AQA GCSE Maths: Higher

Your notes

Functions

Contents

Introduction to Functions

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Introduction to Functions

What is a function?

- A **function** is a combination of **one or more mathematical operations** that takes a **set of numbers** and **changes** them into **another set of numbers**
- The numbers being **put into the function** are often called the **inputs**
- The numbers **coming out of the function** are often called the **outputs**
- A function may be thought of as a mathematical "**machine**"
  - For example, for the function "double the number and add 1", the two mathematical operations are "multiply by 2 (×2)" and "add 1 (+1)"
    - Putting 3 in to the function would give 2 × 3 + 1 = 7
    - Putting -4 in would give 2 × (-4) + 1 = -7
    - Putting x in would give $2x + 1$

What is function notation?

- A function, f, with input x can be written as $\text{f}(x) = \ldots$
  - Letters other than f can be used
    - The letters g, h and j are common but any letter can be used
    - Typically, a **new letter** will be used to define a **new function** in a question
- For example, the function with the rule "triple the number and subtract 4" would be written
  - $\text{f}(x) = 3x - 4$
- In such cases, "x" is the **input** and "$\text{f}(x)$" is the **output**
- Sometimes functions don't have names like f and are just written as $y = \ldots$
  - E.g. $y = 3x - 4$

How does a function work?

- A function has an **input** x and an **output** $\text{f}(x)$

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   - If the input is 2, then the output is $\mathrm{f}(2)$
   - If the input is m, then the output is $\mathrm{f}(m)$
   - If the input is $t+5$, then the output is $\mathrm{f}(t+5)$
      - You cannot simplify this output any further
- If the **function is known**, the **output can be calculated**
   - For example, given the function $\mathrm{f}(x)=2x+1$
      - $\mathrm{f}(3)=2\times3+1=7$
      - $\mathrm{f}(-4)=2\times(-4)+1=-7$
      - $\mathrm{f}(a)=2a+1$
- If the **output is known**, an **equation** can be formed and solved to **find the input**
   - For example, given the function $\mathrm{f}(x)=2x+1$
      - If $\mathrm{f}(x)=15$, then form an equation by replacing $\mathrm{f}(x)$ with $2x+1$
      - $2x+1=15$
      - Solving this equation gives an input of 7
- Note that $\mathrm{f}(x)=15$ and $\mathrm{f}(15)$ are very different things:
   - $\mathrm{f}(x)=15$ means an input of x gives an output of 15
   - $\mathrm{f}(15)$ means substitute the input 15 into the function

What is a mapping diagram?

- A **mapping diagram** shows a **set of different inputs** going into the function to become a **set of different outputs**
   - Transforming inputs into outputs is called **mapping**
- For example, a mapping diagram for the function $\mathrm{f}(x)=x+3$ where $x\geq3$ could be shown as:

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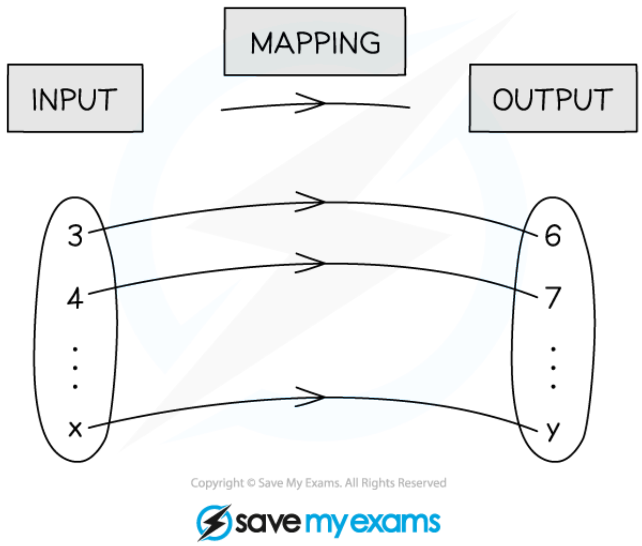

Worked Example

A function is defined as $\mathrm{f}(x) = 3x^2 - 2x + 1$.

(a) Find $\mathrm{f}(7)$.

The input is $x = 7$, so substitute 7 into the expression everywhere you see an x

$$\mathrm{f}(7) = 3(7)^2 - 2(7) + 1$$

Calculate

$$\mathrm{f}(7) = 3(49) - 14 + 1$$
$$= 147 - 14 + 1$$

$$\mathbf{f(7) = 134}$$

(b) Find $\mathrm{f}(x+3)$, giving your answer in the form $ax^2 + bx + c$ where a, b and c are integers to be found.

The input is $(x+3)$ so substitute $(x+3)$ into the expression everywhere you see an x
This is like replacing x with $(x+3)$

$$f(x+3) = 3(x+3)^2 - 2(x+3) + 1$$

Expand the brackets and simplify

Use that $(x+3)^2 = (x+3)(x+3)$

Be careful with negative signs

$$\begin{aligned} f(x+3) &= 3(x^2+6x+9) - 2(x+3) + 1 \\ &= 3x^2 + 18x + 27 - 2x - 6 + 1 \\ &= 3x^2 + 16x + 22 \end{aligned}$$

$$\mathbf{f(x+3) = 3x^2 + 16x + 22}$$

(a = 3, b = 16, c = 22)

A second function is defined by $g(x) = 3x - 4$.

(c) Find the value of x for which $g(x) = -16$.

This is not saying substitute 16 into the function

It says that an input x is substituted into g giving the output -16

To find the input, form an equation by replacing $g(x)$ with $3x - 4$

$$3x - 4 = -16$$

Solve the equation (for example, by adding 4 to both sides, then dividing by 3)

$$\begin{aligned} 3x - 4 &= -16 \\ 3x &= -12 \\ x &= -\frac{12}{3} \end{aligned}$$

$$\mathbf{x = -4}$$

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Composite Functions

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Composite Functions

What is a composite function?

- A **composite function** is a function applied to the **output** of another function
  - The input goes through the 1st function to become an output
  - This output goes through the 2nd function to become a new output

What notation is used for composite functions?

- If $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are two functions, then
  - $\mathrm{g}(\mathrm{f}(x))$ is a **composite** function
  - It means the input x goes through **function** f **first**
    - This gives the output $\mathrm{f}(x)$
    - Then this output, $\mathrm{f}(x)$, becomes the **input** of function g, giving $\mathrm{g}(\mathrm{f}(x))$
  - $\mathrm{gf}(x)$ is the shorthand **notation used** for $\mathrm{g}(\mathrm{f}(x))$
    - It means do f first, then g
    - The **order** of applying the functions goes from **right to left**
    - (the letter nearest the bracket goes first)
    - This is often the opposite of what people expect!
  - $\mathrm{fg}(x)$ means do $\mathrm{g}(x)$ **first** then $\mathrm{f}(x)$ **second**
  - $\mathrm{ff}(x)$ means apply $\mathrm{f}(x)$ twice!
    - This can be written $\mathrm{f}^2(x)$
    - This does not mean the same as $[\mathrm{f}(x)]^2$

Examiner Tips and Tricks

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> A good trick in the exam is to write brackets around $\mathrm{gf}(x)$ to make it $\mathrm{g}(\mathrm{f}(x))$, to see that it is "g" of "f(x)".

How do I substitute numbers into composite functions?

- If you are putting a **number** into a composite function
  - put the number into the function closest to (x)
  - then make the **output** of the **first** function the **input** of the **second** function
- For example, if $\mathrm{f}(x) = 2x + 1$ and $\mathrm{g}(x) = \dfrac{1}{x}$
  - to find $\mathrm{gf}(2)$:
    - Put the 2 in as the input of f first
    - $\mathrm{f}(2) = 2(2) + 1 = 5$
    - Then put 5 in as the input of g
    - So $\mathrm{gf}(2) = \mathrm{g}(\mathrm{f}(2)) = \mathrm{g}(5) = \dfrac{1}{5}$
  - to find $\mathrm{fg}(2)$:
    - Put the 2 in as the input of g first
    - $\mathrm{g}(2) = \dfrac{1}{2}$
    - Then put $\dfrac{1}{2}$ in as the input of f
    - So $\mathrm{fg}(2) = \mathrm{f}(\mathrm{g}(2)) = \mathrm{f}\left(\dfrac{1}{2}\right) = 2\left(\dfrac{1}{2}\right) + 1 = 2$
  - to find $\mathrm{ff}(2)$:
    - $\mathrm{f}(2) = 2 \times 2 + 1 = 5$
    - $\mathrm{f}(5) = 2 \times 5 + 1 = 11$

- so $\mathrm{ff}(2) = 11$

How do I find composite functions algebraically?

- If you are using algebra, **substitute** the whole **algebraic expression** as your input
  - For example, if $\mathrm{f}(x) = 2x + 1$ and $\mathrm{g}(x) = \dfrac{1}{x}$
    - $\mathrm{fg}(x) = \mathrm{f}(\mathrm{g}(x)) = \mathrm{f}\left(\dfrac{1}{x}\right) = 2 \times \left(\dfrac{1}{x}\right) + 1 = \dfrac{2}{x} + 1$
    - $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x)) = \mathrm{g}(2x + 1) = \dfrac{1}{2x + 1}$
    - $\mathrm{ff}(x) = \mathrm{f}(\mathrm{f}(x)) = \mathrm{f}(2x + 1) = 2(2x + 1) + 1$ which simplifies to $\mathrm{ff}(x) = 4x + 3$

Worked Example

In this question, $\mathrm{f}(x) = 2x - 1$ and $\mathrm{g}(x) = (x + 2)^2$.

(a) Find $\mathrm{fg}(4)$.

"g" is on the inside of the composite function, so apply g first

$$\mathrm{g}(4) = (4 + 2)^2 = 6^2 = 36$$

Now apply the function "f" to 36

$$\begin{aligned}\mathrm{f}(36) &= 2(36) - 1 \\ &= 72 - 1\end{aligned}$$

$$\mathbf{fg(4) = 71}$$

(b) Find $\mathrm{gf}(x)$.

"f" is on the inside of the composite function so substitute the function f(*x*) into g(*x*)
It can help to write $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$

$$\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x)) = \mathrm{g}(2x - 1) = ((2x - 1) + 2)^2$$

Simplify inside the bracket

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$\mathrm{gf}(x) = (2x - 1 + 2)^2$

$$\mathbf{gf}(\boldsymbol{x}) = (\mathbf{2}\boldsymbol{x} + \mathbf{1})^{\mathbf{2}}$$

You do not need to expand the answer

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Inverse Functions

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Inverse Functions

What is an inverse function?

- An **inverse function** does the **opposite (reverse) operation** of the function it came from
  - E.g. If a function "doubles the number then adds 1"
  - Then its inverse function "subtracts 1, then halves the result"
    - The same inverse operations are used when solving an equation or rearranging a formula
- An inverse function performs the **inverse operations** in the **reverse order**

What notation is used for inverse functions?

- The inverse function of $\mathrm{f}(x)$ is written as $\mathrm{f}^{-1}(x) = ...$
  - For example, if $\mathrm{f}(x) = 2x + 1$
  - The inverse function is $\mathrm{f}^{-1}(x) = \frac{x-1}{2}$ or $\mathrm{f}^{-1}\colon x \mapsto \frac{x-1}{2}$
- If $\mathrm{f}(a) = b$ then $\mathrm{f}^{-1}(b) = a$
  - For example
    - $\mathrm{f}(3) = 2 \times 3 + 1 = 7$ (inputting 3 into f gives 7)
    - $\mathrm{f}^{-1}(7) = \frac{7-1}{2} = 3$ (inputting 7 into f^{-1} gives back 3)

How do I find an inverse function algebraically?

- The **process** for finding an inverse function is as follows:
  - **Write** the function as $y = ...$
    - E.g. The function $\mathrm{f}(x) = 2x + 1$ becomes $y = 2x + 1$
  - **Swap** the xs and ys to get $x = ...$
    - E.g. $x = 2y + 1$

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- The letters change but no terms move
- **Rearrange** the expression to **make y the subject** again
  - E.g. $x = 2y + 1$ becomes $x - 1 = 2y$ so $y = \frac{x-1}{2}$
- **Replace y** with $\mathrm{f}^{-1}(x) = \ldots$ (or $\mathrm{f}^{-1}: x \mapsto \ldots$)
  - E.g. $\mathrm{f}^{-1}(x) = \frac{x-1}{2}$
  - This is the inverse function
  - y **should not appear** in the final answer

How are inverse functions and composite functions related?

- The **composite function** of f followed by f^{-1} (or the other way round) **cancels out**
  - $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$
    - If you apply a function to *x*, then apply its inverse function, you get back *x*
    - Whatever happened to *x* gets **undone**
    - f and f^{-1} **cancel each other out** when applied together
- For example, solve $\mathrm{f}^{-1}(x) = 5$ where $\mathrm{f}(x) = 2^x$
  - Finding the inverse function $\mathrm{f}^{-1}(x)$ algebraically in this case is tricky
    - (It is impossible if you haven't studied logarithms!)
  - Instead, you can take f of both sides of $\mathrm{f}^{-1}(x) = 5$ and use the fact that ff^{-1} cancel each other out:
    - $\mathrm{ff}^{-1}(x) = \mathrm{f}(5)$ which cancels to $x = \mathrm{f}(5)$ giving $x = 2^5 = 32$

Worked Example

A function is given by $\mathrm{f}(x) = 5 - 3x$. Use algebra to find $\mathrm{f}^{-1}(x)$.

Write the function in the form $y = 5 - 3x$ and then swap the x and y

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$$y = 5 - 3x$$
$$x = 5 - 3y$$

Rearrange the expression to make y the subject again

$$x = 5 - 3y$$
$$x + 3y = 5$$
$$3y = 5 - x$$
$$y = \frac{5 - x}{3}$$

Rewrite the answer using inverse function notation

$$\mathbf{f^{-1}(x) = \frac{5 - x}{3}}$$