AQA GCSE Maths: Higher

Your notes

Pythagoras & Trigonometry

Contents

- Pythagoras Theorem
- SOHCAHTOA
- Angles of Elevation & Depression
- Exact Trig Values

Your notes

Pythagoras Theorem

Who is Pythagoras?

- **Pythagoras** was a **Greek mathematician** who lived over 2500 years ago
- He is most famous for **Pythagoras' theorem**, which includes the important formula for **right-angled triangles**

What is Pythagoras' theorem?

- **Pythagoras' theorem** is a formula that links the **lengths** of the **three sides** of a **right-angled triangle**
- The **longest side** of a right-angled triangle is called the **hypotenuse**
  - The hypotenuse will always be the side **opposite** the right angle
- **Pythagoras' theorem** states that $a^2 + b^2 = c^2$
  - c is the **length** of the **hypotenuse**
  - a and b are the **lengths** of the **two shorter sides**
    - It does **not matter** which is labelled a and which is labelled b

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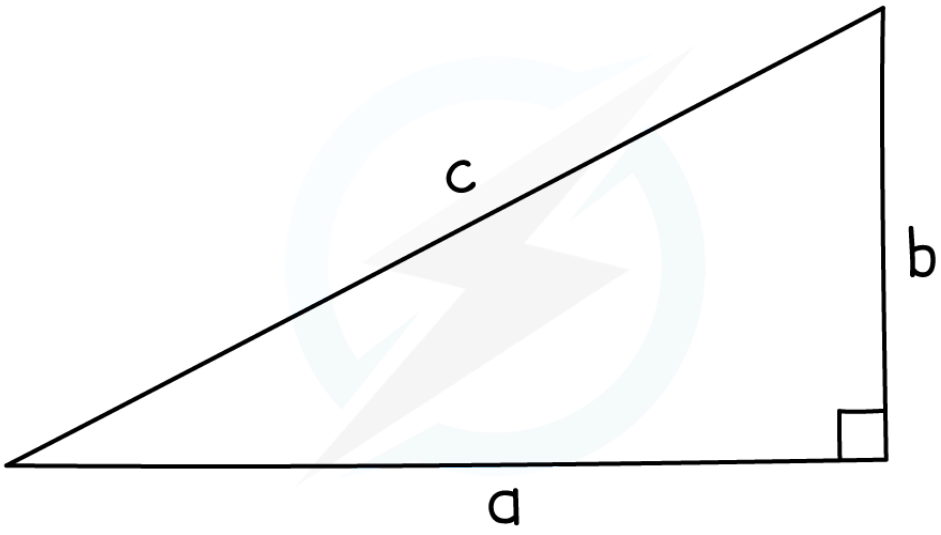

How do I use Pythagoras' theorem to find the length of the hypotenuse?

- To find the length of the **hypotenuse**
  - **Square** the lengths of the **two shorter sides**
  - **Add** these two numbers together
  - Take the positive **square root**
- This can be written as $c = \sqrt{a^2 + b^2}$
  - This is just a **rearrangement** of the formula $a^2 + b^2 = c^2$ to make c the subject
  - Note that when finding the hypotenuse you **add** inside the square root

How do I use Pythagoras' theorem to find the length of a shorter side?

- To find the length of a **shorter side**
  - **Square** the lengths of the **hypotenuse** and the **other shorter side**
  - **Subtract** these numbers to find the difference

- Take the positive **square root**

- This can be written as $a = \sqrt{c^2 - b^2}$
  - This is just a **rearrangement** of the formula $a^2 + b^2 = c^2$ to make a the subject
  - Note that when finding one of the shorter sides you **subtract** inside the square root

Your notes

Examiner Tips and Tricks

- If the hypotenuse ends up being shorter than another side in your answer then you have made a **mistake** somewhere
- Make sure that you subtract the **smaller value from the bigger value** when finding the length of a shorter side
  - Otherwise you will get a "**Math Error**" when trying to find the square root of a negative number
- In questions with **multiple steps**:
  - Leave your answer as an **exact answer**
  - **Do not round** until the very **end of the question**

Worked Example

In the following diagram:

$AB = 12$ cm

AC is a straight line, with $AD = 9$ cm and $AC = 22$ cm

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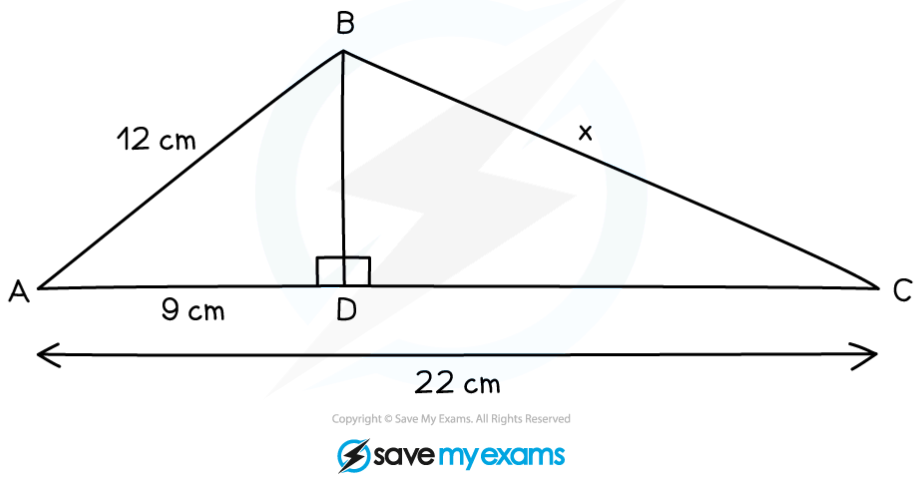


Find x, the length of side BC. Give your answer to 1 decimal place.

To find x, we first need to find the length of BD using triangle ABD

Note that BD is a shorter side

Apply Pythagoras' theorem, $a = \sqrt{c^2 - b^2}$

$$BD = \sqrt{12^2 - 9^2} = \sqrt{63} = 7.93725...$$

It is best to leave rounding until the very end, use $\sqrt{63}$ (or $3\sqrt{7}$ if this is what your calculator has given you) in subsequent working

Find the length of DC by subtracting the length of AD from the length of AC

$$DC = 22 - 9 = 13 \text{ cm}$$

Now we can find x using triangle BCD

Note that BC is the hypotenuse

Apply Pythagoras' theorem, $c = \sqrt{a^2 + b^2}$

$$x = \sqrt{BD^2 + DC^2} = \sqrt{\left(\sqrt{63}\right)^2 + 13^2} = \sqrt{63 + 169}$$

$$x = \sqrt{232} = 15.23154621...$$

Round to 1 decimal place

$x = 15.2$ cm

Your notes

Your notes

SOHCAHTOA

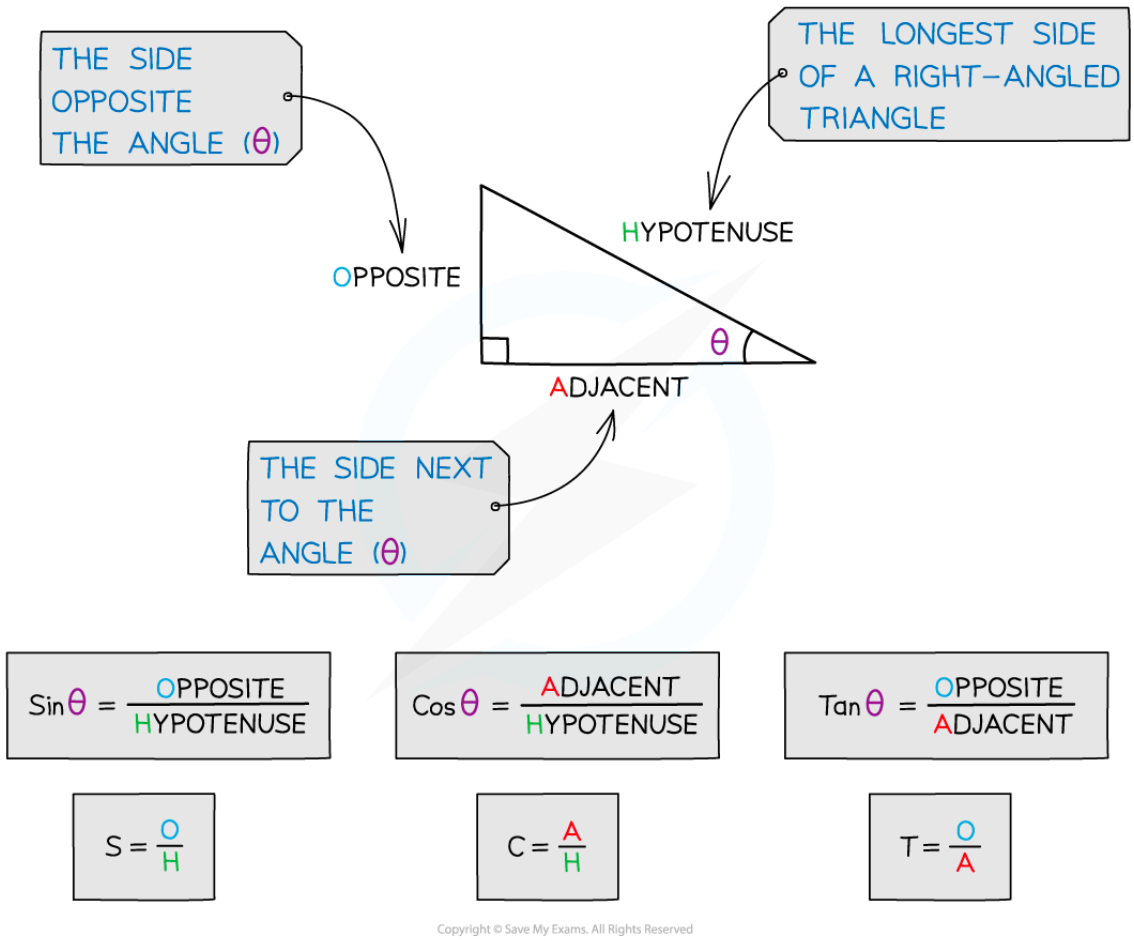
What is trigonometry?

- Trigonometry is the mathematics of **angles** in triangles
- It looks at the relationship between **side lengths** and **angles of triangles**

What are sin, cos and tan?

- The **three** trigonometric functions **sine**, **cosine** and **tangent**
  - They come from **ratios of side lengths** in **right-angled** triangles
- You must **label the sides** of a **right-angled triangle** in relation to a **chosen angle** θ
  - The **hypotenuse, H,** is the **longest side** in a right-angled triangle
    - It will always be **opposite** the right angle
  - The side **opposite θ** will be labelled **opposite**, **O**
  - The side **next to θ** will be labelled **adjacent**, **A**
- The functions sine, cosine and tangent are the ratios of the lengths of these sides as follows
  - $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{O}}{\text{H}}$
  - $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$
  - $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$

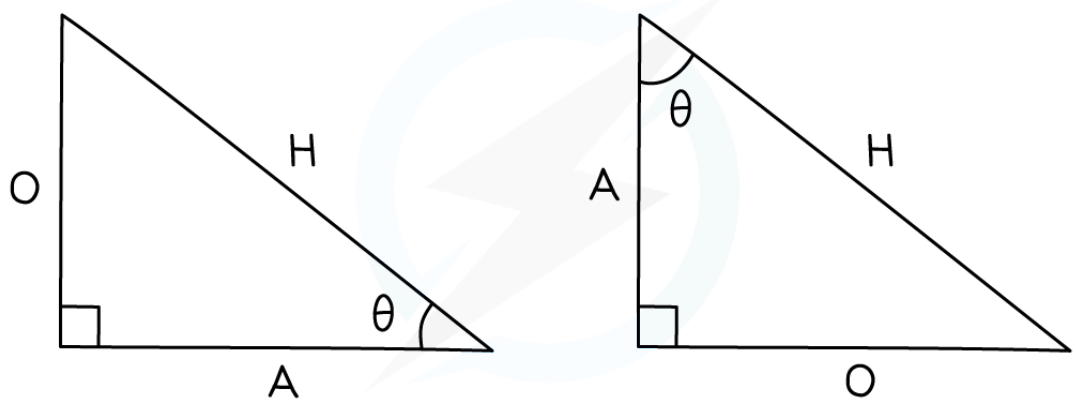
Your notes



What is SOHCAHTOA?

- **SOHCAHTOA** is a mnemonic often used to remember which ratio is which
  - **S**in is **O**pposite over **H**ypotenuse
  - **C**os is **A**djacent over **H**ypotenuse
  - **T**an is **O**pposite over **A**djacent
- **H** is always the same but **O** and **A** change depending on which angle is labelled as θ

Your notes



How can I use SOHCAHTOA to find missing lengths?

- **STEP 1**
  Label the sides of the triangle as H, O and A
- **STEP 2**
  Identify which t**rigonometric ratio** to use: sin, cos or tan
  - Write down the letter of the **length** you are **given**
  - Write down the letter of the **length** you **want to find**
  - Find the two letters in **SOHCAHTOA** to identify which ratio to use
    - If you have **A** and **H** then use **cos**
- **STEP 3**
  Substitute the values into the relevant trigonometric formula
  - Remember to put brackets around the angle
    - $\sin(50) = \frac{A}{7}$ or $\cos(40) = \frac{3}{H}$
- **STEP 4**
  Rearrange and **solve** for the **unknown letter**
  - You will either need to **multiply** or **divide**

Your notes

- $\sin(50) = \frac{A}{7}$ leads to $A = 7 \times \sin(50)$
- $\cos(40) = \frac{3}{H}$ leads to $H = \frac{3}{\cos(40)}$

- **STEP 5**
  Type the expression **into your calculator**
  - The **question** might ask you to **round your answer**
  - If not then round to **three significant figures**

How can I use SOHCAHTOA to find missing angles?

- **STEP 1**
  Label the sides of the triangle as H, O and A
- **STEP 2**
  Identify which t**rigonometric ratio** to use: sin, cos or tan
  - Write down the letters of the **lengths** you are **given**
  - Find the two letters in **SOHCAHTOA** to identify which ratio to use
    - If you have **O** and **A** then use **tan**
- **STEP 3**
  Substitute the values into the relevant trigonometric formula
  - The **angle** will be **unknown**
    - $\tan(\theta) = \frac{3}{4}$
- **STEP 4**
  Substitute the fraction into the **inverse trigonometric function**
  - You normally need to press **SHIFT** on your calculator first
    - $\tan(\theta) = \frac{3}{4}$ leads to $\theta = \tan^{-1}\left(\frac{3}{4}\right)$
- **STEP 5**
  Type the expression **into your calculator**
  - The **question** might ask you to **round your answer**
  - If not then round to **one decimal place**

How do I find the shortest distance from a point to a line?

Your notes

- The shortest distance from any point to a line will always be the **perpendicular** distance
- Form a right-angled triangle and then use SOHCAHTOA to find the relevant distance

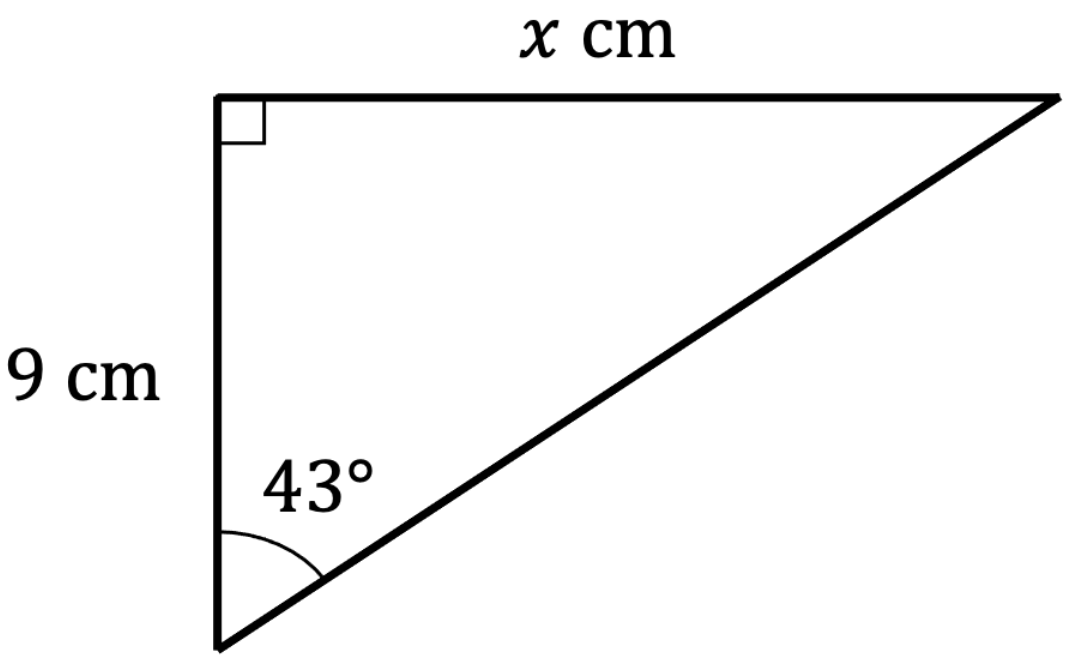
Examiner Tips and Tricks

- SOHCAHTOA (like Pythagoras) can only be used in **right-angled triangles**
- Ensure your calculator is set to measure angles in **degrees**
  - You should see the letter **D** or the word **Deg** at the top of your screen

Worked Example

Find the length of the side x cm in the following triangle.

Give your answer to 3 significant figures.



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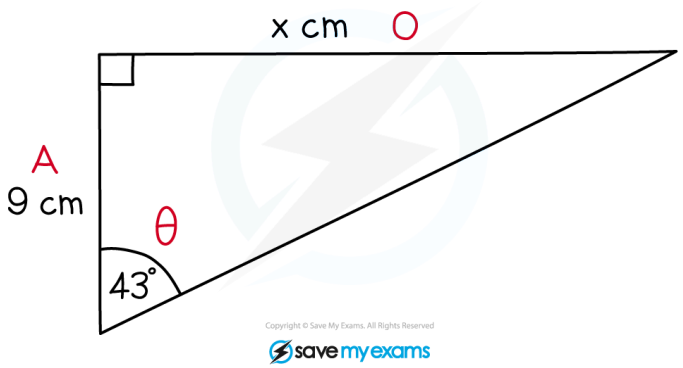
First label the triangle



We know A and we want to know O - that's TOA or $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

$$\tan(43) = \frac{x}{9}$$

Multiply both sides by 9

$$9 \times \tan(43) = x$$

Enter on your calculator

$$x = 8.3926...$$

Round to 3 significant figures

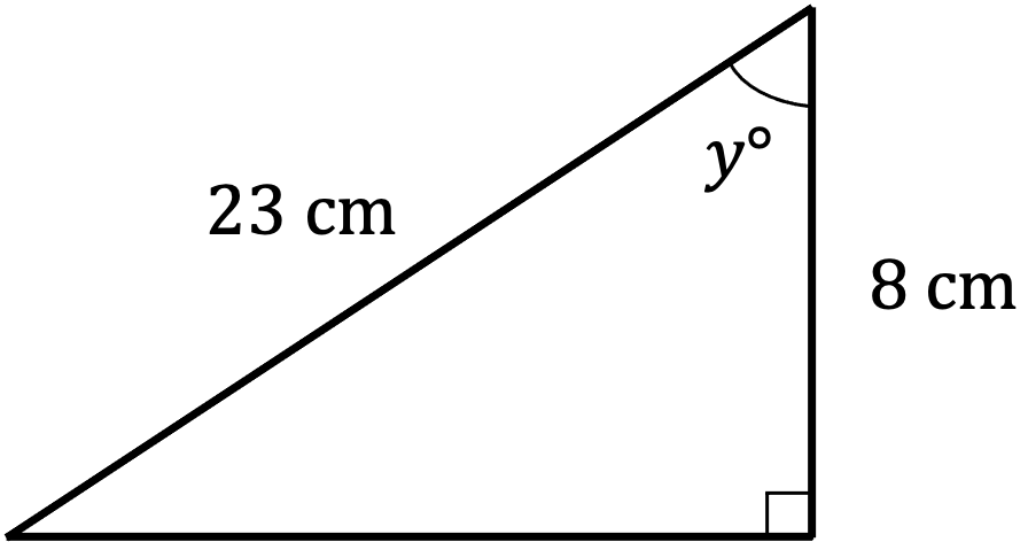
$$x = 8.39 \text{ cm}$$

Worked Example

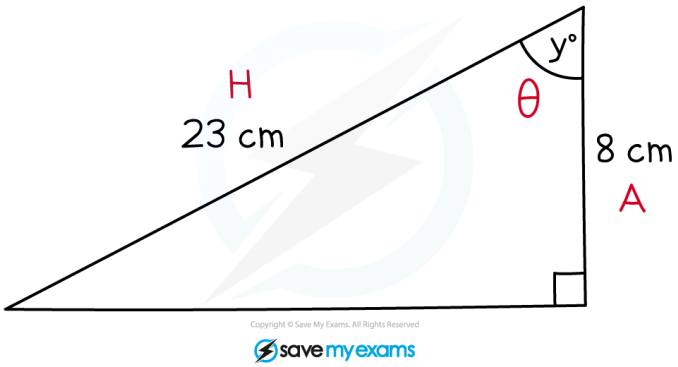
Find the value of the angle $y°$ in the following triangle.

Give your answer to 1 decimal place.

Your notes



First label the triangle



We know A and H - that's CAH or $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$$\cos(y) = \frac{8}{23}$$

Use inverse cos to find y

$$y = \cos^{-1}\left(\frac{8}{23}\right)$$

Enter on your calculator

$$y = 69.6455...$$

Round to 1 decimal place

$$y = 69.6°$$


Your notes

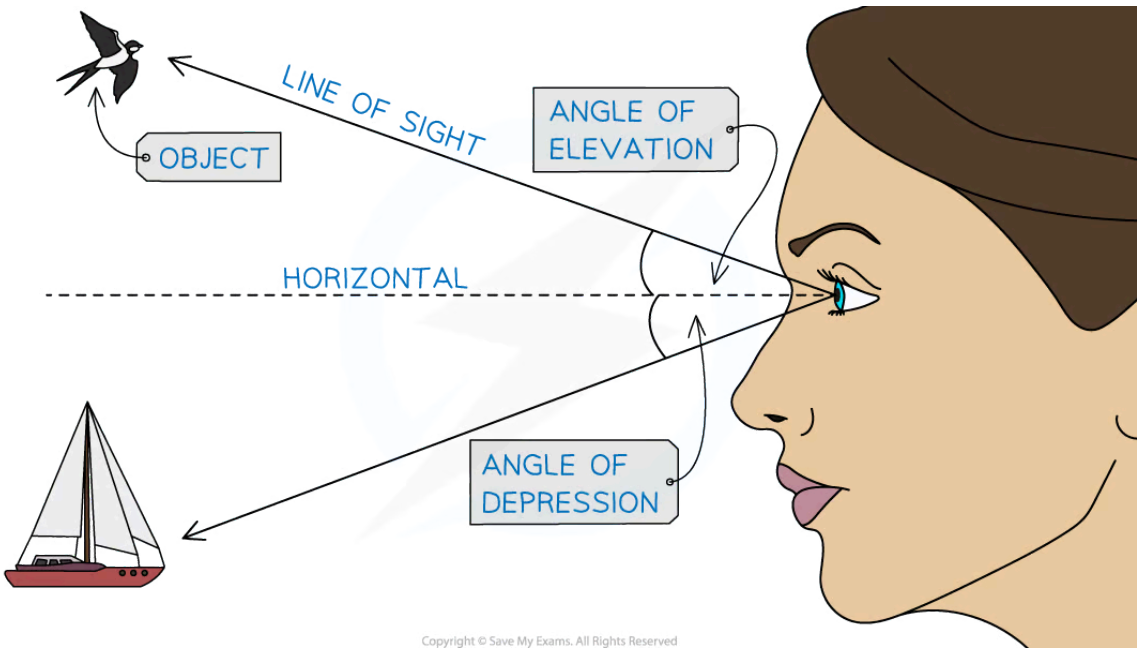
Angles of Elevation & Depression

Your notes

Elevation & Depression

What are angles of elevation and depression?

- An **angle of elevation** or **depression** is the angle measured between the **horizontal** and the **line of sight**
  - Looking **up** at an object creates an angle of **elevation**
  - Looking **down** at an object creates an angle of **depression**
- **Right-angled trigonometry** can be used to find
  - an **angle** of elevation or depression
  - or a missing **distance**
- The **tan ratio** is often used in real-life scenarios
  - You may **know the height** of an object and want to **find the distance** you are from it
  - You may **know the distance** you are from an object and want to **find its height**



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Examiner Tips and Tricks

It may be useful to draw more than one diagram if the triangles that you are interested in overlap one another.

Worked Example

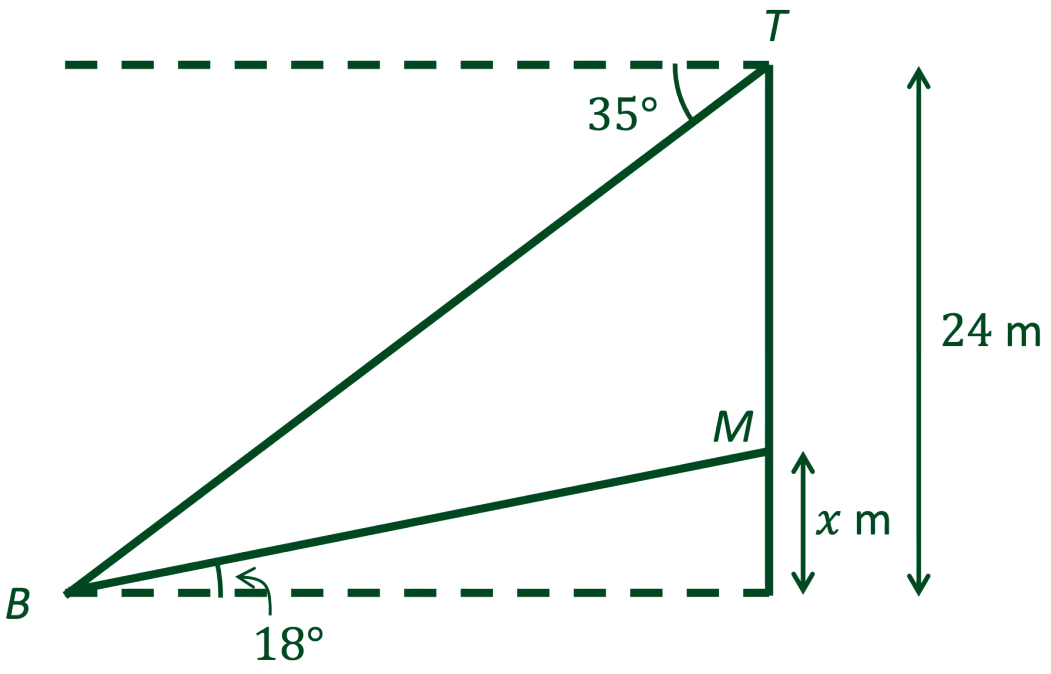
A cliff is perpendicular to the sea and the top of the cliff, *T*, stands 24 metres above the level of the sea.

The angle of depression from the top of the cliff to a boat at sea is 35°.

At a point $\boldsymbol{X}$ metres vertically up from the foot the cliff is a flag marker, *M*.

The angle of elevation from the boat, *B*, to the flag marker is 18°.

(a) Draw a diagram of the situation. Label all the angles and distances given above.



(b) Find the distance from the boat to the foot of the cliff.

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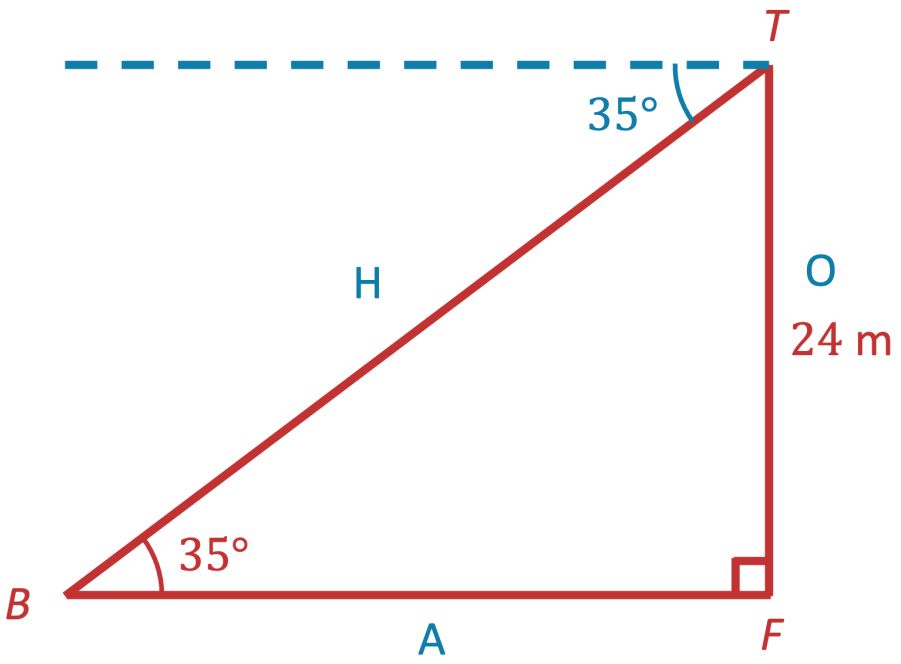
Consider triangle *TBF* where *F* is the foot of the cliff
Angle *TBF* = 35° because of alternate angles

Use SOHCAHTOA to find the missing distance

We know the opposite (*TF*) and we want to find the adjacent (*BF*), so use $\tan\theta = \frac{O}{A}$



$$\tan 35 = \frac{24}{BF}$$

$$BF = \frac{24}{\tan 35}$$

$$BF = 34.27555...$$

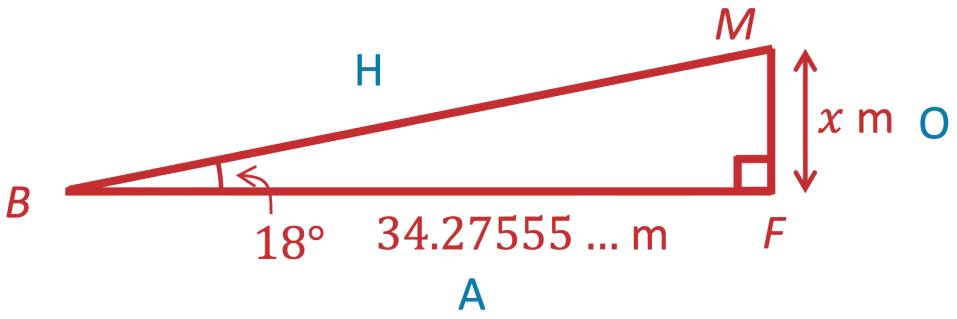
BF = 34.3 m (3 s.f.)

(c) Find the value of $\boldsymbol{X}$.

Consider triangle *MBF*

Use SOHCAHTOA to find the missing distance

We know the adjacent (*BF*) and we want to find opposite (*MF*), so use $\tan\theta = \frac{O}{A}$



$$\tan 18 = \frac{x}{34.27555...}$$

$$34.27555... \tan 18 = x$$

$$x = 11.1368...$$

$$\boldsymbol{x = 11.1} \text{ m (3 s.f.)}$$

Your notes

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Exact Trig Values

What are exact values in trigonometry?

- For **certain angles** the values of sin θ, cos θ and tan θ can be written **exactly**
  - This means using fractions and surds
- You are **expected to know** the exact values of sin, cos and tan for
  - 0°, 30°, 45°, 60°, 90°, 180° and their multiples

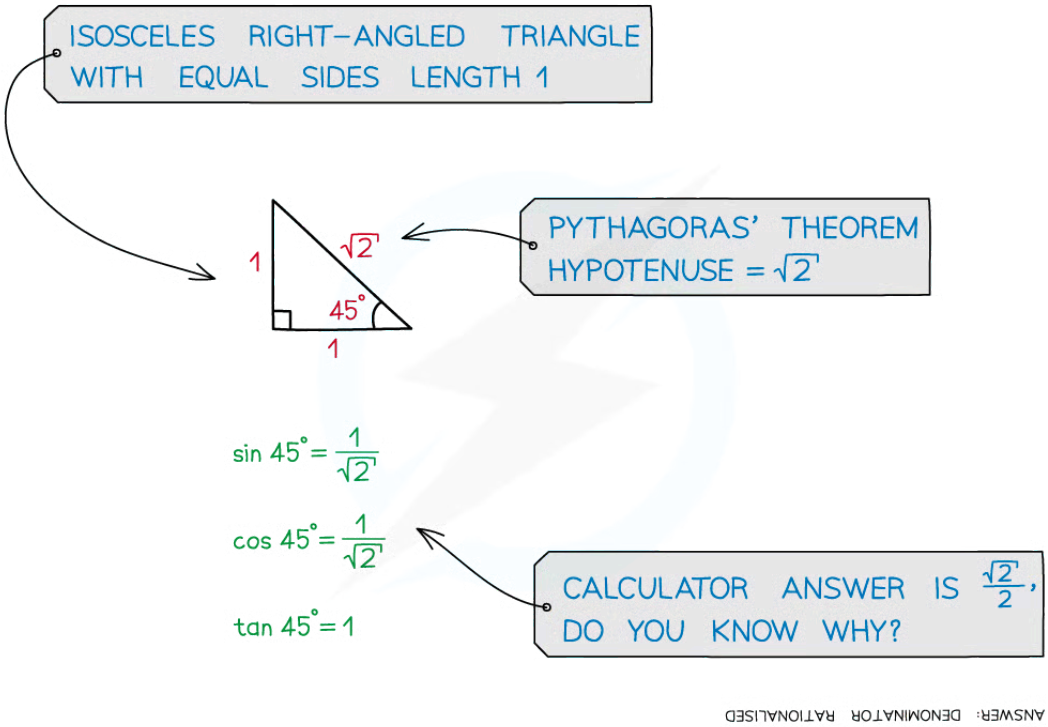
θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

How can I remember these exact trig values?

- Look at **patterns** in the table
  - Note the values of sin θ from 0° to 90° match cos θ in reverse, from 90° to 0°
  - Some people remember sin θ using the trick $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$ which simplifies to $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$
- Two special **right-angled triangles** below can help you to find the exact values for 30°, 45° and 60°

Your notes

- Remember that by rationalising the denominator, $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$



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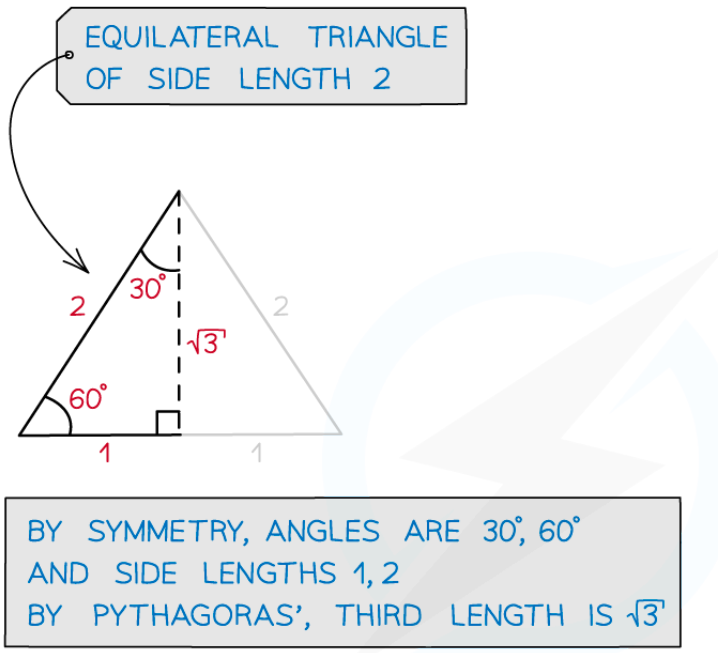
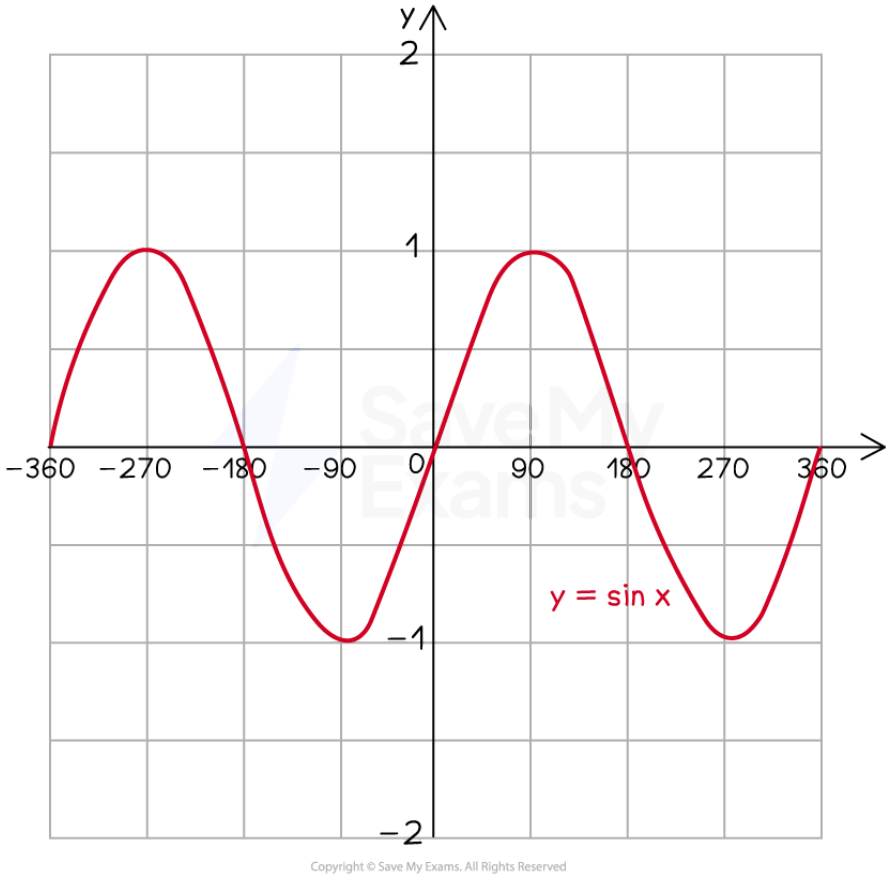


$\sin 30^\circ = \frac{1}{2}$ $\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$ $\cos 60^\circ = \frac{1}{2}$

$\tan 30^\circ = \frac{1}{\sqrt{3}}$ $\tan 60^\circ = \sqrt{3}$

- The **trig graphs** can help you to remember the exact values for 0° and multiples of 90°



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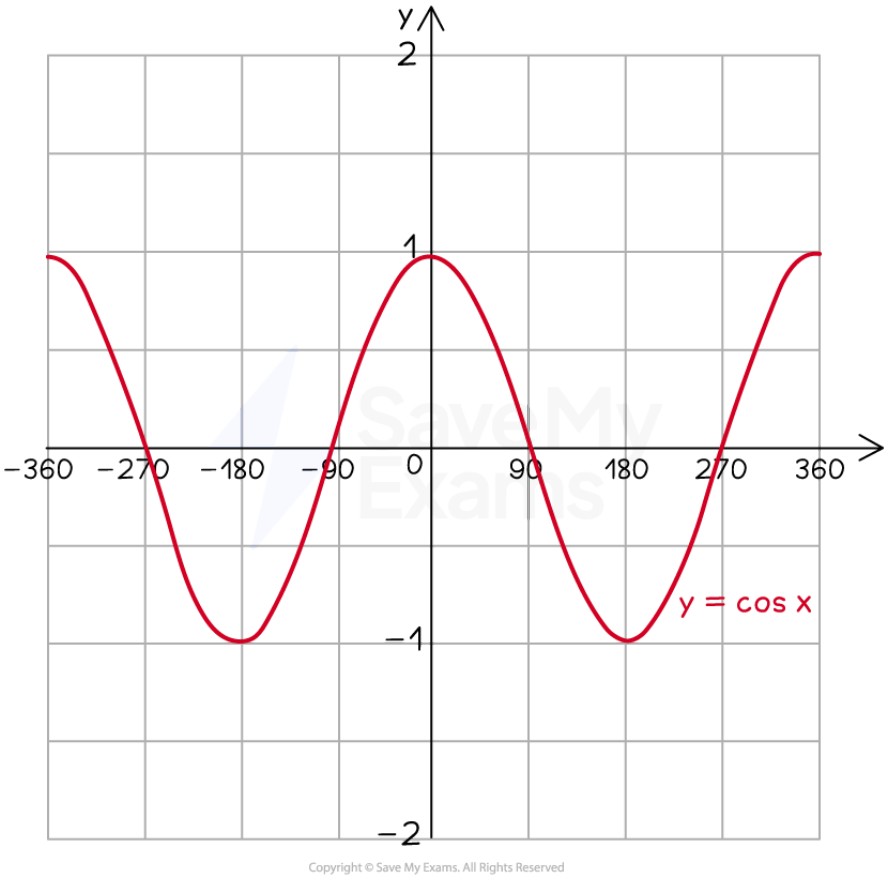

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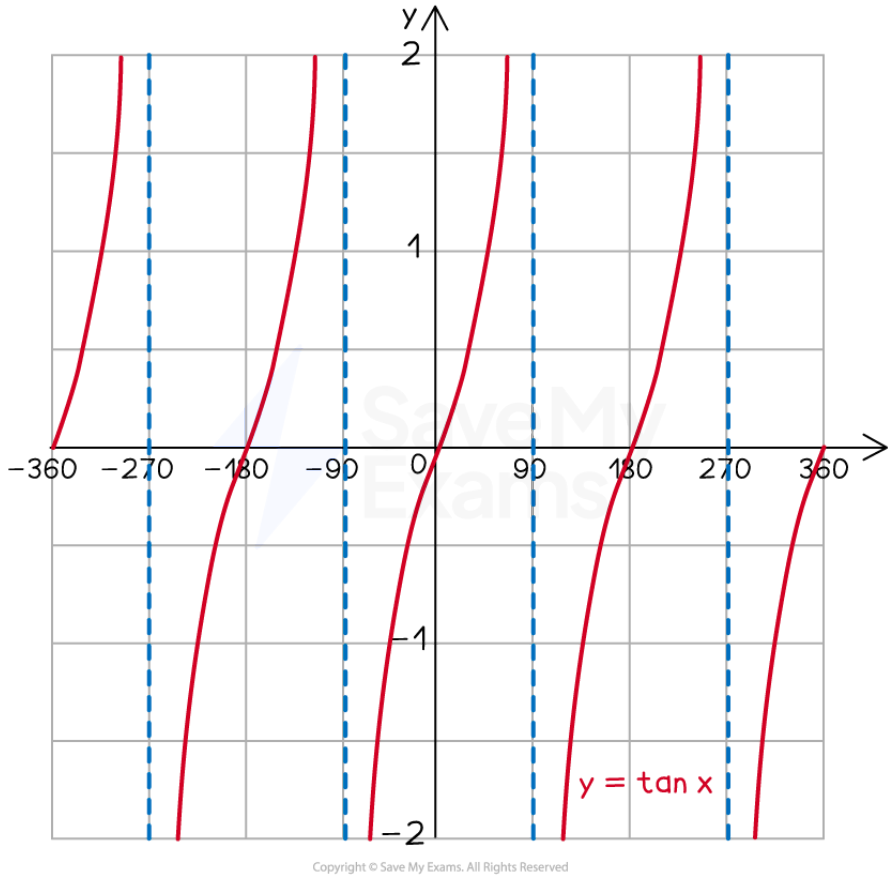


How do I use exact trig values?

- You may come across trig questions in a **non-calculator** question
- In trig calculations, **substitute** in the **exact trig values** and solve as usual
  - E.g. Solve the equation $\cos 45 = \frac{x}{12}$
    - Replace $\cos 45$ with $\frac{\sqrt{2}}{2}$ to give $\frac{\sqrt{2}}{2} = \frac{x}{12}$
    - Then you can solve for x
- On trig graphs, you may be expected to find a coordinate
  - E.g. The coordinates $(30, k)$ lie on the graph $y = \tan x$, find k

- k will be equal to $\tan 30$
- The exact value of $\tan 30$ is $\frac{\sqrt{3}}{3}$
- Therefore $k = \frac{\sqrt{3}}{3}$

Your notes

Examiner Tips and Tricks

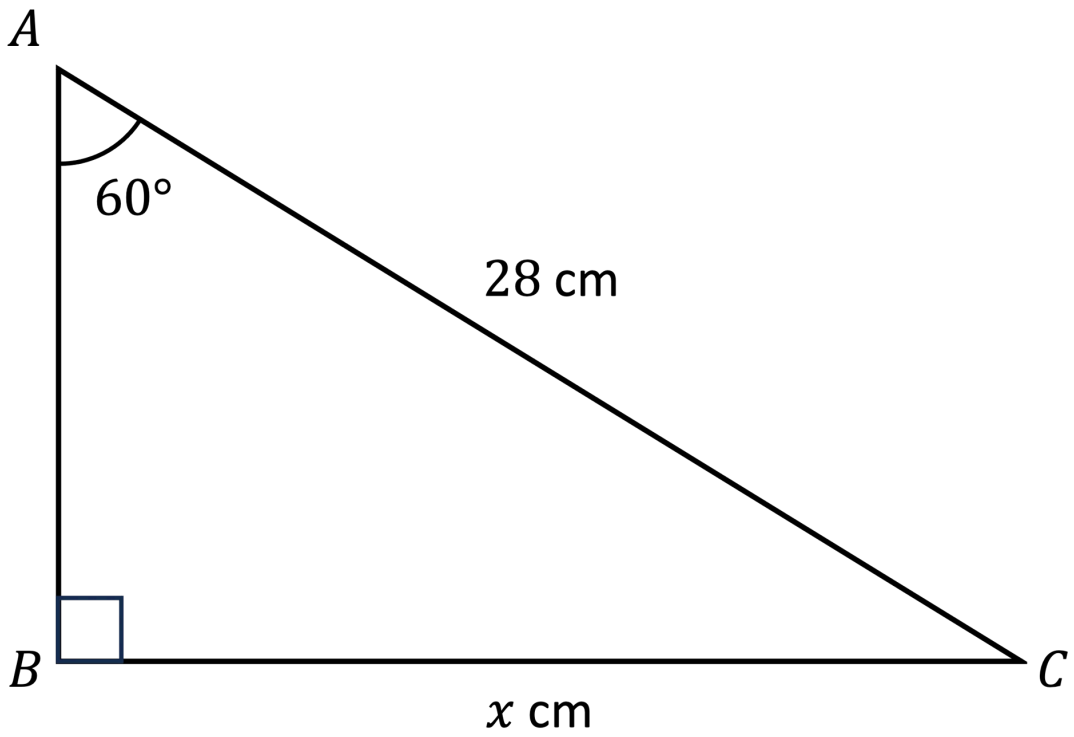
- Writing these out (or sketching the triangles/graphs) on your paper at the beginning of the exam means that you can use them as many times as you need to during the exam!

Worked Example

Find the value of x in the diagram below.

Give your answer as an exact value.

Your notes



Triangle *ABC* is a right-angled triangle, so use SOHCAHTOA

We know the hypotenuse (*AC*) and we want to calculate the opposite (*BC*), so use $\sin\theta = \frac{O}{H}$

$$\sin 60 = \frac{x}{28}$$

Remember that $\sin 60 = \frac{\sqrt{3}}{2}$

So,

$$\frac{\sqrt{3}}{2} = \frac{x}{28}$$

$$28 \times \frac{\sqrt{3}}{2} = x$$

$$14 \times \sqrt{3} = x$$

Leave in exact (surd) form

$$x = 14\sqrt{3} \text{ cm}$$

Your notes