LINKED SPACES AND EXIT PATHS

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Abstract. We introduce explicit exit path ∞-categories (EPCs) for linked spaces, i.e., spans $M \xleftarrow{\pi} L \xrightarrow{\iota} N$ of spaces where π is a fibration and ι is a cofibration, and prove that this induces a fully faithful functor from an ∞-category of linked spaces to the ∞-category of all ∞-categories whose essential image includes the EPCs of conically smooth stratified spaces (CSSs) of depth 1, reducing the stratified topology of depth-1 CSSs to the ordinary topology of linked spaces. Finally, using linked smooth manifolds we resolve various versions of a conjecture of Ayala–Francis–Rozenblyum in the negative by exhibiting finite ∞-categories with conservative functors to $\{0 < 1\}$ with certain further restrictions which are not equivalent to EPCs of CSSs.

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1. Introduction

Many singular spaces naturally decompose into collections of non-singular strata which are glued together in interesting ways. A main goal of stratified topology is to understand the stratified topology of singular spaces in terms of the ordinary topology of the strata and the gluing information. Typically, the latter is again parametrised by spaces, called (homotopy) links, which are not explicit in the original singular space but can be extracted from it. This process of decomposition is well-understood for many types of spaces.

The present paper is concerned with the inverse problem of explicitly reconstructing stratified homotopy types given systems of non-stratified spaces that are interpreted as strata and links. We achieve this 'in depth 1' in a way that recovers and extends a large class of singular spaces. It is natural to expect that the methods we introduce extend to higher depth, but we leave this to future work.

Let $\mathfrak{S} = (M \xleftarrow{\pi} L \xrightarrow{\iota} N)$ be a span of spaces. We call $\mathfrak{S}$ a *linked space* if π is a fibration and ι is a cofibration. If we disregard the condition on ι, such a span would arise immediately from any nice depth-1 filtered space $X = (M \subset \overline{N})$ by setting $N = \overline{N} \smallsetminus M$ and taking L to be the space of paths in X that start in M and end in N, and which exit into N immediately after time 0. Such paths are called *exit paths*.[1] Similarly, paths $\gamma\colon [0,1] \to X$ such that $\gamma(t) \in N$ for $t < 1$ and $\gamma(1) \in M$ are called *enter paths*. An example of central importance to geometric topology and proper homotopy theory is the end space $e(Y)$ of an open manifold Y, the space of all proper maps $[0,\infty) \to Y$. It is equivalent to the space of all paths $\gamma\colon [0,\infty] \to Y^c = X$ to the one-point compactification of Y such that $\gamma(\infty) = \infty$. If Y is the interior of a compact manifold with boundary ∂, then there is a homotopy equivalence $\partial \simeq e(Y)$.[2] At the other extreme, the most general type of span we will consider in this paper is a *linked ∞-category*, where M, L and N are ∞-categories (by which we mean quasi-categories) and π is a right fibration.[3]

If X is a conically smooth stratified space (CSS) in the sense of Ayala, Francis and Tanaka [AFT17b], then M and N are smooth manifolds, and this path space is equivalent to an embedded normally-framed submanifold $\iota\colon L \hookrightarrow N$ of codimension 1 which is simultaneously a fibre bundle $\pi\colon L \twoheadrightarrow M$, producing a linked manifold. This is a very large class of stratified spaces which, by a result of Nocera and Volpe [NV23], includes Whitney-stratified spaces [Whi65] such as analytic varieties (which may have higher depth). We give two prototypical examples: If $X = (\partial M \subset M)$ is a smooth manifold

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[1] See Quinn [Qui88]. The essential idea goes back at least to Fadell [Fad65].

[2] See Hughes–Ranicki [HR96] for a textbook account; also Porter [Por95].

[3] Though in practice we will assume that ι is a cofibration, this assumption can be lifted in a straightforward manner as explained in Remark 3.16.

with boundary, then $L \simeq \partial M$, reconstructing the end space of the interior. If $X = (M \subset \overline{N})$ as above is such that $\overline{N}$ is a smooth manifold and M is a closed smooth submanifold of positive codimension, then $L = \mathbb{S}\mathrm{N}_M$ can be taken to be the sphere bundle of the normal bundle of M, and ι to be induced by the choice of a tubular neighbourhood. In essence, a CSS X of depth n is a singular manifold with an atlas consisting of opens of type $C(Z) \times \mathbb{R}^k$ where $C(Z)$ is the open cone of a compact CSS Z of depth $n-1$, with smooth transition maps in an appropriate sense. A CSS of depth 0 is a (paracompact and Hausdorff) smooth manifold.

The treatment of such stratified spaces and their stratified homotopy types in even greater generality started with unpublished work of MacPherson inspired by Goresky–MacPherson intersection homology and was developed by Treumann [Tre09], Woolf [Woo09], Miller [Mil09]; [Mil13], and Lurie [Lur17]. Subsequently, Ayala, Francis, Rozenblyum and Tanaka [AFT17b]; [AFT17a]; [AFR18a]; [AFR18b] have developed Lurie's treatment of factorisation homology, a topological version of Beilinson–Drinfeld's chiral homology [BD04], in the conically smooth setting and with arbitrary tangential structure. In practice, it is vital to have a precise understanding of the stratified topology of a CSS X, and one would hope to apply non-stratified methods to it. Ideally, upon passing from X to the span $\mathfrak{S}$, the ordinary topology of $\mathfrak{S}$ should give information about the stratified topology of X. The main purpose of the present paper is to provide an explicit inverse way to pass from $\mathfrak{S}$ to X at the level of stratified homotopy types.

1.1. **Main results and applications.** Just as a locally constant sheaf on a space is tantamount to a functor out of the complex of its singular chains, a constructible (stratum-wise locally constant) sheaf on a conically stratified space is tantamount, by a result of Lurie [Lur17, §A], to a functor out of its exit path ∞-category (EPC). Accordingly, the EPC should be thought of as the stratified homotopy type.[4] In the EPC, only those paths which stay within a single stratum or which are exit paths are considered. The latter are by construction non-invertible. Ayala, Francis and Rozenblyum [AFR18a] showed that the EPC construction defines a fully faithful functor

$$\mathbf{E}\mathrm{xit}\colon \mathbf{S}\mathrm{trat}^{\mathrm{cs}} \hookrightarrow \mathbf{C}\mathrm{at}_\infty$$

from the ∞-category of conically smooth stratified spaces and conically smooth maps into the ∞-category of all ∞-categories.[5]

This raises the following question: How can one understand the EPC of X in terms of the corresponding span $\mathfrak{S}$? For $\mathfrak{P}$ a poset, Douteau [Dou21]

[4] Another EPC-like construction in the literature is Tamaki's topological face categories from [Tam17, §5] which, while built using an underlying (cylindrically-normal) stratified space, employs a similar idea.

[5] See Lurie [Lur09, §3] for the latter. We will recall it in Section 6.2.

gave a Quillen equivalence between a certain model category of $\mathfrak{P}$-stratified spaces and a model category of diagrams of simplicial sets indexed over non-degenerate sequences in $\mathfrak{P}$. This bypasses EPCs.[6] In a similar vein, Barwick, Glasman and Haine [BGH20, §2] gave, based on a well-known result of Joyal and Tierney [JT07], an equivalence

$$\mathrm{N}_{\mathfrak{P}}\colon \mathbf{S}\mathrm{tr}_{\mathfrak{P}} \xrightarrow{\sim} \mathbf{D}\mathrm{\acute{e}c}_{\mathfrak{P}},$$

a nerve-type construction, from *abstract $\mathfrak{P}$-stratified homotopy types* to *spatial décollages* over $\mathfrak{P}$. Here, $\mathbf{S}\mathrm{tr}_{\mathfrak{P}} \subset (\mathbf{C}\mathrm{at}_\infty)_{/\mathrm{N}(\mathfrak{P})}$ is the full sub-∞-category of the over-∞-category of $\mathbf{C}\mathrm{at}_\infty$ over the nerve of the poset $\mathfrak{P}$, generated by *conservative* structure morphisms

$$s\colon \mathbf{C}\to\mathrm{N}(\mathfrak{P})$$

which, by definition, have the property that the *strata* $s^{-1}\{p\}$, $p \in \mathfrak{P}$, are ∞-groupoids. As for the target, when $\mathfrak{P} = \{0 < 1\}$, a spatial décollage is a span $(M \leftarrow L \rightarrow N)$ of spaces with no conditions on the maps. In higher depth, spatial décollages are special versions of spatial diagrams in the sense of Douteau. Moreover, Haine [Hai23] proved that the EPC construction of Lurie–MacPherson induces an equivalence

$$\mathbf{E}\mathrm{xit}\colon \mathbf{T}\mathrm{op}^{\mathrm{ex}}_{/\mathfrak{P}}[W^{-1}] \xrightarrow{\sim} \mathbf{S}\mathrm{tr}_{\mathfrak{P}}$$

from a localisation of the category of $\mathfrak{P}$-stratified topological spaces whose EPCs are ∞-categories.[7] For our purposes, the problem is that the functor $\mathrm{N}_{\mathfrak{P}}$ is an equivalence by virtue of being fully faithful and essentially surjective, so specifying a homotopy-inverse, which would assign a stratified homotopy type to every span, a priori requires the axiom of choice.

In contrast, we construct an explicit exit path ∞-category $\mathbf{EX}(\mathfrak{S})$ for every linked space $\mathfrak{S}$ and prove that it defines a fully faithful functor from an ∞-category $\mathbf{LS}$ of linked spaces to $\mathbf{C}\mathrm{at}_\infty$. Our methods are necessarily completely independent of those of the works mentioned above.

Theorem (6.14)**.** *The EPC construction induces a fully faithful functor*

$$\mathbf{EX}\colon \mathbf{LS} \hookrightarrow \mathbf{C}\mathrm{at}_\infty$$

of ∞-categories.

As such, $\mathbf{EX}$ is in particular a restricted inverse to the equivalence of Barwick–Glasman–Haine in depth 1. Here $\mathbf{LS}$ is the homotopy coherent nerve of the topological category LS where

$$\mathrm{Hom}_{\mathrm{LS}}(\mathfrak{S}, \mathfrak{S}') = \mathcal{F}_0 \amalg \mathcal{F}_1 \amalg \mathcal{F}_{01},$$

[6] See also Douteau–Waas [DW21, Recollection 2.53 ff.].
[7] See Waas [Waa25] for improvements and further development.

for $\mathfrak{S}' = (M' \leftarrow L' \to N')$, is simply the disjoint union of the ordinary iterated pullback spaces

$$\begin{aligned}
\mathcal{F}_0 &= [M, M'] \times_{L,M'} [L, M'] \times_{[L,M']} [M, M'],\\
\mathcal{F}_1 &= [M, N'] \times_{[L,N']} [L, N'] \times_{[L,N']} [N, N'],\\
\mathcal{F}_{01} &= [M, M'] \times_{[L,M']} [L, L'] \times_{[L,N']} [N, N']
\end{aligned}$$

where $[-,-] = \mathrm{Hom}_{\mathrm{Top}}(-,-)$ takes the space of continuous maps with the compact-open topology. These model, respectively, the spaces of maps that map only to the lower stratum M', only to the higher stratum N', and to both strata. A systematic construction of **LS** which includes more general linked spaces is given in Section 6.1. For simplicity, we will restrict to linked spaces of type $(M \leftarrow L \to N)$ and to CSSs stratified over $\{0 < 1\}$.

The basic idea of the construction is that a path γ in N that starts in $\iota(L)$ may be adjoined formally as a non-invertible 1-morphism to the Kan complex $\mathrm{Sing}(M) \amalg \mathrm{Sing}(N)$ as one that starts at $\pi(\gamma(0))$ and ends at $\gamma(1)$. Higher-dimensional simplices that connect M and N may be adjoined in a similar fashion, but one must keep track of the *exit index* of an exit path. We do this by means of $(1, n-1)$-shuffles. Such a shuffle is determined uniquely by a number in $\{1, \dots, n\}$, the exit index, which corresponds to the time at which an n-path exits into N. The fact that we only consider shuffles of this type corresponds to our restriction to depth 1. In depth k, one must use $(k, n-1)$-shuffles, whose 'exit index' is a subposet $[k-1] \hookrightarrow \{1 < \cdots < n\}$. This is an incarnation of the idea of generalised links as treated by Douteau and Henriques (see [Dou21]; [Hen]) so that one may interpret **EX** as a way of (combinatorially) specifying a point-set EPC directly from such data. We leave a treatment of arbitrary depth to future work.

The EPC of the span $\mathfrak{S}_X$ naturally associated with a depth-1 CSS X is equivalent to $\mathbf{E}\mathrm{xit}(X)$, the EPC constructed by Ayala–Francis–Rozenblyum [AFR18a]. We show this by means of an explicit comparison map $\mathbf{EX}(\mathfrak{S}_X) \to \mathbf{E}\mathrm{xit}(X)$ of quasi-categories to the Lurie–MacPherson model from [Lur17]. It is equivalent to that of AFR by [AFR18a, Lemma 3.3.9].

Theorem (5.1)**.** *If X is a CSS of depth 1, then $\mathbf{EX}(\mathfrak{S}_X) \simeq \mathbf{E}\mathrm{xit}(X)$.*

The proof uses the unzip construction of [AFT17b]; more generally, it holds for the Lurie–MacPherson EPCs of all depth-1 spaces 'of unzip type' as in (5.2).

Since **EX** applies to linked ∞-categories, it can be iterated to model the EPCs of higher-depth spaces without requiring the natively higher-depth treatment that was sketched above.

A main result of [AFR18a] is that **E**xit is a fully faithful functor from conically smooth spaces into ∞-categories, where $\mathrm{Hom}(X, X')$ for two such spaces is the space of conically smooth maps $X \to X'$ in the sense of Ayala–Francis–Tanaka [AFT17b].[8] These results together imply an improvement on the depth-1 conically smooth approximation conjecture [AFT17b, Conjecture 1.5.1] which states that there is an equivalence $\mathrm{Hom}_{\mathbf{S}\mathrm{trat}^{\mathrm{cs}}}(X, X') \simeq \mathrm{Hom}_{\mathbf{S}\mathrm{trat}}(X, X')$, where **S**trat is the ∞-category of C^0-stratified spaces. The following reduces conically smooth maps, up to weak equivalence, further to non-stratified topological maps.

Corollary 1.1. *Let X and X' be conically smooth stratified spaces over $[1]$, and let $\mathfrak{S}_X = (M \leftarrow L \to N)$, $\mathfrak{S}_{X'} = (M' \leftarrow L' \to N')$ be the associated linked spaces. Then the space of conically smooth maps $X \to X'$ satisfies a weak equivalence*

$$\mathrm{Hom}_{\mathbf{S}\mathrm{trat}^{\mathrm{cs}}}(X, X') \simeq \mathcal{F}_0 \amalg \mathcal{F}_1 \amalg \mathcal{F}_{01} \simeq \mathrm{Hom}_{\mathbf{S}\mathrm{trat}}(X, X').$$

This reduces the stratified topology of such spaces to the ordinary topology of the corresponding spans. We expect that an analogous result holds for conically smooth spaces of arbitrary depth. The first equivalence is conditional on the fully-faithfulness of AFR's **E**xit, and the second follows from Haine's main theorem. The original form of the approximation conjecture, the outer equivalence, follows from results of Waas [Waa25] given AFR's theorem.

We will now list the basic properties of **EX**, suppressing the cofibration. Due to the explicit definition of **EX**, the constructions involved in the proofs of the following statemens are likewise explicit. Given two maps $A, B \to C$ of simplicial sets, we write $(A \downarrow B) = (A \downarrow B)^C = A \times_{C^{\Delta\{0\}}} C^{\Delta[1]} \times_{C^{\Delta\{1\}}} B$ for the space of all morphisms in C that start in the image of A and end in the image of B.[9] It is the homotopy link between A and B.

Proposition (4.1, 4.9, 4.11, 4.4). *Let $\mathfrak{S} = (M \twoheadleftarrow L \hookrightarrow N)$ be a linked ∞-category.*

(1) *The objects of $\mathbf{EX}(\mathfrak{S})$ are those of M and N.*
(2) *The ∞-categories M and N are full sub-∞-categories of $\mathbf{EX}(\mathfrak{S})$.*
(3) *There are no morphisms in $\mathbf{EX}(\mathfrak{S})$ from N to M.*
(4) *For $p \in M$ and $q \in N$, the morphism space $\mathrm{Hom}_{\mathbf{EX}(\mathfrak{S})}(p, q) \simeq \mathbf{P}_{L_p, q}$ is equivalent to the space of morphisms in N that start in the embedded fibre L_p and end in q.*
(5) *There is an isomorphism $(M \downarrow N)^{\mathbf{EX}} \cong (L \downarrow N)^N$.*
(6) *There is a constant exit loop inclusion $\square \colon L \hookrightarrow (M \downarrow N)$.*
(7) *If $\mathfrak{S}$ is a linked ∞-groupoid, then $\square$ is an equivalence: $(M \downarrow N) \simeq L$.*

[8] An explicit description of $\mathrm{Hom}(X, X')$ as a Kan complex is given in [AFR18a, Corollary 2.4.3].

[9] This is in contrast to the ∞-categorical homotopy fibre product which will be empty in cases of interest.

These properties are analogous to facts already known in different versions of stratified space theory. In the context of homotopically stratified spaces in the sense of Quinn [Qui88], Miller showed in [Mil13, Theorem 6.3] that stratified homotopy equivalences between such spaces are exactly those maps which induce weak equivalences on strata and homotopy links. An analogous result for abstract poset-stratified homotopy types was obtained by Barwick–Glasman–Haine [BGH20, Theorem 2.7.4], as discussed above. In the conically smooth setting, AFR idenitifed path spaces between strata in terms of links in [AFR18a, Lemma 3.3.5]. That equivalences of EPCs of CSSs are checked on strata and (generalised) links is also implied by Haine's main theorem in [Hai23].

1.2. **Counterexamples.** Using Theorems 5.1 and 6.14, we obtain three classes of counterexamples to versions of a conjecture of Ayala–Francis–Rozenblyum [AFR18a, Conjecture 0.0.8] which characterises the essential image of **E**xit as those finite ∞-categories in which every endomorphism is an isomorphism.[10] Recall that an ∞-category with a conservative functor to (the nerve of) a poset P is called *P-layered*. The finiteness of the ∞-categories constructed below follows immediately from a result of Volpe [Vol24, Proposition 2.11]. The *(homotopy) link* of a [1]-layered ∞-category $\mathbf{C}$ is the space $(\mathbf{C}_0 \downarrow \mathbf{C}_1)$ as above, and its *local links* are the homotopy fibres of the evaluation map $(\mathbf{C}_0 \downarrow \mathbf{C}_1) \to \mathbf{C}_0$. An example similar to the first one was already given by Volpe [Vol24, Remark 2.14].

Corollary (6.15)**.** *Let B be a smooth manifold without boundary whose fundamental group is not purely torsion, let $\widetilde{B} \to B$ be its universal cover, and let $\widetilde{B} \to Y$ be a cofibration. Then the ∞-category $\mathbf{EX}(B \leftarrow \widetilde{B} \to Y)$ is not equivalent to the EPC of a CSS. If $B, \widetilde{B}$ and Y are finite, then so is $\mathbf{EX}(B \leftarrow \widetilde{B} \to Y)$.*

Corollary (6.17)**.** *Let Λ be compact smooth manifold with a non-vanishing Stiefel–Whitney class, and let $\Lambda \hookrightarrow \mathbb{R}^K$ be a closed embedding. Then the finite [1]-layered ∞-category $\mathbf{EX}(* \leftarrow \Lambda \hookrightarrow \mathbb{R}^K)$ has contractible strata, finite local links, and is not equivalent to the EPC of a CSS.*

Corollary (6.18)**.** *Let Y be a closed smooth manifold which is not a homology sphere and let $Y \hookrightarrow \mathbb{R}^K$ be a closed embedding. Then the finite [1]-layered category $\mathbf{EX}(Y \xleftarrow{\mathrm{id}} Y \hookrightarrow \mathbb{R}^K)$ has contractible local links and is not equivalent to the EPC of a compact CSS.*

Remark 1.2. There exist smooth manifolds whose homotopy types are not finite so [AFR18a, Conjecture 0.0.8] is wrong as stated already in depth 0. It

[10] A (homotopically) finite space is one that has the homotopy type of a finite CW complex. See Volpe [Vol24] for a salient treatment in the ∞-categorical setting and concerning EPCs.

is understood to express the problem of characterising the essential image of the EPC functor on CSSs by finding appropriate finiteness conditions. The examples listed above disprove, respectively, the following weaker versions.

(1) Every finite [1]-layered ∞-category is equivalent to the EPC of a CSSs.
(2) Every finite [1]-layered ∞-category with finite local links (and contractible strata) is equivalent to the EPC of a CSS.
(3) Every finite [1]-layered ∞-category with contractible local links (and contractible higher stratum) is equivalent to the EPC of a compact CSS.

Such finiteness conditions are therefore insufficient to characterise the EPCs of CSSs.

1.3. **Classification of conically smooth bundles in depth 1.** Finally, in the companion paper [Tet25] it shown that for every pair $n, m \geq 0$ of natural numbers there exists a fully faithful functor

$$\mathbf{EX}(B\mathrm{O}(n,m)) \hookrightarrow \mathbf{V}^{\hookrightarrow}$$

of quasi-categories where the domain is the EPC of the linked space

$$B\mathrm{O}(n,m) = (B\mathrm{O}(n) \twoheadleftarrow B\mathrm{O}(n) \times B\mathrm{O}(m) \overset{\oplus}{\hookrightarrow} B\mathrm{O}(n+m))$$

which we call the (infinite) (n,m)-*Grassmannian*, and the target is a quasi-category model of the complete Segal space $\mathbf{V}^{\mathrm{inj}}$, the stratified infinite Grassmannian of Ayala–Francis–Rozenblyum [AFR18b]. Conically smooth vector bundles on a CSS X are classified by ∞-functors $\mathbf{E}\mathrm{xit}(X) \to \mathbf{V}^{\mathrm{inj}}$. Consequently, if X is stratified over $\{0 < 1\}$ and $\mathfrak{S}_X = (M \leftarrow L \to N)$ is the associated linked space, then we have the following classification of vector bundles on X which does not depend on the fully-faithfulness of AFR's $\mathbf{E}$xit:

Corollary (of Theorems 5.1 and 6.14, and [Tet25])**.** *Let X be a depth-1 CSS and $\mathfrak{S} = (M \leftarrow L \to N)$ the associated linked space. Then the moduli space*

$$\mathbf{E}_{n,m}(X) \subset \mathrm{Fun}^{\simeq}(\mathbf{E}\mathrm{xit}(X), \mathbf{V}^{\mathrm{inj}}) = \mathrm{Hom}_{\mathbf{Cat}_\infty}(\mathbf{E}\mathrm{xit}(X), \mathbf{V}^{\mathrm{inj}})$$

of conically smooth vector bundles E on X satisfying $\mathrm{rk}(E|_M) = n$ and $\mathrm{rk}(E|_N) = n + m$ is weakly equivalent to the space of topological span maps $\mathfrak{S} \to B\mathrm{O}(n,m)$:

$$\mathbf{E}_{n,m}(X) \simeq [\mathfrak{S}, B\mathrm{O}(n,m)].$$

Similarly, the full moduli space admits a weak equivalence

$$\mathbf{E}(X) = \mathrm{Hom}_{\mathbf{Cat}_\infty}(\mathbf{E}\mathrm{xit}(X), \mathbf{V}^{\mathrm{inj}}) \simeq \coprod_{n,m\geq 0} [\mathfrak{S}, B\mathrm{O}(n,m)].$$

In particular, the tangent classifier $\mathrm{T}_X \colon \mathbf{E}\mathrm{xit}(X) \to \mathbf{V}^{\mathrm{inj}}$ of a depth-1 CSS X satisfying $\dim(M) = n$ and $\dim(N) = n + m$ is equivalent in this sense to a particular map $\mathrm{T}_{\mathfrak{S}}\colon \mathfrak{S} \to B\mathrm{O}(n,m)$ that is constructed explicitly in [Tet25] following an idea of AFR.

As for a classification of conically smooth principal bundles, let $\{G(n)\}_{n\geq 0}$ be a collection of topological groups together together with an operation $\boxplus\colon G(n)\times G(m) \to G(n+m)$ which is a cofibration, and which induces an $\mathbb{E}_\infty$-structure on $BG_{\amalg} = \coprod_{n\geq 0} BG(n)$. There is a strictification to a topological monoid $(BG_{\amalg}^{\infty}, \boxplus)$ explained in [Tet25, §4.1], the idea behind which is similar to Schwede's treatment of symmetric monoid-valued orthogonal spaces in [Sch18, §2]. Analogously to $\mathbf{V}^{\hookrightarrow}$, we can construct the classifying ∞-category $\mathbf{G}^{\hookrightarrow} = */\mathrm{N}^{\mathrm{hc}}(B^{\boxplus}G)$ of conically smooth G-bundles (of arbitrary depth) as the under-∞-category of the single-object homotopy-coherent nerve of the delooping $B^{\boxplus}G$ of $(BG_{\amalg}^{\infty}, \boxplus)$. If there are maps $G(n) \to \mathrm{O}(n)$ such that the induced operation $BG(n) \times BG(m) \to BG(n+m)$ covers $\oplus\colon B\mathrm{O}(n) \times B\mathrm{O}(m) \to B\mathrm{O}(n+m)$ (i.e., if the system $\{G(n) \to \mathrm{O}(n)\}$ is *multiplicative*), then there is an induced map $\mathbf{G}^{\hookrightarrow} \to \mathbf{V}^{\hookrightarrow}$ ∞-categories. Classical structure groups provide examples.

We now obtain as a corollary that principal (multiplicative) G-bundles on a depth-1 CSS X are classified exactly by topological span maps $\mathfrak{S}_X \to BG(n,m) = (BG(n) \leftarrow BG(n) \times BG(m) \to BG(n+m))$ and that the classification of G-structures on a conically smooth vector bundle E reduces to the study of span-map lifts of $E\colon \mathfrak{S}_X \to B\mathrm{O}(n,m)$ along $BG(n,m) \to B\mathrm{O}(n,m)$ for appropriate $n, m \geq 0$. Precise statements are given in [Tet25, Corollaries 1.2 and 1.3]. The corresponding obstruction theory is left to future work.

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Conventions. The set $\mathbb{N}$ of natural numbers includes zero. We denote the real line by $\mathbb{R}$. The standard simplicial n-simplex is $\Delta[n] \coloneqq \mathrm{Hom}_{\mathbf{\Delta}}(-,[n])$, where $\mathbf{\Delta}$ is the simplex category, and $[n] = \{0 < \cdots < n\}$. The standard topological n-simplex is denoted by Δ^n. For a sub-poset $S \subseteq [n]$, we sometimes abbreviate $\Delta[|S|] \hookrightarrow \Delta[n]$, specified by the inclusion, to ΔS or to S. By an ∞-category we mean a quasi-category in the sense of Joyal [Joy02], that is, a simplicial set satisfing the weak Kan property. By an ∞-groupoid we mean a Kan complex. Spaces are compactly generated and weakly Hausdorff. A cofibration of simplicial sets is a monomorphism, and a right fibration $C \to D$ satisfies the right lifting property for horns $\Lambda^n_i \to D$ where $0 < i \leq n$.

2. Shuffles and exit indices

Let $M \xleftarrow{\pi} L \xrightarrow{\iota} N$ be a span of simplicial sets. In this section, we will provide the combinatorial background that will enable adjoining non-invertible paths from M to N using L.

Definition 2.1. The simplicial set $\mathbf{P} \coloneqq \mathbf{P}_\iota \coloneqq L \times_{N^{\{0\}}} N^{\Delta[1]}$ is called the *mapping cocylinder* of ι.

Remark 2.2. Recall how the mapping cocylinder appears in classical topology: in the analogous construction with spaces L, N and ι a continuous map, the target evaluation $P_\iota \to N$, the path space fibration, is a fibration replacement for ι in view of a homotopy equivalence $L \simeq P_\iota$.

There are two induced maps $\pi, \tau \colon \mathbf{P} \to M, N$ defined as the compositions $\mathbf{P} \to L \to M$ and $\mathbf{P} \to N^{\Delta[1]} \to N^{\{1\}}$, respectively.

Remark 2.3. A vertex of $\mathbf{P}$ is a path of N that starts at a point in $\iota(L)$. One may view this as a path which starts in M, by projecting to M via π, and which, analogously, ends in N via τ. For higher morphisms, however, a direct generalisation of this idea requires unnatural choices. For instance, a 1-morphism in $\mathbf{P}$ may be depicted as

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] \arrow[ur, dashed] & \bullet \arrow[u] \end{tikzcd} \tag{2.4}$$

where the bottom edge is in $\iota(L)$, and the top edge is in N. The two possibly non-degenerate 2-simplices of N we may extract the two triangles corresponding to the two $(1,1)$-shuffles $\Delta[2] \hookrightarrow \Delta[1] \times \Delta[1]$ à la Eilenberg–Mac Lane–Zilber [EZ53]; [EM53] (see also Lurie [Lur25, 00RF], and below). If we were to add (2.4) as a 2-morphism to $M \amalg N$, say with source edge the bottom one, then we would have to choose the hypotenuse of the left triangle as the target edge, and the vertical edge as the intermediate $\overline{12}$-edge. But we may equally well make the analogous choice with the right triangle, declaring the left vertical edge the source. However, both types of triangle are required for composition: if we wish to compose a path in M with a (non-invertible) 1-morphism in $\mathbf{P}$, then we need (assuming there is a lift to L) a triangle of the first type. Similarly, composing a non-invertible 1-morphism with a path in N, requires a triangle of the second type.

Definition 2.5. Any pair $1 \le j \le k$ of natural numbers determines a $(1, k-1)$-shuffle $\mathcal{S}_j^k = \mathcal{S}_j \colon \Delta[k] \hookrightarrow \Delta[1] \times \Delta[k-1]$ given by the non-degenerate element of $(\Delta[1] \times \Delta[k-1])_k$ induced in $\mathbf{\Delta}$ by the poset map $[k] \to [1] \times [k-1]$

defined by

$$i \mapsto \begin{cases} (0,i), & i < j \\ (1,i-1), & i \geq j. \end{cases}$$

We call $\mathcal{S}_j$ an *exit shuffle*, and j its *exit index*. It has multiple left inverses, but we will use a particular one, $\mathcal{C}_j^k = \mathcal{C}_j$, induced in $\mathbf{\Delta}$ by the poset map $[1] \times [k-1] \twoheadrightarrow [k]$ given by

$$(0,i) \mapsto \begin{cases} i, & i < j \\ j-1, & i \geq j \end{cases}, \quad (1,i) \mapsto \begin{cases} j, & i < j \\ i+1, & i \geq j \end{cases}.$$

This choice is justified below by Lemmas 2.9 and 2.10 which otherwise fail.

Definition 2.6. Let $\iota\colon L \to N$ be a map of simplicial sets. For $k \geq 1$, we define

$$\mathbf{P}_{k-1}^{\Delta} \subset N_k \times \{1, \ldots, k\}$$

to be the subset consisting of pairs (γ, j) such that in the diagram

$$\begin{tikzcd} \Delta[k] \arrow[rr, "\gamma"] \arrow[dr, bend right, "\mathcal{S}_j"'] & & N \\ & \Delta[1] \times \Delta[k-1] \arrow[ul, two heads, bend right, "\mathcal{C}_j"'] \arrow[ur, dashed, "\Gamma=\gamma\circ\mathcal{C}_j"'] & \end{tikzcd}$$

the arrow Γ is in the image of the natural map $\mathbf{P} \to N^{\Delta[1]}$. A pair $(\gamma, j) \in \mathbf{P}_{k-1}^{\Delta}$ is an *exit k-path of index j*.

Definition 2.5 corresponds to the following phenomenon: a stratified k-chain $\Delta^k \to X$ of X is a map of poset-stratified spaces, where $\Delta^k = \overline{C}^k(\mathrm{pt})$ is the k-fold closed cone on the point. The closed cone $\overline{C}(Y)$ (see also Construction 5.3) of a stratified space $Y \to \mathfrak{P} = \mathfrak{P}_Y$, where $\mathfrak{P}$ is the stratifying poset (equipped with the Alexandrov topology where downward-closed subsets are closed) has $\mathrm{pt} \coprod_{\{0\}\times Y} [0,1] \times Y$ as its underlying space, and $\mathfrak{P}_{\overline{C}(Y)} = \mathfrak{P}_Y^{\triangleleft}$, i.e., $\mathfrak{P}_Y$ with a minimal element adjoined, as its stratifying poset, together with the obvious stratification $\overline{C}(Y) \to \mathfrak{P}_Y^{\triangleleft}$. Now, the stratified map $\Delta^k \to X$ is a commutative topological square

$$\begin{tikzcd} \Delta^k \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \mathfrak{P}_{\Delta^k} \arrow[r, "s_f"'] & \mathfrak{P}_X \end{tikzcd}.$$

Clearly we have $\mathfrak{P}_{\Delta^k} \simeq [k]$ as posets. If $P_X \simeq \{a \prec b\}$, then the poset map s_f is determined by a unique minimal 'exit index' $j \in [k]$. Namely, let $j = 0$ if s_f is constant, or else let j be the smallest number such that $s_f(j-1 \prec j) = a \prec b$.

Definition 2.7. Let $k \geq 1$, $(\gamma, j) \in \mathbf{P}^{\Delta}_{k-1}$. Then $d_i(\gamma)$ is *low* if

$$\mathcal{S}_j \circ \partial_i \colon \Delta[k-1] \hookrightarrow \Delta[k] \to \Delta[1] \times \Delta[k-1]$$

factors through $\{0\} \times \Delta[k-1]$; *upper* if it factors through $\{1\} \times \Delta[k-1]$; and *vertical* if it is neither low nor upper.

The terms 'low' and 'upper' are borrowed from [EZ50]. In the exit path ∞-category of Definition 3.1 below, vertical faces will remain non-invertible, low faces will become simplices in M, and upper faces in N. Writing '$d_i(\gamma)$ is vertical', etc., is slightly redundant as well as abusive, since these properties depend only on i and j.

Definition 2.8. Let $k \geq 1$. We write

$$\flat^k_{j,i} = \flat_{j,i} \in [k-1], \text{ resp. } \sharp^k_{j,i} = \sharp_{j,i} \in [k]$$

for the smallest number whose image under $\mathcal{S}_j \partial_i \colon [k-1] \to [1] \times [k-1]$, resp. under $\mathcal{S}_j \sigma_i \colon [k+1] \to [1] \times [k-1]$ has first coordinate 1. We leave $\flat^k_{k,k}$ undefined.

See (3.3) and (3.8) below for formulas. The following is a direct check:

Lemma 2.9. *Let $k \geq 2$ and suppose $(j, i) \neq (k, k)$. The maps*

$$\mathcal{S}_j \circ \partial_i \circ \mathcal{C}_{\flat^k_{j,i}} \colon \Delta[1] \times \Delta[k-2] \to \Delta[1] \times \Delta[k-1]$$

and

$$\mathcal{S}_j \circ \sigma_i \circ \mathcal{C}_{\sharp^k_{j,i}} \colon \Delta[1] \times \Delta[k] \to \Delta[1] \times \Delta[k-1]$$

preserve the first coordinate.

Lemma 2.10. *Let $k \geq 2$. If $(\gamma, j) \in \mathbf{P}^{\Delta}_{k-1}$ and $d_i(\gamma)$ is vertical, then $d_i(\gamma, j) \coloneqq (d_i\gamma, \flat^k_{j,i}) \in \mathbf{P}^{\Delta}_{k-2}$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\Delta[k-1] \arrow[r, hook, "\partial_i"] \arrow[ddrr, bend right, "\mathcal{S}_\flat"'] & \Delta[k] \arrow[rr, "\gamma"] \arrow[dr, bend right, "\mathcal{S}_j"'] & & N \\
& & \Delta[1] \times \Delta[k-1] \arrow[ul, two heads, "\mathcal{C}_j"'] \arrow[ur, dashed, "\Gamma"] & \\
& & \Delta[1] \times \Delta[k-2] \arrow[uull, two heads, "\mathcal{C}_\flat"] \arrow[u, dashed, "d'"'] \arrow[uur, dashed, bend right, "\Gamma'"'] &
\end{tikzcd} \tag{2.11}$$

Lemma 2.9 implies in particular that the restriction of $d' = \mathcal{S}_i \partial_i \mathcal{C}_\flat$ to $\{0\} \times \Delta[k-2]$ factors through $\{0\} \times \Delta[k-1]$, which implies that $\Gamma' = \Gamma d'$ lifts to $\mathbf{P}_{k-2}$, as desired. The case $j = i = k$ is precluded by verticality. $\square$

The following variant for degeneracies is analogous.

Lemma 2.12. *Let $k \geq 1$. If $(\gamma, j) \in \mathbf{P}^{\Delta}_{k-1}$, then $s_i(\gamma, j) \coloneqq (s_i\gamma, \sharp^k_{j,i}) \in \mathbf{P}^{\Delta}_k$.*

3. Exit paths of linked quasi-categories

We are now ready to give the main construction of this paper. First, note that if $\iota\colon L \hookrightarrow N$ is a cofibration, then an exit path $(\gamma, j) \in \mathbf{P}^{\Delta}_{k-1}$ determines a canonical $(k-1)$-simplex $\Delta[k-1] \to L$ of L, that is, the restriction of $\Gamma = \gamma \circ \mathcal{C}_j$ along $\{0\} \times \Delta[k-1] \hookrightarrow \Delta[1] \times \Delta[k-1]$ factors then through L *uniquely*. Remark 3.15 will lift this assumption.

Definition 3.1. Let $\mathfrak{S} = (M \xleftarrow{\pi} L \xhookrightarrow{\iota} N)$ be a span of simplicial sets be given, where ι is a cofibration. We define a new simplicial set $\mathbf{EX} = \mathbf{EX}(\mathfrak{S})$ as follows:

- $\mathbf{EX}_0 = M_0 \amalg N_0$.
- $\mathbf{EX}_k = M_k \amalg \mathbf{P}^{\Delta}_{k-1} \amalg N_k$ for $k \geq 1$.
- Face and degeneracy maps restricted to M_k and N_k are those of M and N.
- For $k = 1$ and $\gamma = (\gamma, 1) \in \mathbf{P}^{\Delta}_0 \subset N_1$, we set
  - $d_1(\gamma, 1) = \pi(d_1\gamma) \in M_0$,
  - $d_0(\gamma, 1) = d_0\gamma \in N_0$.
- For $k \geq 2$, $(\gamma, j) \in \mathbf{P}^{\Delta}_{k-1}$, and d_i a face map:
  - if $d_i\gamma$ is vertical, then $d_i(\gamma, j) = (d_i\gamma, \flat_{j,i}) \in \mathbf{P}^{\Delta}_{k-2}$.
  - if $d_i\gamma$ is low, then $d_i(\gamma, j) = \pi(d_i\gamma) \in M_{k-1}$.
  - if $d_i\gamma$ is upper, then $d_i(\gamma, j) = d_i\gamma \in N_{k-1}$.
- If $k \geq 1$, $(\gamma, j) \in \mathbf{P}^{\Delta}_{k-1}$, and s_i is a degeneracy, then $s_i(\gamma, j) = (s_i\gamma, \sharp_{j,i}) \in \mathbf{P}^{\Delta}_k$.

An equivalent description of $\mathbf{EX}$ without using shuffles is given in Remark 3.17.

Lemma 3.2. $\mathbf{EX}(\mathfrak{S})$ *is a simplicial set.*

Proof. We will verify the simplicial identities. Let $(\gamma, e) \in \mathbf{P}^{\Delta}_{k-1}$. We will assume $k \geq 2$ or $k \geq 3$ depending on the applicability of the identity in question, and leave the case $k = 1$ to the reader.
$\underline{d_i d_j = d_{j-1} d_i \text{ for } i < j}$: Using

$$\flat^k_{e,j} = \begin{cases} e, & j \geq e \\ e-1, & j < e \end{cases} \tag{3.3}$$

it is straightforward to see that

$$\flat^{k-1}_{\flat^k_{e,j},i} = \flat^{k-1}_{\flat^k_{e,i},j-1} \tag{3.4}$$

when $i < j$. This finishes the verification if all involved faces of (γ, e) are vertical. Otherwise, Lemma 2.9 and Diagram (2.11) imply the statement; in any of the cases where the case excluded in Lemma 2.9 is involved, the face in question is low. We will give this argument here once and will not repeat it in the verification of the remaining simplicial identities below.

Consider the diagram

$$\begin{tikzcd}[column sep=large] \Delta[k-2] \arrow[r, hook, "\partial_i"] \arrow[d, bend right, "\mathcal{S}_{\flat'}"'] & \Delta[k-1] \arrow[r, hook, "\partial_j"] \arrow[d, bend right, "\mathcal{S}_{\flat}"'] & \Delta[k] \arrow[r, "\gamma"] \arrow[d, bend right, "\mathcal{S}_{e}"'] & N \\ \Delta[1] \times \Delta[k-3] \arrow[r, dashed] \arrow[u, bend right, "\mathcal{C}_{\flat'}"'] & \Delta[1] \times \Delta[k-2] \arrow[r, dashed] \arrow[u, bend right, "\mathcal{C}_{\flat}"'] & \Delta[1] \times \Delta[k-1] \arrow[u, bend right, "\mathcal{C}_{e}"'] \arrow[ur, dashed] & \end{tikzcd} \tag{3.5}$$

Without loss of generality, say $d_i(d_j(\gamma)) = (\partial_j\partial_i)^*\gamma$ is low, so we need to show that so is $d_{j-1}d_i(\gamma)$. That $\mathcal{S}_\flat\partial_i$ factors through $\{0\} \times \Delta[k-2]$ is equivalent to $\mathcal{S}_e\partial_j\mathcal{C}_\flat\mathcal{S}_\flat\partial_i$ factoring thusly by Lemma 2.9. Now, $\mathcal{S}_e\partial_j\mathcal{C}_\flat\mathcal{S}_\flat\partial_i = \mathcal{S}_e\partial_j\partial_i$ by the construction of $\mathcal{C}_\flat$, and similarly $\mathcal{S}_e\partial_j\partial_i = \mathcal{S}_e\partial_j\partial_i\mathcal{C}_{\flat'}\mathcal{S}_{\flat'}$. Together with the same calculation for ∂_i and ∂_j replaced respectively by ∂_{j-1} and ∂_i in Diagram (3.5), we see that

$$\mathcal{S}_e\partial_j\mathcal{C}_\flat\mathcal{S}_\flat\partial_i = \mathcal{S}_e\partial_j\partial_i\mathcal{C}_{\flat'}\mathcal{S}_{\flat'} \quad \text{and} \quad \mathcal{S}_e\partial_i\mathcal{C}_\flat\mathcal{S}_\flat\partial_{j-1} = \mathcal{S}_e\partial_i\partial_{j-1}\mathcal{C}_{\flat'}\mathcal{S}_{\flat'}. \tag{3.6}$$

The indices $\flat^{(\prime)}$ in the two equations are a priori *not* the same since they are calculated for different pairs of indices, but we just showed above in Equation (3.4) that the primed flats on the right hand sides do coincide. Combined with the same simplicial identity for N, this means that the right hand sides in (3.6) agree, which yields the statement.

$d_is_j = s_{j-1}d_i$ for $i < j$: Similarly, we obtain

$$\flat^{k-1}_{\sharp^k_{e,j},i} = \sharp^{k-1}_{\flat^k_{e,i},j-1} \tag{3.7}$$

using

$$\sharp^k_{e,j} = \begin{cases} e, & j \geq e \\ e+1, & j < e. \end{cases} \tag{3.8}$$

Now, Lemma 2.9 and Diagrams (2.11) and (3.5) (mutatis mutandis; e.g., using (3.7) instead of (3.4) for (3.6)) again finish the verification, analogously to the above. We no longer mention this below.

$d_is_j = \mathrm{id}$ for $i = j$ or $i = j+1$: This translates to the identity $\flat^{k-1}_{\sharp^k_{e,j},i} = e$.

$d_is_j = s_jd_{i-1}$ for $i > j+1$: This translates to the identity $\flat^{k-1}_{\sharp^k_{e,j},i} = \sharp^{k-1}_{\flat^k_{e,i-1},j}$.

$s_is_j = s_{j+1}s_i$ for $i \leq j$: This translates to the identity $\sharp^{k-1}_{\sharp^k_{e,j},i} = \sharp^{k-1}_{\sharp^k_{e,i},j+1}$. □

Theorem 3.9. *If M, L, N are ∞-categories, $\pi\colon L \to M$ is a right fibration, and $\iota\colon L \to N$ is a cofibration, then $\mathbf{EX}(\pi, \iota)$ is an ∞-category.*

Proof. We will directly verify the weak Kan property. We call a sub-simplex $(\Delta[\ell] \hookrightarrow \Lambda^k_i \to \mathbf{EX}) \in \mathbf{EX}_\ell$ of a horn *low* if it is in M_ℓ, *vertical* if in $\mathbf{P}^\Delta_{\ell-1}$, and *upper* if in N_ℓ. Sometimes we will not distinguish L from $\iota(L)$ in notation.

Let $h\colon \Lambda^n_i \to \mathbf{EX}$ be an inner horn, i.e., $0 < i < n$, which does not factor through M or N. Then h either has a low face, an upper face, or all of its faces are vertical.

Suppose that h has a low face. Then the low face is necessarily $h|_{0\ldots n-1} \in M_{n-1}$; $h|_n \in N_0$ is the sole upper vertex, and all other faces are vertical. Let us write $h|_{\widehat{j}}$ for $h|_{0\ldots\widehat{j}\ldots n} = h \circ \partial_j\colon \Delta[n-1] \to \mathbf{EX}$ whenever $j \neq i$. Each vertical face $h|_{\widehat{k}} \in \mathbf{P}^\Delta_{n-2} \subset \mathbf{EX}_{n-1}$ for $i \neq k < n$ has its $(n-2)$-face $h|_{\widehat{k}\widehat{n}}$ in common with the low face $h|_{\widehat{n}}$, which therefore gives a lift $h_{\widehat{k}\widehat{n}} \in L_{n-2}$ to L thereof. Here we wrote $h|_{\widehat{k}} = (h_{\widehat{k}}, e)$ and $(h_{\widehat{k}})_{\widehat{n}} = \iota(h_{\widehat{k}\widehat{n}})$. As each $h|_{\widehat{k}}$ itself has a low face, its exit index is necessarily maximal, i.e., $e = n-1$. Now, we obtain the intermediate lifting problem

$$\begin{tikzcd}
\Lambda^{n-1}_i \arrow[r, "\bigcup_{i\neq k\in[n-1]} h|_{\widehat{k}\widehat{n}}"] \arrow[d, hook] & L \arrow[d, "\pi"] \\
\Delta[n-1] \arrow[r, "h|_{\widehat{n}}"'] \arrow[ur, dotted, "H_{\widehat{n}}"] & M
\end{tikzcd} \tag{3.10}$$

with solution $H_{\widehat{n}}$. (It is imperative here that π be a right fibration and not merely an inner one since $i = n-1$ is allowed.) This yields the horn

$$\iota(H_{\widehat{n}}) \cup \bigcup_{i\neq k\in[n-1]} \iota(h|_{\widehat{k}\widehat{n}})\colon \Lambda^n_i \to N$$

which has a filler $H \in N_n$.

We now claim that (H, n) fills h. The restriction of $H \circ \mathcal{C}_n\colon \Delta[1]\times\Delta[n-1] \to N$ to $\{0\} \times \Delta[2]$ is $\iota(H_{\widehat{n}})$, which factors through L by construction. Further, $\mathcal{S}_n \circ \partial_n\colon [n-1] \to [1] \times [n-1]$ sends $n > j \mapsto (0, j)$, so $d_n H$ is low, hence $d_n(H, n) = \pi(H_{\widehat{n}}) = h|_{\widehat{n}}$, as desired. Finally, if $k < n$, then $\mathcal{S}_n \circ \partial_j$ hits both $\{0\} \times [n-1]$ and $\{1\} \times [n-1]$, so $d_k H$ is vertical. Since $\flat_{n,k} = n-1$ for $k < n$, we obtain $d_k(H, n) = (h_{\widehat{k}}, n-1) = h|_{\widehat{k}}$, also as desired.

Suppose now that h has an upper face. Then it is necessarily $h|_{\widehat{0}} \in N_{n-1}$; $h|_0 \in M_0$ is the sole low vertex, and all other faces $h|_{\widehat{k}} = (h_{\widehat{k}}, 1) \in \mathbf{P}^\Delta_{n-2}$ are vertical with exit index necessarily minimal. Now, h is given by a horn $\widetilde{h}\colon \Lambda^n_i \to N$ with $\widetilde{h}|_0 \in \iota(L_0)$. Taking a filler H of $\widetilde{h}$, we claim that $(H, 1)$ fills h. The restriction of $\mathcal{C}_1\colon [1]\times[n-1] \to [n]$ to $\{0\}\times[n-1]$ hits only 0, so $H \circ \mathcal{C}_1$ factors through the mapping cocylinder by construction, independently of the choice of filler H. Further, $\mathcal{S}_1\colon [n] \to [1] \times [n-1]$ sends only 0 to $\{0\} \times [n-1]$ while $\mathcal{S}_1 \circ \partial_0$ factors through $\{1\} \times [n-1]$. This means $d_0 H$ is upper, so $d_0(H, 1) = h|_{\widehat{0}}$, as desired. Finally, $d_k(H, 1) = (d_k H, \flat_{1,k}) = (h_{\widehat{k}}, 1) = h|_{\widehat{k}}$ for every $k \geq 1$, also as desired.

Suppose, finally, that all faces of h are vertical. Then $h|_0 \in M_0$ is low and $h|_n \in N_0$ upper, and moreover there must exist an index $1 \leq e \leq n$ such that $h|_j \in M_0$ for $j < e$ and $h|_j \in N_0$ for $j \geq e$ as otherwise there would exist a pair $0 < j < j' < n$ such that $h|_j \in N_0$ while $h|_{j'} \in M_0$, which is absurd since the edge $h|_{jj'}$ would be of type $N \to M$. Moreover, $e = 1$ resp. $e = n$ are impossible, since then $h|_{\widehat{0}}$ resp. $h|_{\widehat{n}}$ would be low resp. upper. We have obtained $1 < e < n$. Now, we claim that the exit indices of the faces $h|_{\widehat{j}} \in \mathbf{P}^{\Delta}_{n-2}$, $j \neq i$, are determined by e via the formula

$$h|_{\widehat{j}} = \begin{cases} (h_{\widehat{j}}, e), & j \geq e, \\ (h_{\widehat{j}}, e-1), & j < e. \end{cases} \tag{3.11}$$

Indeed, that $\mathcal{C}^{n-1}_{\ell}(\{0\} \times [n-2]) = \{0, \dots, \ell - 1\}$ for any $1 \leq \ell \leq n-1$ implies that if $j \geq e$, then $h_{\widehat{j}} \circ \mathcal{C}^{n-1}_{e}$ factors through $\mathbf{P}$, as does $h|_{\widehat{j}} \circ \mathcal{C}^{n-1}_{e-1}$ if $j < e$. Conversely, suppose $h|_{\widehat{j}}$ has index e': $(h|_{\widehat{j}})|_{0,\dots,e-1}$ must be low, which implies, by the definition of $\mathcal{S}_{e'}$, that $e' \geq e$, and since there are no further low vertices, we have $e' \leq e$ and thus $e = e'$.

Now, as h is underlied by a horn $\widetilde{h} \colon \Lambda^n_i \to N$, we may choose a filler $H \in N_n$. We claim that (H, e) fills h. In order to ensure that $H \circ \mathcal{C}_e \colon \Delta[1] \times \Delta[n-1] \to N$ factors through $\mathbf{P}$, it suffices to observe that the missing face $h|_{\widehat{i}}$ cannot be low for then the choice of filler H does not affect the factorisation property in that $\widetilde{h}$ needs filling only away from $\iota(L)$. Indeed, the only such case would be when $i = n$, but h is inner. Finally, we check that the exit indices of the faces of H are correct: since $1 < e < n$, no face of H is low or upper, and (3.3) implies $d_j(H, e) = h|_{\widehat{j}}$ due to (3.11), as desired. □

Definition 3.12. We call a span $M \xleftarrow{\pi} L \xrightarrow{\iota} N$ of ∞-groupoids or ∞-categories, with π a right fibration, and ι a cofibration, a *linked ∞-groupoid* or *linked space* resp. *linked ∞-category*, of *depth* 1. We call **EX** its *exit path ∞-category.*

Remark 3.13. Every EPC is naturally equipped with a conservative functor to $\mathrm{N}([1])$.

Remark 3.14. For $\mathfrak{S} = (M \leftarrow L \to N)$ a linked ∞-category, the core $\mathbf{EX}(\mathfrak{S})^{\simeq}$ of its exit path ∞-category is canonically isomorphic to $M^{\simeq} \amalg N^{\simeq}$.

Remark 3.15 (arbitrary depth-1 posets). Definition 3.12 admits a straightforward generalisation to linked ∞-categories indexed over a poset P of depth 1. More precisely, let $\mathfrak{S} \to P$ denote a collection consisting of an ∞-category M_p for each $p \in P$, an ∞-category $L_{p \prec q}$ for every non-trivial arrow $p \prec q$, and right fibrations $L_{p \prec q} \to M_p$ and cofibrations $L_{p \prec q} \to M_q$. Then, Definition 3.1 and Theorem 3.9 generalise in the evident way to yield an ∞-category $\mathbf{EX}(\mathfrak{S} \to P)$. In the rest of this paper we will mostly consider spans and leave this level of generality implicit. See Section 6 for a systematic treatment. In the language of [BGH20], $\mathfrak{S} \to P$ is a quasi-categorical décollage over P.

Remark 3.16 (arbitrary ι). In a homotopically-stratified set à la Quinn [Qui88], links are defined as path spaces, in which case the link map ι, the target evaluation, is typically not a cofibration. Generally, one may simply wish to adjoin a path space as the space of non-invertible paths between two otherwise unrelated spaces or categories. We will now discuss a variation on Definition 3.1 that can handle such input.

Let $\mathfrak{S} = (M \xleftarrow{\pi} L \xrightarrow{\iota} N)$ be a span of ∞-categories where π is a right fibration, and suppose $k \geq 1$ and $j \in \{1, \ldots, k\}$. Given $(\gamma, j) \in \mathbf{P}^{\Delta}_{k-1}$, we have that $\mathbf{b}(\gamma, j) := \gamma|_{0,\ldots,j-1}$, its *maximal low face* or *base*, lifts, albeit non-uniquely, to L. Let us now set

$$\widehat{\mathbf{P}^{\Delta}_{k-1}} := \coprod_{j=1}^{k} L_{j-1} \times_{N_{j-1}} \mathbf{P}^{\Delta}_{k-1,j},$$

where $\mathbf{P}^{\Delta}_{k-1,j} \subset N_k$ denotes the set of exit k-simplices of index j, the map $\mathbf{P}^{\Delta}_{k-1,j} \to N_{j-1}$ is given by $\mathbf{b}$, and $L_{j-1} \to N_{j-1}$ by ι.

There is a simplicial set $\mathbf{EX} = \mathbf{EX}(\mathfrak{S})$ whose simplices are given by $\mathbf{EX}_0 = M_0 \amalg N_0$, and $\mathbf{EX}_k = M_k \amalg \widehat{\mathbf{P}^{\Delta}_{k-1}} \amalg N_k$ for $k \geq 1$. We call the face of a simplex that is not wholly within either stratum *low*, *upper*, or *vertical* by referring to its second coordinate. Now, let $(\widehat{\mathbf{b}}, \gamma, j) \in \widehat{\mathbf{P}^{\Delta}_{k-1}}$. If $d_i\gamma$ is low, then we set $d_i(\widehat{\mathbf{b}}, \gamma, j) = \pi(\widehat{\mathbf{b}})$; if it is upper, then $d_i(\widehat{\mathbf{b}}, \gamma, j) = (d_i\gamma)$; and if it is vertical, then $d_i(\widehat{\mathbf{b}}, \gamma, j) = (\widehat{\mathbf{b}}|_{\widehat{i}}, d_i\gamma, \flat_{j,i})$, noting that $\mathbf{b}(d_i\gamma, \flat_{j,i}) = d_i\gamma|_{0,\ldots,\flat-1} = \mathbf{b}|_{\widehat{i}}$. Here, $\mathbf{b}|_{\widehat{i}}$ is $\mathbf{b}$ if $i > \flat - 1$ and $d_i\mathbf{b}$ if $i \leq \flat - 1$, that is, the restriction of $\mathbf{b}$ along $[\flat - 1] \hookrightarrow [k-1] \xhookrightarrow{\partial_i} [k]$; similarly for $\widehat{\mathbf{b}}$. Finally, we set $s_i(\widehat{\mathbf{b}}, \gamma, i) = (\widehat{\mathbf{b}}_{i+}, s_i\gamma, \sharp_{j,i})$, where $\widehat{\mathbf{b}}_{i+}$ is the pullback of $\widehat{\mathbf{b}}$ along $[\sharp_{j,i}] \hookrightarrow [k+1] \xrightarrow{\sigma_i} [k]$.

The proofs of Lemma 3.2 and Theorem 3.9 apply mutatis mutandis to prove that $\mathbf{EX}$ is an ∞-category. The further generalisation in the situation of Remark 3.15 is immediate. Finally, if ι is a cofibration, the resulting ∞-category is canonically isomorphic to the $\mathbf{EX}$ of Definition 3.1.

Remark 3.17. It is possible to define exit paths without referring to shuffles. Namely, saying $(\gamma, j) \in \mathbf{P}^{\Delta}_{k-1}$ is equivalent to asking that $\gamma \in N_k$ and that $\mathbf{b}(\gamma, j) := \gamma|_{0,\ldots,j-1}$ lifts to L. The face $d_i\gamma$ being low is equivalent to $d_i\gamma \subset \mathbf{b}(\gamma, j)$, and it being upper to $d_i\gamma \subset \gamma|_{j,\ldots,k}$. Now we can *define* $\flat$ and $\sharp$ by the formulas (3.3) and (3.8). Using Definition 3.1 verbatim yields an equivalent definition of $\mathbf{EX}$. This version, less motivated and more combinatorial, is easier to formulate, but would make some proofs below more challenging. Shuffles will prove especially useful in later sections.

3.1. **Homotopy categories.** Recall that for S a simplicial set, and $h \in S_2$, $f \in S_1$, $g \in S_1$ such that $f, g\colon x \to y$ for vertices $x, y \in S_0$, the 2-simplex h is called a homotopy from f to g if $d_0h = \mathrm{id}_y$, $d_1h = g$ and $d_2h = f$. If $S = \mathbf{C}$ is an ∞-category, the existence of a homotopy from f to g is equivalent to the existence of a 2-simplex h such that $d_0h = f$, $d_1h = g$ and $d_2h = \mathrm{id}_x$:

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}"] \arrow[rr, "g"] & & y \end{tikzcd} \tag{3.18}$$

This latter version affords us, by Theorem 3.9, the following useful fact:

Lemma 3.19. *Let $\mathbf{C} = \mathbf{EX}(M \xleftarrow{\pi} L \xrightarrow{\iota} N)$ be the exit path ∞-category of a linked ∞-category and let $x \in M$, $y \in N$. Then an index-2 exit 2-path $h = (\widetilde{h}, 2) \in \mathbf{EX}_2$ satisfies $d_2^N\widetilde{h} \in \iota(L_x)$ witnesses a homotopy between d_0h and d_1h. Conversely, every homotopy can be witnessed by such an exit 2-path.*

Lemma 4.1 below shows that the space $\mathrm{Hom}_{\mathbf{EX}}(p, q)$ of paths from $p \in M$ to $q \in N$ is equivalent to the space $\mathbf{P}_{L_p,q}$ of paths in N that start in the embedded fibre $\iota(L_p)$ and end in $q \in N$. Thus Lemma 3.19 and Lemma 4.1 combine to imply a fibrewise gauge invariance (where we suppress ι and write $\circ$ to denote any composition):

Lemma 3.20. *Let $\mathbf{C} = \mathbf{EX}(M \xleftarrow{\pi} L \xrightarrow{\iota} N)$ be the exit path ∞-category of a linked ∞-category and let $x \in M$, $y \in N$. Then if $f = (\widetilde{f}, 1)\colon x \to y$ is an exit path with $\widetilde{f}(0) = l \in L_x$ and $\gamma\colon l' \to l$ is a path in L_x, then f is homotopic to the exit path $g = (f \circ \gamma, 1)\colon x \to y$.*

Proof. Take $h = (\widetilde{h}, 2)$ as in (3.18) where $\widetilde{h}$ is a filler for $d_2^N\widetilde{h} = \gamma$ and $d_0^N\widetilde{h} = \widetilde{f}$. □

Corollary 3.21. *Let $\mathfrak{S} = (M \xleftarrow{\pi} L \to N)$ be a linked space, let $p \in M$, $q \in N$ be points, and let $\Pi_1 = \Pi_1(\mathfrak{S}; \{p, q\})$ be the homotopy category of the full sub-∞-category of $\mathbf{EX}(\mathfrak{S})$ generated by p and q. If M and N are path-connected, then the isomorphism class of the category Π_1 is independent of the choice of p and q.*

Proof. We will write square brackets to indicate homotopy classes and continue suppressing ι. Let $\Gamma\colon p \to p'$ be a path in M and let $\Delta\colon q \to q'$ be a path in N. These induce the classical bijections $[\mathrm{ad}_\Gamma]\colon \mathrm{End}(p) \to \mathrm{End}(p')$ and $[\mathrm{ad}_\Delta]\colon \mathrm{End}(q) \to \mathrm{End}(q')$ between the respective endomorphism sets in the categories $\Pi_1(\mathfrak{S}; \{p, q\})$ and $\Pi_1(\mathfrak{S}; \{p', q'\})$. Without loss of generality, we may assume $q = q'$ and suppress Δ. In order to construct a map $\Phi\colon \mathrm{Hom}(p, q) \to \mathrm{Hom}(p', q)$, consider, as in the classical proof that the fibres of π are homotopy equivalent, the map $L_p \times I \to M$, $(l, t) \mapsto \Gamma(t)$, and fix a lift $H\colon L_p \times I \to L$ along the fibre inclusion $L_p \times \{0\} \to L$. (Recall that $H(-, 1)\colon L_p \to L_{p'}$ is a homotopy equivalence.) Now, let $\alpha \in \mathrm{Hom}(p, q)$ and let a representative $(\widetilde{\alpha}, 1) \in \mathbf{EX}_1$ be given. Set

$$\Phi(\alpha) = [(H(\widetilde{\alpha}(0), -)^{-1} * \widetilde{\alpha}, 1)].$$

By Lemma 3.20 on $\mathrm{Hom}(p, q)$, all representatives of α, up to homotopy in N, are of type $(\epsilon * \widetilde{\alpha}, 1)$ where $\epsilon\colon l \to \widetilde{\alpha}(0)$ is a path in L_p. In order to show well-definedness, it suffices, by Lemma 3.20 on $\mathrm{Hom}(p', q)$, to find a homotopy $H(l, -)^{-1} * \epsilon \sim \epsilon' * H(\widetilde{\alpha}(0), -)^{-1}$ in N for some path $\epsilon'\colon H(l, 1) \to H(\widetilde{\alpha}(0), 1)$ in $L_{p'}$, but this is provided by taking $\epsilon' = H(\epsilon(-), 1)$ since $H(\epsilon(-), -)$ provides such a homotopy. Thus, $[(H(l, -)^{-1} * \epsilon * \widetilde{\alpha}, 1)] = [(\epsilon' * H(\widetilde{\alpha}(0), -) * \widetilde{\alpha}, 1)] = [(H(\widetilde{\alpha}(0), -)^{-1} * \widetilde{\alpha}, 1)]$ and so Φ is well-defined.

It is clear that the resulting map $\Phi\colon \Pi_1(\mathfrak{S}; \{p, q\}) \to \Pi_1(\mathfrak{S}; \{p', q\})$ is functorial with respect to post-composition with loops at q. Let now $\theta\colon p \to p$ be a loop at p and let $\Theta\colon l \to \widetilde{\alpha}(0)$ in L be a lift so that the composition $\alpha \circ [\theta]$ is represented by the exit path $(\Theta * \widetilde{\alpha}, 1)$ and we have $\Phi(\alpha \circ [\theta]) = [(H(l, -)^{-1} * \Theta * \widetilde{\alpha}, 1)]$. On the other hand, $\Phi(\alpha) \circ \Phi([\theta]) = [(H(\widetilde{\alpha}(0), -)^{-1} * \widetilde{\alpha}, 1)] \circ [\mathrm{ad}_\Gamma(\theta)]$, in order to represent which let $\mathrm{AD}_\Gamma(\theta)\colon l' \to H(\widetilde{\alpha}(0), 1)$ in L be a lift of $\mathrm{ad}_\Gamma(\theta)$. Then $\Phi(\alpha) \circ \Phi([\theta]) = [(\mathrm{AD}_\Gamma(\theta) * H(\widetilde{\alpha}, -)^{-1} * \widetilde{\alpha}, 1)]$. But we may take $\mathrm{AD}_\Gamma(\theta) = H(l, -)^{-1} * \Theta * H(\widetilde{\alpha}(0), -)$ which gives $[(H(l, -)^{-1}) * \Theta * H(\widetilde{\alpha}(0), -) * H(\widetilde{\alpha}(0), -)^{-1} * \widetilde{\alpha}, 1)] = \Phi(\alpha \circ [\theta])$, proving the functoriality of $\Phi\colon \Pi_1(\mathfrak{S}; \{p, q\}) \to \Pi_1(\mathfrak{S}; \{p', q\})$.

In order to show that Φ is an isomorphism, let $K\colon L_{p'} \times I \to L$ be a lift of $L_{p'} \times I \to M$, $(l, t) \mapsto \Gamma(1 - t)$ along the fibre inclusion $L_{p'} \times \{0\} \to L$ and let $\Psi\colon \Pi_1(\mathfrak{S}; \{p', q\}) \to \Pi_1(\mathfrak{S}; \{p, q\})$ be defined by $\Psi|_{\mathrm{End}(p')} = [\mathrm{ad}_{\Gamma^{-1}}]$ and $\Psi(\beta) = [(K(\widetilde{\beta}(0), -)^{-1} * \widetilde{\beta}, 1)]$ where $\beta = [(\widetilde{\beta}, 1)] \in \mathrm{Hom}(p', q)$. We observe $\Psi\Phi(\alpha) = [(K(H(\widetilde{\alpha}(0), 1), -)^{-1} * H(\widetilde{\alpha}(0), -)^{-1} * \widetilde{\alpha}, 1)]$, but the first concatenation is over $\Gamma * \Gamma^{-1}$, so let $I \times I \to M$ be a homotopy $\Gamma * \Gamma^{-1} \to \mathrm{const}_p$ and $\Xi\colon I \times I \to L$ be a homotopy extension of $K(H(\widetilde{\alpha}(0), 1), -)^{-1} * H(\widetilde{\alpha}(0), -)^{-1}\colon I \times \{0\} \to L$. Then the latter is homotopic to $\Xi(-, 1)$, a path contained within L_p, hence $\Psi\Phi(\alpha) = [(\Xi(-, 1) * \widetilde{\alpha}, 1)] = [(\widetilde{\alpha}, 1)] = \alpha$ using Lemma 3.20. The converse $\Phi\Psi = \mathrm{id}$ is analogous. □

The bijections $\mathrm{Hom}(p, q) \to \mathrm{Hom}(p', q')$ in the context of Corollary 3.21 need not be given by conjugation, as Example 3.27 illustrates. We note the following simple description:

Lemma 3.22. *Let Π_1 be as in Corollary 3.21, and suppose $q \in L_p$ and that L_p is connected. Then Π_1 is a category with objects p and q, satisfying* $\mathrm{End}(p) \cong \pi_1(M)$, $\mathrm{End}(q) \cong \pi_1(N)$, *and* $\mathrm{Hom}(p, q) \cong \pi_1(N)/\pi_1(L_p)$.

Proof. This follows from the homotopy long exact sequence of the path space fibration $\mathrm{ev}_1\colon P_{L_p,-} \to N$. □

3.2. **Examples.** Any right fibration π alone, or any cofibration ι alone gives an example with a trivial choice for the other leg of the span: the identity cofibration $M \xleftarrow{\pi} L \xrightarrow{\mathrm{id}} L$ or the final fibration to the point $* \leftarrow L \xrightarrow{\iota} N$. In the latter case, there is one more trivial option: $\partial_\iota = (L \xleftarrow{\mathrm{id}} L \xrightarrow{\iota} N)$. In view of Example 3.23 below, we interpret ∂_ι as a *linked bordism* with boundary L and interior N. Not all linked bordisms arise from ordinary bordisms, e.g. if $\dim L < \dim M - 1$. In view of Remark 3.16, ι need not even be a cofibration. In a similar vein, any ∞-category X gives a linked ∞-category $\emptyset \leftarrow \emptyset \to X$ with $\mathbf{EX}(\emptyset \leftarrow \emptyset \to X) \cong X$. The other trivial construction $* \leftarrow X \xrightarrow{\mathrm{id}} X$ corresponds to taking the open cone of X in the sense that $* \in \mathbf{EX}(* \leftarrow X \xrightarrow{\mathrm{id}} X) \simeq X^{\triangleleft}$ is initial. We note this in Corollary 4.2.

Example 3.23. The linked space corresponding to a bordism $(M, \partial M)$ has lower stratum ∂M, higher stratum $M^\circ = M \setminus \partial M$, link $L = \partial M$, $\pi = \mathrm{id}_{\partial M}$, and $\iota \colon \partial M \hookrightarrow M^\circ$ given by the flow along a nowhere-vanishing inward-pointing vector field along the boundary for a chosen non-zero time.

Example 3.24. With a closed submanifold $\Sigma \subset M$ we may associate a linked space with lower stratum Σ, higher stratum $M \setminus \Sigma$, and link $\mathbb{S}\mathrm{N}(\Sigma)$, the sphere bundle of the normal bundle of Σ, together with the projection π and the choice ι of a tubular neighbourhood.

Example 3.25. For $n, m \in \mathbb{N}$, consider the linked space

$$B\mathrm{O}(n,m) \coloneqq (B\mathrm{O}(n) \twoheadleftarrow B\mathrm{O}(n) \times B\mathrm{O}(m) \overset{\oplus}{\hookrightarrow} B\mathrm{O}(n+m)),$$

the (infinite) (n,m)-*Grassmannian*, where $\oplus$ is induced by taking direct sums of vector spaces and the choice of an isomorphism $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$. Similar linked spaces exist for classical structure groups. It is shown in the companion paper [Tet25] that $\mathbf{EX}(B\mathrm{O}(n,m))$ embeds as a full sub-∞-category into a strictified quasi-category model of the stratified Grassmannian of Ayala–Francis–Rozenblyum [AFR18b]. Analogously, the Stiefel manifolds of the same ranks assemble into a linked space $E\mathrm{O}(n,m)$ over $B\mathrm{O}(n,m)$, and $\mathbf{EX}(E\mathrm{O}(n,m))$ embeds into a similar stratified Stiefel ∞-category. Analogous statements hold for classical structure groups.

Using Lemma 3.22 we will calculate the homotopy categories of some linked spaces.

Example 3.26. Consider $\mathbb{C}_0 = (\{0\} \leftarrow S^1 \to \mathbb{C}^*)$, the linked space corresponding to the complex plane stratified by the origin. Let $\Pi_1(\mathbb{C}_0; \{0, i\})$ denote the full sub-category of the homotopy category $\mathrm{h}\mathbf{EX}(\mathbb{C}_0)$ generated by the objects 0 and i. Then $\mathbf{P}_{S^1,i}$ is contractible, hence $\mathrm{Hom}_{\Pi_1(\mathfrak{S};\{0,i\})}(0, i) = *$. Consequently, we obtain an isomorphism $\Pi_1(\mathbb{C}_0; \{0, i\}) \cong (\mathrm{B}\mathbb{Z})^{\triangleleft}$.

Similarly, let $(\mathbb{C} \setminus \{1\})_0 = (\{0 \leftarrow S^1 \to \mathbb{C} \setminus \{0, 1\}\})$, let $\mathbb{Z} * \mathbb{Z}$ denote the free product of $\mathbb{Z}$ with itself, and finally let $e \colon \mathbb{Z} \hookrightarrow \mathbb{Z} * \mathbb{Z}$ denote one of the two subgroup inclusions. In the homotopy long exact sequence of the path space fibration $\mathrm{ev}_1 \colon \mathbf{P}_{S^1,-} \to \mathbb{C} \smallsetminus \{0,1\}$, the map $\mathbb{Z} \cong \pi_1(\mathbf{P}_{S^1,-}) \to \pi_1(\mathbb{C} \smallsetminus \{0,1\}) \cong \mathbb{Z} * \mathbb{Z}$ is given by $\pi_1(\mathrm{ev}_1) = e$, which implies $\pi_0(\mathbf{P}_{S^1,i}) \cong (\mathbb{Z} * \mathbb{Z})/e$. This is a mere set since $\mathrm{Im}(e)$ is not normal. We conclude that the two-object category $\Pi_1((\mathbb{C} \smallsetminus \{1\})_0; \{0,i\})$ satisfies $\mathrm{End}(0) = *$, $\mathrm{End}(i) = \mathbb{Z} * \mathbb{Z}$, $\mathrm{Hom}(0,i) = (\mathbb{Z} * \mathbb{Z})/\mathbb{Z}$, and the composition $\mathrm{End}(i) \times \mathrm{Hom}(0,i) \to \mathrm{Hom}(0,i)$ is given by $(w, w'\mathbb{Z}) \mapsto ww'\mathbb{Z}$.

Example 3.27. Let $K \colon S^1 \hookrightarrow S^3$ be a knot with knot group $W = \pi_1(S^3 \smallsetminus K)$. Since the normal bundle ν of any knot is trivialisable, the linked space which Example 3.24 associates with K can be taken to be $S^3_K = (S^1 \leftarrow S^1 \times S^1 \to S^3 \smallsetminus K)$ where the link embedding $\iota \colon S^1 \times S^1 \to S^3 \smallsetminus K$ is essentially determined by K and the normal framing. Let now $p \in S^1$ and $q \in S^3 \smallsetminus K$, and note $L_p = S^1$. Analogously to Example 3.26, the target evaluation $\mathbf{P}_{S^1,-} \to S^3 \smallsetminus K$ composed with the constant loop inclusion $S^1 \simeq \mathbf{P}_{S^1,-}$ induces a group homomorphism $e \colon \mathbb{Z} \to W$ on fundamental groups and we obtain $\mathrm{Hom}_{\Pi_1(S^3_K;\{p,q\})}(p,q) = \pi_0(\mathbf{P}_{S^1,q}) \cong W/e$. The rest of the two-object category $\Pi_1(S^3_K; \{p,q\})$ is then given similarly to the one in Example 3.26.

Recalling the Wirtinger presentation of W, we observe that the map $S^1 \to S^3 \smallsetminus K$ inducing e specifies a generator ϵ of W, so that e maps onto the subgroup of W generated by ϵ. For instance, for $K = U$ the unknot, we have $W \cong \mathbb{Z}$ and so $W/e \cong *$ for either normal framing (corresponding to the choice $\epsilon = \pm 1$).

If $K = T$ is the trefoil knot, then $W \cong B_3$ is the braid group on 3 strands, one presentation of which is $\langle a, b : a^2 = b^3 \rangle$. Let $a_1, a_2, a_3 \in W$ denote the three generators of W so that $\epsilon = a_i^{\pm 1}$ for some $i \in \{1,2,3\}$. Let us assume $\epsilon = a_i$ upon changing the normal framing (or the plane projection) if necessary. An isomorphism $W \to B_3$ is given by $a_1 \mapsto ab^{-2}aba^{-1}$, $a_2 \mapsto ab^{-1}$, $a_3 \mapsto b^2a^{-1}$. Given this identification, Corollary 3.21 implies that the quotients B_3/a_i for all $i \in \{1,2,3\}$ are in bijection with one another. There are two cases depending on the choice of the basepoints p and q:

- For $\epsilon \in \{a_2, a_3\}$, b^2a^{-1} and ab^{-1} are conjugate, so there is a canonical bijection $B_3/b^2a^{-1} \cong B_3/ab^{-1}$.
- If $\epsilon = a_1 = ab^{-2}aba^{-1}$, then $W/\epsilon \cong B_3/b^{-1}a$ upon conjugation. Although ab^{-1} is not conjugate to $b^{-1}a$, a bijection $B_3/b^{-1}a \cong B_3/ab^{-1}$ can be given as follows: there is an isomorphism $R \colon F_2 = \langle a, b \rangle \to F_2$ on the free group on two letters given by reversing the order of letters in words. Since the relation $a^2 = b^3$ is invariant under it, R induces a bijection $B_3/ab^{-1} \to b^{-1}a \backslash B_3$. The latter is in canonical bijection with $B_3/b^{-1}a$.

Example 3.28. It is well known that $c = a^2 = b^3$ in $B_3 \cong \langle a, b : a^2 = b^3 \rangle$ generates the infinite cyclic centre of B_3 and that B_3 is the universal central extension of the modular group: $B_3/c \cong \mathrm{PSL}(2;\mathbb{Z})$. Now, diverging from the context of Example 3.24 slightly, consider a linked space of type $\mathfrak{S}_{K,\iota} = (* \leftarrow S^1 \xhookrightarrow{\iota} S^3 \smallsetminus K)$ where K is a knot. Taking $K = T$ to be the trefoil knot, $p = *$, $q \in S^3 \smallsetminus \{K \cup \iota\}$ and $\iota \sim (a_3 a_2)^{*3}$ a smooth embedding representative of c written in terms of the Wirtinger generators, we obtain $\mathrm{Hom}_{\Pi_1}(p, q) \cong \mathrm{PSL}(2; \mathbb{Z})$. Thus, the two-object category Π_1 is determined by the modular group together with the action of B_3 thereon. Generalising slightly, any group quotient G/H, where H is a (not necessarily normal) finitely generated subgroup can be realised as a morphism set $\mathrm{Hom}_{\Pi_1}(*, q)$ by considering $* \leftarrow (S^1)^{\vee k} \xhookrightarrow{\iota} K(G, 1)$ for ι representing the generators of H.

4. Morphism spaces and constant exit loops

Notation. Given an embedding $\iota\colon \Sigma \hookrightarrow N$ and a point $q \in N$, we let $P(N)_{\Sigma,q} = P_{\Sigma,q}$ denote the space of paths in N that start in $\iota(\Sigma)$ and end in the point q, equipped with the compact-open topology. We use analogous notation when we work with a cofibration ι of simplicial sets.

The following result formalises and confirms the intuition that the link represents an infinitesimal expansion of the lower stratum into the higher stratum, i.e., the boundary of its unzip (see Theorem 5.1). More precisely, it is a pointwise version of that fact.

Lemma 4.1. *Let $\mathfrak{S} = (M \xleftarrow{\pi} L \xrightarrow{\iota} N)$ be a linked ∞-category, and $p \in M$ and $q \in N$ points in the two strata. There is an equivalence $\mathrm{Hom}_{\mathbf{EX}(\mathfrak{S})}(p, q) \simeq \mathbf{P}_{L_p,q}$ between the morphism space in $\mathbf{EX}$ from p to q and the space of paths in N that start in the embedded fibre $\iota(L_p)$, where $L_p = \{p\} \times_M L$, and end in q.*

Proof. We will work with a model for morphism spaces that makes the proof particularly simple: by [Lur25, 01L5], the morphism space in $\mathbf{EX}$ is equivalent to the right-pinched morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathbf{EX}}(p, q) \coloneqq \{p\} \times_{\mathbf{EX}} (\mathbf{EX}/q)$. We will observe that $\{p\} \times_{\mathbf{EX}} (\mathbf{EX}/q)$ is isomorphic to $\mathbf{P}_{L_p,q} = L_p \times_N (N/q)$. Indeed, at vertex level, the bijection $(\{p\} \times_{\mathbf{EX}} (\mathbf{EX}/q))_0 \cong (L_p \times_N (N/q))_0$ is clear: recalling that non-invertible 1-morphisms in $\mathbf{EX}$ are elements of $\mathbf{P}_0^{\Delta} \subset N_1$, let $(\gamma, 1) \in \mathbf{P}_0^{\Delta}$. For $p = d_1^{\mathbf{EX}}(\gamma, 1) = \pi(d_1^N(\gamma))$ to hold, we must have $d_1^N(\gamma) \in \iota(L_p)$. Similarly, $d_0^{\mathbf{EX}}(\gamma, 1) = \iota(d_0^N(\gamma))$, which yields the bijection.

Let now $k > 0$ and consider an exit $(k+1)$-path $(\gamma\colon \Delta[k] \star \Delta[0] \to N, j)$ in $(\mathbf{EX}/q)_k \subset \mathbf{EX}_{k+1}$. Asking that (γ, j) be in $\{p\} \times_{\mathbf{EX}} (\mathbf{EX}/q)$ is equivalent to asking that (1) its N-face $\Delta[k] \hookrightarrow \Delta[k] \star \Delta[0] \xrightarrow{\gamma} N$, which is $d_{k+1}^N(\gamma)$ under the standard identification $\Delta[k] \star \Delta[0] \cong \Delta[k+1]$, is low, as by construction only then can the corresponding $\mathbf{EX}$-face be in $M_k \subset \mathbf{EX}_k$; and that (2) it lies in particular in $\iota(L_p)$.

Condition (1) implies that $d_\ell^N(\gamma) \in N_k$ is vertical for $\ell < k+1$ since all other faces contain $q\colon \Delta[0] \hookrightarrow \Delta[k] \star \Delta[0]$, whence they are not low; if some $d_\ell(\gamma)$ was upper, that would contradict the lowness of its unique common $(k-1)$-face with $d_{k+1}^N(\gamma)$. In fact, (γ, j) has no upper n-face once $n > 0$: given $\Delta[n] \hookrightarrow \Delta[k+1]$, there is necessarily a vertex in $d_{d+1}^N(\gamma)$ in its image. But then the exit index j must be maximal: $j = k+1$. For if not, then there would exist at least one upper n-face for $n > 0$, the largest such, with $n = k+1-j$, for instance, being specified by $[n] \hookrightarrow [k+1]$, $\alpha \mapsto \ell+\alpha$. We thus obtain a bijection $(\{p\} \times_{\mathbf{EX}} (\mathbf{EX}/q))_k \cong (L_p \times_N (N/q))_k$ by reducing exit paths (γ, j) on the left-hand side to those of index $k+1$, and so to only a subset of N_{k+1}, and specifically those such that $d_{k+1}^N(\gamma) \in L_p$. Finally, $(\{p\} \times_{\mathbf{EX}} (\mathbf{EX}/q))_* \to (L_p \times_N (N/q))_*$, thus defined, is functorial: any vertical face of $(\gamma, k+1)$ is again of maximal index since, using (3.3) and (3.8), we have $\flat_{k+1,i}^{k+1} = k$ and, for degeneracies, we have $\sharp_{k+1,i}^{k+1} = k+2$ for all $i < k+1$. $\square$

Corollary 4.2. *For a linked space of type* $* \leftarrow N \xrightarrow{\mathrm{id}} N$ *we have* $\mathrm{Hom}_{\mathbf{EX}}(*, q) \simeq *$. *Consequently, when* $N = \mathrm{Sing}_\bullet(N)$ *for* N *a smooth manifold, we have* $\mathbf{EX} \simeq \mathrm{Exit}(C(N))$, *where the left-hand side is the exit path* ∞*-category of the open cone* $C(N) = * \amalg_{\{0\}\times N} ([0,1) \times N)$ *on* N *with its canonical stratification over* $\{0 < 1\}$.

Proof. We have $\mathbf{P}_{L_*,q} \simeq N/q \simeq *$ since N is an ∞-groupoid, so $\mathbf{EX} \simeq N^{\triangleleft}$. The second statement follows from [AFR18a, Proposition 3.3.8].[11] $\square$

Remark 4.3. Since the link of the cone locus $*$ and the interior of $C(N)$ is N itself, one could consider $* \twoheadleftarrow N \hookrightarrow N \times \mathbb{R}$ the natural linked space model for the open cone. The proof of Corollary 4.2 implies that this modification changes the exit path ∞-category only up to equivalence. More generally, homotopy equivalent linked spaces have equivalent EPCs: see Definition 6.5 for the topological category of linked spaces and Construction 6.11 for $\mathbf{EX}$ as a topological functor into the Kan-enriched category of ∞-categories.

Next, we will generalise Lemma 4.1 in for linked spaces and identify L, up to equivalence, with the space of paths in $\mathbf{EX}$ from M to N. Namely, for $p \in M$ and $q \in N$, observe that $(L_p \downarrow q)^N = P_{L_p,q} \simeq \mathrm{Hom}(p,q) = (p \downarrow q)^{\mathbf{EX}}$. Formally, varying p should give an equivalence $(L \downarrow q)^N = P_{L,q} \simeq (L \downarrow q)^{\mathbf{EX}} \simeq (M \downarrow q)^{\mathbf{EX}}$ and then varying q should give $L \simeq (L \downarrow N)^N = P_L \simeq (M \downarrow N)^{\mathbf{EX}}$.

This is indeed true, but we will start with the 'constant exit loop inclusion' which is well-defined more generally for all linked ∞-categories.

[11] This is an equivalence of quasi-categories for Lurie's model from [Lur17], or, after translating to the complete Segal space model and using [AFR18a, Lemma 3.3.9], with that of Ayala et al.

Lemma 4.4. *There is an ∞-functor $\square\colon L \to \mathrm{Ar}(\mathbf{EX})$ which factors through $(M \downarrow N) \subset \mathrm{Ar}(\mathbf{EX})$.*

Proof. Let $n \geq 0$ and $\gamma \in L_n$. We define the restriction of $\boxed{\gamma}\colon \Delta[1] \times \Delta[n] \to \mathbf{EX}$ along $\Delta[n] \simeq \Delta\{1\} \times \Delta[n] \hookrightarrow \Delta[1] \times \Delta[n]$, its *upper side*, to be

$$\boxed{\gamma}_{\Delta\{1\}\times\Delta[n]} := \iota(\gamma) \in M_n \subset \mathbf{EX}_n, \tag{4.5}$$

and, its low side, the restriction along $\Delta[n] \simeq \Delta\{0\} \times \Delta[n] \hookrightarrow \Delta[1] \times \Delta[n]$ to be

$$\boxed{\gamma}_{\Delta\{0\}\times\Delta[n]} := \pi(\gamma) \in N_n \subset \mathbf{EX}_n. \tag{4.6}$$

Further, given $i \in [n]$, we define the restriction along $\Delta[1] \simeq \Delta[1] \times \Delta\{i\} \hookrightarrow \Delta[1] \times \Delta[n]$ to be

$$\boxed{\gamma}_{\Delta[1]\times\Delta\{i\}} := (s_0\iota(\gamma|_i), 1) = (s_0\iota((\{i\} \hookrightarrow [n])^*\gamma), 1) \in \mathbf{P}_0^\Delta \subset \mathbf{EX}_1.$$

It is easily seen that these definitions are consistent.

We define the restriction along the exit shuffle $\mathcal{S}_j$, $j \in \{1, \dots, n+1\}$, to be

$$\boxed{\gamma}_{\mathcal{S}_j} := (s_{j-1}\iota(\gamma), j)\,. \tag{4.7}$$

Let us check that this is consistent with the definitions of $\boxed{\gamma}_{\Delta\{0/1\}\times\Delta[n]}$ and $\boxed{\gamma}_{\Delta[1]\times\Delta\{i\}}$ we have given. We must have that the low part[12] of $\boxed{\gamma}_{\mathcal{S}_j}$ coincides with the appropriate face of $\boxed{\gamma}_{\Delta\{0\}\times\Delta[n]}$. The low part is the restriction along the identity inclusion $\Delta[j-1] \hookrightarrow \Delta[n+1]$. Since $(\Delta[j-1] \hookrightarrow \Delta[n+1])^* s_{j-1} = (\sigma_{j-1} \circ ([j-1] \hookrightarrow [n+1]))^*$ and since $[j-1] \hookrightarrow [n+1] \xrightarrow{\sigma_{j-1}} [n]$ coincides with the identity inclusion $[j-1] \hookrightarrow [n]$, we have that $(\Delta[j-1] \hookrightarrow \Delta[n+1])^*(s_{j-1}\iota(\gamma), j) = \pi(\gamma|_{0,\dots,j-1})$, as desired. Similarly, the upper part[13] is the restriction along $\Delta[n+1-j] = \Delta\{j, \dots, n+1\} \hookrightarrow \Delta[n+1]$. Since $\{j, \dots, n+1\} \hookrightarrow [n+1] \xrightarrow{\sigma_{j-1}} [n]$ is $\Delta[n+1-j] = \Delta\{j-1, \dots, n\} \hookrightarrow \Delta[n]$, we have $(\Delta\{j, \dots, n+1\} \hookrightarrow \Delta[n+1])^*(s_{j-1}\iota(\gamma), j) = \iota(\gamma|_{j-1,\dots,n})$. This is as desired, since $\mathcal{S}_j(k) = (1, i-1)$ for $k \geq j$, so that the upper part as picked out in the simplex category $\mathbf{\Delta}$ by the image of $[n+1-j] \xhookrightarrow{+j} [n+1] \xhookrightarrow{\mathcal{S}_j} [1] \times [n]$, which is $\{1\} \times \{j-1, \dots, n\}$, is precisely $\boxed{\gamma}_{\{1\}\times\{j-1,\dots,n\}} = \iota(\gamma|_{j-1,\dots,n})$.

Finally, we must show that the restrictions along the $\mathcal{S}_j$, thus defined, glue. Let a pair of distinct exit indices $j < j'$ in $\{1, \dots, n+1\}$ be given. The intersection of the images of $\mathcal{S}_j, \mathcal{S}_{j'}\colon [n+1] \hookrightarrow [1] \times [n]$ as picked out within $\mathbf{\Delta}$ consists of a purely low part, $\{0\} \times [j-1]$, and a purely upper part, $\{1\} \times \{j'-1, \dots, n+1\}$. Let us write $\delta := j' - j$ and consider the map

$$\mathcal{S}_{j\cap j'}\colon \Delta[n+1-\delta] \hookrightarrow \Delta[1] \times \Delta[n]$$

[12] This means the sub-simplicial set generated by the low vertices of the exit path.

[13] generated by the upper vertices

induced by the map $\mathcal{S}_{j\cap j'}\colon [n+1-\delta] \hookrightarrow [1]\times[n]$ given by

$$i \mapsto \begin{cases} (0,i), & i<j \\ (1,i-1+\delta), & i \geq j. \end{cases}$$

In other words, $\mathcal{S}_{j\cap j'}$ is like $\mathcal{S}_j$ except with upper part shifted by $j'-j$, and so that (the images in $[1]\times[n]$ of) its low and upper parts coincide precisely with those of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$, respectively. That is, $\mathcal{S}_{j\cap j'}$ picks out precisely the intersection of the images of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ within $\Delta[1]\times\Delta[n]$. Moreover, its factorisation through $\mathcal{S}_j$ as well as through $\mathcal{S}_{j'}$ is given by identifying

$$\Delta[n+1-\delta] = \Delta\{0,\dots,j-1,j',\dots,n+1\},$$

in the sense that

$$\begin{tikzcd} \Delta\{0,\dots,j-1,j',\dots,n+1\} \arrow[r, hook, "\mathcal{S}_{j\cap j'}"] \arrow[d, hook, "\Theta"'] & \Delta[1]\times\Delta[n] \\ \Delta[n+1] \arrow[ur, hook, "\mathcal{S}_j \text{ or } \mathcal{S}_{j'}"'] & \end{tikzcd}$$

commutes. This can be checked within $\mathbf{\Delta}$: since $j<j'$, the restrictions of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ along $[0,\dots,j-1]\hookrightarrow[n+1]$ are both $i\mapsto(0,i)$, which coincides with $\mathcal{S}_{j\cap j'}$. For $[n+1-\delta]\ni i\geq j$, we have $\Theta(i)=i+\delta\geq j'>j$ and so $\mathcal{S}_j\Theta(i)=(1,\Theta(i)-1)=\mathcal{S}_{j'}\Theta(i)$. This coincides with $\mathcal{S}_{j\cap j'}(i)=(1,i-1+\delta)$, which proves the commutativity of the diagram.

We must show, therefore, that the two maps

$$\Theta^*\boxed{\gamma}_{\mathcal{S}_j}, \Theta^*\boxed{\gamma}_{\mathcal{S}_{j'}} : \Delta[n+1-\delta]\to\mathbf{EX}$$

coincide. This will reduce to a fact about the behaviour of $\flat(-,-) \coloneqq \flat_{-,-}$. We observe

$$\begin{aligned}\Theta &= \partial_{j'-1}\partial_{j'-2}\cdots\partial_{j+1}\partial_j \\ &: \Delta[n+1-\delta]\hookrightarrow\Delta[n+1-\delta+1]\hookrightarrow\cdots\hookrightarrow\Delta[n+1]\end{aligned}$$

and note that $\Theta^* = d_jd_{j+1}\cdots d_{j'-2}d_{j'-1} = d_jd_j\cdots d_jd_j$ by repeated application of the simplicial identitiy $d_\alpha d_\beta = d_{\beta-1}d_\alpha$ for $\alpha<\beta$. This implies

$$\begin{aligned}\Theta^*\boxed{\gamma}_{\mathcal{S}_j} &= (d_jd_j\cdots d_j\mathrm{id}\iota\gamma, \flat(\flat(\cdots\flat(j,j'-1),j+1),j)) \\ &= (d_jd_{j+1}\cdots d_{j'-3}d_{j'-2}\iota\gamma, j)\end{aligned}$$

using the simplicial identitiy $d_js_{j-1}=\mathrm{id}$ and then by repeated un-application of the previous identity. For the exit index, we used that $\flat(\alpha,\beta)=\alpha$ if $\beta\geq\alpha$, so that

$$\begin{aligned}\flat(\flat(\cdots\flat(j,j'-1),j+1),j) &= \flat(\flat(\dots\flat(j,j'-2),j+1),j) = \dots \\ &= \flat(j,j) = j.\end{aligned}$$

On the other hand, using $\flat(\alpha,\beta)=\alpha-1$ if $\beta<\alpha$, we have

$$\begin{aligned}\flat(\flat(\cdots\flat(j',j'-1),j+1),j) &= \flat(\flat(\cdots\flat(j'-1,j'-2),j+1),j)=\dots\\ &= \flat(j+1,j)=j.\end{aligned}$$

Now, since $d_{j'-1}s_{j'-1}=\mathrm{id}$, we obtain

$$\begin{aligned}\Theta^*\boxed{\gamma}_{\mathcal{S}_{j'}} &= (d_j\cdots d_{j'-1}s_{j'-1}\iota\gamma,\flat(\flat(\cdots\flat(j',j'-1),j+1),j))\\ &= (d_j\cdots d_{j'-2}\iota\gamma,j)\\ &= \Theta^*\boxed{\gamma}_{\mathcal{S}_j},\end{aligned}$$

as desired. This concludes the construction of $\Box$. The triple $(\pi,\Box,\iota)\colon L\to M\times\mathbf{EX}^{\Delta[1]}\times N$ factors through $(M\downarrow N)$ by (4.5) and (4.6). $\Box$

Remark 4.8. Even though $\Box$ is the 'constant exit loop inclusion,' the simplices $\boxed{\gamma}_{\mathcal{S}_j}=(s_{j-1}(\gamma),j)$ are *not* degenerate. Indeed, $(s_{j-1}(\gamma),j)=s_{j-1}(\gamma,e)$ requires $j=\sharp_{e,j-1}$, so $e\in\{j-1,j\}$, but $\sharp_{j-1,j-1}=j-1$ and $\sharp_{j,j-1}=j+1$.

Proposition 4.9. *Let $(M\leftarrow L\rightarrow N)$ be a linked space. There is an equivalence $L\simeq(M\downarrow N)$ of spaces realised by the box map $\Box\colon L\hookrightarrow(M\downarrow N)$.*

We start with the observation that Remark 2.2 holds with ∞-groupoids just as it does with topological spaces. We include a proof for completeness.

Lemma 4.10. *Let P_ι be the mapping cocyclinder as in Definition 2.1. If L and N are ∞-groupoids, then $P_\iota\simeq L$.*

Proof. The source evaluation $N^{\Delta[1]}\to N^{\{0\}}$ is a Kan fibration between Kan complexes. Moreover, each fibre $N_p^{\Delta[1]}\simeq p/N$, $p\in N_0$, is contractible by virtue of being an under-∞-groupoid. This verifies Condition (4) of [Lur25, 00X2], which implies that $N^{\Delta[1]}\to N^{\{0\}}$ is an equivalence, or, equivalently, a trivial Kan fibration. Kan fibrations are stable under pullback by [Lur25, 00T5], so the natural map $s\colon P_\iota\to L$ is a Kan fibration. Finally, as trivial Kan fibrations pull back to trivial Kan fibrations, s is one such. As it is in particular a Kan fibration, [Lur25, 00X2] implies that s is an equivalence. $\Box$

The preceding lemma is a generalisation of the fact that under-∞-groupoids are contractible, which is the special case when L is a point. Since under-∞-categories can be far from contractible, there is no reason to expect that Proposition 4.9 holds for linked ∞-categories. Indeed, most linked ∞-categories where N contains a non-invertible morphism from $\iota(L)$ to N provide counterexamples, e.g., $\{0\}\leftarrow\{0\}\hookrightarrow\Delta[1]\amalg_{\{0\}}\Delta[1]$. However, the following weaker statement holds for any linked ∞-category:

Lemma 4.11. *There is an isomorphism $(L\downarrow N)\cong(M\downarrow N)$.*

Proof. Observe $(M \downarrow N)_0 = \{\alpha \in \mathbf{EX}_1 : d_1\alpha \in M,\ d_0\alpha \in N\} = \mathbf{P}_0^\Delta$ so that $\alpha \in (M \downarrow N)_0$ iff $\alpha = (\Gamma, 1)$ with $\Gamma \in (P_L)_0 = (L \downarrow N)_0$. Thus, we have the map $(M \downarrow N)_0 \to (L \downarrow N)_0$, $(\Gamma, 1) \mapsto \Gamma$. This gives a bijection $(M \downarrow N)_0 \cong (L \downarrow N)_0$.

Generally, let $\alpha\colon \Delta[1]\times\Delta[n] \to \mathbf{EX}$ be an element of $(M \downarrow N)_n$, i.e., $\mathrm{ev}_0\alpha \in M_n$, $\mathrm{ev}_1\alpha \in N_n$. Then its restriction along any exit shuffle $\mathcal{S}_j\colon \Delta[n+1] \hookrightarrow \Delta[1] \times \Delta[n]$ where $j \in \{1, \ldots, n+1\}$ is vertical. Thus, $\alpha|_{\mathcal{S}_j} = (\alpha_j, j) \in \mathbf{P}_n^\Delta$ with $\alpha_j|_{0,\ldots,j-1} \in \iota(L)_{j-1}$. We will observe that the collection $\{\alpha_j\} \subset N_{j+1}$ assembles to give a map $A\colon \Delta[1] \times \Delta[n] \to N$ which lies within $(L \downarrow N)_n$. Indeed, setting $A|_{\mathcal{S}_j} \coloneqq \alpha_j$ defines A on every non-degenerate $(n+1)$-simplex of $\Delta[1] \times \Delta[n]$ consistently since α itself is well-defined. More precisely, let $\Theta\colon \Delta[\theta] \hookrightarrow \Delta[n+1]$ be some common simplicial subset of $\mathcal{S}_j$ and $\mathcal{S}_{j'}$ in $\Delta[1] \times \Delta[n]$. We have $\Theta^*A|_{\mathcal{S}_j} = \Theta^*A|_{\mathcal{S}_{j'}}$ since α satisfies $\Theta^*\alpha|_{\mathcal{S}_j} = \Theta^*\alpha|_{\mathcal{S}_{j'}}$ so that in particular $\Theta^*\alpha_j = \Theta^*\alpha_{j'}$ in $N_{n+1-(j'-j)}$. This yields $A \in N_n^{\Delta[1]} = (N \downarrow N)_n$. Since $\mathrm{ev}_0 A|_{\mathcal{S}_j} = \alpha_j|_{0,\ldots,j-1} \in \iota(L)_{j-1}$ for any exit index j as remarked above, we have $\mathrm{ev}_0 A \in \iota(L)_n$, giving $A \in (L \downarrow N)_n$. We have thus constructed a map $\Phi\colon (M \downarrow N) \to (L \downarrow N),\ \alpha \mapsto A$.

As for the inverse, let $\beta\colon \Delta[1] \times \Delta[n] \to N$ be an element of $(L \downarrow N)_n$, i.e., $\mathrm{ev}_0\beta \in \iota(L)_n$, and let $j < j'$ be exit indices as above. Set $B|_{\mathcal{S}_j} \coloneqq (\beta|_{\mathcal{S}_j}, j) \in \mathbf{P}_n^\Delta$. This is well-defined, since $\mathrm{ev}_0\mathcal{C}_j\beta|_{\mathcal{S}_j} = \beta|_{\mathcal{S}_j}|_{0,\ldots,j-1} = (\mathrm{ev}_0\beta)|_{0,\ldots,j-1} \in \iota(L)_{j-1}$, so that $(\beta|_{\mathcal{S}_j}, j)$ is indeed an exit path of index j. We have $\Theta^*B|_{\mathcal{S}_j} = \Theta^*B|_{\mathcal{S}_{j'}}$ since $\Theta^*\beta|_{\mathcal{S}_j} = \Theta^*\beta|_{\mathcal{S}_{j'}}$. Thus, $\{B|_{\mathcal{S}_j}\}$ assembles to give a map $B\colon \Delta[1] \times \Delta[n] \to \mathbf{EX}$. Moreover, B descends to $(M \downarrow N)_n$, since the initial vertex of every $(\beta|_{\mathcal{S}_j}, j)$ – or of any exit path – is low, so that all vertices of $\mathrm{ev}_0 B$ are low. We have thus constructed a map $\Psi\colon (L \downarrow N) \to (M \downarrow N),\ \beta \mapsto B$, the inverse to Φ. $\square$

Proof of Proposition 4.9. We have $P_\iota = L \times_{N\{0\}} N^{\Delta[1]} \cong (L \downarrow N)$. The stated equivalence follows by composing Lemma 4.10 and Lemma 4.11. In order to show that it is realised by $\square$, we will use notation from the proof of Lemma 4.11. Let $b \in L_n$, and let $\beta\colon \Delta[1] \times \Delta[n] \to N$ be the degenerate composition $\Delta[1] \times \Delta[n] \overset{\mathrm{pr}}{\twoheadrightarrow} \Delta[n] \xrightarrow{\iota(b)} N$, so that $\beta \in (L \downarrow N)$. For $j \in \{1, \ldots, n+1\}$, we have $\boxed{b}_{\mathcal{S}_j} = (s_{j-1}\iota(b), j)$, and

$$\Psi(\beta)|_{\mathcal{S}_j} = (\beta|_{\mathcal{S}_j}, j) = \left(\left(\Delta[n+1] \overset{\mathcal{S}_j}{\hookrightarrow} \Delta[1] \times \Delta[n] \twoheadrightarrow \Delta[n] \xrightarrow{\iota(b)} N\right), j\right).$$

The underlying simplex map $\mathrm{pr} \circ \mathcal{S}_j\colon [n+1] \to [n]$ is $i \mapsto (0, i) \mapsto i$ if $i \leq j-1$ and $i \mapsto (1, i-1) \mapsto (i-1)$ if $i \geq j$, so $\mathrm{pr} \circ \mathcal{S}_j = s_{j-1}$. Thus $\square = \Psi \circ \mathrm{pr}^*$ where $\mathrm{pr}^*\colon L \mapsto (L \downarrow N)$, the ordinary constant loop inclusion, maps b to β. $\square$

5. EPCs of CSSs

Theorem 5.1. *Let X be a conically smooth stratified space of depth 1 and $\mathfrak{S}$ the corresponding linked space. Then* $\mathbf{EX}(\mathfrak{S}) \simeq \mathbf{E}\mathrm{xit}(X)$.

In preparation for the proof, we need a few auxiliary constructions. They will be put to use to define an explicit comparison map $\mathbf{EX}(\mathfrak{S}) \to \mathbf{E}\mathrm{xit}(X)$; see (5.8). Before we proceed, let us note the problem with the obvious Ansatz for such a map that these constructions will solve. Let us assume for simplicity that X has two strata so that $\mathfrak{S} = (M \xleftarrow{\pi} L \xrightarrow{\iota} N)$.

Writing

$$\mathrm{Unzip} = L \times \mathbb{R}_{\geq 0} \amalg_{L \times \mathbb{R}_{>0}} N,$$

recall from [AFT17b] that there is a pullback-pushout square

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, two heads, "\pi"] & \mathrm{Unzip} \arrow[d, hook] \\ M \arrow[r, hook] & X \end{tikzcd}$$

and an isomorphism

$$X \cong |\mathfrak{S}| \coloneqq C(\pi) \amalg_{L \times \mathbb{R}_{>0}} N \tag{5.2}$$

which uses the *fibrewise open cone*

$$C(\pi) = M \amalg_{L \times \{0\}} L \times \mathbb{R}_{\geq 0}$$

and an open embedding $I\colon L \times \mathbb{R}_{>0} \hookrightarrow N$ satisfying $I_1 = \iota$.

Now, given $(\gamma, e) \in \mathbf{EX}_n$, we may see γ within $|\mathfrak{S}|$ by reparametrising Unzip to $\mathrm{Unzip}' = L \times \mathbb{R}_{\geq 1} \amalg_{L \times \mathbb{R}_{>1}} N$ so that γ lies within Unzip and $\gamma|_{0,\dots,e-1}$ within the boundary $\mathrm{Unzip}_0 = L \times \{0\}$. However, in order for this to be a well-defined n-simplex of X, γ outside of $\gamma|_{0,\dots,e-1}$ must lie wholly in the complement $N \smallsetminus L$ per condition (5.7), but this is not imposed on (γ, e). We must therefore first elongate $\gamma|_{0,\dots,e-1}$ appropriately before embedding into X. The technical difficulty consists therefore in defining any functor $\mathbf{EX}(\mathfrak{S}) \to \mathbf{E}\mathrm{xit}(X)$ at all; that it is an equivalence will follow essentially from Proposition 4.9 and the corresponding result in the conically smooth context.

Construction 5.3. The standard topological simplices may be alternatively constructed recursively, starting with the empty set, by taking closed cones. We will present a slightly different but equivalent version of [AFR18a, Notation 2.3.2] and spell out how they assemble into a cosimplicial space $\Delta^\bullet\colon \mathbf{\Delta} \to \mathrm{Top}$. The details are provided for completeness.

For a topological space X, set $D(X) = *$ if $X = \emptyset$ and

$$D(X) \coloneqq X \times [0,1] \amalg_{X \times \{1\}} *$$

otherwise, defining the *closed cone* on X. We may define Δ^n for $n \geq 0$ iteratively by setting $\Delta^n = D(\Delta^{n-1})$ and $\Delta^{-1} = \emptyset$. We will write n for the cone point, the equivalence class of $* \in D(\Delta^{n-1})$. The face maps can be defined as follows: $\partial_n^n \colon \Delta^{n-1} \hookrightarrow \Delta^n$ is the inclusion of the base,

$$\partial_n^n(x) = (x, 0), \tag{5.4}$$

and for $i < n$ we define $\partial_i^n \colon \Delta^{n-1} = D(\Delta^{n-2}) \hookrightarrow D(\Delta^{n-1}) = \Delta^n$ iteratively by $\partial_i^n(x,t) = (\partial_i^{n-1}(x), t)$. When $n = 1$, we define $\partial_0^1 \colon \Delta^0 \hookrightarrow \Delta^1$ to be $* \mapsto (*, 1)$. The degeneracy maps can be defined similarly: $\sigma_0^1 \colon \Delta^1 \to \Delta^0$ is unique, so let $n \geq 2$. We define $\sigma_{n-1}^n \colon \Delta^n \to \Delta^{n-1}$ by

$$\sigma_{n-1}^n(x,t) = \begin{cases} x, & t < 1 \\ n-1, & t = 1. \end{cases}$$

For $j < n-1$, we define $\sigma_j^n \colon \Delta^n \to \Delta^{n-1}$ by $\sigma_j^n(x,t) = (\sigma_j^{n-1}(x), t)$. Note that this implies $\sigma_j^n(n) = n-1$.

Construction 5.5. For $n \geq 1$ and $e \in \{1, \dots, n\}$, let

$$\Delta_e^n = \Delta^n \amalg_{\Delta^{e-1} \times \Delta^{\{1\}}} \Delta^{e-1} \times \Delta^1$$

where the gluing map $\Delta^{e-1} \times \Delta^{\{1\}} \hookrightarrow \Delta^n$ is the standard inclusion. The *elongation map* is

$$\begin{aligned} E_n \colon \Delta^n &\to \Delta_n^n, \\ (x,t) &\mapsto \begin{cases} (x, 2t), & t \leq \frac{1}{2} \\ (x, 2t-1), & t \geq \frac{1}{2} \end{cases} \end{aligned}$$

where we wrote $(x,t) \in D(\Delta^{n-1}) = \Delta^n$ according to Construction 5.3. This is well-defined since at $t = \frac{1}{2}$ the gluing identifies $\Delta^{n-1} \times \Delta^{\{1\}} \ni (x,1) \sim (x,0) \in \Delta^n$ per (5.4).

Construction 5.6. For $n \geq 1$ and $e \in \{1, \dots, n\}$, the *collapse map*

$$K_e = K_e^n \colon \Delta_n^n \to \Delta_e^n$$

is defined as follows: its restriction to $\Delta^n \subset \Delta_n^n$ is the identity, and its restriction to $\Delta^{n-1} \times \Delta^1$ is induced by the map

$$k_e \colon \Delta^{n-1} \times \Delta^1 \to \Delta^{n-1} \times \Delta^1$$

induced, using the fact that geometric realisation commutes with cartesian products, by the poset map, $[n-1] \times [1] \to [n-1] \times [1]$,

$$(i, 0) \mapsto \begin{cases} (i, 0), & i \leq e-1 \\ (i, 1), & i \geq e \end{cases}$$

and $(i,1) \mapsto (i,1)$ for all i. We obtain a map to Δ_e^n by composing as follows:

$$\Delta^{n-1} \times \Delta^1 \xrightarrow{k_e} \Delta^{n-1} \amalg_{\Delta^{e-1} \times \Delta^{\{1\}}} \Delta^{e-1} \times \Delta^1 \hookrightarrow \Delta_e^n$$

where k_e maps onto the second component and is well-defined on the union since $k_e|_{\Delta^{n-1}\times\Delta^{\{1\}}} = \mathrm{id}$, and the second map is given by $\partial_n\colon \Delta^{n-1} \hookrightarrow \Delta^n$ and $\mathrm{id}_{\Delta^{e-1}\times\Delta^1}$.

Proof of Theorem 5.1. Without loss of generality, let X and $\mathfrak{S}$ be as discussed above Construction 5.3. We will exhibit an equivalence

$$\mathbf{R}\colon \mathbf{EX}(\mathfrak{S}) \xrightarrow{\sim} \mathbf{E}\mathrm{xit}(|\mathfrak{S}|).$$

It is the identity on objects, so let $n \geq 1$. We will use Lurie's quasi-category model for the target, where $\mathbf{E}\mathrm{xit}_n(|\mathfrak{S}|)$ consists of those paths $\gamma\colon \Delta^n \to |\mathfrak{S}|$ for which there exists a sequence $p_0 \leq \cdots \leq p_n$ in $[1]$ satisfying

$$s(\gamma(t_0, \ldots, t_i, 0, \ldots, 0)) = p_i \tag{5.7}$$

where $t_i \neq 0$ and $s\colon |\mathfrak{S}| \to [1]$ is the stratification (see [Lur17, §A.6]). In terms of Construction 5.3, condition (5.7) reads $s(\gamma(x,t)) = p_n$ for $t > 0$ and $s(\partial_n^n(\gamma)(x',t')) = p_{n-1}$ for $t' > 0$, and so on.

An exit path $(\gamma, e) \in \mathbf{EX}_n$ induces a map

$$\Delta^n_e \to |\mathfrak{S}|$$

as follows. Since $\mathbf{b} := \mathbf{b}(\gamma, e) = \gamma|_{0,\ldots,e-1} \in L_{e-1}$ (recall Remark 3.16), the map $\mathbf{b} \times \mathrm{id}\colon \Delta^{e-1} \times \Delta^1 \to L \times \mathbb{R}_{\geq 0}$ is well-defined and factors through $L \times [0,1]$. Using $I_1 = \iota$, this yields the desired map

$$\gamma \cup (\mathbf{b} \times \mathrm{id})\colon \Delta^n_e \to \mathrm{Unzip} = L \times \mathbb{R}_{\geq 0} \amalg_{L\times\mathbb{R}_{>0}} N \to |\mathfrak{S}|$$

where the final quotient projection identifies the $(e-1)$-path $\mathbf{b} \times \{0\}$ with its projection in M. We now define $\mathbf{R}(\gamma, e)$ to be the composition

$$\mathbf{R}(\gamma, e)\colon \Delta^n \xrightarrow{E_n} \Delta^n_n \xrightarrow{K_e} \Delta^n_e \xrightarrow{\gamma\cup(\mathbf{b}\times\mathrm{id})} |\mathfrak{S}|. \tag{5.8}$$

It is a direct check that $\mathbf{R}$ is functorial.

It now suffices to prove that the derivative $\mathbf{R}_*\colon (M \downarrow N)^{\mathbf{EX}} \to (M \downarrow N)^{\mathbf{E}\mathrm{xit}}$ is an equivalence. To this end, consider the composition

$$\mathbf{R}_* \circ \square\colon L \to (M \downarrow N)^{\mathbf{E}\mathrm{xit}}.$$

For $\delta \in L_n$ and $j \in \{1, \ldots, n+1\}$, we have

$$\mathbf{b}(s_{j-1}\delta, j) = (s_{j-1}\delta)|_{0,\ldots,j-1} = \delta|_{0,\ldots,j-1},$$

so, recalling (4.7), we obtain

$$(\mathbf{b}\boxed{\delta}_{\mathcal{S}_j} \times \mathrm{id}) = \delta|_{0,\ldots,j-1} \times \mathrm{id}\colon \Delta^{j-1} \times \Delta^1 \to |\mathfrak{S}|.$$

Hence

$$\mathbf{R}\boxed{\delta}_{\mathcal{S}_j} = (K_{j-1}E_n)^*(\boxed{\delta}_{\mathcal{S}_j} \cup (\delta|_{0,\ldots,j-1} \times \mathrm{id})) \in \mathbf{E}\mathrm{xit}_n(|\mathfrak{S}|).$$

Now, an equivalence $\Xi\colon (M \downarrow N)^{\mathbf{E}\mathrm{xit}} \xrightarrow{\sim} L$ is given by sending a prism to its source face and restricting the second coordinate along $L \times \{0\} \hookrightarrow \mathrm{Unzip} \to$

$|\mathfrak{S}|$. In order to obtain the source of the prism $\mathbf{R}_*\boxed{\delta}$, we observe that if we take $j = n+1$, then

$$d_{n+1}(\boxed{\delta}_{\mathcal{S}_{n+1}}) = d_{n+1}(s_n\delta, n+1) = \pi(d_{n+1}s_n\delta) = \pi(\delta)$$

and that $K_n = \mathrm{id}$, $E_n|_{d_n\Delta^n} = \mathrm{id}_{\Delta^{n-1}}$. Consequently,

$$\Xi(\mathbf{R}_*\boxed{\delta}) = \delta, \tag{5.9}$$

whence $\mathbf{R}_* \circ \square$ is an equivalence and so $\mathbf{R}_*$ itself is also an equivalence by Lemma 4.4. As $\mathbf{R}$ is thus fully faithful, it is an equivalence. $\square$

6. Functoriality

Ayala, Francis and Rozenblyum showed in [AFR18a, Theorem 4.2.8] that the EPC construction embeds conically smooth stratified spaces fully faithfully into the ∞-category of ∞-categories (see [Lur09, §3] or [Lur25, 01YV]). In this section, we will prove the same for **EX**.

6.1. **The quasi-category of linked spaces.** Let $\mathrm{Poset}_{\leq 1}$ denote the ordinary category of posets of depth (at most) 1, and Cat the ordinary (and not the 2-)category of categories. Readers familiar with décollages in the sense of [BGH20] can safely skip to Definition 6.5 and are referred to Remark 6.7.

Definition 6.1. The *linking functor*

$$\mathbf{L}\colon \mathrm{Poset}_{\leq 1} \to \mathrm{Cat}$$

is defined as follows. Let $P \in \mathrm{Poset}_{\leq 1}$. For every object and every morphism of P, $\mathbf{L}P$ has an object. We denote the object corresponding to $p \in P$ by $m_p \in \mathbf{L}P$, and the object corresponding to $f\colon p_0 \to p_1$ in P by $l_f \in \mathbf{L}P$. Besides the identities, $\mathbf{L}P$ has two arrows as in

$$m_{p_0} \leftarrow l_f \to m_{p_1}$$

for every morphism $f\colon p_0 \to p_1$ in P. Note in particular that for every $p \in P$ there is a nontrivial span $m_p \leftarrow l_{\mathrm{id}_p} \to m_p$ at m_p, not merely the identity at m_p. There are no further objects or morphisms in $\mathbf{L}P$.

Given a morphism $F\colon P \to P'$ of posets, we define the functor $\mathbf{L}F\colon \mathbf{L}P \to \mathbf{L}P'$ by setting $\mathbf{L}F(m_p) = m_{F(p)}$, $\mathbf{L}F(l_f) = l_{F(f)}$ on objects and

$$\mathbf{L}F(m_{p_0} \leftarrow l_f \to m_{p_1}) = (m_{F(p_0)} \leftarrow l_{F(f)} \to m_{F(p_1)})$$

on the nontrivial morphisms.

Remark 6.2. The linking functor is faithful but not fully so; e.g., the constant functors $(m_* \leftarrow l_{\mathrm{id}_*} \to m_*) = \mathbf{L}* \to \mathbf{L}*'$ given by $m_*, l_{\mathrm{id}_*} \mapsto m_{*'}$ and $m_*, l_{\mathrm{id}_*} \mapsto l_{\mathrm{id}_{*'}}$ are not in the image of $\mathbf{L}|_{\mathrm{Hom}_{\mathrm{Poset}_{\leq 1}}(*,*')}$ which consists solely of the isomorphism $\mathbf{L}* \cong \mathbf{L}*'$

Definition 6.3. The auxiliary category $\widehat{\mathbf{LS}} = \widehat{\mathbf{LS}}_{\leq 1} \subset \mathrm{Diag}(\mathbf{T}\mathrm{op})$ of *quasi-linked spaces* of depth (at most) 1 is the subcategory of the category of topological diagrams whose shapes are linking posets, i.e.,

$$\widehat{\mathbf{LS}}_{\leq 1} = \mathrm{Poset}_{\leq 1} \times_{\mathrm{Cat}} \mathrm{Diag}(\mathbf{T}\mathrm{op}).$$

where $\mathrm{Diag}(\mathbf{T}\mathrm{op}) \to \mathrm{Cat}$, $(D \to \mathbf{T}\mathrm{op}) \mapsto D$, maps a diagram to its shape. It is naturally a topological category by setting

$$\mathrm{Hom}_{\widehat{\mathbf{LS}}}((P, \mathfrak{S}), (Q, \mathfrak{T})) = \coprod_{\Phi\colon P \to Q} \prod_{f\colon p_0 \to p_1} \Big([\mathfrak{S}_{p_0}, \mathfrak{T}_{\mathbf{L}\Phi(p_0)}] \times_{[\mathfrak{S}_f, \mathfrak{T}_{\mathbf{L}\Phi(p_0)}]} [\mathfrak{S}_f, \mathfrak{T}_{\mathbf{L}\Phi(f)}] \times_{[\mathfrak{S}_f, \mathfrak{T}_{\mathbf{L}\Phi(p_1)}]} [\mathfrak{S}_{p_1}, \mathfrak{T}_{\mathbf{L}\Phi(p_1)}] \Big)$$

where the disjoint union is over poset maps, the product is over arrows in P, and $[-,-] = \mathrm{Hom}_{\mathbf{T}\mathrm{op}}(-,-)$.

Concretely, a quasi-linked space is a pair $(P, \mathfrak{S}\colon \mathbf{L}P \to \mathbf{T}\mathrm{op})$ where $P \in \mathrm{Poset}_{\leq 1}$ and $\mathfrak{S}$ is a functor. Given $f\colon p_0 \to p_1$ in P, we write $\mathfrak{S}(m_{p_0} \leftarrow l_f \to m_{p_1}) = (\mathfrak{S}_{p_0} \xleftarrow{\pi_f} \mathfrak{S}_f \xrightarrow{\iota_f} \mathfrak{S}_{p_1}) = (M_{p_0} \xleftarrow{\pi_f} L_f \xrightarrow{\iota_f} M_{p_1})$. A morphism $\phi\colon (P, \mathfrak{S}\colon \mathbf{L}P \to \mathbf{T}\mathrm{op}) \to (Q, \mathfrak{T}\colon \mathbf{L}Q \to \mathbf{T}\mathrm{op})$ is a pair $(\Phi\colon P \to Q, \phi\colon \mathfrak{S} \to \mathfrak{T})$ where Φ is a morphism in $\mathrm{Poset}_{\leq 1}$ and ϕ is a natural transformation $\phi\colon \mathfrak{S} \Rightarrow \mathfrak{T} \circ \mathbf{L}\Phi$. We will sometimes suppress the posets in notation.

Remark 6.4. Due to the parametrisation over poset maps mediated by the linking functor, $\widehat{\mathbf{LS}}$ differs substantially from the category of topological spans. For instance, a P-stratified quasi-linked space consists, for $P = [1]$, of five spaces organised into three spans:

$$\mathfrak{S} = (L_{\mathrm{id}_0} \rightrightarrows M_0 \leftarrow L_{0<1} \to M_1 \leftleftarrows L_{\mathrm{id}_1}).$$

A map $\phi\colon ([1], \mathfrak{S}) \to ([1], \mathfrak{S}')$ need not map the spans in $\mathfrak{S}$ to the corresponding spans in $\mathfrak{S}'$; rather, when the accompanying poset map $\Phi\colon [1] \to [1]$ factors, say, through $\{0\}$, then $\phi\colon \mathfrak{S} \Rightarrow \mathfrak{S}' \circ \mathbf{L}\Phi$ consists of three topological span maps

$$(L_{\mathrm{id}_0} \rightrightarrows M_0), (M_0 \leftarrow L_{0<1} \to M_1), (M_1 \leftleftarrows L_{\mathrm{id}_1}) \to (L'_{\mathrm{id}_0} \rightrightarrows M'_0)$$

such that the maps out of M_0 and M_1 coincide in all three but with no further conditions.

Definition 6.5. The category $\mathbf{LS} \subset \widehat{\mathbf{LS}}$ of *linked spaces* of depth (at most) 1 is the full topological subcategory of $\widehat{\mathbf{LS}}$ consisting of those quasi-linked spaces $(P, \mathfrak{S}\colon \mathbf{L}P \to \mathbf{T}\mathrm{op})$ which satisfy the following conditions:

(1) For every non-identity morphism f in P, the map π_f is a fibration and the map ι_f is a cofibration.
(2) For every object p in P, we have $L_{\mathrm{id}_p} = M_p$ and $\pi_{\mathrm{id}_p} = \iota_{\mathrm{id}_p} = \mathrm{id}_{M_p}$.

Remark 6.6. Definition 6.5 removes the extravagance of Definition 6.3 pointed out in Remark 6.4. A map $\phi\colon ([1], \mathfrak{S}) \to ([0], \mathfrak{S}')$ as discussed there is, in **LS**, determined by a single span map $(M_0 \leftarrow L_{0<1} \to M_1) \to (M_0' \rightrightarrows M_0')$. Such a span map is tantamount to a map $M_0 \amalg_{L_{0<1}} M_1 \to M_0'$ so in particular every conically smooth map from a depth-1 space into a smooth manifold is of this type. This is analogous to the condition that the lower stratum in an ordinary stratified space be in the closure of the higher.

Remark 6.7. The secondition condition of Definition 6.5 makes **LS** a subcategory of $\mathrm{Diag}_P = \mathrm{Fun}(R(P)^{\mathrm{op}}, \mathrm{Top})$, the topological version of the category of diagrams over P in the sense of Douteau: see [Dou21, Definition 1.10]. Over depth-1 posets, these coincide with spatial décollages in the sense of Barwick–Glasman–Haine [BGH20, §2.6]. Removing said condition amounts to allowing degenerate subdivisions of P. We have chosen to circumvent the construction of R for concreteness as we only consider depth-1 posets.

Example 6.8. Consider the linked space $\mathrm{sp} = ([1], \mathrm{sp})$ where $\mathrm{sp} = (* \leftarrow * \to *)$. A map $\mathrm{sp} \to M = ([0], \mathrm{const}_M)$ is determined by a point in M. In contrast, a map $\phi\colon \mathrm{sp} \to ([1], \mathfrak{S})$ with $\Phi = \mathrm{id}_{[1]}$ is determined by a link point $l \in L = \mathfrak{S}_{0<1}$, which can be interpreted as an 'infinitesimal exit path' $\pi(l) \to \iota(l)$ via the box map of Lemma 4.4.

Example 6.9. Consider the poset $I = \{\pm <_{\pm} 1\}$ and the linked space $\mathfrak{I} = (I, \mathfrak{I}\colon \mathbf{L}I \to \mathbf{T}\mathrm{op})$ where $\mathfrak{I}_- = * = \mathfrak{I}_+$, $\mathfrak{I}_1 = \mathbb{R}$, $\mathfrak{I}_{<_\pm} = *$, and $\iota_{<_-} = 0$, $\iota_{<_+} = 1$. Let now $\phi\colon \mathfrak{I} \to ([1], \mathfrak{S})$ be a map satisfying $\Phi(0) = 0$ and $\Phi(+) = \Phi(1) = 1$. Then $\phi\colon \mathfrak{I} \Rightarrow \mathfrak{S} \circ \mathbf{L}\Phi$ is a diagram of type

$$\begin{tikzcd}
 & * \arrow[dl] \arrow[dd] \arrow[dr, "0"] & & * \arrow[dl, "1"'] \arrow[dd] \arrow[dr] & \\
* \arrow[dd] & & \mathbb{R} \arrow[dd, "\gamma"] & & * \arrow[dd] \\
 & L \arrow[dl, "\pi"] \arrow[dr, "\iota"'] & & N \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
M & & N & & N
\end{tikzcd}$$

so γ specifies an exit path in $\mathbf{EX}(\mathfrak{S})$ from $\pi(\gamma(0))$ to $\gamma(1)$, namely $(\gamma|_{[0,1]}, 1) \in \mathbf{P}_0^{\Delta}$. Conversely, any exit path may be extended to a map $\mathbb{R} \to N$ constantly at its endpoints to define a map $\phi\colon (I, \mathfrak{I}) \to ([1], \mathfrak{S})$ with Φ as above.

Definition 6.10. The *∞-category of linked spaces of depth at most* 1 is the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}(\mathbf{LS})$ of the topological category $\mathbf{LS} = \mathbf{LS}_{\leq 1}$ of linked spaces of depth at most 1. We will denote it by **LS** if there is no risk of confusion.

6.2. **Fully-faithfulness.** Let $\mathbf{C}\mathrm{at}_{\mathrm{Top}}$ denote the locally-Kan simplicial category of all ∞-categories, where $\mathrm{Hom}_{\mathbf{C}\mathrm{at}_{\mathrm{Top}}}(\mathbf{C}, \mathbf{D}) = \mathrm{Fun}^{\simeq}(\mathbf{C}, \mathbf{D})$ is the core of the functor ∞-category $\mathrm{Fun}_\bullet(\mathbf{C},\mathbf{D}) = \mathrm{Hom}_{\mathrm{sSet}}(\mathbf{C}\times\Delta^\bullet, \mathbf{D})$ generated by those natural transformations which are natural isomorphisms. As in [Lur09, §3], we will denote by $\mathbf{C}\mathrm{at}_\infty = \mathrm{N}^{\mathrm{hc}}(\mathbf{C}\mathrm{at}_{\mathrm{Top}})$ the homotopy-coherent nerve of $\mathbf{C}\mathrm{at}_{\mathrm{Top}}$.

Construction 6.11. We will promote the EPC construction to a topological functor

$$\mathbf{EX}\colon \mathbf{LS} \to \mathbf{C}\mathrm{at}_{\mathrm{Top}}.$$

It will suffice to consider maps between posets of at most two elements, so let $([1], \mathfrak{S}), ([1], \mathfrak{T}), ([0], \mathfrak{U}) \in \mathbf{LS}$. A map $\phi\colon ([1], \mathfrak{S}) \to ([1], \mathfrak{T})$ with $\Phi = \mathrm{id}_{[1]}$ is an ordinary span map (cf. Remark 6.6) and naturally induces an ∞-functor

$$\phi_!\colon \mathbf{EX}(\mathfrak{S}) \to \mathbf{EX}(\mathfrak{T}).$$

The case $\Phi \neq \mathrm{id}_{[1]}$ is tantamount to a map $\phi\colon ([1], \mathfrak{S}) \to ([0], \mathfrak{U})$. We set

$$\mathbf{EX}([0], \mathfrak{U}) = \mathfrak{U}_0,$$

suppressing the distinction between a space and its complex of singular chains. Now, $\phi\colon \mathfrak{S} \Rightarrow \mathfrak{U} \circ \mathbf{L}\Phi$ is a span map of type $(M \leftarrow L \rightarrow N) \to (M' \rightrightarrows M')$ which is already determined by the map $\phi_1\colon N \to M'$ (cf. Remark 6.6). This induces an ∞-functor

$$\phi_!\colon \mathbf{EX}(\mathfrak{S}) \to M'$$

by mapping an exit path $(\gamma, e) \in \mathbf{P}^{\Delta}_{k-1} \subset \mathbf{EX}_k$ to

$$\phi_!(\gamma, e) = \phi_1(\gamma).$$

It is easily seen that this is functorial since $\phi_0(\pi(\mathbf{b}(\gamma, e))) = \phi_1(\mathbf{b}(\gamma, e))$. Finally, a map of type $\phi\colon ([0], \mathfrak{U}) \to ([1], \mathfrak{S})$ is tantamount to an ordinary map $M' \to M$ or $M' \to N$ depending on $\Phi\colon [0] \to [1]$.

This defines $\mathbf{EX}$ as an ordinary functor. We will now define its extension

$$\mathrm{Hom}_{\mathbf{LS}}(([1], \mathfrak{S}), ([1], \mathfrak{T}))_\bullet \to \mathrm{Hom}_{\mathbf{C}\mathrm{at}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T}))_\bullet$$

onto the full morphism spaces. Recalling Definition 6.3, let us write $\mathrm{Hom}_{\mathbf{LS}} = \mathcal{F}_0 \amalg \mathcal{F}_1 \amalg \mathcal{F}_{01}$ where $\mathcal{F}_i$ is the connected component at $\Phi = \sigma_i\colon [1] \to [1]$ for $i = 0, 1$ and $\mathcal{F}_{01}$ is the connected component at $\Phi = \mathrm{id}_{[1]}$. If $L = \mathfrak{S}_{0<1} = \emptyset$, we set $(\phi_i)_!|_X = (\phi_i)|_X$ for $i = 0, 1$ and $X = M, N$.

Suppose now that $L \neq \emptyset$. By construction, an n-simplex in $\mathcal{F}_0$ is of type

$$H = (H_M, H_L, H_N) \in [M, M']_n \times_{[L,M']_n} [L, M']_n \times_{[L,M']_n} [N, M']_n.$$

where $H_-\colon - \times \Delta[n] \to M$. We define the restriction to the core by $H_!|_{\mathbf{EX}(\mathfrak{S})^{\simeq}} = H_M \amalg H_N\colon \mathbf{EX}(\mathfrak{S})^{\simeq} = M \amalg N \to M' \subset \mathbf{EX}(\mathfrak{T})$. Now let $(\gamma, e) \in \mathbf{P}^{\Delta}_{k-1} \subset \mathbf{EX}_k(\mathfrak{S})$ and let $\Gamma\colon \Delta[k] \to \mathbf{EX}(\mathfrak{S}) \times \Delta[n]$ be given such that $\Gamma_1 = (\gamma, e)$. Then Γ induces

$$\widetilde{\Gamma} = (\gamma, \Gamma_2)\colon \Delta[k] \to N \times \Delta[n]$$

and thus $H_N(\widetilde{\Gamma}) = H_N \circ \widetilde{\Gamma} \colon \Delta^k \to M'$. We set

$$H_!(\Gamma) = H_N(\widetilde{\Gamma}). \tag{6.12}$$

This gives a map $H_! \colon \mathbf{EX}(\mathfrak{S}) \times \Delta[n] \to M'$ which defines $\mathrm{Hom}_{\mathbf{LS}} \supset \mathcal{F}_0 \to \mathbf{F}_0 \subset \mathrm{Hom}_{\mathbf{C}\mathrm{at}_{\mathrm{Top}}}$, $H \mapsto H_!$. The restriction $\mathcal{F}_1 \to \mathbf{F}_1$ is induced analogously by replacing M' with $N' = \mathfrak{T}_1$ throughout. Similarly, given an n-simplex $H = (H_M, H_L, H_N) \in [M, M']_n \times_{[L,M']_n} [L, L']_n \times_{[L,N']_n} [N, N']_n$ in $\mathcal{F}_{01}$, where $L = \mathfrak{T}_{0<1}$, and given $\Gamma = ((\gamma, e), \Gamma_2) \in (\mathbf{EX}(\mathfrak{S}) \times \Delta^n)_k$, $\widetilde{\Gamma} = (\gamma, \Gamma_2) \in (N \times \Delta[n])_k$ as above, we set

$$H_!(\Gamma) = (H_N(\widetilde{\Gamma}), e) \tag{6.13}$$

which defines the restriction $\mathcal{F}_{01} \to \mathbf{F}_{01}$.

Theorem 6.14. *The topological EPC functor of Construction 6.11 induces a fully faithful functor*

$$\mathbf{EX} \colon \mathbf{LS}_{\leq 1} \hookrightarrow \mathbf{C}\mathrm{at}_\infty.$$

of ∞-categories.

The reason for our passing to homotopy-coherent nerves is that $\mathbf{EX} = (-)_! \colon \mathrm{Hom}_{\mathbf{LS}} \to \mathrm{Hom}_{\mathbf{C}\mathrm{at}_{\mathrm{Top}}}$ is merely a weak equivalence, as we will show. Since the morphism spaces in homotopy-coherent nerves are equivalent to those in the original locally-Kan category (this is due to Joyal; see also Hebestreit–Krause [HK20] for a direct proof), Theorem 6.14 will follow.

Before we prove Theorem 6.14, we will use it to produce three classes of counterexamples to the various versions of the conically smooth stratified homotopy hypothesis discussed in Remark 1.2.

6.2.1. *Class I: non-finite local links.* Let B be a connected smooth manifold without boundary whose fundamental group is not purely torsion so that the fibres of its universal cover $\pi \colon \widetilde{B} \to B$ have infinitely many connected components. Let $\widetilde{B} \to Y$ be any cofibration. Each can be chosen finite, e.g., $(S^1 \leftarrow \mathbb{R} \hookrightarrow \mathbb{R}^2)$. The idea is to violate the compactness of the local links in a CSS.

Corollary 6.15. *The $[1]$-layered ∞-category $\mathbf{EX}(B \leftarrow \widetilde{B} \to Y)$ is not equivalent to the EPC of a CSS. If $B, \widetilde{B}$ and Y are finite, then so is $\mathbf{EX}(B \leftarrow \widetilde{B} \to Y)$.*

Proof. Suppose X is a CSS with associated linked manifold $(M \xleftarrow{\pi'} L \to N)$ such that $\mathbf{E}\mathrm{xit}(X) \simeq \mathbf{EX}(B \leftarrow \widetilde{B} \to Y)$. By Theorem 5.1 we have $\mathbf{EX}(M \leftarrow L \to N) \simeq \mathbf{EX}(B \leftarrow \widetilde{B} \to Y)$ and by Theorem 6.14 we have a homotopy equivalence $(B \leftarrow \widetilde{B} \to Y) \simeq (M \leftarrow L \to N)$ of spans. In particular, we obtain the commutative square

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\overline{f}", "\sim"'] \arrow[d, "\pi"] & L \arrow[d, "\pi'"] \\ B \arrow[r, "f", "\sim"'] & M \end{tikzcd}$$

where the vertical maps are fibre bundles and $\overline{f}$ is a homotopy equivalence covering the homotopy equivalence f, i.e., a fibre-homotopy equivalence. Therefore, the fibres of π and π' are homotopy equivalent. The fibres of π' are compact since around each point $q \in M$ there exists a basic open of type $U \cong \mathbb{R}^k \times C(Z)$ such that $(\pi')^{-1}(q) \cong Z$ where Z is a closed smooth manifold. But the fibres of π have infinitely many connected components by assumption, contradicting the compactness of Z. The second statement follows from [Vol24, Proposition 2.11]. □

Remark 6.16. The local links of $\mathbf{EX}(B \leftarrow \widetilde{B} \to Y)$ are not finite. A similar example was given by Volpe in [Vol24, Remark 2.14].

6.2.2. *Class II: finite local links.* Besides the compactness of the spaces Z that appear in the proof of Corollary 6.15, another characteristic property of a CSS over [1] is that the link embedding $\iota \colon L \hookrightarrow N$ has a (rank-1) normal framing. This leads to a second class of examples.

Recall that a smooth n-manifold Y is called *stably (normally) frameable* if there exists a natural number $k \geq 0$ together with a bundle isomorphism $\mathrm{T}Y \oplus \varepsilon^k \cong \varepsilon^{n+k}$, where ε denotes the trivial real bundle of rank 1. A closed embedding $Y \hookrightarrow \mathbb{R}^K$ whose normal bundle is trivialisable makes Y stably frameable. The Stiefel–Whitney classes of a stably frameable smooth manifold vanish. Contractible smooth manifolds are stably frameable, and if there is a homotopy equivalence $Y \simeq Y'$ of compact manifolds and the Stiefel–Whitney classes of Y vanish, then so do those of Y' using Wu's formula and the homotopy invariance of Steenrod operations.

Corollary 6.17. *Let Λ be compact smooth manifold with a non-vanishing Stiefel–Whitney class, and let $\Lambda \hookrightarrow \mathbb{R}^K$ be a closed embedding. Then the finite* [1]*-layered ∞-category $\mathbf{EX}(* \leftarrow \Lambda \hookrightarrow \mathbb{R}^K)$ has contractible strata, finite local links, and is not equivalent to the EPC of a CSS.*

Proof. Suppose that X is a CSS with associated linked manifold $(M \leftarrow L \to N)$ such that $\mathbf{E}\mathrm{xit}(X) \simeq \mathbf{EX}(* \leftarrow \Lambda \to \mathbb{R}^K)$. By Theorem 5.1 we have $\mathbf{EX}(M \leftarrow L \to N) \simeq \mathbf{EX}(* \leftarrow \Lambda \to \mathbb{R}^K)$ and by Theorem 6.14 we have a homotopy equivalence $(M \leftarrow L \to N) \simeq (* \leftarrow \Lambda \to \mathbb{R}^K)$ of spans. In particular, N is contractible, and since X is a depth-1 CSS, the map $L \hookrightarrow N$ has a normal framing. Hence L stably frameable. Moreover, the map $L \to M$ is a smooth fibre bundle with closed smooth fibre Z over the contractible smooth manifold M, making Z also stably frameable.[14] But then $Z \simeq Z \times M \simeq L \simeq \Lambda$ – a contradiction. $\square$

6.2.3. *Class III: contractible local links.* Recall that a space is called a *homology sphere* if its singular homology coincides with that of a sphere. Lefschetz duality implies that the boundary of a compact contractible manifold is a homology sphere. This leads to the following:

Corollary 6.18. *Let Y be a closed smooth manifold which is not a homology sphere and let $Y \hookrightarrow \mathbb{R}^K$ be a closed embedding. Then the finite $[1]$-layered category $\mathbf{EX}(Y \xleftarrow{\mathrm{id}} Y \hookrightarrow \mathbb{R}^K)$ has contractible local links and is not equivalent to the EPC of a compact CSS.*

Proof. For simplicity, let us assume Y connected. Suppose X is a compact CSS with associated linked space $(M \leftarrow L \rightarrow N)$. Using Theorems 5.1 and 6.14 we obtain a homotopy equivalence $(Y \leftarrow Y \rightarrow \mathbb{R}^K) \simeq (M \leftarrow L \rightarrow N)$ of spans. In particular, M is connected and N is contractible. Let now $x \in M$ be given and let $U \cong \mathbb{R}^{n-1} \times C(Z)$ be a basic open centred at x. Then $L \rightarrow M$ is a smooth Z-bundle homotopy equivalent to Y over a base that is homotopy equivalent to Y, so Z is contractible. Since Z is a closed smooth manifold, we obtain $Z = *$. But then $U \cong \mathbb{H}^n$ is the n-dimensional upper half-space, and the conically smooth atlas on X gives it the structure of a compact smooth manifold with boundary such that $M = \partial X$ and $N = X^\circ$. But then M is a homology sphere – a contradiction. $\square$

6.2.4. *The proof of Theorem 6.14.* Let us observe that the existence of different classes of maps of linked spaces depending on the accompanying poset maps is also reflected in maps between EPCs. Throughout, let $([1], \mathfrak{S}), ([1], \mathfrak{T}) \in \mathbf{LS}$.

Notation. We write $\mathbf{F}_0 \subset \mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T}))$ for the subspace consisting of those functors $F\colon \mathbf{EX}(\mathfrak{S}) \rightarrow \mathbf{EX}(\mathfrak{T})$ which factor through $M' = \mathfrak{T}_0$; $\mathbf{F}_1$ for those which factor through $N' = \mathfrak{T}_1$, and $\mathbf{F}_{01} = \mathrm{Hom}_{\mathbf{Cat}_\infty}(\mathbf{EX}(\mathfrak{S}, \mathbf{EX}(\mathfrak{T}))) \smallsetminus (\mathbf{F}_0 \cup \mathbf{F}_1)$.

Lemma 6.19. $\mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T})) = \mathbf{F}_0 \amalg \mathbf{F}_1 \amalg \mathbf{F}_{01}$.

Proof. The paths in question are natural isomorphisms. $\square$

Definition 6.20. Let $\mathfrak{S}$ and $\mathfrak{T}$ be two linked spaces. We call the image of $\mathbf{EX}\colon \mathrm{Hom}_{\mathbf{LS}}(\mathfrak{S}, \mathfrak{T}) \rightarrow \mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T}))$ the space of *tame maps* (and higher tame homotopies), and will denote it by $\mathrm{Hom}^\tau(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T}))$.

[14] Let $\mathrm{T}L \oplus \varepsilon^k \cong \varepsilon^N$ be a stable framing on L. Over a contractible trivialising neighbourhood $U \subset M$ we have a bundle isomorphism $\mathrm{T}(Z \times U) \cong \mathrm{T}Z \oplus \varepsilon^{\dim M}$. Since $Z \times U$ is open in L, we have $\varepsilon^N \cong \mathrm{T}L|_{Z \times U} \oplus \varepsilon^k \cong \mathrm{T}(Z \times U) \oplus \varepsilon^k \cong \mathrm{T}Z \oplus \varepsilon^{\dim M} \oplus \varepsilon^k$, yielding a stable framing on Z.

Remark 6.21. Let $F\colon \mathbf{EX}(\mathfrak{S}) \to \mathbf{EX}(\mathfrak{T})$ be given and suppose, without loss of generality, that $F \in \mathbf{F}_{01}$. The induced map $F_*\colon \mathbf{EX}(\mathfrak{S})^{\Delta[1]} \to \mathbf{EX}(\mathfrak{T})^{\Delta[1]}$, which we will call the *derivative* of F, restricts to the map

$$F_*\colon (M \downarrow N) \to (M' \downarrow N').$$

Note, however, that F induces another, similar map:

$$(F|_N)'_*\colon (M \downarrow N) \xrightarrow{\Phi} (L \downarrow N) \xrightarrow{(F|_N)_*} (N')^{\Delta^1}$$

where Φ is the isomorphism from Lemma 4.11, with inverse Ψ. If F is tame, then (recall (6.12) and (6.13)) $(F|_N)_*$ factors through $(L' \downarrow N') \hookrightarrow (N')^{\Delta^1}$ and F_* is in fact equal to the resulting map

$$(M \downarrow N) \xrightarrow{\Phi} (L \downarrow N) \xrightarrow{(F|_N)_*} (L' \downarrow N') \xrightarrow{\Psi} (M' \downarrow N').$$

If F is not tame, this need not be true, as the following example illustrates in the simplest case.

Example 6.22. Recall from Example 6.8 the linked space $\mathrm{sp} = (* \leftarrow * \to *)$ and that a map $\mathrm{sp} \to \mathfrak{S}$ is determined by a single link point $l \in L$, which is in turn determined by the exit path $\boxed{l} = (s(l), 1)$ with degenerate underlying path in N. Conversely, all tame maps are of this type, but not every map $F\colon \mathbf{EX}(\mathrm{sp}) = *^{\triangleleft} \to \mathbf{EX}(\mathfrak{S})$ is of this type; instead, F is determined by an arbitrary exit path $F(s(*,1)) = (\gamma, 1)$ where γ need not be constant. Nevertheless, F is homotopic to the tame map $F^t = (\pi(\gamma(0)), \gamma(0), \gamma(0))_!\colon \mathbf{EX}(\mathrm{sp}) \to \mathbf{EX}(\mathfrak{S})$.

In order to prove Theorem 6.14, we will show that the map on morphism spaces is an equivalence by showing that it in turn is essentially surjective and fully faithful. We start with the former:

Lemma 6.23. *Every map $F\colon \mathbf{EX}(\mathfrak{S}) \to \mathbf{EX}(\mathfrak{T})$ is homotopic to a tame map.*

Proof. Suppose $F \in \mathbf{F}_{01}$; the cases $F \in \mathcal{F}_0$ and $F \in \mathcal{F}_1$ are analogous. The induced map

$$h^F\colon L \overset{\square}{\hookrightarrow} (M \downarrow N) \xrightarrow{F_*} (M' \downarrow N') \overset{\Phi}{\cong} (L' \downarrow N'),$$

fits into the diagram

$$\begin{tikzcd} L \arrow[r, "h^F"] \arrow[d, hook, "\iota"'] & (N')^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ N \arrow[r, "F|_N"'] \arrow[ur, dotted, "H^F"] & N' \end{tikzcd}$$

where the lift H^F exists because ι is a cofibration. Let us write $h_0^F\colon L \to L'$ for the evaluation at 0, and similarly for $H_0^F\colon N \to N'$. Now, the triple

$$F^\tau = (F|_M, h_0^F, H_0^F)\colon \mathfrak{S} \to \mathfrak{T}$$

is a span map, and we claim that there is a homotopy

$$F \sim F_!^\tau$$

in $\mathrm{Hom}_{\mathbf{C}\mathrm{at}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T}))$.

On M, we take the homotopy to be constant, and on N we take it to be given by H itself. It remains to extend it to exit paths. First, observe that such an extension can be extended from a homotopy of the following type. There are two induced derivatives

$$F'_*, (H_0^F)_* \colon (L \downarrow N) \to (L' \downarrow N')$$

where we write

$$F'_* = \Phi' F_* \Psi \colon (L \downarrow N) \xrightarrow{\Psi} (M \downarrow N) \xrightarrow{F_*} (M' \downarrow N') \xrightarrow{\Phi'} (L' \downarrow N').$$

Note that $(\gamma, e) = \mathcal{S}_e^* \mathcal{C}_e^*(\gamma, e) = \mathcal{S}_e^* \Psi \Phi \mathcal{C}_e^*(\gamma, e)$, thus

$$\begin{aligned} F(\gamma, e) &= \mathcal{S}_e^* F_* \mathcal{C}_e^*(\gamma, e) \\ &= \mathcal{S}_e^* \Psi' F'_* \Phi \mathcal{C}_e^*(\gamma, e) \end{aligned}$$

and similarly for H_0^F. Consequently, if there is a homotopy

$$(\mathbf{H} \colon F'_* \to (H_0^F)_*) \colon (L \downarrow N) \to (L' \downarrow N')^{\Delta[1]}$$

which it covers the homotopy $(\mathrm{id}_{h_0^F}, H^F) \colon (L \times N) \to \mathrm{Map}(\Delta^1, L' \times N')$ along the evaluations at 0 and 1, it would induce, with a slight abuse of notation, the full homotopy on exit paths:

$$\begin{aligned} \mathbf{H} \colon \mathbf{EX}(\mathfrak{S}) &\to \mathrm{Fun}(\Delta[1], \mathbf{EX}(\mathfrak{T})) \\ (\gamma, e) &\mapsto (\mathbf{H} \colon \mathcal{S}_e^* \Psi' F'_* \Phi \mathcal{C}_e^*(\gamma, e) \to \mathcal{S}_e^* \Psi' (H_0^F)_* \Phi \mathcal{C}_e^*(\gamma, e)). \end{aligned}$$

Here, the latter $\mathbf{H}$ stands more precisely for Ψ' applied to the map

$$\Delta[k] \times \Delta[1] \xrightarrow{\mathcal{S}_e \times \mathrm{id}} \Delta[1] \times \Delta[k-1] \times \Delta[1] \xrightarrow{\mathbf{H}(\Phi \mathcal{C}_e^*(\gamma, e))} N' \tag{6.24}$$

where we used $\mathbf{H}(\Phi \mathcal{C}_e^*(\gamma, e)) \in ((L' \downarrow N')^{\Delta[1]})_{k-1} \subset \mathrm{Hom}(\Delta[1] \times \Delta[k-1], \mathrm{Fun}(\Delta[1], N')) \cong \mathrm{Hom}(\Delta[1] \times \Delta[k-1] \times \Delta[1], N')$. The condition that $\mathbf{H}$ cover $(\mathrm{id}_{h_0^F}, H^F)$ ensures that the above, together with its restrictions to M and N as already specified, is functorial. Since $(F^\tau)'_* = (H_0^F)_*$ holds by construction (recall (6.12) and (6.13)), this would be a homotopy of the required type.

It remains to construct the desired homotopy $\widehat{H} \colon F'_* \to (H_0^F)_*$.[15] We will perform the construction in two steps, by first giving a homotopy $K_1 \colon F'_* \to F''_*$ to an auxiliary map F''_* that is implicit in F itself, and then a homotopy $K_2 \colon F''_* \to (H_0^F)_*$ that will be induced by H^F.

[15] Of course F'_* and $(H_0^F)'_*$ are homotopic since their restrictions along the equivalence $\Phi\Box \colon L \hookrightarrow (L \downarrow N)$ coincide, but a priori not over $(\mathrm{id}_{h_0^F}, H^F)$. This will be evident in our construction.

In order to construct K_1, let $(\gamma, 1) \in (M \downarrow N)_0 \subset \mathbf{EX}_1(\mathfrak{S})$ be an exit 1-path starting at L. We first observe that $(\gamma, 1)$ is canonically a composition of $\boxed{\gamma_0} = (s_0\gamma(0), 1)$ and γ, and this is witnessed by the exit 2-path $(s_0\gamma, 1) \in \mathbf{EX}_2(\mathfrak{S})$. Consequently, $F(\gamma, 1)$ is canonically the composition of the image of a constant exit loop and the image of a path under $F|_N$:

$$(s_0\gamma, 1) = \left(\begin{tikzcd} \bullet \arrow[r, "\gamma"] & \bullet \\ \bullet \arrow[u, "\boxed{\gamma_0}"] \arrow[ur, "(\gamma,1)"'] & \end{tikzcd}\right) \mapsto F(s_0\gamma, 1) = \left(\begin{tikzcd} \bullet \arrow[r, "F|_N\gamma"] & \bullet \\ \bullet \arrow[u, "F_*\boxed{\gamma_0}"] \arrow[ur, "F(\gamma,1)"'] & \end{tikzcd}\right).$$

Note that $F_*\boxed{\gamma_0} = h^F(\gamma(0))$ by definition. Now, the map

$$\begin{aligned} F''_*\colon (L \downarrow N) &\xrightarrow{(h^F \circ \mathrm{ev}_0) \times (F|_N)_*} \mathrm{Map}(\Delta^1, N') \times_{N'} \mathrm{Map}(\Delta^1, N') \\ &\cong \mathrm{Map}(\Delta^1_1, N') \\ &\xrightarrow{E_1^*} \mathrm{Map}(\Delta^1, N') \end{aligned}$$

factors by construction through $(L' \downarrow N') \subset \mathrm{Map}(\Delta^1, N')$. Here, the left leg of the pullback is evaluation at 1 and the right leg is evaluation at 0, and the map $E_1\colon \Delta^1 \to \Delta^1_1 = \Delta^1 \vee \Delta^1$ is the natural map defined in Construction 5.5. We have thus defined the map $F''_*\colon (L \downarrow N) \to (L' \downarrow N')$. The consideration above shows that it is homotopic to F'_*; an explicit homotopy $K_1\colon F'_* \to F''_*$ is given by

$$\begin{aligned} K_1\colon (L \downarrow N) &\to \mathrm{Map}(\Delta^1, (L' \downarrow N')) \\ \gamma &\mapsto F'_*\mathcal{C}_1^* s_0\gamma. \end{aligned}$$

Indeed, the map $\Phi(\gamma, 1) \mapsto F'_*\Phi\mathcal{C}_1^*(s_0\gamma, 1)$ is the desired homotopy, but $\Phi(\gamma, 1) = \gamma$ and it is a direct check that $\Phi\mathcal{C}_1^*(s_0\gamma, 1) = \mathcal{C}_1^* s_0\gamma$ using the identities $\mathcal{S}_1^*\mathcal{C}_1^* = \mathrm{id}$ and $\mathcal{S}_2^*\mathcal{C}_1^* = s_0 d_1$ to show that the two non-degenerate 2-simplices of the square $\Phi\mathcal{C}_1^*(s_0\gamma, 1)$ are both given by $s_0\gamma$.[16]

Finally, $K_2\colon F''_* \to (H_0^F)_*$ is given by gluing the homotopy $h^F \circ \mathrm{ev}_0 \to h_0^F$, given by $h^F = H^F|_L$ itself, with the homotopy $(F|_N)_* \to (H_0^F)_*$, likewise given by H^F itself (or more precisely by $(H^F)^{-1}$). Concatenating K_1 and K_2 yields the desired homotopy $F'_* \to (H_0^F)'_*$. Since both K_1 and K_2 cover $(\mathrm{id}_{h_0^F}, H^F)$ by construction, we obtain the full homotopy $\mathbf{H}\colon F \to F_!^\tau$. $\square$

The following natural construction will let us generalise the method of proof of Lemma 6.23.

Construction 6.25. Recall the bijections

$$\mathrm{Hom}_{\mathrm{sSet}}(A \times B, C) \cong \mathrm{Hom}_{\mathrm{sSet}}(A, \mathrm{Fun}(B, C))$$

[16] This shows that $\mathcal{S}_2^*\mathcal{C}_1^*(s_0\gamma, 1) = (s_0\gamma, 2)$, hence Φ applied to it yields $s_0\gamma$.

which induce an isomorphism $\mathrm{Fun}(A, \mathrm{Fun}(B,C)) \cong \mathrm{Fun}(A \times B, C)$ of simplicial sets for any three simplicial sets A, B, C. Consequently, any map $F\colon \mathbf{C} \times X \to \mathbf{D}$ induces a map

$$F_*\colon \mathrm{Ar}(\mathbf{C}) \times X \to \mathrm{Ar}(\mathbf{D})$$

by writing F as $\mathbf{C} \to \mathrm{Fun}(X, \mathbf{D})$, obtaining $\mathrm{Ar}(\mathbf{C}) \to \mathrm{Ar}(\mathrm{Fun}(X,\mathbf{D})) \cong \mathrm{Fun}(\Delta[1] \times X, \mathbf{D}) \cong \mathrm{Fun}(X, \mathrm{Ar}(\mathbf{D}))$ and finally rewriting as $\mathrm{Ar}(\mathbf{C}) \times X \to \mathrm{Ar}(\mathbf{D})$.

Before we proceed to the proof of Theorem 6.14, we will prove a generalisation of Lemma 6.23 which is a corollary to its proof technique. Given a space X and a linked space $\mathfrak{S}$, let us denote by $\mathfrak{S} \times X$ the linked space given by term-wise cartesian product with X.

Lemma 6.26. *Given a space X and a map $\gamma\colon \mathbf{EX}(\mathfrak{S}) \times X \to \mathbf{EX}(\mathfrak{T})$, there exists a homotopy*

$$\mathbf{H}\colon \mathbf{EX}(\mathfrak{S}) \times X \times \Delta^1 \to \mathbf{EX}(\mathfrak{T})$$

from γ to $\gamma^\tau_! = \mathbf{H}|_1$, where

$$\gamma^\tau\colon \mathfrak{S} \times X \to \mathfrak{T}$$

is a tame map. Moreover, for every point $x \in X$ at which the restriction $\gamma|_x\colon \mathbf{EX}(\mathfrak{T}) \to \mathbf{EX}(\mathfrak{T})$ is tame, the restricted homotopy

$$(\mathbf{H}|_x\colon \gamma|_x \to (\gamma^\tau|_x)_!)\colon \mathbf{EX}(\mathfrak{S}) \times \Delta^1 \to \mathbf{EX}(\mathfrak{T})$$

is also tame.

Proof. Suppose $\gamma \in \mathbf{F}_{01}$ in the sense that the restriction $\gamma|_{Y \times X}$ factors through Y' for $Y = M, N$. The remaining cases are analogous. Using Construction 6.25 we obtain the map

$$\gamma_*\colon \mathrm{Ar}(\mathbf{EX}(\mathfrak{S})) \times X \to \mathrm{Ar}(\mathbf{EX}(\mathfrak{T}))$$

which induces the map

$$h^\gamma\colon L \times X \xhookrightarrow{\square \times \mathrm{id}} (M \downarrow N) \times X \xrightarrow{\gamma_*} (M' \downarrow N') \cong (L' \downarrow N') \subset (N')^{\Delta^1}$$

which in turn yields the commuting square

$$\begin{tikzcd} L \times X \arrow[r, "h^\gamma"] \arrow[d, hook, "\iota \times \mathrm{id}"'] & (N')^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ N \times X \arrow[r, "\gamma|_N"'] \arrow[ur, dotted, "H^\gamma"] & N' \end{tikzcd}$$

with a homotopy extension H^γ. We set

$$\gamma^\tau = (\gamma|_{M \times X}, h^\gamma_0, H^\gamma_0)\colon \mathfrak{S} \times X \to \mathfrak{T}.$$

The desired homotopy in the form

$$\mathbf{H}\colon \mathbf{EX}(\mathfrak{S}) \times X \to \mathrm{Fun}(\Delta[1], \mathbf{EX}(\mathfrak{T}))$$

can now be constructed similarly to Lemma 6.23. We will provide the details for completeness.

We claim that the derivatives $\gamma'_*, (H_0^\gamma)_*\colon (L \downarrow N) \times X \to (L' \downarrow N')$ are homotopic over $(\mathrm{id}_{h_0^\gamma}, H^\gamma)\colon L \times N \times X \to (L' \times N')^{\Delta^1}$ via a homotopy

$$\mathbf{H}\colon (L \downarrow N) \times X \to (L' \downarrow N')^{\Delta^1}$$

which will, for $((\gamma, e), \alpha) \in \mathbf{EX}_k(\mathfrak{S}) \times X_k$, yield the desired homotopy

$$\Psi' \mathcal{S}_e^* \mathbf{H}(\Phi \mathcal{C}_e^*(\gamma, e), \alpha)\colon \mathcal{S}_e^* \Psi' \gamma'_*(\Phi \mathcal{C}_e^*(\gamma, e), \alpha) \to \mathcal{S}_e^* \Psi' (H_0^\gamma)_*(\Phi \mathcal{C}_e^*(\gamma, e), \alpha)$$

with $\mathcal{S}_e^* \mathbf{H}$ defined as in (6.24). Its first half $K_1\colon \gamma'_* \to \gamma''_*$ is constructed as follows. The maps $(L \downarrow N) \times X \xrightarrow{\mathrm{ev}_0 \times \mathrm{id}} L \times X \xrightarrow{h^\gamma} \mathrm{Map}(\Delta^1, N')$ and $(L \downarrow N) \times X \xrightarrow{(\gamma|_{N \times X})_*} \mathrm{Map}(\Delta^1, N')$ yield

$$\gamma''_*\colon (L \downarrow N) \times X \xrightarrow{(h^\gamma \circ (\mathrm{ev}_0 \times \mathrm{id})) \vee (\gamma|_{N \times X})_*} \mathrm{Map}(\Delta^1_1, N') \xrightarrow{E_1^*} \mathrm{Map}(\Delta^1, N').$$

For each $\delta \in (L \downarrow N)$ we have $\mathcal{C}_1^* s_0 \delta \in \mathrm{Map}(\Delta^1, (L \downarrow N))$, applying $\gamma'_*\colon (L \downarrow N) \to \mathrm{Map}(X, (L' \downarrow N'))$ to which yields $K_1\colon (L \downarrow N) \to \mathrm{Map}(\Delta^1, \mathrm{Map}(X, (L' \downarrow N')))$, $\delta \mapsto \gamma'_* \mathcal{C}_1^* s_0 \delta$. Finally, the second half $K_2\colon \gamma''_* \to (H_0^\gamma)_*$ is induced by h^γ and $(H^\gamma)^{-1}$ exactly as before.

As for the second statement, suppose

$$\gamma|_x = (\gamma|_{M \times \{x\}}, (\gamma|_x)_L, \gamma|_{N \times \{x\}})\colon \mathfrak{S} \to \mathfrak{T}$$

is a span map and observe, using the fact that $\Phi \circ \square = \mathrm{const}$ is the ordinary constant loop inclusion, that

$$\begin{tikzcd} L' \arrow[r, "\mathrm{const}"] & (L')^{\Delta^1} \arrow[d, "\iota'"] \\ L \times \{x\} \arrow[u, "(\gamma|_x)_L"] \arrow[r, "h^\gamma|_x"] & (N')^{\Delta^1} \end{tikzcd}$$

commutes, as does therefore

$$\begin{tikzcd} L \times \{x\} \arrow[r, "\mathrm{const} \circ (\gamma|_x)_L"] \arrow[d, "\iota"] & (L')^{\Delta^1} \arrow[d, "\iota'"] \\ N \times \{x\} \arrow[r, "H^\gamma|_x"] & (N')^{\Delta^1} \end{tikzcd}$$

using the defining property of H^γ. In particular we have $h_0^\gamma|_x = (\gamma|_x)_L$, implying

$$(\gamma^\tau|_x) = (\gamma|_{M \times \{x\}}, (\gamma|_x)_L, H_0^\gamma|_x).$$

Consequently, since $H_1^\gamma|_x = \gamma|_{N \times \{x\}}$ by construction, we obtain the homotopy

$$\mathcal{H}_x = (\mathrm{id}, \mathrm{id}, H^\gamma|_x)\colon \gamma^\tau|_x \to \gamma|_x \tag{6.27}$$

of spans maps $\mathfrak{S} \to \mathfrak{T}$. We claim not only that $\mathbf{H}|_x$ is tame, but that it is tamely homotopic to $(\mathcal{H}_x)^{-1}$.

Now, the map

$$h^\gamma \circ (\mathrm{ev}_0 \times \mathrm{id})|_x \colon (L \downarrow N) \times \{x\} \to \mathrm{Map}(\Delta^1, N')$$

coincides with $\mathrm{const} \circ (\gamma|_x)_L \circ \mathrm{ev}_0$ and so γ''_* coincides with $(\gamma|_{N\times X})_*$ up to reparametrisation (due to the pre-composition with E_1), hence $K_1|_x$ is the constant homotopy up to reparametrisation. Similarly, $K_2|_x$ is induced only by H^γ up to reparametrisation, hence so is $\mathbf{H}|_x$. In particular, writing $\mathbf{H}^\tau|_x$ for the underlying tame homotopy of $\mathbf{H}|_x = (\mathbf{H}^\tau|_x)_!$, reparametrisation provides a homotopy

$$(\mathcal{H}_x)^{-1} \sim \mathbf{H}^\tau|_x, \tag{6.28}$$

a path in $\mathrm{Hom}_{\mathrm{Hom}(\mathfrak{S},\mathfrak{T})}(\gamma|_x, \gamma^\tau|_x)$. □

Proof of Theorem 6.14. By Lemma 6.23, it remains to show that the map $\mathrm{Hom}_{\mathbf{LS}}(\mathfrak{S},\mathfrak{T}) \to \mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T}))$ itself is fully faithful, that is, that for all $F, G \colon \mathfrak{S} \to \mathfrak{T}$ the map

$$\mathrm{Hom}_{\mathrm{Hom}_{\mathbf{LS}}(\mathfrak{S},\mathfrak{T})}(F, G) \to \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}),\mathbf{EX}(\mathfrak{T}))}(F_!, G_!)$$

is an equivalence. We will first show that it is essentially surjective.

Let $\gamma \colon \mathbf{EX}(\mathfrak{S}) \times \Delta^1 \to \mathbf{EX}(\mathfrak{T})$ be a path $F_! \to G_!$, an object of the target. By Lemma 6.26 we obtain a homotopy

$$(\mathbf{H} \colon \gamma \to \gamma^\tau_!) \colon \mathbf{EX}(\mathfrak{S}) \times \Delta^1 \times \Delta^1 \to \mathbf{EX}(\mathfrak{S}),$$

where $\gamma^\tau = (\gamma|_{M\times\Delta^1}, h^\gamma_0, H^\gamma_0) \colon \gamma^\tau_0 \to \gamma^\tau_1$ is a path in $\mathrm{Hom}_{\mathbf{LS}}(\gamma^\tau_0, \gamma^\tau_1)$, and in fact $\gamma^\tau_0 = (F_M, F_L, H^\gamma_0|_0)$ and $\gamma^\tau_1 = (G(m), G_L, H^\gamma_0|_1)$. Moreover, $\mathbf{H}$ restricts to the paths

$$\mathbf{H}_0 = (\mathbf{H}|_{\{0\}\times\Delta[1]} \colon F_! \to (\gamma^\tau_0)_!),\ \mathbf{H}_1 = (\mathbf{H}|_{\{1\}\times\Delta[1]} \colon G_! \to (\gamma^\tau_1)_!)$$

in $\mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T}))$ and provides equivalently a path

$$\mathbf{H} \colon \gamma \to (\mathbf{H}_0 * \gamma^\tau_! * \mathbf{H}_1^{-1}) \tag{6.29}$$

in $\mathrm{Hom}_{\mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}),\mathbf{EX}(\mathfrak{T}))}(F_!, G_!)$.

On the other hand, we have the homotopies $\mathcal{H}_0 \colon \gamma^\tau_0 \to F$ and $\mathcal{H}_1 \colon \gamma^\tau_1 \to G$ as in (6.28), yielding the homotopy equivalence

$$\Upsilon \colon \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{LS}}(\mathfrak{S},\mathfrak{T})}(\gamma^\tau_0, \gamma^\tau_1) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{LS}}(\mathfrak{S},\mathfrak{T})}(F, G),$$

$\delta \mapsto \mathcal{H}_0^{-1} * \delta * \mathcal{H}_1$. This yields the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Hom}_{\mathbf{LS}}(\mathfrak{S},\mathfrak{T})}(F, G) \arrow[r] \arrow[d, "\Upsilon^{-1}"] & \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{T}),\mathbf{EX}(\mathfrak{T}))}(F_!, G_!) \\
\mathrm{Hom}_{\mathrm{Hom}_{\mathbf{LS}}(\mathfrak{S},\mathfrak{T})}(\gamma^\tau_0, \gamma^\tau_1) \arrow[r] & \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{T}),\mathbf{EX}(\mathfrak{T}))}((\gamma^\tau_0)_!, (\gamma^\tau_1)_!) \arrow[u, "\Upsilon_!"']
\end{tikzcd}$$

which homotopy-commutes due to the functoriality of $\mathbf{EX}$.

Let now $\Gamma^\tau\colon F \to G$ be a path such that $\Upsilon(\gamma^\tau) \sim \Gamma^\tau$. We claim that $\Gamma^\tau_! \sim \gamma$, proving the essential surjectivity of the map $\mathrm{Hom}(F,G) \to \mathrm{Hom}(F_!, G_!)$. Indeed, since $(\mathcal{H}_x^{-1})_! \simeq \mathbf{H}_x$ for $x = 0, 1$ by (6.28), we observe, using (6.29), the homotopy $\Gamma^\tau_! \sim \Upsilon_!(\gamma^\tau_!) = (\mathcal{H}_0^{-1})_! * \gamma^\tau_! * (\mathcal{H}_1^{-1})_! \sim \gamma$.

What we have proved so far can be reformulated as follows. Lemma 6.23 shows that

$$\pi_0\mathbf{EX}\colon \pi_0\mathrm{Hom}_{\mathbf{LS}}(\mathfrak{S}, \mathfrak{T}) \to \pi_0\mathrm{Hom}_{\mathbf{Cat}_{\mathrm{Top}}}(\mathbf{EX}(\mathfrak{S}), \mathbf{EX}(\mathfrak{T}))$$

is surjective, and we have proved that for any $F, G \in \mathrm{Hom}(\mathfrak{S}, \mathfrak{T})$, the map

$$\Pi_1\mathbf{EX}\colon \pi_0\mathrm{Hom}(F,G) \to \pi_0\mathrm{Hom}(F_!, G_!)$$

is also surjective. In particular this implies that $\pi_0\mathbf{EX}$ is injective and so bijective, and that $\pi_1\mathbf{EX}$, with respect to any basepoint, is surjective. The fact that $\mathbf{EX}$ induces isomorphisms on all homotopy groups can now be seen in a similar fashion.

Let $n \geq 1$ and assume by induction that $\pi_i\mathbf{EX}$ is an isomorphism for $i \in \{0, \ldots, n-1\}$ and that $\pi_n\mathbf{EX}$ is surjective. It suffices to prove, for every $F, G \in \Omega^n\mathrm{Hom}(\mathfrak{S}, \mathfrak{T})$ in the n-fold loop space of $\mathrm{Hom}(\mathfrak{S}, \mathfrak{T})$ with respect to any basepoint, that the map

$$\Pi_{n+1}\mathbf{EX}\colon \pi_0\mathrm{Hom}_{\Omega^n\mathrm{Hom}(\mathfrak{S},\mathfrak{T})}(F,G) \to \pi_0\mathrm{Hom}_{\Omega^n\mathrm{Hom}(\mathbf{EX}(\mathfrak{S}),\mathbf{EX}(\mathfrak{T}))}(F_!, G_!)$$

is surjective, yielding in particular the injectivity of $\pi_n\mathbf{EX}$.

Let us write $\mathbb{S}^n$ for the simplicial n-sphere and let $\gamma\colon \mathbf{EX}(\mathfrak{S}) \times \mathbb{S}^n \times \Delta^1 \to \mathbf{EX}(\mathfrak{T})$ be a path $F_! \to G_!$, representing an element in the target of $\Pi_{n+1}\mathbf{EX}$. By Lemma 6.26 we obtain a homotopy

$$(\mathbf{H}\colon \gamma \to \gamma^\tau_!)\colon \mathbf{EX}(\mathfrak{S}) \times \mathbb{S}^n \times \Delta^1 \times \Delta^1 \to \mathbf{EX}(\mathfrak{T}),$$

where the path $\gamma^\tau = (\gamma|_{M \times \mathbb{S}^n \times \Delta^1}, h_0^\gamma, H_0^\gamma)\colon \gamma_0^\tau \to \gamma_1^\tau$ is tame. Note that the second statement of Lemma 6.26 also holds for subspaces with the same proof: if $Y \subset X$ is a subspace such that the restriction $\gamma|_Y\colon \mathbf{EX}(\mathfrak{S}) \times Y \to \mathbf{EX}(\mathfrak{T})$ is tame in the sense that $\gamma = \gamma^\tau_!$ for a tame map $\gamma^\tau\colon \mathfrak{S} \times Y \to \mathfrak{T}$, then the restricted homotopy $\mathbf{H}_Y\colon \mathbf{EX}(\mathfrak{S}) \times Y \times \Delta^1 \to \mathbf{EX}(\mathfrak{T})$ is also tame. The construction of a homotopy-preimage Γ^τ in $\mathrm{Hom}_{\Omega^n\mathrm{Hom}(\mathfrak{S},\mathfrak{T})}(F,G)$ proceeds now as above upon replacing $x = 0, 1 \in \Delta^1$ everywhere by the subspaces $Y = \mathbb{S}^n \times \{i\} \subset X = \mathbb{S}^n \times \Delta^1$, $i \in \{0, 1\}$. This proves that $\Pi_{n+1}\mathbf{EX}$ is surjective and that $\pi_n\mathbf{EX}$ is injective. $\square$

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