Advanced Probabilistic Machine Learning and Applications

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1 Tutorial 9: Bethe approximation and BP

Exercise 1: representing models using factor graphs

Write the following problems (i) in terms of a probability distribution and (ii) in terms of a graphical model by drawing an example of the corresponding factor graph.

(a) **p-spin model**

One model that is commonly studied in physics is the so-called Ising 3-spin model. The Hamiltonian of this model is written as

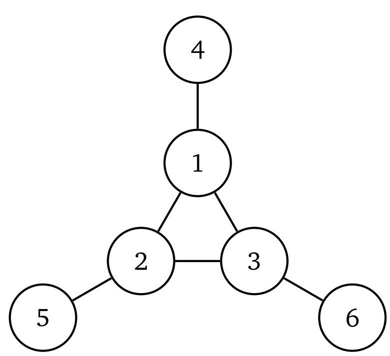
$$H(\mathbf{s}) = -\sum_{(ijk)\in E} J_{ijk} s_i s_j s_k - \sum_{i=1}^{N} h_i s_i \tag{1}$$

where E is a given set of (unordered) triplets $i \neq j \neq k$, J_{ijk} is the interaction strength for the triplet $(ijk) \in E$, and h_i is a magnetic field on spin i. The spins are Ising, which in physics means $s_i \in \{+1, -1\}$.

(b) **Independent set problem**

Independent set is a problem defined and studied in combinatorics and graph theory. Given a (unweighted, undirected) graph $G(V,E)$, an independent set $S \subseteq V$ is defined as a subset of nodes such that if $i \in S$ then for all $j \in \partial i$ we have $j \notin S$. In other words in for all $(ij) \in E$ only i or j can belong to the independent set.

For example, suppose we have the following graph:



Draw the corresponding factor graph.

(iii) Write a probability distribution that is uniform over all independent sets on a given graph.
(iv) Write a probability distribution that gives a larger weight to larger independent sets, where the size of an independent set is simply $|S|$.

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Exercise 2: graph coloring problem and BP

Coloring is another classical problem of graph theory. Given a (unweighted, undirected) graph $G(V, E)$ a coloring $M \subseteq E$ is an assignment of labels, called colors, to the vertices of a graph such that no two adjacent vertices share the same color.

(a) Write a probability distribution for the coloring problem.

(b) Consider a "soft" constraint instead which relaxes the "hard" one and write the corresponding interaction function of the factor node. Hint: the soft constraint allows two neighboring nodes to have the same color, but penalized this a lot.

(c) Draw a factor graph corresponding to it.

(d) Using BP to model marginals of the coloring assignment, denote as:

- $\nu_{s_i}^{(ij)\to i}$ the messages from function node (ij) to variable node i.
- $\chi_{s_i}^{i\to(ij)}$ the message from variable node i to function node (ij).

Note that they are both functions of the state s_i of variable node i.
Write BP equations for this model.

(e) Find a fix point of these equations. Hint: what would a random guess do?
PS: recall that there might be more than one fixed point.

(f) Write the equation for the one-point marginal $P(s_i)$ and the two-point marginal $P(s_i, s_j)$ obtained from BP.