AQA GCSE Maths: Higher

Your notes

Congruence, Similarity & Geometrical Proof

Contents

Your notes

Congruence

What is congruence?

- Two shapes are **congruent** if they are **identical** in **shape** and **size**
  - One may be a **reflection**, **rotation**, or **translation** of the other
- If one shape is an **enlargement** of the other, then they are **not identical in size** and so are **not** congruent

How do we prove that two shapes are congruent?

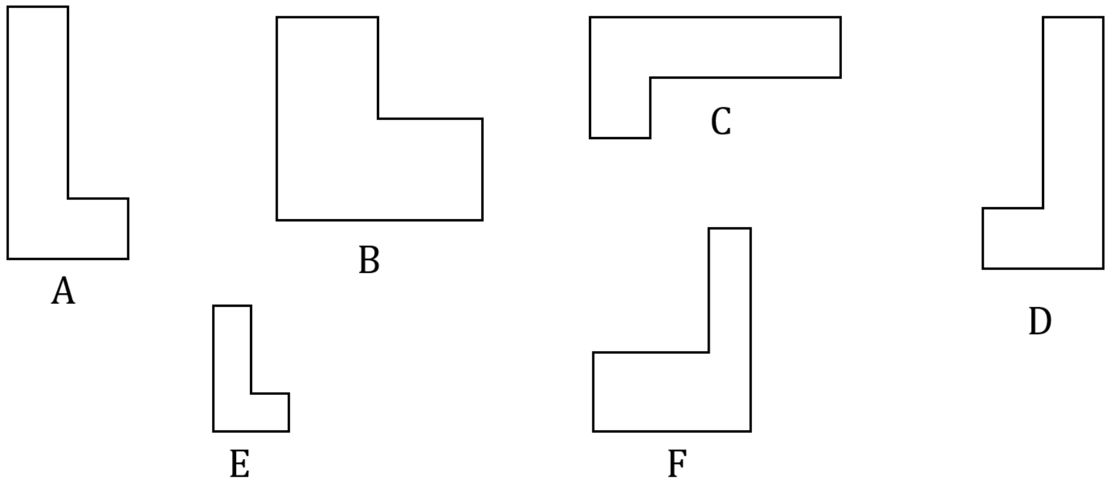
- To show that two shapes are **congruent** you need to show that they are both the **same shape** and the **same size**
  - If a shape has been reflected, rotated or translated, then its image is **congruent** to it
- Show that **corresponding sides** are the **same length**
- Show that **corresponding angles** are the **same size**
- You do **not** need to show that they are facing in the **same direction**

Examiner Tips and Tricks

- **Tracing paper** can help in the exam if you are unsure whether two shapes are congruent
  - Trace over one shape and then see if it fits **exactly** on top of the other
  - Only do this if the image is drawn to **scale**

Worked Example

Write down the letters of the two shapes below which are congruent to A.



Shapes C and D are congruent to A

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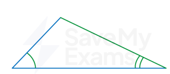
Congruent Triangles

What are congruent triangles?

- Two triangles are **congruent** if they are the same **size** and **shape**
  - Although they may be reflections, translations or rotations of each other
- All **three angles** and all **three sides** must be the same in both triangles

How do I prove that two triangles are congruent?

- We only need to show that **3 of the 6 things** are the same for both triangles
  - as long as they are the right three!
- To do this we must use **one** of the **5 standard tests**

Name	Description	Diagram
SAS **S**ide **A**ngle **S**ide	Two sides and the angle between them	
ASA **A**ngle **S**ide **A**ngle	Two angles and the side between them	
AAS **A**ngle **A**ngle **S**ide	Any two angles and any side	
SSS **S**ide **S**ide **S**ide	All three sides	

RHS **R**ight-angle **H**ypotenuse **S**ide	The hypotenuse and any other side for a right-angled triangle	

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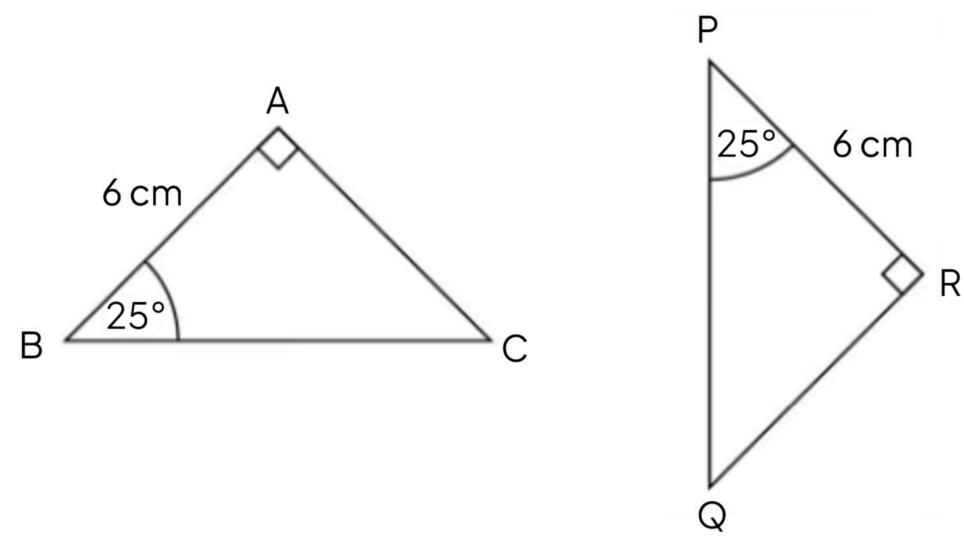
Examiner Tips and Tricks

- AAA and SSA are **not congruent** conditions

Worked Example

Prove that triangles *ABC* and *PQR* are congruent.



Angle *ABC* and angle *RPQ* are both 25°

Angle *BAC* and angle *PRQ* are both 90°

Line *PR* and line *AB* are both 6cm

Two angles are the same, and the lengths between them are the same

Triangles are congruent by the ASA condition

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Similarity

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Similarity

What are similar shapes?

- Two shapes are **similar** if they have the same **shape** and their **corresponding sides** are in **proportion**
  - One shape is an **enlargement** of the other

How do we prove that two triangles are similar?

- To show that two **triangles** are **similar** you need to show that their **angles are the same**
  - If the angles are the same then **corresponding lengths** of a triangle will automatically be **in proportion**
- You can use angle properties to **identify equal angles**
  - Look out for for **isosceles triangles**, **vertically opposite angles** and **angles on parallel lines**
- If a question asks you to **prove two triangles** are **similar**
  - For **each pair** of corresponding angles
    - **State** that they are of **equal size**
    - Give a **reason** for why they are equal

How do we prove that two shapes are similar?

- To show that two **non-triangular shapes** are **similar** you need to show that their **corresponding sides** are **in proportion**
  - Divide the length of one side by the length of the corresponding side on the other shape to find the **scale factor**
- If the **scale factor** is the **same** for **all corresponding sides**, then the shapes are **similar**

Examiner Tips and Tricks

- A pair of **similar triangles** can often be **opposite** each other in an hourglass formation.
  - Look out for the **vertically opposite**, equal angles.

- It may be helpful to sketch the triangles **next** to each other and facing in the **same direction**.

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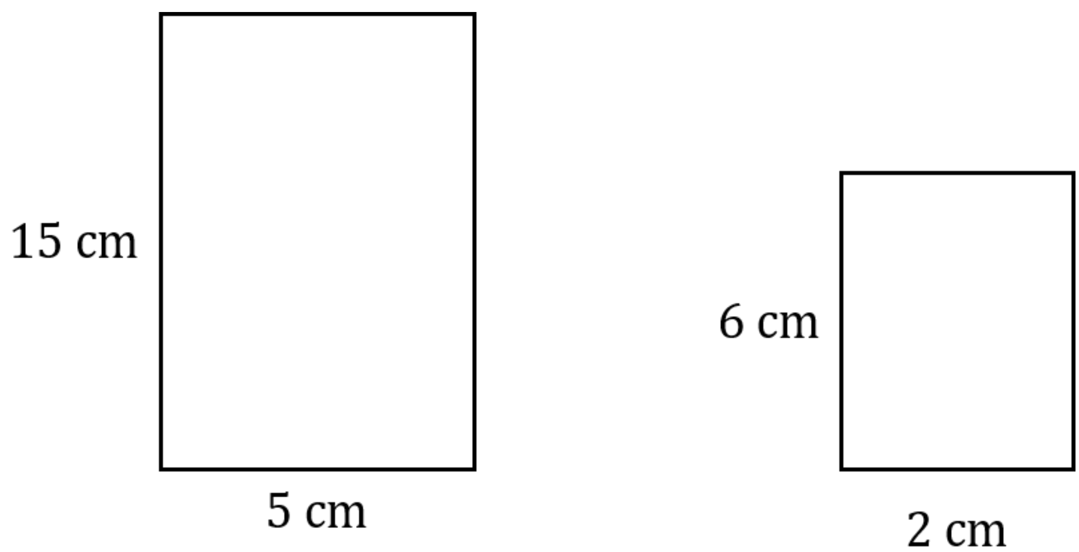
Worked Example

(a) Prove that the two rectangles shown in the diagram below are similar.



Use the corresponding lengths (15 cm and 6 cm) to find the scale factor

$$\frac{15}{6} = 2.5$$

Use the corresponding width (5 cm and 2 cm) to find the scale factor for the other pair of sides

$$\frac{5}{2} = 2.5$$

The two rectangles are similar, with a scale factor of 2.5

(b) In the diagram below, *AB* and *CD* are parallel lines.
Show that triangles *ABX* and *CDX* are similar.

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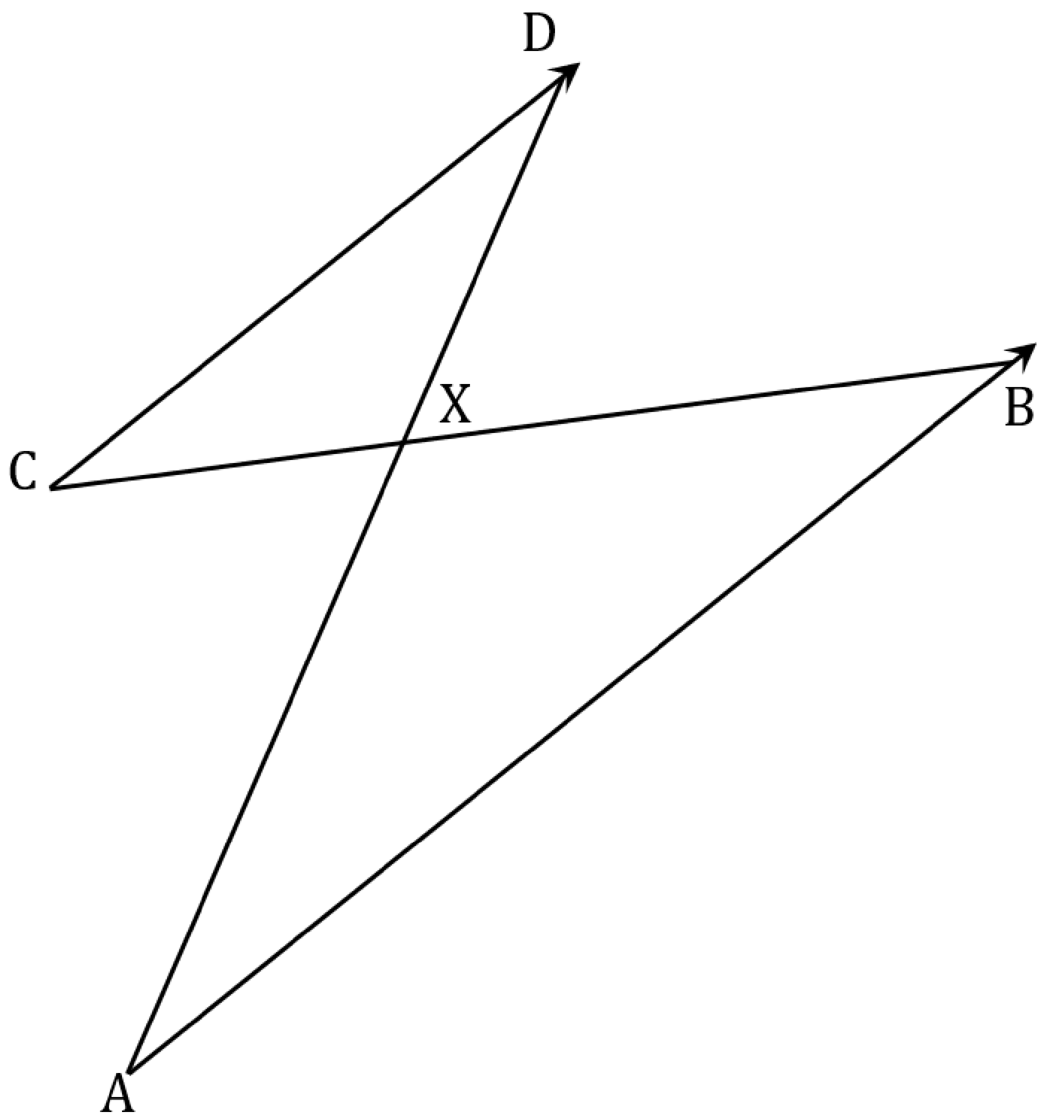

State the equal angles by name, along with clear reasons
Don't forget to state that similar triangles need to have equal corresponding angles

Angle AXB = angle CXD (vertically opposite angles are equal)
Angle ABC = angle BCD (alternate angles on parallel lines are equal)
Angle BAD = angle ADC (alternate angles on parallel lines are equal)

All three corresponding angles are equal, so the two triangles are similar

Similar Lengths

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Similar Lengths

How do I solve problems that involve similar lengths?

- Equivalent **lengths** in two similar shapes will be in the same ratio and are linked by a **scale factor**
  - Identify **known** lengths of **corresponding sides**
  - Establish the **type** of enlargement
    - If the shape is **getting bigger**, then the **scale factor** is **greater than 1**
    - If the shape is **getting smaller**, then the **scale factor** is **greater than 0** but **less than 1**
  - Find the **scale factor**
    - **Divide** a **known length** on the **second shape** by the **corresponding known length** on the **first shape**
  - Use the scale factor to find the **length** you need
    - **Multiply** a known length by the **scale factor** on the **first shape** to find the **corresponding length** on the **second shape**
    - **Divide** a known length on the **second shape** by the **scale factor** to find the **corresponding length** on the **first shape**

Examiner Tips and Tricks

- If similar shapes overlap on the diagram (or are not clear) draw them separately.
  - For example, in this diagram the triangles ABC and APQ are similar:

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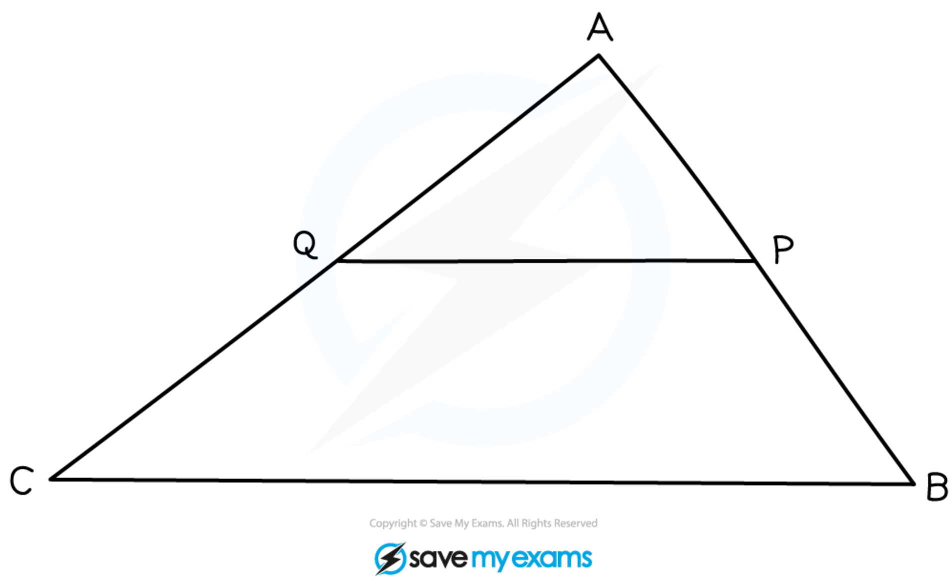
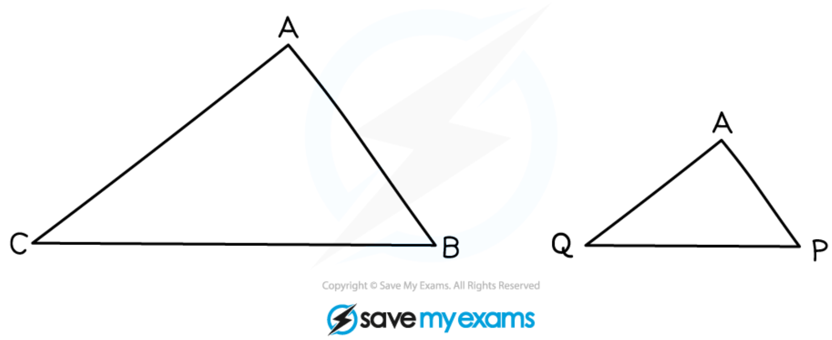


- So redraw them separately before starting:



Worked Example

ABCD and *PQRS* are similar shapes.
Find the length of *PS*.

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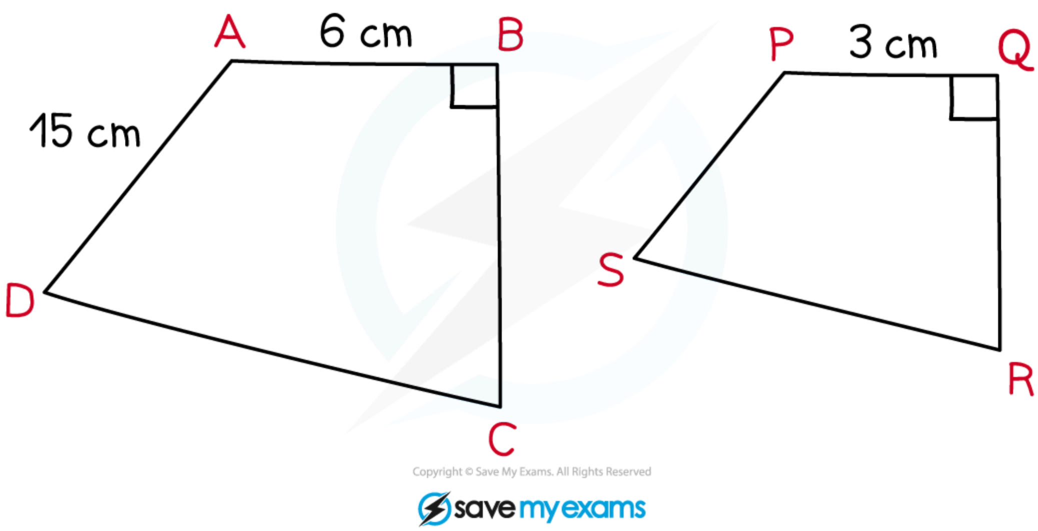

The two shapes are mathematically similar
Each length on the first shape can be multiplied by a scale factor to find the corresponding length on the second shape

Identify two known corresponding sides of the similar shapes

AB and PQ are corresponding sides

The second shape is smaller than the first shape so the scale factor will be between 0 and 1
Divide the known length on the second shape by the corresponding length on the first shape to find the scale factor

$$\text{Scale Factor} = \frac{3}{6} = \frac{1}{2}$$

Multiply the length AD by the scale factor to find its corresponding length PS on the second shape

$$PS = \frac{1}{2} \times 15 = \frac{15}{2}$$

$$\boldsymbol{PS = 7.5 \text{ cm}}$$

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Geometrical Proof

What is a geometrical proof?

- **Geometric proof** involves using known rules about geometry to prove a new statement about geometry
- A proof question might start with "Prove..." or "Show that ..."
- The rules that you might need to use to complete a proof include;
  - **Properties of 2D shapes**
    - Especially triangles and quadrilaterals
  - **Basic angle properties**
  - **Angles in polygons**
  - **Angles in parallel lines**
  - **Congruence and similarity**
  - **Pythagoras theorem**
- You will need to be familiar with the vocabulary of the topics above, in order to fully answer many geometrical proof questions

How do I write a geometrical proof?

- Usually you will need to write down two or three **steps** to prove the statement
- At each step, you should write down a **fact and a reason**
  - For example, "*AB = CD, opposite sides of a rectangle are equal length*"
- The proof is complete when you have **written down all the steps clearly**
  - Use the diagram!
  - **Add key information** such as angles or line lengths to the **diagram** as you work through the steps
    - but you must write them down in your written answer too

What geometric notation should I use?

- **Points** or **vertices of a shape** are labelled with **capital letters**

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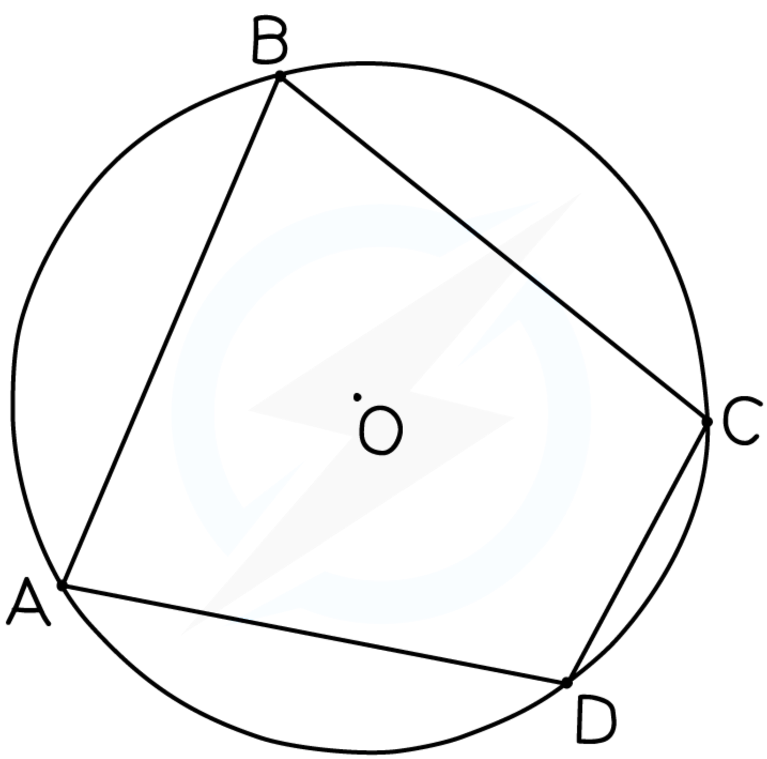
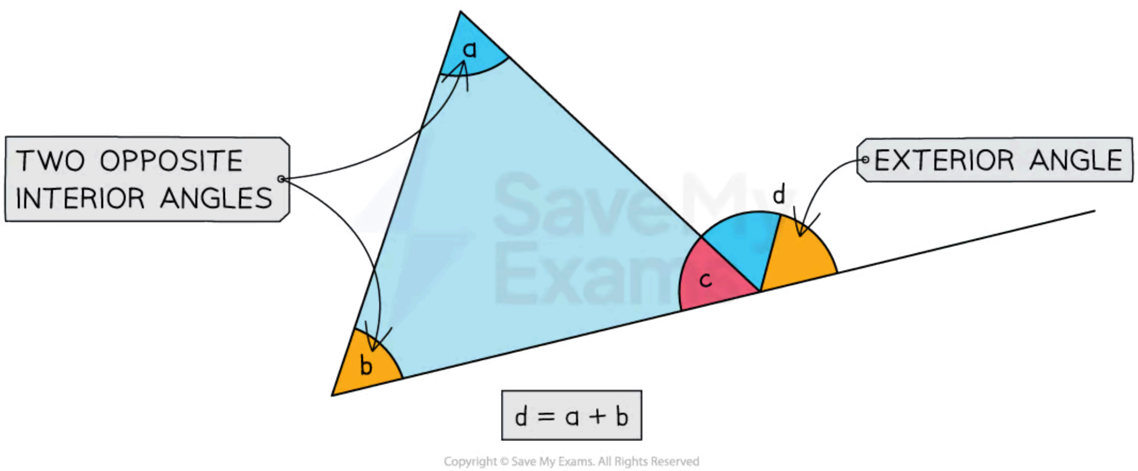
  - *A*, *B*, *C* and *D* are the vertices of the quadrilateral
  - *O* is the centre of the circle
- **Two letters** are used to represent the **line between the points**
  - *AB* is the line between points *A* and *B*
- **Three letters** are used to represent the **angle formed by the three points**
  - Angle *ABC* is the angle between lines *AB* and *BC*
  - The letter in the middle is the point where the angle is at
- **Multiple letters** are used to represent the **whole shape**
  - *ABCD* is a quadrilateral
  - The letters are written down so that they go **clockwise** around the shape
- If you use a **variable** to represent a **length** or an **angle** then write it down
  - Angle *ABC* = x



How can I prove that the exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices?

- Let a, b and c be the **three interior angles** in a triangle
- Let d be the **exterior angle** next to the interior angle c
- Split d into **two angles** by **drawing a parallel line** to the other side of the triangle
  - There will be an angle **alternate** to angle a
  - There will be an angle **corresponding** to angle b
- Therefore the **exterior angle** is the **sum** of the **two opposite interior angles**



What are common geometric reasons I can use?

- There are **common phrases** that are sufficient as explanations and should be **learnt**
  - These will be what mark schemes look for
- For **triangles** and **quadrilaterals**
  - **Angles** in a **triangle add** up to **180°**
  - **Base angles** of an **isosceles triangle** are **equal**
  - **Angles** in an **equilateral triangle** are **equal**
  - **Angles** in a **quadrilateral add** up to **360°**
  - An **exterior angle** of a triangle is **equal** to the **sum** of the **interior opposite angles**

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- For **straight lines**
  - **Vertically opposite** angles are **equal**
  - **Angles** on a **straight line add** up to **180°**
  - **Angles** at a **point add** up to **360°**
- For **parallel lines**
  - **Alternate** angles are **equal**
  - **Corresponding** angles are **equal**
  - **Allied** (or **co-interior**) angles **add** up to **180°**
- For **polygons**
  - **Exterior angles** of a polygon **add** up to **360°**
  - The **interior and exterior angle** of any polygon **add** up to **180°**

Examiner Tips and Tricks

- DO show all the key steps
  - If in doubt, include it
- DON'T write in full sentences
  - For each step, just write down the fact, followed by the key mathematical reason that justifies it

Worked Example

In the diagram below, *AC* and *DG* are parallel lines. *B* lies on *AC*, *E* and *F* lie on *DG* and triangle *BEF* is isosceles.

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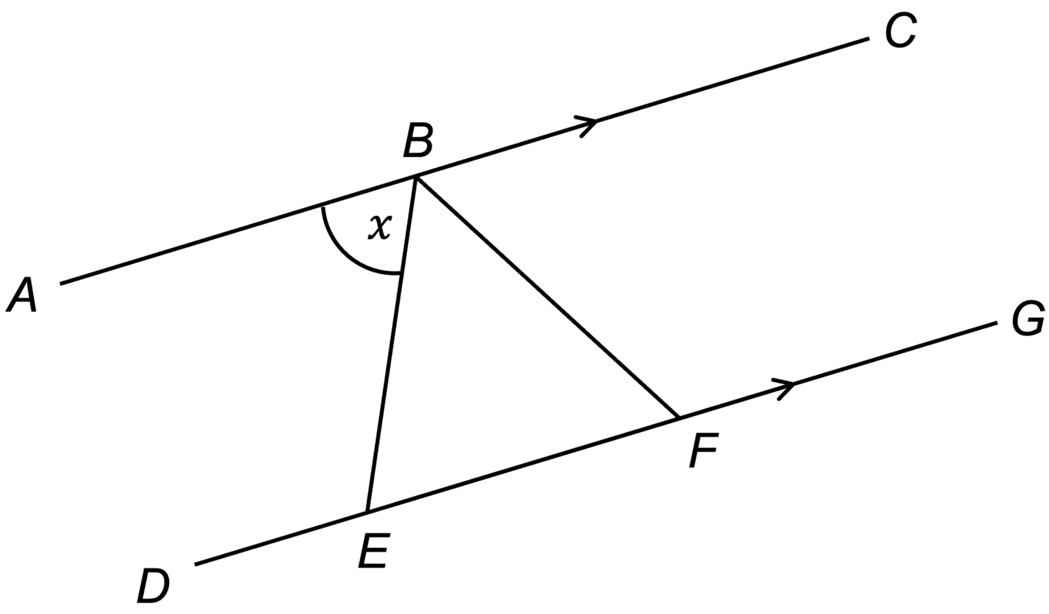
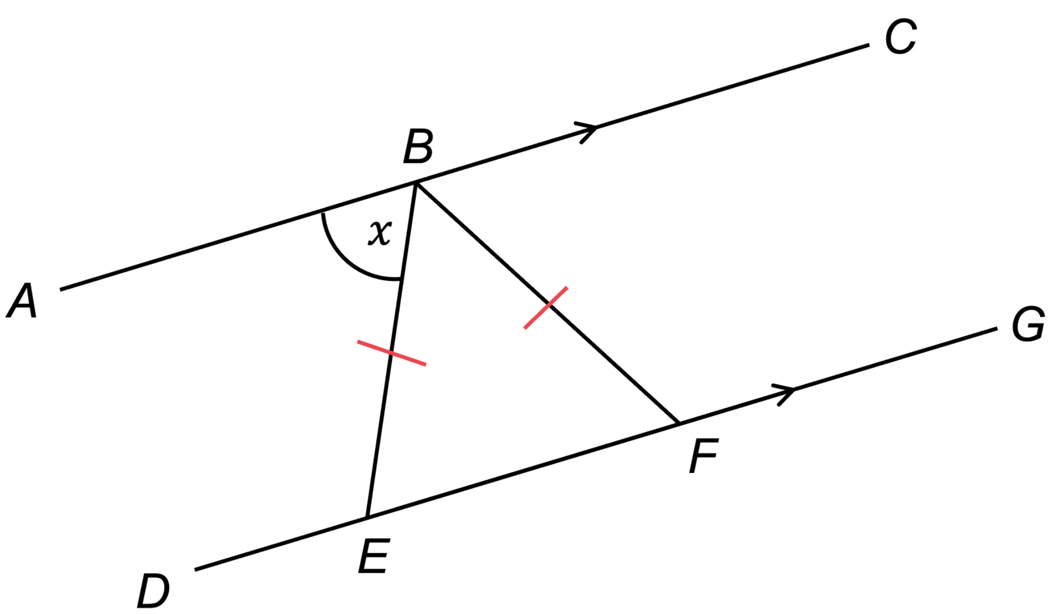
Your notes



Prove that angle *EBF* is $180 - 2x$. Give reasons for each stage of your working.

Mark on the diagram that triangle *BEF* is isosceles



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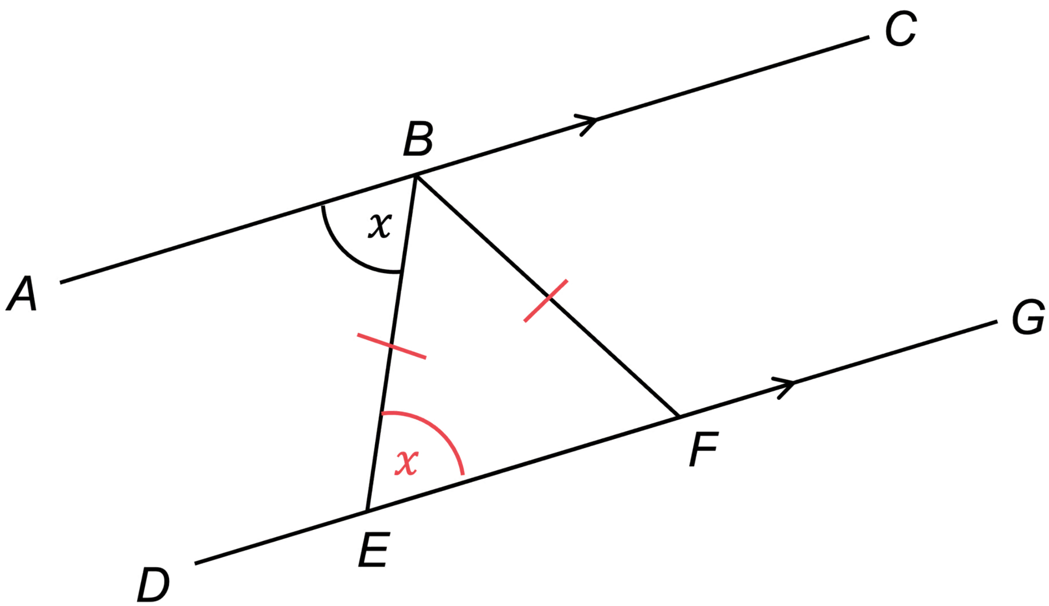
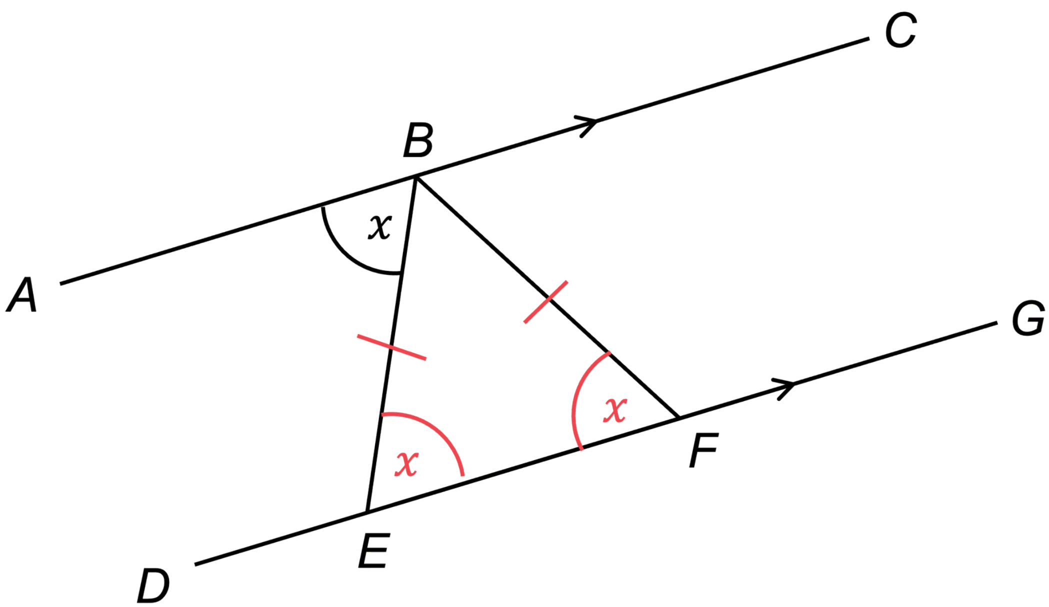
AC and *DG* are parallel lines, so using alternate angles we know that angle *BEF* = x
Mark this on the diagram



Write the fact, and the reason using the key mathematical vocabulary

angle *BEF* = x, alternate angles are equal

Now using the fact that triangle *BEF* is isosceles, we can see that angle *BFE* = x
Mark this on the diagram



Write the fact, and the reason using the key mathematical vocabulary

angle *BFE* = x, base angles of an isosceles triangle are equal

Now we can see that angle *EBF* is the last remaining angle in a triangle, and as the angles in a triangle sum to 180, angle *EBF* = $180 - 2x$

Write the fact, and the reason using the key mathematical vocabulary

angle *EBF* = $180 - 2x$, angles in a triangle sum to 180

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