AQA GCSE Maths: Higher

Your notes

Transformations

Contents

Your notes

Translations

Translations

What are transformations in maths?

- There are **four transformations** to learn
  - **translations**, **rotations**, **reflections** and **enlargements**
- A transformation can **change** the **position**, **orientation** and/or **size** of a **shape**
  - The **original shape** is called the **object**
  - The **transformed shape** is called the **image**
- Vertices are labelled to show **corresponding points**
  - Vertices on the **object** are labelled A, B, C, etc.
  - Vertices on the **image** are labelled A', B', C' etc.

What is a translation?

- A **translation moves** a shape
- The **size** and **orientation** (which way up it is) of the shape **stays the same**
  - The **object** and **image** are **congruent**

What is a translation vector?

- The movement of a translation is described by a **vector**
- You need to know how to write a translation using a vector (rather than words)
- Vectors are written as **column vectors** in the form $\begin{pmatrix} x \\ y \end{pmatrix}$ where:
  - x is the distance moved **horizontally**
    - **Negative** means move to the **left**
    - **Positive** means move to the **right**
  - y is the distance moved **vertically**

Your notes

- **Negative** means move **down**
- **Positive** means move **up**

How do I translate a shape?

- **STEP 1**

  Interpret the translation vector

  - $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ means 3 to the **right** and 1 **down**
- **STEP 2**
  Move each vertex on the **original object** according to the vector
- **STEP 3**
  Connect the **new vertices** and label the **translated image**
  - It should look **identical** to the **original object** just in a **different position**
- In some cases the **image** can overlap the **object**

How do I describe a translation?

- To describe a **translation**, you must:
  - State that the transformation is a **translation**
  - Give the **column vector** that describes the movement
- To find the **vector**:
  - **Pick** a **point** on the **original** shape
  - **Identify** the **corresponding point** on the **image**
  - Count how far **left or right** (x) you need to go **from the object** to get **to the image**
    - If you go to the **left** then x will be a **negative number**
  - Count how far **up or down** (y) you need to go **from the object** to get **to the image**
    - If you go **down** then y will be a **negative number**
  - Write these numbers as a vector
    - $\begin{pmatrix} x \\ y \end{pmatrix}$

Your notes

How do I reverse a translation?

- To **return a shape to its original position** after a translation
  - the **horizontal** and **vertical** translations must **both** be **reversed**
- The **column vector** to **reverse a translation** is simply the same as the original vector, but with the **sign of both values changed**
  - E.g. For a translation described by the column vector $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$
  - The column vector for the reverse translation is $\begin{pmatrix} 2 \\ -7 \end{pmatrix}$

Examiner Tips and Tricks

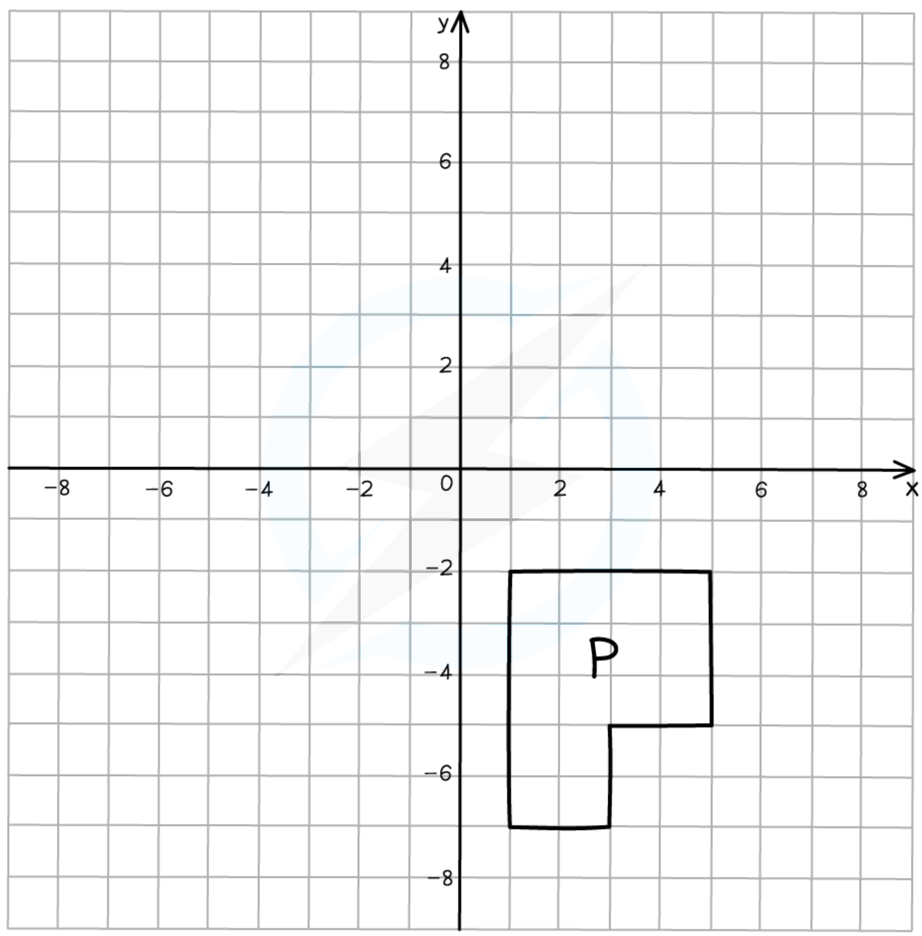
- The vector is how the shape moves **not** the size of the gap between the object and the image
  - Watch out for this common error!
- Use **tracing paper** to check your answer

Worked Example

(a) On the grid below translate shape P using the vector $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$.

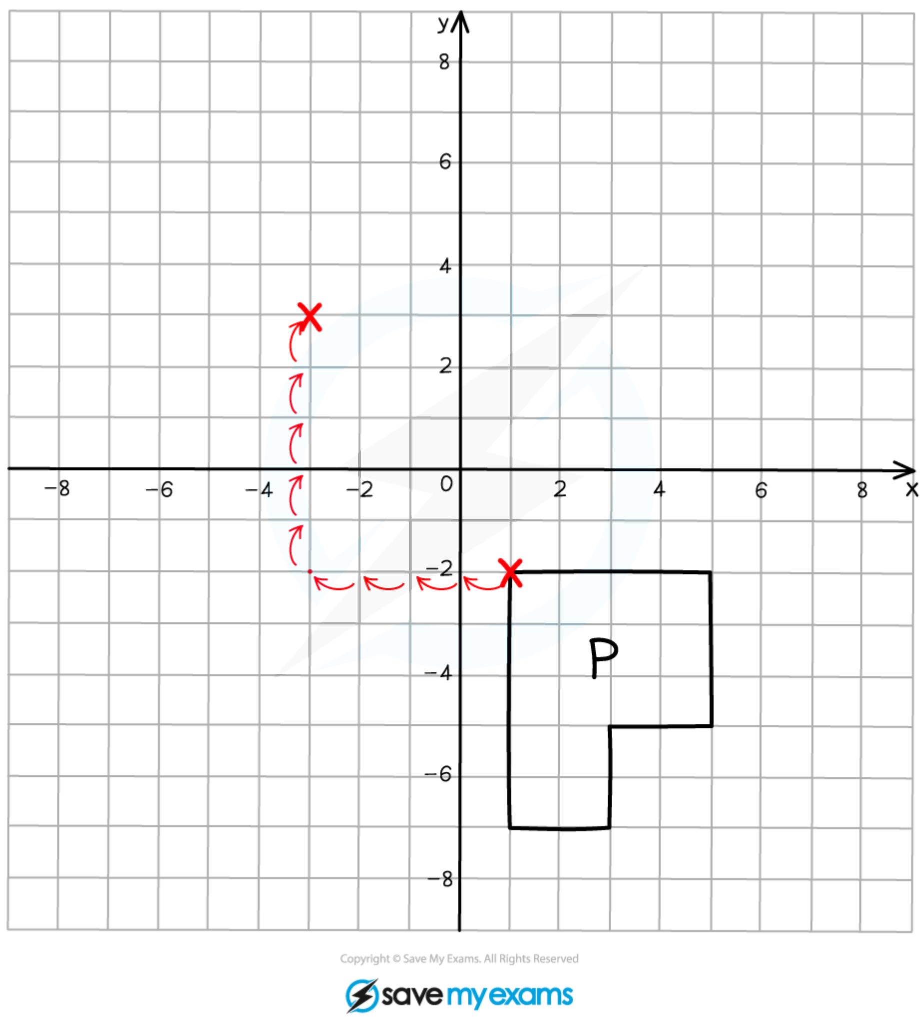
Label your translated shape P'.

Your notes



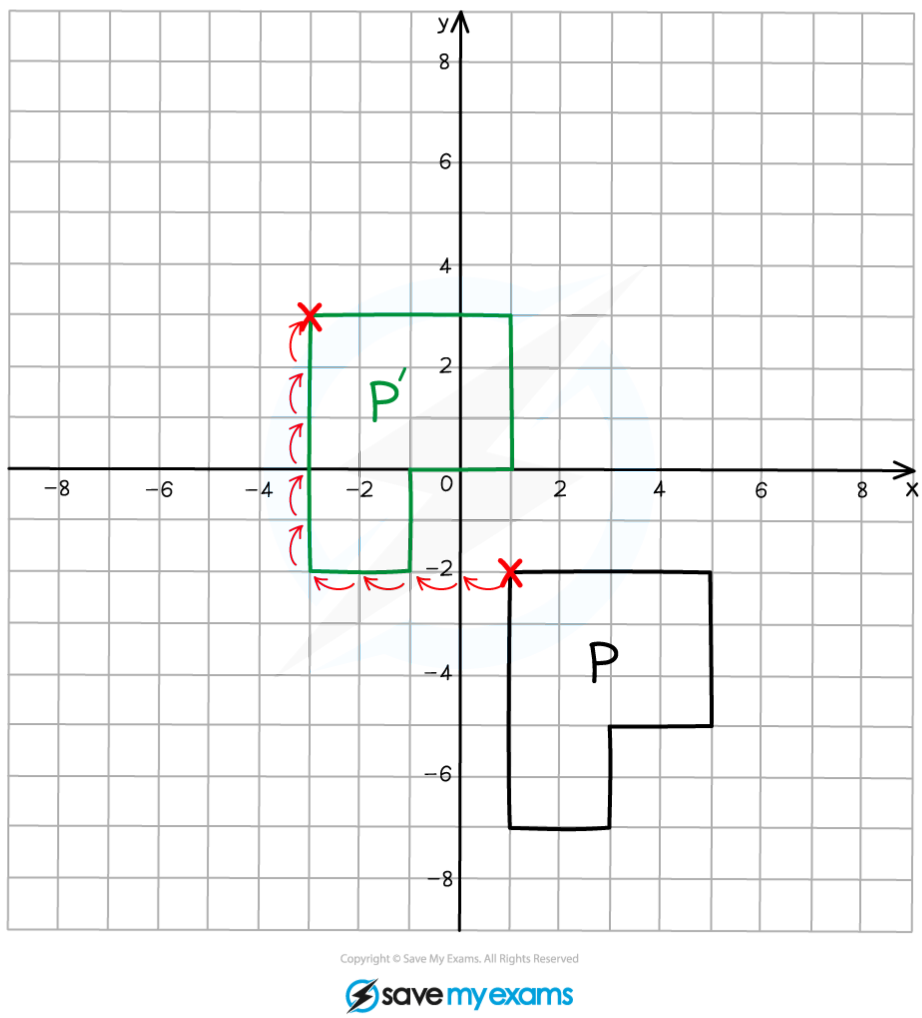
The vector means "4 to the left" and "5 up"
You don't have to draw in any arrows but it is a good idea to mark your paper after counting across and up a couple of times to check that you are in the correct place

Your notes



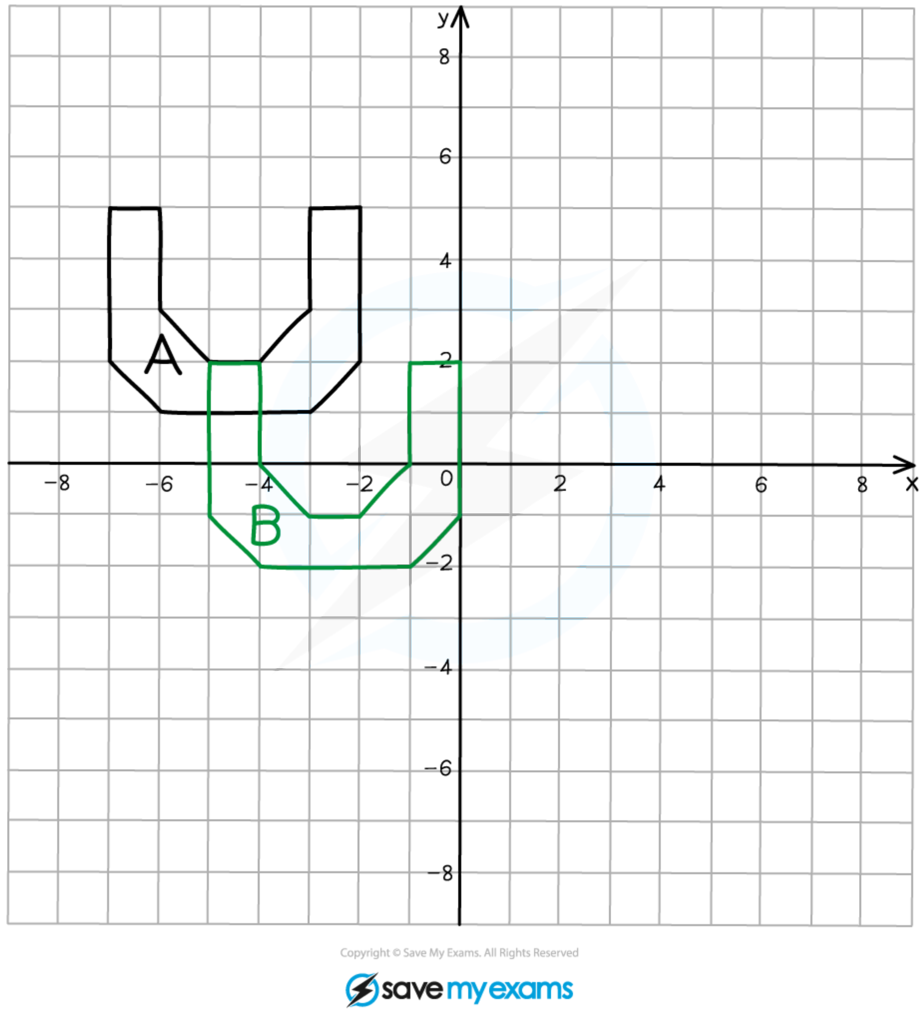
Translating one vertex and then following around the shape one vertex at a time makes it easier to get the shape in exactly the right position!

Your notes



(b) Describe fully the single transformation that creates shape B from shape A.

Your notes



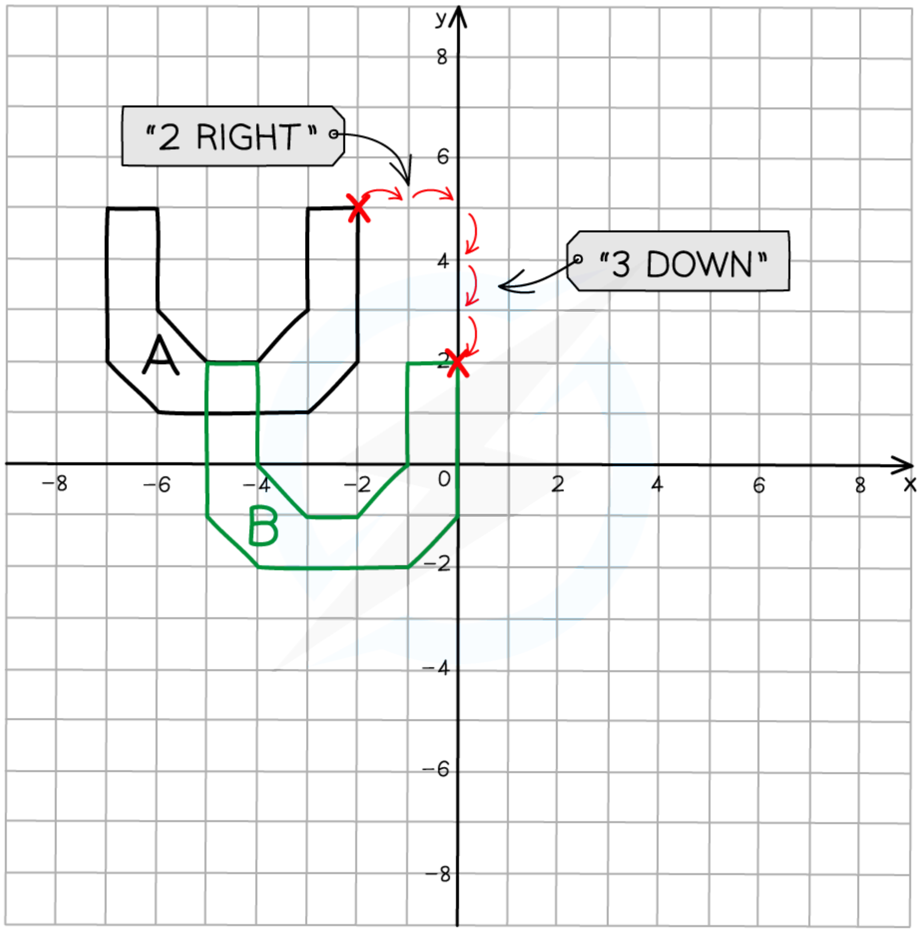
This is a case where the image overlaps the object

You should still see that the shape is the same size and the same way up so it is a **translation**

Start at a vertex on the **original object** that is well away from any overlap area to avoid confusion and count the number of position left/right and up/down that you need to move to reach the **corresponding vertex** on the translated image

Take care when counting around the axes!

Your notes



Shape A has been translated using the vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(c) A shape has been translated from A to B using the translation vector $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$.

Write down the vector used to reverse the translation from B to A.

To reverse the translation the shape must be moved in the opposite direction by the same amount

Reverse the signs in the translation vector

$$\begin{pmatrix} -6 \\ 8 \end{pmatrix}$$

Your notes

Reflections

Your notes

Reflections

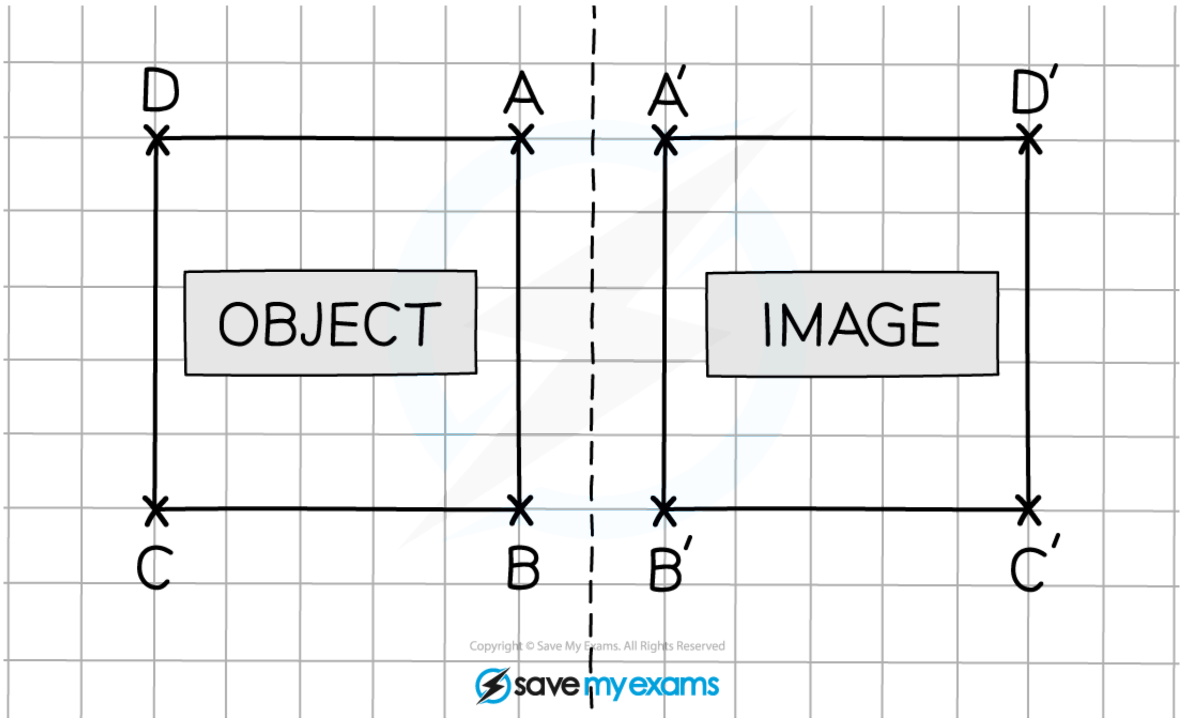
What is a reflection?

- A **reflection flips** a shape across a **mirror line**
  - This is called the **line of reflection**
- The **reflected image** is the **same size** as the **original object**
  - It has been **flipped** across the mirror line to a **new position** and **orientation**
- The following two **distances will be equal** for each point:
  - The **perpendicular distance** between the **original point and the mirror line**
  - The **perpendicular distance** between the **reflected point and the mirror line**
- Any **points** that are **on the mirror line** do not move
  - These are called **invariant points**

How do I reflect a shape?

- STEP 1
  Draw the **line of reflection**
  - This will usually be a **vertical** line ($x = k$) or a **horizontal** line ($y = k$)
  - A **diagonal** line will either be $y = x$ or $y = -x$
- STEP 2
  From **each vertex** on the **original object** measure the **perpendicular distance** to the **mirror line**
  - You can usually do this by **counting squares** on the grid
  - If the line is **diagonal** then count the **diagonals of the squares**
- STEP 3

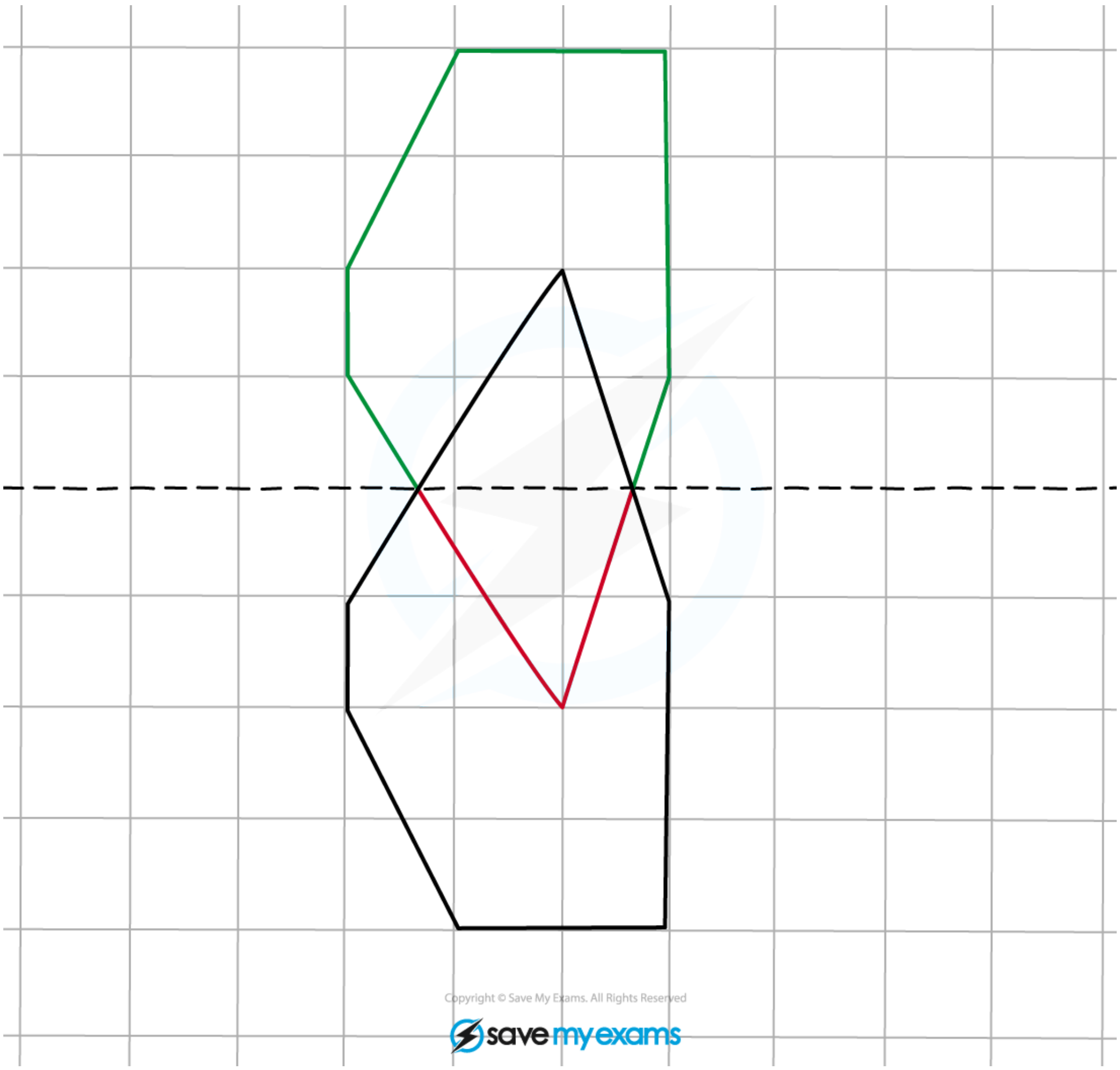
  Find the **reflected** point by **measuring** the **same distance** in the **same direction** from the point on the mirror line
- STEP 4
  Join together the reflected points and label the reflected image



Your notes

How do I reflect a shape when the line of reflection goes through the shape?

- You follow the **same steps** as above
- Part of the shape gets reflected on **one side** of the mirror line, and the other part gets reflected on the other side



Your notes

How do I describe a reflection?

- To describe a **reflection**, you must:
  - State that the transformation is a **reflection**
  - Give the mathematical **equation of the mirror line**
- To find the **equation** of the **reflection line**:
  - **Horizontal** lines are of the form $y = k$

Your notes

- k is the number that the **line passes through on the y-axis**
- **Vertical** lines are of the form $x = k$
  - k is the number that the **line passes through on the x-axis**
- A **diagonal** line with a **positive gradient** will be $y = x$
- A **diagonal** line with a **negative gradient** will be $y = -x$

How do I reverse a reflection?

- If a shape has been reflected to a new position, you can perform a single transformation to **return the shape** to its **original position**
  - You can **reverse the reflection**
- The transformation to **reverse a reflection**, is the **same transformation** as the **original**
  - E.g. If a shape is reflected in the *x*-axis, then reflecting it again in the *x*-axis will return it to its original position

Examiner Tips and Tricks

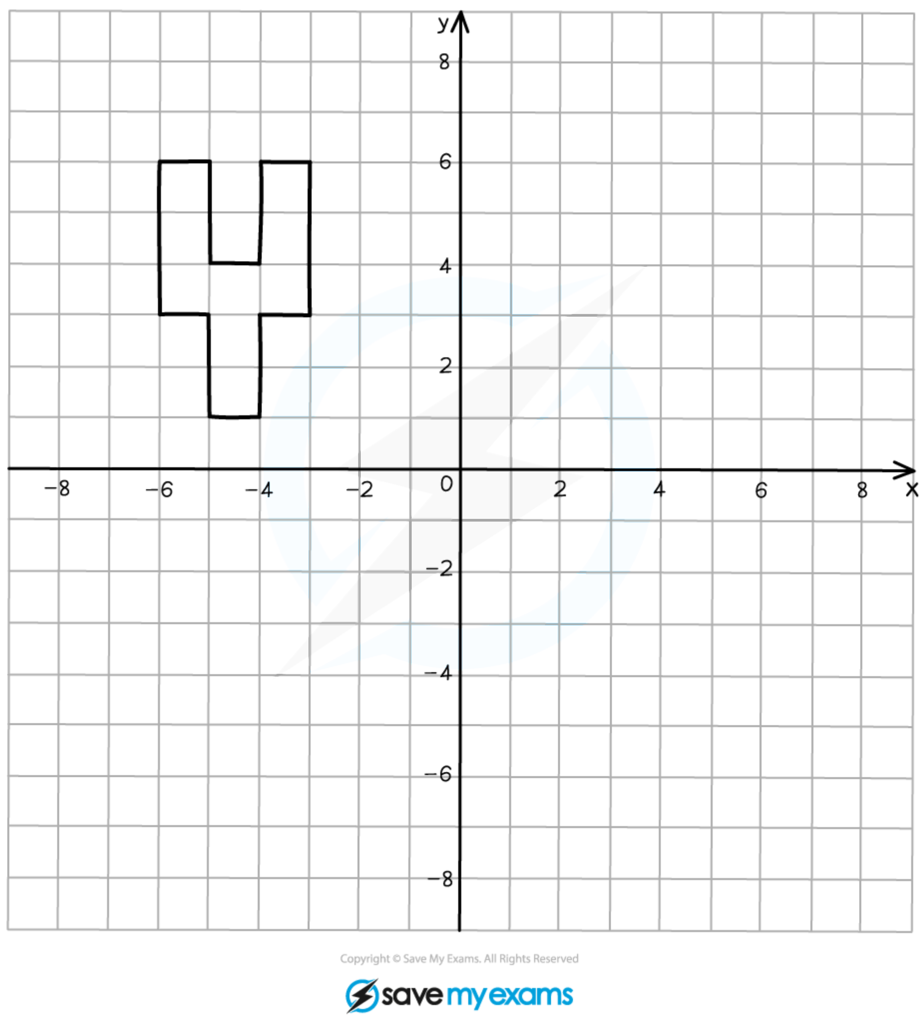
- It is very easy to muddle up the equations for **horizontal** and **vertical** lines, remember:
  - If the line crosses the **x-axis** then it will be $x = k$
  - If the line crosses the **y-axis** then it will be $y = k$
- You can use **tracing paper** to check that your object has remained the same shape

Worked Example

(a) On the grid below, reflect shape S in the line $x = -1$.

State the coordinates of all of the vertices of your reflected shape.

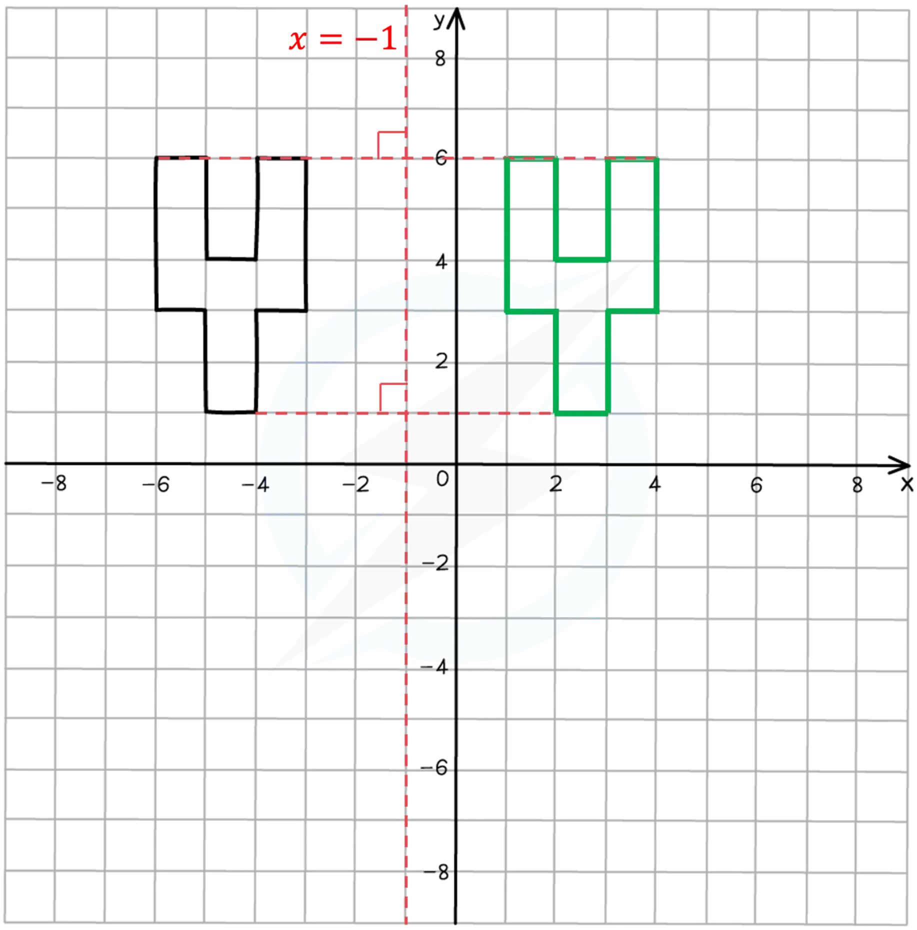
Your notes



Draw in the mirror line; $x = -1$ will be a vertical line passing through -1 on the *x*-axis

Measure or count the number of units from the shape "diagonals" on the other side of the mirror line to find the position of the corresponding vertex on the reflected image

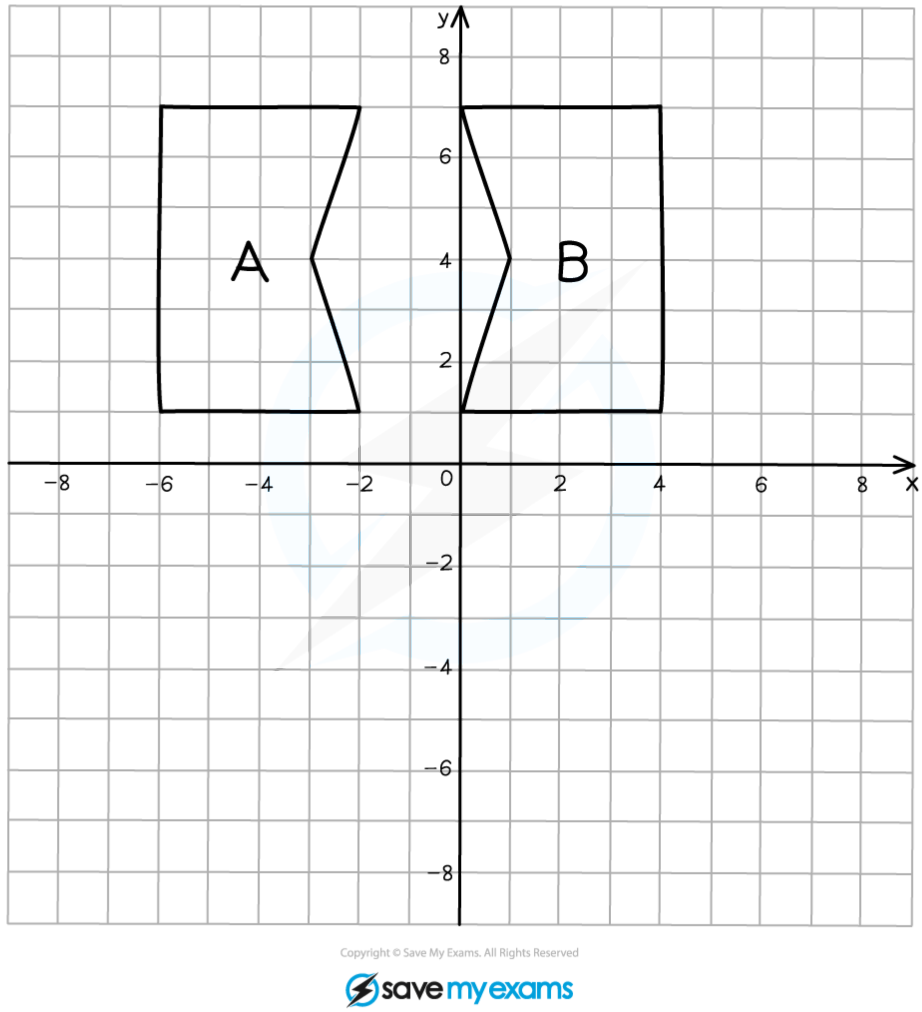
Your notes



List the vertices of the reflected image.
Work your way around the shape vertex by vertex so that you don't miss any out as there are quite a few!

Vertices of the reflected shape: (1, 6), (2, 6), (2, 4), (3, 4), (3, 6), (4, 6), (4, 3), (3, 3), (3, 1), (2, 1), (2, 3), (1,3)

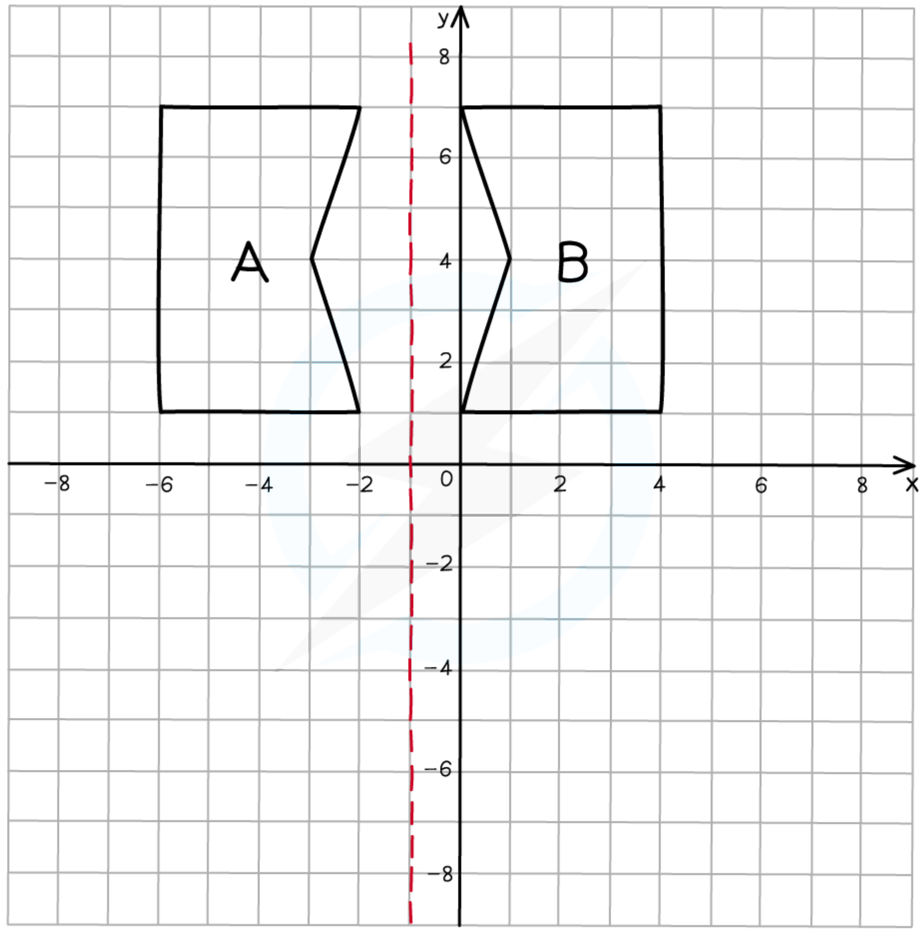
(b) Describe fully the single transformation that creates shape B from shape A.



Your notes

You should be able to "see" where the mirror line should be without too much difficulty.
Draw the mirror line on the diagram.
You can check that it is in the correct position by measuring/counting the perpendicular distance from a pair of corresponding points on the original object and the reflected image to the same point on the mirror line.
Be careful with mirror lines near axes as it is easy to miscount.

Your notes



Write down that the transformation was a reflection and the equation of the mirror line.

Shape A has been reflected in the line $x = -1$ to create shape B

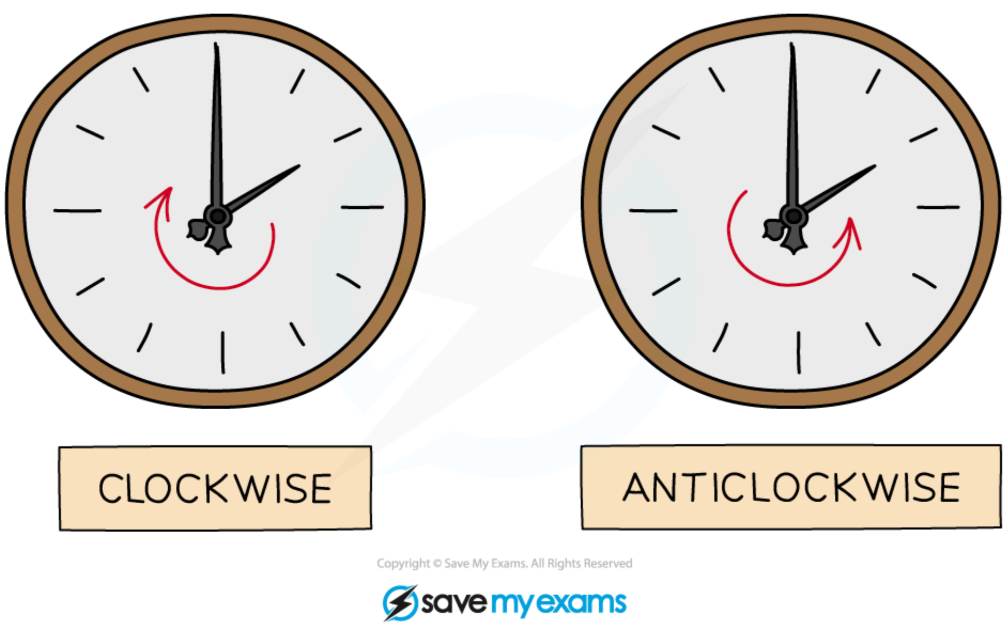
Rotations

Your notes

Rotations

What is a rotation?

- A **rotation turns** a shape **around a point**
  - This is called the **centre of rotation**
- The **rotated image** is the **same size** as the **original image**
  - It will have a **new position and orientation**
- If the centre is a point on the original shape then that point is **not changed** by the rotation
  - It is called an **invariant point**



How do I rotate a shape?

- **STEP 1**

  Place the **tracing paper** over page and draw over the original object

- **STEP 2**

Your notes

Place the point of your pencil on the **centre of rotation**

- **STEP 3**

  Rotate the tracing paper by the **given angle** in the **given direction**

  - The angle will be **90°**, **180°** or **270°**

- **STEP 4**
  Carefully draw the image onto the coordinate grid in the **position shown by the tracing paper**

How do I describe a rotation?

- To describe a **rotation**, you must:
  - State that the transformation is a **rotation**
  - State the **centre of rotation**
  - State the **angle of rotation**
    - This will be 90°, 180° or 270°
  - State the **direction of rotation**
    - Clockwise or anti-clockwise
    - A direction is not required if the angle is 180°
    - 90° clockwise is the same as 270° anti-clockwise
- To find the **centre of rotation**:
  - If the rotation is **90° or 270°**
    - Use **tracing paper** and **start on the original shape**
    - **Try a point** as the centre and rotate the original shape
    - If the **rotated shape matches the image** then that point is the centre
    - Otherwise keep picking points until one works
  - If the rotation is **180°**
    - Draw **lines connecting** each vertex on the **original shape** with the corresponding vertices on the **image**
    - These lines will **intersect** at the centre of rotation

How do I reverse a rotation?

- If a shape has been **rotated** to a new position, you can perform a single transformation to **return the shape** to its **original position**
- A rotation can be **reversed** by simply **reversing the direction** of rotation
  - The angle of rotation is the same
  - The centre of rotation is the same
- For a shape rotated by 45° in a clockwise direction about the point (0, 3)
  - The **reverse transformation** is
    - a rotation of 45°
    - in an anti-clockwise direction
    - about the point (0, 3)

Your notes

Examiner Tips and Tricks

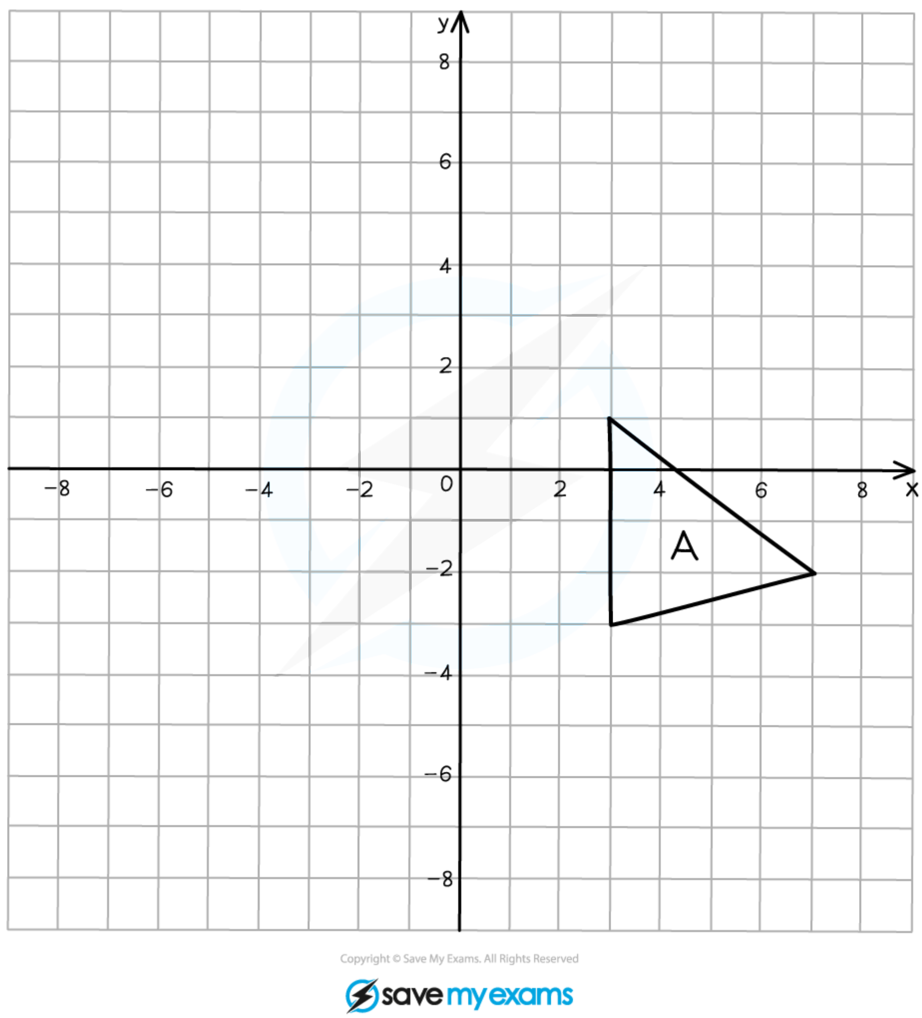
- When you first go into the exam room, make sure there is some **tracing paper** on your desk ready for you
  - If there isn't ask for some before the exam begins
- Draw an **arrow facing up** on your tracing paper
  - The arrow will be **facing left or right** when you have turned **90° or 270°**
  - The arrow will be **facing down** when you have turned **180°**
- **Double-check** that you have copied the rotated image into the correct position
  - Put the tracing paper over the original object and rotate it again to see that it lines up with your image

Worked Example

(a) On the grid below rotate shape A by 90° anti-clockwise about the point (0, 2).

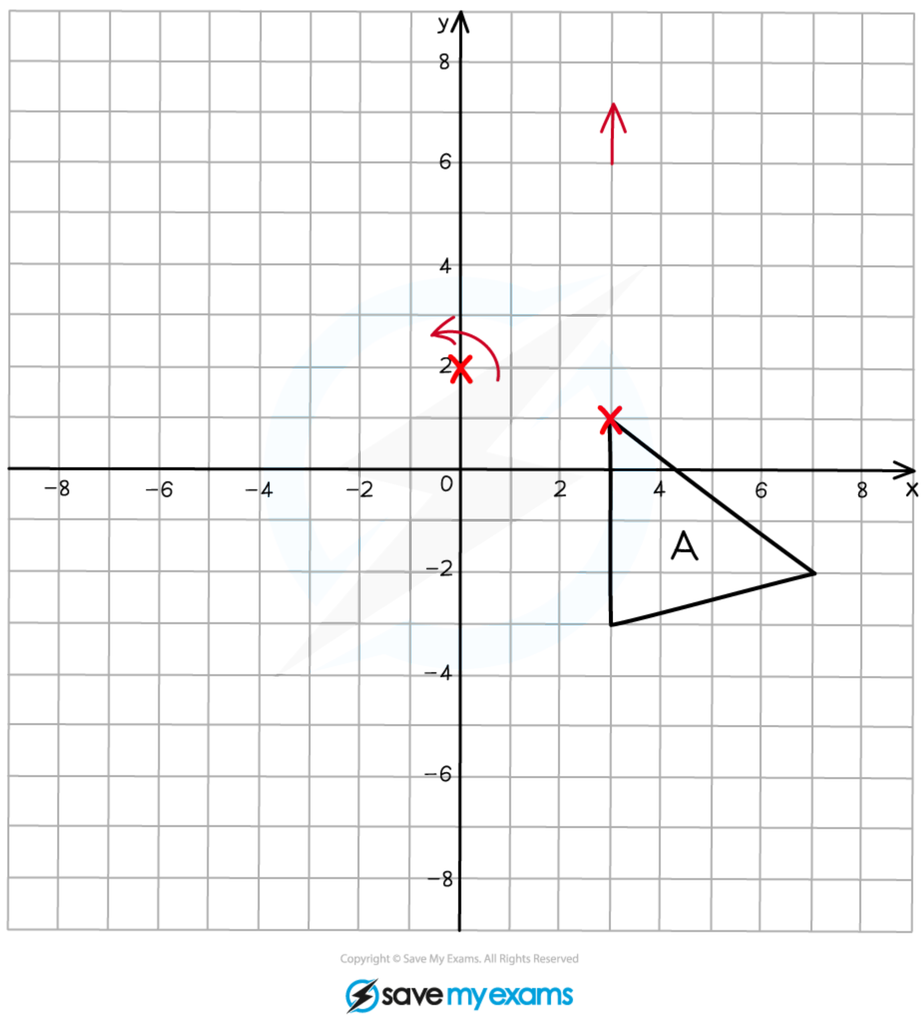
Label your answer A'.

Your notes



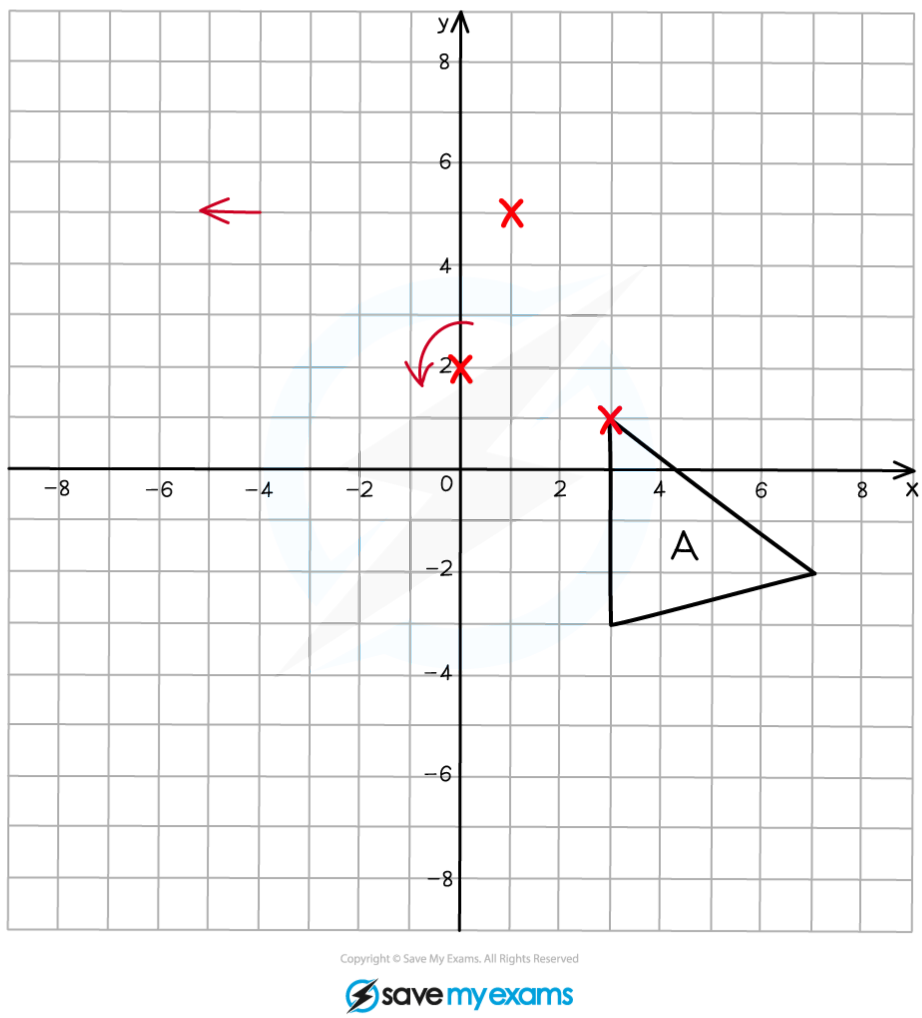
Using tracing paper, draw over the original object and mark one vertex.
Mark on the centre of rotation.
Draw an arrow pointing vertically upwards on the paper.

Your notes



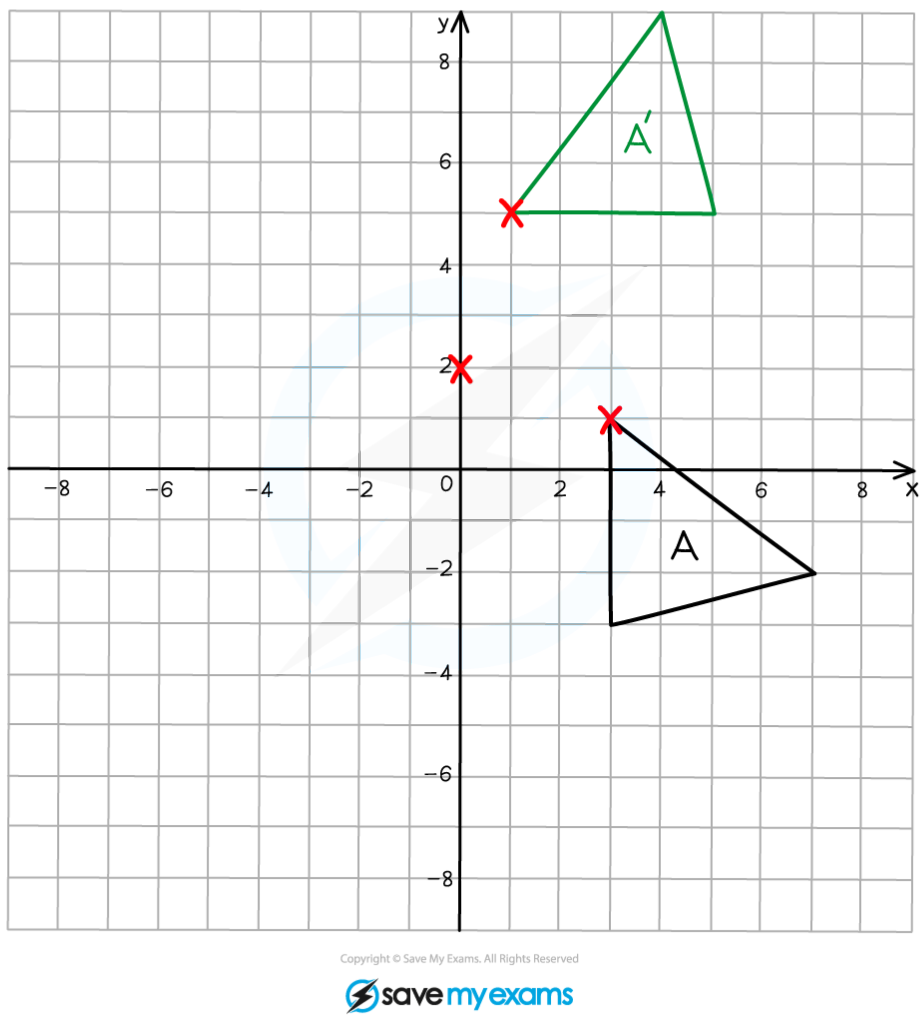
With your pencil fixed on the point of rotation, rotate the tracing paper 90° anti-clockwise, the arrow that you drew should now be pointing left.
Make a mental note of the new coordinates of the vertex that you marked on your tracing paper.
Draw the new position of this vertex onto the grid.

Your notes



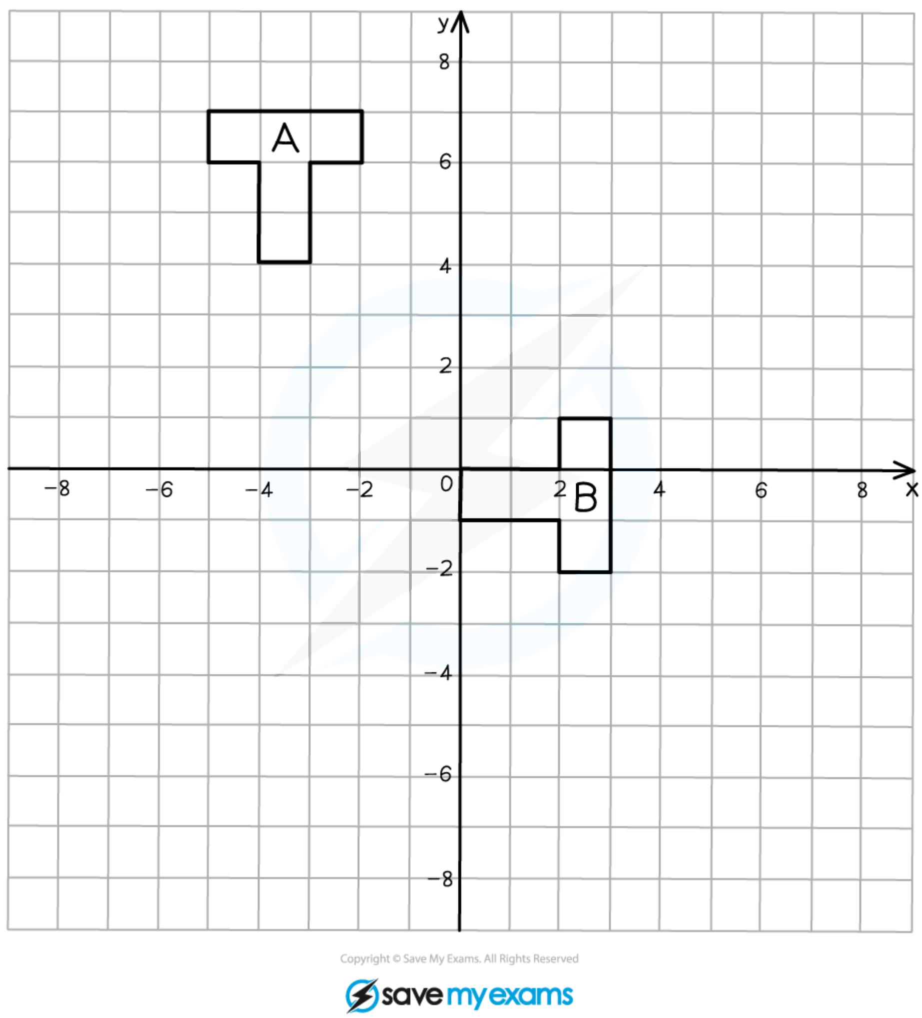
Repeat this process for the other two vertices on the triangle.
Connect the vertices together to draw the rotated image.

Your notes

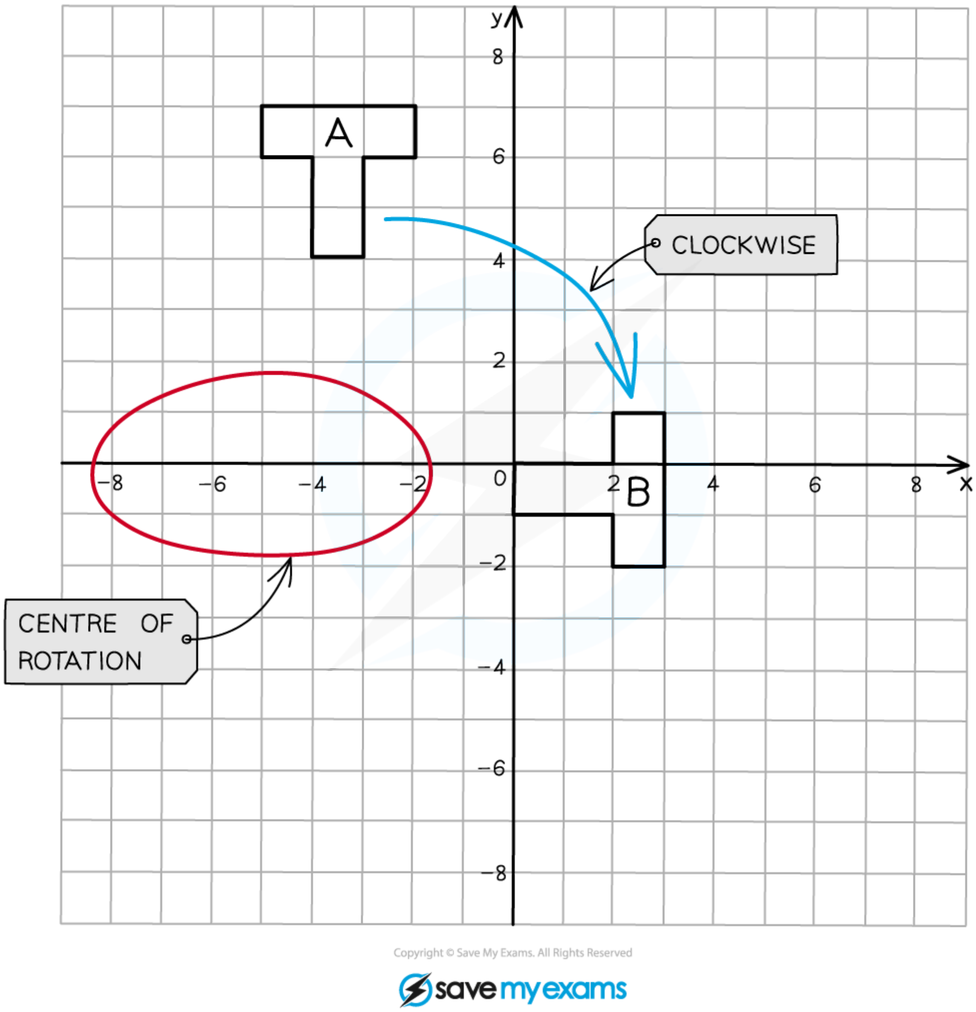


(b) Describe fully the single transformation that creates shape B from shape A.

Your notes



You should be able to see that the object has been rotated 90° clockwise (or 270° anti-clockwise). You are likely to be able to see roughly where the centre of rotation is but it may take a little time to find its position exactly.

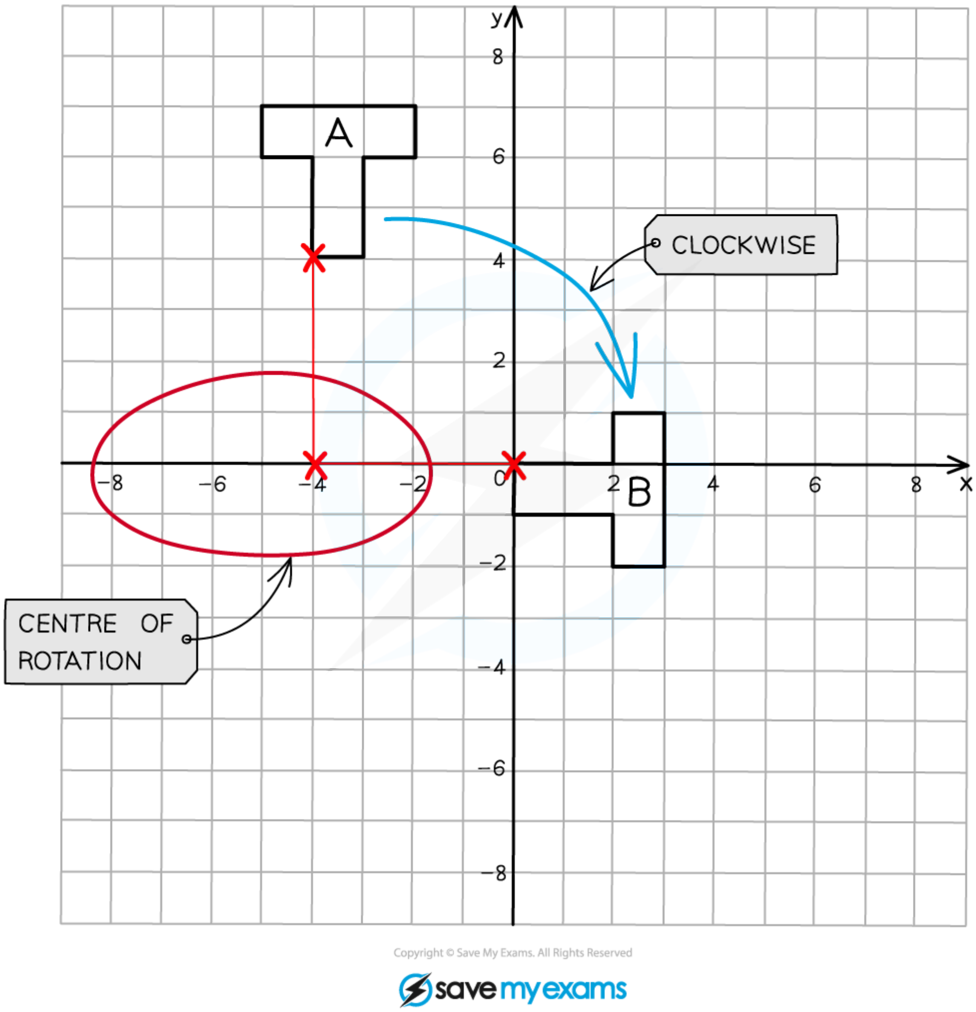


Your notes

To find the exact coordinates of the centre of rotation you can play around with tracing paper.

Draw over shape A on tracing paper, then try out different locations for the centre of enlargement by placing your pencil on a point, rotating the paper 90° clockwise and seeing if it lines up with shape B.

Your notes



Write down the all of the elements required to fully describe the transformation: the type of transformation, the centre of rotation, the angle and the direction.

Rotation, 90° clockwise with centre (-4, 0)

Your notes

Enlargements

Enlargements

What is an enlargement?

- An **enlargement changes** the **size** and **position** of a shape
- The **length of each side** of the shape is **multiplied** by a **scale factor**
  - If the scale factor is **greater than 1** then the **enlarged image** will be **bigger** than the **original object**
  - If the scale factor is **between 0 and 1 (fractional)** then the **enlarged image** will be **smaller** than the **original object**
- The **centre of enlargement** determines the **position** of the enlarged image
  - If the scale factor is **greater than 1** then the **enlarged image** will be **further away** from the centre of enlargement
  - If the scale factor is **between 0 and 1** then the **enlarged image** will be **closer to** the centre of enlargement

How do I enlarge a shape?

- **STEP 1**

  Pick a **vertex** of the shape and count the **horizontal and vertical distances** from the **centre of enlargement**
- **STEP 2**
  Multiply **both** the horizontal and vertical distances by the given **scale factor**
- **STEP 3**
  Start at the centre of enlargement and **measure the new distances** to find the **enlarged vertex**
- **STEP 4**
  Repeat the steps for the **other vertices**
  - You might be able to draw the enlarged shape from the first vertex by **multiplying the original lengths** by the scale factor
    - This can be done **quickly** if the shape is made up of **vertical and horizontal lines**
- **STEP 5**
  Connect the vertices on the **enlarged image** and label it

How do I describe an enlargement?

Your notes

- To describe an **enlargement**, you must:
  - State that the transformation is an **enlargement**
  - State the **scale factor**
    - This may be an integer or a fraction
  - Give the coordinates of the **centre of enlargement**
- To find the **scale factor**:
  - **Pick a side** of the **original shape**
  - Identify the **corresponding side** on the **enlarged image**
    - For a fractional enlargement, the side on the enlarged image will be **smaller** than the corresponding side on the original image
  - **Divide** the **length of the enlarged side** by the **length of the original side**
- To find the **centre of enlargement**:
  - **Pick a vertex** of the **original shape**
  - Identify the **corresponding vertex** on the **enlarged image**
  - Draw a **line going through these two vertices**
  - **Repeat** this for the **other vertices** of the original shape
  - These lines will **intersect** at the centre of **enlargement**

How do I reverse an enlargement?

- If a shape has been **enlarged**, you can perform a single transformation to **return the shape** to its **original size and position**
- An enlargement can be **reversed** by multiplying the enlarged shape by the **reciprocal of the original scale factor**
  - The centre of enlargement is the same
- For a shape enlarged by a scale factor of 3 with centre of enlargement (-1, 6)
  - The **reverse transformation** is
    - an enlargement of scale factor $\frac{1}{3}$
    - with centre of enlargement (-1, 6)

Your notes

Examiner Tips and Tricks

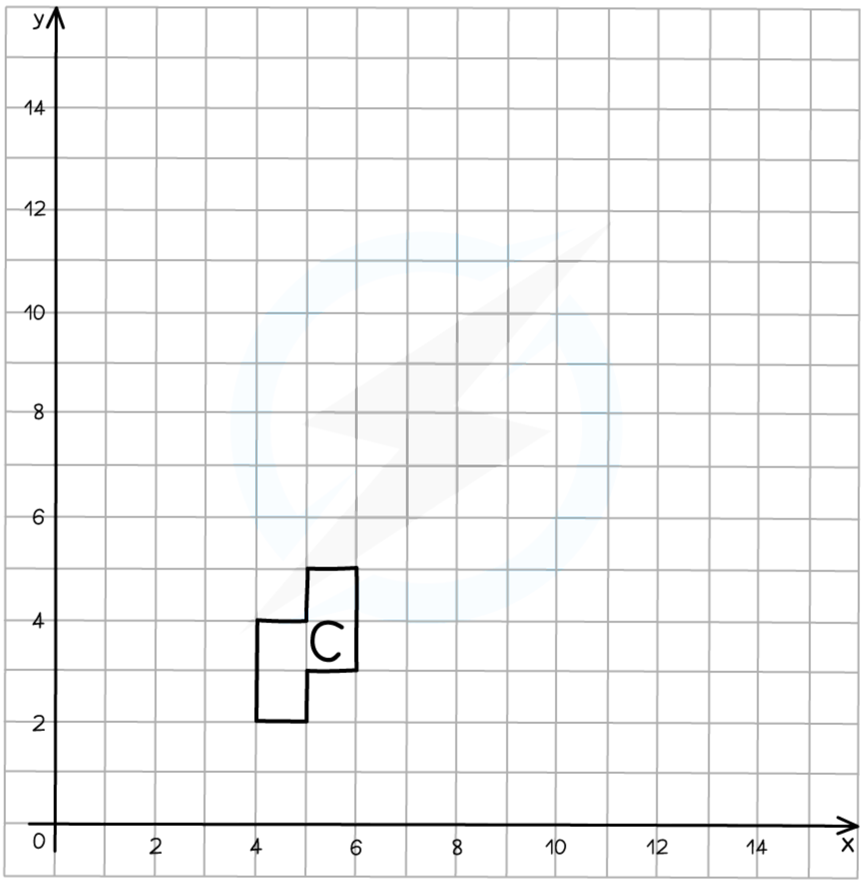
- To **check** that you have enlarged a shape **correctly**:
  - **Draw lines** going **from the centre of enlargement** to **each of the vertices** of the original shape
  - **Extend** these lines
  - The **lines should go through** the **corresponding vertices** of the enlarged image

Worked Example

(a) On the grid below enlarge shape C using scale factor 2 and centre of enlargement (2, 1).

Label your enlarged shape C'.

Your notes



Start by marking on the centre of enlargement (CoE)

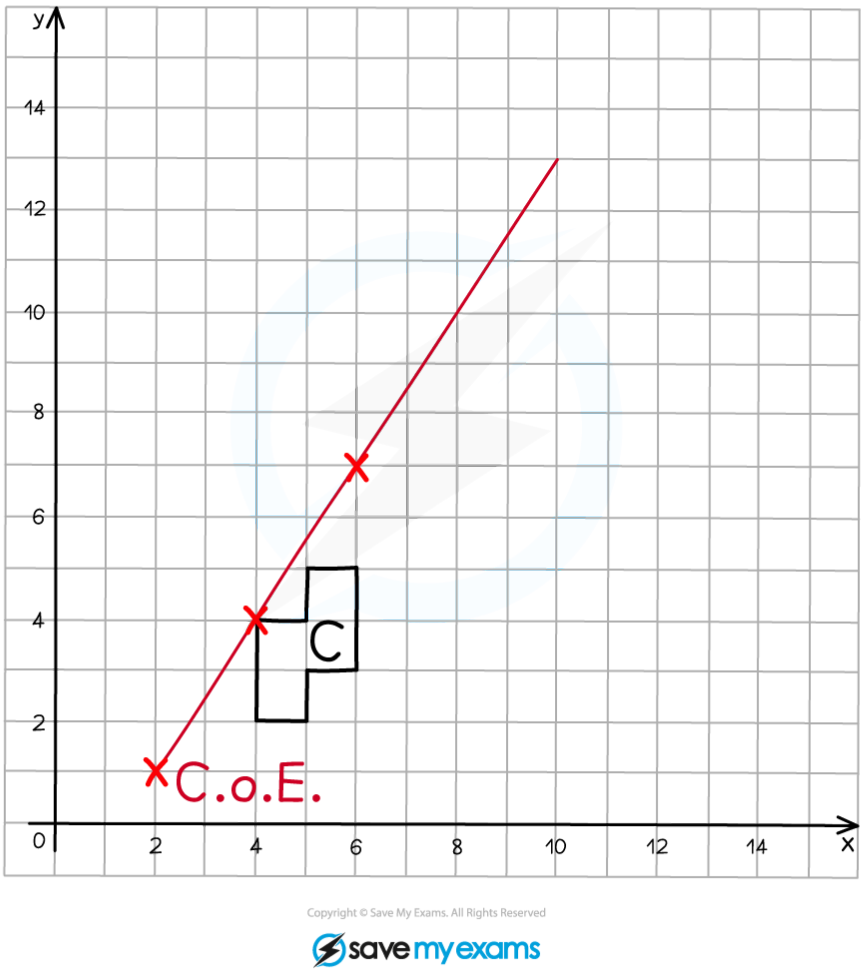
Count the number of squares in both a horizontal and vertical direction to go from the CoE to one of the vertices on the original object, this is 2 to the right and 3 up in this example

As the scale factor is 2, multiply these distances by 2, so they become 4 to the right and 6 up

Count these new distances from the CoE to the corresponding point on the enlarged image and mark it on

Draw a line through the CoE and the pair of corresponding points, they should line up in a straight line

Your notes

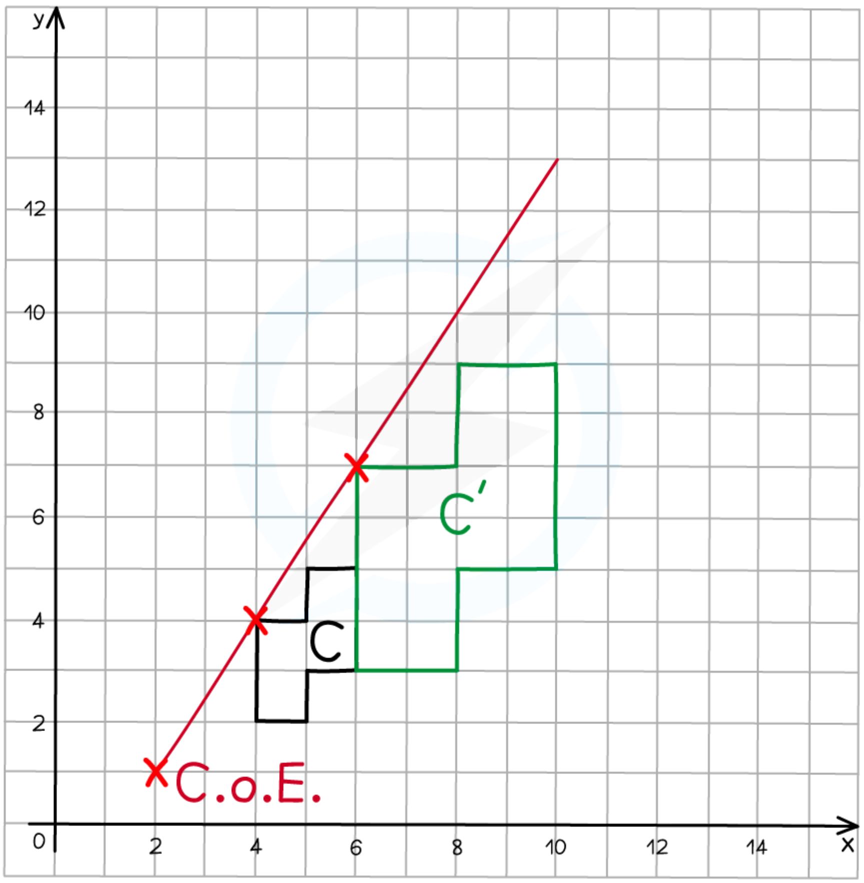


Repeat this process for each of the vertices on the original object (or at least 2)

Join adjacent vertices on the enlarged image as you go

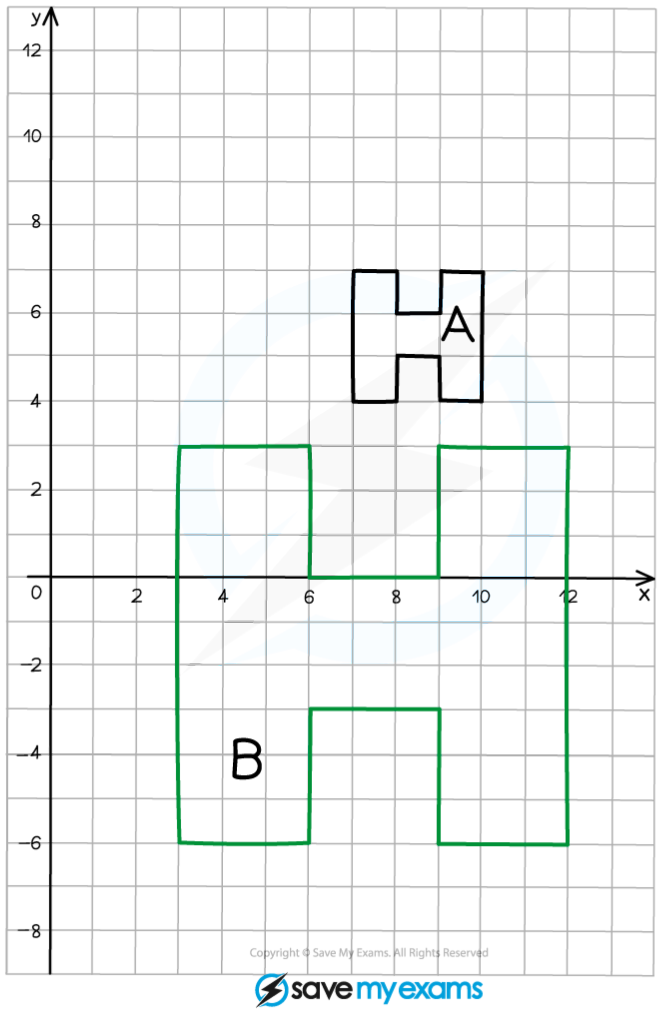
Label the enlarged image C'

Your notes



(b) Describe fully the single transformation that creates shape B from shape A.

Your notes



We can see that the image is larger than the original object, therefore it must be an **enlargement**

As the enlarged image is bigger than the original object, the scale factor must be greater than 1

Compare two corresponding edges on the object and the image to find the scale factor

The height of the original "H" is 3 squares
The height of the enlarged "H" is 9 squares

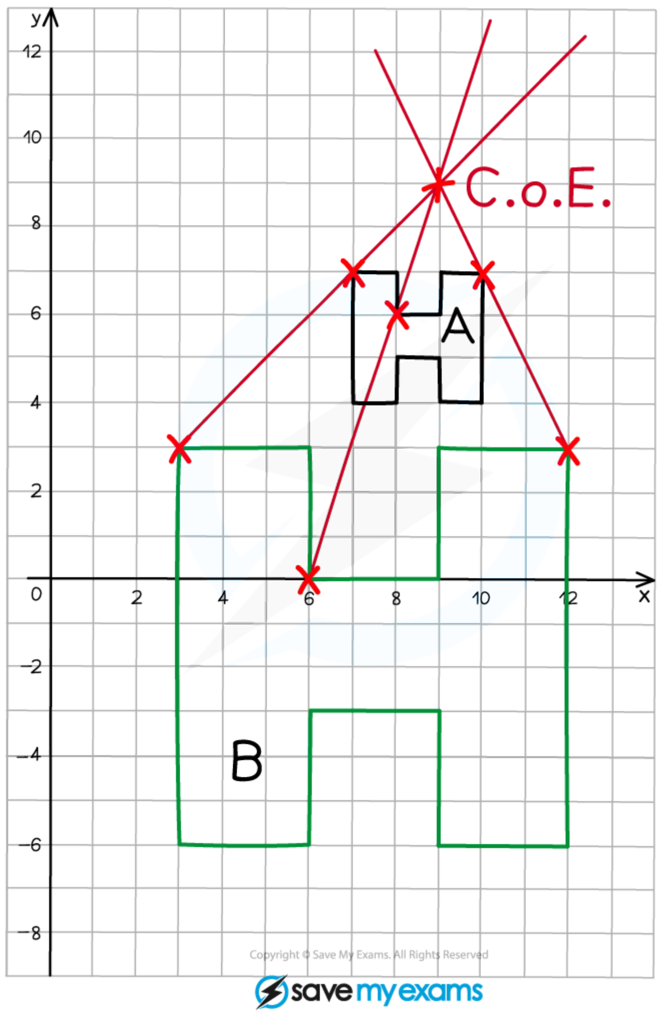
$$\therefore \text{Scale Factor} = \frac{9}{3} = 3$$

Draw a straight line through the CoE and a pair of corresponding points on the original object and the enlarged image

Your notes

Repeat this step for as many vertices as you feel you need to so you can confidently locate the CoE
Do this for all pairs of vertices to be sure!

The point of intersection of the lines is the CoE



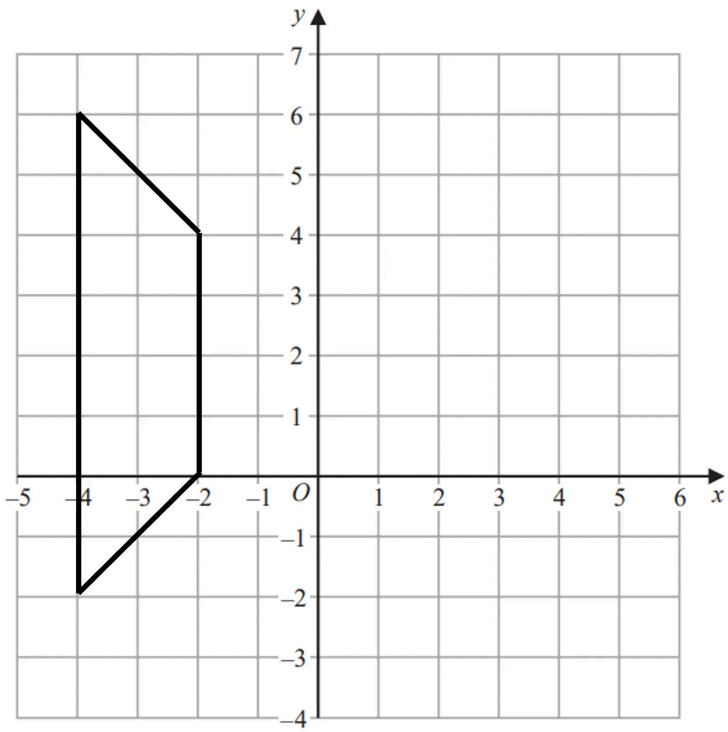
Shape A has been enlarged using a scale factor of 3 and a centre of enlargement (9, 9) to create shape B

Worked Example

Your notes

(a) On the grid below enlarge shape C using scale factor $\frac{1}{2}$ and centre of enlargement (4, 2).

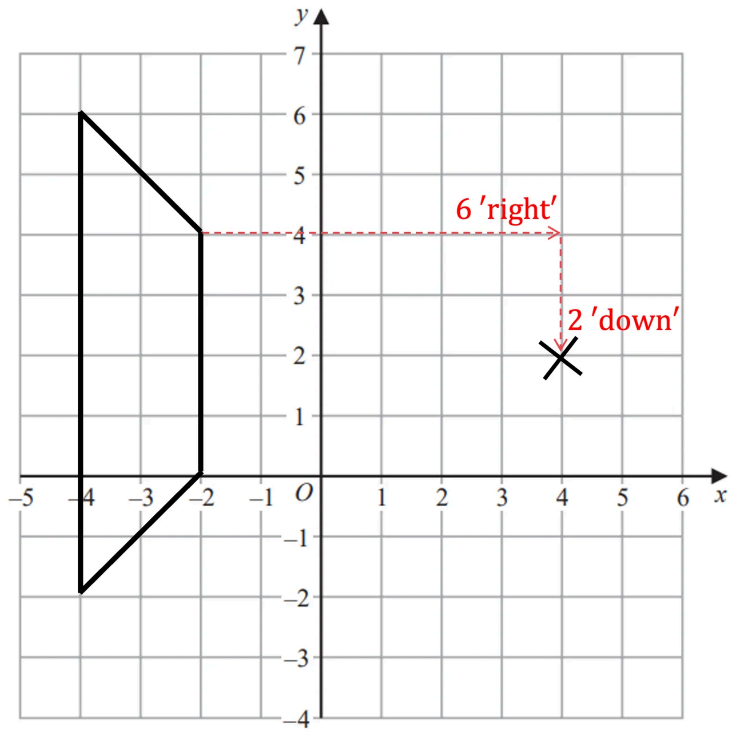
Write down the four vertices of your enlarged shape.



Mark the centre of enlargement at (4, 2)

Count the number of squares horizontally and vertically to any vertex - we've chosen the vertex at (-2, 4)

Your notes



Multiply these distance by the scale factor, $\frac{1}{2}$

$$6 \text{ 'right} \times \frac{1}{2} = 3 \text{ 'right'}$$

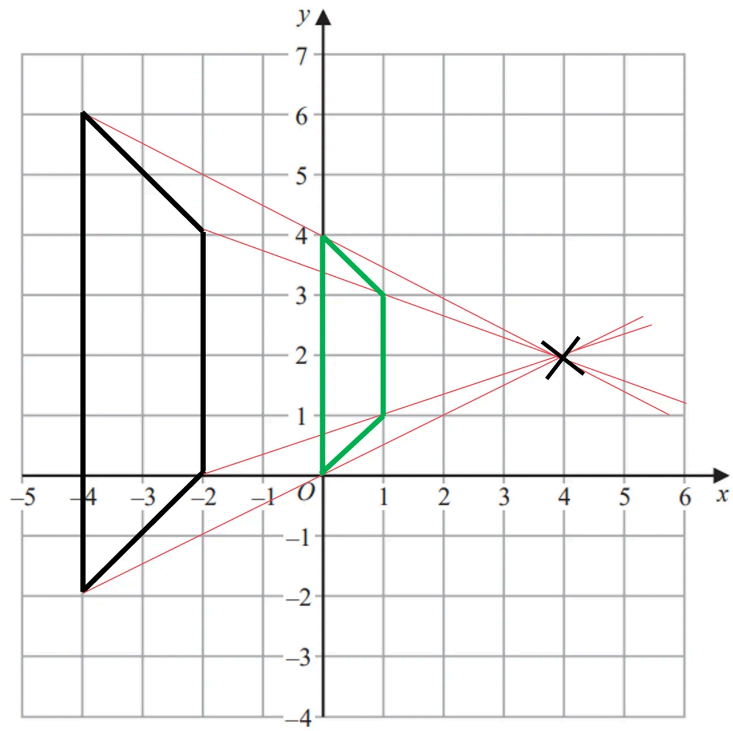
$$2 \text{ 'down'} \times \frac{1}{2} = 1 \text{ 'down'}$$

Count these new distances (which should be smaller than the originals) from the CoE to find the corresponding point on the new image and mark it on

Repeat as required and draw lines through corresponding vertices and the CoE as a check

Use a logical order, working your way round the shape slowly, to ensure you do miss any vertices out

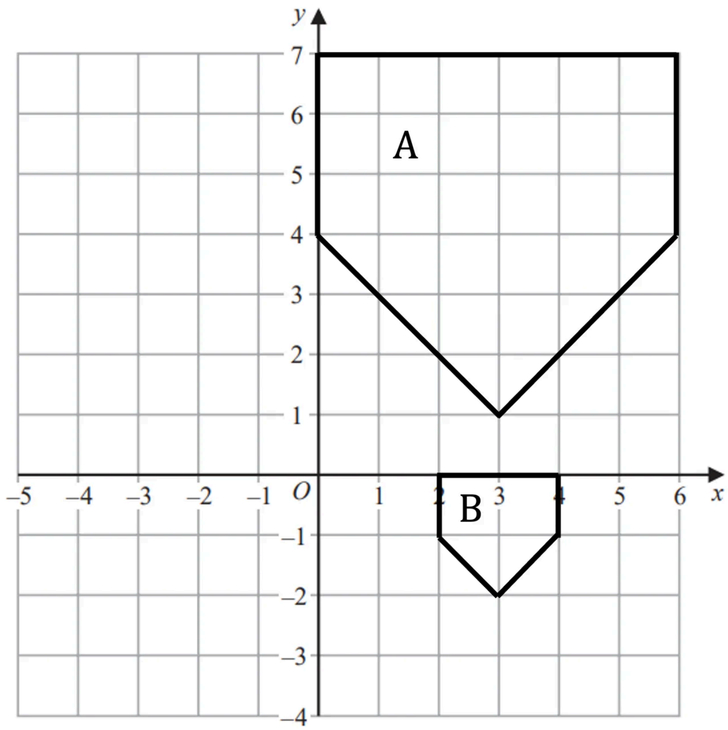
Your notes



The four vertices of the enlarged shape are (0, 0), (0, 4), (1, 3) and (1, 1)

(b) Describe fully the single transformation that creates shape B from shape A.

Your notes



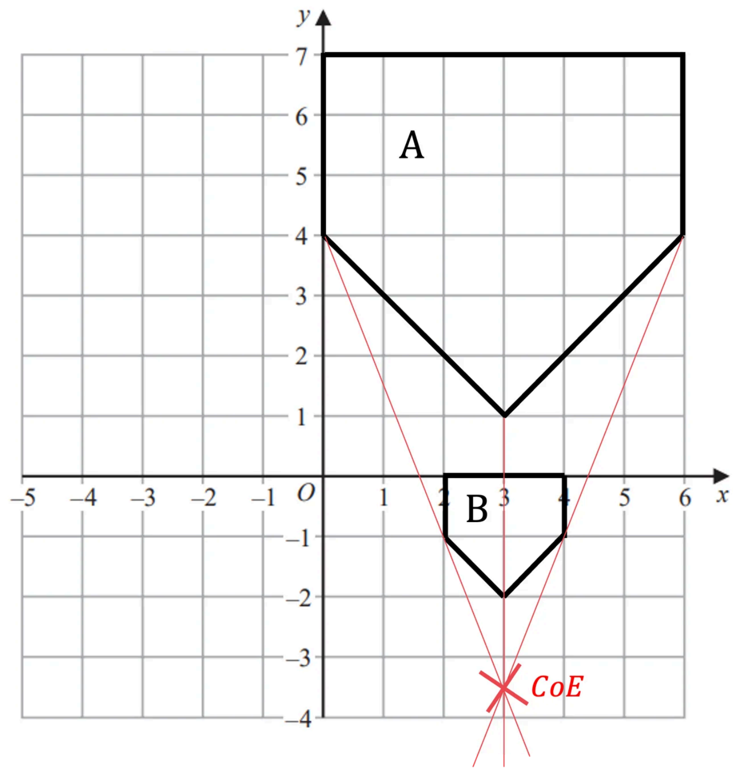
We can see the image is smaller than the original so it is a **fractional enlargement**

Compare two corresponding edges to find the scale factor - we've used the top edge

scale factor = $\frac{\text{new edge}}{\text{old edge}} = \frac{2}{6} = \frac{1}{3}$

Draw straight lines through corresponding vertices on the original shape
Repeat this 3–4 times and you should find the lines intersect at the same point
This point will be the CoE

Your notes



Shape A has been enlarged using a scale factor of $\frac{1}{3}$ and a centre of enlargement (3, -3.5) to create shape B

Negative Enlargements

How do I enlarge a shape if it has a negative scale factor?

- You will still need to **perform enlargements** with **negative scale factors**
  - it is possible but unusual to be asked to **identify** one
- Follow the **same process** as you would for a positive scale factor enlargement, the key things to look out for are:
  - the **orientation** of the object is changed, it is **rotated by 180°**
  - the **distance** between the **centre of enlargement (CoE) and the enlarged image,** is measured on the **opposite side of the CoE**

How do I reverse a negative enlargement?

Your notes

- If a shape has undergone a **negative enlargement**, you can perform a single transformation to **return the shape** to its **original size and position**
- An enlargement can be **reversed** by multiplying the enlarged shape by the **reciprocal of the original scale factor**
  - the sign of the scale factor remains negative
  - the centre of enlargement is the same
- For a shape enlarged by a scale factor of $-\frac{1}{2}$ with centre of enlargement (0, 5)
  - The **reverse transformation** is
    - an enlargement of scale factor -2
    - with centre of enlargement (0, 5)

Examiner Tips and Tricks

- Draw lines **through the CoE** and a **vertex on the original object**
  - this will remind you that the distances away from the CoE carry on in the **opposite direction** for a negative scale factor
- Watch out, exam questions are quite keen on **combining** both **negative and fractional** scale factors!

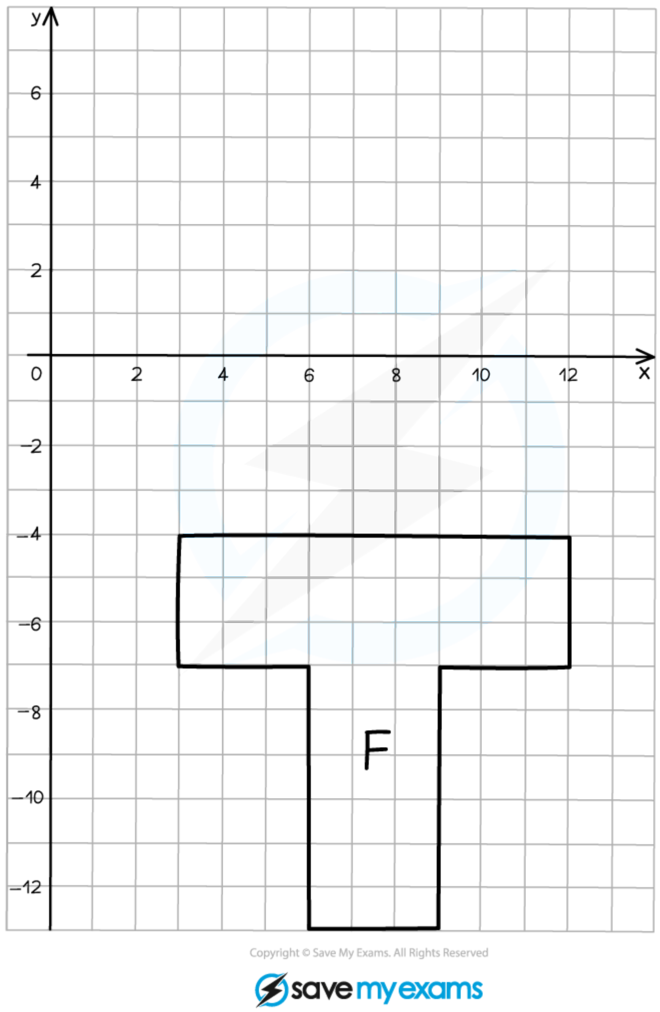
Worked Example

On the grid below enlarge shape F using scale factor $-\frac{1}{3}$ and centre of enlargement $(6, -1)$.

Label this shape F'.

If the area of F is 45 cm^2 write down the area of F'.

Your notes



Start by marking the centre of enlargement (CoE) (6, -1) and selecting a starting vertex

Count the horizontal and vertical distances from the vertex to the CoE
Multiply those distances by the scale factor

Vertex at (-4, 3)

Distance to CoE from vertex on original object: 3 to the right and 3 up

Distances from CoE to corresponding vertex on enlarged image: $3 \times \frac{1}{3} = 1$ to the right and

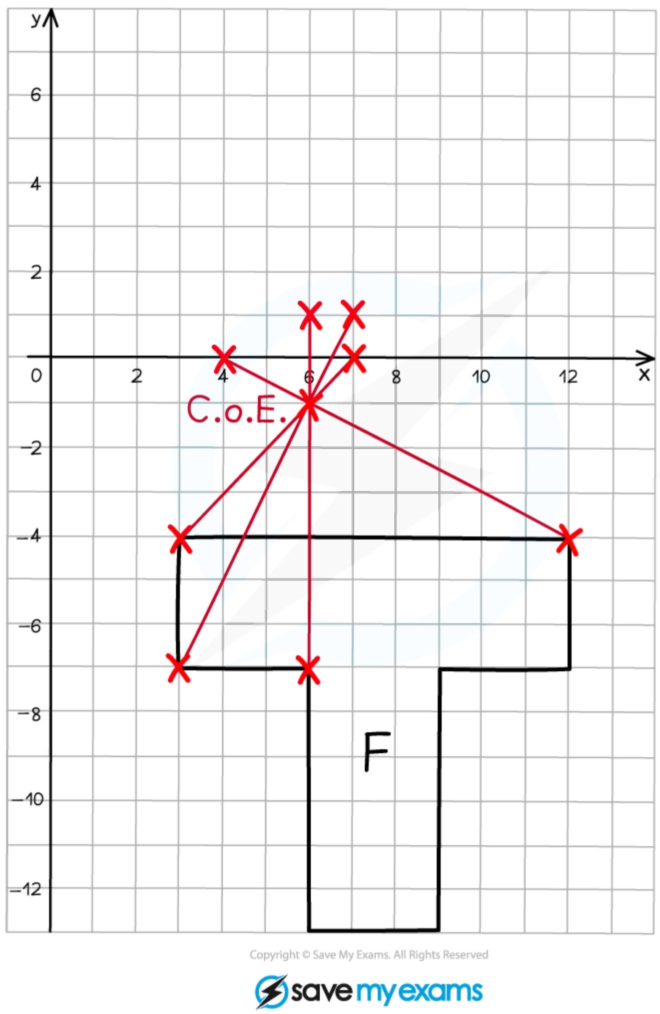
$3 \times \frac{1}{3} = 1$ up

Your notes

Count the new distances from the CoE, on the other side from the original object
Mark on the position of the corresponding point on the enlarged image

Draw a straight line through the corresponding vertices and the CoE to check that they line up

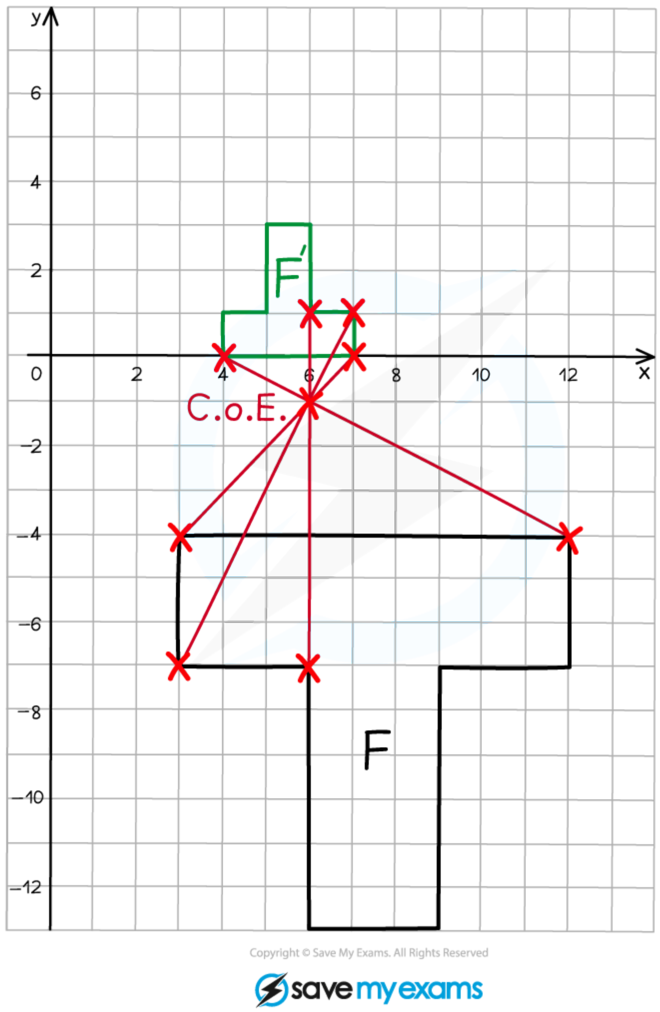
Repeat this process for each vertex in turn



Connect the vertices as you go around so that you don't forget which should connect to which

Remember, your enlarged image will be rotated by 180°

Your notes



The length scale factor is $\frac{1}{3}$, meaning that each edge of the enlarged image is $\frac{1}{3}$ the length of the corresponding edge on the original object

Find the area scale factor by squaring the length scale factor

$$\text{Area scale factor } = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

Multiply the area of the original object by the area scale factor to find the area of the enlarged image

Your notes

$$45 \times \frac{1}{9}$$

5 cm^2

Combination of Transformations

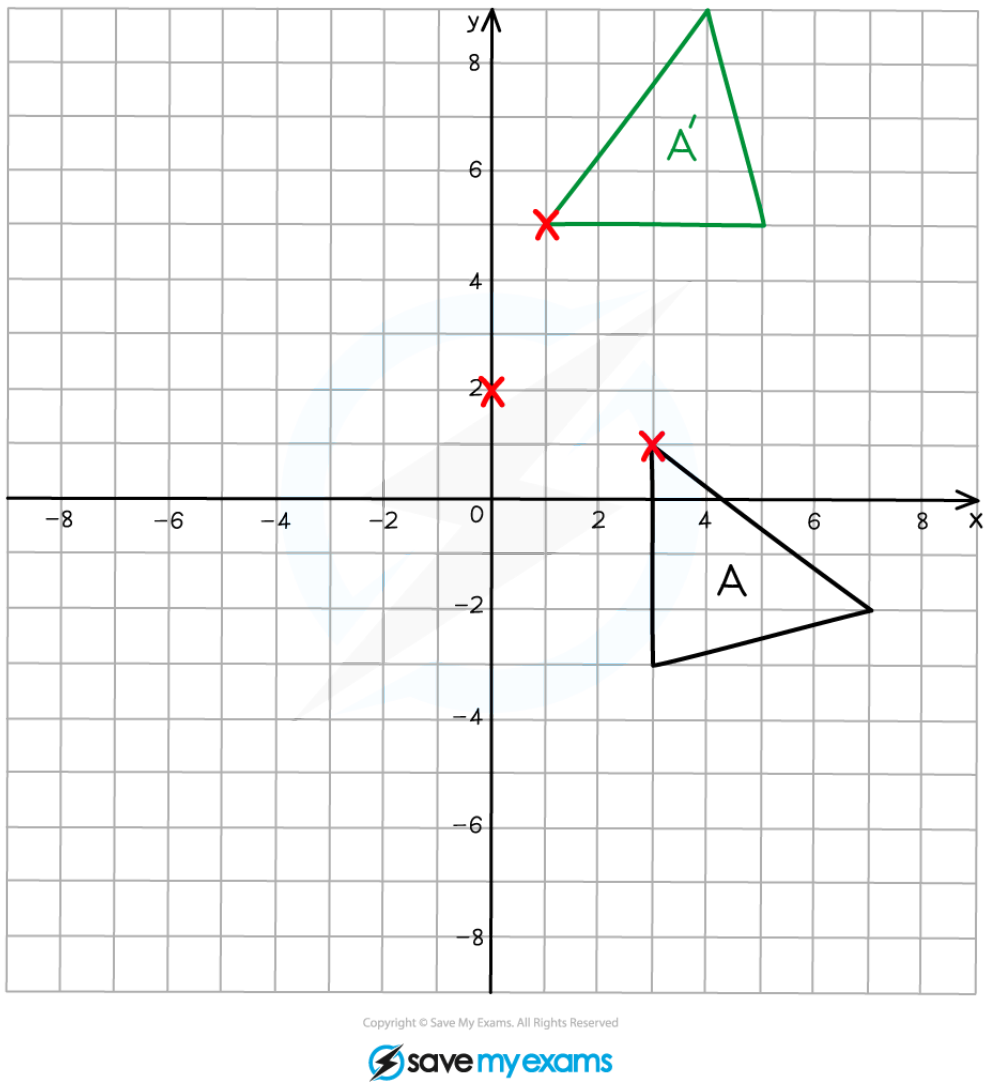
Your notes

Combination of Transformations

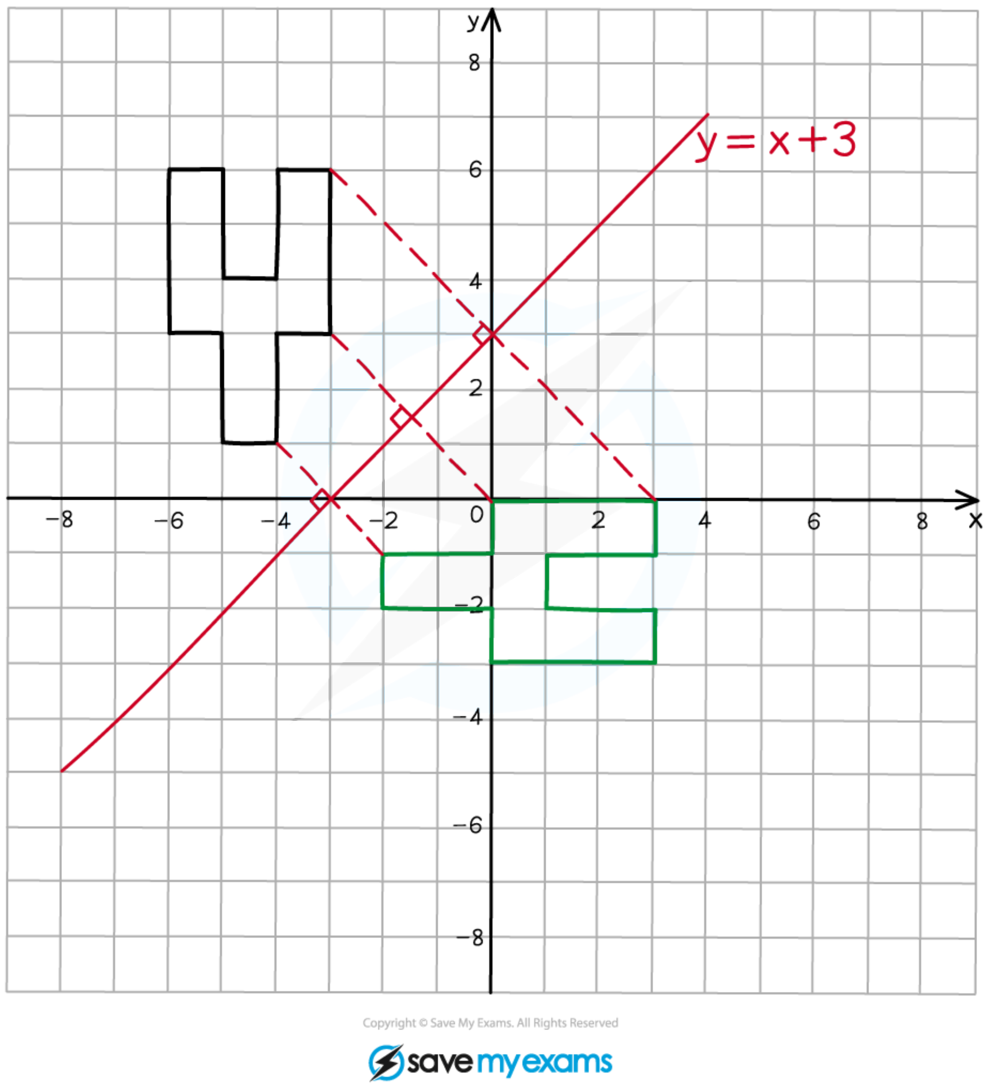
What do I need to know about combined transformations?

- **Combined transformations** are when more than one transformation is performed, one after the other
- In many cases, two transformations can be equivalent to one alternative **single** transformation
  - Finding this single transformation is a common exam question
- **Rotation**
  - Requires an angle, direction and centre of rotation
  - It is usually easy to tell the angle from the orientation of the image
  - You can use trial and error and tracing paper to find the centre of enlargement

Your notes



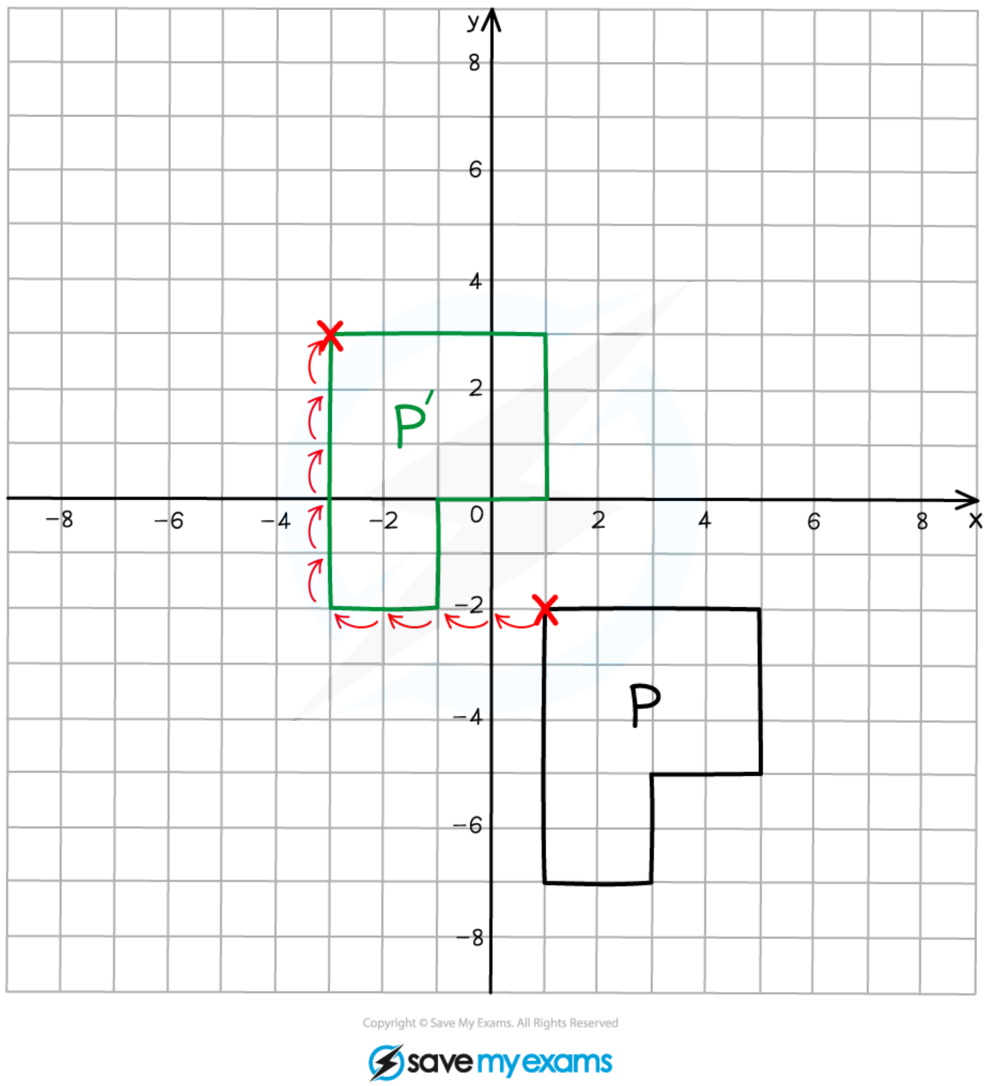
- **Reflection**
  - A reflection will be in a mirror line which can be vertical (x = k), horizontal (y = k) or diagonal (y = mx + c)
  - Points on the mirror line do not move
  - It is possible for a mirror line to pass through the object



Your notes

- **Translation**
  - A translation is a movement which does not change the orientation or size of the shape, it simply moves location
  - A translation is described by a vector in the form $\begin{pmatrix} x \\ y \end{pmatrix}$
    - This represents a movement of *x* units to the right and *y* units vertically upwards

Your notes



What are common combinations of transformations?

- A combination of **two reflections** can be the same as a **single rotation**
  - One reflection using the line $x = a$ and the other using the line $y = b$
  - This is the same as a **180° rotation** about the centre $(a,\ b)$
- The **order of the combination** can be important to the **overall effect**

Your notes

- A reflection in the line ***y* = *x* followed by** a reflection in the ***x*-axis** is the same as a 90° rotation **clockwise** about the origin
- A reflection in the ***x*-axis followed by** a reflection in the line ***y* = *x*** is the same as a 90° rotation **anticlockwise** about the origin

How do I undo a transformation to get back to the original shape?

- After transforming shape A to make shape B you could be asked to describe the transformation that maps B to A

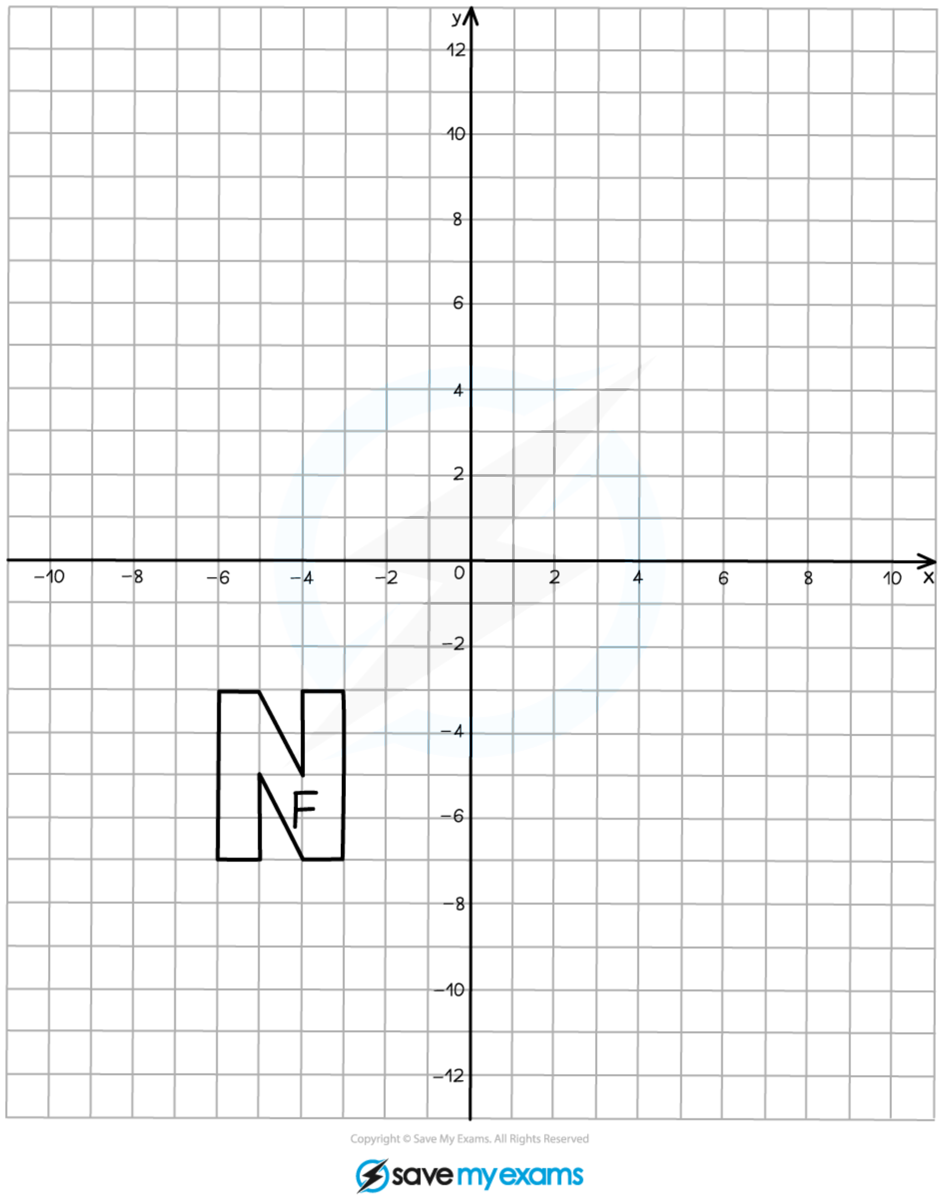
Transformation from A to B	Transformation from B to A
Translation by vector $\begin{pmatrix} x \\ y \end{pmatrix}$	Translation by vector $\begin{pmatrix} -x \\ -y \end{pmatrix}$
Reflection in a given line	Reflection in the same line
Rotation by θ° in a direction about the centre (x, y)	Rotation by θ° in the opposite direction about the centre (x, y)
Enlargement of scale factor k about the centre (x, y)	Enlargement of scale factor $\frac{1}{k}$ about the centre (x, y)

Worked Example

(a) On the grid below rotate shape F by 180° using the origin as the centre of rotation.

Label this shape F'.

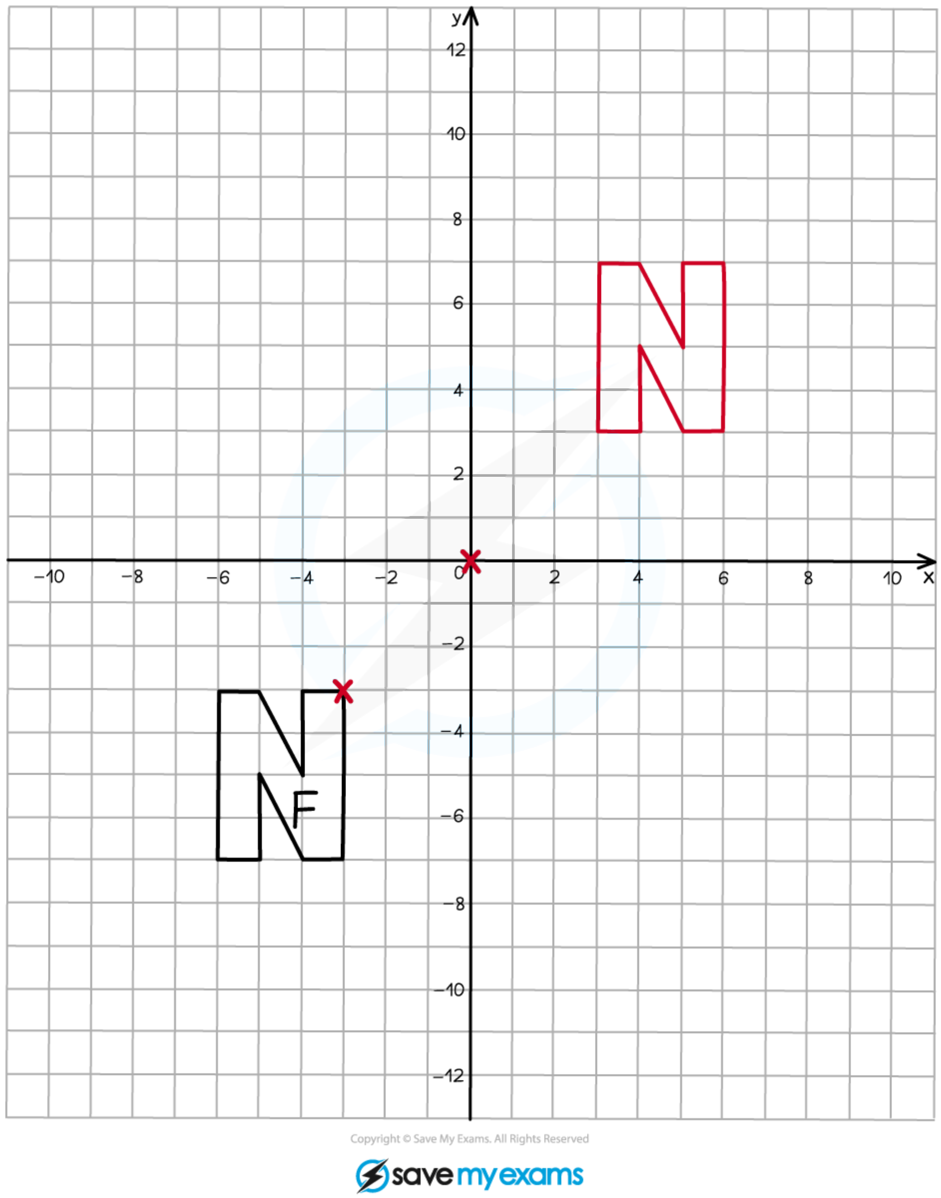
Your notes



Using tracing paper, draw over the original object then place your pencil on the origin and rotate the tracing paper by 180°
Mark the position of the rotated image onto the coordinate grid

Label the rotated image F'

Your notes



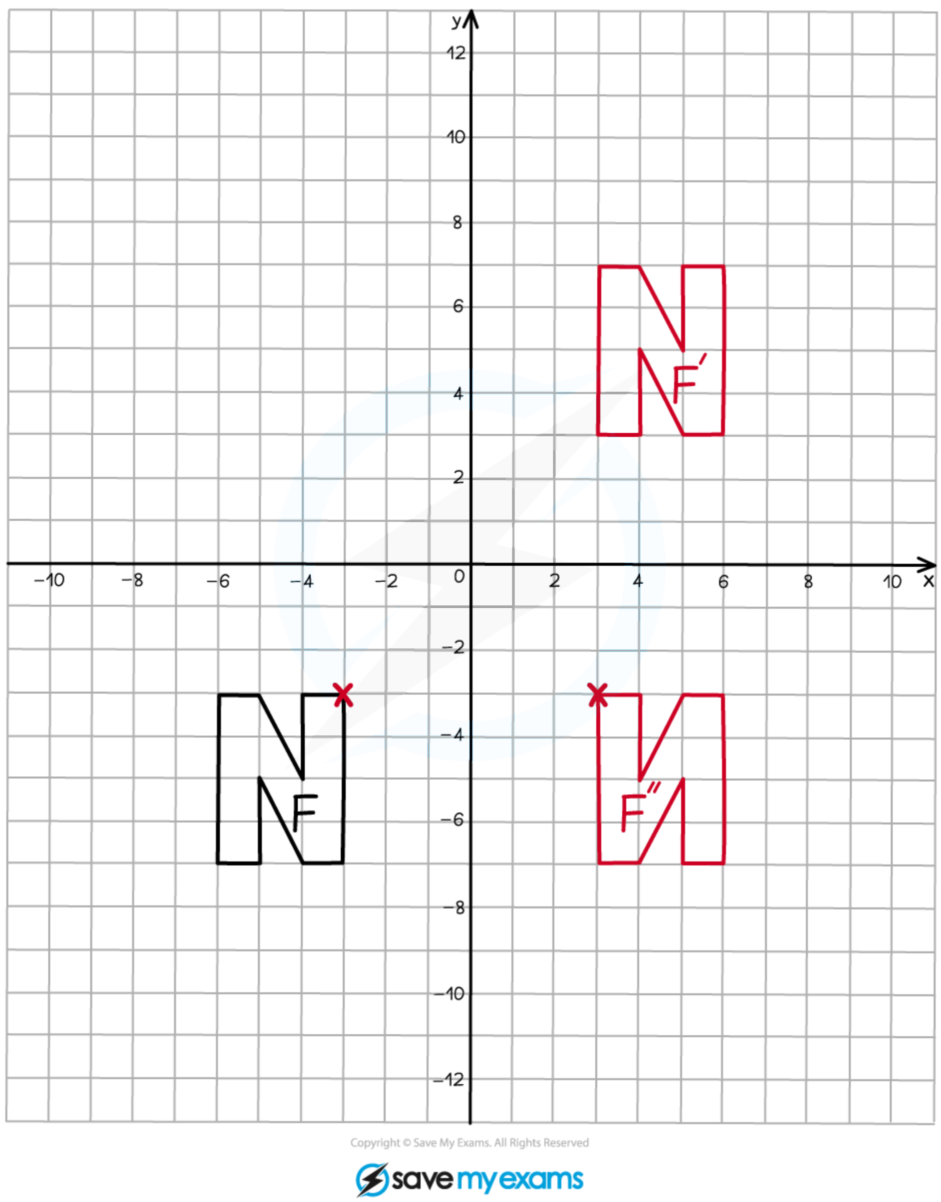
(b) Reflect shape F' in the line $y = 0$. Label this shape F''.

The line $y = 0$ is the x-axis

Measure the perpendicular distance (the vertical distance) between each vertex on the original object and the x-axis, then measure the same distance on the other side of the mirror line and mark on the corresponding vertex on the reflected image

Repeat this for all of the vertices and join them together to create the reflected image

Label the reflected image F''

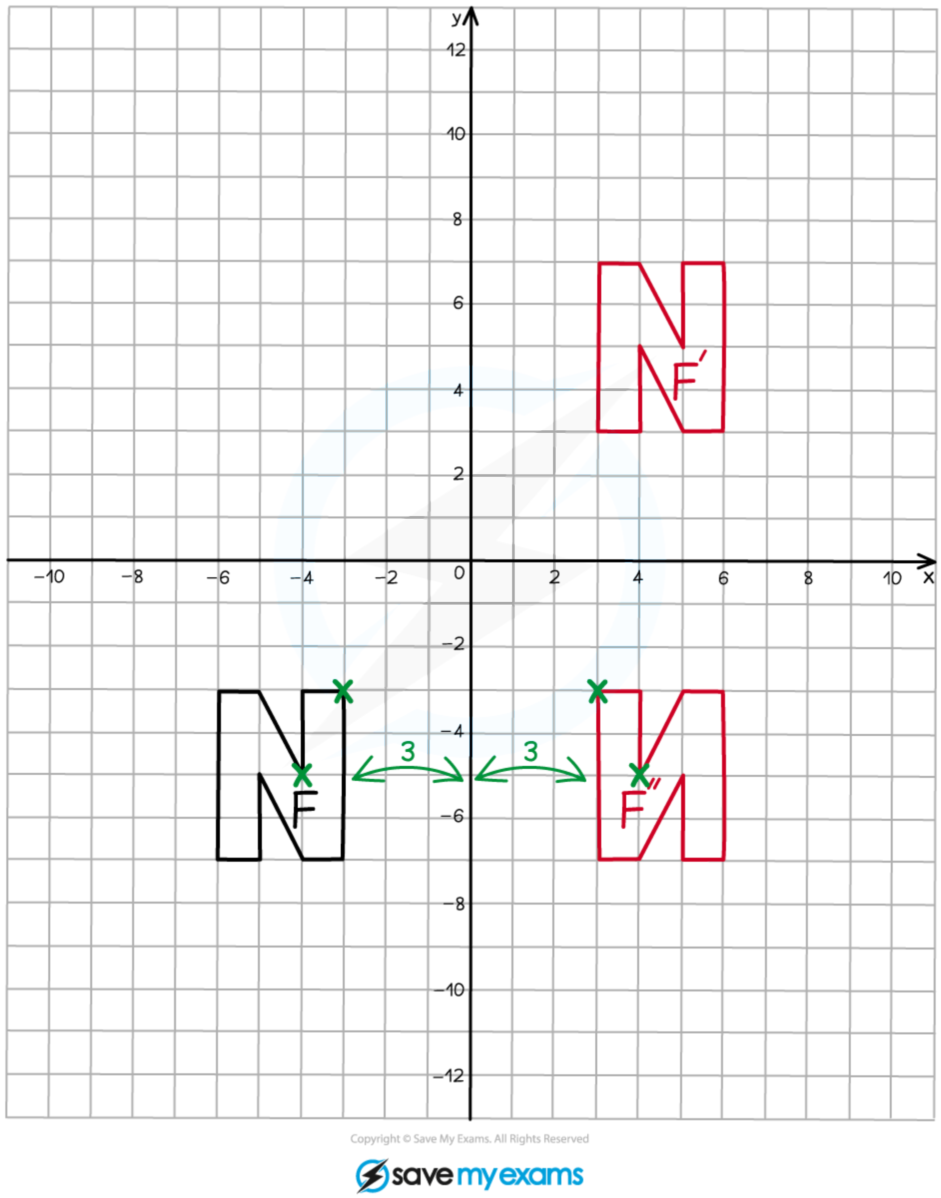


(c) Fully describe the single transformation that would create shape F'' from shape F.

The object (F) and image (F'') are reflections of each other in the y-axis

Your notes

Your notes



The single transformation from F to F'' is a reflection in the y-axis

Stating "the y-axis" or writing equation $x = 0$ are both acceptable