AQA GCSE Maths: Higher

Your notes

Quadratic Equations

Contents

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Solving Quadratics by Factorising

Solving Quadratics by Factorising

How do I solve a quadratic equation using factorisation?

- **Rearrange** it into the form $ax^2 + bx + c = 0$
  - Zero must be on one side
  - It is easier if you rearrange so that *a* is positive
- **Factorise** the quadratic and **solve** each bracket equal to zero
  - If $(x + 4)(x - 1) = 0$, then either $x + 4 = 0$ or $x - 1 = 0$
    - Because if two things multiply together to give zero,
      - then one or the other of them must be equal to zero
- To solve $(x-3)(x+7)=0$
  - …solve **first bracket = 0**:
    - $x - 3 = 0$
    - add 3 to both sides: $x = 3$
  - …and solve **second bracket = 0**
    - $x + 7 = 0$
    - subtract 7 from both sides: $x = -7$
  - The **two solutions** are $x = 3$ or $x = -7$
    - The solutions in this example are the numbers in the brackets, but with opposite signs

What if there are numbers in front of the x's in the brackets?

- The process is **the same**
  - There's a bit more work to find the solutions
  - You can't just write down the answers by changing the signs
- To solve $(2x-3)(3x+5)=0$
  - …solve **first bracket = 0**

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- $2x - 3 = 0$
- add 3 to both sides: $2x = 3$
- divide both sides by 2: $x = \frac{3}{2}$
- ...solve **second bracket = 0**
  - $3x + 5 = 0$
  - subtract 5 from both sides: $3x = -5$
  - divide both sides by 3: $x = -\frac{5}{3}$
- The **two solutions** are $x = \frac{3}{2}$ or $x = -\frac{5}{3}$

What if x is a factor?

- The process is **the same**
  - Just be sure to handle the x correctly
  - That 'x as a factor' gives one of the solutions
- To solve $x(x-4) = 0$
  - it may help to think of x as $(x - 0)$ or (x)
  - ...solve **first bracket = 0**
    - $(x) = 0$, so $x = 0$
  - ...solve **second bracket = 0**
    - $x - 4 = 0$
    - add 4 to both sides: $x = 4$
  - The **two solutions** are $x = 0$ or $x = 4$
- It is a common **mistake** to divide (cancel) both sides by x at the beginning
  - If you do this you will **lose a solution** (the $x = 0$ solution)

How can I use my calculator to help with solving quadratics by factorising?

Your notes

- You can use your calculator to **help you to factorise**
  - A calculator gives solutions to $6x^2 + x - 2 = 0$ as $x = -\frac{2}{3}$ and $x = \frac{1}{2}$
    - **Reverse** the method above to factorise!
    - $6x^2 + x - 2 = (3x + 2)(2x - 1)$
  - **Be careful**: a calculator also gives solutions to $12x^2 + 2x - 4 = 0$ as $x = -\frac{2}{3}$ and $x = \frac{1}{2}$
    - But $12x^2 + 2x - 4 \neq (3x + 2)(2x - 1)$
    - The right-hand side expands to $6x^2 + \ldots$ not $12x^2 + \ldots$
    - **Multiply** outside the brackets by 2 to correct this
    - $12x^2 + 2x - 4 = 2(3x + 2)(2x - 1)$

Examiner Tips and Tricks

- Remember that you can check your solutions by either
  - substituting them back into the original equation
  - using a different quadratic method
  - or using a calculator

Worked Example

(a) Solve $(x - 2)(x + 5) = 0$

Set the first bracket equal to zero

$$x - 2 = 0$$

Add 2 to both sides

$$x = 2$$

Set the second bracket equal to zero

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$$x + 5 = 0$$

Subtract 5 from both sides

$$x = -5$$

Write both solutions together using "or"

$$\boldsymbol{x = 2 \text{ or } x = -5}$$

(b) Solve $(8x + 7)(2x - 3) = 0$

Set the first bracket equal to zero

$$8x + 7 = 0$$

Subtract 7 from both sides

$$8x = -7$$

Divide both sides by 8

$$x = -\frac{7}{8}$$

Set the second bracket equal to zero

$$2x - 3 = 0$$

Add 3 to both sides

$$2x = 3$$

Divide both sides by 2

$$x = \frac{3}{2}$$

Write both solutions together using "or"

$$\boldsymbol{x = -\frac{7}{8} \text{ or } x = \frac{3}{2}}$$

(c) Solve $x(5x - 1) = 0$

Do not divide both sides by x (this will lose a solution at the end)
Set the first "bracket" equal to zero

$$(x) = 0$$

Solve this equation to find x

$$x = 0$$

Set the second bracket equal to zero

$$5x - 1 = 0$$

Add 1 to both sides

$$5x = 1$$

Divide both sides by 5

$$x = \frac{1}{5}$$

Write both solutions together using "or"

$$x = 0 \text{ or } x = \frac{1}{5}$$

Your notes

The Quadratic Formula

Your notes

Quadratic Formula

What is the quadratic formula?

- A **quadratic equation** has the form $ax^2 + bx + c = 0$ (where $a \neq 0$)
  - you need "**= 0**" on one side
- The **quadratic formula** is a formula that gives both solutions to a quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examiner Tips and Tricks

- Make sure the quadratic equation has "= 0" on the right-hand side
  - Otherwise it needs rearranging first

How do I use the quadratic formula to solve a quadratic equation?

- Read off the **values** of a, b and c from the equation
- **Substitute** these into the formula
  - **Write** this line of working in the exam
  - Put **brackets** around any **negative numbers** being substituted in
- To solve $2x^2 - 8x - 3 = 0$ using the quadratic formula:
  - $a = 2$, $b = -8$ and $c = -3$
  - $$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 2 \times (-3)}}{2 \times 2}$$
  - Either type this into a calculator or simplify by hand
    - Type it once using + for ± then again using - for ±

Your notes

- The **solutions** are $x = 4.3452078...$ or $x = -0.34520787....$
  - To 3 **decimal places**: $x = 4.345$ or $x = -0.345$
  - To 3 **significant figures**: x = 4.35 or x = -0.345

Examiner Tips and Tricks

- Always look for how the question wants you to leave your final answers
  - For example, correct to 2 decimal places

How do I write the solutions in an exact (surd) form?

- You may be asked to give answers in an **exact** (**surd**) form
- In the example above, work out the **number** under the **square root** sign
  - Be careful with negatives!
    - $(-8)^2 - 4 \times 2 \times (-3) = 64 + 24 = 88$
  - Now square root this number and use **surd rules** to simplify
    - $\sqrt{88} = \sqrt{4 \times 22} = \sqrt{4} \times \sqrt{22} = 2\sqrt{22}$
  - **Substitute** this back into the formula and simplify
    - $x = \dfrac{8 \pm 2\sqrt{22}}{4} = \dfrac{2\left(4 \pm \sqrt{22}\right)}{4} = \dfrac{4 \pm \sqrt{22}}{2}$
    - The solutions in **exact** (**surd**) form are $x = \dfrac{4+\sqrt{22}}{2}$ or $x = \dfrac{4-\sqrt{22}}{2}$
- **Calculators** that can solve quadratics will give solutions in **exact** (**surd**) form

What is the discriminant?

- The part of the formula under the square root ($b^2 - 4ac$) is called the **discriminant**
- The **sign** of this value tells you if there are 0, 1 or 2 solutions
  - If $\boldsymbol{b^2 - 4ac > 0}$ (positive)
    - then there are 2 different solutions

- If b^2 **– 4ac = 0**
  - then there is only 1 solution
  - sometimes called "two repeated solutions"
- If b^2 **– 4ac < 0** (negative)
  - then there are no solutions
  - If your calculator gives you solutions with i terms in, these are "complex" and are not what we are looking for
- Interestingly, if $b^2 - 4ac$ is a perfect square number (1, 4, 9, 16, ...) then the quadratic expression could have been factorised!

Can I use my calculator to solve quadratic equations?

- If your calculator solves quadratic equations, use it to **check** your **final answers**
  - But a correct method and working must still be shown

Worked Example

Use the quadratic formula to find the solutions of the equation $3x^2 - 2x - 4 = 0$.
Give each solution as an exact value in its simplest form.

Write down the values of *a*, *b* and c

$$a = 3, b = -2, c = -4$$

Substitute these values into the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Put brackets around any negative numbers

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-4)}}{2 \times 3}$$

Simplify the expressions

$$x = \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$

Simplify the surd

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$$x = = \frac{2 \pm \sqrt{4 \times 13}}{6} = \frac{2 \pm 2\sqrt{13}}{6}$$

Simplify the fraction

$$x = \frac{1 \pm \sqrt{13}}{3}$$

Your notes

Completing the Square

Your notes

Solving by Completing the Square

How do I solve a quadratic equation by completing the square?

- To solve $x^2 + bx + c = 0$
  - **replace** the first two terms, $\boldsymbol{x^2 + bx}$, with $\boldsymbol{(x + p)^2 - p^2}$ where ***p* is half of *b***
  - This is **completing the square**
    - $x^2 + bx + c = 0$ becomes $(x + p)^2 - p^2 + c = 0$
    - (where p is half of b)
  - rearrange this equation to **make *x* the subject** (using ±√)
- For example, solve $x^2 + 10x + 9 = 0$ by completing the square
  - $x^2 + 10x$ becomes $(x + 5)^2 - 5^2$
  - so $x^2 + 10x + 9 = 0$ becomes $(x + 5)^2 - 5^2 + 9 = 0$
  - make x the subject (using ±√)
    - $(x + 5)^2 - 25 + 9 = 0$
    - $(x + 5)^2 = 16$
    - $x + 5 = \pm\sqrt{16}$
    - $x + 5 = \pm 4$
    - $x = -5 \pm 4$
    - $x = -1$ or $x = -9$
- It also works with numbers that lead to **surds**
  - The answers found will be in **exact** (**surd**) form

Examiner Tips and Tricks

- When making x the subject to find the solutions, don't expand the squared bracket back out again!
  - Remember to use $\pm\sqrt{}$ to get **two** solutions

How do I solve by completing the square when there is a coefficient in front of the x^2 term?

- If the equation is $ax^2 + bx + c = 0$ with a **number** (other than 1) **in front of x^2**
  - you can **divide both sides by *a*** first (before completing the square)
    - For example $3x^2 + 12x + 9 = 0$
    - Divide both sides by 3
      - $x^2 + 4x + 3 = 0$
    - Complete the square on this easier equation
- This trick **only** works when completing the square to **solve** a quadratic equation
  - i.e. it has an "=0" on the right-hand side
- **Don't** do this when using completing the square to rewrite a **quadratic expression** in a new form
  - i.e. when there is no "=0"
  - For that, you must **factorise out** the a (but not divide by it)
    - $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x\right] + c$ and so on

How does completing the square link to the quadratic formula?

- The **quadratic formula** actually comes from **completing the square** to solve $ax^2 + bx + c = 0$
  - a, b and c are left as letters when completing the square
    - This makes it as general as possible
- You can see hints of this when you solve quadratics
  - For example, solving $x^2 + 10x + 9 = 0$
    - by completing the square, $(x + 5)^2 = 16$ so $x = -5 \pm 4$ (as above)

- by the quadratic formula, $x = \dfrac{-10 \pm \sqrt{64}}{2} = -5 \pm \dfrac{8}{2}$ = -5 ± 4 (the same structure)

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Worked Example

Solve $2x^2 - 8x - 24 = 0$ by completing the square.

Divide both sides by 2 to make the quadratic start with x^2

$$x^2 - 4x - 12 = 0$$

Halve the middle number, -4, to get -2
Replace the first two terms, $x^2 - 4x$, with $(x - 2)^2 - (-2)^2$

$$(x-2)^2 - (-2)^2 - 12 = 0$$

Simplify the numbers

$$(x-2)^2 - 4 - 12 = 0$$
$$(x-2)^2 - 16 = 0$$

Add 16 to both sides

$$(x-2)^2 = 16$$

Take the square root of both sides
Include the ± sign to get two solutions

$$x - 2 = \pm\sqrt{16} = \pm 4$$

Add 2 to both sides

$$x = 2 \pm 4$$

Work out each solution separately

$x = 6$ or $x = -2$

Deciding the Quadratic Method

Your notes

Deciding the Quadratic Method

If you have to solve a quadratic equation but are not told which method to use, here is a guide for what to do.

When should I solve by factorisation?

- Use factorisation when the question asks to **solve by factorisation**
  - For example
    - part (a) Factorise $6x^2 + 7x - 3$
    - part (b) Solve $6x^2 + 7x - 3 = 0$
- Use factorisation when solving **two-term quadratic equations**
  - For example, solve $x^2 - 4x = 0$
    - Take out a **common factor** of x to get $x(x - 4) = 0$
    - So $x = 0$ and $x = 4$
  - For example, solve $x^2 - 9 = 0$
    - Use the **difference of two squares** to factorise it as $(x + 3)(x - 3) = 0$
    - So $x = -3$ and $x = 3$
    - (Or rearrange to $x^2 = 9$ and use $\pm\sqrt{}$ to get $x = \pm 3$)
- Factorising can often be the **quickest** way to solve a quadratic equation

When should I use the quadratic formula?

- Use the quadratic formula when the question says to leave solutions correct to a **given accuracy** (2 decimal places, 3 significant figures etc)
  - This is a hint that the equation will not factorise
- Use the quadratic formula when it may be **faster than factorising**
  - It's quicker to solve $36x^2 + 33x - 20 = 0$ using the quadratic formula than by factorisation
- Use the quadratic formula **if in doubt**, as it **always works**

When should I solve by completing the square?

- Use completing the square when part (a) of a question says to **complete the square** and part (b) says to use part (a) to solve the equation
- Use completing the square when making x the **subject of harder formulae** containing both x^2 and x terms
  - For example, make x the subject of the formula $x^2 + 6x = y$
    - Complete the square: $(x + 3)^2 - 9 = y$
    - Add 9 to both sides: $(x + 3)^2 = y + 9$
    - Take square roots and use ±: $x + 3 = \pm\sqrt{y+9}$
    - Subtract 3: $x = -3 \pm \sqrt{y+9}$
- Completing the square **always works**
  - But it's not always quick or easy to do

Your notes

Examiner Tips and Tricks

- If your calculator solves quadratic equations, use it to check your solutions
- If the solutions on your calculator are whole numbers or fractions (with no square roots), this means the quadratic equation does factorise

Worked Example

(a) Solve $x^2 - 7x + 2 = 0$, giving your answers correct to 2 decimal places.

“Correct to 2 decimal places” suggests using the quadratic formula
Substitute $a = 1$, $b = -7$ and $c = 2$ into the formula
Put brackets around any negative numbers

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 1 \times 2}}{2 \times 1}$$

Use a calculator to find each solution

$x = 6.70156...$ or $0.2984...$

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Round your final answers to 2 decimal places

$x = 6.70$ or $x = 0.30$ (2 d.p.)

(b) Solve $16x^2 - 82x + 45 = 0$.

Method 1
If you cannot spot the factorisation, use the quadratic formula
Substitute $a = 16$, $b = -82$ and $c = 45$ into the formula
Put brackets around any negative numbers

$$x = \frac{-(-82) \pm \sqrt{(-82)^2 - 4 \times 16 \times 45}}{2 \times 16}$$

Use a calculator to find each solution

$x = \frac{9}{2}$ or $x = \frac{5}{8}$

Method 2
If you do spot the factorisation, $(2x - 9)(8x - 5)$, then use that method instead

$$(2x - 9)(8x - 5) = 0$$

Set the first bracket equal to zero

$$2x - 9 = 0$$

Add 9 to both sides then divide by 2

$$2x = 9$$
$$x = \frac{9}{2}$$

Set the second bracket equal to zero

$$8x - 5 = 0$$

Add 5 to both sides then divide by 8

$$8x = 5$$
$$x = \frac{5}{8}$$

$x = \dfrac{9}{2}$ or $x = \dfrac{5}{8}$

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(c) By writing $x^2 + 6x + 5$ in the form $(x + p)^2 + q$, solve $x^2 + 6x + 5 = 0$.

This question wants you to complete the square first
Find p (by halving the middle number)

$$p = \frac{6}{2} = 3$$

Write $x^2 + 6x$ as $(x + p)^2 - p^2$

$$\begin{aligned} x^2 + 6x &= (x + 3)^2 - 3^2 \\ &= (x + 3)^2 - 9 \end{aligned}$$

Replace $x^2 + 6x$ with $(x + 3)^2 - 9$ in the equation

$$\begin{aligned} (x + 3)^2 - 9 + 5 &= 0 \\ (x + 3)^2 - 4 &= 0 \end{aligned}$$

Now solve it
Make x the subject of the equation (start by adding 4 to both sides)

$$(x + 3)^2 = 4$$

Take square roots of both sides (include a ± sign to get both solutions)

$$x + 3 = \pm\sqrt{4} = \pm 2$$

Subtract 3 from both sides

$$x = -3 \pm 2$$

Find each solution separately using + first, then - second

$x = -1$ or $x = -5$

Even though the quadratic factorises to $(x + 5)(x + 1)$, this is not the method asked for in the question