AQA GCSE Maths: Higher

Your notes

Introduction to Probability

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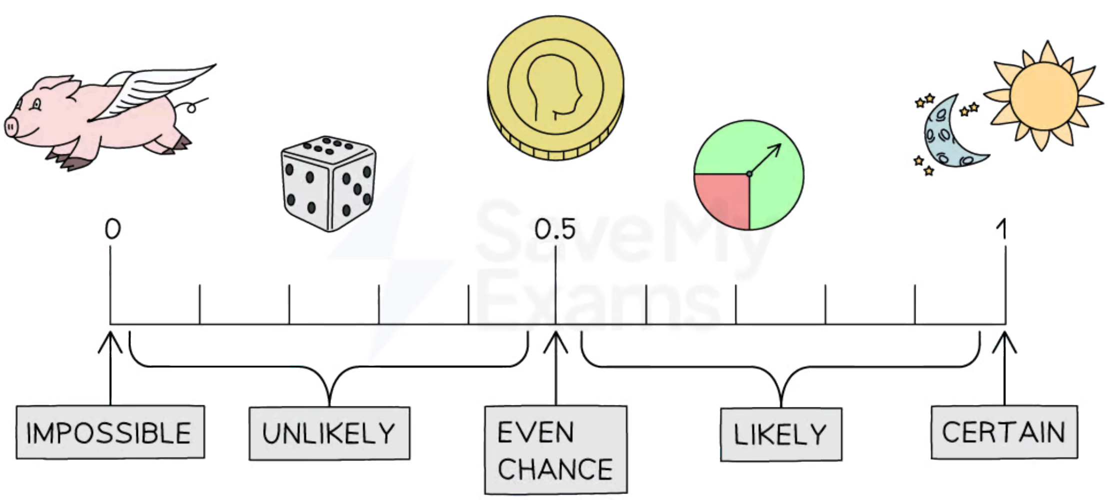
Basic Probability

Basic Probability

What is probability?

- Probability describes the **likelihood** of something happening
  - In real-life you might use words such as **impossible, unlikely** and **certain**
- In maths we use the **probability scale** to describe probability
  - This means giving it a number between **0 and 1**
    - **0** means **impossible**
    - Between **0 and 0.5** means **unlikely**
    - **0.5** means **even chance**
    - Between **0.5 and 1** means **likely**
    - **1** means **certain**
- Probabilities can be given as **fractions**, **decimals** or **percentages**



What key words and terminology are used in probability?

- An **experiment** is an activity that is **repeated** to produce a set of **results**

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    - Results can be observed (seen) or recorded
    - Each repeat is called a **trial**
- An **outcome** is a possible result of a trial
- An **event** is an outcome (or a collection of outcomes)
    - For example:
        - a dice lands on a six
        - a dice lands on an even number
    - Events are usually given capital letters
    - n(*A*) is the number of possible outcomes from event *A*
        - *A* = a dice lands on an even number (2, 4 or 6)
        - n(*A*) = 3
- A **sample space** is the set of **all** possible outcomes of an experiment
    - It can be represented as a **list** or a **table**
- The **probability** of event *A* is written P(*A*)
- An event is said to be **fair** if there is an **equal chance** of achieving each outcome
    - If there is not an equal chance, the event is **biased**
    - For example, a fair coin has an equal chance of landing on heads or tails

How do I calculate basic probabilities?

- If all outcomes are **equally likely** then the probability for each outcome is the same
    - The probability for each outcome is $\frac{1}{\text{Total number of outcomes}}$
        - If there are 50 marbles in a bag then the probability of selecting a specific one is $\frac{1}{50}$
- The **theoretical probability** of an event can be calculated by dividing the number of outcomes of that event by the total number of outcomes
    - $\mathrm{P}(A) = \frac{\text{Total number of outcomes for the event}}{\text{Total number of outcomes}}$
    - This can be calculated without actually doing the experiment

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- If there are 50 marbles in a bag and 20 are blue, then the probability of selecting a blue marble is $\frac{20}{50}$

How do I find missing probabilities?

- The probabilities of **all** the outcomes **add up to 1**
  - If you have a table of probabilities with one **missing**, find it by making them all add up to 1
- The **complement of event *A*** is the event where ***A* does not happen**
  - This can be thought of as **not *A***
  - P(event does **not** happen) = 1 - P(event **does** happen)
    - For example, if the probability of rain is 0.3, then the probability of **not** rain is 1 - 0.3 = 0.7

What are mutually exclusive events?

- Two events are **mutually exclusive** if they **can not both happen at once**
  - When rolling a dice, the events "getting a prime number" and "getting a 6" are mutually exclusive
- If *A* and *B* are mutually exclusive events, then the probability of **either A or B happening** is **P(A) + P(B)**
- **Complementary events** are mutually exclusive

Examiner Tips and Tricks

- If you are not told in the question how to leave your answer, then fractions are best for probabilities.

Worked Example

Emilia is using a spinner that has outcomes and probabilities as shown in the table.

Outcome	**Blue**	**Yellow**	**Green**	**Red**	**Purple**
Probability		0.2	0.1		0.4

The spinner has an equal chance of landing on blue or red.

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(a) Complete the probability table.

The probabilities of **all** the outcomes **should add** up to **1**

$$1 - 0.2 - 0.1 - 0.4 = 0.3$$

The probability that it lands on blue or red is 0.3
As the probabilities of blue and red are **equal** you can halve this to get each probability

$$0.3 \div 2 = 0.15$$

Now complete the table

Outcome	**Blue**	**Yellow**	**Green**	**Red**	**Purple**
Probability	**0.15**	0.2	0.1	**0.15**	0.4

(b) Find the probability that the spinner lands on green or purple.

As the spinner cannot land on green and purple at the same time they are **mutually exclusive**
This means you can **add** their probabilities together

$$0.1 + 0.4 = 0.5$$

P(Green or Purple) = 0.5

(c) Find the probability that the spinner does not land on yellow.

The probability of **not** landing on yellow is equal to 1 minus the probability of landing on yellow

$$1 - 0.2 = 0.8$$

P(Not Yellow) = 0.8

Sample Space Diagrams

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Sample Space

What is a sample space diagram?

- In probability, the **sample space** means **all the possible outcomes**
- In simple situations it can be given as a **list**
  - For flipping a coin, the sample space is: Heads, Tails
    - the letters H, T can be used
  - For rolling a six-sided dice, the sample space is: 1, 2, 3, 4, 5, 6
- If there are **two** sets of outcomes, a **grid** can be used
  - These are called **sample space diagrams** (or possibility diagrams)
  - For example, roll two six-sided dice and add their scores
  - A list of all the possibilities would be very long
    - You might miss a possibility
    - It would be hard to spot any patterns in the sample space

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2nd dice

1st dice

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- Combining **more than two** sets of outcomes must be done by **listing** the possibilities
  - For example, flipping three coins
    - The sample space is HHH, HHT, HTH, THH, HTT, THT, TTH, TTT (8 possible outcomes)

How do I use a sample space diagram to calculate probabilities?

- Probabilities can be found by **counting** the **number** of possibilities you **want**, then **dividing** by the **total** number of possibilities in the sample space
  - In the sample space 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, there are four prime numbers (2, 3, 5 and 7)

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    - The probability of getting a prime number is $\frac{4}{10} = \frac{2}{5}$
  - Using the sample space diagram above for rolling two dice, the probability of getting an eight is $\frac{5}{36}$
    - There are 5 eights in the grid, out of the total 36 numbers
- **Be careful**, this **counting** method **only** works if all possibilities in the sample space are **equally likely**
  - For a **fair** six-sided dice: 1, 2, 3, 4, 5, 6 are all equally likely
  - For a fair (**unbiased**) coin: H, T are equally likely
  - Winning the lottery: Win, Lose are are not equally likely!
    - You cannot count possibilities here to say the probability of winning the lottery is $\frac{1}{2}$

Examiner Tips and Tricks

- Some harder questions may not say "by drawing a sample space diagram" so you may have to do it on your own.

Worked Example

Two fair six-sided dice are rolled.

(a) Find the probability that the sum of the numbers showing on the two dice is an odd number greater than 5, giving your answer as a fraction in simplest form.

Draw a sample space diagram to show all the possible outcomes

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2nd dice

1st dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Circle the possibilities that are odd numbers greater than 5
(5 is not included)

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2nd dice

1st dice	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Count the number of possibilities that are circled (12) and divide them by the total number of possibilities in the diagram (36)

$$\frac{12}{36}$$

Cancel the fraction

$$\frac{12}{36} = \frac{12 \times 1}{12 \times 3} = \frac{1}{3}$$

$$\mathbf{\frac{1}{3}}$$

(b) Given that the sum of the numbers showing on the two dice is an odd number greater than 5, find the probability that one of the dice shows the number 2. Give your answer as a fraction in simplest form.

From part (a) you already know there are 12 ways to get an odd number greater than 5
Out of these 12 possibilities, only two possibilities had the number 2 on a dice: (2, 5) and (5, 2)
So the probability we are looking for is 2 divided by 12

$$\frac{2}{12}$$

Cancel the fraction

$$\frac{2}{12} = \frac{2 \times 1}{2 \times 6} = \frac{1}{6}$$

$$\mathbf{\frac{1}{6}}$$

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Relative & Expected Frequency

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Relative Frequency

What is relative frequency?

- **Relative frequency** is an **estimate of a probability** using results from an **experiment**
  - For a certain number of trials of that experience, the probability of 'success' is:

$$\frac{\text{Number of successful outcomes}}{\text{Total number of trials}}$$

- If you flip an unfair coin 50 times and it lands on heads 20 times, an **estimate** for the probability of the coin landing on heads is $\frac{20}{50}$ (its relative frequency)
  - That is the **best estimate** we can make, given the data we have
  - We do **not** know the **actual** probability
- The **more trials** that are carried out, the **more accurate** relative frequency becomes
  - It gets closer and closer to the **actual** probability

When will I be asked to use relative frequency?

- Relative frequency is used when **actual probabilities** are **unavailable** (or **not possible to calculate**)
  - For example, if you do not know the actual probability of being left-handed, you can run an experiment to find an estimate (the relative frequency)
- Sometimes actual probabilities are known, as they can be calculated in theory (called **theoretical probabilities**)
  - The **theoretical** probability of a **fair** coin landing on heads is 0.5
  - The **theoretical** probability of a **fair** standard six-sided dice landing on a six is $\frac{1}{6}$
- **Relative frequency** can be **compared** to a **theoretical probability** to test if a situation is **fair** or **biased**
  - If 100 flips of the coin give a relative frequency of 0.48 for landing on heads, the coin is likely to be fair
    - The theoretical probability is 0.5 and 0.48 is close to 0.5

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    - If 100 flips of the coin give a relative frequency of 0.13 for landing on heads, the coin is likely to be biased (not fair)

What else do I need to know about relative frequency?

- Relative frequency assumes that there is an **equal chance of success** on each trial
    - The trials are independent of each other
        - For example, if choosing something out of a bag (a ball, or marble etc), it would need to be **replaced each time** to use relative frequency
- Any experiments used to calculate relative frequency should be **random**
    - If the experiment is not random, this could introduce **bias**

Examiner Tips and Tricks

- Exam questions will not necessarily use the phrase relative frequency
- If you have to choose the best estimate, choose the one with the most trials

Worked Example

There are an unknown number of different coloured buttons in a bag.
Johan selects a button at random, notes its colour and replaces the button in the bag.
Repeating this 30 times, Johan notes that on 18 occasions he selected a red button.

Use Johan's results to estimate the probability that a button drawn at random from the bag is red.

Taking 'red' to be a success, Johan had 18 successes out of a total of 30 trials.

$$\text{P}(\text{red}) = \frac{18}{30} = \frac{3}{5}$$

Expected Frequency

What is expected frequency?

- **Expected frequency** refers to the **number of times** you would **expect** a particular outcome to occur
- It is found by **multiplying** the **probability** by the **number of trials**

- If you flip a fair coin 100 times, you would expect 0.5 × 100 = 50 heads
- Sometimes you need to calculate the **relative frequency** first
  - If you flip a biased coin 40 times and get 10 heads, how many heads would you expect when flipping 100 times?
    - The relative frequency is $\frac{10}{40}$ = 0.25 from the first experiment
    - 0.25 × 100 = 25, you would expect to get heads 25 times from 100 throws

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Examiner Tips and Tricks

- Exam questions will not necessarily use the phrase "expected frequency", but might ask how many you "would expect"

Worked Example

There are 6 blue, 4 red and 5 yellow counters in a bag.
One counter is drawn at random and its colour noted.
The counter is then returned to the bag.

(a) Find the probability that a counter drawn from the bag is yellow.

There are 5 yellow counters out of a total of 6 + 4 + 5 = 15 counters in the bag

$$\text{P(Yellow)} = \frac{5}{15} = \frac{1}{3}$$

(b) How many times would you expect a yellow counter to be drawn, if this experiment is repeated 300 times?

This is expected frequency so multiply the number of trials by the probability from part (a)

$$300 \times \frac{1}{3} = 100$$

We would expect 100 yellow counters