AQA GCSE Maths: Higher

Your notes

Vectors

Contents

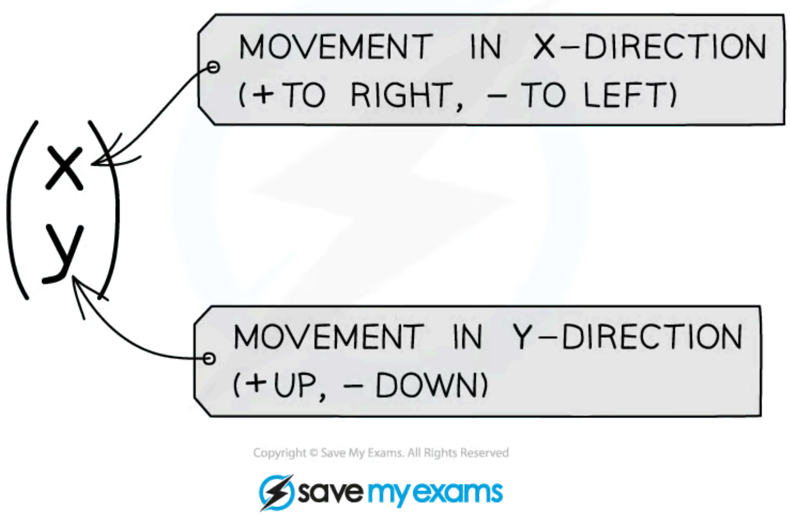
Your notes

Introduction to Column Vectors

Basic Vectors

What are column vectors?

- A **column vector** can be used to **describe** how to get **from one point to another point**
  - This is also called a **translation vector**
  - $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ means **6 units to the right** and **3 units up**



How do I add and subtract column vectors?

- **Adding and subtracting vectors** is done by looking at the **top** numbers and **bottom** numbers **separately**
- To **add** column vectors
  - Add the top numbers together
  - Add the bottom numbers together

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- $\begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}5+3\\2+(-1)\end{pmatrix}=\begin{pmatrix}8\\1\end{pmatrix}$
- To **subtract** column vectors
  - Subtract the second top number from the first
  - Subtract the second bottom number from the first
  - $\begin{pmatrix}5\\2\end{pmatrix}-\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}5-3\\2-(-1)\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$

How do I multiply a vector by a scalar?

- A **scalar** is **number** not a **vector**
  - It does not have a direction
- To **multiply** a column vector by a **scalar**
  - Multiply the top number by the scalar
  - Multiply the bottom number by the scalar
  - $3\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}3\times 2\\3\times(-1)\end{pmatrix}=\begin{pmatrix}6\\-3\end{pmatrix}$

How do I write an expression as a single column vector?

- You need to follow the **order of operations**
  - $2\begin{pmatrix}5\\2\end{pmatrix}+5\begin{pmatrix}3\\-1\end{pmatrix}$
- **STEP 1**
  Multiply each vector by the scalar in front of it
  - $\begin{pmatrix}2\times 5\\2\times 2\end{pmatrix}+\begin{pmatrix}5\times 3\\5\times(-1)\end{pmatrix}=\begin{pmatrix}10\\4\end{pmatrix}+\begin{pmatrix}15\\-5\end{pmatrix}$
- **STEP 2**
  Add or subtract the new column vectors
  - $\begin{pmatrix}10+15\\4+(-5)\end{pmatrix}=\begin{pmatrix}25\\-1\end{pmatrix}$

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Worked Example

$\mathbf{a} = \begin{pmatrix} p \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Given that $2\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} 4 \\ q \end{pmatrix}$, find the value of p and the value of q.

Write the left-side side as one vector
Multiple each vector by the scalar in front of it

$$\begin{pmatrix} 2p \\ 6 \end{pmatrix} + \begin{pmatrix} -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ q \end{pmatrix}$$

Add the vectors together

$$\begin{pmatrix} 2p-6 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ q \end{pmatrix}$$

The top components are equal
Form and solve an equation

$$\begin{aligned} 2p - 6 &= 4 \\ 2p &= 10 \\ p &= 5 \end{aligned}$$

The bottom components are equal

$$9 = q$$

$\boldsymbol{p = 5}$ and $\boldsymbol{q = 9}$

Representing Vectors as Diagrams

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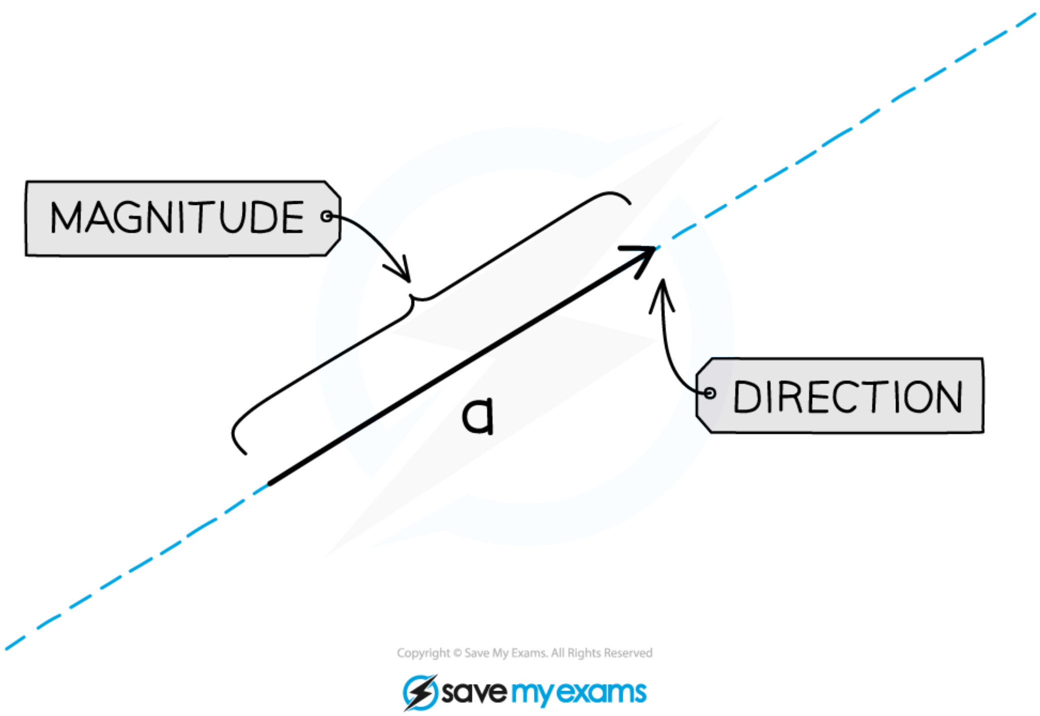
Vector Diagrams

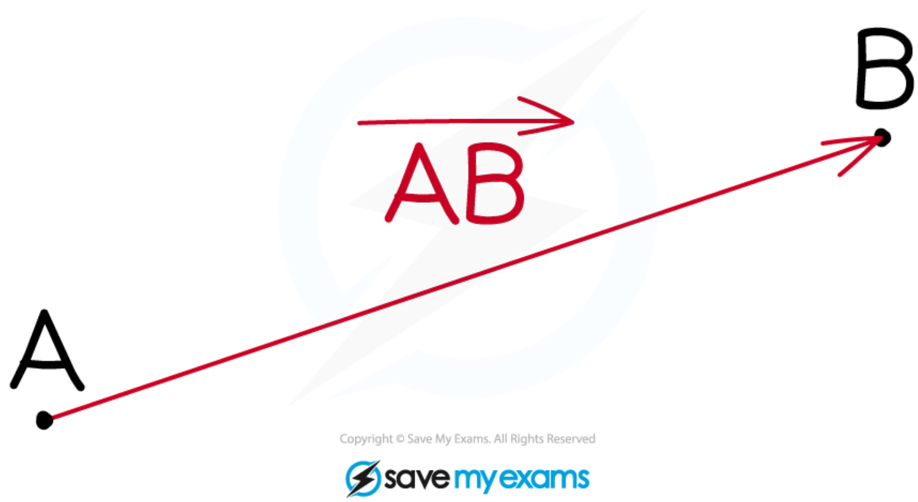
How can I represent a vector visually?

- A **vector** has **both a size (magnitude) and a direction**
  - You need to draw a **line** to show the **size of the vector**
  - You also need to draw an **arrow** to show the **direction of the vector**



- Vectors are written in **bold** when typed to **show** that they are a vector and not a scalar
  - When writing a vector in an exam you should **underline** the letter to show it is a vector
  - $\boldsymbol{a}$ when **typed** and $\underline{a}$ when **handwritten**
    - You will **not lose marks** if you forget to underline vectors

- If a vector **starts at *A*** and **ends at *B*** we can write it as $\overrightarrow{AB}$
  - Here the arrow will **point toward *B***
  - Vector $\overrightarrow{BA}$ will have the **same length** but **point toward *A***



How do I draw a vector on a grid?

- You can draw a vector **anywhere** on a grid
  - Just make sure it has the **correct length** and the **correct direction**
- To draw the vector $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
  - **Pick** a point on the grid and draw a **dot** there
  - Count 3 units to the right and 4 units up and **draw another dot**
  - Draw a **line between the two dots**
  - Put an **arrow** on the line **pointing toward the second dot**
- Look out for **negatives** and **zeroes**
  - $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ goes **2 to the right** and **4 down**

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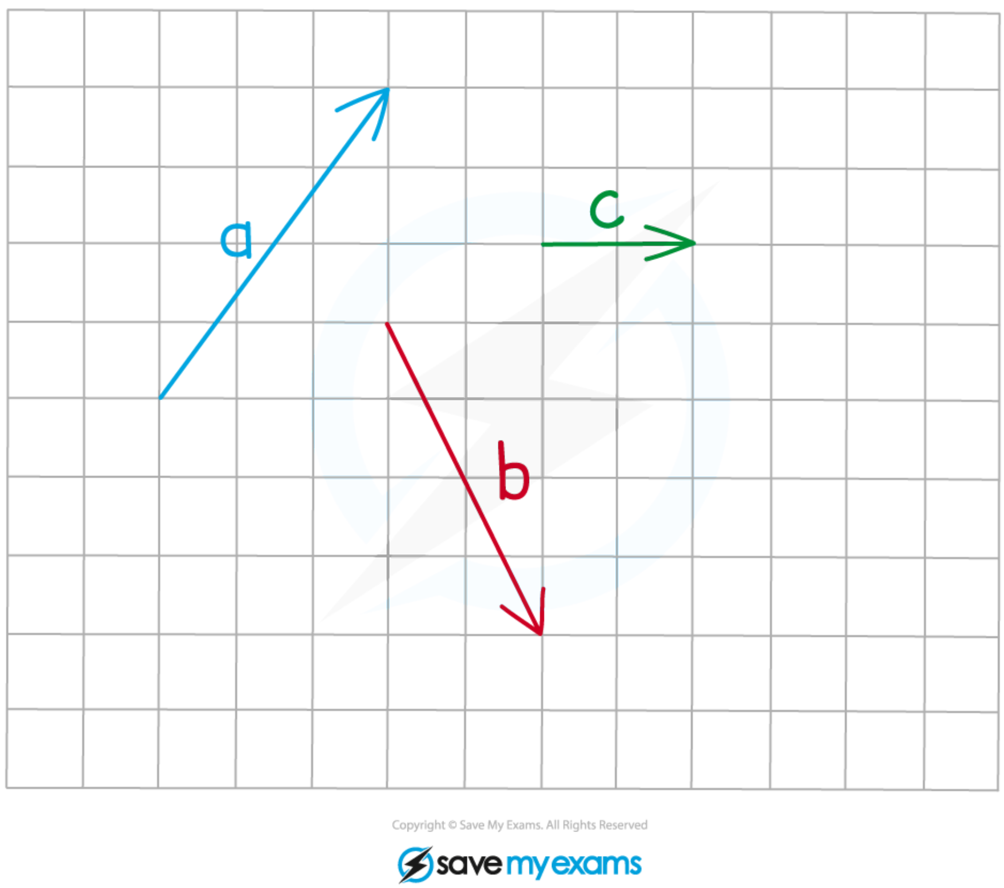
- $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ goes **2 to the right** but **does not go up or down**



What happens when I multiply a vector by a scalar?

- When you **multiply** a vector by a **positive scalar**:
  - The **direction** stays the **same**
  - The **length** of the vector is **multiplied by the scalar**
- For example, $\mathbf{a} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

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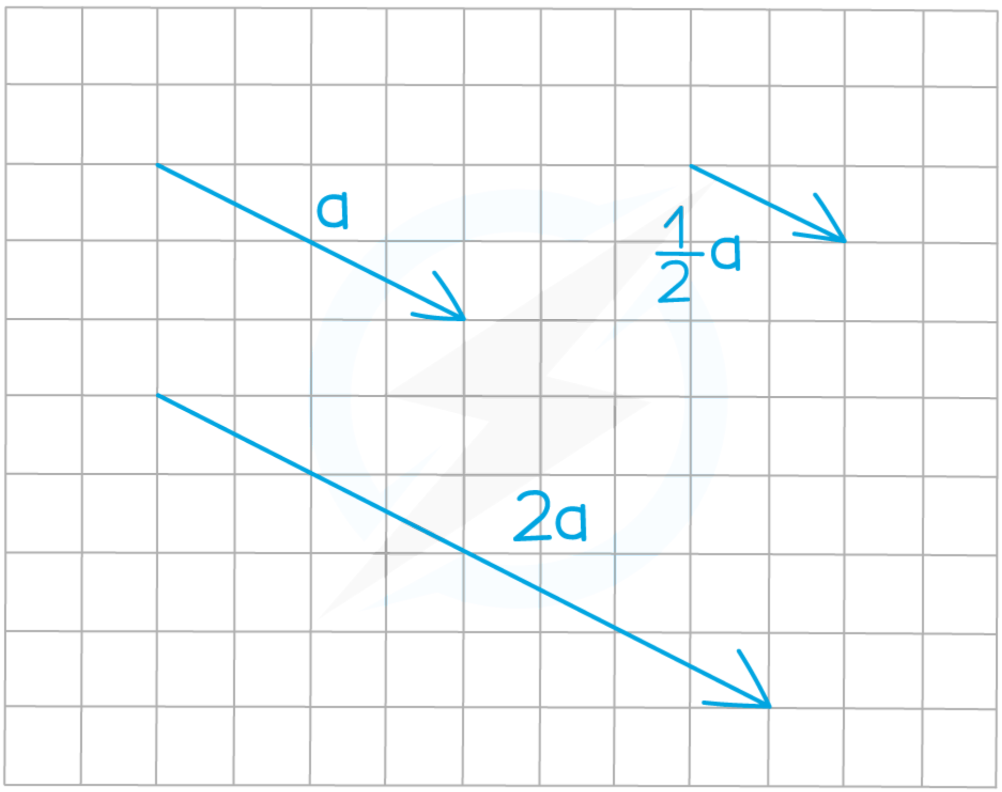
- $2\mathbf{a}=\begin{pmatrix}8\\-4\end{pmatrix}$ will have the **same direction** but **double the length**
- $\frac{1}{2}\mathbf{a}=\begin{pmatrix}2\\-1\end{pmatrix}$ will have the **same direction** but **half the length**



- When you **multiply** a vector by a **negative scalar**:
  - The **direction** is **reversed**
  - The **length** of the vector is **multiplied** by the **number after the negative sign**
- For example, $\mathbf{a}=\begin{pmatrix}4\\-2\end{pmatrix}$

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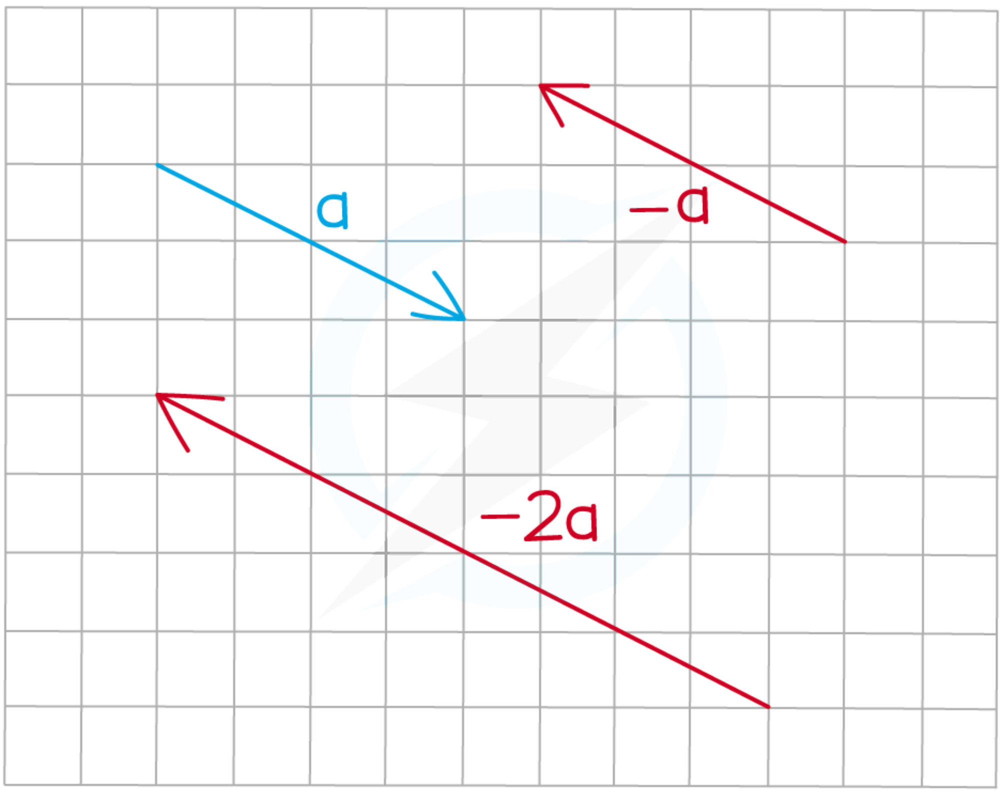
- $-\mathbf{a} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ will be in the **opposite direction** and its **length will be the same**
- $-2\mathbf{a} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$ will be in the **opposite direction** and its **length will be doubled**



What happens when I add or subtract vectors?

- To **draw** the vector $\mathbf{a} + \mathbf{b}$
  - Draw the vector $\mathbf{a}$
  - Draw the vector $\mathbf{b}$ starting at the endpoint of $\mathbf{a}$

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- Draw a line that **starts at the start** of $\mathbf{a}$ and **ends at the end** of $\mathbf{b}$
- To **draw** the vector $\mathbf{a} - \mathbf{b}$
  - Draw the vector $\mathbf{a}$
  - Draw the vector $-\mathbf{b}$ starting at the endpoint of $\mathbf{a}$
  - Draw a line that **starts at the start** of $\mathbf{a}$ and **ends at the end** of $-\mathbf{b}$

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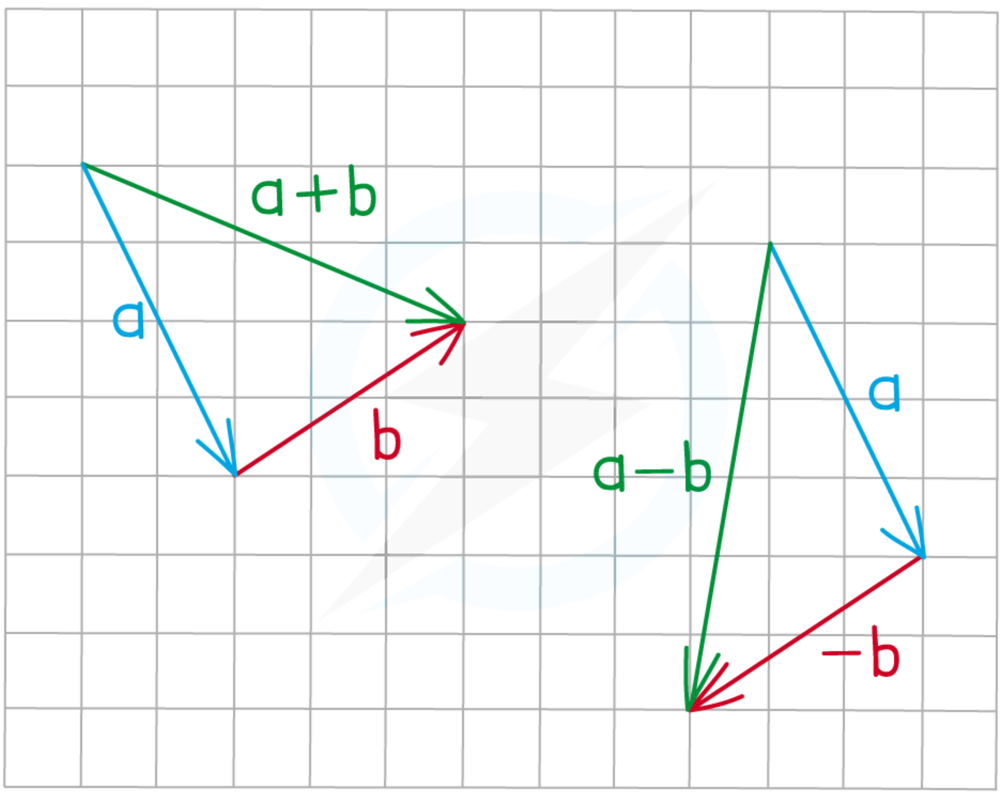


Worked Example

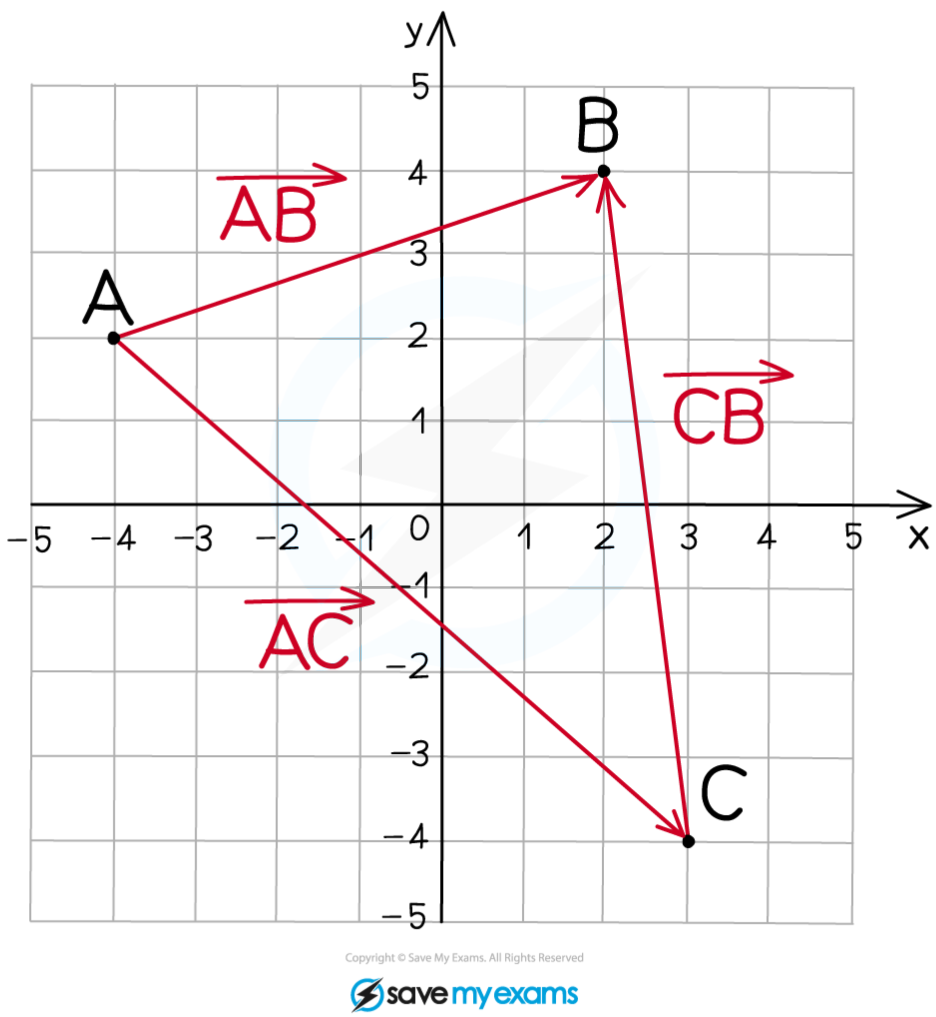
The points *A*, *B* and *C* are shown on the following coordinate grid.

Your notes

(a)

Write the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{CB}$ as column vectors.

Start by drawing the three vectors onto the grid



From *A* to *B*, it is 6 to the right and 2 up

$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

From *A* to C, it is 7 to the right and 6 down

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$$\overrightarrow{\mathbf{AC}} = \begin{pmatrix} 7 \\ -6 \end{pmatrix}$$

From C to *B*, it is 1 to the left and 8 up

$$\overrightarrow{\mathbf{CB}} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

(b)

Without using any calculations, explain why $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The vector goes from A to B, then from B to C, then from C back to A

The vector returns to its starting point

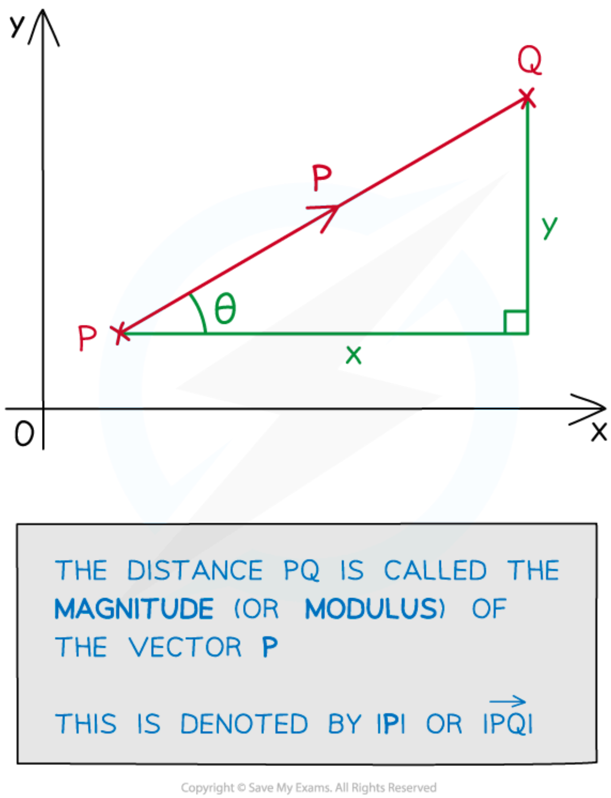
Length of a Vector

Your notes

Length of a Vector

How do I find the magnitude of a vector?

- The **magnitude** of a **vector** is its **length** (distance)
  - It is also called the **modulus**
  - This is always a **positive** value
  - The direction of the vector is irrelevant
- The magnitude of $\overrightarrow{AB}$ is written $\left|\overrightarrow{AB}\right|$
  - The magnitude of **a** is written |**a**|
- Depending on the use of the vector, the magnitude of a vector represents **different quantities**
  - For velocity, magnitude would be speed
  - For a force, magnitude would be the strength of the force (in Newtons)
- In **component** form, the magnitude is the **hypotenuse** of a right-angled triangle
  - Use **Pythagoras' theorem** to find the magnitude
  - The magnitude of $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$
    - $|\mathbf{a}| = \sqrt{x^2 + y^2}$



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Examiner Tips and Tricks

- If there is no diagram, sketch one!
  - You can sketch a vector and use it to form a right-angled triangle

Worked Example

Consider two points $A(-3,\ 5)$ and $B(7,\ 1)$.

(a) Write down the column vector $\overrightarrow{AB}$.

Your notes

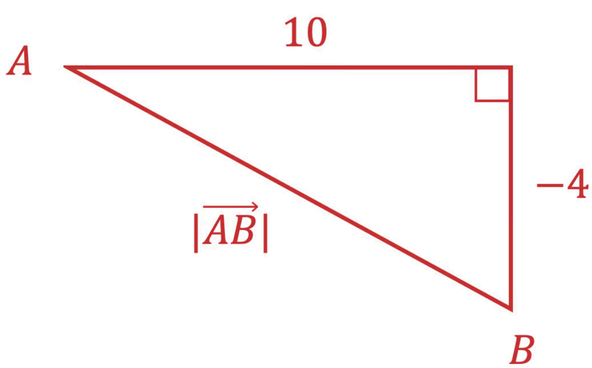
Find the horizontal and vertical distances between the two points
Subtract the x and y components of A from B

$$\overrightarrow{AB} = \begin{pmatrix} 7 - -3 \\ 1 - 5 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$$

(b) Find the modulus of vector $\overrightarrow{AB}$.

Sketching a diagram of the vector $\overrightarrow{AB}$ can help



Apply Pythagoras' theorem to the x and y components of $\overrightarrow{AB}$

$$\begin{aligned} \left|\overrightarrow{AB}\right| &= \sqrt{10^2 + (-4)^2} \\ &= \sqrt{100 + 16} \\ &= \sqrt{116} \end{aligned}$$

$$\left|\overrightarrow{AB}\right| = 2\sqrt{29}$$

(c) Briefly explain why $\left|\overrightarrow{BA}\right| = \left|\overrightarrow{AB}\right|$.

The magnitude of a vector is it's 'size'

Direction of the vector is ignored

$\left|\overrightarrow{BA}\right| = \left|\overrightarrow{AB}\right|$ since both vectors have the same distance

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Another vector, $\overrightarrow{CD}$, has three times the magnitude of vector $\overrightarrow{AB}$.

(d) Write down a possible column vector for $\overrightarrow{CD}$.

Being three times $\left|\overrightarrow{AB}\right|$ means the vector $\overrightarrow{AB}$ is three times longer

One way to find a vector is to multiply each component of the vector $\overrightarrow{AB}$ by 3 or -3

$$\overrightarrow{CD} = 3\overrightarrow{AB} = 3\begin{pmatrix} 10 \\ -4 \end{pmatrix}$$

$$\overrightarrow{CD} = \begin{pmatrix} 30 \\ -12 \end{pmatrix}$$

Another possible answer is $\overrightarrow{CD} = \begin{pmatrix} -30 \\ 12 \end{pmatrix}$

Position & Displacement Vectors

Your notes

Position & Displacement Vectors

What are position vectors?

- A **position vector** describes where a specific **point**, A, is, relative to a fixed **origin**, O
  - **Lower-case bold** (or underlined) letters are used
    - The point A has position vector **a** = $\overrightarrow{OA}$
- Their **components** are equal to their **coordinates**
  - The point with coordinates (3, -2) has position vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ from the origin

What are displacement vectors?

- A **displacement vector** describes the **direction and distance** between **two points**
  - The displacement vector from A to B is $\overrightarrow{AB}$
    - How to get from A to B
- If the points A and B have position vectors **a** and **b** relative to O
  - then A to B is the same as A to O (**-a**) followed by O to B (**b**)
    - $\overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$
    - This is a useful rule to remember

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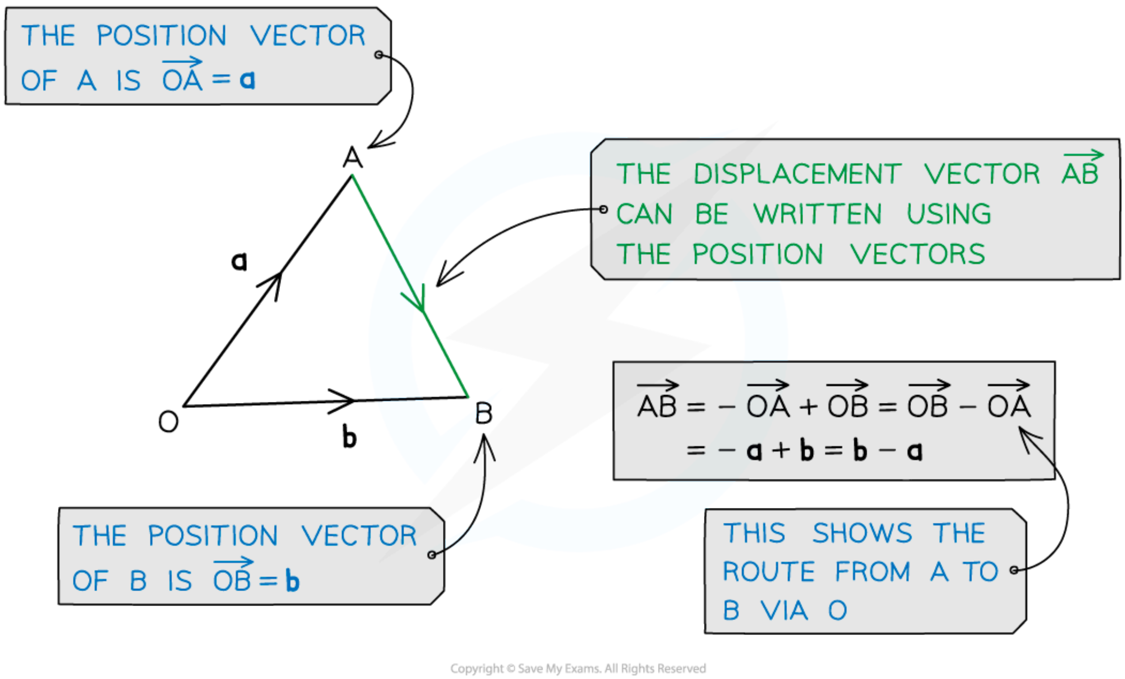


Examiner Tips and Tricks

- You may need to draw an origin, O, on to a diagram to be able to sketch position vectors.

Worked Example

The points P and Q have position vectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -10 \end{pmatrix}$ respectively.

Find and simplify the vector $\overrightarrow{PQ}$.

Let $\mathbf{p}$ and $\mathbf{q}$ be position vectors of P and Q

$\overrightarrow{PQ}$ is the displacement vector from P to Q

Use the rule that $\overrightarrow{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$$

Substitute in $\mathbf{p}$ and $\mathbf{q}$

$$\overrightarrow{PQ} = \begin{pmatrix} 6 \\ -10 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Expand and simplify

$$= \begin{pmatrix} 6-3 \\ -10-2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -12 \end{pmatrix}$$

$$\boldsymbol{\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -12 \end{pmatrix}}$$

You can also get this answer by seeing what vector must be added to $\mathbf{p}$ to get $\mathbf{q}$

Your notes

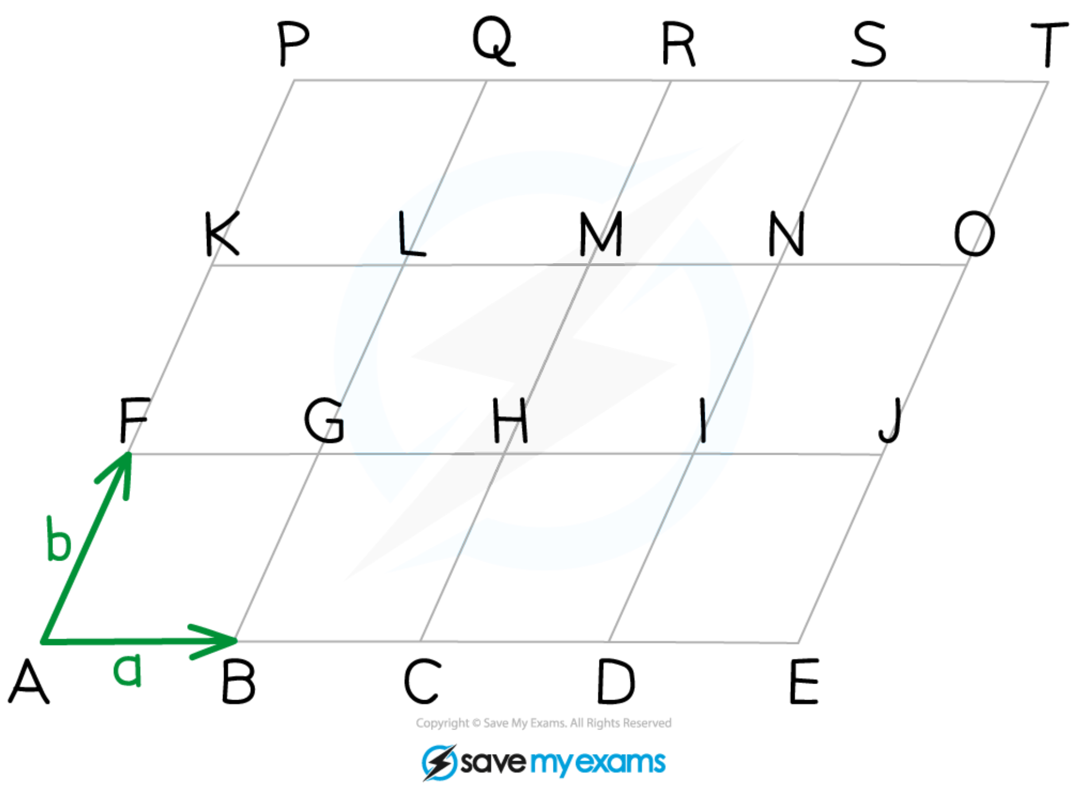
Finding Vector Paths

Your notes

Finding Vector Paths

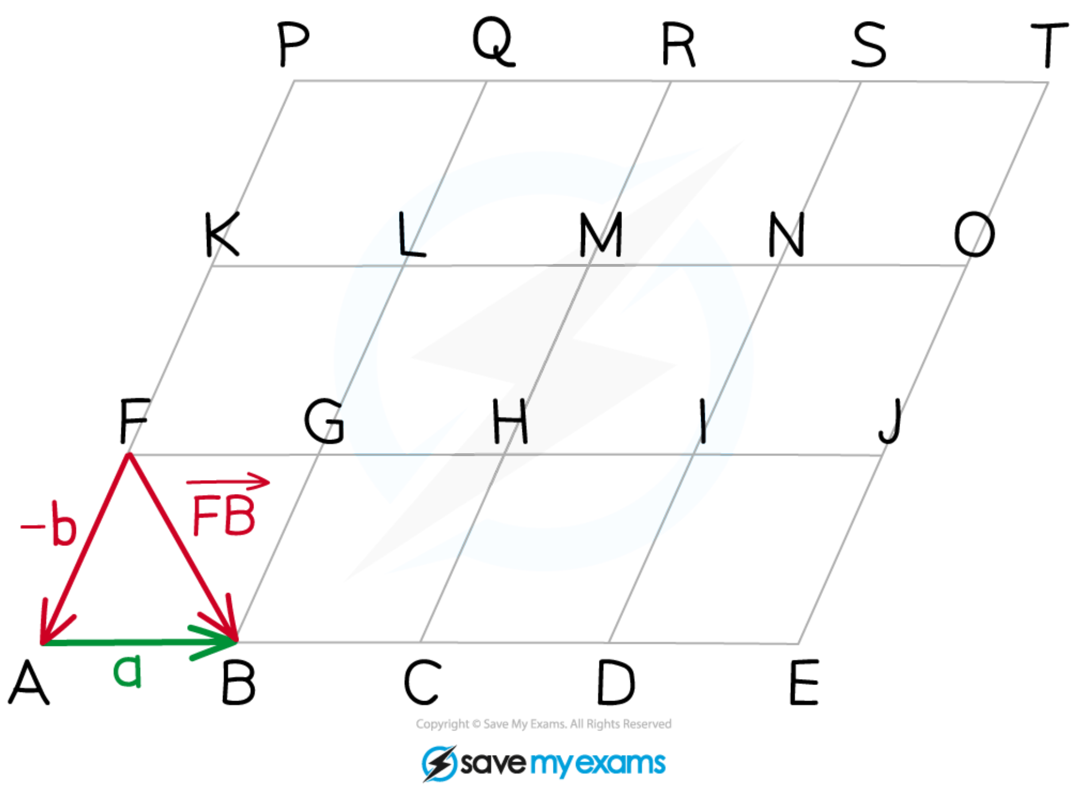
How do I find the vector between two points?

- A **vector path** is a path of vectors taking you from a **start point** to an **end point**
- The following grid is made up entirely of parallelograms
  - The vectors **a** and **b** defined as marked in the diagram:
    - Any vector that goes **horizontally to the right** along a side of a parallelogram will be equal to **a**
    - Any vector that goes **up diagonally to the right** along a side of a parallelogram will be equal to **b**



- To find the vector between two points

- Count how many times you need to go **horizontally to the right**
  - This will tell you how many **a**'s are in your answer
- Count how many times you need to go up **diagonally to the right**
  - This will tell you how many **b**'s are in your answer
- Add the **a**'s and **b**'s together
  - E.g. $\overrightarrow{AR} = 2\mathbf{a} + 3\mathbf{b}$
- You will have to put a **negative** in front of the vector if it goes in the **opposite direction**
  - -**a** is one length **horizontally to the left**
  - -**b** is one length **down diagonally to the left**
    - E.g. $\overrightarrow{FB} = -\mathbf{b} + \mathbf{a}$ or $\overrightarrow{FB} = \mathbf{a} - \mathbf{b}$
    - Likewise, $\overrightarrow{BF} = -\overrightarrow{FB} = -(-\mathbf{b} + \mathbf{a}) = \mathbf{b} - \mathbf{a}$



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- It is possible to describe **any vector** that goes from one point to another in the above diagram in terms of **a** and **b**

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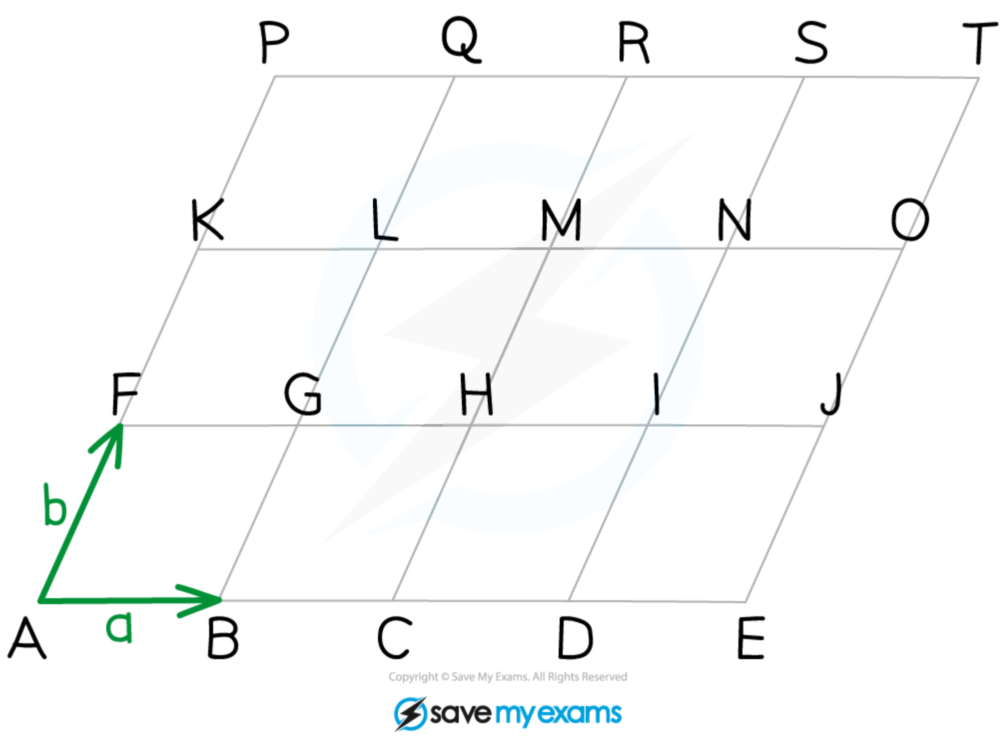
Examiner Tips and Tricks

- Mark schemes will accept different correct paths, as long as the final answer is fully simplified
- Check for symmetries in the diagram to see if the vectors given can be used anywhere else

Worked Example

The following diagram consists of a grid of identical parallelograms.

Vectors **a** and **b** are defined by $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AF}$.



Write the following vectors in terms of **a** and **b**.

Your notes

a) $\overrightarrow{AE}$

To get from *A* to *E* we need to follow vector **a** four times to the right

$$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$$
$$= \mathbf{a} + \mathbf{a} + \mathbf{a} + \mathbf{a}$$

$$\overrightarrow{AE} = 4\mathbf{a}$$

b) $\overrightarrow{GT}$

There are many ways to get from G to *T*
One option is to go from G to Q (**b** twice), and then from Q to *T* (**a** three times)

$$\overrightarrow{GT} = \overrightarrow{GL} + \overrightarrow{LQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST}$$
$$= \boldsymbol{b} + \boldsymbol{b} + \boldsymbol{a} + \boldsymbol{a} + \boldsymbol{a}$$

$$\overrightarrow{GT} = 3\mathbf{a} + 2\mathbf{b}$$

c) $\overrightarrow{EK}$

There are many ways to get from *E* to *K*
One option is to go from *E* to O (**b** twice), and then from O to *K* (-**a** four times)

$$\overrightarrow{EK} = \overrightarrow{EJ} + \overrightarrow{JO} + \overrightarrow{ON} + \overrightarrow{NM} + \overrightarrow{ML} + \overrightarrow{LK}$$
$$= \mathbf{b} + \mathbf{b} - \mathbf{a} - \mathbf{a} - \mathbf{a} - \mathbf{a}$$

$$\overrightarrow{EK} = 2\boldsymbol{b} - 4\boldsymbol{a}$$

$-4\mathrm{a} + 2\mathrm{b}$ is also acceptable

Your notes

Problem Solving with Vectors

Vector Problem Solving

What are vector proofs?

- Vectors can be used to **prove** things that are true in geometrical diagrams
  - **Vector proofs** can be used to find additional information that can help us to **solve problems**

How do I know if two vectors are parallel?

- Two vectors are **parallel** if one is a **scalar multiple** of the other
  - This means if **b** is parallel to **a**, then **b** = *k***a**
    - where *k* is a **constant number** (scalar)
- For example, $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$
  - $\begin{pmatrix} 2 \\ 6 \end{pmatrix} = 2 \times \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ so $\mathbf{b} = 2\mathbf{a}$
  - **b** is a scalar multiple of **a**, so **b** is **parallel** to **a**
- If the scalar multiple is **negative**, then the vectors are **parallel** and in **opposite** directions
  - $\mathbf{c} = \begin{pmatrix} -3 \\ -9 \end{pmatrix} = -3\mathbf{a}$
    - **c** is **parallel** to **a** and in the **opposite** direction
- If two vectors **factorise** with a **common bracket**, then they are parallel
  - They can be written as **scalar multiples**
- For example
  - 9**a** + 6**b** factorises to 3(3**a** + 2**b**)
  - 12**a** + 8**b** factorises to 4(3**a** + 2**b**)
  - This means $12\mathbf{a} + 8\mathbf{b} = \frac{4}{3}(9\mathbf{a} + 6\mathbf{b})$

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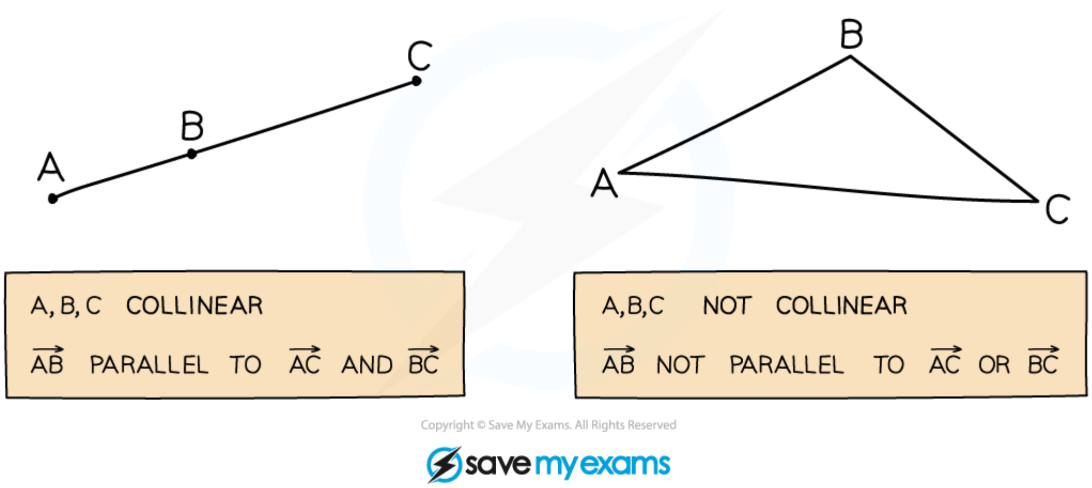
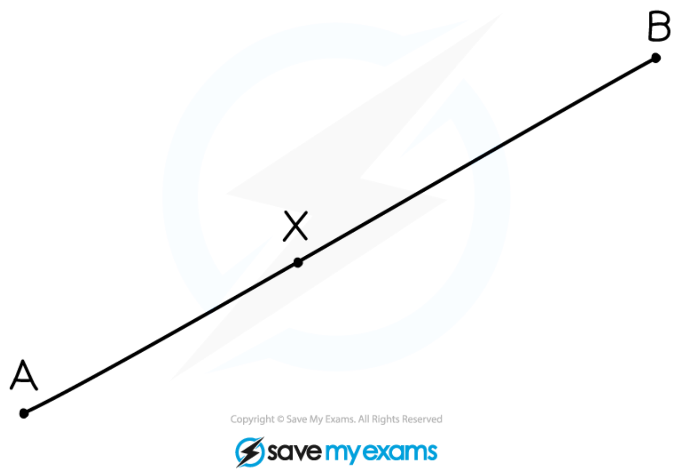
- so they are **scalar multiples** of each other
- and therefore **parallel**

How do I know if three points lie on a straight line?

- You may be asked to prove that three points lie on a straight line
  - Points that lie on a straight line are **collinear**
- To show that the points A, B and C are **collinear**
  - prove that two line segments are **parallel**
  - and show that there is at least **one point** that **lies on both** segments
    - This makes them parallel and **connected** (not parallel and side-by-side)
- For example, if you show that $\overrightarrow{BC} = 2\overrightarrow{AB}$ then
  - the line segments AB and BC are parallel
  - and they have a **common point**, B
    - So A, B and C must be collinear
- Similarly, $\overrightarrow{AC} = 3\overrightarrow{AB}$ means AC and AB are parallel
  - and they have a common point, A
    - so A, B and C must be collinear



How do I use ratios in vector paths?



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- Convert **ratios** into **fractions**
- In the example shown, if $AX : XB = 3:5$ then
  - $\overrightarrow{AX} = \frac{3}{8}\overrightarrow{AB}$
  - $\overrightarrow{XB} = \frac{5}{8}\overrightarrow{AB}$
    - The ratio 3:5 has 3 + 5 = 8 parts
- Always **check** which ratio you are being asked for
  - $\overrightarrow{AX} = \frac{3}{5}\overrightarrow{XB}$
  - $\overrightarrow{XB} = \frac{5}{3}\overrightarrow{AX}$

Worked Example

The diagram shows trapezium *OABC*.

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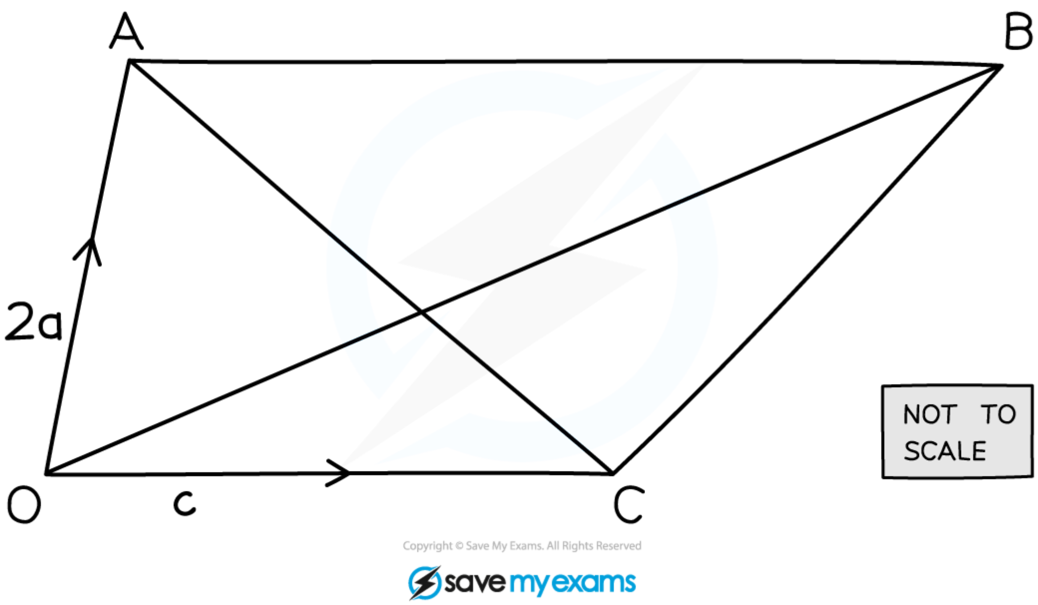
$\overrightarrow{OA} = 2\mathbf{a}$

$\overrightarrow{OC} = \mathbf{c}$

AB is parallel to OC, with $\overrightarrow{AB} = 3\overrightarrow{OC}$.



(a) Find expressions for vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of **a** and **c**.

$\overrightarrow{AB} = 3\overrightarrow{OC}$ and $\overrightarrow{OC} = \mathbf{c}$ so $\overrightarrow{AB} = 3\boldsymbol{c}$.

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\ &= 2\mathbf{a} + 3\mathbf{c}\end{aligned}$$

$$\boldsymbol{\overrightarrow{OB} = 2a + 3c}$$

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= -\overrightarrow{OA} + \overrightarrow{OC}\\ &= -2\mathbf{a} + \mathbf{c}\end{aligned}$$

$$\boldsymbol{\overrightarrow{AC} = c - 2a}$$

(b) Point P lies on AC such that $AP : PC = 3 : 1$.

Find expressions for vectors $\overrightarrow{AP}$ and $\overrightarrow{OP}$ in terms of **a** and **c**.

$AP : PC = 3 : 1$ means that $\overrightarrow{AP} = \frac{3}{3+1}\overrightarrow{AC} = \frac{3}{4}\overrightarrow{AC}$

$$\overrightarrow{AP} = \frac{3}{4}\overrightarrow{AC} = \frac{3}{4}(-2\mathbf{a} + \mathbf{c})$$

$$\boldsymbol{\overrightarrow{AP} = -\frac{3}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$= 2\mathbf{a} + \left(-\frac{3}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}\right)$$

$$\boldsymbol{\overrightarrow{OP} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{c}}$$

(c) Hence, prove that point P lies on line OB, and determine the ratio $\overrightarrow{OP} : \overrightarrow{PB}$.

To show that O, P, and B are colinear (lie on the same line), note that

$$\overrightarrow{OP} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{c} = \frac{1}{4}(2\mathbf{a} + 3\mathbf{c})$$

$$\overrightarrow{OP} = \frac{1}{4}(2\mathbf{a} + 3\mathbf{c}) = \frac{1}{4}\overrightarrow{OB}$$

$\boldsymbol{\overrightarrow{OP} = \frac{1}{4}\overrightarrow{OB}}$ **therefore OP is parallel to OB**

and so P must lie on the line OB

If $\overrightarrow{OP} = \frac{1}{4}\overrightarrow{OB}$ then $\overrightarrow{PB} = \frac{3}{4}\overrightarrow{OB}$

$$\boldsymbol{\overrightarrow{OP} : \overrightarrow{PB} = 1 : 3}$$

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