AQA GCSE Maths: Higher

Your notes

Prime Factors, HCF & LCM

Contents

Types of Numbers

Your notes

Types of Number

You will come across vocabulary such as

- **Integers** and **natural** numbers
- **Rational** and **irrational** numbers
- **Multiples**
- **Factors**
- **Prime** numbers
- **Squares**, **cubes** and **roots**
- **Reciprocals**

Knowing what each of these terms mean is essential.

What are integers and natural numbers?

- **Integers** are **whole** numbers;
  - They can be **positive, negative** and **zero**
  - For example, -3, -2, -1, 0, 1, 2, 3 are all integers
- **Natural** numbers are the **positive integers**
  - They can be thought of as counting numbers
  - 1, 2, 3, 4, ... are the natural numbers
    - Notice that 0 is **not** included

What are multiples?

- A **multiple** is a number which can be divided by another number, without leaving a remainder
  - For example, 12 is a multiple of 3
    - 12 divided by 3 is exactly 4
- A **common multiple** is multiple that is shared by more than one number
  - For example, 12 is a common multiple of 4 and 6

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- **Even** numbers (2, 4, 6, 8, 10, ...) are multiples of 2
- **Odd** numbers (1, 3, 5, 7, 9, ...) are **not** multiples of 2
- Multiples can be **algebraic**
  - For example, the multiples of k would be $k, 2k, 3k, 4k, 5k. ...$

What are factors?

- A **factor** of a given number is a value that divides the given number exactly, with **no remainder**
  - 6 is a factor of 18
    - because 18 divided by 6 is exactly 3
- Every integer greater than 1 has at least two factors
  - The integer itself, and 1
- A **common factor** is a factor that is shared by more than one number
  - For example, 3 is a common factor of both 21 and 18

How do I find factors?

- Finding all the factors of a particular value can be done by finding **factor pairs**
- For example when finding the factors of 18
  - 1 and 18 will be the first factor pair
  - Divide by 2, 3, 4 and so on to test if they are factors
    - 18 ÷ 2 = 9, so 9 and 2 are factors
    - 18 ÷ 3 = 6, so 6 and 3 are factors
    - 18 ÷ 4 = 4.5, so 4 is not a factor
    - 18 ÷ 5 = 3.6, so 5 is not a factor
    - 18 ÷ 6 would be next, but we have already found that 6 was a factor
    - So we have now found all the factors of 18: 1, 2, 3, 6, 9

How do I find factors without a calculator?

- Use a **divisibility test**
  - Some tests are easier to remember, and more useful, than others

- Once you know that the number has a particular factor, you can divide by that factor to find the factor pair
- Instead of a divisibility test, you could use a formal written method to divide by a value
  - If the result is an integer; you have found a factor

What are some useful divisibility tests?

- A number is **divisible by 2** if the last digit is even (a multiple of 2)
- A number is **divisible by 3** if the sum of the digits is divisible by 3 (a multiple of 3)
  - 123
    $1+2+3=6$; 6 is a multiple of 3, so 123 is divisible by 3
  - 134
    $1+3+4=8$; 8 is not a multiple of 3, so 134 is not divisible by 3
- A number is **divisible by 4** if halving the number **twice** results in an integer
- A number is **divisible by 8** if it can be halved **3 times** and the result is an integer
- A number is **divisible by 5** if the last digit is a 0 or 5
- A number is **divisible by 10** if the last digit is a 0

What are prime numbers?

- A **prime** number is a number which has exactly two (**distinct**) **factors;** itself and 1
  - You should remember at least the first ten prime numbers:
    - 2, 3, 5, 7, 11, 13, 17, 19, 23, 29
- 1 is **not** a prime number, because:
  - by definition, prime numbers are integers greater than or equal to 2
  - 1 only has one factor
- 2 is the only even prime number
- If a number has any factors other than itself and 1, it is not a prime number

Worked Example

Show that 51 is **not** a prime number.

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If we can find a factor of 51 (that is not 1 or 51), this will prove it is not prime

51 is not even so is not divisible by 2
Next use the divisibility test for 3

$$5 + 1 = 6;\ 6 \text{ is divisible by 3; therefore 51 is divisible by 3}$$
$$51 \div 3 = 17$$

The factors of 51 are 1, 3, 17 and 51

51 is not prime as it has more than two (distinct) factors

What are square numbers?

- A **square** number is the result of multiplying a number by itself
  - The first square number is $1 \times 1 = 1$, the second is $2 \times 2 = 4$ and so on
- The first 15 square numbers are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225
  - Aim to remember at least the first fifteen square numbers
- In algebra, square numbers can be written using a power of 2
  - $a \times a = a^2$

What are cube numbers?

- A **cube** number is the result of multiplying a number by itself, twice
  - The first cube number is $1 \times 1 \times 1 = 1$, the second is $2 \times 2 \times 2 = 8$ and so on
- The first 5 cube numbers are 1, 8, 27, 64 and 125
  - Aim to remember at least the first five cube numbers
  - You should also remember $10^3 = 1000$
- In algebra, cube numbers can be written using a power of 3
  - $a \times a \times a = a^3$

What are square roots?

- The **square root** of a value, is the number that when multiplied by itself equals that value
  - For example, 4 is the square root of 16
  - It is the opposite of squaring

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- Square roots are indicated by the symbol $\sqrt{\ }$
  - e.g. The square root of 49 would be written as $\sqrt{49}$
- Square roots can be **positive and negative**
  - e.g. The square roots of 25 are 5 and -5
- If a negative square root is required then a - sign would be used
  - e.g. $\sqrt{25} = 5$ but $-\sqrt{25} = -5$
  - Sometimes both positive and negative square roots are of interest and would be indicated by $\pm\sqrt{25}$
- The square root of a non-square **integer** is also called a **surd**
  - e.g. $\sqrt{3}$ is a surd, as 3 is not a square number
  - Surds are **irrational** numbers
  - $\sqrt{64}$ is **rational,** as it is equal to 8
  - However, $\sqrt{2}$ is **irrational**, as 2 is not a square number
- You should aim to remember the square roots of the first 15 square numbers
  - $\sqrt{1}, \sqrt{4}, \sqrt{9}, \sqrt{16}, \sqrt{25}, \sqrt{36}, \sqrt{49}, \sqrt{64}, \sqrt{81}, \sqrt{100}, \sqrt{121}, \sqrt{144}, \sqrt{169}, \sqrt{196}, \sqrt{225}$

What are cube roots?

- The **cube root** of a value, is the number that when multiplied by itself twice equals that value
  - For example, 3 is the cube root of 27
  - It is the opposite of cubing
  - Cube roots are indicated by the symbol $\sqrt[3]{\ }$
    - e.g. The cube root of 64 would be written as $\sqrt[3]{64}$
  - You should remember the values of the following cube roots:
    - $\sqrt[3]{1}, \sqrt[3]{8}, \sqrt[3]{27}, \sqrt[3]{64}, \sqrt[3]{125}, \sqrt[3]{1000}$

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Worked Example

Write down a number which is both a cube number and a square number, and hence express this number in two different ways using index notation.

Listing the first 12 square numbers

1, 4, 9, 16, 25, 36, 49, **64**, 81, 100, 121, 144

Listing the first 5 cube numbers

1, 8, 27, **64**, 125

64 appears in both lists, it is the 8th square number and 4th cube number

64 is both a square and cube number
$64 = 8^2$ and $64 = 4^3$

What is a reciprocal?

- The **reciprocal** of a number is the result of **dividing 1 by that number**
  - Any number multiplied by its reciprocal will be equal to 1
    - The reciprocal of an integer is $\frac{1}{\text{integer}}$
    - The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$
    - The fraction is flipped upside-down!
- The reciprocal of 3 is $\frac{1}{3}$
  - The reciprocal of $\frac{1}{3}$ is 3
  - $3 \times \frac{1}{3} = \frac{1}{3} \times 3 = 1$
- The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$

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- The reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$
- $\frac{2}{3} \times \frac{3}{2} = \frac{3}{2} \times \frac{2}{3} = 1$

- Algebraically the reciprocal of a is $\frac{1}{a}$
  - The reciprocal of $\frac{1}{a}$ is a
  - This can also be written using a power of -1
    - $\frac{1}{a} = a^{-1}$

Worked Example

Write down a fraction that completes this calculation: $\frac{3}{7} \times \frac{...}{...} = 1$

Recall that a number multiplied by its reciprocal is equal to 1

$$\frac{3}{7} \times \boldsymbol{\frac{7}{3}} = 1$$

Prime Factor Decomposition

Your notes

Prime Factor Decomposition

What are prime factors?

- A **factor** of a given number is a value that divides the given number exactly, with no remainder
  - e.g. 6 is a factor of 18
- A **prime** number is a number which has exactly two **factors;** itself and 1
  - e.g. 5 is a prime number, as its only factors are 5 and 1
  - You should remember the first few prime numbers:
    - 2, 3, 5, 7, 11, 13, 17, 19, ...
- The **prime factors** of a number are therefore all the primes which multiply to give that number
  - e.g. The prime factors of 30 are 2, 3, and 5
    - $2 \times 3 \times 5 = 30$

How do I find prime factors?

- Use a **factor tree** to find prime factors
  - Split the number up into a **pair of factors**
  - Then split each of those factors up into **another pair**
  - Continue splitting up factors along each "branch" until you get to a **prime number**
    - These can not be split into anything other than 1 and themselves
    - It helps to **circle the prime** numbers at the end of the branches

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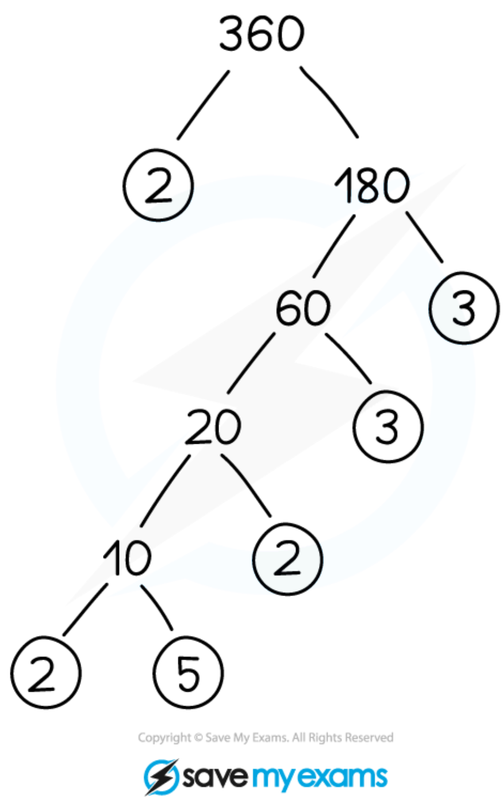

- A number can be uniquely written as a **product of prime factors**
  - Write the prime factors as a multiplication, in ascending order
    - $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$
  - This can then be written more concisely using powers
    - $360 = 2^3 \times 3^2 \times 5$
- A question asking you to do this will usually be phrased as "Express ... as the product of its prime factors"

Worked Example

Write 432 as the product of its prime factors.

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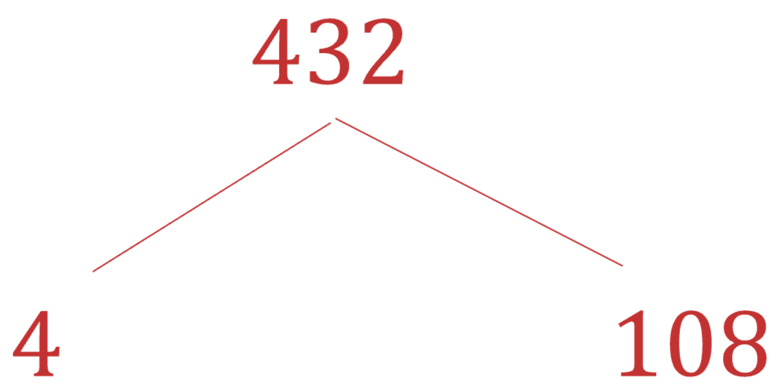
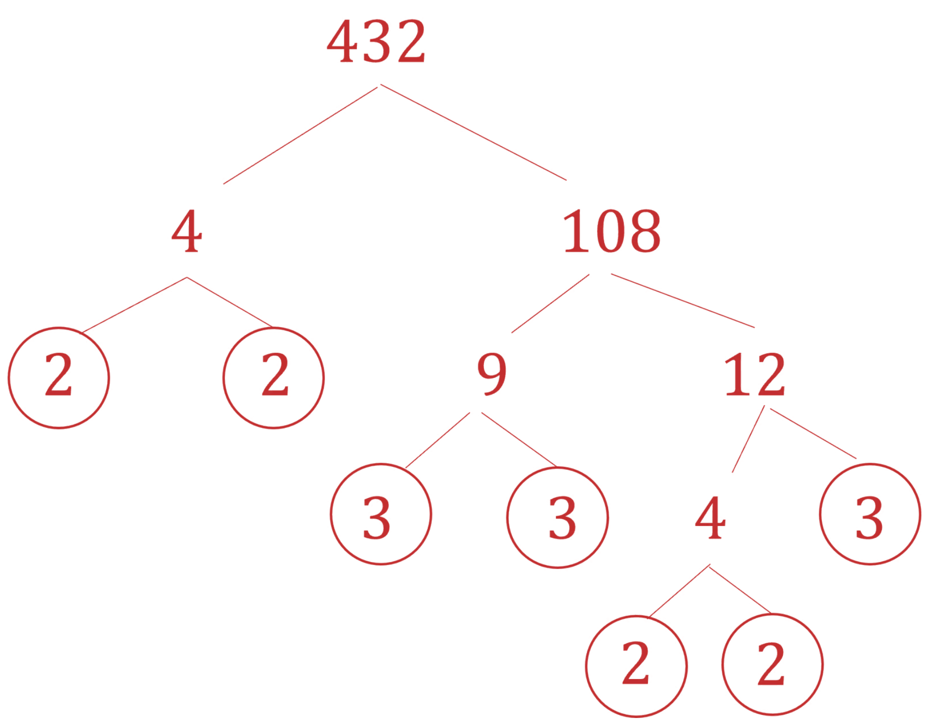
Create a factor tree
Start with 432 and choose any two numbers that multiply together to make 432



Repeat this for the two factors, until all of the values are prime numbers and cannot be broken down any further



The answer will be the same regardless of the factors chosen in the first step

Write the prime numbers out as a product

$$432 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3$$

Any repeated prime factors can be written as a power

$$432 = \mathbf{2^4} \times \mathbf{3^3}$$

Your notes

Uses of Prime Factor Decomposition

Your notes

Uses of Prime Factor Decomposition

When a number has been written as its **prime factor decomposition** (**PFD**), it can be used to find out if that number is a **square** or **cube** number, or to find the **square root** of that number without a calculator.

How can I use PFD to identify a square or cube number?

- If all the **indices** in the prime factor decomposition of a number are **even**, then that number is a **square** number
  - E.g. The prime factor decomposition of 7056 is $2^4 \times 3^2 \times 7^2$
  - All powers are even so it must be a square number
    - It can be written as $(2^2 \times 3 \times 7)^2$
- If all the indices in the prime factor decomposition of a number are **multiples of 3**, then that number is a **cube** number
  - E.g. The prime factor decomposition of 1728000 is $2^9 \times 3^3 \times 5^3$
  - All powers are multiples of 3 so it must be a cube number
    - It can be written as $(2^3 \times 3 \times 5)^3$

How can I use PFD to find the square root of a square number?

- Write the number in its prime factor decomposition
  - All the indices should be **even** if it is a square number
- For example, to find the square root of $144 = 2^4 \times 3^2$
  - **Halve** all of the indices
    - $2^2 \times 3$
    - So $\sqrt{2^4 \times 3^2} = 2^2 \times 3$
- This is the prime factor decomposition of the **square root** of the number
  - To find it as an integer, multiply the prime factors together
  - $2^2 \times 3 = 12$, so the square root of 144 is 12

How can I use PFD to find the exact square root of a number?

Your notes

- If the number is not a square number, its **exact square root** can still be found using its prime factor decomposition
- Write the number in its prime factor decomposition
  - $1440 = 2^5 \times 3^2 \times 5$
- Rewrite the prime factor decomposition with **as many even indices as you can**
  - E.g. $2^3 = 2^2 \times 2$, or $5^7 = 5^6 \times 5$
  - $1440 = 2^4 \times 2 \times 3^2 \times 5$
- Collect the terms with even powers together
  - $1440 = 2^4 \times 3^2 \times 2 \times 5$
- Square root both sides
  - $\sqrt{1440} = \sqrt{2^4 \times 3^2 \times 2 \times 5}$
- Using the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$, apply the square root to the terms with the even indices separately to the terms with odd indices
  - $\sqrt{1440} = \sqrt{2^4 \times 3^2} \times \sqrt{2 \times 5}$
- Simplify to find your answer, remembering that $\sqrt{a^2b^2} = ab$
  - $\sqrt{1440} = 2^2 \times 3 \times \sqrt{10}$
  - $\sqrt{1440} = 12\sqrt{10}$
  - $12\sqrt{10}$ is the **exact** square root of 1440

Worked Example

$N = 2^3 \times 3^2 \times 5^7$ and $AN = B$ where A is an integer and B is a non-zero square number.

Find the smallest value of A.

Substitute $N = 2^3 \times 3^2 \times 5^7$ into the formula $AN = B$

$$A(2^3 \times 3^2 \times 5^7) = B$$

To be a square number, the prime factors of *AN* must all have even powers

Consider the prime factors *A* needs to have to make all the values on the left hand side have even powers

$$(2 \times 5)(2^3 \times 3^2 \times 5^7) = B$$

$$2^4 \times 3^2 \times 5^8 = B$$

So *A*, when written as a product of its prime factors, is 2×5

Make sure you write *A* as an integer value in the answer

_A_ = 10

Your notes

Your notes

HCF & LCM

Highest Common Factor (HCF)

What is the highest common factor (HCF) of two numbers?

- A **common factor** of two numbers is a value that **both numbers** can be divided by, leaving no remainder
  - **1** is always a common factor of any two numbers
  - Any **factor** of a **common factor** will also be a common factor of the original two numbers
    - 6 is a common factor of 24 and 30
    - Therefore 1, 2 and 3 are also common factors of 24 and 30
- The **highest** common factor is the largest common factor of the two numbers
  - The highest common factor is useful when simplifying fractions or factorising expressions

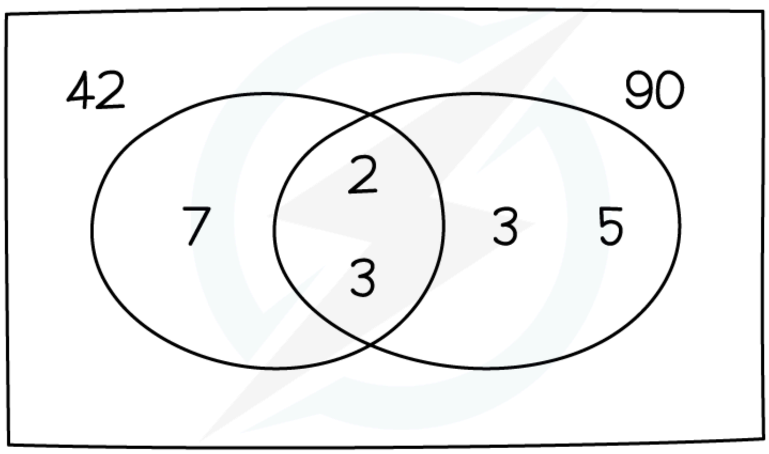
How do I find the highest common factor (HCF) of two numbers?

- To find **common factors**:
  - write out the **factors** of each number in a list
  - identify the numbers that appear in **both** lists
- The **highest** common factor will be the **largest factor** that appears in **both** lists

How can I use a Venn diagram to find the highest common factor (HCF) of two numbers?

- Write each number as a **product of its prime factors**
  - $42 = 2 \times 3 \times 7$ and $90 = 2 \times 3 \times 3 \times 5$
- Find the prime factors that are **common** to **both numbers** and put these in the **centre of the Venn diagram**
  - 42 and 90 both have a prime factor of 2
    - Put 2 in the centre of the diagram
  - Although 3 appears twice in the prime factors of 90, it appears once in the prime factors of 42
    - Put a single 3 in the centre of the diagram

- If there are no common prime factors, put a 1 in the centre of the diagram
- Put the **remaining prime factors** in the respective regions
  - 7 would go in the region for 42
  - 3 and 5 would go in the region for 90
- The **highest common factor** is the **product** of the **numbers in the centre**
  - The HCF of 42 and 90 is 2×3, which is 6
- If there are no common prime factors then the HCF is 1



How can I use the powers of prime factors to find the highest common factor (HCF) of two numbers?

- Write each number as a **product of the powers of its prime factors**
  - $24 = 2^3 \times 3$ and $60 = 2^2 \times 3 \times 5$
- Find all **common** prime factors and identify the **highest power** that appears in both numbers
  - The highest power of 2 in both is 2^2
    - 2^2 is a common factor
  - The highest power of 3 in both is 3^1
    - 3 is a common factor

Your notes

- No other prime number appears in both
- The **highest common factor** is the **product** of the common powers of primes
  - The HCF of 24 and 60 is $2^2 \times 3$ which is 12

Examiner Tips and Tricks

- The highest common factor of two numbers could be one of the numbers!
  - The highest common factor of 4 and 12 is 4

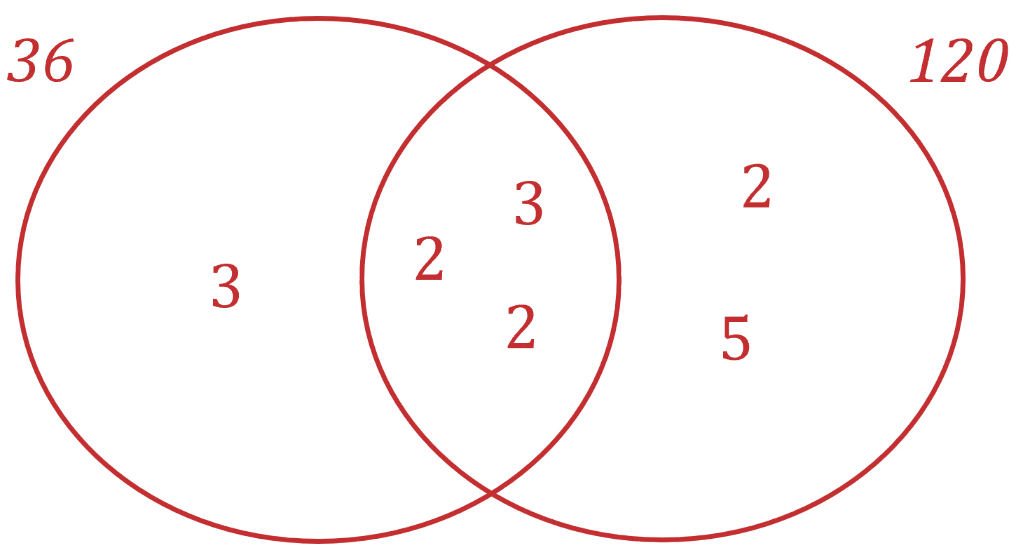
Worked Example

Find the highest common factor of 36 and 120.

Write both numbers as a product of prime factors

$$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$$
$$120 = 2 \times 2 \times 2 \times 3 \times 5 = 2^3 \times 3 \times 5$$

Write the prime factors in a Venn diagram



Multiply the common prime factors in the centre

Your notes

Your notes

$$HCF = 2 \times 2 \times 3$$

Alternatively, list the factors for each number

36: 1, 2, 3, 4, 6, 9, 12, 18, 36
120: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120

Another alternative is to find the highest common powers of primes

2^2 and 3^1 are the highest common powers of primes
$$HCF = 2^2 \times 3^1$$

HCF = 12

Lowest Common Multiple (LCM)

What is the lowest common multiple (LCM) of two numbers?

- A **common multiple** of two numbers is a number that appears in **both** of their times tables
  - The **product** of the original two numbers is always a common multiple (but not necessarily the lowest)
  - **Any multiple** of a **common multiple** will **also be a common multiple** of the original two numbers
    - 30 is a common multiple of 3 and 10
    - Therefore 60, 90, 120, ... are also common multiples of 3 and 10
- The **lowest common multiple** is the smallest common multiple between two numbers
  - This is useful when finding a common denominator and when adding or subtracting fractions

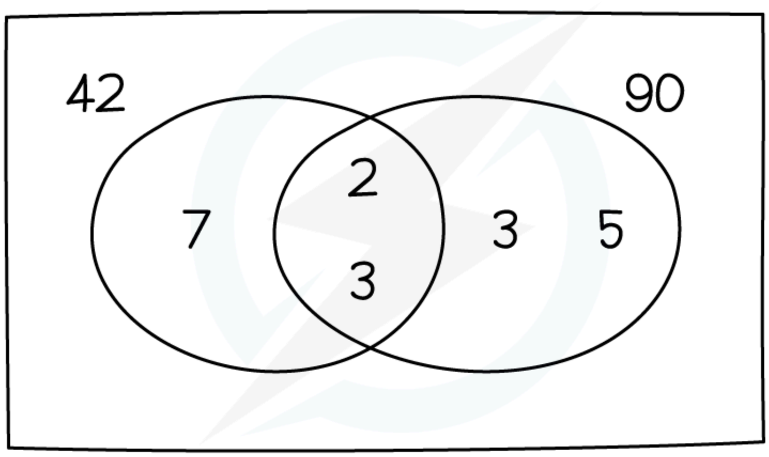
How do I find the lowest common multiple (LCM) of two numbers?

- To find the **lowest common multiple** of two numbers:
  - write out the first few **multiples** of each number
  - identify the multiples that appear in **both** lists
    - If there are **none** then write out the **next few multiples** of each number until you find a common multiple
- The **lowest common multiple** will be the **smallest** multiple that appears in **both** lists

How can I use a Venn diagram to find the lowest common multiple (LCM) of two numbers?

Your notes

- Write each number as a **product of its prime factors**
  - 42 = 2×3×7 and 90 = 2×3×3×5
- Find the prime factors that are **common** to **both numbers** and put these in the **centre of the Venn diagram**
  - 42 and 90 both have a prime factor of 2
    - Put a 2 in the centre of the diagram
  - Although 3 appears twice in the prime factors of 90, it appears once in the prime factors of 42
    - Put a single 3 in the centre of the diagram
  - If there are no common prime factors then put a 1 in the centre of the diagram
- Put the **remaining prime factors** in the respective regions
  - 7 would go in the region for 42
  - 3 and 5 would go in the region for 90
- The **lowest common multiple** is the **product** of **all the numbers in the Venn diagram**
  - The LCM of 42 and 90 is 7×2×3×3×5, which is 630



How can I use the powers of prime factors to find the lowest common multiple (LCM) of two numbers?

Your notes

- Write each number as a **product of the powers of its prime factors**
  - $72 = 2^3 \times 3^2$ and $540 = 2^2 \times 3^3 \times 5$
- Find the **highest power** of **each and every** prime that appears in either number (they do **not** have to be **common** primes)
  - 2^3 is the highest power of 2 shown (from 72)
  - 3^3 is the highest power of 3 shown (from 540)
  - 5^1 is the highest power of 5 shown (from 540)
    - Note: 5 is not a common prime in 72 and 540, but it is still needed for the LCM
- The **lowest common multiple** is the **product** of these **highest powers**
  - The LCM of 72 and 540 is $2^3 \times 3^3 \times 5$, which is 1080

Examiner Tips and Tricks

- The lowest common multiple of two numbers could be one of the numbers!
- The lowest common multiple of 4 and 12 is 12

Worked Example

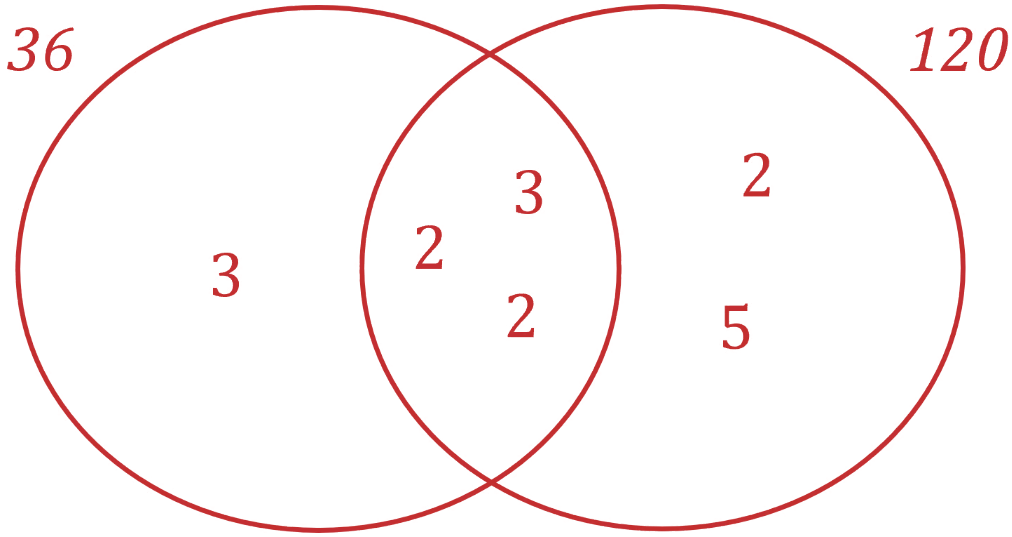
Find the lowest common multiple of 36 and 120.

Write both numbers as a product of prime factors

$$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$$
$$120 = 2 \times 2 \times 2 \times 3 \times 5 = 2^3 \times 3 \times 5$$

Write the prime factors in a Venn diagram

Your notes



Multiply all the prime factors in the diagram

$$\text{LCM} = 3 \times 2 \times 2 \times 3 \times 2 \times 5$$

An alternative method is to write out the multiples

36: 26, 72, 108, 144, 180, 216, 252, 288, 324, <u>360</u>, 396, ...
120: 120, 240, <u>360</u>, 480, ...

Another alternative method is to find the highest powers of each and every prime that appear
Then multiply these together

$$2^3 \times 3^2 \times 5^1$$

LCM = 360