AQA GCSE Maths: Higher

Your notes

Circle Theorems

Contents

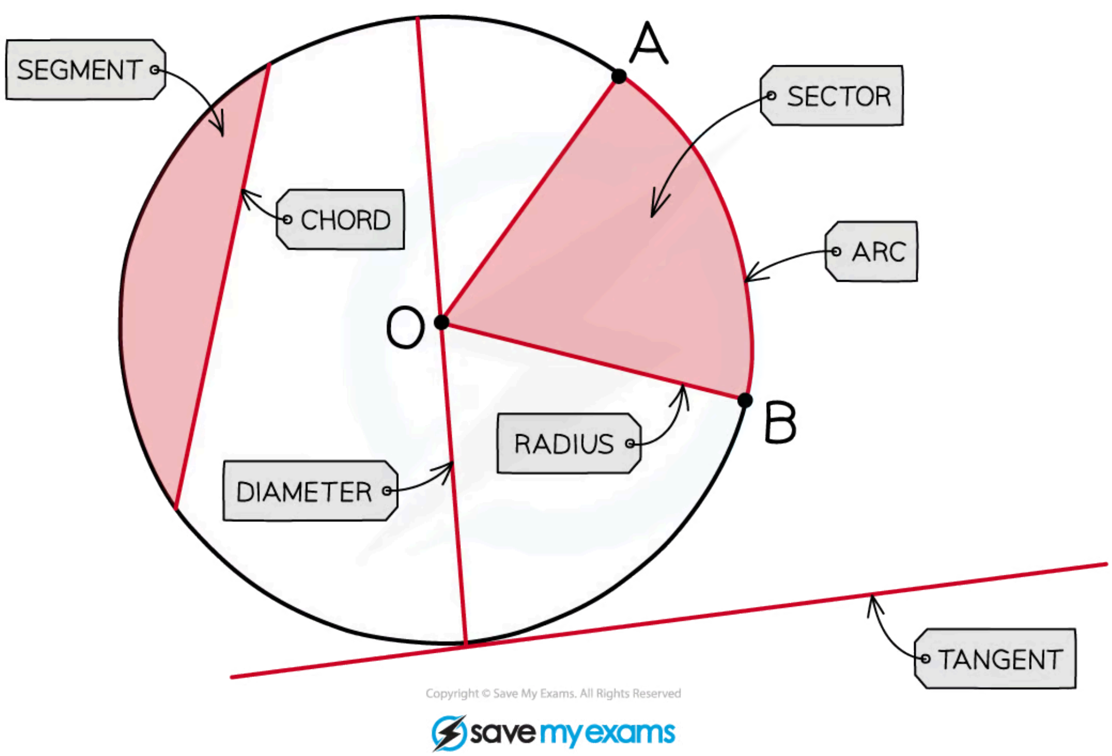
Your notes

Angles at Centre & Circumference

Angles at Centre & Circumference

What are circle theorems?

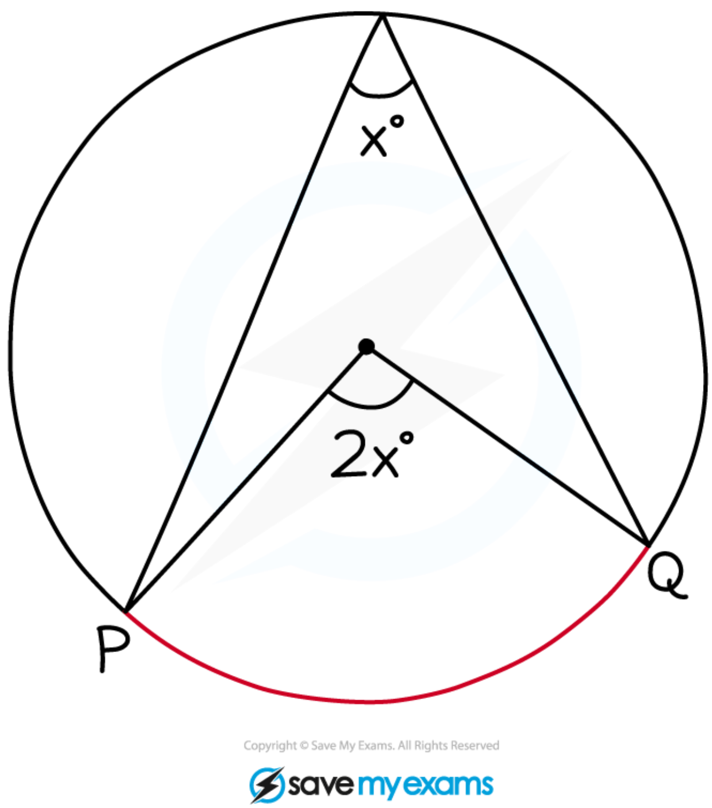
- **Circle Theorems** deal with angles that occur when lines are drawn within (and connected to) a circle
- You may need to use other **facts and rules** such as:
  - basic properties of lines and angles
  - properties of triangles and quadrilaterals
  - angles in parallel lines or polygons



Circle Theorem: The angle at the centre is twice the angle at the circumference

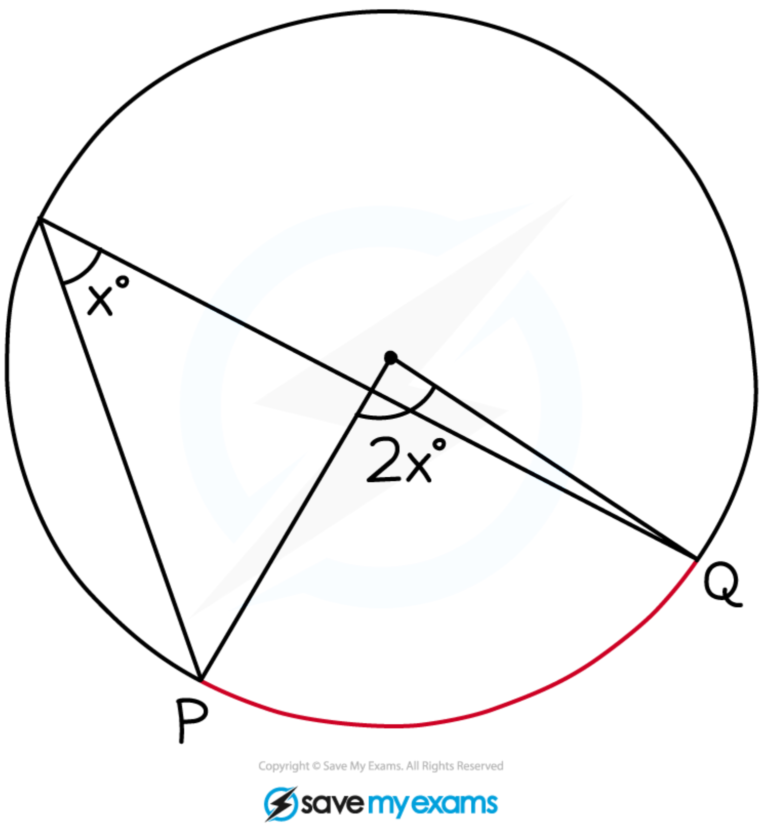
Your notes

- In this theorem, the **chords (radii) to the centre** and the **chords to the circumference** are both drawn from (subtended by) the ends of the **same arc**



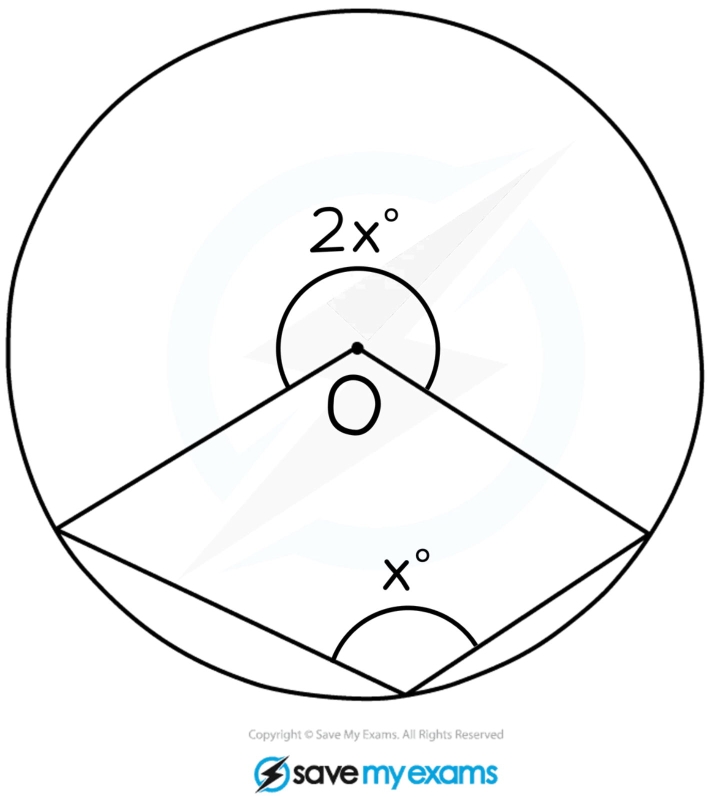
- To spot this circle theorem on a diagram
  - find any **two radii** in the circle and follow them to the **circumference**
  - see if there are **lines from those points** going to **any other point** on the **circumference**
  - it may look like the shape of an **arrowhead**
- When explaining this theorem in an exam you must use the keywords:
  - **The angle at the centre is twice the angle at the circumference**
- This theorem is still true when the **'triangle parts' overlap**

Your notes



- It is also true when the lines form a **diamond shape**
  - You need to compare the **reflex angle at the centre** with the angle at the circumference
  - **Common mistakes** are to
    - compare the wrong angles
    - confuse it with a different circle theorem on cyclic quadrilaterals

Your notes



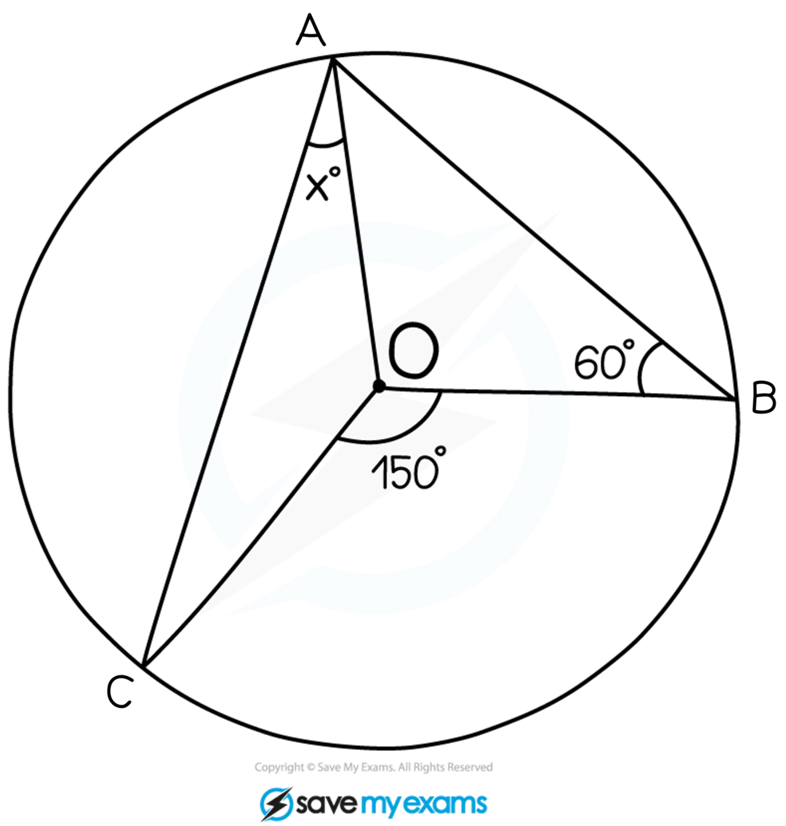
Examiner Tips and Tricks

- Questions often say to give "reasons" for your answer
  - Quote an **angle fact** or **circle theorem** for **every** angle you find (not just one for the final answer)

Worked Example

Find the value of X in the diagram below.

Your notes



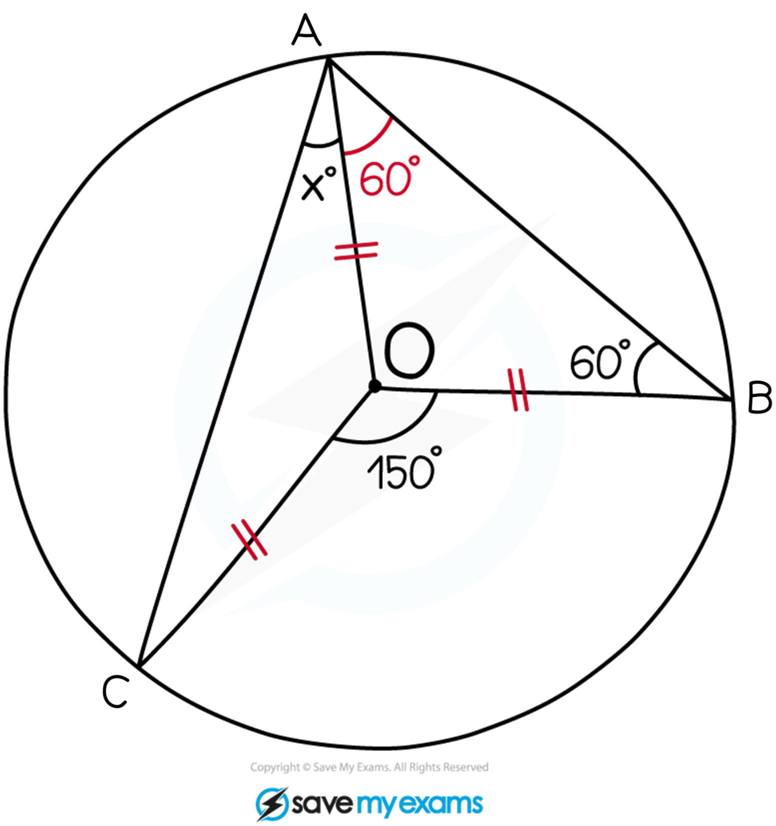
Give a reason for each step of your working.

There are three radii in the diagram, *AO*, *BO* and *CO*
Mark these as equal length lines

Notice how they create two isosceles triangles
Base angles in isosceles triangles are equal

Angle *OAB* = angle *OBA* = 60° (isosceles triangle)

Your notes



Use the circle theorem:

The angle at the centre is twice the angle at the circumference

Form an equation for x

$$2(x + 60) = 150$$

Expand the brackets and solve the equation

$$2x + 120 = 150$$
$$2x = 30$$
$$x = 15$$

$$\boldsymbol{x = 15}$$

Base angles in isosceles triangles are equal

The angle at the centre is twice the angle at the circumference

Your notes

Angle in a Semicircle

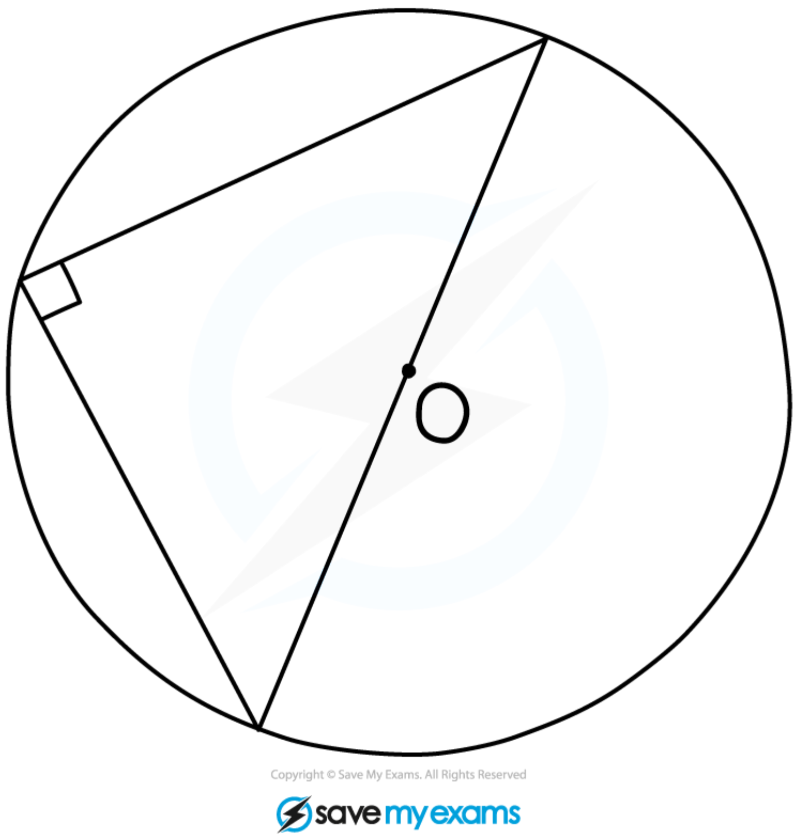
Your notes

Angle in a Semicircle

Circle Theorem: The angle in a semicircle is 90°

- The lines drawn from a **point** on the **circumference** to either end of a **diameter** are **perpendicular**
  - The angle at that point on the circumference is **90°**
  - This circle theorem only uses half of the circle
    - The right-angle is called the angle in a **semicircle**
- This is a **special case** of the circle theorem "the angle at the centre is twice the angle at the circumference"
  - The angle on the diameter is 180°
  - The angle at the circumference is halved, giving 90°

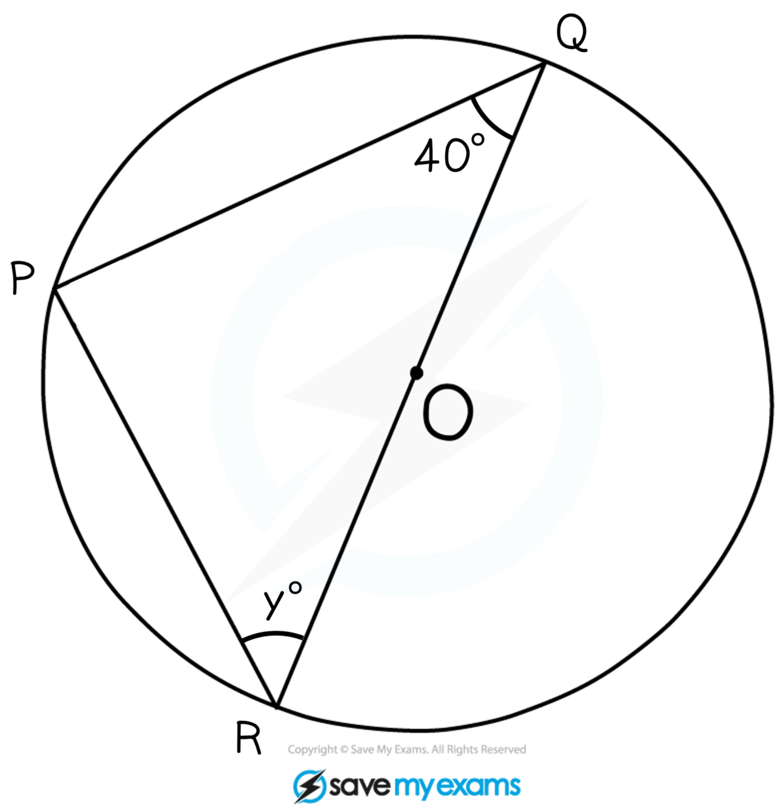
Your notes



- To spot this circle theorem on a diagram look for a triangle where
  - **one side** is the **diameter**
    - Remember that a diameter always goes through the **centre**
  - **all three vertices** are on the **circumference**
- The 90° angle will always be the angle **opposite** the diameter
- When explaining this theorem in an exam you must use the **keywords:**
  - **The angle in a semicircle is 90°**
- Questions that use this theorem may
  - appear in **whole circles** or in **semicircles**
  - require the use of **Pythagoras' Theorem** to find a missing length

Your notes

Worked Example



P, *Q* and *R* are points on a circle.
RQ is a diameter.

Find the value of y.

Give a reason for your answer.

Use the fact that angles in a triangle add up to 180° and the circle theorem

The angle in a semicircle is 90°

Write an equation for y

$$y + 90 + 40 = 180$$

Solve for y

$$y = 180 - 90 - 40$$
$$y = 50$$

$$\boldsymbol{y = 50}$$

The angle in a semicircle is 90°

Your notes

Your notes

Theorems with Chords & Tangents

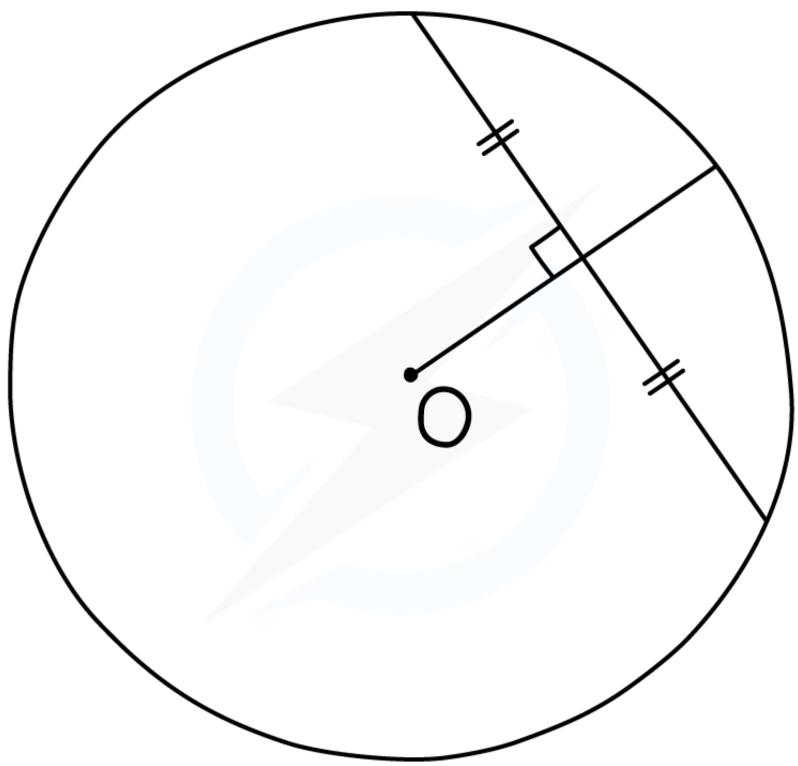
Circles & Chords

What is a chord?

- A **chord** is any straight line is a circle that joins any **two points** on the **circumference**
  - Chords of **equal length** are **equidistant** (the same distance) from the centre

Circle Theorem: The perpendicular bisector of a chord passes through the centre

- If a **line through the centre** (such as a **radius or diameter**) goes through the **midpoint** of **chord**
  - it will **bisect** (cut in half) that chord at **right angles** to it



- To spot this circle theorem on a diagram
  - look for a **radius** and see if it **intersects** any **chords**
  - or look to see if you could **draw** a radius that bisects a chord
- When explaining this theorem in an exam you can use either phrase below:
  - **A radius bisects a chord at right angles**
  - **The perpendicular bisector of a chord passes through the centre**

Your notes

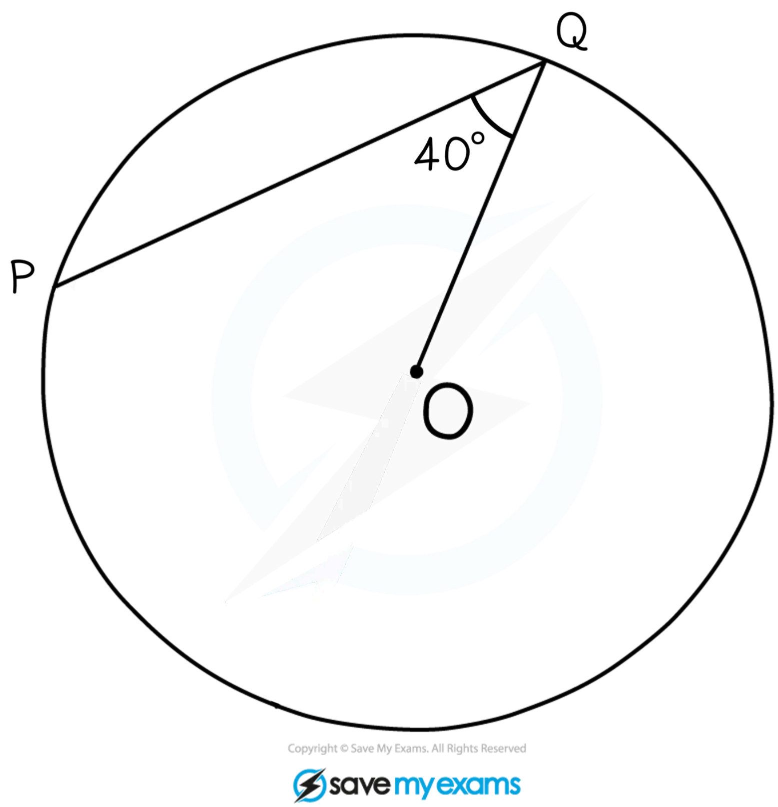
Examiner Tips and Tricks

- Look out for **isosceles triangles** formed by a chord and two radii
  - Two angles in the triangle will be equal and there will be at least one line of symmetry

Worked Example

The diagram below shows a circle with centre, *O*.
Two points, *P* and Q, lie on its circumference.
The radius of the circle is 6 cm.
Angle *OPQ* = 40°.

Find the length *PQ*.

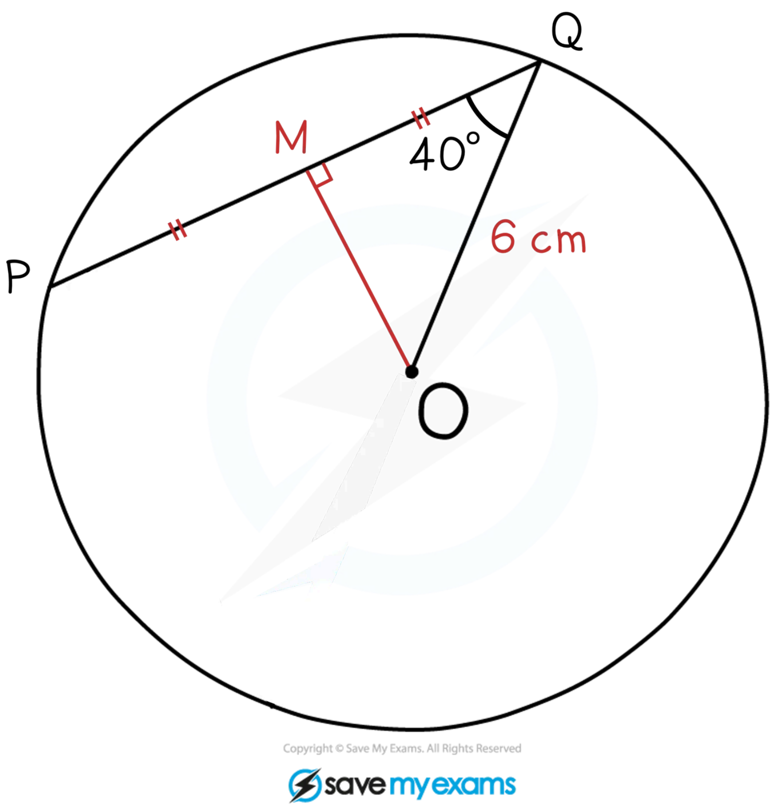


Label the radius on the diagram 6 cm

Draw a line from O to the midpoint, *M*, of the line *PQ*
The angle formed between the *OM* and *PQ* will be a right angle

Your notes

Your notes



Use SOHCAHTOA on triangle *OMQ* to find the length *MQ*

$$\cos 40 = \frac{MQ}{6}$$
$$6\cos 40 = MQ$$
$$MQ = 4.59626...$$

Double *MQ* to find the length *PQ*

$$4.59626... \times 2 = 9.19253...$$

Round to 3 significant figures

PQ = 9.19 cm (3 s.f.)

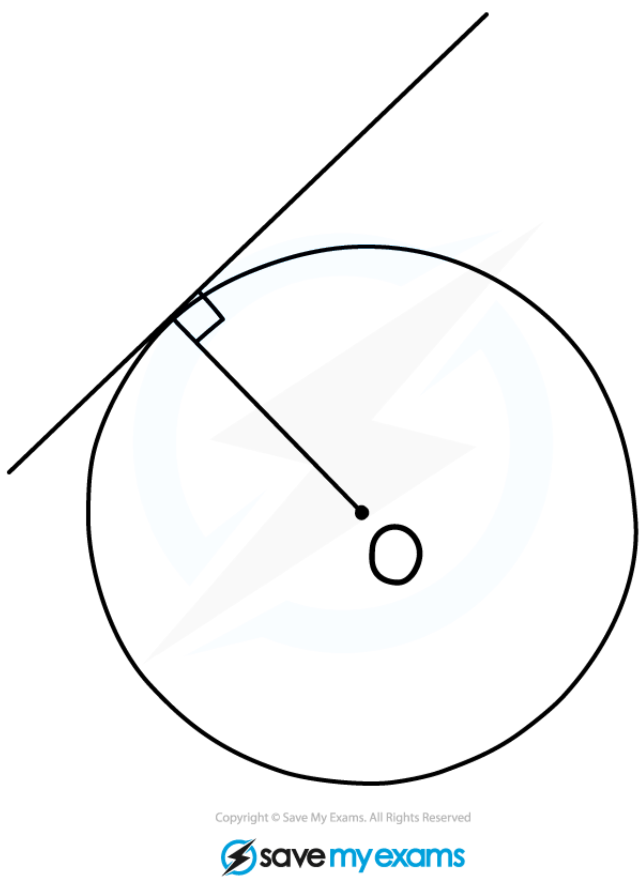
Circles & Tangents

What is a tangent?

- A tangent to a circle is a straight line **outside of the circle** that touches its **circumference** at exactly **one point**

Circle Theorem: A radius and a tangent meet at right angles

- If a **radius and a tangent** meet at a point on the **circumference** of a circle, the **angle** formed between them will be **90°**
  - They are **perpendicular** to each other



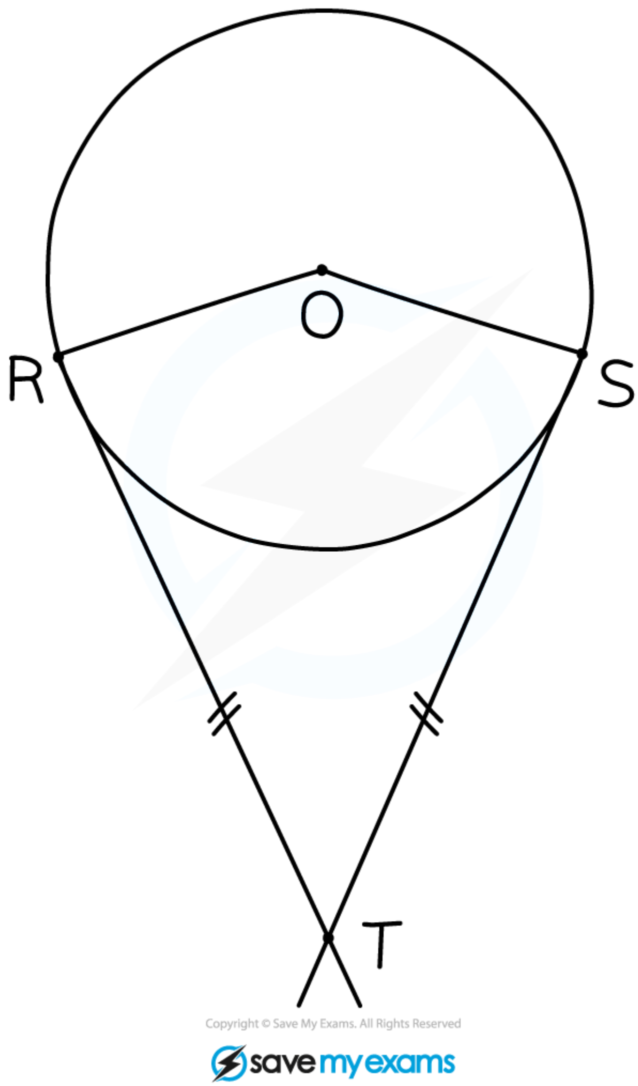
- When explaining this theorem in an exam you must use the keywords:
  - **A radius and a tangent meet at right angles**

Circle Theorem: Tangents from an external point are equal in length

- **Two tangents** from the **same external point** are equal in length

Your notes

- This means that a **kite** can be formed by two tangents meeting a circle
  - The kite below has a vertical line of symmetry
    - It is formed from two **congruent** triangles back-to-back
  - The kite will have **two right angles** where the tangents meet the radii
    - You can use **Pythagoras** and **SOHCAHTOA** on each of these triangles



Your notes

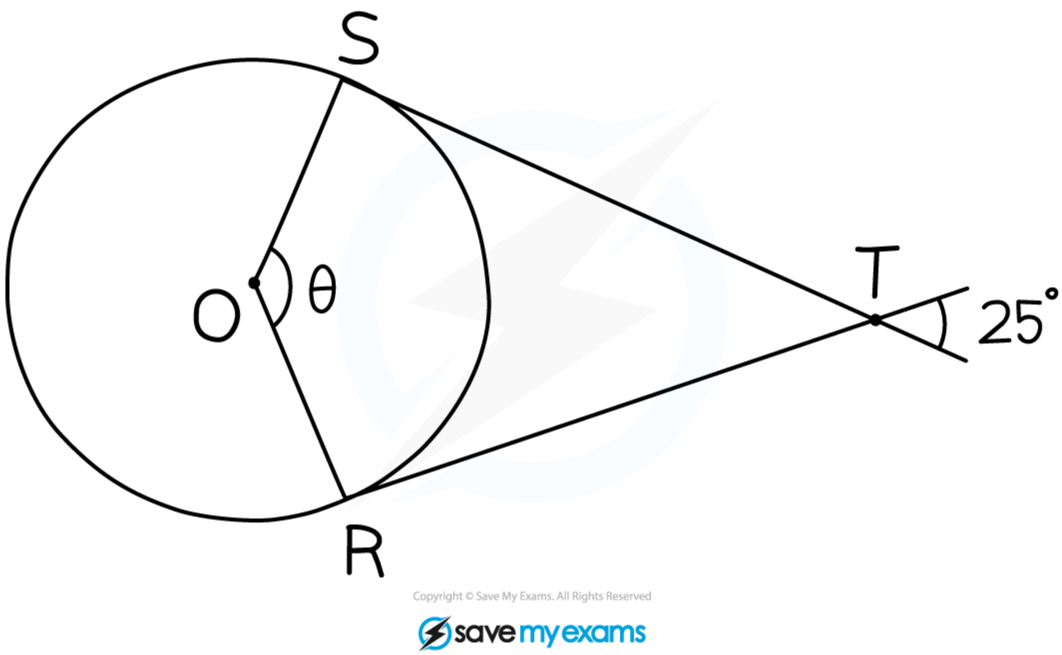
Your notes

Examiner Tips and Tricks

- Look for tangents in the exam and draw on the radius at right angles to see if it helps

Worked Example

Find the value of θ in the diagram below.



The lines *ST* and *RT* are both tangents to the circle
They meet the two radii on the circumference at the points *S* and *T*

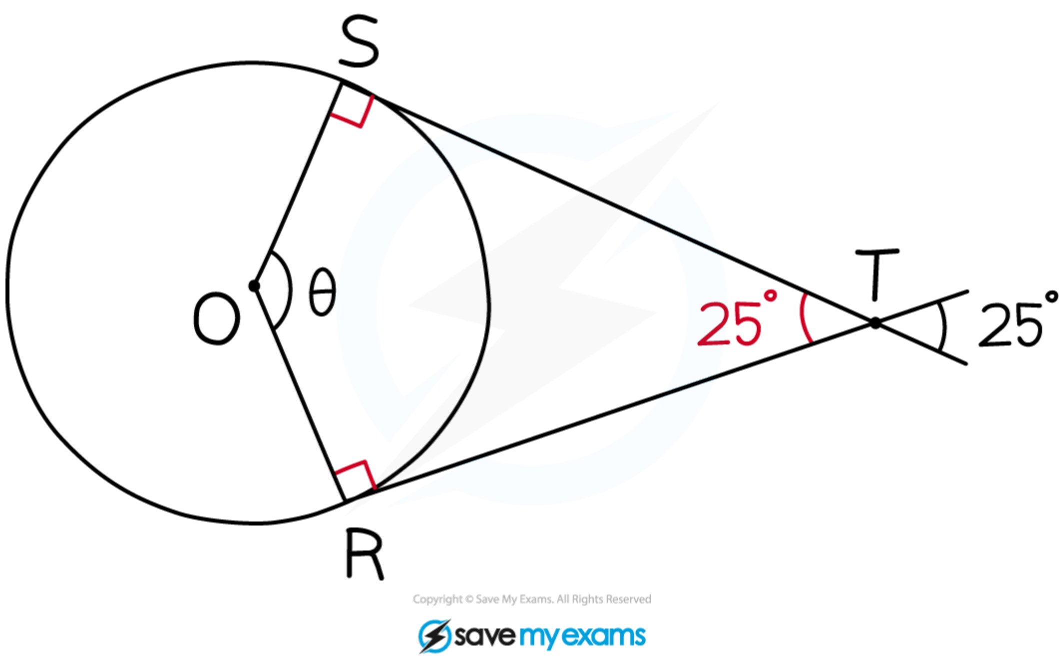
Angle *TSO* = angle *TRO* = 90°
A radius and a tangent meet at right angles

Use vertically opposite angles to find the value of angle *RTS*

Angle *RTS* = 25°
Vertically opposite angles

Mark these angles clearly on the diagram

Your notes



Angles in a quadrilateral add up to 360°
Use this to form an equation for θ

$$\theta + 90 + 90 + 25 = 360$$

Angles in a quadrilateral sum to 360°

Simplify

$$\theta + 205 = 360$$

Solve

$$\theta = 360 - 205$$

$$\boldsymbol{\theta = 155°}$$

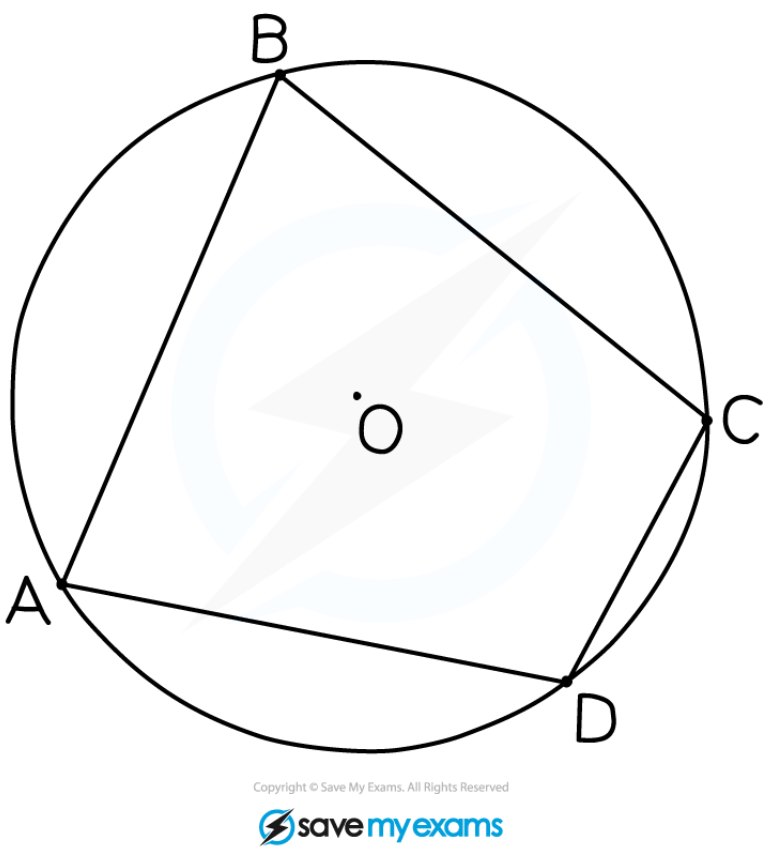
Your notes

Angles in Cyclic Quadrilaterals

Cyclic Quadrilaterals

Circle theorem: Opposite angles in a cyclic quadrilateral add up to 180°

- A quadrilateral that is formed by **four points** on the **circumference** of a circle, (a **cyclic quadrilateral**), will have pairs of **opposite angles** that add up to **180°**



- To spot this theorem in a diagram
  - look for **quadrilaterals** that have all four points on the **circumference**
- When explaining this theorem in an exam you must use the keywords:
  - **Opposite angles in a cyclic quadrilateral add up to 180°**

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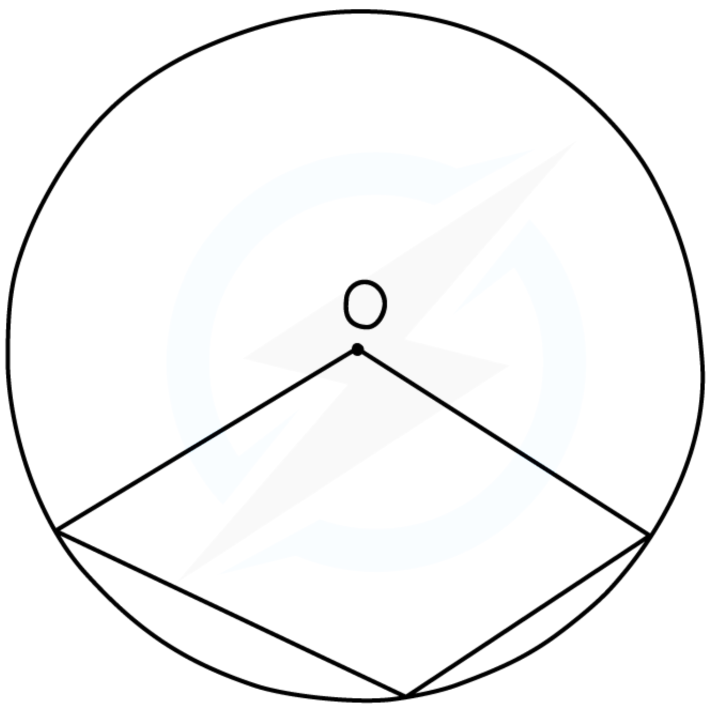
- The theorem only works for **cyclic quadrilaterals**
  - The diagram below shows a common scenario that is **not** a cyclic quadrilateral



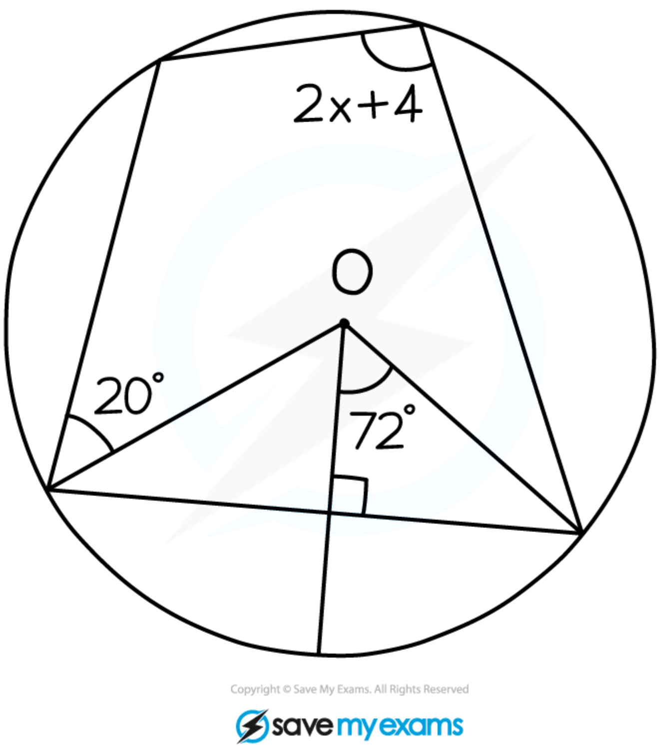
Examiner Tips and Tricks

- Cyclic quadrilaterals are often easy to spot in a busy diagram
  - Mark on their angles (even if you think you don't need them) as they may help you later on!

Worked Example

The circle below has centre, O.

Find the value of x.

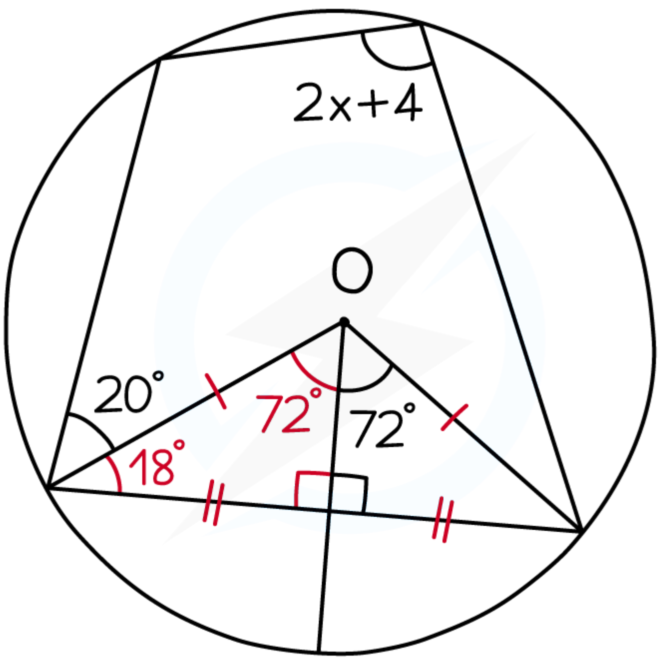


Identify both the cyclic quadrilateral and the radius perpendicular to the chord

Add to the diagram as you work through the problem

Your notes

Your notes



The radius bisects the chord and so creates two congruent triangles

Use this to work out 72° (equal to the equivalent angle in the other triangle)
And 18° (angles in a triangle add up to 180°)

Then use that **opposite angles in a cyclic quadrilateral add up to 180°**

$$2x + 4 + 20 + 18 = 180$$
$$2x = 138$$

$$\boldsymbol{x = 69°}$$

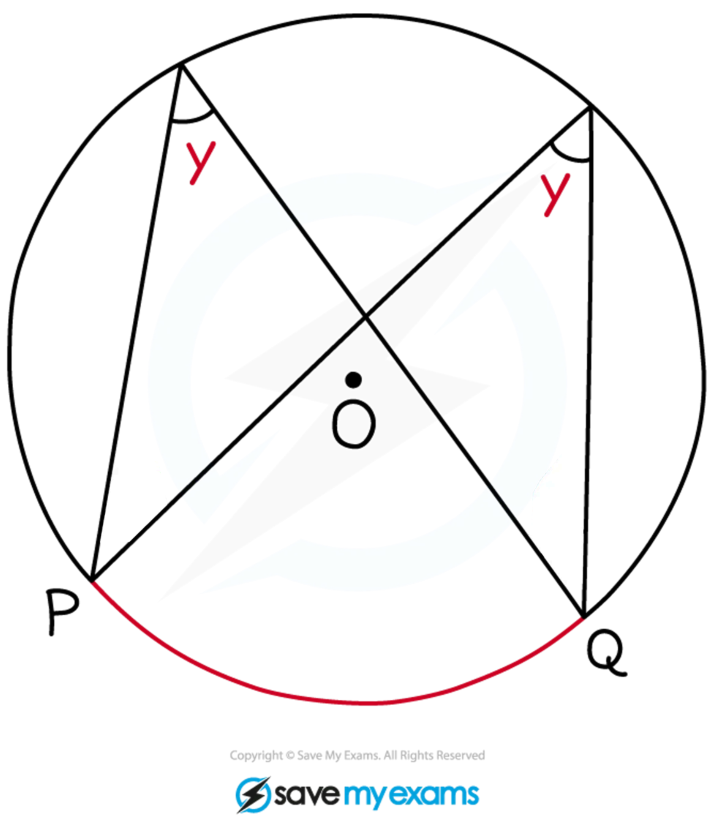
Angles in the Same Segment

Your notes

Circles & Segments

Circle Theorem: Angles in the same segment are equal

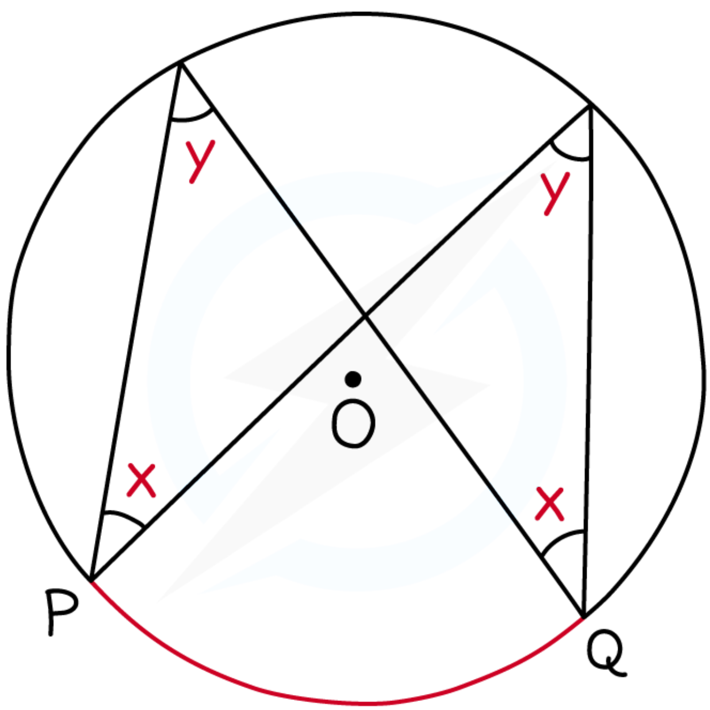
- **Any two angles** on the **circumference** of a circle that are formed from the **same two points** on the circumference are **equal**
  - These two angles are in the **same segment** of the circle
    - To see this, add the chord *PQ* below to split the circle into two segments



- To spot this circle theorem on a diagram
  - Find **two points** on the **circumference** that meet at a **third point**

Your notes

    - See if there are **any other pairs of lines** from the **same two original points** that meet at a different point on the circumference
- When explaining this theorem in an exam you must use the keywords:
    - **Angles in the same segment are equal**
- Look out for a **bowtie shape**
    - The theorem works **upside down**, in that the angles at *P* and *Q* are **also equal**



Examiner Tips and Tricks

- An exam question diagram may have **multiple equal angles**
    - Look for as many as possible by seeing how many pairs of lines start from the same two points on the circumference

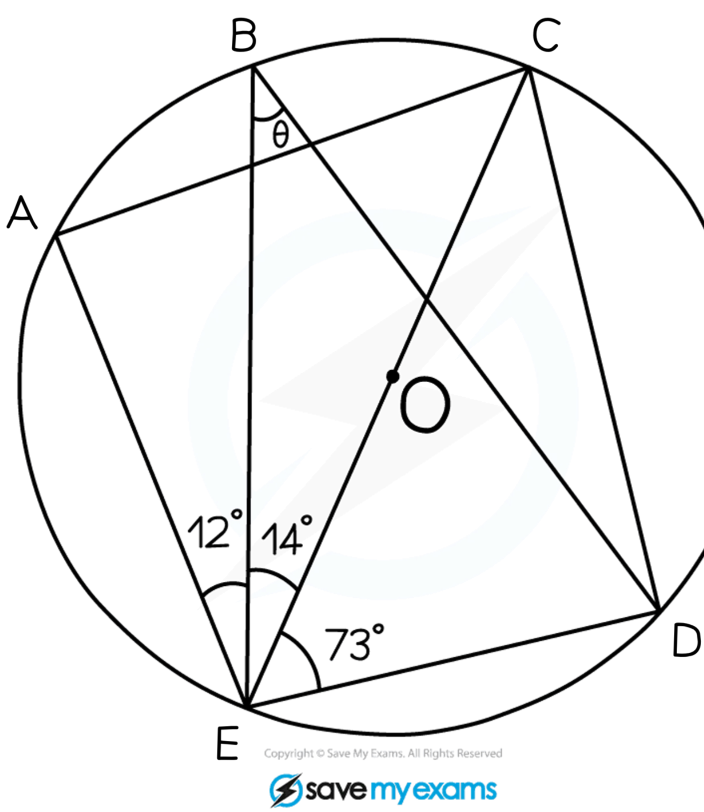
Your notes

Worked Example

The diagram below shows a circle with centre, *O*.
A, *B*, *C*, *D* and *E* are five points on the circumference on the circle.

Angle *AEB* = 12°.
Angle *BEC* = 14°.
Angle *CED* = 73°.
Angle *EBD* = θ°.

Find the value of θ .



CE is a diameter
This means that triangles *EAC* and *CED* are both triangles in a semicircle

Angle *EAC* = 90°
Angle *CED* = 90°

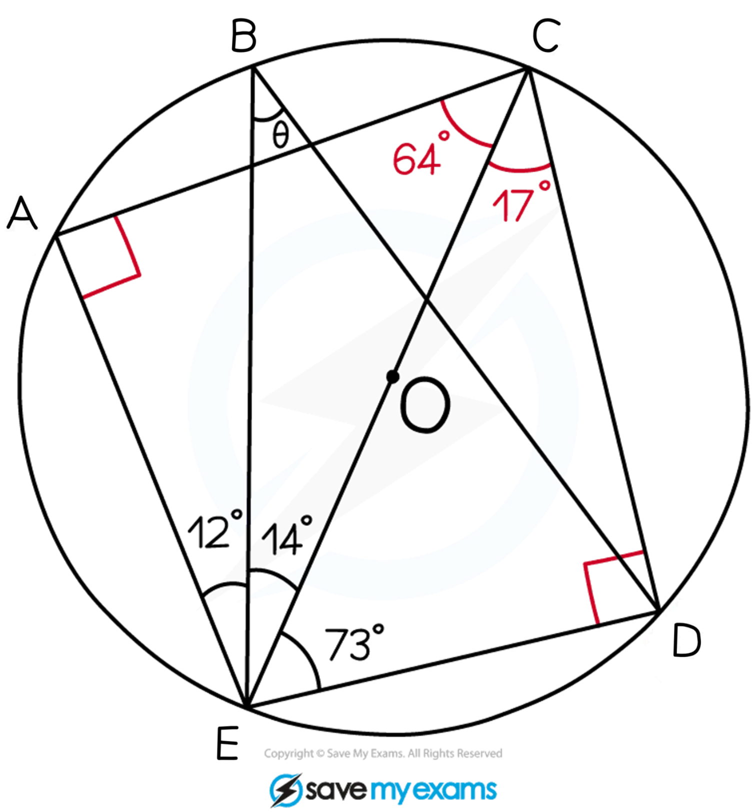
Your notes

Angle in a semicircle = 90°

Find the other angles in the triangles

Angle *ECA* = 64°
Angle *ECD* = 17°
Angles in a triangle = 180°

Label these angles on the diagram



Angle θ is formed by two lines coming from either end of the chord *ED*
Angle *ECD* is also formed by two lines coming from either end of the chord *ED*

Angle θ = angle ECD = 17°
Angles in the same segment are equal

Angle θ = 17°

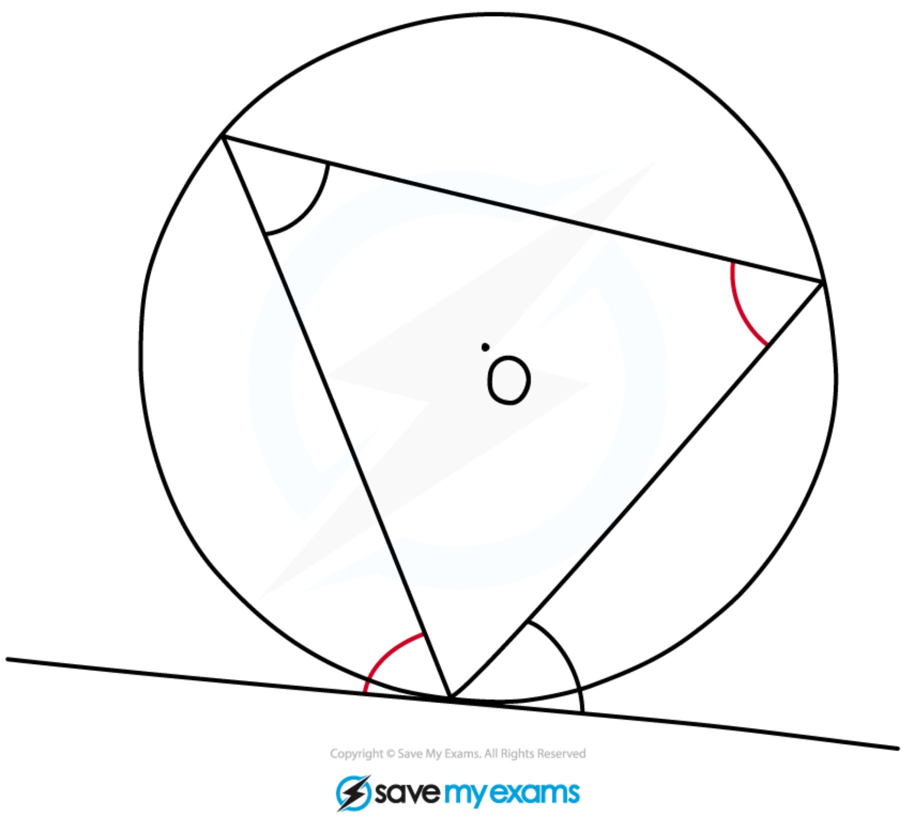
The Alternate Segment Theorem

Your notes

Alternate Segment Theorem

Circle theorem: The Alternate Segment Theorem

- The **angle** between a **chord** and a **tangent** is **equal** to the **angle** in the **alternate segment**



- To spot this circle theorem on a diagram
  - look for a **cyclic triangle**
    - where all three vertices of the triangle lie on the circumference
  - **one vertex** of the triangle meets a **tangent**
- To identify which angles are equal

- mark the **angle** between the **tangent** and the **side** of the **cyclic triangle**
- the angle **inside** the triangle at the corner **opposite the side** of the triangle that forms the first angle is the **equal angle**

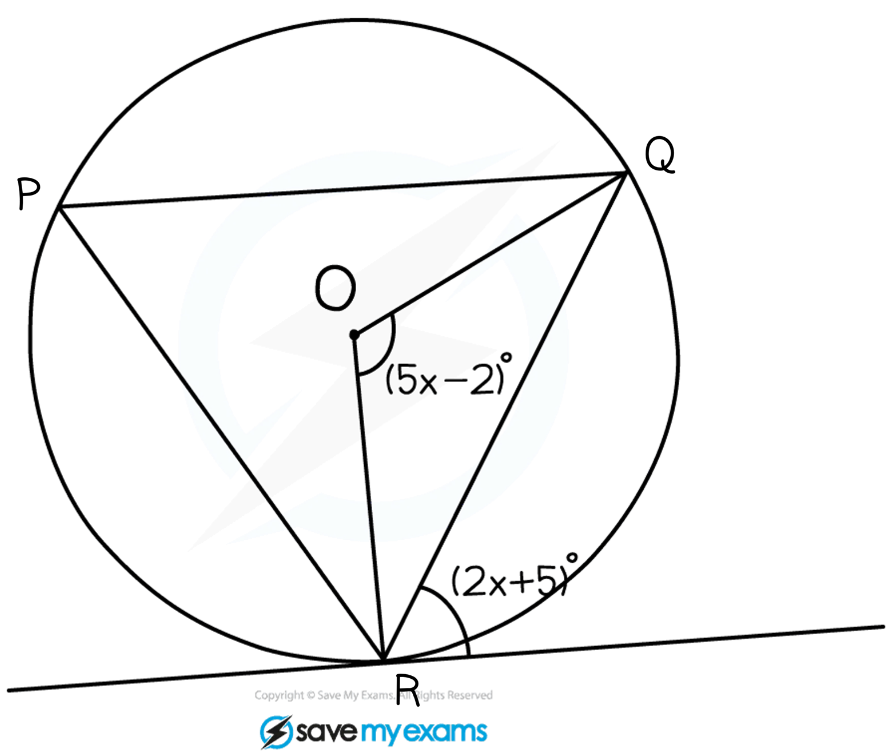
- When explaining this theorem in an exam you can just say the phrase:
  - **The Alternate segment theorem**

Examiner Tips and Tricks

- Look for **cyclic triangles** and **tangents** in busy diagrams
  - Questions involving the alternate segment theorem frequently appear in exams!

Worked Example

Find the value of x.

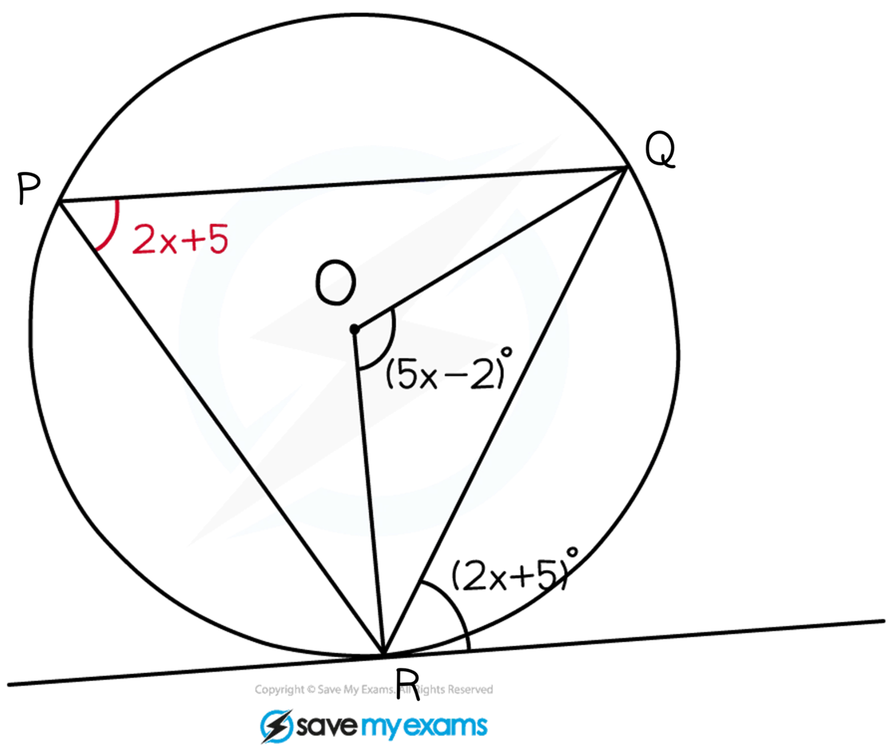


Your notes

Identify the cyclic quadrilateral (triangle in the circle with all three vertices at the circumference)

One vertex of this triangle meets a tangent at point *R*
The angle between one of its sides (*QR*) and the tangent is given
Find the angle inside the triangle, opposite to the same side (*QR*)

Angle between *QR* and the tangent = Angle *RQP* = $(2x+5)$
Alternate segment theorem



Notice that angle *RPQ* and angle *ROQ* both come from the same two points on the circumference

Angle *ROQ* = $2(2x+5)$
Angle at the centre is twice the angle at the circumference

Form an equation using the two expressions for angle *ROQ*

$$2(2x+5)=5x-2$$

Expand the brackets and solve

$$4x + 10 = 5x - 2$$
$$12 = x$$

Your notes

$$\boldsymbol{x = 12}$$

Your notes

Circle Theorem Proofs

How do I prove circle theorems using radii to form isosceles triangles?

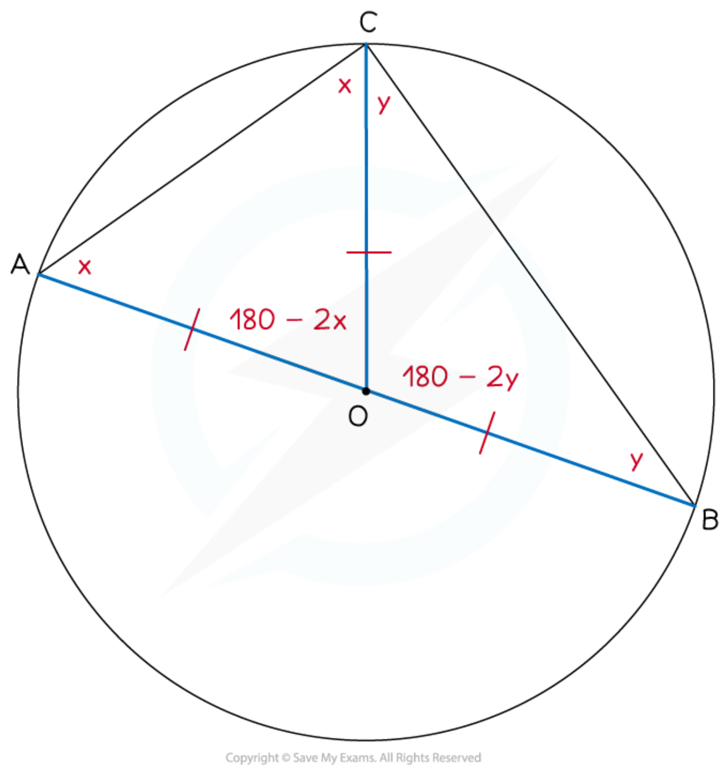
- This type of proof can be used to prove the following circle theorems
  - The **angle** in a **semicircle** is always **90°**
  - The **angle** at the **centre** is **twice** the angle at the **circumference**
  - **Angles** in the **same segment** are **equal**
  - **Opposite angles** in a **cyclic quadrilateral** add up to **180°**

How do I prove that the angle in a semicircle is 90°?

- This circle theorem states that an **angle subtended** at the **circumference** of a **semicircle** is always a **right angle**
  - Although it is a special case of the angle at the centre and circumference circle theorem, it can be proved without using any other circle theorems
- STEP 1
  Draw a radius from the centre of the circle to the angle subtended at the circumference
  - This will form two **isosceles triangles**
- STEP 2
  Label the two angles formed at the angle subtended at the circumference x and y.
  - The angle you are trying to prove is 90° is now $x + y$
- STEP 3
  Label the remaining angles in each of the isosceles triangles with algebraic expressions in terms of x and y
  - **Angles** at the **base** of an **isosceles triangle** are equal
    - Therefore the two remaining angles at the circumference are also x and y
  - **Angles** in a **triangle** add up to **180°**

Your notes

- Therefore each angle **at the centre** will be labelled with the expression $180 - 2x$ and $180 - 2y$



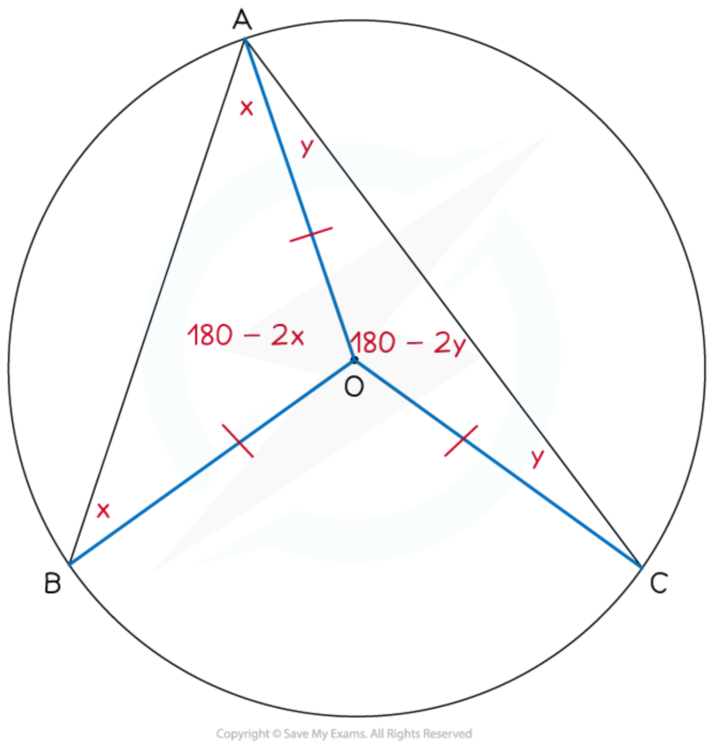
- **STEP 4**
  The angles at the centre lie on a **diameter,** which is a straight line, therefore
  $180 - 2x + 180 - 2y = 180$
  - Rearrange this equation to show that $x + y = 90°$
- In **STEP 2** you already labeled the angle at the circumference as $x + y$, so this proves that the angle at the circumference equals 90°
  - Give clear reasons throughout your proof, using the key words given in **bold**

How do I prove that the angle at the centre is twice the angle at the circumference?

- This circle theorem states that an **angle subtended** at the **centre** of a circle is **twice** the angle subtended at the **circumference** of a circle from the same **arc**
  - It does not need any other circle theorems to prove it

Your notes

- **STEPS 1**, **2** and **3** are the same as above

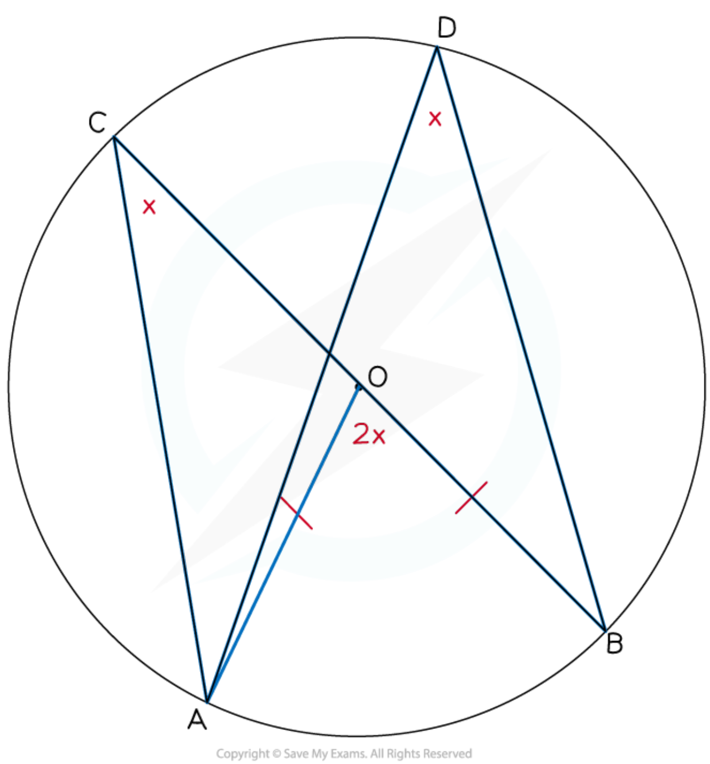


- **STEP 4**
  The three angles formed at the centre lie at a **point,** therefore they will add to 360°
  - Label the third angle at the centre θ, then you can form the equation
    $\theta + 180 - 2x + 180 - 2y = 360$
  - Rearrange this equation to show that $\theta = 2(x + y)$
- In **STEP 2** you already labelled the angle at the circumference as $x + y$, so this proves that the angle at the centre is twice the angle at the circumference
  - Give clear reasons throughout your proof, using the keywords given in **bold**
- This circle theorem is also a more generic version of the circle theorem the **angle in a semicircle is always 90°**
  - You could be asked to prove either without the use of any circle theorems

How do I prove that angles at the circumference from the same arc are equal?

Your notes

- This circle theorem states that any **angles subtended** at the **circumference** of a circle from the same **arc** are **equal**
- This theorem is proved using the circle theorem an **angle subtended** at the **centre** of a circle is **twice** the angle subtended at the **circumference** of a circle
- Draw the radii from the centre of the circle to the points on the circumference forming the arc which the angles are subtended from
  - This will form an angle at the centre, label this angle $2x$
- By the circle theorem "The angle and the centre is twice the angle at the circumference", any angle at the circumference can be labelled x



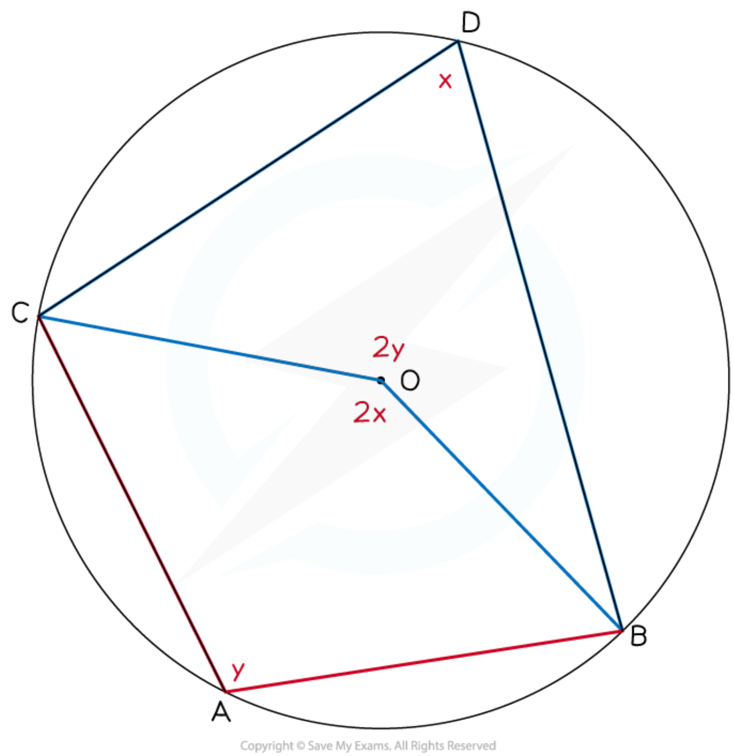
- You do not need to prove the circle theorem you have used in this proof, but you must give clear reasons

Your notes

How do I prove that opposite angles in a cyclic quadrilateral add up to 180°?

- This theorem is proved using the circle theorem "An **angle subtended** at the **centre** of a circle is **twice** the angle subtended at the **circumference** of a circle"
- **STEP 1**
  Draw the radii from the centre of the circle to **any two** of the vertices of the cyclic quadrilateral that are **opposite** each other
  - This will form two angles at the centre, label these angles $2x$ and $2y$
  - The angles $2x$ and $2y$ are at a point, so they add up to 360°
  - Therefore $2x + 2y = 360°$ which can be simplified to $x + y = 180°$
- **STEP 2**
  By the circle theorem "The angle and the centre is twice the angle at the circumference", the two angles at the circumference can be labelled $\frac{1}{2}(2x) = x$ and $\frac{1}{2}(2y) = y$
  - We have already shown that $x + y = 180°$

Your notes



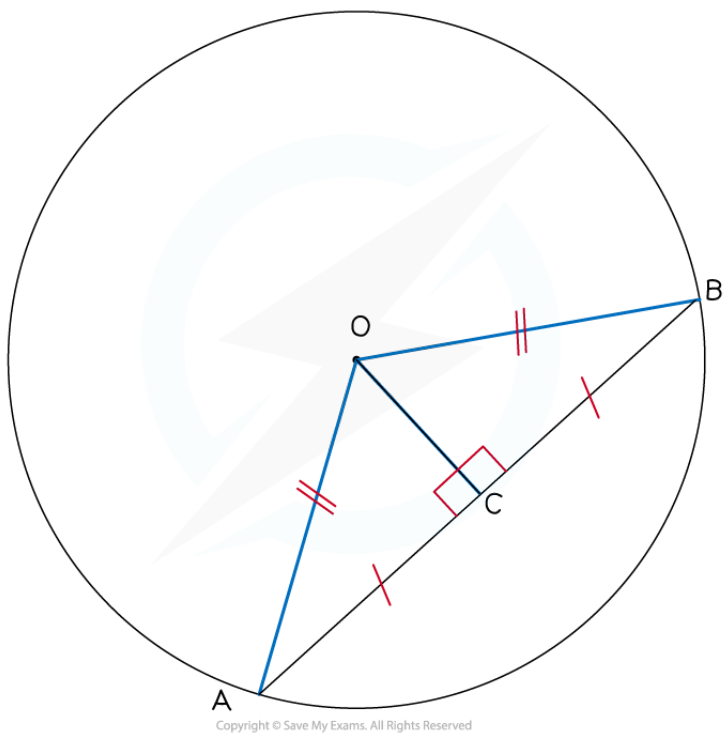
- You do not need to prove the circle theorem you have used in this proof, but you must give clear reasons

How do I prove circle theorems involving chords and tangents?

- This type of proof can be used to prove the following circle theorems
  - The **perpendicular** from the **centre** of a circle **bisects** a **chord**
  - The **tangent** to a circle **meets** the **radius** at **90°**
  - The **alternate segment theorem**
- These proofs can be more tricky and will use other circle theorems within them
- The questions will often guide you through the proof
- The circle theorem **"The perpendicular from the centre of a circle bisects a chord'** can be proved using **congruent triangles**
- If the radius is perpendicular to the chord, two right-angled triangles will be formed

Your notes

- Prove that these two triangle are **congruent** using the RHS (right angle, hypotenuse, side) rule
  - The chord and the radius are **perpendicular**, therefore both triangles have a **right angle**
  - The **hypotenuse** is the line from the centre to the circumference, therefore both triangles have an **equal hypotenuse**
  - The line joining the chord to the centre is shared between both triangles, therefore this is a **same side** in both triangles
  - Therefore, by RHS, the two triangles are **congruent**



- If the two triangles are congruent, then all three sides will be the same and so the perpendicular must **bisect** the chord
- The proof for the circle theorem **"The tangent to a circle meets the radius at a right angle"** uses proof by contradiction and involves assuming that they do not meet at 90° and proving that this is not possible
- The proof for the **alternate segment theorem** uses the circle theorems 'the angle in a semicircle is always 90°' and 'the tangent to a circle meets the radius at 90°'

Your notes

Examiner Tips and Tricks

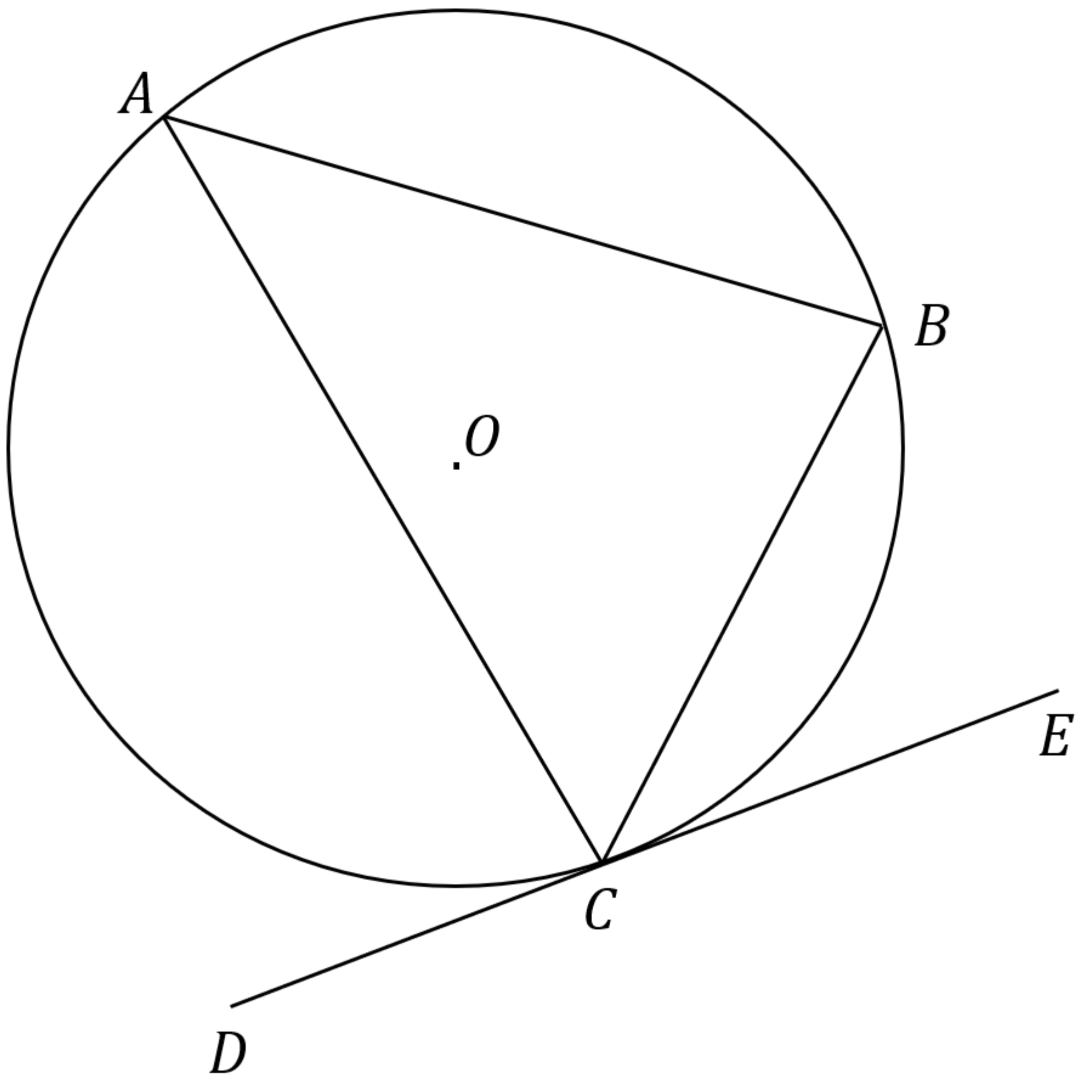
- If you are unsure of how to start a proof question, begin by drawing in the radii from the centre to any significant point on the circumference and look for isosceles triangles
- The question may tell you not to use any circle theorems in your proof, in this case you will most likely be looking for isosceles triangles

Worked Example

In the diagram below, A, B and C are points on the circumference of a circle, centre O.
DCE is a tangent to the circle.

Prove that angle BCE and angle BAC are equal.

Your notes

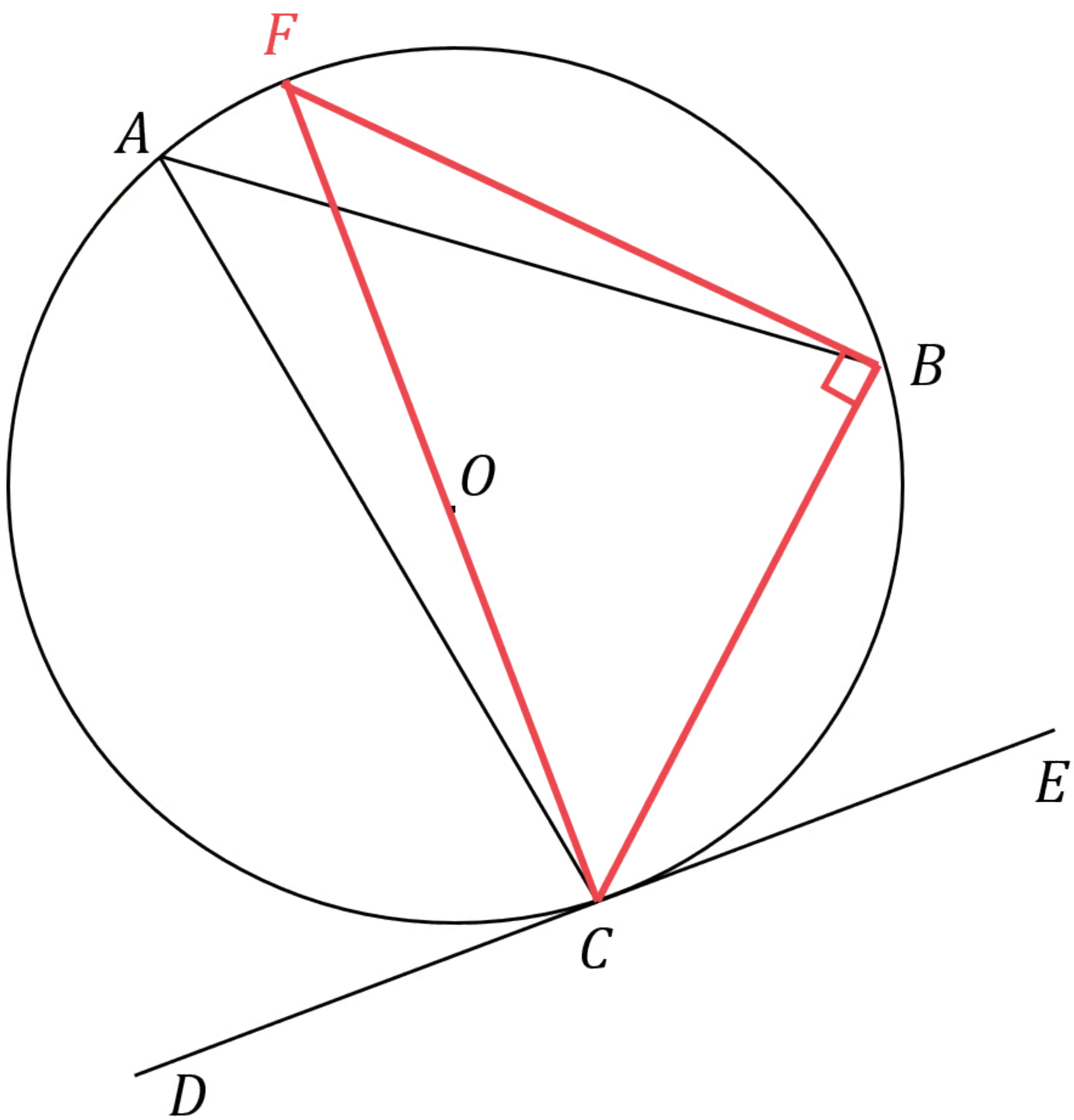


Begin by joining the point *O* to the point *C* and continue the line through to the circumference so that a diameter is drawn on the diagram.

Label this new point on the circumference *F*.

Join the point *F* to the point *B* on the circumference.

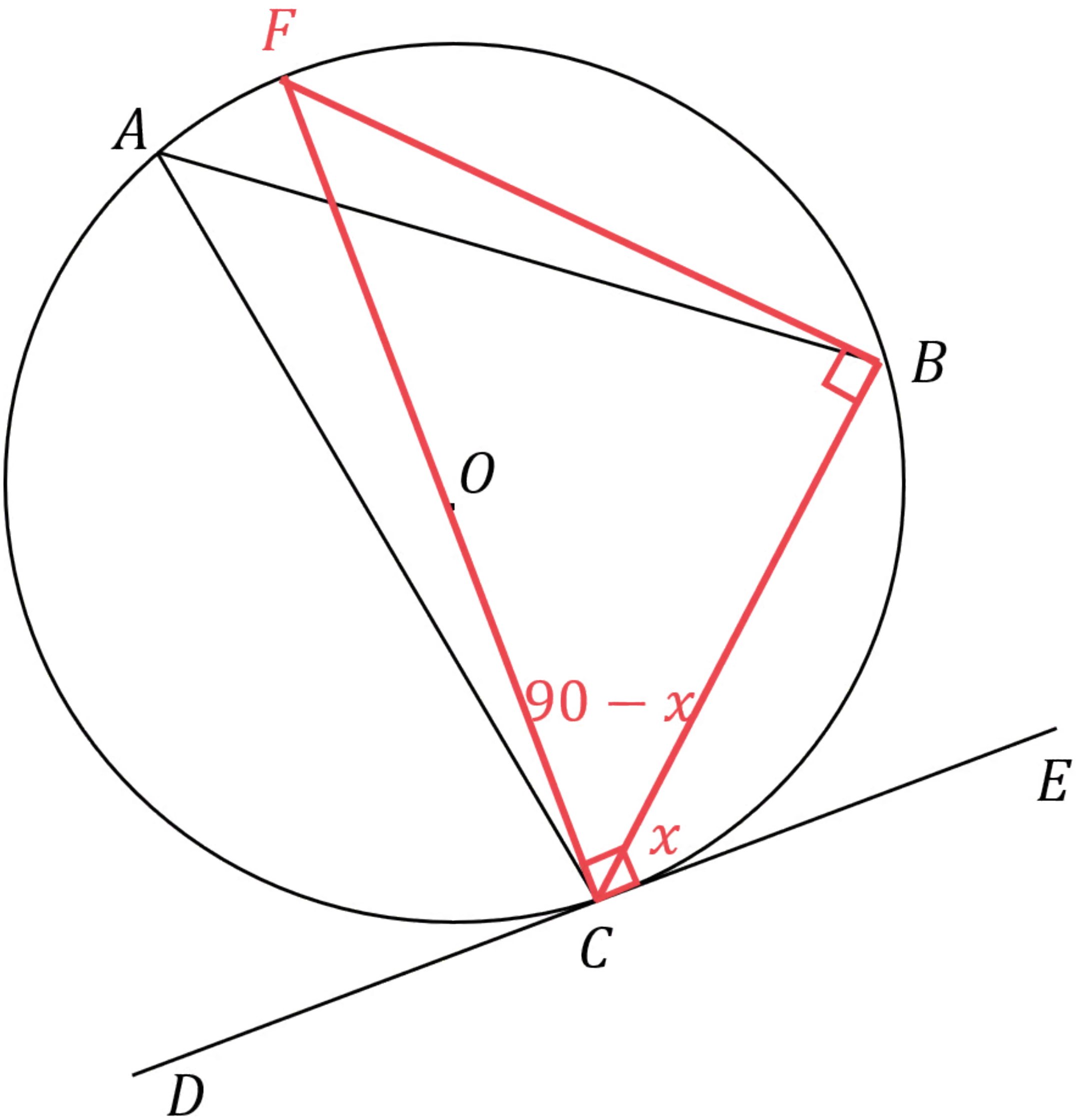
Your notes



Angle CBF = 90°
The angle in a semicircle is always 90°

The line OC is a radius, so it will meet the tangent *DE* at 90°.
Let angle $BCE = x$.

Your notes



Angle $FCB = 90 - x$
The radius meets a tangent at 90°

Angles in a triangle add up to 180°, use this to find the angle *CFB* in terms of x.

Your notes

$$\begin{aligned}\text{Angle } CFB &= 180 - \text{angle } FCB - \text{angle } CBF \\ &= 180 - (90 - x) - 90 \\ &= 180 - 90 + x - 90 \\ &= x\end{aligned}$$

The angles in a triangle add up to 180°

Angles subtended at the circumference from the same arc are equal.

Use this to find an expression for angle *BAC* in terms of x.

$$\text{angle } BAC = \text{angle } CFB = x°$$

We have already stated that $BCE = x$.

Therefore **Angle *BCE* = Angle *BAC***