Types, universes, univalence

Ödül Tetik

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1 Trinitarianism, quaternitarianism

There are multiple reasons to be dissatisfied with set-theoretical foundations for ordinary mathematics. For most practising mathematicians in our post-Bourbaki era, objects are typically sets-with-structure and maps thereof, where 'structures' are definable set-theoretically (e.g., *an underlying set/map of sets...*) and using typically first-order logic (*... such that P*). We typically do not think about constructions, theorems and proofs in these terms, but are content with the translatability of our mathematics into set-theoretical terms, in case the foundational police ask for our papers.[1] In fact, any underlying set theory tends to be irrelevant to our constructions and *arguments* in proofs. This is best illustrated by category-theoretical proofs: if a result (or rather its proof) is general enough, it is now ordinarily formulated in categorical terms, and is then valid in all categories where the constructions and arguments apply. At that stage, any underlying set-theoretical 'mental picture' is no longer necessary. In fact, the result may hold in non-concrete categories, i.e. those that are not categories of structured sets and structure-preserving maps. It has been recognised for a long time now that we don't need sets to do topology, we don't need triangles to do geometry, and we don't need truth tables to do logic. Indeed, logic is naturally subsumable into geometry (or topology), once the latter is taken in sufficient generality.

So far I have been advertising a version of R. Harper's 'computational trinitarianism' (by affinity, also the quaternitarianism of Baez–Stay [BS10]), as cited by Corfield [Cor20]:

> The central dogma of computational trinitarianism holds that Logic, Languages, and Categories are but three manifestations of one divine notion of computation. There is no preferred route to enlightenment: each aspect provides insights that comprise the experience of computation in our lives. [...] Imagine a world in which logic, programming, and mathematics are unified, in which every proof corresponds to a program, every program to a mapping, every mapping to a proof!

[1] No pun intended.

> Imagine a world in which the code is the math, in which there is no separation between the reasoning and the execution, no difference between the language of mathematics and the language of computing. Trinitarianism is the central organizing principle of a theory of computation that integrates, unifies, and enriches the language of logic, programming, and mathematics. It provides a framework for discovery, as well as analysis, of computational phenomena. An innovation in one aspect must have implications for the other; a good idea is a good idea, in whatever form it may arise. If an idea does not make good sense logically, categorially, and typically... then it cannot be a manifestation of the divine.[2]

An operational kind of ontology for mathematics underlies, or is necessary for, this picture. This can be seen as a consequence of the vague 'foundational' commitment to the notion that the extent of the meaning of a (mathematical) object is bounded by the ways in which it can be manipulated, by itself or in combination with other objects – a common notion, and a vague version, restricted to mathematics, of Peirce's pragmati(ci)sm [LH21], or Wittgenstein's declaration [Wit35, Ts-309, 7] that

> [] if we had to name anything which is the life of the sign, we should have to say that it was its *use*.

In our time, we are in a position to delegate execution (and even some original reasoning) to computers. *Lean* is, as are other similar tools, one way of realising an equation of reasoning and execution. It happens to be quite useful, to conform naturally to ordinary mathematical practise as a result of such preconceptions as mentioned above common to human practise and *Lean*, and to have a large community currently building on it. Before we consider type theory, on which *Lean* is based, let us treat and motivate some antecedent points of view relating logic, categories, and geometry/topology. In other words, we would like to elaborate type theory by regarding some special cases from logic, category theory and geometry/topology.

One might read into this approach also a delegation of mathematical *ontology* to computers. After all, are we not also equating meaning and/or existence with operation and/or implementation, and delegating the latter to the computer? Whether this is a valid counterattack on the foundational sceptic is an infamously subtle question, in view of Wittgenstein's rule-following problem, extended to computers; cf. [Kri82]. Addressing this issue is beyond the scope of this note.

[2] Harper, being a computer scientist, talks about a divine notion of computation, but we may substitute pure mathematics, for instance like in geometry or topology, for it.

2 Categorical logic

There are many ways to approach this topic. We will take but one possible route. For more complete accounts of classic results and points of view, we refer the reader to [MM12], [LS88], [Bor94], [Uni13], [Cor20], and references therein.

Recall the logical connectives $\wedge, \vee, \to, \neg$, and the quantifiers $\forall, \exists$. Classically, due to the law of excluded middle (LEM: $\forall A, A \vee \neg A$)) and the declared truth table for an implication $A \to B$, such an implication is equivalent to $\neg A \vee B$. We will start with an account of the algebraic behaviour of these things, excepting the quantifiers.

2.1 Order 0

A ***bounded lattice*** is a poset with finite products and coproducts, respectively called ***meets*** and ***joins***, denoted by $\wedge$ and $\vee$. All our lattices will be bounded. As special cases, an empty product, i.e. a terminal object, as well as an empty coproduct, i.e. an initial object, exist, denoted respectively by $\mathbf{1}$ or $\top$ and $\mathbf{0}$ or $\bot$. We will write $\vdash, \dashv$ for the arrows in a lattice, rather than the customary $\leq, \geq$. In a ***distributive lattice***, $\wedge$ distributes over finite $\vee$'s.[3] In a ***complemented lattice***, each object x has a negation ***negation*** or ***complement***, $\neg x$, such that $x \wedge \neg x = \bot$ and $x \vee \neg x = \top$. That is to say, $\neg x$ 'as far away from x as possible', in both directions. A complemented distributive lattice is also called a ***boolean algebra***.

All our lattices will be bounded and distributive.

Besides classical propositional logic (0th order logic), the main example of a boolean algebra is the category $\mathcal{P}(X)$ of subsets of a set X, where arrows correspond to inclusions.[4] Clearly, $\emptyset \simeq \bot$, $X \simeq \top$, $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $\neg A = A^c$, and $\neg$ behaves as required (Venn diagrams). In fact, as we may not need all subsets to have a boolean algebra, we may loosen this example to that of a ***field of sets***, which may be defined to be a boolean subalgebra of the boolean algebra $\mathcal{P}(X)$ for some set X. If one has not just finite but countable $\wedge$'s, or equivalently (due to the properties $\neg$) countable $\vee$'s, this is precisely a σ-algebra. It is a theorem of Stone that any boolean algebra is isomorphic to a field of sets [Sto36]. Specifically, it is isomorphic to the field of clopens of the associated Stone space, a compact Hausdorff totally disconnected space. (Conversely clopens in any topological space make up a boolean algebra.)

Remark 2.1.1. Incidentally, The category of such spaces is equivalent to that of profinite spaces (cofiltered limits of finite discrete spaces), which also happens to be the so-called pro-étale site[5] of a point. Sheaves on this site are called ***condensed sets*** [CS21] (a particular generalisation of ordinary topology) and are the basic objects of condensed mathematics. (Dually speaking, condensed

[3] It is known that this property is independent of the others.
[4] We will write '=' instead of '$\simeq$' here.
[5] The covers are finite jointly surjective families.

mathematics is about sheaves on boolean algebras.) Recall that the ordinary sheaf category $\mathrm{Sh}(*)$ on a point is the same as the category of sets.

In order to reconcile the term 'field' with its use in algebra, we may analogise $\wedge$ to $\cdot$, $\vee$ to $+$, $\neg$ to $(-)^{-1}$, $\top$ to 1 and $\perp$ to 0. This is not a perfect analogy, since $(-)^{-1}$ is never an inverse both multiplicatively and additively. But it's pretty good if one dispenses with the law of excluded middle $A \vee \neg A = \top$. Also, what is even $\geq$ or $\vdash$ (morphisms) here?

Example 2.1.2.

(i) This calls for a better algebraic example: consider $\mathbb{N}$ as the set of objects of a poset D where $n \vdash m$ iff $n \mid m$. Clearly, $n \wedge m = \gcd(n, m)$, $n \vee m = \mathrm{lcm}(n, m)$, and in particular $\top = 0$ is the 'empty gcd', $\perp = 1$ the 'empty lsd'. It is easy to check that D is distributive. Here, given n, any coprime m satisfies $n \wedge m = \perp$, while there is no m such that $n \vee m = \top$, except when $n = 0$. We see that this almost-boolean algebra is naturally 'intuitionistic' in the sense that it does not satisfy LEM. In a sense, it's as far away from LEM as it gets. One way to 'truncate' D into a boolean algebra is to choose a nice subcategory.

(ii) If we restrict, for some n, to the subcategory $\mathsf{D}_{|n}$ of the divisors of n, we remove 0 by construction and obtain $\top = n$, while $\perp = 1$ remains. We have the same $\wedge$-inverses, but if, say $n = p^2$, then $p \vee p^2 = p^2 = \top$, but $p \wedge p^2 = p \neq \perp$. More generally, if $k \mid n$, then the existence of a divisor $m \mid n$ coprime to k such that $k \vee m = n$ is equivalent to saying $k \nmid n/k$, as necessarily $m = n/k$. Thus, $\mathsf{D}_{|\mathsf{n}}$ is boolean iff n is square-free.

Entailment, $A \vdash B$, and ***implication***, $A \to B$ or $A \Rightarrow B$, have different meanings. From the categorical point of view, the difference is a familiar one: it is that between arrows in the category, and the internal-homs. In logical terms, consider, roughly, a category of 'propositions' (for instance, well-formed formulas of symbols/strings in some formal system) where each *syntactic derivation* $P \vdash Q$, a transformation of P according to the rules (syntax) of the system, corresponds to an arrow. Let us simply write $P \vdash Q$ again for the corresponding arrow. There may or may not be, in this system, a way to form out of two proposition P, Q a new proposition (a new object in the category), $P \to Q$ which, syntactically and/or semantically, behaves like a familiar logical implication.

In any case, for instance in a distributive lattice one would require of $P \to Q$ that it at least satisfies modus ponens : $P \wedge (P \to Q) \vdash Q$, which is a special case of the defining 'hom-tensor adjunction' $- \wedge Q \dashv (-)^Q$ for internal homs:

$$P \wedge Q \vdash R \quad \text{iff} \quad P \vdash (Q \to R).$$

The 'iff' is understood to mean that such an arrow exists iff the other kind of arrow exists, which, if our category is locally small, is customarily written as a bijection

$$\hom(P \wedge Q, R) \simeq \hom(P, Q \to R).$$

This is to be natural in a way we will not articulate here [6]. Internal homming is also called ***exponentiation***, and one writes $R^Q := Q \to R$. In a boolean algebra, one may declare $P \to Q := \neg P \vee Q$.

Remark 2.1.3. Giving a modus ponens arrow is then equivalent to giving an arrow $P \vdash Q^{P \to Q}$. Any sensible definition of $P \to Q$ should allow such an arrow, in terms of the 'evaluation of a function at a given point', though we cannot, at least easily, talk about 'points' of objects in a category. But the condition on any category that every object have an identity arrow already gives a canonical modus ponens arrow. Namely, notice, using $Q^P \wedge P$ instead of $P \wedge Q^P$, that

$$\hom(Q^P \wedge P, Q) \simeq \hom(Q^P, Q^P),$$

so id_{Q^P} corresponds uniquely to an arrow $Q^P \wedge P \vdash Q$, called ***evaluation***.

Definition 2.1.4. A category with finite products and and exponentials is called ***cartesian closed***. A ***Heyting algebra*** is a cartesian closed bounded lattice. Boolean algebras are Heyting.

Let us briefly turn back to our example D. An exponential m^n (not the usual one) is characterised by requiring $k \wedge n \mid m$ iff $k \mid m^n$. For instance, m^n must be divisible by any k coprime to n. This is absurd due to the infinitude of primes. If $n = 0$, due to $k \wedge \top = k$, we can set $m^0 = m$. Thus, only 0 is exponentiable, and exponentiates to the identity. On the other hand, in $\mathsf{D}_{|n}$ for n square-free, we may set $m^k = \neg k \vee m = (n/k) \vee m$.

Continuing our general discussion, we note in sum that $\vdash$ and $\to$, though quite distinct, are intimately connected. We can further observe that $\hom(P, Q) \simeq \hom(\top \wedge P, Q) \simeq \hom(\top, Q^P)$, i.e. derivations $P \vdash Q$ correspond to '$\top$-points' of Q^P, also called ***global points*** of Q^P. In particular, global points of any object P correspond to global points of $P^\top$. In fact, again using $Q \wedge \top \simeq Q$, we have $\hom(Q \wedge \top, P) \simeq \hom(Q, P)$, implying $P^\top \simeq P$, as they satisfy the same universal property. Note that this is *not* to say '$\hom(\top, P) \simeq P$', an expression that might or might not make sense. If this and/or similar further conditions are true in an appropriate sense, the category is called ***concrete***. In the category Set of sets, P^Q is necessarily the set of functions from Q to P, and one may check that the evaluation map is the usual evaluation of functions. We note that all this *follows* from the abstract requirements on internal homs unpacked in Set.

Indeed, we should keep in mind that $\top$-points of an object c might not 'exhaust' c, if our objects do not behave like plain sets. We saw this already in D and in $\mathsf{D}_{|n}$. In both, $\hom(\top, k)$ is empty for $k \neq 0$. In particular in $\mathsf{D}_{|n}$, $\emptyset = \hom(\top, k) \neq k^\top = k$ is an absurd comparison, in addition to being meaningless. In a more geometric context, this phenomenon is most dramatically illustrated as follows: Consider $\widehat{*} := \mathsf{PSh}(*) := \mathsf{Set}^{*^{\mathrm{op}}}$. Trivially, $\mathsf{Set} \simeq \widehat{*} \simeq \mathsf{Sh}(*)$. Consider now $\mathsf{Sh}(X)$, the category of Set-valued sheaves on a topological space X, a topos.

[6] It is in some contexts called ***currying*** (after Curry)

Maps $\widehat{*} \to \mathsf{Sh}(X)$ induced by maps $* \to X$ (if you like, the right adjoints/direct images in the induced geometric morphisms $\mathsf{Sh}(X) \rightleftarrows \mathsf{Sh}(*)$), sometimes called the 'concrete points' of the sheaf topos $\mathsf{Sh}(X)$, pick out only the skyscraper sheaves on X. This is a continuous version of the difference between picking out basis vectors of a vector space versus picking out linear combinations of them.

Remark. To be sure, this is not exactly analogous to the discussion on propositions above. In what category are we, and is $\widehat{*}$ a terminal object there?

The main non-logical example of a Heyting algebra is the poset $\mathcal{O}(X)$[7] of open subsets of a topological space X, where arrows correspond to inclusion. If opens and closeds coincide, this is a boolean algebra, since the set-theoretical complement serves as negation. An nontrivial example is the ***Sorgenfrey line*** $\mathbb{R}_l$, whose topology is generated by the basis given by half-open intervals $[a, b)$. This is a totally disconnected Hausdorff space, just like any Stone space.

In general, we have $\wedge = \cap$, $\vee = \cup$, $\top = X$ and $\bot = \emptyset$. Note that $\mathcal{O}(X)$ has all colimits, though not necessarily all limits. For $U \in \mathcal{O}(X)$, $(X \setminus U)^\circ$, the interior of the set-theoretical complement, satisfies $U \wedge (X \setminus U)^\circ = \bot$, but it is not a $\vee$-inverse, as the difference to $\top$ is the boundary of U. In general, $\mathcal{O}(X)$ has no negation in this classical sense. Still, it has all exponentials: we may set $(U \to V) = \bigcup_{U \cap W \subseteq V} U \cap W$. More economically, consider the overcategory $\mathcal{O}(X)_{/V}$, and consider

2.2 Order 1

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[7] Not to be confused with the structure sheaf notation $\mathcal{O}$.

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