Logic, categories, types
$\int$
"spaces"

Boolean alg's

(classical) Propositional logic = 0^{th} order

$P, Q, R, \ldots$

$(\vdash,)\ \wedge, \vee, \neg$

$$P \to Q := \neg P \vee Q.$$

behave like posets w/

products, coproducts

(& if one allows empty ones):

"truth" $\top$, $\bot$ "falsity"

$\top$ = empty product = terminal

$\bot$ = empty coproduct = initial

Collectively called complete bounded lattices.

Negation characterised by

$x \wedge \neg x = \bot$, $x \vee \neg x = \top$.

$\rightsquigarrow$ boolean algebra.

Standard "non-logical" example:

$\mathcal{P}(X)$, X any set

$\vdash = \subseteq$, $\wedge = \cap$, $\vee = \cup$, $\neg$ = complement

$\top = X$, $\bot = \emptyset$.

Stone duality = converse:

A boolean $\Rightarrow$ $\exists X$ set,

$A \underset{\text{b.a.}}{\cong}$ a subalgebra of $\mathcal{P}(X)$

In general, a subalg of some $\mathcal{P}(X)$ is called a field or algebra of sets. (+ countable $\vee$'s $\rightsquigarrow$ σ-alg)

Specifically: $\exists$ Stone sp X, a compact, totally disc. Hausd. sp, equivalently a profinite [discrete] sp w/ $A \cong$ clopens of X.

Rmk $\underline{\text{Stone}}$ is naturally a site,

$\simeq$ pro-étale site on $*$.

$$Sh(*_{\text{proét}}) =: \text{condensed sets}.$$

This is an enlargement of

$$Sh(*) \cong \underline{\text{Set}}.$$

Arithmetic ex Name "field" suggests

$$\wedge = \cdot, \quad \vee = +, \quad \neg = (\ )^{-1},$$

$$\top = 1, \quad \bot = 0,$$

but $x \cdot x^{-1} = 1$ AND $x + x^{-1} = 0$

is silly.

(spoiler: LEM is the problem)

Better: $\underline{D}$ ("divisibility"):

obj : $\mathbb{N}$

morph : $n \vdash m$ iff $n \mid m$.

$\Rightarrow \underline{D}$ poset.

$n \wedge m = \gcd(n, m)$

$$n \vee m = \mathrm{lcd}(n, m)$$

$$\top = 0 \quad , \quad \bot = 1$$ (try not to get confused. good luck)

$n \wedge m = \bot$ for any m coprime to n, but

$n \vee m = \top$ impossible (if $n \neq 0$).

$\underline{D}$ is "as intuitionistic as it gets"

Truncate $\underline{D}$ to get sth boolean?

Problem w/ $\underline{D}$ is: 0 as "at infinity".

Can make 0 "finite":

$\underline{D}_{|n}$:= (full subcat on) divisors of n.

$\rightsquigarrow \quad \top = n \quad$ (still $\bot = 1$),

Same $\wedge$, so $k \wedge \neg k = \bot$ still ok.

Last issue: $k \vee \neg k = \top$ implies

$\neg k = \frac{n}{k}$, but

[illegible] iff $k \nmid \frac{n}{k}$.

Simultaneously $k \wedge \frac{n}{k} = \bot$ iff [illegible] k

$\Rightarrow$ $\underline{D}_n$ boolean iff n square-free.

Entailment vs implication

$P \vdash Q$: $\dfrac{\text{(syntactic) derivation}}{\text{arrow in cat}}$

$P \to Q$: $\dfrac{\text{new proposition}}{\text{object in cat}}$

internal hom / exponential Q^P

At least must have modus ponens:

$$P \wedge (P \to Q) \vdash Q .$$

More generally, want currying:

$$P \wedge Q \vdash R \quad \text{iff} \quad P \vdash (Q \to R)$$

modus ponens corresponds to $P = Q \to R$:

$$(Q \to R) \wedge R \vdash R$$

$$\text{iff}$$

$$(Q \to R) \vdash (Q \to R)$$

the identity arrow on RHS gives a canonical modus ponens, called evaluation.

A cartesian closed compl bdd lattice is called a Heyting algebra, i.e. one w/ all exponentials, suitably functorial.

Modus ponens is just

"$\hom(P \wedge Q, R) \simeq \hom(P, R^Q)$"

Boolean $\Rightarrow$ Heyting w/

$$R^Q := \neg Q \vee R$$

[In propositional logic, customarily "proven" via truth tables ("semantically")]

Arith'c ex cont'd

An exp'l m^n must satisfy

$k \wedge n \mid m \iff k \mid m^n$.

In particular, $k \mid m^n$ whenever

k coprime to n,

absurd due to the infinitude of primes.

EXCEPT when $n = 0 = \top$:

$k \wedge 0 = k$, so $m^0 = m$.

$\Rightarrow$ only 0 exponentiable.

In $\underline{D}_{\ln}$ for n square-free,

may set $k^\ell := \neg \ell \vee k$

$= \frac{n}{\ell} \vee k$.

Global pt's / concreteness

[illegible] $\Rightarrow$ more concrete

Modus ponens

Connection betw $\vdash$, $\to$:

$T \wedge P = P$ implies

$\hom(P, Q) \simeq \hom(T, Q^P)$.

Maps $T \vdash Q^P$: T-points

or global pt's of Q^P.

From $T \wedge P = P$, we also

have $P^T = P$.

WARNING :

"$\hom(T, P) \simeq P$"

might or might not make sense!

True in Set / "concrete" categories,

but for instance not in

$\underline{D}$ or $\underline{D}|n$.

Ex More dramatically :

Consider a point $* \xrightarrow{\gamma} X$

of a top'l sp.

⇝ a "geometric morphism"

$$\gamma^* : Sh(X) \rightleftarrows Sh(*) : \gamma_*$$

$$\begin{array}{c} Sh(*) \\ \cong \\ \underline{Set} \end{array}$$

an adjoint pair, called a

concrete morphism between

these sheaf "topoi".

"Direct image" γ_* hits only

skyscraper sheaves.

Open/frames/locales

Main "non-logical" ex of Heyting:

$\mathcal{O}(X)$, opens of top sp X.

Na..d.. with

Namely,

$$V^U = \mathrm{int}(U^c \cup V)$$

(check modus ponens!)

Rmk

If [open $\Leftrightarrow$ closed] $\Rightarrow$ $\mathcal{O}(X)$ bool.

Non-silly ex.: Sorgenfrey line

$\mathbb{R}_\ell$: basics: $[a, b)$.

It's totally disc. & Hausdorff, like any Stone sp.

In $\mathcal{O}(X)$, more is true:

$\exists$ <u>all</u> products (interior of $\bigcap$)

& $\exists$ <u>all</u> coproducts

Such Heyting alg's are called

complete.

We will call them frames.

$\underline{\text{Locales}} := \underline{\text{Frames}}^{op}$.

Frame morphisms preserve finite $\wedge$'s, arbitrary $\vee$'s.

We have a functor

$$\underline{\text{Top}}^{op} \xrightarrow{\mathcal{O}} \underline{\text{Locales}}$$
$$X \mapsto \mathcal{O}(X)$$
$$X \xrightarrow{f} Y \mapsto \mathcal{O}(Y) \xrightarrow{f^{-1}} \mathcal{O}(X).$$

There a right adjoint

$$\underline{\text{Locales}}^{op} \xrightarrow{\text{Spec}} \underline{\text{Top}}$$
$$L \mapsto \text{"space of pt's"}$$
$$= (\text{as sets})$$
$$\{P(*) \to L\}$$

Restricting to the subcategories where $\mathcal{O}$, Spec are inverses, may define an equiv of sober sp's and spatial locales.

NOT all locales come from top'l spaces!

(one can still restrict Stone duality appropriately though)

Corresponding "natural" general'n of Space : topoi.

In 2022, this is ancient history, so just a sketch

Topoi

Heyting algebras are

" " topoi

internal logics of topoi.

Geometrically, any topos $\mathcal{E}$ gives a locale $L(\mathcal{E})$, and any locale L gives a sheaf topos $Sh(L)$.

Not every topos $\mathcal{E}$ is of the form $Sh(L)$, but almost: $\exists L$ with "covering map"

$$\begin{array}{c} Sh(L) \\ \downarrow \\ \mathcal{E} \end{array} \qquad (\text{"surj \& open"})$$

So what is a topos?

Idea: a cat that behaves like $Sh(X)$.

... the "point" in

Set thy is the [illegible]

topos thy :

$$Sh(*) = \underline{Set} .$$

In a sense, it is natural to set

$$\text{"} Sh(\emptyset) = Prop \text{"} ,$$

where the sheaves are either empty or singletons.

Slightly generalized : X a cat w/ a "topology"

[dist. coverings = sieves = subfunctors of $hom(-, x) \in PSh(X)$, with nice collective properties.]

$\rightsquigarrow$ can define $Sh(X)$:

formulate locality etc. using
dist. sieves.

Such a cat Sh(X), where
morph's = geom morph's,
is a **Grothendieck topos**.

Directly from logic, another def
is in the air:
a cat $\underline{C}$ w/
- finite (co)limits
- cartesian closed
- **subobject classifier** Ω

is called an (**elementary**) **topos**.

Here Ω serves to internalise
"higher-order-logical descriptions of subobjects":

In Set, given X, a subset U
described (externally to Set!)

may desc...

linguistically (via properties, quantifiers $\forall, \exists \ldots$), and equivalently as an arrow

$$X \xrightarrow{\chi_U} \underline{2} = \{\top, \bot\},$$

so that $U = \chi_U^{-1} \top$.

In diagramatic terms, a pullback

$$\begin{tikzcd}
U \arrow[r, "\text{final}"] \arrow[d, hook] & * \arrow[d, hook, "\top"] \\
X \arrow[r, "\chi_U"] & \underline{2}
\end{tikzcd}$$

Possible for every "subobject" (equiv classes of monics $U \rightarrowtail X$) of every object

$\Rightarrow$ $\underline{2}$ called "the" subobject classifier of $\underline{\text{Set}}$.

Internal logic = Heyting alg L

... $*$ - terminal obj.

of subobj's of ...

$\Leftrightarrow$ global pt's of Ω:

$$* \xrightarrow{P} \Omega .$$

Now, let $\top$, $\perp$ stand for terminal, initial in L.

Internal hom $\perp^P$ serves as $\neg P$, w/out LEM: $P \vee \neg P = \top$.

<u>Set</u> has LEM, but $Sh(X)$, X top sp, in general does not.

In $Sh(X)$, $*$ is the constant sheaf $U \mapsto *$, same as the representable $hom(-, X) \in Sh(X)$.

$\rightsquigarrow$ Subobjects = of the form $hom(-U)$.

$\rightsquigarrow$ $L(Sh(X))$ is just

$\mathcal{O}(X)$ as above.

In particular,

$$\neg U = \emptyset^U = \mathrm{int}(U^c \cup \emptyset) = \mathrm{int}(U^c)$$

$\Rightarrow$ no LEM.

Type theory

Very roughly, a constructive version of category theory.

Can talk about "elements", terms, of types

$$a : A$$

(like global pt's $* \xrightarrow{a} A$).

New types can be defined not only by universal properties, but also using constructors and eliminators

(how it receives & emits arrows)

and computation rules

(what happens when we emit what we receive)

These are possible in cartesian closed cat's + refinements

E.g.

$$\begin{tikzcd} & * \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & A\times B \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

amounts to the characterisation of a product type $A \times B$ and its terms, as well as all the rules.

In some sense, there is a

syntax / semantics

$\wr$ $\wr$

type thy / category thy

duality.

Roughly speaking: (some are guesses)

classical first-order	$\sim$	Boolean cat $\supset$ some topoi
constructive first-order	$\sim$	Heyting cat $\supset$ topoi
constructive simply-typed λ-calc	$\sim$	cartesian closed
Lean [(extensional) dependent type thy	$\sim$	locally cart closed
HoTT	$\sim$	loc cart closed

$(\infty, 1)$ - cat

HoTT univalence $\sim$ $(\infty, 1)$ - topos

Lastly we have correspondences like:

unit type $\sim$ terminal

empty type $\sim$ initial

prop $\sim$ (-1)-trunc'd obj.

β - reduction $\sim$ counit for hom-tensor

η - conversion $\sim$ unit —"—

We will see what the LHS is over the next few weeks.