AQA GCSE Maths: Higher

Your notes

Factorising

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Factorising Out Terms

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Basic Factorising

What is factorisation?

- A **factorised** expression is one written as the **product** (**multiplication**) of two, or more, terms (**factors**)
    - $3(x+2)$ is factorised
        - It is $3 \times (x+2)$
    - $3x+6$ is **not** factorised
    - $3xy$ is factorised
        - It is $3 \times x \times y$
    - Numbers can also be factorised
        - 12 = 2 x 2 x 3
- In algebra, factorisation is the **reverse** of **expanding brackets**
    - It's putting it into brackets, rather than removing brackets

How do I factorise two terms?

- To factorise $12x^2+18x$
    - Find the highest common factor of the **number** parts
        - 6
    - Find the highest common factor of the **algebra** parts
        - x
    - Multiply both to get the overall **highest common factor**
        - $6x$
    - $12x^2+18x$ is the same as $6x \times 2x + 6x \times 3$
        - Using the highest common factor
    - **Take out** the highest common factor
        - Write it **outside** a set of **brackets**

- Put the remaining terms, $2x+3$, inside the brackets
- This gives the answer
  - $6x(2x+3)$
- To factorise an expression containing multiple variables, e.g. $2a^3b - 4a^2b^2$
  - Use the same approach as above
  - Find the highest common factor of the **number** parts
    - 2
  - Find the highest common factor of the **algebra** parts
    - a and b appear in both terms
    - The highest common factor of a^3 and a^2 is a^2
    - The highest common factor of b and b^2 is b
    - a^2b
  - Multiply both to get the overall **highest common factor**
    - $2a^2b$
  - $2a^3b - 4a^2b^2$ is the same as $2a^2b \times a - 2a^2b \times 2b$
    - Using the highest common factor
  - **Take out** the highest common factor
    - Write it **outside** a set of **brackets**
    - Put the remaining terms, $a - 2b$, inside the brackets
  - This gives the answer
    - $2a^2b(a - 2b)$

Examiner Tips and Tricks

- In the exam, check that your factorisation is correct by **expanding** the brackets!
- **Factorise** mean factorise **fully.**
  - $x(6x+10)$ is not fully factorised but $2x(3x+5)$ is.

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Worked Example

(a) Factorise $5x + 15$

Find the highest common factor of 5 and 15

$$5$$

There is no x in the second term, so no highest common factor in x is needed
Think of each term as $5 \times$ something

$$5 \times x + 5 \times 3$$

Take out the 5 and put $x + 3$ in brackets

$$5(x + 3)$$

$5(x + 3)$

(b) Factorise fully $30x^2 - 24x$

Find the highest common factor of 30 and 24

$$6$$

Find the highest common factor of x^2 and x

$$x$$

Find the overall highest common factor by multiplying these together

$$6x$$

Think of each term as $6x \times$ something

$$6x \times 5x - 6x \times 4$$

Take out the $6x$ and put $5x - 4$ in brackets

$$6x(5x - 4)$$

$6x(5x - 4)$

Your notes

Factorising by Grouping

How do I factorise expressions with a common bracket?

- Look at the expression $3x(t+4)+2(t+4)$
  - Both terms have a **common bracket**, $(t+4)$
  - The whole bracket, $(t+4)$, can be "taken out" like a common factor:
    - $(t+4)(3x+2)$
- This is like factorising $3xy+2y$ to get $y(3x+2)$
  - y represents $(t+4)$ above

How do I factorise by grouping?

- Some questions may require you to **form a common bracket yourself**
  - For example $xy+3x+5y+15$
    - The first two terms have a common factor of x
    - The second two terms have a common factor of 5
  - Factorising fully the **first pair** of terms, and the **last pair** of terms:
    - $x(y+3)+5(y+3)$
  - You can now spot a common bracket of $(y+3)$
    - $(y+3)(x+5)$
- This is called **factorising by grouping**

Does it matter what order I group in?

- You can often **rearrange terms** to factorise in a **different order**
  - Rewriting the same example, $xy+3x+5y+15$, but in a different order:
    - $xy+5y+3x+15$
    - The first pair of terms have a common factor of y
    - The second pair of terms have a common factor of 3

Your notes

- Factorising gives $y(x + 5) + 3(x + 5)$
  - You can now spot a common bracket, this time of $(x + 5)$
- $(x+5)(y+3)$
  - This gives the **same result** as found previously
- Some rearrangements **cannot** be factorised as "first pair" then "second pair"
  - For example, rewriting the above example as $xy + 15 + 3x + 5y$

Examiner Tips and Tricks

Once you have factorised something, expand it by hand to check your answer is correct.

Worked Example

Factorise $ab + 3b + 2a + 6$.

Method 1:
Notice that ab and $3b$ have a common factor of b
Notice that $2a$ and 6 have a common factor of 2

Factorise the first two terms, using b as a common factor

$$b(a + 3) + 2a + 6$$

Factorise the second two terms, using 2 as a common factor

$$b(a + 3) + 2(a + 3)$$

$(a + 3)$ is a common bracket
We can now factorise out the bracket $(a + 3)$

$$\mathbf{(a + 3)(b + 2)}$$

Method 2:
Notice that ab and $2a$ have a common factor of a
Notice that $3b$ and 6 have a common factor of 3

Rewrite the expression, grouping these terms together

$$ab + 2a + 3b + 6$$

Factorise the first two terms, using a as a common factor

$$a(b+2)+3b+6$$

Factorise the second two terms, using 3 as a common factor

$$a(b+2)+3(b+2)$$

$(b+2)$ is a common bracket
We can now factorise out the bracket $(b+2)$

$$\mathbf{(b+2)(a+3)}$$

Your notes

Factorising Simple Quadratics

Your notes

Factorising Simple Quadratics

What is a quadratic expression?

- A **quadratic expression** is in the form:
  - $ax^2 + bx + c$ (where $a \neq 0$)
- If there are any **higher powers** of x (like x^3 say) then it is **not** a quadratic

How do I factorise quadratics by inspection?

- This is shown most easily through an example: factorising $x^2 - 2x - 8$
- We need a **pair of numbers** that for $x^2 + bx + c$
  - **multiply** to give c
    - which in this case is -8
  - and **add** to give b
    - which in this case is -2
  - +2 and -4 satisfy these conditions
    - $2 \times (-4) = -8$ and $2 + (-4) = -2$
  - **Write** these numbers in a **pair of brackets** like this:
    - $(x+2)(x-4)$

How do I factorise quadratics by grouping?

- This is shown most easily through an example: factorising $x^2 - 2x - 8$
- We need a **pair of numbers** that for $x^2 + bx + c$
  - **multiply** to give c
    - which in this case is -8
  - and **add** to give b
    - which in this case is -2

Your notes

- +2 and -4 satisfy these conditions
  - 2 × (-4) = -8 and 2 + (-4) = -2
- **Rewrite** the middle term by using +2*x* and -4*x*
  - $x^2 + 2x - 4x - 8$
- **Group** and **factorise** the **first two terms**, using *x* as the common factor
- and **group** and **factorise** the **last two terms**, using -4 as the common factor
  - $x(x+2) - 4(x+2)$
- Note that these both now have a **common factor** of $(x + 2)$ so this **whole bracket** can be factorised out
  - $(x+2)(x-4)$

How do I factorise quadratics using a grid?

- This is shown most easily through an example: factorising $x^2 - 2x - 8$
- We need a **pair of numbers** that for $x^2 + bx + c$
  - **multiply** to give c
    - which in this case is -8
  - and **add** to give *b*
    - which in this case is -2
  - +2 and -4 satisfy these conditions
    - 2 × (-4) = -8 and 2 + (-4) = -2
  - Write the quadratic equation in a **grid** (as if you had used a grid to expand the brackets)
    - splitting the middle term as +2*x* and -4*x*
- The grid works by multiplying the row and column headings, to give a product in the boxes in the middle

	x^2	$-4x$
	$+2x$	-8

Your notes

- Write a heading for the first row, using x as the highest common factor of x^2 and $-4x$

x	x^2	$-4x$
	$+2x$	-8

- You can then use this to find the headings for the columns
  - e.g. "What does x need to be multiplied by to give x^2?"
  - and "What does x need to be multiplied by to give $-4x$?"

	x	**-4**
x	x^2	$-4x$
	$+2x$	-8

- We can then fill in the remaining row heading using the same idea
  - e.g. "What does x need to be multiplied by to give $+2x$?"
  - or "What does -4 need to be multiplied by to give -8?"

	x	**-4**
x	x^2	$-4x$
$+2$	$+2x$	-8

- We can now read off the factors from the column and row headings
  - $(x+2)(x-4)$

Which method should I use for factorising simple quadratics?

- The first method, by **inspection**, is by far the **quickest**
  - So this is recommended in an exam for simple quadratics (where $a = 1$)

- However the other two methods (grouping, or using a grid) can be used for **harder quadratic equations** where $a \neq 1$
  - So you should learn at least one of them too

Your notes

Examiner Tips and Tricks

As a check, expand your answer and make sure you get the same expression as the one you were trying to factorise.

Worked Example

(a) Factorise $x^2 - 4x - 21$.

We will factorise by inspection

We need two numbers that multiply to give -21, and sum to give -4
+3 and -7 satisfy this

$$3 \times (-7) = -21$$

$$3 + (-7) = -4$$

Write down the brackets

$$(x + 3)(x - 7)$$

(b) Factorise $x^2 - 5x + 6$.

We will factorise by splitting the middle term and grouping

We need two numbers that multiply to 6, and sum to -5
-3 and -2 satisfy this

$$(-3) \times (-2) = 6$$

$$(-3) + (-2) = -5$$

Split the middle term

$$x^2 - 2x - 3x + 6$$

Your notes

Factorise x out of the first two terms

$$x(x - 2) - 3x + 6$$

Factorise -3 out of the last two terms

$$x(x - 2) - 3(x - 2)$$

These have a common factor of $(x - 2)$ which can be factored out

$$(x - 2)(x - 3)$$

(c) Factorise $x^2 - 2x - 24$.

We will factorise by using a grid

We need two numbers that multiply to -24, and sum to -2
+4, and -6 satisfy this

$$4 \times (-6) = -24$$

$$4 + (-6) = -2$$

Use these to split the $-2x$ term and write in a grid

	x^2	$+4x$
	$-6x$	-24

Write a heading using a common factor for the first row

x	x^2	$+4x$
	$-6x$	-24

Work out the headings for the rows
"What does x need to be multiplied by to make x^2?"
"What does x need to be multiplied by to make $+4x$?"

	x	$+4$
x	x^2	$+4x$
	$-6x$	-24

Your notes

Repeat for the heading for the remaining row
"What does x need to be multiplied by to make $-6x$?"
(Or "What does +4 need to be multiplied by to make -24?")

	x	**+4**
x	x^2	$+4x$
-6	$-6x$	-24

Read off the factors from the column and row headings

$(x + 4)(x - 6)$

Your notes

Factorising Harder Quadratics

How do I factorise a quadratic expression where a ≠ 1 in ax^2 + bx + c?

Method 1: Factorising by grouping

- This is shown most easily through an example: factorising $4x^2 - 25x - 21$
- We need a **pair of numbers** that, for $ax^2 + bx + c$
  - both **multiply** to give *ac*
    - *ac* in this case is 4 × -21 = -84
  - and both **add** to give *b*
    - *b* in this case is -25
  - -28 and +3 satisfy these conditions
  - **Rewrite** the middle term using -28*x* and +3*x*
    - $4x^2 - 28x + 3x - 21$
  - **Group** and fully **factorise** the **first two terms**, using 4*x* as the common factor
  - and **group** and fully **factorise** the **last two terms**, using 3 as the common factor
    - $4x(x-7)+3(x-7)$
  - These terms now have a **common factor** of $(x-7)$
    - This **whole bracket** can be factorised out
    - This gives the answer $(x-7)(4x+3)$

Method 2: Factorising using a grid

- Use the same example: factorising $4x^2 - 25x - 21$
- We need a **pair of numbers** that for $ax^2 + bx + c$
  - **multiply** to give *ac*

Your notes

  - ac in this case is $4 \times -21 = -84$
- and **add** to give b
  - b in this case is -25
- -28 and +3 satisfy these conditions
- Write the quadratic equation in a **grid**
  - (as if you had used a grid to expand the brackets)
  - splitting the middle term up as $-28x$ and $+3x$ (either order)
- The grid works by multiplying the row and column headings, to give a product in the boxes in the middle

	$4x^2$	$-28x$
	$+3x$	-21

- Write a heading for the first row, using $4x$ as the highest common factor of $4x^2$ and $-28x$

4x	$4x^2$	$-28x$
	$+3x$	-21

- You can then use this to find the headings for the columns, e.g. "What does $4x$ need to be multiplied by to give $4x^2$?"

	x	**-7**
4x	$4x^2$	$-28x$
	$+3x$	-21

- We can then fill in the remaining row heading using the same idea, e.g. "What does x need to be multiplied by to give $+3x$?"

	x	**-7**
4x	$4x^2$	$-28x$
+3	$+3x$	-21

- We can now read off the brackets from the column and row headings:
  - $(x-7)(4x+3)$

Your notes

Worked Example

(a) Factorise $6x^2 - 7x - 3$.

We will factorise by grouping

We need two numbers that:

multiply to 6 × -3 = -18
and sum to -7

-9, and +2

Split the middle term up using these values

$6x^2 + 2x - 9x - 3$

Factorise $2x$ out of the first two terms

$2x(3x + 1) - 9x - 3$

Factorise -3 of out the last two terms

$2x(3x + 1) - 3(3x + 1)$

These have a common factor of $(3x + 1)$ which can be factorised out

$(3x + 1)(2x - 3)$

(b) Factorise $10x^2 + 9x - 7$.

We will factorise using a grid

We need two numbers that:

multiply to $10 \times -7 = -70$
and sum to $+9$

-5, and $+14$

Use these values to split the $9x$ term and write in a grid

	$10x^2$	$-5x$
	$+14x$	-7

Write a heading using a common factor of $5x$ from the first row

5x	$10x^2$	$-5x$
	$+14x$	-7

Work out the headings for the rows, e.g. "What does $5x$ need to be multiplied by to make $10x^2$?"

	2x	**-1**
5x	$10x^2$	$-5x$
	$+14x$	-7

Repeat for the heading for the remaining row, e.g. "What does $2x$ need to be multiplied by to make $+14x$?"

	2x	**-1**
5x	$10x^2$	$-5x$
+7	$+14x$	-7

Read off the brackets from the column and row headings

$(2x - 1)(5x + 7)$

Your notes

Difference of Two Squares

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Difference of Two Squares

What is the difference of two squares?

- When a "squared" quantity is subtracted from another "squared" quantity, you get the **difference of two squares**
  - For example:
    - $a^2 - b^2$
    - $9^2 - 5^2$
    - $(x + 1)^2 - (x - 4)^2$
    - $4m^2 - 25n^2$, which is $(2m)^2 - (5n)^2$

How do I factorise the difference of two squares?

- **$a^2 - b^2$ factorises to $(a + b)(a - b)$**
  - This can be shown by expanding the brackets
    - $(a+b)(a-b) = a^2 - ab + ba - b^2 = a^2 - b^2$
  - The brackets can swap order
    - **$a^2 - b^2 = (a + b)(a - b) = (a - b)(a + b)$**
    - (but terms inside a bracket cannot swap order)
- For example, $x^2 - 9 = (x+3)(x-3)$
  - This **is the same** as $(x-3)(x+3)$
  - But **not the same** as $(3+x)(3-x)$
    - which expands to $9 - x^2$

How can the difference of two squares be made harder?

- You may find it used with:
  - **numbers**
    - $7^2 - 3^2 = (7+3)(7-3) = (10)(4) = 40$

Your notes

- A combination of **square numbers** and **squared variables**
  - $4m^2 - 9n^2 = (2m)^2 - (3n)^2 = (2m + 3n)(2m - 3n)$
- Any **other powers** which can be written as a **difference of two squares**
  - $a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 + b^2)(a^2 - b^2)$
  - $r^8 - t^6 = (r^4)^2 - (t^3)^2 = (r^4 + t^3)(r^4 - t^3)$
- You may also need to **take out a common factor first**
  - $2x^2 - 18 = 2(x^2 - 9)$ giving $2(x + 3)(x - 3)$
    - The 2 comes out in front

Can I use the difference of two squares to expand?

- Using the difference of two squares to **expand** is **quicker** than expanding double brackets and collecting like terms
- Brackets of the form **$(a + b)(a - b)$ expand to $a^2 - b^2$**
  - For example $(2x + 3)(2x - 3)$ expands to $4x^2 - 9$

Examiner Tips and Tricks

- The difference between two squares is often the trick required to complete a harder algebraic question in the exam
  - Make sure you are able to spot it!

Worked Example

(a) Factorise $9x^2 - 16$.

Recognise that $9x^2$ and 16 are both squared terms

Therefore you can factorise using the difference of two squares

Rewrite as a difference of two squared terms

$$9x^2 - 16 = (3x)^2 - (4)^2$$

Use the rule $a^2 - b^2 = (a+b)(a-b)$

$$(3x+4)(3x-4)$$

(b) Factorise $4r^2 - t^4$.

Recognise that $4r^2$ and t^4 are both squared terms

Therefore you can factorise using the difference of two squares

Rewrite as a difference of two squared terms

$$4r^2 - t^4 = (2r)^2 - (t^2)^2$$

Use the rule $a^2 - b^2 = (a+b)(a-b)$

$$(2r+t^2)(2r-t^2)$$

(c) Factorise $2y^2 - 50$

This does not appear to be in the form $a^2 - b^2$

There is a common factor of 2, so take this factor out

$$2(y^2 - 25)$$

You can now see $y^2 - 25$ which has the form $y^2 - 5^2$

Use the rule $a^2 - b^2 = (a+b)(a-b)$

$$y^2 - 25 = (y+5)(y-5)$$

Now multiply this answer by 2 (leaving the 2 on the outside)

$$2(y+5)(y-5)$$

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Deciding the Factorisation Method

Quadratics Factorising Methods

How do I know if an expression factorises?

- The easiest way to check if $ax^2 + bx + c$ factorises is to check if you can find a pair of integers which:
  - Multiply to give ac
  - Sum to give b
  - If you can find integers to satisfy this, the expression must factorise
- There are some alternate methods to check:
  - Method 1: Use a **calculator** to solve the quadratic expression equal to 0
    - Only some calculators have this functionality
    - If the **solutions are integers or fractions** (without square roots), then the quadratic expression will factorise
  - Method 2: Find the value under the square root in the quadratic formula
    - $\boldsymbol{b^2 - 4ac}$
    - If this number is a **square number**, then the quadratic expression will factorise

Which factorisation method should I use for a quadratic expression?

- Does it have **2 terms only**?
  - Yes, like $x^2 - 7x$
    - Factorise out the highest common factor, x
    - $x(x-7)$
  - Yes, like $x^2 - 9$
    - Use the "difference of two squares" to factorise
    - $(x+3)(x-3)$

Does it have **3 terms**?

Your notes

- Yes, starting with x^2 like $x^2 - 3x - 10$
  - Use "factorising simple quadratics" by finding two numbers that add to -3 and multiply to -10
  - $(x+2)(x-5)$
- Yes, starting with ax^2 like $3x^2 + 15x + 18$
  - Check to see if the 3 in front of x^2 is a **common factor** for **all three terms** (which it is in this case), then **factorise it out** of all three terms
  - $3(x^2+5x+6)$
  - The quadratic expression inside the brackets is now $x^2 + \ldots$, which factorises more easily
  - $3(x+2)(x+3)$
- Yes, starting with ax^2 like $3x^2 - 5x - 2$
  - The 3 in front of x^2 is not a common factor for all three term
  - Use "factorising harder quadratics", for example factorising by grouping or factorising using a grid
  - $(3x+1)(x-2)$

What other expressions should I be able to factorise?

- You may have a **cubed term** like $x^3 - 3x^2 - 10x$
  - Check to see if x is a **common factor** for **all three terms** (which it is in this case), so **factorise it out** of all three terms
  - $x(x^2-3x-10)$
  - The remaining quadratic can then be factorised
  - $x(x+2)(x-5)$
- It can also be useful to spot a quadratic in the form $x^2 + 2ax + a^2$
  - This factorises to $(x+a)^2$
  - E.g. $x^2+6x+9 \ = \ (x+3)^2$

Examiner Tips and Tricks

- A common **mistake** in the exam is to divide **expressions** by numbers, e.g. $2x^2+4x+2$ becomes x^2+2x+1 (which is incorrect)
  - This can only be done with equations
  - e.g. $2x^2+4x+2=0$ becomes $x^2+2x+1=0$ (dividing "both sides" by 2)

Your notes

Worked Example

Factorise $-8x^2+100x-48$.

Spot the common factor of -4 and factorise it out

$$-8x^2+100x-48=-4(2x^2-25x+12)$$

Check to see if the quadratic in the bracket will factorise using b^2-4ac

$$\begin{aligned}&(-25)^2-(4\times 2\times 12)\\&=625-96\\&=529\end{aligned}$$

529 is a square number (23^2) so the expression will factorise

Factorise $2x^2-25x+12$

We require a pair of numbers which multiply to *ac*, and sum to *b*

$$a\times c=2\times 12=24$$

The only numbers which multiply to 24 and sum to -25 are

-24 and -1

Split the $-25x$ term into $-24x-x$

$$2x^2-24x-x+12$$

Group and factorise the first two terms, using $2x$ as the common factor
Group and factorise the last two terms using -1 as the common factor

$$2x(x-12)-1(x-12)$$

These factorised terms now have a common term of $(x - 12)$, so this can be factorised out

$$(2x - 1)(x - 12)$$

Recall that -4 was factorised out at the start

$$-8x^2 + 100x - 48 = -4(2x^2 - 25x + 12) = -4(2x - 1)(x - 12)$$

$$\mathbf{-4(2x - 1)(x - 12)}$$

Your notes