } x_1^2 \le x_2 x_1 + x_2 \le 2$$

$$L(x_1, x_2, \mu_1, \mu_2) = x_1^2 + x_2^2 - 4x_1 - 4x_2 + \mu_1(x_1^2 - x_2) + \mu_2(x_1 + x_2 - 2) x_1^2 - x_2 \le 0 x_1 + x_2 - 2 \le 0 \mu_1 \ge 0 \mu_2 \ge 0$$

$$2x_1 + 2\mu_1 x_1 + \mu_2 - 4 = 0 2x_2 - \mu_1 + \mu_2 - 4 = 0 \mu_1(x_1^2 - x_2) = 0 \mu_2(x_1 + x_2 - 2) = 0 \mu_1, \mu_2 \ge 0$$

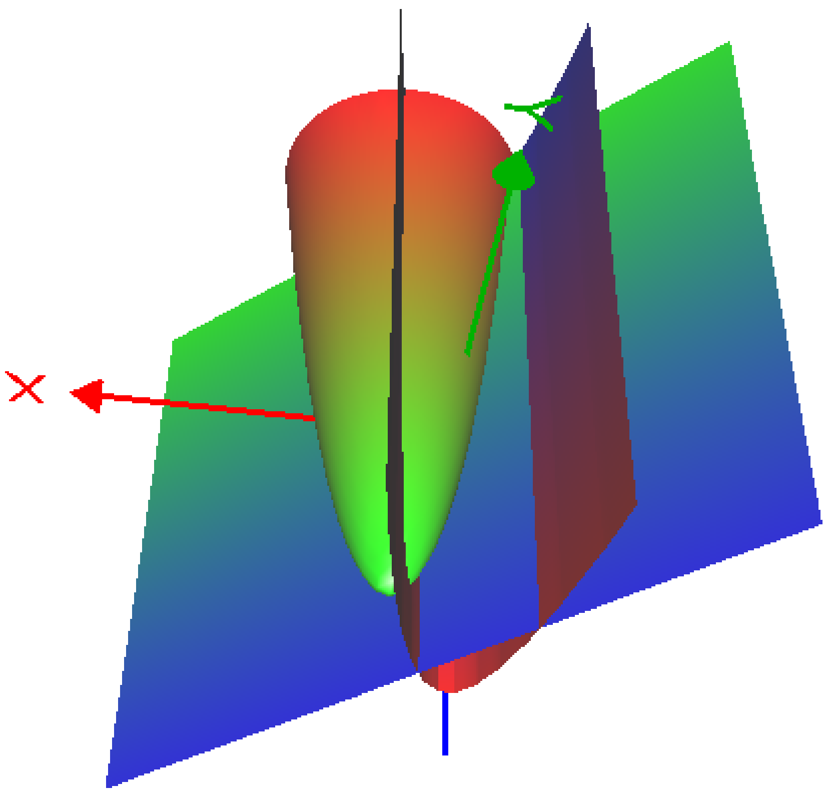
- $\mu_1 = 0, x_1 + x_2 - 2 = 0 \to x_2 = 1, x_1 = 1, \mu_2 = 2$
- \$ _2 = 0, x_1^2 = x_2 →x_1 = -2, x_2 = 4, _1 = 4 \$
- $\mu_1 = \mu_2 = 0 \to x_1 = 2, x_2 = 2, x_2 = x_1^2$

- $x_1^2 - x_2 = 0, x_1 + x_2 - 2 = 0 \to x_1 = 1, x_1 = -2 \cdots$

  f(1, 1) = -6

$$f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 4x_2 = (x_1 - 2)^2 + (x_2 - 2)^2 - 8 \quad x_1, x_2$$

x y



(2,2) (1, 1) (2,2),

Chapter 14

{Conjugate gradient}

$Ax = b$, Jacobi Method Gauss–Seidel method

A $(A^T = A)$, $\forall \vec{x} \neq 0, \vec{x}^T A \vec{x} > 0$

$$Ax = b$$

14.1 Gradient descent

Gradient descent Machine Learning

‘ ’ $f : \mathbb{R}^n \to \mathbb{R}$

P P

$$f(\vec{x}) = \frac{1}{2}\vec{x}^T A \vec{x} - \vec{b}^T \vec{x} + c$$

$f(\vec{x})$

$$\nabla f(\vec{x}) = A\vec{x} - \vec{b}$$

- $\vec{d}_k = -\nabla f(\vec{x}_{k-1}) = \vec{b} - A\vec{x}_{k-1}$
- $\vec{x}_k = \vec{x}_{k-1} + \alpha_k \vec{d}_k,$ α_k $f(\vec{x}_k) < f(\vec{x}_{k-1})$

α_k:

$$
\begin{aligned}
g(\alpha) &= f(\vec{x} + \alpha\vec{d}) \\
&= \frac{1}{2}(\vec{x} + \alpha\vec{d})^T A(\vec{x} + \alpha\vec{d}) - \vec{b}^T(\vec{x} + \alpha\vec{d}) + c \\
&= \frac{1}{2}(\vec{x}^T A\vec{x} + 2\alpha\vec{x}^T A\vec{d} + \alpha^2\vec{d}^T A\vec{d}) - \vec{b}^T\vec{x} - \alpha\vec{b}^T\vec{d} + c \\
&= \frac{1}{2}\alpha^2\vec{d}^T A\vec{d} + \alpha(\vec{x}^T A\vec{d} - \vec{b}^T\vec{d}) + const
\end{aligned}
$$

α

$$
\begin{aligned}
\frac{dg(\alpha)}{d\alpha} &= \alpha\vec{d}^T A\vec{d} + (\vec{x}^T A\vec{d} - \vec{b}^T\vec{d}) \\
&= \alpha\vec{d}^T A\vec{d} + \vec{d}^T A\vec{x} - \vec{d}^T\vec{b} \\
&= \alpha\vec{d}^T A\vec{d} + \vec{d}^T(A\vec{x} - \vec{b})
\end{aligned}
$$

0

$$\alpha = \frac{\vec{d}^T(\vec{b} - A\vec{x})}{\vec{d}^T A\vec{d}}$$

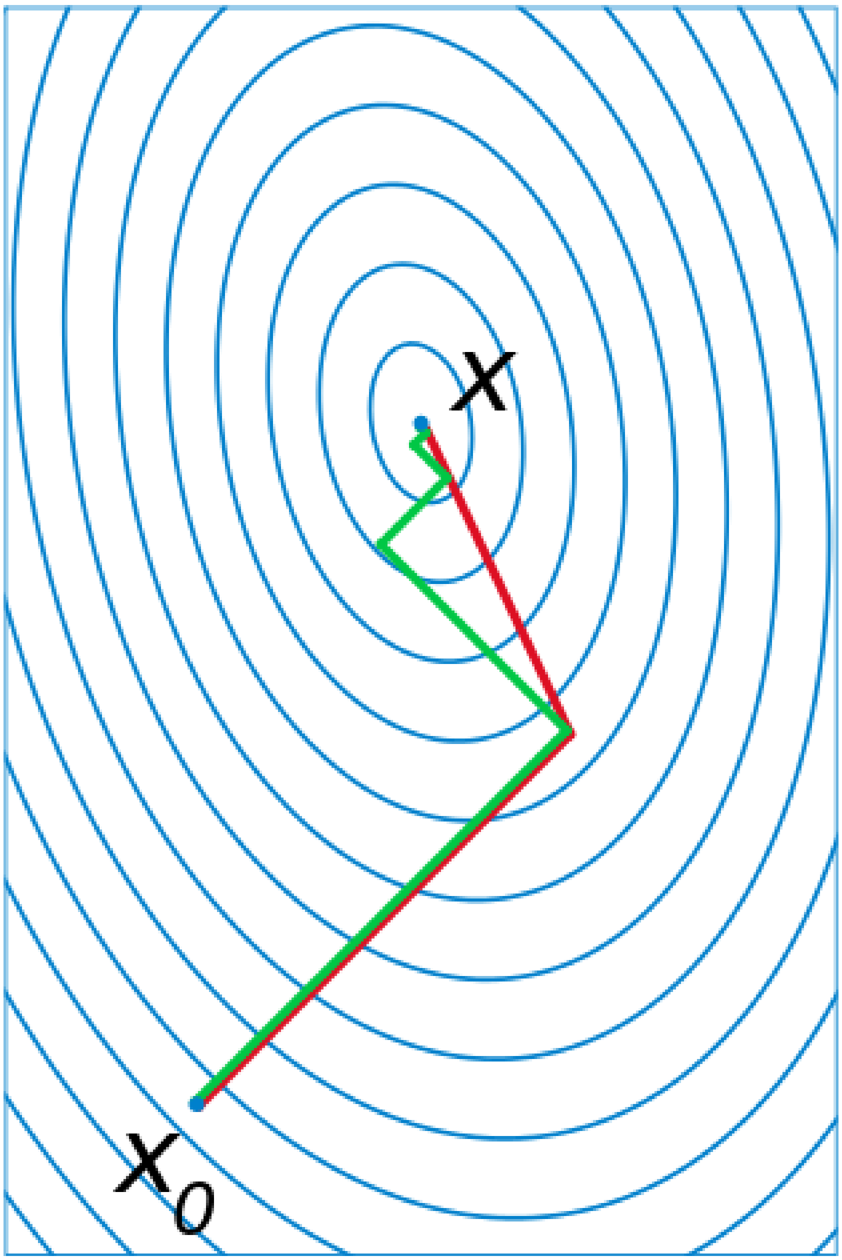
$\vec{d}_k = \vec{b} - A\vec{x}_{k-1}$

$$\alpha_k = \frac{\vec{d}_k^T\vec{d}_k}{\vec{d}_k^T A\vec{d}_k}$$

$$\vec{d}_k = \vec{b} - A\vec{x}_{k-1}\alpha_k = \frac{\vec{d}_k^T\vec{d}_k}{\vec{d}_k^T A\vec{d}_k}\vec{x}_k = \vec{x}_{k-1} + \alpha_k\vec{d}_k$$

AA^T

14.2 Conjugate gradient



/

z

n = 2,

call API

‘ (conjugate)’ $\vec{u}$ $\vec{v}$ A

$$\vec{u}^{\mathsf{T}}\mathbf{A}\vec{v} = 0.$$

‘ ’

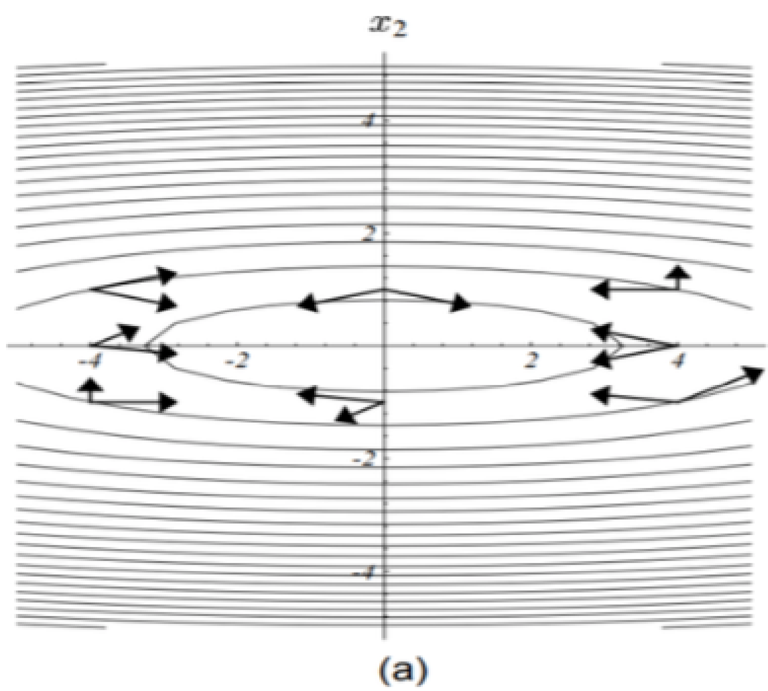
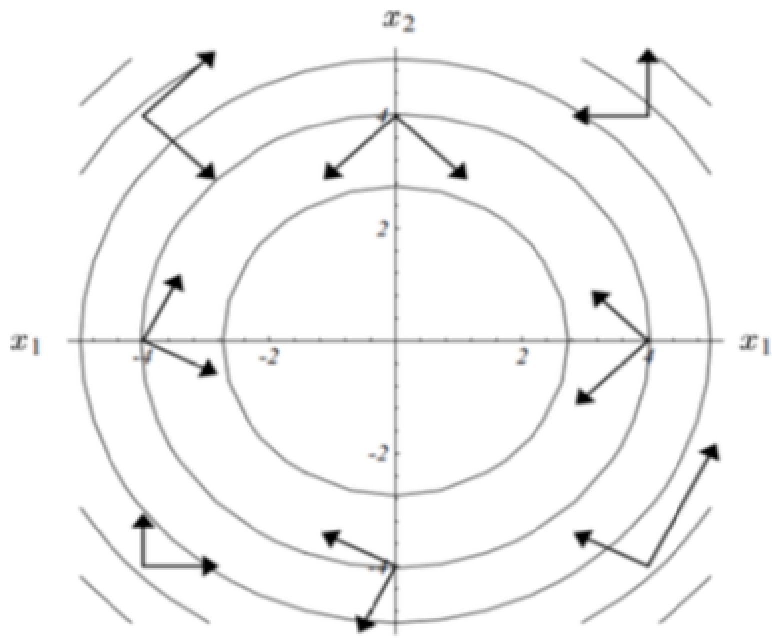


Figure 14.1: conjugate_02.png

‘ ’



(b)

-from wikipedia

$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$
if $\mathbf{r}_0$ is sufficiently small, then return $\mathbf{x}_0$ as the result
$\mathbf{p}_0 := \mathbf{r}_0$
$k := 0$
repeat

$\alpha_k := \dfrac{\mathbf{r}_k^\mathsf{T}\mathbf{r}_k}{\mathbf{p}_k^\mathsf{T}\mathbf{A}\mathbf{p}_k}$
$\mathbf{x}_{k+1} := \mathbf{x}_k + \alpha_k\mathbf{p}_k$
$\mathbf{r}_{k+1} := \mathbf{r}_k - \alpha_k\mathbf{A}\mathbf{p}_k$
if $\mathbf{r}_{k+1}$ is sufficiently small, then exit loop
$\beta_k := \dfrac{\mathbf{r}_{k+1}^\mathsf{T}\mathbf{r}_{k+1}}{\mathbf{r}_k^\mathsf{T}\mathbf{r}_k}$
$\mathbf{p}_{k+1} := \mathbf{r}_{k+1} + \beta_k\mathbf{p}_k$
$k := k + 1$

end repeat
return $\mathbf{x}_{k+1}$ as the result

wikipedia MATLAB

https://www.zhihu.com/question/27157047/answer/121950241

14.3

$$Ax = b$$

A

- dense and/or small
- large and sparse, or not available ex- plicitly
- narrow-banded

Tridiagonal matrix:

$$\begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix}$$

Pentadiagonal matrix

$$\begin{pmatrix} c_1 & d_1 & e_1 & 0 & \cdots & \cdots & 0 \\ b_1 & c_2 & d_2 & e_2 & \ddots & & \vdots \\ a_1 & b_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & a_2 & \ddots & \ddots & \ddots & e_{n-3} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & d_{n-2} & e_{n-2} \\ \vdots & & \ddots & a_{n-3} & b_{n-2} & c_{n-1} & d_{n-1} \\ 0 & \cdots & \cdots & 0 & a_{n-2} & b_{n-1} & c_n \end{pmatrix}.$$

, 0

- symmetric positive definite dense and/or small Cholesky
- symmetric positive definite large and sparse
- symmetric indefinite, dense and/or small Bunch–Kaufman
- symmetric indefinite, large and sparse MINRES
- nonsymmetric, large and sparse GMRES BiCGSTAB or IDR

- https://en.wikipedia.org/wiki/Gradient_descent
- https://en.wikipedia.org/wiki/Conjugate_gradient_method
- Solution of Linear Systems via Chen Greif

Chapter 15

{Interpolate}

B

/ $\phi_1, \phi_2, \cdots$

$$f(x) = \sum_i a_i \phi_i(x)$$

a

k $(x_1, y_1), \cdots, (x_k, y_k)$ $x_1 < x_2 < \cdots < x_k$

15.1

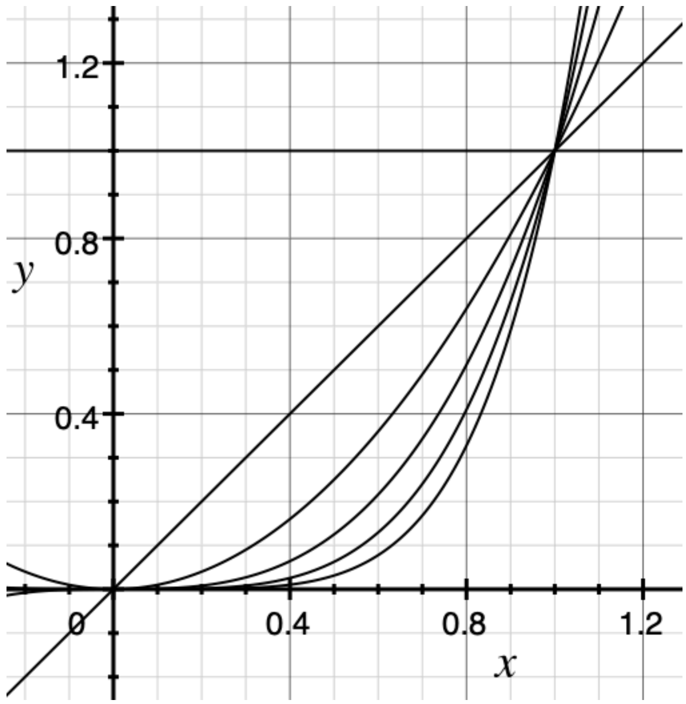
:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1}$$

$$\begin{bmatrix} 1 & x_1 & \dots & x_1^{k-1} \\ 1 & x_2 & \dots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{k-1} & \dots & x_k^{k-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

x,y a,

$$\{1, x, x^2, \cdots, x^{k-1}\}$$



B

15.2

$$\phi_i(x) = \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)}$$

$$\phi_i(x_l) = \begin{cases} 1, l = i \\ 0, otherwise \end{cases}$$

*

$x_i \approx x_j,$

a

15.3

$$\phi_i(x) = \prod_{j=1}^{i-1}(x - x_j)$$

$$\phi_1(x) = 1$$

$$\phi_i(x_l) = 0, l < i$$

$$f(x) = \sum_i a_i\phi_i(x)$$

$$\begin{bmatrix} \phi_1(x_1) & 0 & ... & 0 \\ \phi_1(x_1) & \phi_2(x_2) & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1(x_1) & \phi_2(x_2) & ... & \phi_k(x_k) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$$

$$f(x) = \frac{p_0 + p_1x + p_2x^2 + \dots + p_mx^m}{q_0 + q_1x + q_2x^2 + \dots + q_nx^n}$$

15.4

spline local / /

(interpolate) extrapolate linear bilinear trilinear bicubic mesh

Chapter 16

{Numerial Intergration}

16.1

$f : \mathbb{R} \to \mathbb{R}$

16.1.1

/

o( )o

[a,b] f, f $ x_{0},...,x_{n} t_{0},...,t_{n-1}$

$$\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$$

$ x_{i+1}-x_{i}$ t_i $f(t_i)$ t_i X

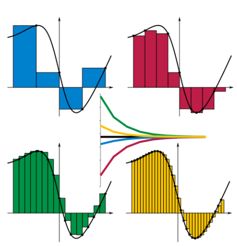
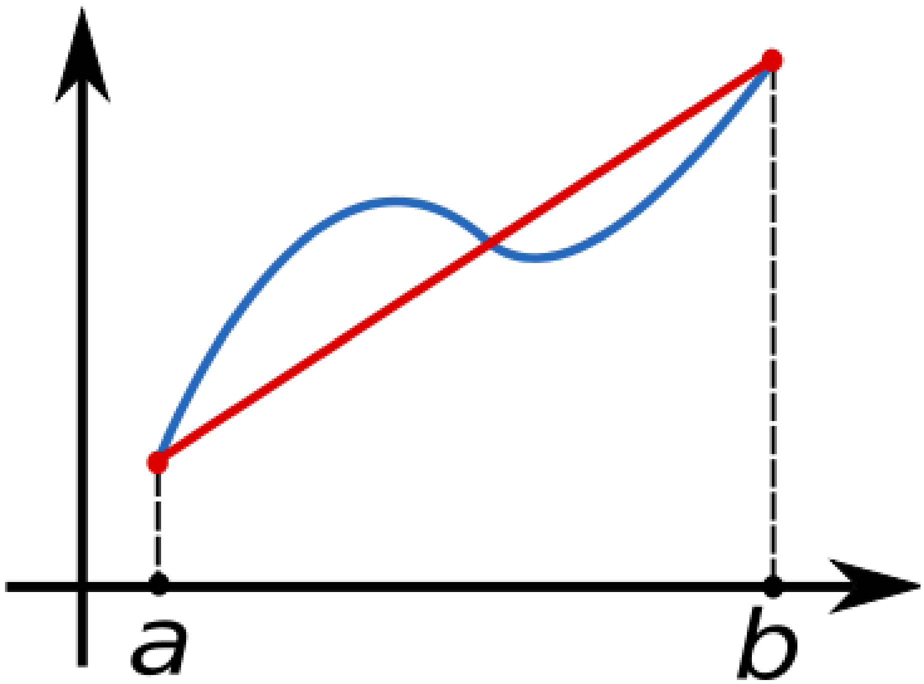


Figure 16.1: wikipedia

$f(t_i)$

16.1.2

‘ ’

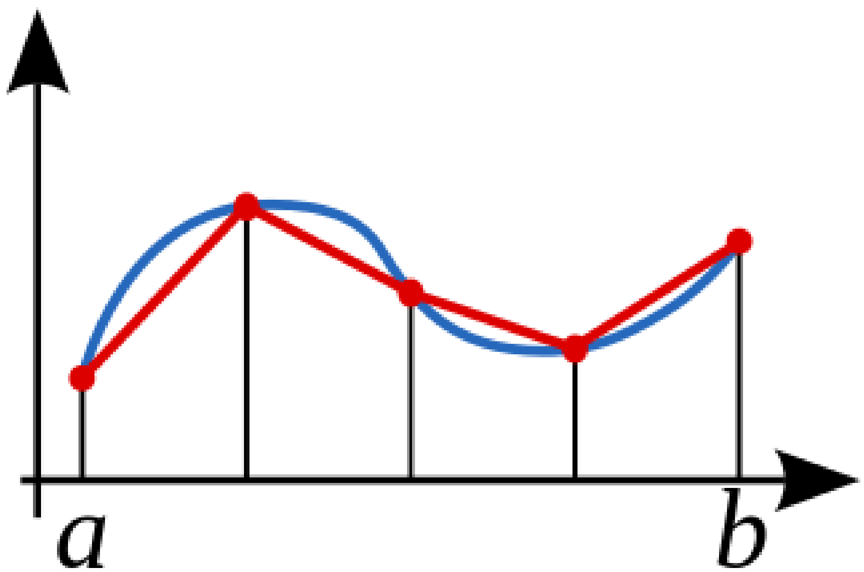


$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a)+f(b)]$$

$$\int_a^b f(x)dx \approx (b-a)f(\frac{a+b}{2})$$

Trapezoidal rule

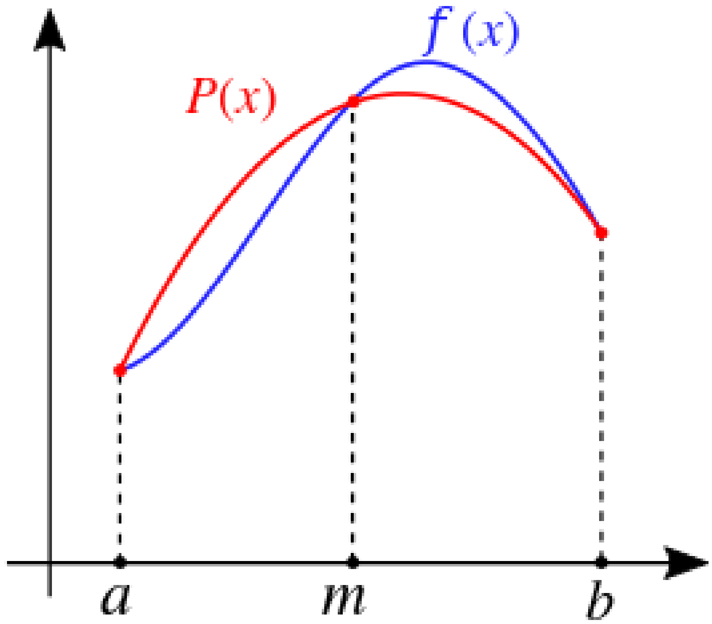
1.
2.



16.1.3

Simpson's rule -

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$



$f(x) = Ax^2 + Bx + C$

- Newton-Cotes rule / Newton-Cotes formula

16.1.4 -

n -

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i\, f(x_i)$$

- $f(x_0), f(x_1), \ldots, f(x_n)$
- n+1 f(x)
- n

$\int_a^b f(x)\,dx$ n w_i

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b \left[\sum_i a_i\phi_i(x)\right]dx\\ &= \sum_i a_i\left[\int_a^b \phi_i(x)\right]dx\\ &= \sum_i c_i a_i,\ \text{for } c_i \equiv \int_a^b \phi_i(x)dx\end{aligned}$$

$$w_1\cdot 1+w_2\cdot 2+\cdots+w_n\cdot 1=\int_a^b 1dx = b-a w_1\cdot x_1+w_2\cdot x_2+\cdots+w_n\cdot x_n=\int_a^b xdx=(b^2-a^2)/2 \vdots w_1\cdot x_1^{n-1}+w_2\cdot x_2^{n-1}+\cdots+w_n\cdot x$$

16.1.5

$O(\Delta x^3)$, $O(\Delta x^4)$

$f:\mathbb{R}^k\to\mathbb{R}$ /

16.2

$$f'(x)=\sum_i a_i\phi_i'(x)$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

h

$$f'(x)\approx\frac{f(x+h)-f(x)}{h}$$

x

$$f'(x)\approx\frac{f(x)-f(x-h)}{h}$$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \cdots$$

O(h):

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \cdots \tag{1}$$

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + \cdots \tag{2}$$

$$f(x+h) - f(x-h) = 2f'(x) + \frac{1}{3}f'''(x)h^3 + \cdots$$

$O(h^2)$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

1 2 O(h)

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$D(h) = \frac{f(x+h) - f(x)}{h} = f'(x) + \frac{1}{2}f''(x)h + O(h^2)$$

α $D(h), D(\alpha h)$:

$$D(h) = f'(x) + \frac{1}{2}f''(x)h + O(h^2) D(\alpha h) = f'(x) + \frac{1}{2}f''(x)\alpha h + O(h^2)$$

$$\begin{pmatrix} D(h) \\ D(\alpha h) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}h \\ 1 & \frac{1}{2}\alpha h \end{pmatrix} \begin{pmatrix} f'(x) \\ f''(x) \end{pmatrix} + O(h^2)$$

$$\begin{pmatrix} f'(x) \\ f''(x) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}h \\ 1 & \frac{1}{2}\alpha h \end{pmatrix}^{-1} \begin{pmatrix} D(h) \\ D(\alpha h) \end{pmatrix} + O(h^2)$$

$O(h^2)$ f'(x):

$$f'(x) = \frac{1}{1-\alpha}(-\alpha D(h) + D(\alpha h)) + O(h^2)$$

-
-
- -
- Taylor series
- Riemann_sum
- Trapezoidal rule
- Simpson's rule

/

Scipy integrate SciPy

Chapter 17

{ODE}

- ordinary differential equation ODE)

$$m\frac{d^2x}{dt^2} = F(x)$$

logistic function:

$$\frac{dP}{dt} = P$$

17.1

$F(t) : \mathbb{R} \to \mathbb{R}^n$satifying $: F[t, f(t), f'(t), f''(t), \cdots, f^{(k)}(t)] = 0$Given $f(0), f'(0), f''(0), \cdots, f^{(k-1)}(0)$

$f(t), f'(t), \cdots, f^{(k)}(t)$ t

- vs

n implicit explicit

$$f^{(k)}(t) = F[t, f(t), f'(t), f''(t), \cdots, f^{(k-1)}(t)]$$

- vs

autonomous f

$$\frac{d}{dt}x(t) = f(x(t))$$

$$\frac{d}{dt}x(t) = g(x(t), t)$$

-

.

17.2 ODE

ODE

$$y''' = 3y'' - 2y' + y$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$z = \frac{dy}{dt} w = \frac{d^2y}{dt^2}$$

$$\frac{d}{dt}\begin{pmatrix} y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} y \\ z \\ w \end{pmatrix}$$

ODE ODE ODE:

$$f^{(k)}(t) = F[t, f(t), f'(t), f''(t), \cdots, f^{(k-1)}(t)]$$

ODE:

$$\frac{d}{dt}\begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_{k-1}(t) \\ g_k(t) \end{pmatrix} = \begin{pmatrix} g_2(t) \\ g_3(t) \\ \vdots \\ g_k(t) \\ F[t, g_1(t), g_2(t), \cdots, g_{(k-1)}(t)] \end{pmatrix}$$

$g_2(t) = g_1'(t), g_3(t) = g_2'(t) = g_1''(t).$

$$f'(t) = F[t, f(t)], \qquad g(t) = t, \quad f'(t)$$

$$\frac{d}{dt}\begin{pmatrix} g(t) \\ f(t) \end{pmatrix} = \begin{pmatrix} 1 \\ F[g(t), f(t)] \end{pmatrix}$$

ODE

$$f'(t) = F[f(t)]$$

17.3

ODE

- slope field

  A slope field is a collection of short line segments, whose slopes match that of a solution of a first-order differential equation passing through the segment's midpoint. The pattern produced by the slope field aids in visualizing the shape of the curve of the solution. This is especially useful when the solution to a differential equation is difficult to obtain analytically.

$$\frac{dy}{dx} = x^2 - x - 2$$

$$x^3/3 - x^2/2 - 2x + 4x^3/3 - x^2/2 - 2xx^3/3 - x^2/2 - 2x - 4$$

ODE

- phase diagram

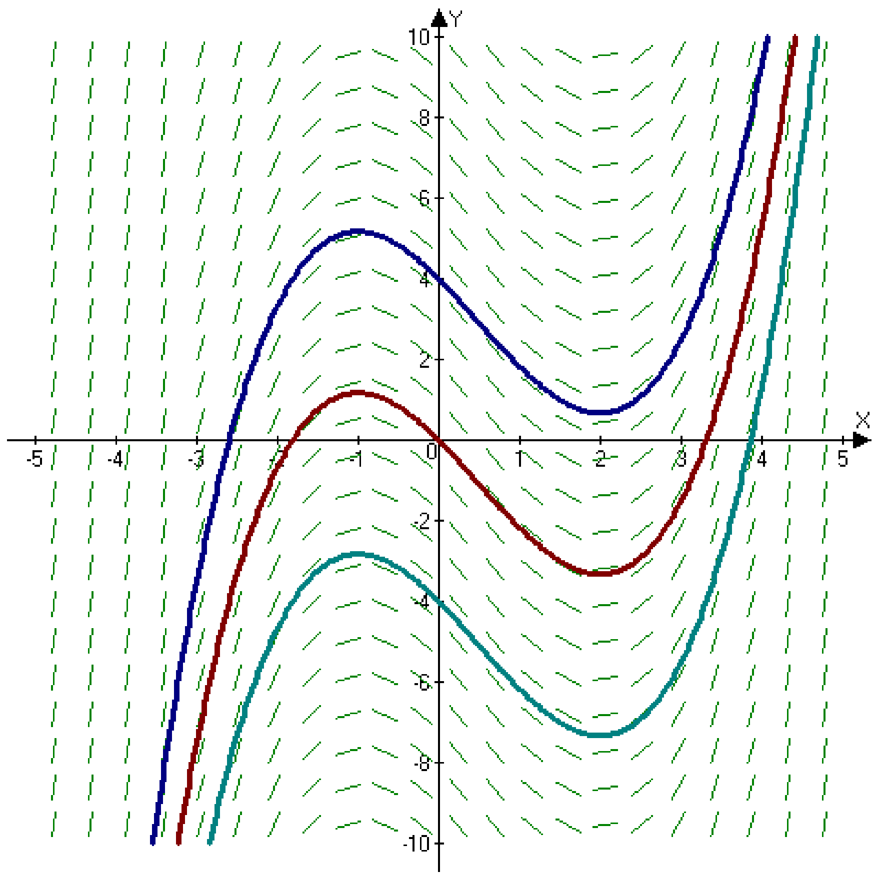
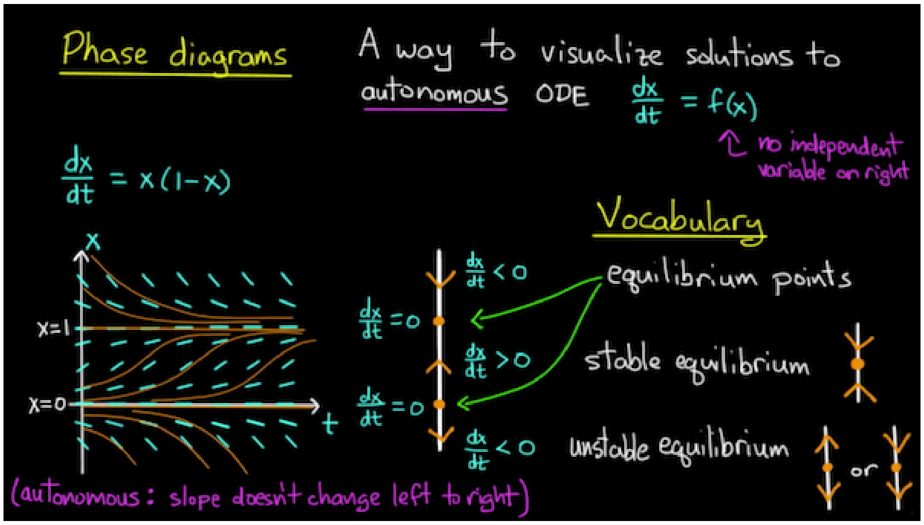


Figure 17.1: Slope_Field.png



$dx/dt = f(x)$:

$$\frac{dx}{dt} = x(1-x)$$

x = 0 x = 1 $dx/dt = 0$, phase line

17.4

$y' = 2y/t$:

$$\frac{dy}{dt} = \frac{2y}{t}$$

$$ln|y| = 2lnt + c$$

$y = Ct^2$

- $y(0) \neq 0$
- $y(0) = 0$, $C \in \mathbb{R}$
-

F Lipschitz continuity $|F[\vec{y}] - F[\vec{x}]|_2 \leq L|\vec{y} - \vec{x}|_2$ $f'(t) = F[f(t)]$

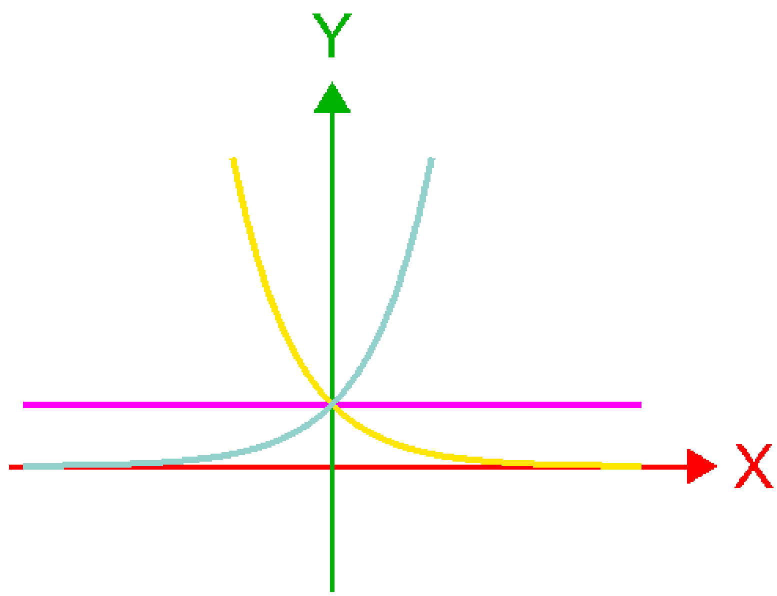
17.5 ODE

ODE

$$y' = ay$$

$$y(t) = Ce^{at}$$

a



- a = 0: y(t) t
- a < 0: t y(t) 0
- a > 0: t y(t)

ODE

$$\vec{y}' = A\vec{y}$$

$\vec{y}_1, \cdots, \vec{y}_k$ A $\lambda_1, \cdots, \lambda_k$ $\vec{y}(0) = c_1\vec{y}_1 + \cdots + c_k\vec{y}_k$

$$\vec{y}(t) = c_1 e^{\lambda_1 t}\vec{y}_1 + \cdots + c_k e^{\lambda_k t}\vec{y}_k$$

ODE $\vec{y}' = F[\vec{y}]$, F

$$F[\vec{y}] = F[\vec{y}_0] + J_F(\vec{y}_0)(\vec{y} - \vec{y}_0)$$

t_k $\vec{y}_k$, $\vec{y}' = F[\vec{y}]$ $\vec{y}_{k+1}$

17.6

17.6.1

$$y_{k+1} = y_k + hF[y_k]$$

y' = ay $a < 0, 0 \leq h \leq \frac{2}{|a|}$ h h

17.6.2

$$y_k = y_{k+1} - hF[y_{k+1}]$$

y_{k+1}.

Chapter 18

{PDE}

ODE partial differential equation)

18.1

ODE Lipschitz continuity PDE finite element method o( )o

18.2

:

$f : \mathbb{R}^3 \to \mathbb{R}, \vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$Gradient: $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3})$Divergence: $\nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$Curl: $\nabla \times \vec{v} = (\frac{\partial v_3}{\partial x_2} - \frac{\partial v}{\partial x}$

operator	operand	result
Gradient	Multivariate function $f : \mathbb{R}^3 \to \mathbb{R}$	Vector $\mathbb{R} \to \mathbb{R}^3$
Divergence	Vector Field $\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$	scalar $\mathbb{R}^3 \to \mathbb{R}$
Curl	Vector Field $\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$	Vector Field $\mathbb{R}^3 \to \mathbb{R}^3$
Laplacian	Multivariate function $f : \mathbb{R}^3 \to \mathbb{R}$	scalar $\mathbb{R}^3 \to \mathbb{R}$

-
-

$f(t; x, y, z)$ nabla $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, t

18.3 - Navier-Stokes equations

Navier-Stokes equations PDE

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \mu\nabla^2\vec{v} + \vec{f} t \in [0, \infty) : \text{time} \quad \vec{v}(t) : \Omega \to \mathbb{R}^3 : \text{velocity} \quad \rho(t) : \Omega \to \mathbb{R} : \text{density} \quad p(t)$$

wikipedia

$$\overbrace{\rho\Big(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\substack{\text{Unsteady}\\\text{acceleration}}} + \underbrace{(\mathbf{v}\cdot\nabla)\mathbf{v}}_{\substack{\text{Convective}\\\text{acceleration}}}\Big)}^{\text{Inertia}} = \underbrace{-\nabla p}_{\substack{\text{Pressure}\\\text{gradient}}} + \underbrace{\mu\nabla^2\mathbf{v}}_{\text{Viscosity}} + \underbrace{\mathbf{f}}_{\substack{\text{Other}\\\text{forces}}}$$

Navier-Stokes equations

> Prove or give a counter-example of the following statement:
>
> In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

$1,000,000

- P vs NP
-
-
-
- ...

18.4 Maxwell's equations

PDE -

$$\text{Gauss's law: } \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \text{Gauss's law for magnetism: } \nabla\cdot\mathbf{B} = 0 \text{Maxwell–Faraday equation: } \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

18.5 Laplace's equation

$$\nabla^2 f(\vec{x}) = 0$$

harmonic function

harmonic (overtone)

- $f : \mathbb{R} \to \mathbb{R}$: ,

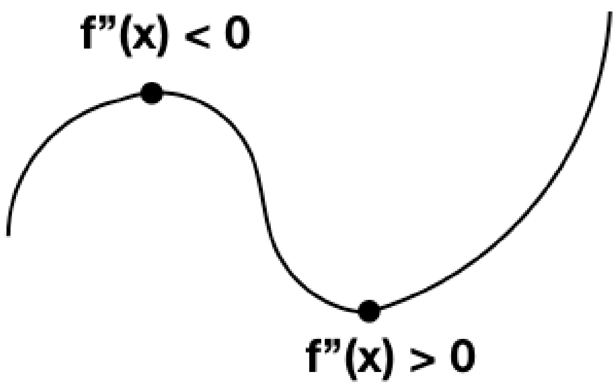


Figure 18.1: 2nd_derivative.png

- $f : \mathbb{R}^n \to \mathbb{R}$:

harmonic function: $f(x, y) = e^x siny$

,

Hodge

…… ……

, setup

$$\text{minimize}_f \int_\Omega \| \nabla f(\vec{x}) \|_2^2 \, d\vec{x} \text{such that } f(\vec{x}) = g(\vec{x}), \forall \vec{x} \in \partial\Omega$$

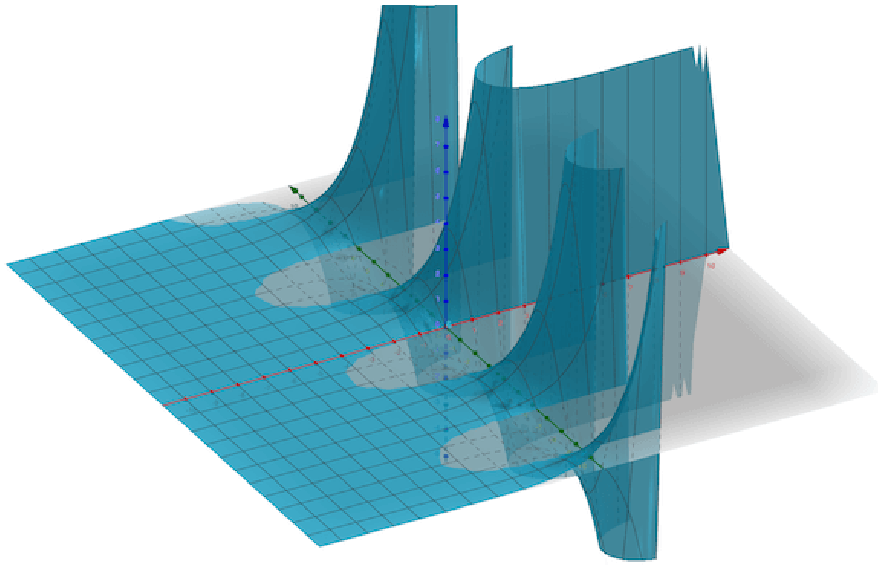


Figure 18.2: harmonic_f2.png

Ω $\partial\Omega$ $\partial\Omega$ $g(\vec{x})$ $f(\vec{x}) = g(\vec{x})$

$\text{minimize}_f \int_\Omega \parallel \nabla f(\vec{x}) \parallel_2^2 d\vec{x}$ - (Dirichlet's energy)

$$E[f] = \int_\Omega \parallel \nabla f(\vec{x}) \parallel_2^2 d\vec{x}$$

energy function

l2 norm -

- f = g
- f

f 'as smooth as possible'. 'as rigid as possible')

f

h

$$E[f+h] \geq E[f]$$

$E[f+\epsilon h]$:

$$\begin{aligned} E[f+\epsilon h] &= \int_\Omega \parallel \nabla f(\vec{x}) + \epsilon \nabla h(\vec{x}) \parallel_2^2 d\vec{x} \\ &= \int_\Omega (\parallel \nabla f(\vec{x}) \parallel_2^2 + 2\epsilon f(\vec{x}) \cdot \nabla h(\vec{x}) + \epsilon^2 \parallel \nabla h(\vec{x}) \parallel_2^2) d\vec{x} \end{aligned}$$

ϵ

$$\frac{d}{d\epsilon}E[f+\epsilon h] = \int_\Omega \left(2f(\vec{x})\cdot\nabla h(\vec{x}) + 2\epsilon \parallel \nabla h(\vec{x}) \parallel_2^2\right)d\vec{x}$$
$$\frac{d}{d\epsilon}E[f+\epsilon h]|_{\epsilon=0} = 2\int_\Omega \left(f(\vec{x})\cdot\nabla h(\vec{x})\right)d\vec{x}$$

h $h(\vec{x}) = 0, \vec{x} \in \partial\Omega$

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega \;=\; \int_\Gamma v\,\nabla u\cdot\hat{\mathbf{n}}\,d\Gamma - \int_\Omega v\,\nabla^2 u\,d\Omega$$

$$\frac{d}{d\epsilon}E[f+\epsilon h]|_{\epsilon=0} = -2\int_\Omega \left(h(\vec{x})\nabla^2 f(\vec{x})\right)d\vec{x}$$

0

$$\nabla^2 f(\vec{x}) = 0, x \in \Omega \quad \partial\Omega$$

PDE

$$\nabla^2 f(\vec{x}) = 0$$
$$f(\vec{x}) = g(\vec{x}), \forall \vec{x} \in \partial\Omega$$

Dirichlet problem

$\mathbb{R}^n$ g f f f = g

18.6 Harmonic analysis

$$\nabla^2 f = \lambda f$$

PDE

18.7 Boundary Value Problems

Dirichlet problem

- Dirichlet conditions: $f(\vec{x}) = g(\vec{x})$on $\partial\Omega$
- Neumann conditions: $\nabla f(\vec{x}) = g(\vec{x})on\partial\Omega$
- Robin boundary condition: $af(\vec{x}) + b\nabla f(\vec{x}) = g(\vec{x})$on $\partial\Omega$

18.8 PDE

PDE

$$\sum_{ij} a_{ij}\frac{\partial f}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial f}{\partial x_i} + cf = 0$$

$$(\nabla^T A \nabla + \nabla \cdot \vec{b} + c)f = 0$$

- A elliptic
- A 0 parabolic
- A hyperbolic
- ultrahyperbolic

18.9 PDE

- &
- C^∞
- / $\nabla^2 f = g$

18.10 PDE

- /
- $\frac{\partial f}{\partial t} = \alpha \nabla^2 f$
-

18.11 PDE

- : $\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$
- :

18.12

$$y_k'' \quad :$$

$$y_k'' = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}$$

f(x) [0,1]

$$y_0 = f(0), y_0 = f(h), y_2 = f(2h), \cdots, y_n = f(nh)$$

‘ ’ :

$$y_k'' = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}$$

$$y_k'' h^2 = y_{k+1} - 2y_k + y_{k-1}$$

y_k $\vec{y} \in \mathbb{R}^{n+1}$ y_k'' $\vec{w} \in \mathbb{R}^{n+1}$:

$$h^2 \vec{w} = L_1 \vec{y}$$

L_1

- Dirichlet

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

- Neumann

$$\begin{pmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix}$$

- f(0) = f (1)

$$\begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}$$

2D

$$(\nabla^2 y)_{k,l} = \frac{1}{h^2}(y_{(k-1),l} + y_{k,(l-1)} + y_{(k+1),l} + y_{k,(l+1)} - 4y_{k,l})$$