AQA GCSE Maths: Higher

Your notes

Simultaneous Equations

Contents

Linear Simultaneous Equations

Your notes

Linear Simultaneous Equations

What are linear simultaneous equations?

- When there are **two unknowns** (**x and y**), we need **two equations** to find them both
  - For example, $3x + 2y = 11$ and $2x - y = 5$
    - The values that work are $x = 3$ and $y = 1$
- These are called **linear simultaneous equations**
  - **Linear** because there are no terms like x^2 or y^2

How do I solve linear simultaneous equations by elimination?

- **Elimination** removes one of the variables, x or y
- To eliminate the x's from $3x + 2y = 11$ and $2x - y = 5$, make the number in front of the x (the **coefficient**) in both equations the same (the sign may be different)
  - Multiply **every term** in the **first** equation **by 2**
    - $6x + 4y = 22$
  - Multiply **every term** in the **second** equation **by 3**
    - $6x - 3y = 15$
  - **Subtracting** the second equation from the first **eliminates *x***
    - When the **sign** in front of the term you want to eliminate is the **same**, **subtract** the equations

$$\begin{array}{rr} & 6x+4y=22 \\ - & 6x-3y=15 \\ \hline & 7y=7 \end{array}$$

- The y terms have become $4y - (-3y) = 7y$ (be careful with **negatives**)
  - **Solve** the resulting equation to find y
  - $y = 1$
- Then **substitute** $y = 1$ into one of the **original** equations to find x

Your notes

- $3x + 2 = 11$, so $3x = 9$, giving $x = 3$
- Write out **both solutions** together, $x = 3$ and $y = 1$
- **Alternatively**, you could have eliminated the y's from $3x + 2y = 11$ and $2x - y = 5$ by making the **coefficient** of y in both equations the same
  - Multiply every term in the second equation by 2
  - **Adding** this to the first equation **eliminates *y*** (and so on)
    - When the **sign** in front of the term you want to eliminate is **different**, **add** the equations

$$\begin{array}{rl} & 3x + 2y = 11 \\ + & 4x - 2y = 10 \\ \hline & 7x \qquad = 21 \end{array}$$

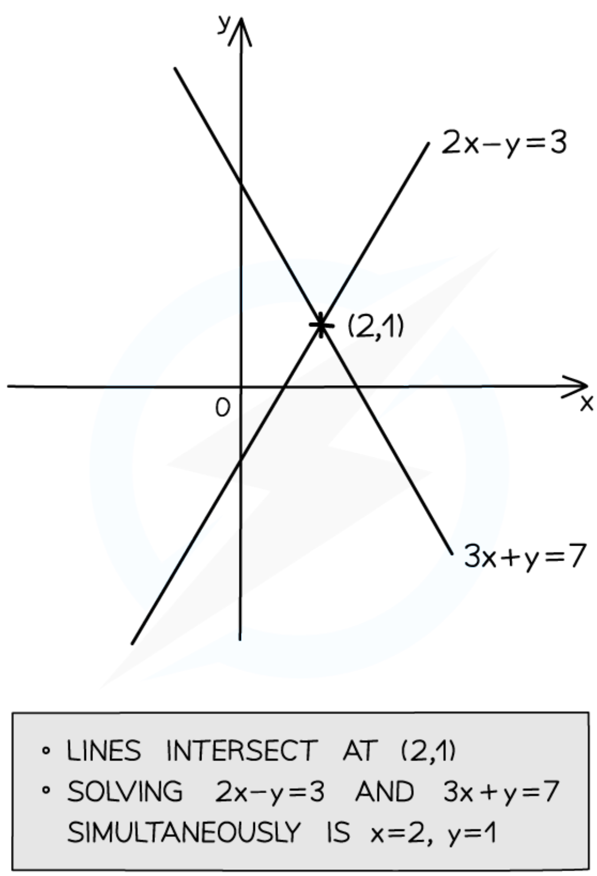
How do I solve linear simultaneous equations by substitution?

- **Substitution** means substituting one equation into the other
  - This is an **alternative** method to **elimination**
    - You can still use elimination if you prefer
- To solve $3x + 2y = 11$ and $2x - y = 5$ by substitution
  - **Rearrange** one of the equations into ***y* =** ... (or x = ...)
    - For example, the second equation becomes $y = 2x - 5$
  - **Substitute** this into the first equation
    - This means **replace** all y's with $2x - 5$ in brackets
    - $3x + 2(2x - 5) = 11$
  - **Solve** this equation to find x
    - $x = 3$
  - Then **substitute** $x = 3$ into $y = 2x - 5$ to find y
    - $y = 1$

How do I solve linear simultaneous equations graphically?

- **Plot** both equations on the same set of axes
  - To do this, you can use a **table of values**

- or rearrange into $y = mx + c$ if that helps
- Find where the lines **intersect** (cross over)
  - The x and y **solutions** to the simultaneous equations are the x and y **coordinates** of the point of **intersection**
- For example, to solve $2x - y = 3$ and $3x + y = 4$ simultaneously
  - First **plot** them both on the **same axes** (see graph)
  - Find the **point of intersection**, (2, 1)
  - The solution is $x = 2$ and $y = 1$



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Examiner Tips and Tricks

- Always check that your final solutions satisfy **both** original simultaneous equations!
- Write out both solutions (*x* and *y*) together at the end to avoid examiners missing a solution in your working

Worked Example

Solve the simultaneous equations

$$5x + 2y = 11$$
$$4x - 3y = 18$$

It helps to number the equations

$$5x + 2y = 11 \quad (1)$$
$$4x - 3y = 18 \quad (2)$$

We will choose to eliminate the *y* terms
Make the *y* terms equal by multiplying all parts of equation 1 by 3 and all parts of equation 2 by 2

$$15x + 6y = 33 \quad (3)$$
$$8x - 6y = 36 \quad (4)$$

The 6*y* terms have different signs, so they can be eliminated by adding equation 4 to equation 3

$$\begin{array}{rrcl} & 15x + 6y & = & 33 \\ + & 8x - 6y & = & 36 \\ \hline & 23x \quad\quad & = & 69 \end{array}$$

Solve the equation to find *x* (divide both sides by 23)

$$x = \frac{69}{23} = 3$$

Substitute *x* = 3 into either of the two original equations

$$(1) \quad 5(3) + 2y = 11$$

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Solve this equation to find *y*

$$\begin{aligned} 15 + 2y &= 11 \\ 2y &= 11 - 15 \\ 2y &= -4 \\ y &= \frac{-4}{2} = -2 \end{aligned}$$

Substitute $x = 3$ and $y = -2$ into the other equation to check that they are correct

$$\begin{aligned} (2) \quad 4x - 3y &= 18 \\ 4(3) - 3(-2) &= 18 \\ 12 - (-6) &= 18 \\ 18 &= 18 \end{aligned}$$

Write out both solutions together

$$\boldsymbol{x = 3, \quad y = -2}$$

This question can also be done by eliminating *x* first (multiplying equation 1 by 4 and equation 2 by 5 then subtracting)

How do I form simultaneous equations?

- Introduce **two letters**, *x* and *y*, to represent **two unknowns**
  - Make sure you know exactly what they stand for (and any units)
- Create **two different equations** from the words or contexts
  - 3 apples and 2 bananas cost \$1.80, while 5 apples and 1 banana cost \$2.30
    - $3x + 2y = 180$ and $5x + y = 230$
      x is the **price** of an apple, in **cents**
      y is the **price** of a banana, in **cents**
    - This question could also be done in dollars, \$
- **Solve** the equations **simultaneously**
- Give answers **in context** (relate them to the story, with units)
  - $x = 40, y = 30$
  - In context: an apple costs 40 cents and a banana costs 30 cents

Your notes

- Some questions don't ask you to solve simultaneously, but you still need to
  - Two numbers have a **sum** of 19 and a **difference** of 5, what is their **product**?
    - $x + y = 19$ and $x - y = 5$
    - Solve simultaneously to get $x = 12$, $y = 7$
    - The product is $xy = 12 \times 7 = 84$

Examiner Tips and Tricks

- Always check that you've answered the question! Sometimes finding x and y isn't the end
  - E.g. you may have to state a conclusion

Worked Example

At a bakery a customer pays £9 in total for six bagels and twelve sausage rolls.

Another customer buys nine bagels and ten sausage rolls, which costs £12.30 in total.

Find the cost of 5 bagels and 15 sausage rolls.

The two variables are the price of bagels, b, and the price of sausage rolls, s

Write an equation for the first customer's purchases, and label it equation 1

$$(1) \quad 6b + 12s = 9$$

Write an equation for the second customer's purchases, and label it equation 2

$$(2) \quad 9b + 10s = 12.3$$

We will choose to eliminate the b terms
Make the b terms equal by multiplying all parts of equation 1 by 3 and all parts of equation 2 by 2
Label these as equations 3 and 4

$$(1) \times 3 \quad 18b + 36s = 27 \qquad (3)$$
$$(2) \times 2 \quad 18b + 20s = 24.6 \qquad (4)$$

To eliminate b, subtract equation 4 from equation 3

$$(3) - (4) \quad 16s = 2.4$$

Solve for s

$$s = \frac{2.4}{16} = 0.15$$

Substitute this into either equation to find b, we will use equation 1

$$\begin{aligned} (1) \quad 6b + 12(0.15) &= 9 \\ 6b + 1.8 &= 9 \\ 6b &= 7.2 \\ b &= 1.2 \end{aligned}$$

So sausage rolls cost £0.15 each and bagels cost £1.20 each

Use these values to find the price of 5 bagels and 15 sausage rolls

$$(5 \times 1.2) + (15 \times 0.15) = 8.25$$

£8.25

Your notes

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Quadratic Simultaneous Equations

What are quadratic simultaneous equations?

- When there are two unknowns (e.g. x and y) in a problem, we need two equations to be able to find them both; these are called **simultaneous equations**
- If there is an x^2 or y^2 or xy in one of the equations then they are **quadratic** (or **non-linear**) simultaneous equations

How do I solve quadratic simultaneous equations?

- Use **substitution**
  - **Substitute** the **linear** equation, $y = ...$ (or $x = ...$), into the quadratic equation
    - Do not try to substitute the quadratic equation into the linear equation
- E.g. To solve $x^2 + y^2 = 25$ and $y - 2x = 5$
  - **Rearrange** the **linear** equation into $y = 2x + 5$
  - **Substitute** this into the quadratic equation, replacing all y's with $(2x + 5)$
    - $x^2 + (2x + 5)^2 = 25$
- Expand and **solve** this **quadratic** equation
  - $x^2 + 4x^2 + 20x + 25 = 25$
  - $5x^2 + 20x = 0$
  - $5x(x + 4) = 0$
  - $x = 0$ and $x = -4$
- Substitute **each** value of x into the **linear** equation, $y = 2x + 5$, to find the corresponding y values
  - $y = 2(0) + 5 = 5$
  - $y = 2(-4) + 5 = -3$
- Present your solutions in a way that makes it obvious which x belongs to which y

Your notes

- $x = 0, y = 5$ or $x = -4, y = -3$
- **Check** your final solutions satisfy both equations

What if the quadratic has repeated roots or no roots?

- If the resulting quadratic after substituting has a **repeated root**,
  - then the line is a **tangent** to the curve
    - i.e. the curve and the line intersect in one place only
  - There is only **one solution** for x and y
- If the resulting quadratic to be solved has **no roots**,
  - then the line does not intersect with the curve
  - There are **no solutions** to the simultaneous equations
  - If this happens it *may* be an indicator that your working is wrong!

What if I can't substitute one equation into the other straight away?

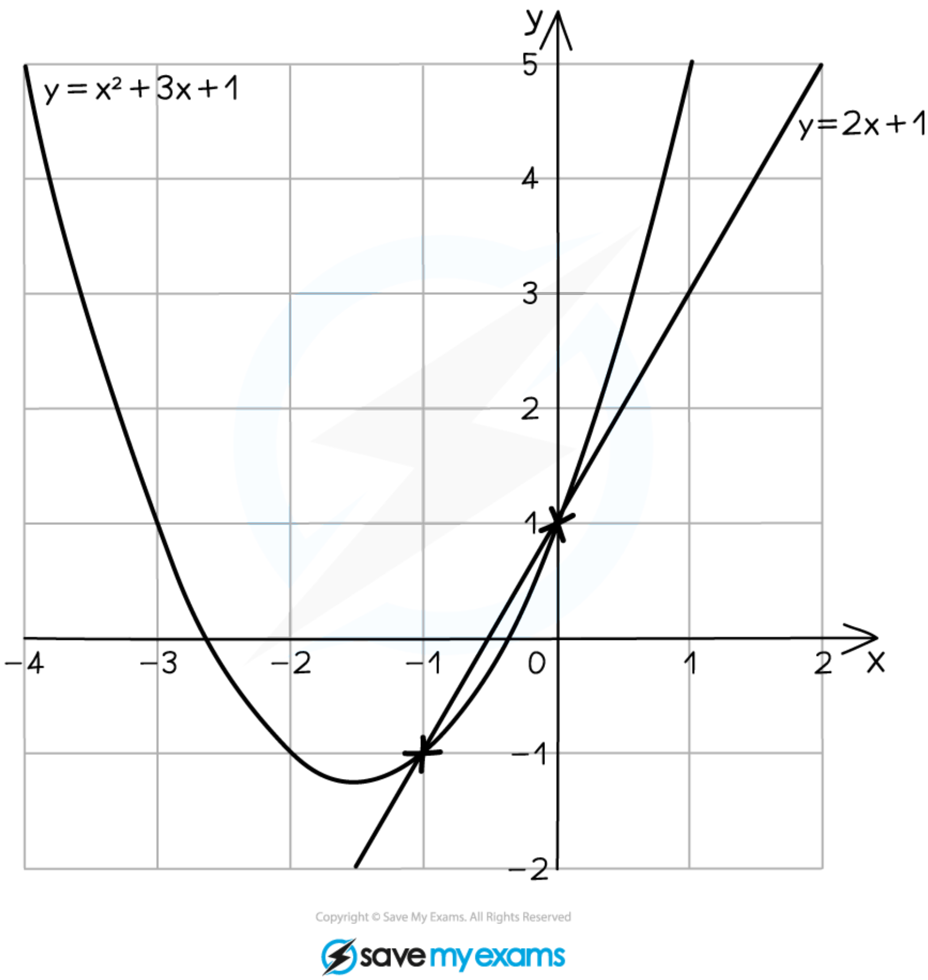
- If the linear equation is **not** in the form $y = \ldots$ or $x = \ldots$
  - You will need to **rearrange** it first, so that it can be **substituted** into the quadratic equation
- Consider solving $xy = 3$ and $x + y = 4$
- Either:
  - Rearrange the second equation to $y = 4 - x$ and substitute into $xy = 3$
    - $x(4 - x) = 3$
    - Expanding produces a quadratic that can be solved for x
    - $4x - x^2 = 3$
  - Or rearrange the first equation to $y = \frac{3}{x}$ and substitute into $x + y = 4$
    - $x + \frac{3}{x} = 4$
    - Multiplying both sides by x produces a quadratic that can be solved for x

- $x^2 + 3 = 4x$

How do I use a graph to solve quadratic simultaneous equations?

- **Plot** both equations on the same set of axes
  - To do this, you can use a table of values
  - Or for straight lines it can help to rearrange into $y = mx + c$
- Find the point where the lines **intersect**
  - The x and y **solutions** to the simultaneous equations are the x and y **coordinates** of the point of **intersection**
- E.g. To solve $y = x^2 + 3x + 1$ and $y = 2x + 1$ simultaneously
  - First **plot** them both (see graph below)
  - Then find the points of **intersection**, (-1, -1) and (0, 1)
  - So the **solutions** are $x = -1$ and $y = -1$ or $x = 0$ and $y = 1$

Your notes



Examiner Tips and Tricks

- When giving your final answer, make sure you indicate which x and y values go together

Worked Example

Solve the equations

Your notes

$$x^2 + y^2 = 36$$
$$x - 2y = 6$$

Number the equations

$$x^2 + y^2 = 36 \qquad (1)$$
$$x - 2y = 6 \qquad (2)$$

There is one quadratic equation and one linear equation so this must be done by substitution

Equation 2 can be rearranged to make x the subject, which can then be substituted into equation 1

You could rearrange to make y the subject instead, but this results in a fraction which can be more tricky to deal with

Rearranging equation 2

$$x = 2y + 6$$

Substituting into equation 1

$$(2y + 6)^2 + y^2 = 36$$

Expand the brackets
Remember that a bracket squared should be treated the same as double brackets

$$(2y + 6)(2y + 6) + y^2 = 36$$
$$4y^2 + 6(2y) + 6(2y) + 6^2 + y^2 = 36$$

Simplify

$$4y^2 + 12y + 12y + 36 + y^2 = 36$$
$$5y^2 + 24y + 36 = 36$$

Rearrange to form a quadratic equation that is equal to zero
Do this by subtracting 36 from both sides

$$5y^2 + 24y = 0$$

Take out the common factor of y

$$y(5y + 24) = 0$$

Solve to find the values of y by equating each factor to zero

Your notes

$$y = 0 \text{ or } 5y + 24 = 0$$

Solve the linear equation above

$$y = -\frac{24}{5}$$

So the two y values are

$$y_1 = 0$$

$$y_2 = -\frac{24}{5}$$

Substitute the values of y into one of the equations (the linear equation is easiest) to find the values of x

$$x = 2y + 6$$

$$x_1 = 2(0) + 6 = 6 \qquad x_2 = 2\left(-\frac{24}{5}\right) + 6 = -\frac{18}{5}$$

Write the final solutions in clear pairs

$$\boldsymbol{x_1 = 6, \ \ y_1 = 0}$$

$$\boldsymbol{x_2 = -\frac{18}{5}, \ \ y_2 = -\frac{24}{5}}$$