AQA GCSE Maths: Higher

Your notes

Sequences

Contents

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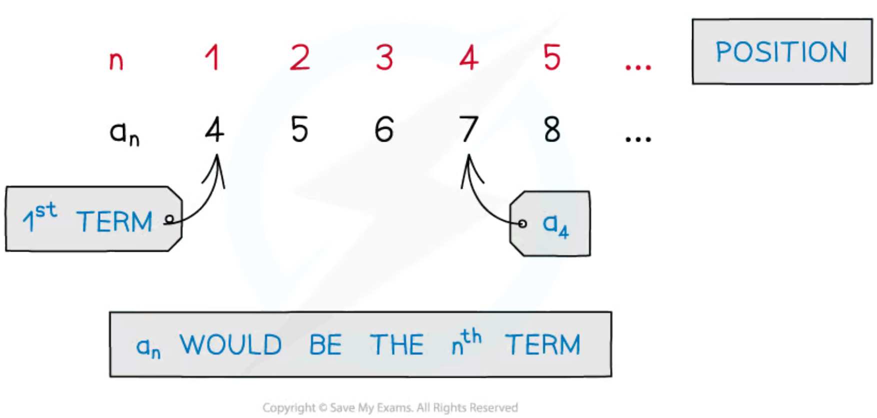
Introduction to Sequences

Introduction to Sequences

What are sequences?

- A **sequence** is an ordered set of numbers that follow a **rule**
  - For example 3, 6, 9, 12...
    - The rule is to add 3 each time
- Each number in a sequence is called a **term**
- The **location** of a term within a sequence is called its **position**
  - The letter n is used for position
    - $n = 1$ refers to the **1st term**
    - $n = 2$ refers to the **2nd term**
    - If you do not know its position, you can say the ***n*** **th term**
- Another way to show the position of a term is using **subscripts**
  - A general sequence is given by $a_1, a_2, a_3, \ldots$
    - a_1 represents the **1st term**
    - a_2 represents the **2nd term**
    - a_n represents the ***n*th term**



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How do I write out a sequence using a term-to-term rule?

- **Term-to-term rules** tell you how to get the **next term** from the term you are on
  - It is what you do **each time**
  - For example, starting on 4, add 10 each time
    - 4, 14, 24, 34, ...

How do I write out a sequence using a position-to-term rule?

- A **position-to-term rule** is an **algebraic expression** in n that lets you find **any term** in the sequence
  - This is also called the **n th term formula**
- You need to know what **position** in the sequence you are looking for
  - To get the **1st term**, substitute in $n = 1$
  - To get the **2nd term**, substitute in $n = 2$
- You can jump straight to the 100th term by substituting in $n = 100$
  - You do not need to find all 99 previous terms
- For example, the n th term is $8n + 2$
  - The 1st term is $8 \times 1 + 2 = 10$
  - The 2nd term is $8 \times 2 + 2 = 18$
  - The 100th term is $8 \times 100 + 2 = 802$

How do I know if a value belongs to a sequence?

- If you know the n th term formula, set the value equal to the formula
  - This creates an **equation** to **solve** for n
- For example, a sequence has the n th term formula $8n + 2$
  - Is 98 in the sequence?

$$8n + 2 = 98$$
$$8n = 96$$
$$n = \frac{96}{8}$$
$$n = 12$$

- It is in the sequence, it is the 12th term
- Is 124 in the sequence?

$$8n + 2 = 124$$
$$8n = 122$$
$$n = \frac{122}{8}$$
$$n = 15.25$$

- n is **not** a **whole number**, so it is **not** in the sequence

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Examiner Tips and Tricks

- In the exam, it helps to write the position number (the value of n) above each term in the sequence.

Worked Example

A sequence has the n th term formula $3n + 2$.

(a) Find the first three terms in the sequence.

Substitute $n = 1$, $n = 2$ and $n = 3$ into the formula

$$3 \times 1 + 2 = 5$$
$$3 \times 2 + 2 = 8$$
$$3 \times 3 + 2 = 11$$

5, 8, 11

(b) Find the 80th term.

Substitute $n = 80$ into the formula

$$3 \times 80 + 2$$

The 80th term is 242

(c) Determine whether the number 96 is in the sequence.

Set the formula equal to 96

$$3n + 2 = 96$$

Solve to find *n*
If *n* is a whole number, it is a term in the sequence

$$3n = 96 - 2$$
$$3n = 94$$
$$n = \frac{94}{3}$$

94 is not divisible by 3
The nearest possible value is 95 ((95–2) ÷ 3 = 31, the 31st term)

96 is not in the sequence

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Types of Sequences

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Types of Sequences

What other sequences are there?

- **Linear** and **quadratic** sequences are particular types of sequence covered in previous notes
- Other sequences include **geometric** and **Fibonacci** sequences, which are looked at in more detail below
- Other sequences include cube numbers and triangular numbers
- Another common type of sequence in exam questions, is fractions with combinations of the above
  - Look for anything that makes the position-to-term and/or the term-to-term rule easy to spot

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TYPES OF SEQUENCES

LINEAR

e.g. $2 \xrightarrow{+4} 6 \xrightarrow{+4} 10 \xrightarrow{+4} 14 \xrightarrow{+4} 18 \; \dots$

FIRST DIFFERENCES CONSTANT

QUADRATIC

e.g. $5 \xrightarrow{+4} 9 \xrightarrow{+6} 15 \xrightarrow{+8} 23 \xrightarrow{+10} 33 \; \dots$

$+4 \xrightarrow{+2} +6 \xrightarrow{+2} +8 \xrightarrow{+2} +10$

SECOND DIFFERENCES CONSTANT

GEOMETRIC

e.g. $4 \xrightarrow{\times 3} 12 \xrightarrow{\times 3} 36 \xrightarrow{\times 3} 108 \xrightarrow{\times 3} 324 \; \dots$

CONSTANT MULTIPLIER (COMMON RATIO)

FIBONACCI

e.g. 2 4 6 10 16 26 42 ...

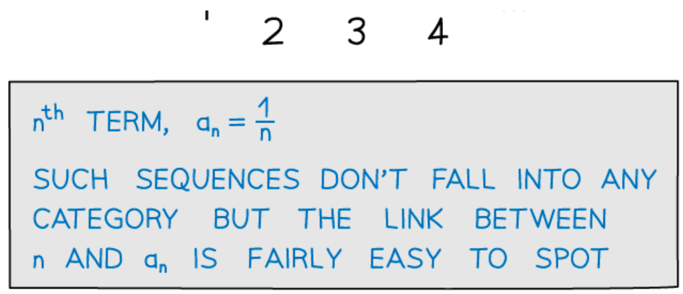
$2 + 4 = 6$, $4 + 6 = 10$, $10 + 16 = 26$, $16 + 26 = 42$

ADD THE PREVIOUS TWO TERMS

OTHER

e.g.

n	1	2	3	4	
	1	1	1	1	...



What is a geometric sequence?

- A geometric sequence can also be referred to as a **geometric progression** and sometimes as an **exponential sequence**
- In a geometric sequence, the term-to-term rule would be to multiply by a constant, ***r***
  - $a_{n+1} = r.a_n$
- ***r*** is called the **common ratio** and can be found by dividing any two consecutive terms, or
  - $r = a_{n+1} / a_n$
- In the sequence 4, 8, 16, 32, 64, ... the common ratio, *r*, would be 2 (8 ÷ 4 or 16 ÷ 8 or 32 ÷ 16 and so on)

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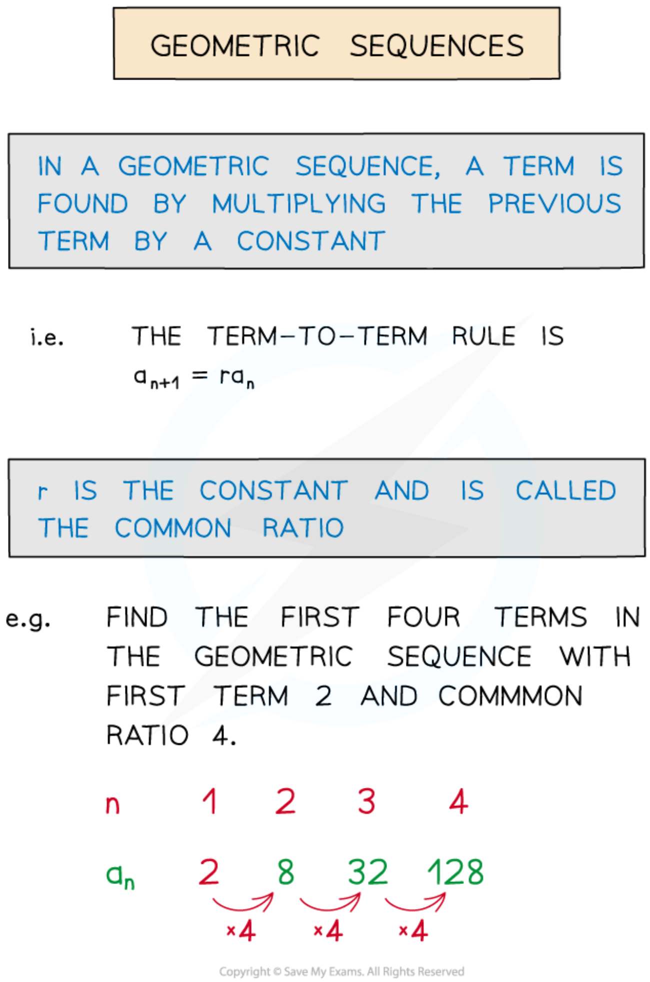

What is a Fibonacci sequence?

- **The** Fibonacci sequence is **1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...**
- The sequence starts with the **first two terms** as **1**
- Each subsequent term is the **sum** of the **previous two**
  - ie The term-to-term rule is $\mathbf{a_{n+2} = a_{n+1} + a_n}$
  - Notice that two terms are needed to start a Fibonacci sequence
- Any sequence that has the term-to-term rule of adding the previous two terms is called **a** Fibonacci sequence but the first two terms will not both be 1
- Fibonacci sequences occur a lot in nature such as the number of petals of flowers

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FIBONACCI SEQUENCES

IN A FIBONACCI SEQUENCE, A TERM IS FOUND BY ADDDING THE PREVIOUS TWO TERMS TOGETHER

i.e. THE TERM-TO-TERM RULE IS

$$a_{n+2} = a_{n+1} + a_n$$

NOTICE THAT TWO TERMS WILL BE NEEDED TO START OFF WITH

e.g. FIND THE FIRST SIX TERMS OF A FIBONACCI SEQUENCE THAT HAS FIRST TERM 2 AND SECOND TERM 9

n	1	2	3	4	5	6
a_n	2	9	11	20	31	51

$a_3 = a_2 + a_1$
$9 + 2 = 11$

$a_5 = a_4 + a_3$
$20 + 11 = 31$

Problem solving and sequences

- When the type of sequence is known it is possible to find unknown terms within the sequence
- This can lead to problems involving setting up and solving equations
  - Possibly simultaneous equations
- Other problems may involve sequences that are related to common number sequences such as square numbers, cube numbers and triangular numbers

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e.g. IN A FIBONACCI SEQUENCE THE 4^{th} TERM IS $2a$, AND THE 5^{th} TERM IS $b + 1$

a) WRITE DOWN EXPESSIONS FOR THE 6^{th} AND 7^{th} TERMS

6^{th} TERM $= (b + 1) + 2a$

$a_6 = a_5 + a_4$

6^{th} TERM $= 2a + b + 1$

7^{th} TERM $= (2a + b + 1) + (b + 1)$

$a_7 = a_6 + a_5$

7^{th} TERM $= 2a + 2b + 2$

b) GIVEN $a_6 = 20$ AND $a_7 = 32$
FIND THE VALUES OF a AND b

$2a + b + 1 = 20$
$2a + 2b + 2 = 32$

SOLVE AS SIMULTANEOUS EQUATIONS

$2a + b = 19$
$2a + 2b = 30$

$b = 11$

SUBTRACT

$2a + 11 = 19$
$2a = 8$
$a = 4$

SUBSTITUTE VALUE OF b INTO ANY EQUATION USED

$a = 4$ AND $b = 11$

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Worked Example

a)

Identify the types of sequence below;

i) 4, 5, 9, 14, 23, 37, 60, ...

ii) 6, 10, 16, 24, 34, ...

iii) 12, 7, 2, -3, ...

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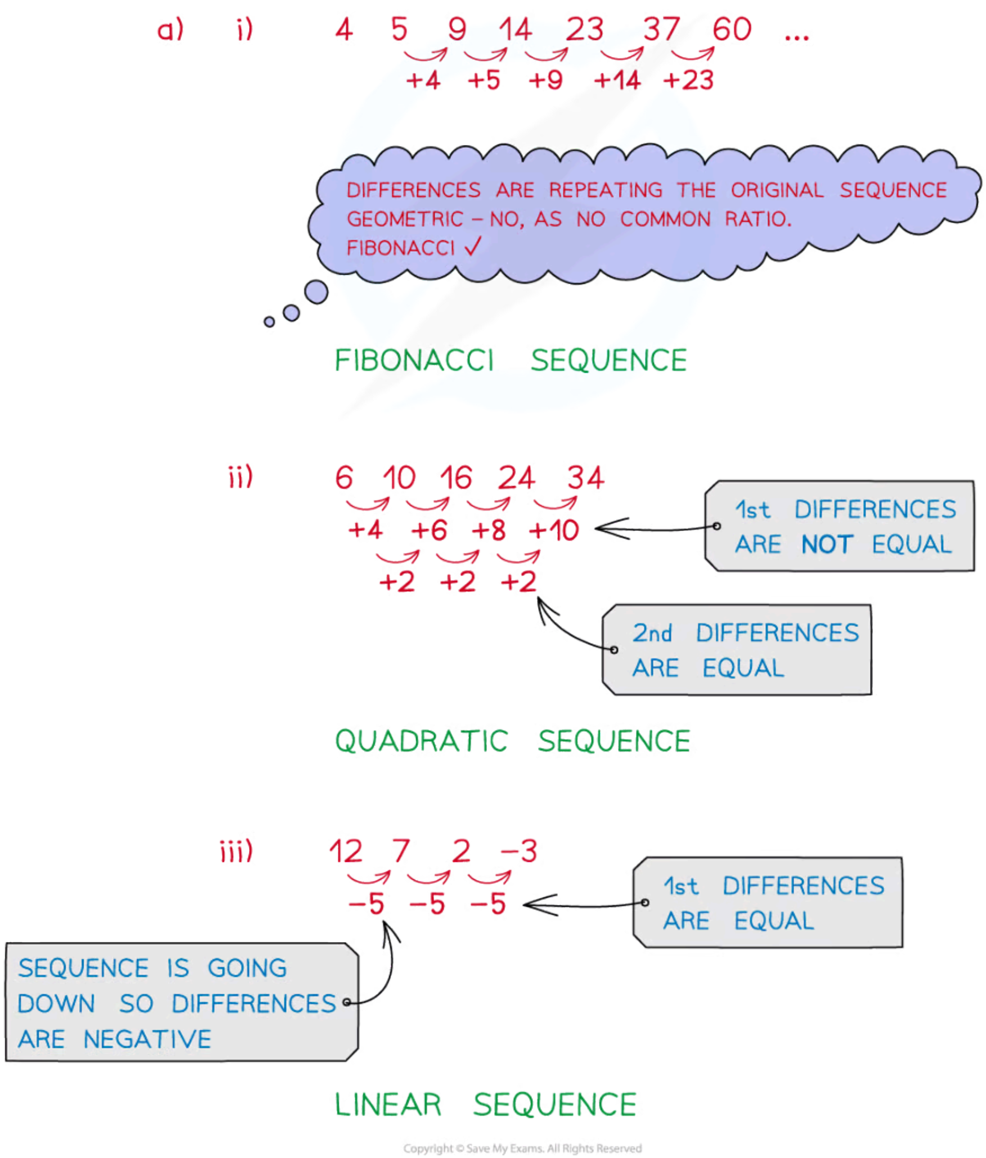

b)

The 3rd and 6th terms in a Fibonacci sequence are 7 and 31 respectively.

Find the 1st and 2nd terms of the sequence.

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b)

WRITE AT WHAT YOU DO KNOW ABOUT THE SEQUENCE

n	1	2	3	4	5	6	7
a_n		x	7	$x+7$	$x+14$	31	

$a_{n+2} = a_{n+1} + a_n$

FIBONACCI – TOLD IN QUESTION

$a_3 = a_2 + a_1$ $\quad\quad a_2 + a_1 = 7$

$a_6 = a_5 + a_4$ $\quad\quad a_5 + a_4 = 31$

LET $a_2 = x$, THEN, $a_4 = x + 7$

AND $a_5 = (x + 7) + 7$

$= x + 14$

SO $(x + 14) + (x + 7) = 31$ $\quad\quad a_6 = a_5 + a_4 = 31$

$2x + 21 = 31$

$2x = 10$

$x = 5$ $\quad\quad a_2 = 5$

$a_3 = a_2 + a_1$

$7 = 5 + a_1$ $\quad\quad a_1 = 2$

nth Terms of Linear Sequences

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Linear Sequences

What is a linear sequence?

- A **linear** sequence goes **up** (or **down**) by the **same amount** each time
- This amount is called the **common difference, *d***
  - For example:
    1, 4, 7, 10, 13, ...(adding 3, so $d = 3$)
    15, 10, 5, 0, -5, ... (subtracting 5, so $d = -5$)
- Linear sequences are also called **arithmetic** sequences

How do I find the n^{th} term formula for a linear sequence?

- The formula is ***n* th term = *dn* + *b***
  - ***d*** is the **common difference**
    - The amount it goes up by each time
  - ***b*** is the value **before** the **first** term (sometimes called the **zero** term)
    - Imagine going **backwards**
- For example 5, 7, 9, 11, ...
  - The sequence adds 2 each time
    - $d = 2$
  - Now continue the sequence backwards, from 5, by one term
    - (3), 5, 7, 9, 11, ...
    - $b = 3$
  - So the n th term = $2n + 3$
- For example 15, 10, 5, ...
  - Subtracting 5 each time means $d = -5$
  - Going backwards from 15 gives 15 + 5 = 20
    - (20), 15, 10, 5, ... so $b = 20$

- The n th term = $-5n + 20$

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Worked Example

Find a formula for the n^{th} term of the sequence -7, -3, 1, 5, 9, ...

The n th term is $dn + b$ where d is the common difference and b is the term before the 1st term
The sequence goes up by 4 each time

$$d = 4$$

Continue the sequence backwards by one term (-7–4) to find b

$$(-11), -7, -3, 1, 5, 9, \ldots$$

$$b = -11$$

Substitute $d = 4$ and $b = -11$ into $dn + b$

n^{th} term = $4n - 11$

Quadratic Sequences

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Quadratic Sequences

What is a quadratic sequence?

- A **quadratic** sequence has an ***n* th term formula** that involves n^2
- The **second differences** are **constant** (the same)
  - These are the differences between the first differences
  - For example, 3, 9, 19, 33, 51, ...
    1st Differences: 6, 10, 14, 18, ...

    2nd Differences: 4, 4, 4, ...
- The sequence with the *n* th term formula n^2 are the **square numbers**
  - 1, 4, 9, 16, 25, 36, 49, ...
    - From $1^2, 2^2, 3^2, 4^2, \ldots$

How do I find the n^{th} term formula for a simple quadratic sequence?

- **STEP 1**
  Work out the **sequences** of **first** and **second differences**
  - e.g. for the sequence 1, 10, 23, 40, 61

sequence	1	10	23	40	61
first difference		+9	+13	+17	+21
second difference			+4	+4	+4

- **STEP 2**
  Divide the **second difference by 2** to find the coefficient of n^2
  - e.g. $a = 4 \div 2 = 2$
- **STEP 3**
  Write out the **first three** or **four** terms of an^2 and **subtract** the terms from the **corresponding terms** of the given sequence

- e.g. for the sequence 1, 10, 23, 40, 61

sequence	1	10	23	40
$2n^2$	2	8	18	32
difference	-1	2	5	8

- **STEP 4**
  Work out the ***nth* term** of these **differences** to find the $bn + c$
  - e.g. the nth term of -1, 2, 5, 8, ... is $bn + c = 3n - 4$
- **STEP 5**
  Find $an^2 + bn + c$ by adding together this linear nth term to an^2
  - e.g. $an^2 + bn + c = 2n^2 + 3n - 4$

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Examiner Tips and Tricks

- You must learn the square numbers from 1^2 to 15^2

Worked Example

For the sequence 5, 7, 11, 17, 25, ...

(a) Find a formula for the n^{th} term.
Start by finding the first and second differences

Sequence: 5, 7, 11, 17, 25

First differences: 2, 4, 6, 8, ...

Second difference: 2, 2, 2, ...

Hence

$$a = 2 \div 2 = 1$$

Now write down an^2 (just n^2 in this case as $a = 1$) and subtract the terms from the original sequence

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$$\text{sequence: } 5, 7, 11, 17, \ldots$$

$$an^{2}: 1, 4, 9, 16, \ldots$$

$$\text{difference: } 4, 3, 2, 1, \ldots$$

Work out the nth term of these differences to give you $bn + c$

$$bn + c = -n + 5$$

Add an^2 and $bn + c$ together to give you the n^{th} term of the sequence

n^{th} term = $n^2 - n + 5$

(b) Hence find the 20^{th} term of the sequence.

Substitute $n = 20$ into $n^2 - n + 5$

$$(20)^2 - 20 + 5 = 400 - 15$$

20^{th} term = 385