Tutorial 7: Mean Field approach

Nicolò Ruggeri and Caterina De Bacco

Exercise 1: Curie-Weiss model again

The Hamiltonian of the Curie-Weiss model can be written as:

$$H(\mathbf{s}) = -\frac{J}{2N}\sum_{i,j} s_i s_j - h\sum_i s_i \,.$$

From the lecture, we know that the Mean Field free variational energy is:

$$F[Q] = -\frac{J}{2}m^2 - hm + \left[\frac{1+m}{2}\log\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\log\left(\frac{1-m}{2}\right)\right] \tag{1}$$

where m is the expected total magnetisation $\mathbb{E}_Q\left[\sum_i s_i/N\right]$.

(a) Plot the function $F[Q]$ as a function of m in the two following cases: (i) at $h = 0$ for different values of J larger and lower than 1 and (ii) at a value of J larger than 1 for different values (positive and negative) of h. Describe what you see in both cases.

(b) The minimizer m^* is also the solution of the "self-consistent equation":

$$m = \tanh\left[\frac{J}{2}m + h\right].$$

Compute the value of m^* in the three following cases:
(i) $h = 10^{-6}$ and J between 0 and 2;
(ii) $h = -10^{-6}$ and J between 0 and 2; and
(iii) $J = 1.5$ and h between -1 and 1.

(c) Focusing on the case $J = 1.5$. With $h = 0.1, 0.2$, how many solutions to the self-consistency equations are there? Which one is the correct one?
Plot the function $F[Q]$ to answer to these questions.

Exercise 2: sampling from the Curie-Weiss model

Consider again the Hamiltonian of the Curie-Weiss model.
A practical way to sample configurations of N spins from the Gibbs probability distribution:

$$P(\mathbf{s}) = \frac{e^{-\beta H(\mathbf{s})}}{Z} \quad ,$$

with

$$H(\mathbf{s}) = -\frac{J}{2N}\sum_{i,j} s_i s_j - h\sum_i s_i \, ,$$

is the Monte-Carlo-Markov-Chain (MCMC) method, and in particular the Metropolis-Hastings algorithm.
It works as follows:

1. Choose a starting configuration for the N spins values $s_i = \pm 1$ for $i = 1, \ldots, N$.
2. Choose a spin i at random. Compute the current value of the energy H_{now} and the value of the energy H_{flip} if the spins i is flipped (that is if $s_i^{\text{new}} = -s_i^{\text{old}}$).
3. Sample a number r uniformly in $[0,1]$ and, if $r < e^{\beta(H_{\text{now}} - H_{\text{flip}})}$ perform the flip (i.e. $s_i^{\text{new}} = -s_i^{\text{old}}$) otherwise leave it as it is.
4. Goto step 2.

If one is performing this program long enough, it is guarantied that the final configuration ($\mathbf{s}$) will have been chosen with the correct probability.

(a) Write a code to perform the MCMC dynamics, and start by a configuration where all spins are equal to 1. Take $h = 0$, $J = 1$, $\beta = 1.2$ and try your dynamics for a long enough time (say, with $t_{\max} = 100N$ attempts to flips spins) and monitor the value of the magnetization per spin $m = \sum_i s_i / N$ as a function of time. Make a plot for $N = 10, 50, 100, 200, 1000$ spins. Compare with the exact solution at $N = \infty$. Comment.

(b) Start by a configuration where all spins are equal to 1 and take $h = -0.1$, $J = 1$, $\beta = 1.2$. Monitor again the value of the magnetization per spin $m = \sum_i s_i / N$ as a function of time. Make a plot for $N = 10, 50, 100, 200, 1000$ spins. Compare with the exact solution at $N = \infty$. Comment.