Seminar on loop quantum gravity
Talk 7
4d: BF with cosmological constant is equivalent to Ooguri–Crane–Yetter

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29th April 2020

It turns out that what we saw in 3d in Talk 4 can be recast into 4d. Whether (or in what sense) this is in 'extension' should I think be a matter of controversy.

In a certain sense, the 3d version was about finding a state-sum model for 3d BF theory. The Crane–Yetter theory plays the same role for 4d BF theory (with cosmological constant).

In terms of spin network states, this is directly linked to the Plebanski formalism from Michele's talk and the loop representation. Due to time constraints, I couldn't fit a discussion of this into this note. Instead, let me refer (or defer) to Baez [2].

Contents

1 Quick meta discussion

We know, given a spherical fusion category $\mathcal{C}$, how to build a 2-3d functorial TQFT (Turaev–Viro(–Barrett–Westbury–Balsam–Kirillov)). That construction does not require any kind braiding on $\mathcal{C}$. Most of the structure is in the 'fusion part', which includes $\oplus$, $\otimes$, ev, coev, hence also the left and right traces tr_L, tr_R, and assumptions on simple objects – these are the ingredients that let us colour and evaluate diagrams coming from the dual of a triangulation (à la Penrose/Kirillov–Reshetikhin/Temperley–Lieb 'recoupling theory'/diagrammatics). Sphericality says that $\mathrm{tr}_L = \mathrm{tr}_R$, *which resolves an ambiguity in the reading off of a diagram from a triangulation.* In 2-3d TV theory, this ambiguity comes from the choice of a point on the sphere from which we stereographically project to get a *planar diagram from the original diagram on a sphere.*

The Reshetikhin–Turaev TQFT, which have not covered, requires a braiding such that $\mathcal{C}$ is a ***ribbon*** (also called ***tortile*** by Baez and some others) fusion category, meaning there is a braiding as well as a twist, such that the braiding, the twist, and duality play well with each other. This extra structure removes the need for the sphericality assumption, resolving the ambiguity by compatibility instead. Also, it is like adding square roots to the algebra: indeed, the RT theory is a kind of square root of the TV theory. (See Balsam–Kirillov for a precise statement.)

In 4d, the Ooguri–Crane–Yetter construction[1] has the RT theory as its boundary. For geometric reasons in 4d, the construction requires a ribbon structure. In fact, CR talk about modular tensor categories, which means ribbon fusion + the ***modularity*** condition, which is a nondegeneracy condition on the braiding; it means that the matrix whose ij-entry is the evaluation of the diagram

is nondegenerate. Here, the indices make up a complete list of representatives of simple objects ($\sim$ simple spin representations).

According to nLab, modular tensor categories originated in the study of CFT (they encode Moore–Seiberg data), which is not surprising. They lift to 3d to construct state-sum input data for a TQFT, which we saw in Talk 4. This would be the Chern–Simons/CFT 2-3d duality ('holography'). Indeed, it was recently shown that MTCs with a little bit of extra structure essentially encode

[1] This is what I'll call the Crane–Yetter construction, because, as they also say, it is a slight modification of Ooguri's state sum model (which diverges on its face).

all 1-2-3d FTQFTs [4]. The result that standard MTCs encode 2-3d FTQFTs is apparently a much earlier result due to Moore–Seiberg [8]. Crane–Yetter lift this structure yet again to 4d, basically without modification. At this point, things become unsatisfactory. One would expect that if one has an input category for 3d, the 4d version should be some categorification, which would mean a 2-category. In fact, if the input category is appropriate to 2d (CFT), one would expect the 4d lift to accept a 3-category. Crane seems to think similarly [5]. He says that what their construction ought to be is a special case where the actual input category is a 2-category with a single object. In their joint paper [7], Crane and Yetter also suggest that we might be seeing the top-dimensional sector of a 3-category with a single object and a single 1-morphism. We will come back to this later.

2 The Ooguri–Crane–Yetter state sum

Let us first discuss what Ooguri did [9]. I'll be brief and have to hand-wave a little, but that's only because the Crane–Yetter version works perfectly without any of that. Still, I believe that looking at the original construction will help understand things better.

2.1 Ooguri's state sum and some remarks

We may set $G = SU(2)$ but this is not necessary. The constructions should work for any compact Lie group, especially if simply-connected like $SU(2)$, and also for any discrete finite group. The 'fields' are functions

$$\phi\colon G^{\times 4} \to \mathbb{R}$$

which we assume (for finiteness/well-definedness reasons) to be invariant under the right diagonal G-action on $G^{\times 4}$. The 4d action is

$$\begin{aligned} S(\phi) &= \frac{1}{2}\int \prod_{i=1}^{4} dg_i \phi^2(g_1, g_2, g_3, g_4) \\ &\quad + \frac{\lambda}{5!}\int \prod_{i=1}^{10} dg_i \phi(g_1, g_2, g_3, g_4)\phi(g_4, g_5, g_6, g_7)\phi(g_7, g_3, g_8, g_9)\phi(g_9, g_6, g_2, g_{10})\phi(g_{10}, g_8, g_5, g_1) \end{aligned}$$

which is supposed to be imitating Dijkgraaf–Witten theory. (Recall [if you've read the notes for Talk 4] that Turaev–Viro with the category of G-graded vector spaces as input describes Dijkgraaf–Witten theory. This is also the relation to Chern–Simons.)

Let us interpret this model. I will start with why the fields of this gauge-theoretic model cannot be just connections.

Colourings and the gauge-theoretic input. The integral in the interaction term is meant to be over all colourings of a fixed 4-simplex where the group elements are attached to triangles (faces of constituting tetrahedra). [The kinetic term is more subtle; I will give an interpretation for it in a moment.] This is the main point of divergence from the 3d construction, as well as the point of greatest similarity to it. In 3d, one colours edges; here, we are colouring triangles, so everything is just shifted up one dimension. *This is the basic reason why essentially the same algebraic input data as in 3d is enough to construct a similar TQFT in 4d as well.* At the same time, however, this is also the basic reason why this construction is unsatisfactory, and one would expect that we are only seeing the 1-2-dimensional sector of a 2-category that we have cut down to a single object (which we are nevertheless overlooking). This is why it is also physically unsatisfying: the reason we have been colouring edges is that we want to work with spin network states, which are generalisations of Wilson loops, but not in the dimensional sense (i.e. not generalisations like surface operators) but in a linear sense. We are forcing lattice gauge theory onto smooth manifolds, interpreting the group element tagged onto a 1-cell as the holonomy along that 1-cell (of some connection on a principal G-bundle over the manifold – which is why we have been interpreting colourings as field content). All this is because our fields are connections, i.e. locally 1-forms in the affine sense. When we attach a group element to a triangle, however, this interpretation is no longer valid, or, alternatively, this means that the fields of this model aren't just connections. Still, such an assignment is possible if the group G is not the usual gauge group but the morphisms in a Lie 2-group (an internal category in the category of Lie groups), say $\mathcal{G}$, which in toto means that the Ooguri model has as gauge group a Lie 2-group with one object (a singleton regarded as a Lie group). We see that a generalised version of this model that can be a precursor to a Crane–Yetter style FTQFT extended to codimension 2 must evaluate lattice gauge data from a 2-bundle with a 2-connection over the manifold M. In the case where the bundle is trivial (which we will assume later; in any case the tangent bundle of any orientable 3-manifold is trivialisable), such a 2-connection will be a (smooth) 2-functor of type

$$\mathrm{Hol}_\nabla : \mathcal{P}_2(M) \to \mathcal{G}$$

where $\mathcal{P}_2(M)$ is the (smooth) path 2-groupoid of M (2-morphisms are bigons between paths sharing the same endpoints), so that paths are sent to the Lie group $G' := \mathrm{Ob}(\mathcal{G})$ and bigons are sent to $G' \ltimes G = \mathrm{Mor}(\mathcal{G})$, which is essentially Ooguri's Lie group G except that $G' \ltimes G$ keeps track of the source points of maps (cf. [3]).

Field content. The 'field' ϕ is used by Ooguri as a placeholder for a prescription to evaluate a colouring; this is on the level of tetrahedra: ϕ gives a number for each coloured tetrahedron. At first, this seems to contrast our discussion in 3d, as there the evaluation was 'automatic' in that we turned coloured triangulations into coloured diagrams and evaluated them in the usual way (P/KR/TL).

This is due to the fact that in 3d state sums we talked directly about partition functions, bypassing the action.

4-simplices and the interaction term. A 4-simplex consists of 5 tetrahedra, 10 triangles, 10 edges, and 5 vertices. Thus the 5-particle interaction term, besides the coupling factor $\frac{\lambda}{5!}$, is obtained as follows. Just like in 3d where one has a triangle opposite to each vertex, we have here a tetrahedron opposite to each vertex, 5 in number. For each vertex, then, we evaluate ϕ on the 4 coloured faces of the opposing tetrahedron, and multiply these numbers together. In this way, ϕ induces an aggregate function, say $\hat{\phi}$, that can be evaluated on a coloured 4-simplex. The interaction term is then the integral

$$\int_{\substack{\text{admissible}\\ \text{colourings}\\ \subseteq G^{\times 10}}} \hat{\phi} \quad \text{or more bluntly} \quad \int_{G^{\times 10}} \hat{\phi}.$$

In the first option, I mean that one might wish to restrict the integral using some notion of admissability (as in TV theory in 3d). This would require a discussion of analouges of the Clebsch–Gordan rule and permutation symmetries. Ooguri integrates over all of $G^{\times 10}$.

Implicit is a choice of ordering on the vertices of the 4-simplex. Integrating over all of $G^{\times 10}$ might be fixing this. In 3d, Turaev–Viro built invariance under reorderings into the algebraic input data as a condition, which happens to be fulfilled by 6j-symbols. I would think that the problem will be fixed here as well, but again only at the level of the partition function.

Spin reps and partition function. By straightforward but tedious algebraic manipulation, one can show that the partition function on a closed 4-manifold M looks like

$$Z_O(M) = N_{sym}(M)^{-1}\lambda^{N_4(M)} \sum_{\text{colourings } j} \prod_{\text{2-simplices } t} (2j_t{+}1) \prod_{\text{3-simplices}} 6j \prod_{\text{4-simplices}} 15j$$

- The factors in front are not of much interest: the first one normalises for symmetries and the second one for the number of 4-simplices in M.
- The 15j symbols are made up of 6j symbols and are their analogue for 4-simplices. Instead of writing down an expression for them here, I will discuss how they can be computed by diagrammatics when we come to the Crane–Yetter version.

But let us talk first about how to get these j's from what we started with.

The 15j symbols here (apparently, there are several variants) are not symmetric under reordering of the vertices of 4-simplices. The kinetic term comes to the rescue: under exp it gives a multiplicative factor which is essentially the contribution $\prod_{\text{2-simplices } t}(2j_t+1)$, thanks to which Z_M is invariant under reorderings of vertices. Here j stands as always for spin, so the contribution of each triangle t coloured with j_t is the dimension of the spin-j representation.

But we haven't assigned spins to faces, we are only assigning group elements to edges! Indeed. Ooguri goes from functions $G^{\times 4} \to \mathbb{R}$ to expressions with spins by writing the functions as

$$\phi(g_1, g_2, g_3, g_4) = \sum_{j_i, m_i, n_i} \phi^{j_1 \dots j_4}_{m_1 n_1 \dots m_4 n_4} D^{j_1}_{m_1 n_1}(g_1) \cdots D^{j_4}_{m_4 n_4}(g_4)$$

where $D^j_{mn}(g)$ stands for the matrix element of g in the spin-j representation, thus $m_i, n_i = 1, \dots, 2j_i + 1$. (Here we restrict w.l.o.g. to $G = SU(2)$, of course.) In this way one can express the action and the partition function in terms of spin-data and real coefficients $\phi^{j_1 \cdots j_4}_{m_1 n_1 \dots m_4 n_4}$. The only extra assumption on ϕ is that it be invariant under cyclic permutations of triples of arguments. This is analogous to the 3d case.

TV-type interpretation and the kinetic term. Let us discuss how to get the same result by KR/TL diagrammatics, which will be a prelude to Crane–Yetter. Take a 4-simplex, whose boundary consists of 5 tetrahedra. Just as in 3d, we can extract a trivalent graph as follows:

Before saying more on diagrammatics, let us explicate this and find the the kinetic term:

- Each triangle becomes an edge. 2 are incoming, 2 are outgoing. (This is determined by the ordering on the vertices of the triangulation.) This is just imitating the TV construction, only one dimension up. (In fact, Barrett–Westbury do almost exactly this in their paper.)
- Without any choices, the tetrahedron would naturally become the central vertex in the 4-valent graph in the figure. We blow up this vertex into an edge to get a trivalent graph. A colouring on this graph induced on this graph has to be completed with a choice of colouring for the new internal edge, here l.

- (For comparison, recall that in TV edges become edges and the triangles become vertices.)
- The l-edge and its endpoints must also be interpreted as if they come from a triangle and two tetrahedra in the triangulation. This contributes as an extra term in the action, the kinetic term (up to the factor $\frac{1}{2}$).
- Glue in the natural way to get the whole triangulation.

The interpretation in terms of diagrammatics with spin reps is now easy: let the tags a, b, c, d, l stand for simple objects as in TV. Clebsch–Gordan determines a unique intertwiner at the vertices (what Ooguri calls the 3j-symbol). Evaluating this diagram as usual gives the contribution $\prod 6j \prod 15j$.

Remark. This is not exactly what Ooguri does: he does not choose an ordering of the vertices of the triangulation, so he has no canonical way to blow up the 4-valent vertex into two 3-valent ones. He proposes to make an arbitrary choice, and asserts that the result will be independent to the orthogonality property of 6j-symbols. Alternatively, one could perform an outermost sum over all orderings of the vertices, where in each summand one evaluates according to the ordering as specified above, and then normalise by $\frac{1}{\#\{\text{vertices}\}!}$.

2.2 The CR version

As we have discussed, the Ooguri model is essentially TV, only one dimension up. Crane–Yetter formalised this [7].

Start again with a triangulated 4-manifold M, with an ordering on the vertices. This determines at each tetrahedron in the 3-skeleton a way of blowing up the 4-valent vertex into a trivalent graph with two vertices, and gives in total a trivalent graph for the whole surface. As mentioned above, any MTC determines a 2-3d TQFT (via the TV construction). A clean short-cut is thus to thicken this graph into a surface and use the TQFT:

Gluing these pieces along the triangulation, we get a surface Σ_M, which gives a state space for any input MTC. The modular structure (braiding+twist+modularity) ensures that this is well-defined.

The TV-type partition function of Crane–Yetter is this:

$$\sum \mathcal{D}^{\#\text{vertices}} D^{-\#\text{edges}} \prod_{\text{triangles}} d_j \prod_{\text{tetrahedra}} d_p^{-1} \prod_{\text{4-simplices}} 15j.$$

Note that now *the edges also contribute via the dimension $\mathcal{D}$ of the MTC, not just vertices!* Similarly, instead of the edges, the triangles and the tetrahedra contribute via the spins attached to them (or rather their dimensions). (Here, p is the spin attached to the cut circle in the dual surface of a tetrahedron.) In 3d, the only interesting contributions were the 6j-symbols coming from the tetrahedra, and the lower-dimensional contributions via categorial dimensions were there for normalisation. The direct 4d analogue of this holds here also. (The exponentials alternate in sign as in the TV construction, though not exactly in the same way, but this is not too important.)

We can define the 15j symbols generally within the input MTC as the evaluation of the diagram obtained from a 4-simplex, without any recourse to the representation theory of $SU(2)$ or some other Tannaka-dual group. It looks like this:

The case with boundary can be treated exactly as in the TV construction (but we'll see more below).

3 Relation to BF theory & the FTQFT formulation

3.1 The OCR partition function in terms of signature

It turns out that the OCR invariant $Z_{OCR}(M)$, the partition function on a closed 4-manifold can expressed as

$$Z_{OCR}(M) = \kappa_{\mathcal{C}}^{\sigma(M)} y_{\mathcal{C}}^{\chi(M)/2},$$

and upon further normalisation, as

$$Z_{OCR}(M) = y_{\mathcal{C}}^{\sigma(M)}$$

where

- $\kappa_{\mathcal{C}}$ is obtained from the root of unity q at which one quantises the complex semisimple input Lie algebra underlying the input category $\mathcal{C}$;
- σ stands for signature, χ for the Euler characteristic;
- $y_{\mathcal{C}} \in \Bbbk$ is a number obtained from $\mathcal{C}$ as a dimension-weighted sum of the evaluations of framed unknots over the representative set of simple objects.

This is essentially due to [10] but see also [6]. Unfortunately, we have no time to go into how this works.

Baez has shown that any two 3-4d TQFTs Z, Z' such that

$$Z(M) = y^{\sigma(M)} = Z'(M)$$

for closed 4-manifolds M are equivalent. More to the point, he has constructed such a TQFT that can be identified with BF theory with cosmological constant. We will now see how this works.

3.2 FTQFT formulation and equivalence to BF with cosmological constant

As in the setup of BF theory, let G be a Lie group with a symmetric invariant on its Lie algebra $\mathfrak{g}$.

The TQFT version due to Baez [1] is the composition

$$Z_{BF} \colon \Omega^{SO}_{3,4} \xrightarrow{c_1} \tilde{\Omega}^{SO}_{3,4} \xrightarrow{c_2} \mathcal{C}_{3,4} \xrightarrow{\tilde{Z}_{BF}} \mathcal{V}ect$$

where

- $\Omega^{SO}_{3,4}$ is the usual symmetric monoidal 3-4d oriented cobordism category[2];
- $\mathcal{C}_{3,4}$ is the symmetric monoidal 3-4d cobordism category whose objects are trivialisable principal G-bundles $P_\Sigma \to \Sigma$ over compact oriented 3-manifolds Σ, and whose morphisms are (equivalence classes of) G-bundles $P_M \to M$ on compact oriented 4-manifolds M, with equivalence defined by G-bundle isomorphisms. The orientations work as follows: for $\partial M = \Sigma \amalg \Sigma'$ in the topological sense, the morphism $M \colon \Sigma \to \Sigma'$ comes with a commuting diagram

$$\begin{tikzcd} P_{\overline{\Sigma}} \amalg P'_{\Sigma} \arrow[r] \arrow[d] & P_M|_{\partial M} \arrow[d] \\ \overline{\Sigma} \amalg \Sigma' \arrow[r] & \partial M \end{tikzcd}$$

  where the top map is a G-bundle isomorphism and the bottom map is an orientation-preserving diffeomorphism.

[2] This notation is sometimes used for *stable* tangential SO-structure. Here we mean both the 3-manifolds and 4-bordisms are oriented on the nose.

- $\tilde{\Omega}^{SO}_{3,4}$ is $\Omega^{SO}_{3,4}$ except that morphisms $M\colon \Sigma \to \Sigma'$ come with 'lapse and shift': a vector field over ∂M that is nowhere tangent to ∂M and inward-pointing along $\Sigma \hookrightarrow \partial M$ and outward-pointing along $\Sigma' \hookrightarrow \partial M$.

It is the structure on $\mathcal{C}_{3,4}$ that is important. It comes from the construction of states for BF theory with cosmological constant. The assumption of the trivialisability of the bundles is because this is required in the construction of the Chern–Simons action. The extra structure defining $\tilde{\Omega}^{SO}_{3,4}$ is only there to express the procedure of going from $\Omega^{SO}_{3,4}$ to $\mathcal{C}_{3,4}$. The lapse-shift vector fields are choices needed to set up the commuting diagrams in $\mathcal{C}_{3,4}$, i.e. to fill in the extra dimension on the RHS.

The construction is straightforward in the case

$$G = GL(4, \mathbb{R})$$

owing to the fact that $T\Sigma$ for Σ compact and oriented is always trivialisable. The maps defining Z are as follows:

The map c_2 (c stands for *choices*) adds a trivial dimension and takes the oriented frame bundle. That is, an object Σ is sent to the $GL(4, \mathbb{R})$-bundle P_Σ which is the bundle of oriented frames of the vector bundle $T\Sigma \oplus \underline{\mathbb{R}} \to \Sigma$. The horizontal maps in the diagram above are determined by the lapse-and-shift vector fields.

The map c_1 just picks any lapse-and-shift vector field. Although this is not functorial, the composition $Z\colon \Omega^{SO}_{3,4} \to \mathcal{V}ect$ is.

The map $\tilde{Z}_{BF}$ is inspired by BF with cosmological constant. On objects,

$$\tilde{Z}_{BF}(\Sigma) \coloneqq \mathbb{C} \cdot f \subset \mathrm{Fun}(\mathcal{A}_\Sigma) \quad \text{for} \quad f(A_\Sigma) = e^{-(3i/\Lambda) S_{CS}(A_\Sigma)}$$

where $\mathcal{A}_\Sigma$ is the space of connections on P_Σ. We also set $Z(\emptyset) = \mathbb{C}$.

For a morphism $M\colon \emptyset \to \Sigma$, we define the function on connections it gives as

$$\psi(A_\Sigma) \coloneqq \tilde{Z}_{BF}(M)(1)(A_\Sigma) = e^{-(3i/\Lambda) \int_M \mathrm{tr}(F_A \wedge F_A)}$$

where A is any extension of A_Σ to all of M. We will see in a moment why this is in $\tilde{Z}_{BF}(\Sigma)$.

This takes care of general morphisms $M\colon \Sigma \to \Sigma'$, once we show

$$\tilde{Z}_{BF}(\overline{\Sigma} \amalg \Sigma') = \tilde{Z}_{BF}(\Sigma)^* \otimes \tilde{Z}_{BF}(\Sigma').$$

This is immediate using the fact that S_{CS} changes sign upon changing the orientation.

Why is this BF? Recall the BF and Chern–Simons actions:

$$S_{BF}(A, B) = \int_M \mathrm{tr}\left(B \wedge F_A + \frac{\Lambda}{12} B \wedge B\right)$$

and

$$S_{CS}(A_\Sigma) = \int_\Sigma \mathrm{tr}\Big(A_\Sigma \wedge dA_\Sigma + \frac{2}{3} A_\Sigma \wedge A_\Sigma \wedge A_\Sigma\Big),$$

where, for a G-bundle $P \to M$, A is a connection and B is an $\mathrm{Ad}(P)$-valued 2-form, A_Σ lives over Σ. Recall also that S_{CS} defines a state $\psi\colon \mathcal{A}_\Sigma \to \mathbb{C}$ of BF-theory by

$$\psi(A_\Sigma) = e^{-(3i/\Lambda)S_{CS}(A_\Sigma)},$$

using

$$\frac{\delta S_{CS}}{\delta A_{ka}} = \varepsilon^{ijk} F^a_{ik},$$

where i, j, k are spacelike and a is internal (as are b, c below). This is because states ψ must satisfy, besides the Gauss constraint, the equation

$$\Big(F^a_{ij} - i\frac{\Lambda}{6}\varepsilon_{ijk}\frac{\delta}{\delta A_{ka}}\Big)\psi = 0,$$

which is the canonically quantised version of the constraint $F^a_{ij} + \frac{\Lambda}{6}B^a_{ij} = 0$, quantising the classical conjugation relation over Σ, which reads $\{B^a_{ij}(x), A_{kb}(y)\} = \delta^a_b \varepsilon_{ijk}\delta(x,y)$.

What of 4-manifolds M? For $\partial M = \Sigma$ interpreted again as $M\colon \emptyset \to \Sigma$, one finds (heuristically) in terms of path integrals that its boundary state, defined by integrating over all extensions of the field A_Σ to M (recall that we do the same thing with state-sum TQFTs, where we sum over all extensions of a given colouring on the boundary), satisfies

$$\begin{aligned}\psi(A_\Sigma) &= \int_{A|\Sigma = A_\Sigma} \mathcal{D}A \int \mathcal{D}B e^{i\int_M \mathrm{tr}\left(B + F_A + \frac{\Lambda}{12}B\wedge B\right)} \\ &= \int_{A|\Sigma = A_\Sigma} \mathcal{D}A e^{-(3i/\Lambda)\int_M \mathrm{tr}(F_A + F_A)}.\end{aligned}$$

Fix now a reference flat connection A_0 on P_Σ (which is assumed to be trivialisable) and write $F_0 = F_{A_0}$. Because $d\mathrm{CS} = \mathrm{tr}(F \wedge F) \propto$ (Chern form of the connection) and the Stokes theorem (with boundary), we have

$$\int_M \mathrm{tr}(F \wedge F) = S_{CS}(A_\Sigma) + \int_M \mathrm{tr}(F_0 \wedge F_0)$$

which gives $\psi(A_\Sigma) = \exp\big(-(3i/\Lambda)(S_{CS}(A_\Sigma) + \int_M \mathrm{tr}(F_0 \wedge F_0))\big)$, which is the previous version up to a factor. When $\Sigma = \emptyset$, this gives

$$Z_{BF}(M) = e^{-(3i/\Lambda)\int_M \mathrm{tr}(F\wedge F)}.$$

This is how Baez's TQFT can be identified with BF theory with cosmological constant. Note that we've also seen $Z_{BF}(M) \in Z_{BF}(\Sigma)$ for $M\colon \emptyset \to \Sigma$.

Baez's theorem. It is straightforward to see that any two TQFTs $Z, Z' \colon \Omega^{SO}_{3,4} \to \mathcal{V}ect$ are equivalent, if, on closed objects, $Z(M) = y^{\sigma(M)} = Z'(M)$.

The crucial observation is that for any object Σ we have $\dim Z(\Sigma) = Z(S^1 \times \Sigma) = 1$ because $\sigma(S^1 \times \Sigma) = 0$. (In our state sums, the first equation is due to the normalisation factors that we've been carrying around.) Hence there is just one sensible way to define an isomorphism

$$F_\Sigma \colon Z(\Sigma) \to Z'(\Sigma)$$

which is by requiring that

$$F_\Sigma(Z(M)1) = Z'(M)1$$

for *all* $M \colon \emptyset \to \Sigma$. It is routine to check that this is well-defined, natural, and monoidal.

Now, the Hirzebruch signature theorem implies that

$$\int_M \mathrm{tr}(F_A \wedge F_A) = 12\pi^2 \sigma(M)$$

for any connection A, hence $Z_{BF}(M) = e^{-36\pi^2 i \sigma(M)/\Lambda}$. Thus we see that Z_{OCR} and Z_{BF} are equivalent if $y = e^{-36\pi^2 i/\Lambda}$

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