Advanced Probabilistic Machine Learning and Applications

Caterina De Bacco and Nicoló Ruggeri

1 Tutorial 8: TAP approach

Exercise 1: planted SK model

We consider a model similar to the SK model, with the difference that the J_{ij} are generated from a particular realization, called "planted". Specifically, we start from the following generative model for the $P(\mathbf{s},\mathbf{J})$:

$$
\begin{aligned}
s_i &\sim \frac{1}{2}\delta(s_i-1)+\frac{1}{2}\delta(s_i+1) \quad , \quad i=1,\dots,N \qquad (1)\\
J_{ij}\,|\,s_i,s_j &\sim \mathcal{N}\left(\frac{s_i s_j}{\sqrt{N}},\sigma^2\right) \quad , \quad (i,j)\in N^2 \qquad (2)
\end{aligned}
$$

Goal: estimate the planted $\mathbf{s}$ given the $\mathbf{J}$.

(a) Use Bayes' theorem to write the posterior distribution $P(\mathbf{s}|\mathbf{J})$.

(b) Rewrite it as a Boltzmann distribution similar to the SK model with $\beta := \frac{1}{\sigma^2}$.

(c) We would like to estimate the mean of $P(\mathbf{s}|\mathbf{J})$. However, this is not tractable analytically. We will instead use the approximation introduced in the class.
Write in a *jupyter* notebook a function to *sample* an instance of $(\mathbf{s},\mathbf{J})$.

(d) In the same notebook, write a function that implements the TAP equation to approximate the mean $\hat{\mathbf{s}}$ of $P(\mathbf{s}|\mathbf{J})$. This is an iteration that, if it converges, gives a very good approximation for $\hat{\mathbf{s}}$ as $N\to\infty$.

For numerical reasons implement the fixed point iterations as follows:

$$
m_i^{(t+1)} = \tanh\left(\frac{1}{\sigma^2\sqrt{N}}\sum_j J_{ij}\,m_j^{(t)}\right) \qquad \text{Mean Field}
$$

$$
m_i^{(t+1)} = \tanh\left(\frac{1}{\sigma^2\sqrt{N}}\sum_j J_{ij}\,m_j^{(t)} - m_i^{(t-1)}\frac{1}{N\sigma^4}\sum_j J_{ij}^2\left(1-(m_j^{(t)})^2\right)\right) \qquad \text{TAP}
$$

(e) Run some experiments ($N_{real}\in[10,100]$ re-samplings of J,s at your choice) for $N=10,100,1000,5000$ and fixed $\sigma^2=0.1$ and check that the overlap[1] with ground-truth improves with the iterations.

[1] The overlap is defined as: $overlap(\mathbf{m},\mathbf{s}_0) := |\frac{\mathbf{m}\cdot\mathbf{s}_0}{N}|$

(f) i) Run sum experiments ($N_{real} \in [10, 100]$ re-samplings of J, s at your choice) for $N = 10, 100, 1000, 5000$ and varying $\sigma^2 \in [0.1, 2]$.
ii) Repeat the same experiments but using the MF approximation instead.
iii) Plot the performance metrics values at convergence for TAP and MF as a function of the noise σ^2 for various N.
Comment on what you observe.

Exercise 2: sampling from the SK model

Consider again the Hamiltonian of the SK model:

$$H(\mathbf{s}) = -\sum_{i \neq j} J_{ij} s_i s_j \quad . \tag{3}$$

We now want to *sample* configurations of N variables from the corresponding Boltzmann distribution for a particular realization of the couplings and comparing the empirical mean of the magnetizations with the one inferred using TAP and MF.
For sampling, we use the Monte-Carlo-Markov-Chain (MCMC) Metropolis-Hastings algorithm as we learned for the Curie Weiss model in tutorial 7.

(a) Write a code to perform the MCMC dynamics, and start by configurations extracted uniformly at random.
i) Sample a particular realization of $\mathbf{J} \sim \mathcal{N}(0, \sigma^2/\sqrt{N})$, for $\sigma^2 = 1$ and $\beta = 1.1$.
ii) Run your dynamics for a long enough time (say, with $t_{\max} = 10^3 N$ attempts to flips spins) and monitor the value of the magnetization $m = \sum_i s_i / N$ as a function of time. Make a plot for $N = 10, 100, 1000$ spins.

(b) Iterate MF and TAP equations and compare the values obtained from the ground truth MCMC sampler. To do so, draw a scatter plot of the single values m_i obtained from the MCMC sampler and the approximations (one plot per approximation, for a total of 2 plots).
Comment. Similarly to before, implement the consistency equations as

$$m_i^{(t+1)} = \tanh\left(\beta \sum_j J_{ij}\, m_j^{(t)}\right) \qquad \text{Mean Field}$$

$$m_i^{(t+1)} = \tanh\left(\beta \sum_j J_{ij}\, m_j^{(t)} - \beta^2 m_i^{(t-1)} \sum_j J_{ij}^2 \,(1 - (m_j^{(t)})^2)\right) \qquad \text{TAP}$$