QUADRATIC DUALS OF LINKED SPACES
V0.4

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ABSTRACT.

CONTENTS

1. INTRODUCTION

1.1. **Summary of results written down so far.**

Section 2.1. Passage from the simplicial nerve version of the stratified Grassmannian (written $\mathcal{V}^{\hookrightarrow}$) to the homotopy coherent nerve version. No new results.

Section 2.2. Given $n, m \in \mathbb{N}$, the exit path ∞-category $\mathcal{EX}(BO(n, m))$, in the sense of [Tet23], of the (n, m)-*Grassmannian*

$$BO(n, m) = \left(BO(n) \xleftarrow{\mathrm{pr}_1} BO(n) \times BO(m) \xrightarrow{\boxplus} BO(n + m) \right)$$

has objects $BO(n)_0 \amalg BO(n + m)_0$, which canonically map to $\mathcal{V}_0^{\hookrightarrow}$ (cf. ([Tet22])). Moreover, a non-invertible path $(\gamma, 1) \in \mathcal{P}_0^{\Delta} \subset \mathcal{EX}(BO(n, m))_1$ corresponds to a path γ in $BO(n + m)$ of type

$$V_{12} \boxplus V_{01} \to V_{02},$$

where $V_{01} \in BO(n)$ and $V_{12} \in BO(m)$. This datum arranges as the morphism

in $\mathcal{V}_1^{\hookrightarrow}$. Thus, we have an assignment

$$\mathcal{EX}_{\leq 1}(BO(n, m)) \to \mathcal{V}_{\leq 1}^{\hookrightarrow}$$

Theorem (2.2). *This assignment extends to an ∞-functor*

$$\mathbb{U} \colon \mathcal{EX}(BO(n, m)) \to \mathcal{V}^{\hookrightarrow}.$$

Section 2.3. Given a *linked manifold*

$$\mathfrak{S} = \left(M \xrightarrow{\pi} L \xhookrightarrow{\iota} N \right),$$

that is, a triple of smooth manifolds where π is a fibre bundle and ι is an embedding, we construct (Construction 2.10) its tangent bundle in terms of purely smooth data:

$$\mathrm{T}\mathfrak{S} \colon \mathfrak{S} \to BO(n, m).$$

This is informed by an informal discussion present in the conically-smooth setting ([AFR18b]) which becomes completely natural in the linked setting. Namely, we give it as the following span map (cf. Remark 2.11):
(1)

This yields an ∞-functor $\mathcal{EX}(\mathfrak{S}) \to \mathcal{V}^{\hookrightarrow}$ which we also call the tangent bundle of $\mathfrak{S}$.

Section 3.1. Tangential structures in the stratified setting, which are ∞-categories over $\mathcal{V}^{\hookrightarrow}$, pull back to the linked setting:

$$\mathbb{U}^* \colon \mathrm{Cat}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathrm{Cat}_\infty/\mathcal{EX}(BO(n,m)).$$

We introduce the problem of finding a linked space $\mathfrak{B}$ such that

$$\mathfrak{U}^*\mathcal{B} = \mathcal{EX}(\mathfrak{B})$$

for a stratified tangential structure $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, and solve it in some simple cases: framings of fixed rank (Example 3.1); classical tangential structures given by space maps of type $B \to BO(\kappa) \subset \mathcal{V}^{\hookrightarrow}$ (Example 3.2); and variframings, in depth 1 but of arbitrary codimension (Example 3.3). We also discuss variframed point defects (Example 3.4).

Sections 3.2 and 3.3. Given a tangential structure $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, the main problem of performing factorisation homology, given a $\mathcal{B}$-structured disk algebra, on bordisms is that bordisms do not in general have $\mathcal{B}$-structure, but may be asked instead to have a relaxation of it. The most rigid relaxation is the *stable* one (for fixed maximal rank, namely that of the dimension of the disk algebra), used implicitly in [Lur08]. A further relaxation, the *solid* one, was considered in the stratified setting in [AFR18b]. For instance, if a certain structure is determined by some bundle F over X, a solid F-structure is a bundle injection

$$\mathrm{T}X \hookrightarrow F,$$

while a stable F-structure is a solid one together with a trivialisation of the normal bundle of this injection. We first discuss both relaxations in the smooth setting, and express them as tangential structures over a single Grassmannian: Definition 3.9 gives the solid version, and Definition 3.10 a further-restricted one, the simpliest special case of which recovers the stable version.

This justifies the way in which we transfer such structures to the linked setting. We see stratified solid structures as cartesian fibration replacements of the original $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ à la [GHN17] (see Definition 3.11). Cartesian fibration replacements of 'classical' tangential structures, i.e., those of the form $Y \xrightarrow{F} BO(R) \hookrightarrow \mathcal{V}^{\sim} \subset \mathcal{V}^{\hookrightarrow}$ for Y an ∞-groupoid are for us of particular interest. Typically, F classifes a rank-R vector bundle on Y, or Y is of type BG for a group G over $\mathrm{O}(R)$. We denote the replacement of such an (Y, F) by $\overline{Y}$, and set, for $r \leq R$,

$$\overline{Y}|_r := BO(r) \times_{\mathcal{V}^{\hookrightarrow}} \overline{Y},$$

At this point, determining the morphism spaces in $\mathcal{EX}(\mathfrak{S})$, for $\mathfrak{S}$ a linked space, becomes compulsory. We prove:

Theorem (3.5). *If $p \in M$ and $q \in N$, then $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p,q) \simeq \mathcal{P}_{\mathcal{L}_p,q}$.*

Here, $\mathcal{P}_{\mathcal{L}_p,q}$ is the space of paths in N that start in the (the image of the) fibre of π at p (under ι) and end at q. We then explore some immediate consequences of this result in Corollaries 3.6 and 3.7, concerning linked spaces of two certain types. In particular, we recover the stratified homotopy type of the conically-smooth open cone on a smooth manifold N as $\mathrm{Exit}(C(N)) \simeq \mathcal{EX}\left(* \leftarrow N \xrightarrow{\mathrm{id}} N\right)$. We are then in a position to identify the novel morphism spaces in the stratified Grassmannian. For $p \in BO(r)$ and $q \in BO(R)$, we obtain:

Proposition (3.12). *$\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p,q) \simeq \mathrm{Hom}_{\mathcal{EX}(BO(r,R-r))}(p,q).$*

This leads to the following description of the restricted cartesian replacement $\overline{Y}|_r$ as a classical homotopy pullback:

Proposition (3.14, 3.15). *We have $\overline{Y}|_r \simeq (BO(r) \times BO(R-r)) \times^{\mathrm{h}}_{BO(R)} Y$. Consequently, over a smooth manifold, solid F-structures and cartesian F-structures coincide.*

Section 4 introduces the second main topic of the present work: *collars* and *quadratic duals* of linked spaces with a solid or stable $\mathcal{B}$-structure for $\mathcal{B}$ a classical tangential structure. The picture we present is that the datum induces an 'extension'

$$X \hookrightarrow \overline{X} \overset{p}{\twoheadrightarrow} X^!$$

where the middle term is the collar and the last the dual. The relevance to field theory consists in p being the map along which one pushes forward the disk algebra A of observables with $\mathcal{B}$-structure. In sloganic form:

$$\mathcal{Z}_A = \left(\overline{(-)} \twoheadrightarrow (-)^! \right)_* A.$$

This systematically relaxes factorisation homology in the sense that when the argument bordism X is of top dimension and without boundary, then $\overline{X} = X$ and $X^! = *$, so that the pushforward algebra is simply the (0-disk algebra whose underlying object is) $\int_X A$.

The history of this idea and the differences of our approach to previous ones are explained very briefly at the beginning of this section. The relation to previous sections consists in the fact that the dual $X^!$ parametrises the normal bundle of the solid/stable $\mathcal{B}$-structure on X, while the collar $\overline{X}$ is in fact the 'interior' X° given a new stratification induced by that on X and its solid/stable $\mathcal{B}$-structure.

In Sections 4.1 to 4.3 we immediately discuss many examples with numerous illustrations: mostly bordisms of different codimensions, as well as cutting and gluing – all in the linked setting, which simplifies the theory immensely. Although we restrict ourselves to depth 1 in this work, we also give a teaser on depth 2 by giving a quite explicit discussion of $\overline{X}$ and $X^!$ in the case of bordisms with corners.

Section 4.4 gives a belated account of the category of linked spaces. We propose a way of seeing a smooth space as a linked space in a natural way as well as that of a map of linked spaces in terms of smooth data only, in such a way that maps of linked spaces naturally induce maps ∞-functors on exit paths.

1.2. **Prose.** We give a procedure to evaluate a non-singular disk algebra on singular spaces. This suggests a solution to some extent of a problem posed by Ayala–Francis–Tanaka, extending a construction, due to Lurie, Calaque and Scheimbauer, of framed functorial field theories given bulk/point datum in the form of an $\mathbb{E}_n$-algebra. The core novelty consists in the decomposition of a nice-enough stratified space into its strata and their links, organised as a linked space, which uses only smooth manifold data. We transfer some ideas from the theory of tangential structures in stratified geometry, due to Ayala–Francis–Rozenblyum to this setting, by constructing comparison maps from 'linked' Grassmannians to the stratified Grassmannian. This results in a much more accessible theory of bundles and tangential structures in the linked setting.

The main practical advantage of this approach is that it bypasses a good portion of the complicated higher-categorical treatment of bordism categories in terms of iterated Segal spaces. Concretely, this is because in the Lurie–Calaque–Scheimbauer approach, boundaries and corners are specified by cut functions on a manifold (without boundary) of full dimension, embedded, essentially, in a (colimit of) euclidean space(s). One justication for this is the relatively straightforward manner in which one may then introduce tangential structures on boundaries and corners, by simply introducing one on the containing manifold. On the other hand, this approach introduces many technical issues and impracticalities that are not of immediate interest from the point of view of the field theory itself. In our approach, we reverse the places where the difficulty and ease lie: we adopt the more complicated theory of tangential structures on stratified spaces à la Ayala–Francis–Rozenblyum, which we trade for an easier, more intuitive approach to bordisms, in tune with the Atiyah–Segal–Witten paradigm. In fact, we go further and transfer, as mentioned, the theory of stratified tangential structures to the simpler linked setting, which only uses non-singular data. Even though general linked geometry should be at least as complicated as conically-smooth stratified geometry, the linked spaces induced by bordisms are especially simple, whence the difficult part of this approach is also greatly simplified. It seems, therefore, that for the purpose of connecting BV-type field theory with FFT, the linked approach is 'correct'.

The idea, due to Lurie, of completing a bordism into an honestly framed collar and evaluating the disk algebra on the latter in a particular way in order to construct field theory, is generalised to the linked setting by defining factorisation algebras/homology therein in terms of algebraic coupling data, in such a way that it reproduces some well-known classifying statements in the stratified setting.

Our procedure does not depend intrinsically on the chosen non-singular tangential structure being that of a framing. In particular, it does not rely on Dunn–Lurie additivity, as did Lurie's original construction, and so suggests an extension to any tangential structure. In view of a result of Ginot et al. that generalises Lurie's correspondence between locally-constant factorisation algebras on $\mathbb{R}^n$ and $\mathbb{E}_n$-algebras to one between such on a smooth manifold M and $(M, \mathrm{T}M)$-structured disk algebras, we present, as a special case, a way to construct functorial field theories given observables algebras (and correlators) on *any* smooth manifold, not just on the local $\mathbb{R}^n$. This is achieved at least in the case when bordisms are 'stably-$(M, \mathrm{T}M)$-structured'.

The 'evaluation' of a structured disk algebra on a structured collar of a singular space (such as a bordism), when performed à la Lurie, naturally gives rise to representation-theoretic data, as was further systematised by Calaque–Scheimbauer. The datum of compatible actions is parametrised by especially simple spaces when one evaluates bordisms, namely, euclidean spaces with flag-like stratifications. We obtain such spaces directly from the stable tangential structure on bordisms, and call them their *quadratic duals*. They parametrise the Morita categories which provide the coeffcents of the functorial field theories thus obtained. We show that they have the same shape regardless of tangential structure, but when bordisms are relaxed to be solidly-structured rather than stably, the quadratic duals become more complicated. We pose their construction problem, which is a certain stratified surgery problem, and present ad hoc solutions in some simple cases in low dimensions.

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Conventions. We say *smooth manifold* to refer always to one without boundary. In the linked context, it is understood moreover to be connected.

Definition 1.1. A *linked manifold* (of *depth* 1) is a span

$$
\begin{array}{ccc}
 & L & \\
{\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \iota} \\
M & & N
\end{array}
$$

of smooth manifolds, where π is a proper fibre bundle, and ι a closed embedding.

2. Stratified tangent bundles

For technical reasons, we need to modify our convention for the homotopy-coherent nerve from that in [Lur09] to the one in [Lur23, 00KM]. The former was adopted in [Tet22].

2.1. **The stratified Grassmannian via the homotopy-coherent nerve.** Recall the treatment of the stratified Grassmannian of [AFR18b] given in [Tet22]:

$$
\mathcal{V}^{\hookrightarrow} := */\mathcal{B}^{\boxplus}\mathrm{O},
$$

where

- $\mathcal{B}^{\boxplus}\mathrm{O} := \mathrm{N}^{\boldsymbol{\Delta}}(B^{\boxplus}\mathrm{O})$, where
  - $B^{\boxplus}\mathrm{O}$ is the simplicial category given by delooping the topological monoid $(BO_{\amalg}^{\infty}, \boxplus)$ and taking the singular complex,
  - $\mathrm{N}^{\boldsymbol{\Delta}}$ takes the simplicial nerve as in [Lur09] (see below),
- $*/-$ takes the under ∞-category under the single object $*$ of $\mathcal{B}^{\boxplus}\mathrm{O}$.

Specifically, the simplicial set that is the simplicial nerve of a simplicially-enriched category $\mathcal{A}$ is defined levelwise by setting $\mathrm{N}^{\boldsymbol{\Delta}}(\mathcal{A})_k := \mathrm{Fun}(\mathfrak{C}[k], \mathcal{A})$, the set of functors of simplicially-enriched categories from $\mathfrak{C}[k]$ to $\mathcal{A}$, where

- the objects of $\mathfrak{C}[k]$ are the same as those of $[k]$,
- the simplicial hom-set from i to j, if $i \leq j$, is $\mathrm{Hom}_{\mathfrak{C}[k]}(i, j) := \mathrm{N}(P_{i,j})$, the nerve of the poset of subposets of k with least element i and greatest element j, ordered by inclusion (empty if $i > j$),
- composition is given by taking unions.

We will use N^{hc} instead of $\mathrm{N}^{\boldsymbol{\Delta}}$, which is defined levelwise by $\mathrm{N}^{\mathrm{hc}}(\mathcal{A})_k := \mathrm{Fun}(\mathrm{Path}[k], \mathcal{A})$, where $\mathrm{Path}[k] := \mathfrak{C}[k]^{\mathrm{op}}$. (This can be read as $\mathrm{Path}[k] = \mathrm{N}(P_{i,j}^{\mathrm{op}})$.) We will call functors $\mathrm{Path}[k] \to \mathcal{A}$ *k-paths* in $\mathcal{A}$. To illustrate, let us abuse notation and write

$$
\mathcal{V}^{\hookrightarrow} := */\mathrm{N}^{\mathrm{hc}}(B^{\boxplus}\mathrm{O}).
$$

We then have the bijection $\mathcal{V}_1^{\hookrightarrow} \cong \mathrm{Fun}\left(\mathrm{Path}[2], B^{\boxplus}\mathrm{O}\right)$, and so a morphism in $\mathcal{V}^{\hookrightarrow}$ in this version is uniquely determined by data that can be summed up in the diagram

(2)
$$
\begin{array}{ccc}
 & 1 & \\
{\scriptstyle V_{01}}\nearrow & & \searrow{\scriptstyle V_{12}} \\
0 & \xrightarrow{\;\; V_{02} \;\;} & 2 \\
 & \Uparrow & \\
 & V_{12}\boxplus V_{01} &
\end{array}
$$

where $\Rightarrow$ denotes a path from $V_{12} \boxplus V_{01}$ to V_{02} in $BO_{\amalg}^{\infty}$. In the other version of $\mathcal{V}^{\hookrightarrow}$, in contrast, the path is to go in the reverse direction.

Since the two versions are clearly equivalent, we will keep this notation for $\mathcal{V}^{\hookrightarrow}$ to refer to the homotopy-coherent version. In particular, its maximal sub-∞-groupoid is still $BO_{\amalg}^{\infty}$ ([Tet22, Proposition 2.7]). It will become clear momentarily why this is convenient.

2.2. Grassmannians: linked to stratified.

Definition 2.1 ([Tet23, Example 2.7]). Let $n, m \in \mathbb{N}$. We call the linked space

$$BO(n) \times BO(m)$$
$$\overset{\text{pr}}{\swarrow} \qquad \overset{\boxplus}{\searrow}$$
$$BO(n) \qquad\qquad\qquad BO(n+m)$$

the (n, m)-*Grassmannian* and denote it by $BO(n, m)$.

The goal of this section is to construct a map

$$\mathbb{U} : \mathcal{EX}(BO(n, m)) \to \mathcal{V}^{\hookrightarrow}$$

from the exit path ∞-category of the (n, m)-Grassmannian to the stratified Grassmannian. The theory of tangential structures on linked spaces will then be able to interface with the conically-smooth variant, which we expand upon in Section 2.3. We will be completely explicit excepting specification of some trivial choices.

The map restricted to $BO(n)_{\bullet}$ and $BO(n+m)_{\bullet}$ inside $\mathcal{EX}$ is defined to be inclusion into the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$. It remains to define the restriction

$$\mathcal{EX}_{k+1} \supset \mathcal{P}_k^{\Delta} \to \mathcal{V}_{k+1}^{\hookrightarrow} \cong \mathrm{Fun}\left(\mathrm{Path}[k+2], B^{\boxplus}\mathrm{O}\right),$$

for $k \geq 0$. We will explain dimensions 1 and 2 verbosely before giving the full definition without further explanation.

2.2.1. 1-*morphisms.* An element $(\gamma, 1)$ of $\mathcal{P}_0^{\Delta}$ – the exit index in this dimension is necessarily 1 – corresponds to a path γ in $BO(n+m)$ whose starting point is a direct sum $V_{12} \boxplus V_{01}$ with $V_{01} \in BO(n)$, $V_{12} \in BO(m)$. Denoting the endpoint by V_{02}, γ determines a 2-path by arranging the data exactly as in (2). We have thus defined

$$(3) \qquad\qquad\qquad \mathcal{EX}_{\leq 1} \to \mathcal{V}_{\leq 1}^{\hookrightarrow}.$$

Observe that this is compatible with face maps: the source of the image of γ is V_{01}, and the target is V_{02} ([Tet22, §2.2.2]), which are, by construction, the images of the source and target of γ in $\mathcal{EX}$, respectively:

$$d_1^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}(\gamma, 1)) = V_{01} = \mathbb{U}\left(\mathrm{pr}\left(d_1^{BO(n+m)}\gamma\right)\right) = \mathbb{U}\left(d_1^{\mathcal{EX}}(\gamma, 1)\right),$$

$$d_0^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}\gamma) = V_{02} = \mathbb{U}\left(d_0^{BO(n+m)}\gamma\right) = \mathbb{U}\left(d_0^{\mathcal{EX}}(\gamma, 1)\right).$$

Compatibility with degeneracies is also clear.

Theorem 2.2. *For any pair of natural numbers $n, m \in \mathbb{N}$, the assignment* (3) *extends to an ∞-functor*

$$\mathbb{U} : \mathcal{EX}(BO(n, m)) \to \mathcal{V}^{\hookrightarrow},$$

which we call the unpacking map.

First, we will discuss what $\mathbb{U}$ has to do with 2-morphisms at a phenomenological level so as to elucidate the essential issues to be overcome.

2.2.2. *Intermezzo:* 2-*morphisms.* Exit paths in $\mathcal{P}_k^\Delta$ come in $k+1$ classes according to their exit indices, which need to be mapped to $\mathcal{V}^\rightarrow$ in different ways.

First, in order to uniquely determine a 3-path in $B^{\boxplus}\mathrm{O}$, it is enough to map out of the sets $\mathrm{N}_{\leq 2}(P_{i,j}^{\mathrm{op}})$ into $BO_{\leq 2} := (BO_{\mathrm{II}}^\infty)_{\leq 2}$ (and in general, a κ-path in $B^{\boxplus}\mathrm{O}$ is determined in dimensions $\leq \kappa - 1$), since higher dimensions are degenerate. The simple (non-decomposable) morphisms in $\mathrm{Path}[3]$ are of type $\mathrm{N}_0(P_{\alpha\beta}^{\mathrm{op}}) \ni \underline{\alpha\beta} := \{\alpha < \beta\} \subset [3]$, which by a 3-path are mapped to $V_{\alpha\beta} \in BO_0$. When $\beta = \alpha + 2$ (of which type there are two pairs), there are arrows $\underline{\alpha, \alpha+1, \beta} = \underline{\alpha+1, \beta} \cup \underline{\alpha, \alpha+1} > \underline{\alpha\beta}$ in $\mathrm{N}_1(P_{\alpha\beta}^{\mathrm{op}})$, which determine paths

$$V_{\alpha+1,\beta} \boxplus V_{\alpha,\alpha+1} \to V_{\alpha\beta},$$

i.e., $V_{12} \boxplus V_{01} \to V_{02}$ and $V_{23} \boxplus V_{12} \to V_{13}$ in BO_1, namely two of the face 2-paths. The remaining two faces are supplied analogously by considering $(\alpha, \beta) = (0, 3)$ and the compositions $\underline{013} = \underline{13} \cup \underline{01}$ and $\underline{023} = \underline{23} \cup \underline{02}$. Finally, again for $(\alpha, \beta) = (0, 3)$, consider $\underline{0123} = \underline{23} \cup \underline{12} \cup \underline{01}$, which is to be mapped to $V_{0123} = V_{23} \boxplus V_{12} \boxplus V_{01}$. Out of $\mathrm{N}_1(P_{0,3}^{\mathrm{op}})$ we receive paths $V_{0123} \to V_{03}$, $V_{0123} \to V_{013}$, $V_{0123} \to V_{023}$. The two non-degenerate elements $(\underline{0123} > \underline{023} > \underline{03})$ and $(\underline{0123} > \underline{013} > \underline{03})$ in $\mathrm{N}_2(P_{0,3}^{\mathrm{op}})$ are to map in BO_2 to

$$(4) \qquad
\begin{array}{ccc}
& V_{023} & \\
\nearrow & & \searrow \\
V_{0123} & \longrightarrow & V_{03}
\end{array}
\quad = \quad
\begin{array}{ccc}
& V_{23} \boxplus V_{02} & \\
\nearrow & & \searrow \\
V_{23} \boxplus V_{12} \boxplus V_{01} & \longrightarrow & V_{03}
\end{array}$$

and

$$(5) \qquad
\begin{array}{ccc}
& V_{013} & \\
\nearrow & & \searrow \\
V_{0123} & \longrightarrow & V_{03}
\end{array}
\quad = \quad
\begin{array}{ccc}
& V_{13} \boxplus V_{01} & \\
\nearrow & & \searrow \\
V_{23} \boxplus V_{12} \boxplus V_{01} & \longrightarrow & V_{03}
\end{array}$$

We have thus summed up the data needed to provide a functor $\mathrm{Path}[3] \to B^{\boxplus}\mathrm{O}$.

Now, let us start with paths of exit index $2 \in \{1, 2\}$. Such an exit path $(\gamma, 2)$ (in $\mathcal{P}_1^\Delta \subset \mathcal{EX}_2$) consists of a 2-simplex $\gamma \in BO(n+m)_2$ of type

$$(6) \qquad
\begin{array}{ccc}
& & K \\
& \nearrow^{\gamma_\boxplus} & \uparrow{\scriptstyle \gamma_{\boxplus'}} \\
W \boxplus V & \xrightarrow[\gamma_W \boxplus \gamma_V]{} & W' \boxplus V'
\end{array}$$

where the bottom edge comes from $BO(n) \times BO(m)$ (whence it is $\boxplus$ of two paths). The natural choice for the image, visualised as a 3-simplex of $\mathcal{B}^{\boxplus}\mathrm{O}$, is

$$\mathbb{U}(\gamma, 2) = \quad
\begin{array}{c}

\end{array}
\begin{array}{c}
2 \\
\nearrow \uparrow \searrow \\
\scriptstyle 0 \uparrow W' \\
\swarrow | \searrow \\
1 \!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- W \to 3 \\
\nwarrow | \nearrow \\
V V' K \\
\searrow | \swarrow \\
0
\end{array}
\quad \cdot$$

Indeed, the edges in (6) supply the face triangles – in fact, the fact that the bottom edge is of type $\gamma_W \boxplus \gamma_V$ is crucial, since the summand paths supply the triangles

adjacent to the edge decorated by the zero vector space. The only wrinkle is that the upper face requires a path $W' \to W$, which can be taken to be the (standard) inverse of γ_W, which we will denote by γ_W^{-1}. As for (4), i.e.,

$$(7) \qquad \begin{array}{ccc} & W' \boxplus V' & \\ {}^{(i)}\nearrow & & \searrow^{\gamma_{\boxplus'}} \\ W' \boxplus V & \xrightarrow[\text{(ii)}]{} & K \end{array} \quad ,$$

note that we are still free to (and have to) choose the paths $W' \boxplus V \to W' \boxplus V'$ and $W' \boxplus V \to K$ (corresponding to the arrows $\underline{0123} > \underline{023}$ and $\underline{0123} > \underline{03}$). To this end, consider the diagram $\begin{array}{ccc} W' \boxplus V' & \xrightarrow{\gamma_{\boxplus'}} & K \\ {}_{\mathrm{id}_{W'}\boxplus\gamma_V}\uparrow \,\,\,\nwarrow_{\gamma_W\boxplus\gamma_V} & & \uparrow_{\gamma_{\boxplus}} \\ W' \boxplus V & \xrightarrow[\gamma_W^{-1}\boxplus\mathrm{id}_V]{} & W \boxplus V \end{array}$ and choose the obvious fillers.

(By id_A we mean the constant loop at A.) This suggests using $(i) = \mathrm{id}_{W'} \boxplus \gamma_V$, $(ii) = (\gamma_W^{-1} \boxplus \mathrm{id}_V) * \gamma_{\boxplus}$ (we denote concatenation from left to right) whereupon the obvious filler can be chosen. Similarly, for (5), i.e.,

$$(8) \qquad \begin{array}{ccc} & W \boxplus V & \\ {}^{(i)'}\nearrow & & \searrow^{\gamma_{\boxplus}} \\ W' \boxplus V & \xrightarrow[\text{(ii)}']{} & K \end{array} \quad ,$$

consider $\begin{array}{ccc} W \boxplus V & \xrightarrow{\gamma_{\boxplus}} & K \\ {}_{\gamma_W^{-1}\boxplus\mathrm{id}_V}\uparrow \,\,\,\nwarrow_{\gamma_W^{-1}\boxplus\gamma_V^{-1}} & & \uparrow_{\gamma_{\boxplus'}} \\ W' \boxplus V & \xrightarrow[\mathrm{id}_W\boxplus\gamma_V]{} & W' \boxplus V' \end{array}$ and proceed similarly. (We give a systematic ac-

count in Section 2.2.3.) This completes the construction of $N_{\leq 2}(P_{0,3}^{\mathrm{op}}) \to BO_{\leq 2}$ and so in toto of the 3-path $\mathrm{Path}[3] \to B^{\boxplus}O$ associated with the exit path $(\gamma, 2)$. The image of an index-1 exit path

$$(9) \qquad \begin{array}{ccc} K & \xrightarrow{\gamma_K} & K' \\ {}_{\gamma_{\boxplus}}\uparrow & \nearrow_{\gamma_{\boxplus'}} & \\ W \boxplus V & & \end{array}$$
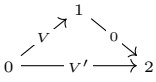
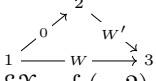is constructed analogously, with its picture as a 3-simplex of $\mathcal{B}^{\boxplus}O$ given by

We have thus defined

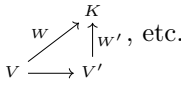$$\mathcal{E}\mathcal{X}_{\leq 2} \to \mathcal{V}^{\hookrightarrow}_{\leq 2}.$$

As for simpliciality, consider again an index-2 exit path $(\gamma, 2)$ as in (6). Its source edge is the path $d_2^{\mathcal{E}\mathcal{X}}(\gamma, 2) = (\gamma_V \colon V \to V') \in BO_1 \subset \mathcal{E}\mathcal{X}_1$. Since its two remaining edges are vertical, they are the elements of $\mathcal{P}_0^{\Delta}$ induced by $\gamma_{\boxplus'}$ and $\gamma_{\boxplus}$. Now, recall that $\mathbb{U}(\gamma, 2)$ is identified with an element of $\mathcal{B}^{\boxplus}O_3 \cong \mathcal{V}_2^{\hookrightarrow}$ due to $\mathcal{V}^{\hookrightarrow}$'s being an under-$\infty$-category, via $\Delta[0] \star \Delta[2] \simeq \Delta[3]$. Accordingly, face maps apply on the

factor $\Delta[2]$ (i.e., ∂ acts as $\mathrm{id}_{\Delta[0]} \star \partial$). In the picture in $\mathcal{B}^{\boxplus}O_3$, this means that when pulling back along a face map $\partial \colon \Delta[1] \hookrightarrow \Delta[2]$, we restrict to the triangle whose top edge is specified by ∂; e.g., $\partial_2 \colon \Delta[1] \hookrightarrow \Delta[2]$, which skips 2, applies to give

$$d_2^{\mathcal{V}^{\rightarrow}} \mathbb{U}(\gamma, 2) = \quad \begin{array}{c} \nearrow^{1} \searrow_{0} \\ {}_{V} \\ 0 \xrightarrow{} V' \xrightarrow{} 2 \end{array} \quad , \text{ which is precisely } \mathbb{U}(\gamma_V). \text{ So as to avoid confusion,}$$

note in particular that the top face $\quad \begin{array}{c} \nearrow^{2} \searrow_{W'} \\ {}_{0} \\ 1 \xrightarrow{} W \xrightarrow{} 3 \end{array} \quad$, where we inverted γ_W, is *not* a face in $\mathcal{V}^{\rightarrow}$, nor is $\gamma_W \in BO(m)_1 \subset \mathcal{EX}_1$ of $(\gamma, 2)$ in $\mathcal{EX}$.

We leave the analogous treatment of the remaining two faces and of the index-1 case to the reader. Now, we proceed to finally give the general construction.

2.2.3. *Proof of Theorem 2.2.* We will first give a systematic account of $\mathcal{P}_1^{\Delta} \to [\mathrm{Path}[3], B^{\boxplus}O]$ in such a way that the ideas generalise to all dimensions. At some places, it will conventient to slightly rearrange the visual representation of exit paths. For $(\gamma, 1) \in \mathcal{P}_0^{\Delta}$, the diagram $W \boxplus V \to K$ depicts $\gamma \in BO_1$. Instead, $V \xrightarrow{(W,\gamma)} K$ or $V \xrightarrow{W} K$ for short, depicts $(\gamma, 1)$ more properly. Similarly, 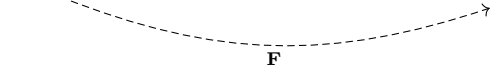we also depict $(\gamma, 2) \in \mathcal{P}_1^{\Delta}$ by $\quad \begin{array}{c} \overset{K}{} \\ W \nearrow \uparrow_{W'} \\ V \xrightarrow{} V' \end{array}$, etc.

Notation 2.3. $[A, B] \coloneqq \mathrm{Fun}(A, B)$.

Definition 2.4. We use $\mathbf{F} \colon BO(n) \times BO(m) \to BO(m)$, for *f*ibre, to denote the second coordinate projection. When we apply $\mathbf{F}$ to a bottom face of an exit path (γ, j), we mean that first the corresponding face of γ is to be taken, which is then (unambiguously) to be identified with a simplex of the link, and then $\mathbf{F}$ is to be applied. Namely, we have, by abuse of notation, a map

$$\mathbf{F} \colon \mathcal{P}_*^{\Delta} \to BO(m)_*$$

for each $* \geq 0$, given by the composition

$$\mathcal{P}_*^{\Delta} \xrightarrow{(\gamma,j) \mapsto \Gamma_j = \gamma \circ \mathcal{C}_j} \mathcal{P}_* \twoheadrightarrow \mathcal{L}_* \xrightarrow{\mathbf{F}} BO(m)_*$$
$$\underset{\mathbf{F}}{\dashrightarrow}$$

where we have not omitted $*$ since $\mathcal{P}^{\Delta}$ is not a simplicial set.

Notation 2.5. For X a space, we denote by Op the canonical isomorphism $X \simeq X^{\mathrm{op}}$ of Kan complexes, by which we mean $\mathrm{Sing}_{\bullet}(X) \simeq \mathrm{Sing}_{\bullet}(X)^{\mathrm{op}}$ ([Lur23, 003R]). This inverts simplices of all dimensions in a compatible fashion.

Notation 2.6. Let $\alpha_0, \ldots, \alpha_{\ell} \in [k]$. By $\mathrm{Path}[\alpha_0, \ldots, \alpha_{\ell}] \cong \mathrm{Path}[\ell]$ we denote the simplicial subcategory of $\mathrm{Path}[k]$ generated by the objects $\alpha_0, \ldots, \alpha_{\ell}$.

Lemma 2.7. $d_i^{\mathcal{V}^{\rightarrow}} = d_{i+1}^{\mathcal{B}^{\boxplus}O}$.

Proof. $(\mathrm{id}_0 \star \partial_i \colon \Delta[0] \star \Delta[1] \hookrightarrow \Delta[0] \star \Delta[2]) = (\partial_{1+i} \colon \Delta[2] \hookrightarrow \Delta[3])$. $\qquad \square$

The exitence of the restriction of $\mathbb{U} \colon \mathcal{EX} = \mathcal{EX}(BO(n,m)) \to \mathcal{V}^{\rightarrow}$ to non-invertible paths will be shown inductively. It is defined on $BO(n), BO(n+m) \subset \mathcal{EX}$ by inclusion into the maximal sub-∞-groupoid of $\mathcal{V}^{\rightarrow}$, and on $\mathcal{P}_*^{\Delta} \subset \mathcal{EX}_{*+1}$, using

$\mathcal{V}_k^{\hookrightarrow} \cong \left[\mathrm{Path}[k+2], B^{\boxplus}\mathrm{O}\right]$ and that a $(k+2)$-path is determined on hom-ssets in dimensions $\leq k-1$, as follows:

$\underline{\mathcal{P}_0^{\Delta} \to \left[\mathrm{Path}[2], B^{\boxplus}\mathrm{O}\right]}$:

$$\mathbb{U}|\mathcal{P}_{\mathbf{1},0}^{\Delta} : (\gamma, 1) = \overset{K}{\underset{W \boxplus V}{\gamma\uparrow}} \mapsto \begin{cases} \mathrm{N}_0(P_{0,a}^{\mathrm{op}}) \to BO_0, & \underline{0,a} \mapsto \{a-1\}^*(\gamma, 1) = \begin{cases} V, & a = 1 \\ K, & a = 2 \end{cases} \\ \mathrm{N}_0(P_{1,2}^{\mathrm{op}}) \to BO_0, & \underline{12} \mapsto \mathbf{F}(\gamma, j) = W \\ \mathrm{N}_1(P_{0,2}) \to BO_1, & (\underline{012} > \underline{02}) \mapsto \gamma \end{cases}$$

$\underline{\mathcal{P}_1^{\Delta} \to \left[\mathrm{Path}[3], B^{\boxplus}\mathrm{O}\right]}$:

Induced faces. Let $(\gamma, j) \in \mathcal{P}_1^{\Delta}$. We first define the faces of $\mathbb{U}(\gamma, j)$. The three faces $d_{0,1,2}^{\mathcal{V}^{\hookrightarrow}}\mathbb{U}(\gamma, 2)$ are defined by $\mathbb{U}|\mathcal{EX}_0$ via simpliciality, i.e., by

$$(10) \qquad d_i^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}(\gamma, j)) := \mathbb{U}(d_i^{\mathcal{EX}}(\gamma, j)).$$

This fixes $\mathbb{U}(\gamma, j)$, by Lemma 2.7, on the subcategories $\mathrm{Path}[0, k, l] \cong \mathrm{Path}[2]$ of $\mathrm{Path}[3]$, for $1 \leq k < l \leq 3$ (Notation 2.5).

The top face. The remaining $\mathcal{B}^{\boxplus}\mathrm{O}$-face $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}(\mathbb{U}(\gamma, j))$ is the restriction to $\mathrm{Path}[1, 2, 3]$. The edges of $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}(\mathbb{U}(\gamma, j)) \in \mathcal{B}^{\boxplus}\mathrm{O}_2 \cong \mathcal{V}_1^{\hookrightarrow}$ are already specified by the $d_i^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}(\gamma, j))$, so only

$$\mathrm{N}_1(P_{1,3}^{\mathrm{op}}) \to BO_1$$
$$(\underline{23} \cup \underline{12} > \underline{13}) \mapsto \left(\mathbb{U}_{(\gamma, j)}(\underline{23}) \boxplus \mathbb{U}_{(\gamma, j)}(\underline{12}) \to \mathbb{U}_{(\gamma, j)}(\underline{13})\right)$$
$$= (\mathbb{U}(d_0 d_0(\gamma, j)) \boxplus \mathbb{U}(d_0 d_2(\gamma, j)) \to \mathbb{U}(d_0 d_1(\gamma, j)))$$

remains. This is determined by $\mathbf{F}$:

$$(11) \qquad \mathbb{U}_{(\gamma, j)}|\mathrm{N}_1(P_{1,3}^{\mathrm{op}}) := \mathrm{Op}\mathbf{F}(\gamma, j).$$

Remark 2.8 (interrupting the proof). We should note that it is immaterial that (11) is 'not functorial' (although $\mathbb{U}$ will be). As noted in Lemma 2.9, the direct sums appearing in the $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}$-face are trivial in that all summands but one are zero, the non-zero one being determined by the exit index j. We use $\mathrm{Op}\mathbf{F}$ to supply *only the path* in $BO(m)$. We have $\mathbb{U}_{\gamma, j}(\underline{23}) = 0$ if $j = 1$, and $\mathbb{U}_{\gamma, j}(\underline{12}) = 0$ if $j = 2$. In the former case, the edges of $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}$ are specified by simpliciality as in

$$\begin{array}{ccc} & \overset{2}{\underset{W \nearrow \quad \searrow 0}{}} & \\ 1 \overset{}{\underset{W}{\longrightarrow}} & & 3 \end{array}$$

, and $\mathrm{Op}\mathbf{F}(\gamma, 1)$ is $\mathrm{Op}(\mathrm{id}_W) = \mathrm{id}_W : W = W \boxplus 0 \to W$. Here, (11) happens to be functorial as $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}$ happens to lie in $\mathcal{V}^{\sim}$. In the latter case, the filler of

$$\begin{array}{ccc} & \overset{2}{\underset{0 \nearrow \quad \searrow W'}{}} & \\ 1 \overset{}{\underset{W}{\longrightarrow}} & & 3 \end{array}$$

is supplied by $\mathrm{Op}\mathbf{F}(\gamma, 2) = \mathrm{Op}(\gamma_W) = \gamma_W^{-1} : W' = 0 \boxplus W' \to W$. This breaks functoriality, since $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}$ is *not* invertible. Still, the path is as desired.

(The proof cont.)
1-paths induced by functoriality. Some 1-paths in the image of $\mathbb{U}(\gamma, j)$ are determined by the data provided thus far and by imposing functoriality. Namely,

we have the following decompositions:

$$(\mathrm{i}) = (\underline{0123} > \underline{023}) = (\underline{23} \cup \underline{012} > \underline{23} \cup \underline{02})$$
$$= \mathrm{id}_{\underline{23}} \cup [\underline{012} > \underline{02}]$$
$$\in \Im\left(\cup : \mathrm{N}_1(P_{2,3}^{\mathrm{op}}) \times \mathrm{N}_1(P_{0,2}^{\mathrm{op}}) \to \mathrm{N}_1(P_{0,3}^{\mathrm{op}})\right),$$
$$(\mathrm{i})' = (\underline{0123} > \underline{013}) = (\underline{123} \cup \underline{01} > \underline{13} \cup \underline{01})$$
$$= [\underline{123} > \underline{13}] \cup \mathrm{id}_{\underline{01}}$$
$$\in \Im\left(\cup : \mathrm{N}_1(P_{1,3}^{\mathrm{op}}) \times \mathrm{N}_1(P_{0,1}^{\mathrm{op}}) \to \mathrm{N}_1(P_{0,3}^{\mathrm{op}})\right).$$

Thus, functoriality imposes

$$\mathbb{U}_{(\gamma,j)}(\mathrm{i}) = \mathrm{id}_{\mathbb{U}_{(\gamma,j)}(\underline{23})} \boxplus \mathbb{U}_{(\gamma,j)}(\underline{012} > \underline{02})$$
$$\mathbb{U}_{(\gamma,j)}(\mathrm{i})' = \mathbb{U}_{(\gamma,j)}(\underline{123} > \underline{13}) \boxplus \mathrm{id}_{\mathbb{U}_{(\gamma,j)}(\underline{01})}$$

where the first non-constant summand is determined by (10) and the second by (11). In particular, the ad hoc assignments in (7) and (8) were correct.

Remining 1-path and 2-paths. It remains to specify $\mathbb{U}(\gamma, j)$ on the 'long path' in $\mathrm{N}_1(P_{0,3}^{\mathrm{op}})$ and on $\mathrm{N}_2(P_{0,3}^{\mathrm{op}})$. In contrast to the paths induced by functoriality, $\underline{03}$ is $\cup$-simple (not a composition), so

$$(\mathrm{ii}) = (\underline{0123} > \underline{03})$$

presents a genuine choice, and was not handled systematically in Section 2.2.2. It is considered most naturally in conjunction with the two non-degenerate elements in $N_2(P_{0,3}^{\mathrm{op}})$ to be mapped, as it is their (necessarily-)common composition:

First, note that, regardless of exit index, this square decomposes into two triangles:

For $j = 2$ (using the labels in (6)), this reads

and for $j = 1$ (using the labels in (9)),

$$0 \boxplus W \boxplus V = W \boxplus V$$

with $\mathrm{id}_0 \boxplus \gamma_{\boxplus} = \gamma_{\boxplus}$ and $\mathrm{id}_0 \boxplus \mathrm{id}_W \boxplus \mathrm{id}_V = \mathrm{id}_W \boxplus \mathrm{id}_V$

$$0 \boxplus K = K \xleftarrow{\quad \gamma_{\boxplus} \quad} W \boxplus V$$

$$\gamma_K \searrow \qquad \swarrow \gamma_{\boxplus'}$$

$$K'$$

For both indices, the bottom triangle is filled by γ itself, and the top one has a canonical filler. This suggest assigning to (ii) the outer-left concatenation, $\mathbb{U}_{(\gamma,j)}(\text{ii}) = \mathbb{U}_{(\gamma,j)}(\underline{0123} > \underline{023}) * \mathbb{U}_{(\gamma,j)}(\underline{023} > \underline{03})$. Accordingly, $\mathbb{U}(\gamma,j)|\mathrm{N}_2(P^{\mathrm{op}}_{0,3})$ is determined by said fillers. This concludes the construction of $\mathbb{U}|\mathcal{P}^{\Delta}_1$.

The induction step. Assume now that the $\mathbb{U}|\mathcal{P}^{\Delta}_{<k}$ have been constructed simplicially.

$\underline{\mathcal{P}^{\Delta}_k \to \left[\mathrm{Path}[k+2], B^{\boxplus}O\right]}$: We have constructed $\mathbb{U}|\mathcal{P}^{\Delta}_1$ as independently of exit indices as possible, and the same ideas apply here mutatis mutandis.

Let $(\gamma, j) \in \mathcal{P}^{\Delta}_k$.

Induced faces. The restriction of $\mathbb{U}(\gamma, j)$ to the subcategories $\mathrm{Path}[0, k_1, \ldots, k_{k+1}] \cong \mathrm{Path}[k+1]$ of $\mathrm{Path}[k+2]$, $1 \le k_\ell < k_{\ell'} \le k+2 \; \forall 1 \le \ell < \ell' \le k+1$, are determined by setting

$$(12) \qquad d^{\mathcal{V}^{\hookrightarrow}}_i(\mathbb{U}(\gamma, j)) = \mathbb{U}(d^{\mathcal{E}\mathcal{X}}_i(\gamma, j)).$$

The top face. The restriction $d^{B^{\boxplus}O}_0(\mathbb{U}(\gamma, j))$ to $\mathrm{Path}[1, \ldots, k+2]$ is given, in view of Lemma 2.9, by $\mathrm{Op}\mathbf{F}(\gamma, j)$.

The rest of $P^{\mathrm{op}}_{0,k+2}$. All arrows in $P^{\mathrm{op}}_{0,k+2}$ out of the initial $\underline{0, \ldots, k+2} = [k+2]$ except for $\underline{0, \ldots, k+2} > \underline{0, k+2}$ are clearly $\cup$-composite and are thus determined by (12) and functoriality. This is well-defined due to the inductive assumption that so is $\mathbb{U}|\mathcal{P}^{\Delta}_{<k}$.

We set

$$\mathbb{U}_{(\gamma,j)}\left([k+2] > \underline{0, k+2}\right) = \mathbb{U}_{(\gamma,j)}\left([k+2] > [k+2] \smallsetminus \{1\}\right)$$
$$* \, \mathbb{U}_{(\gamma,j)}\left([k+2] \smallsetminus \{1\} > [k+2] \smallsetminus \{1, 2\}\right)$$
$$* \cdots$$
$$* \, \mathbb{U}_{(\gamma,j)}\left(\underline{0, k+1, k+2} > \underline{0, k+2}\right).$$

Among the $\cup$-simple ones, there are $k+1$ arrows $\underline{0, i, k+2} > \underline{0, k+2}$. In the $(k+1)$-cube $P_{0,k+2}$, the $(k+1)$-simplex with vertices the domains $\underline{0, i, k+2}$, and finally $\underline{0, k+2}$, is filled by $\gamma \in BO(n+m)_{k+1}$ according to the rule $\alpha \mapsto \underline{0, \alpha+1, k+2}$ for $k+2 > \alpha \in [k+2]$, and $k+2 \mapsto \underline{0, k+2}$. Thus, in any chain in $\mathrm{N}_{k+1}(P^{\mathrm{op}}_{0,k+2})$ from $[k+2]$ to $\underline{0, k+2}$, the concatenation of the images of the short arrows is homotopic to $\mathbb{U}_{(\gamma,j)}([k+2] > \underline{0, k+2})$, which provides $\mathbb{U}(\gamma, j)|\mathrm{N}_{k+1}(P^{\mathrm{op}}_{0,k+2})$.

This concludes the construction of $\mathbb{U}$.

Admittedly, the construction above is more brute-force than elightening. It does not rely on universal properties, which might be explained by the low-level (in the programming sense) status it has in linked space theory, and, as such, does not benefit from higher-level universal-property arguments. Essentially, we force the construction inductively from the lowest non-trivial assignment $\mathcal{P}^{\Delta}_0 \to \left[\mathrm{Path}[2], B^{\boxplus}O\right]$, guided by an explicit systematisation of the first 'induction step',

$\mathcal{P}_1^{\Delta} \to \left[\text{Path}[3], B^{\boxplus}\text{O}\right]$ – whose redressed presentation was, strictly speaking, not necessary – which illustrates the fundamental idea quite well already.

The reason why the construction then works out in all dimensions remains slightly mysterious: the core phenomenon leveraged is buried in the argument that we use to construct the restriction $N_\bullet(P_{0,k+2}^{\text{op}}) \to BO_\bullet$ of $\mathbb{U}(\gamma, j)$, $(\gamma, j) \in \mathcal{P}_k^{\Delta}$, once all $\mathcal{B}^{\boxplus}\text{O}$-faces have been forced by lower dimensions and simpliciality. Here, recognising $P_{0,k+2}^{\text{op}}$ to be a cube of dimension $k+1$ (as was also noted in [Lur09, §1.1.5], [Lur23, 00LM]), we observe that $\gamma \in BO(n+m)_{k+1}$ itself provides a filler for a particular $(k+1)$-simplex within this cube (the one with vertices $\underline{0, i, k+2}$, $1 < i < k+2$, and $\underline{0, k+2}$). The rest of the cube has a canonical filler, which has, crucially, to do with the inversion of the paths originating in the pr-fibre $BO(m)$ (paths like $\gamma_W^{-1}\colon W' \to W$), necessitated above by the edge arrangements in $\mathcal{B}^{\boxplus}\text{O}$. So, as disconcerting as this might have seemed above, it proves absolutely essential.

As a final illustration, consider an exit 3-path of index 3:

The (1d) edges of $\mathbb{U}(\gamma, 3) \in \mathcal{B}^{\boxplus}\text{O}_4$ are given, due to (12), as follows:

$$\begin{cases} \underline{01} \mapsto V_0, \ \underline{02} \mapsto V_1, \ \underline{03} \mapsto V_2, \ \underline{04} \mapsto K, \\ \underline{14} \mapsto W_0, \ \underline{24} \mapsto W_1, \ \underline{34} \mapsto W_2 \end{cases}$$

and the remaining edges are zero. Now, the 3-cube $P_{0,4}^{\text{op}}$ and its image under $\mathbb{U}(\gamma, j)$ look as follows:

Painting in the outer-left concatenation chosen to be the image of $\underline{01234} > \underline{04}$ (in green) and the (edges of the) 'lower' tetrahedron given by $\gamma \in BO_3$ itself (in blue),

we see that

homotopy-commutes by inspection.

Lemma 2.9 (ad $d_0'^{\mathcal{B}^{\boxplus}\circ\mathbb{U}}(\gamma, j)$)**.**

Proof. $\square$

2.3. Classifying maps of linked tangent bundles.

Construction 2.10 (linked tangent bundle)**.** Let $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xrightarrow{\iota} N \right)$ be a linked manifold, with each manifold riemannian. Given this contractible choice, we will show that there is a 'canonical' map

(13) $$\mathrm{T}\mathfrak{S} \colon \mathfrak{S} \to BO(n, m)$$

of linked spaces, which we call the *tangent bundle* of $\mathfrak{S}$.

Since $\mathrm{d}\pi$ surjects, the induced linear dual map $(\pi^*\mathrm{T}M)^\vee \hookrightarrow (\mathrm{T}L)^\vee$, of bundles over L, injects. Using the metrics, this gives an injection $\pi^*\mathrm{T}M \hookrightarrow \mathrm{T}L$. Let us assume all manifolds paracompact, so that vector subbundles split. Then, writing $\mathrm{N}_L M \coloneqq (\pi^*\mathrm{T}M)^\perp \subset \mathrm{T}L$, we have a splitting $\mathrm{T}L \simeq \pi^*\mathrm{T}M \oplus \mathrm{N}_L M$. Similarly, writing $\mathrm{N}_N L \coloneqq (\iota_*\mathrm{T}L)^\perp \subset \iota^*\mathrm{T}N$, we have a splitting $\iota^*\mathrm{T}N \simeq \mathrm{T}L \oplus \mathrm{N}_N L \simeq \pi^*\mathrm{T}M \oplus \mathrm{N}_L M \oplus \mathrm{N}_N L$. This implies precisely that in the diagram

(14)

the back square

commutes, and the front square commutes trivially. This yields (13).

Remark 2.11. $\mathrm{N}_L M \oplus \mathrm{N}_N L \simeq \mathrm{N}_N M \coloneqq (\pi^*\mathrm{T}M)^\perp \subset \iota^*\mathrm{T}N$ with respect to the pulled back metric.

Applying $\mathcal{EX}$ and post-composing with $\mathfrak{U}$, we have the induced map

$$\mathcal{EX}(\mathfrak{S}) \to \mathcal{V}^{\hookrightarrow},$$

which is the linked version of the (conically-smooth) constructible tangent bundle.

Example 2.12. If L is induced by a closed submanifold inclusion $M \subset \overline{N}$ as $L = \mathbb{S}(\mathrm{N}_N M)$, the sphere bundle of the normal bundle ([Tet23, Example 2.6]), then L has dimension $n + m - 1$, $\mathrm{N}_L M$ has rank $m - 1$, and $\mathrm{N}_L N$ has rank 1. More specifically, in the conically-smooth context, the link (of a pair of strata) comes with an open embedding $L \times \mathbb{R} \hookrightarrow N$, which is tantamount to the triviality of the latter normal bundle, i.e., $\mathrm{N}_N L \simeq \varepsilon^1$, or, equivalently, to a diffeomorphism $L \times \mathbb{R} \simeq \mathbb{S}(\mathrm{N}_N M) \times \mathbb{R}$. This $\mathbb{R}$-factor incarnates the extra $\mathbb{E}_1$-structure featuring in the classification of stratified locally-constant factorisation algebras on stratified spaces of type $M \subset \overline{N}$.

Example 2.13. An even simpler situation arises when L (and $\mathfrak{S}$) is induced by a boundary $M = \partial \overline{N} \subset \overline{N}$ as $L \cong M$, the boundary pushed diffeomorphically into the interior $N = \overline{N} \smallsetminus M$ by following the flow of a nowhere-vanishing inward pointing vector field along the boundary (which always exists) for a chosen non-zero time ([Tet23, Example 2.5]); we will denote this closed link embedding later by ι_+. Then $\mathrm{N}_L M = 0$ and $\mathrm{N}_N L \simeq \varepsilon^1$ again. The type of classification statement mentioned in Example 2.12 reduces here to Kontsevich's Swiss-Cheese conjecture (e.g., [Tho16]).

Definition 2.14. We call a linked manifold with M of dimension n and N of dimension $n + m$ *constructible* if L is of dimension $n + m - 1$, and its normal bundle in N is trivialisable.

3. Linked geometry

3.1. **Adapting AFR-type structures.** The ∞-category of *tangential structures* is the over-∞-category $\mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow}$ ([AFR18b]; see [Lur09, §3] for $\mathcal{C}\mathrm{at}_\infty$). Via

$$\mathbb{U}^* \colon \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{EX}(BO(n,m)),$$

these transfer to tangential structures on linked manifolds: given $\mathfrak{S}$, and writing $\mathcal{B}_{(n,m)} \coloneqq \mathbb{U}^*\mathcal{B}$, we may define the space (homotopy type) of $\mathcal{B}$-structures on $\mathfrak{S}$ to be

$$\mathcal{B}\text{-red}(\mathfrak{S}) \coloneqq \mathrm{Map}_{/BO(n,m)}\left(\mathcal{EX}(\mathfrak{S}), \mathcal{B}_{(n,m)}\right),$$

the mapping space in $\mathcal{C}\mathrm{at}_\infty/\mathcal{EX}(BO(n,m))$, where the first argument uses $\mathrm{T}\mathfrak{S}$ (Construction 2.10). Equivalently, $\mathcal{B}\text{-red}(\mathfrak{S}) = \Gamma\left((\mathrm{T}\mathfrak{S})^* \mathcal{B}_{(n,m)}\right)$, the homotopy-sections of $(\mathrm{T}\mathfrak{S})^* \mathcal{B}_{(n,m)} \to \mathcal{EX}(\mathfrak{S})$.

Given $\mathcal{B}$ and (n,m), the natural question is whether

$$\mathcal{B}_{(n,m)} = \mathcal{EX}(\mathfrak{B})$$

for a linked space $\mathfrak{B}$, which would enable us to discuss stratified tangential structures without having to refer to exit paths.

We will first discuss the simplest example imaginable. To begin with, recall that for $\kappa \in \mathbb{N}$, rank-κ framings (κ-*framings*) are expressed by the tangential structure $\boldsymbol{\kappa} \colon * \to \mathcal{V}^{\hookrightarrow}$ that sends the point to $\boldsymbol{\kappa} \coloneqq \mathbb{R}^\kappa$.

Example 3.1 (framings). We have

$$\boldsymbol{\kappa}_{(n,m)} = \begin{cases} \mathcal{EX}\left(\emptyset \leftarrow \emptyset \rightarrow *\right) = *, & n + m = \kappa, \\ \emptyset, & \text{else} \end{cases}$$

with

$$\left(\boldsymbol{\kappa}_{(n,m)} \rightarrow \mathcal{EX}\left(BO(n,m)\right)\right) = \mathcal{EX}\left(\left(\emptyset \leftarrow \emptyset \rightarrow *\right) \rightarrow BO(n,m), * \xrightarrow{\kappa} BO(n+m)\right).$$

This reflects the fact that a nontrivially stratified space does not admit a κ-framing: the else-statement implies that for a linked space to admit a κ-framing its bulk must be κ-dimensional. The first statement implies moreover that for a lift of $T\mathfrak{S}$ to $(\emptyset \leftarrow \emptyset \rightarrow *)$ to exist, the space must be of type $\mathfrak{S} = (\emptyset \leftarrow \emptyset \rightarrow N)$ (if non-empty), and $\dim N = \kappa$.

Similar considerations apply to any *classical tangential structure* $\mathfrak{b} \colon \mathcal{B} \rightarrow \mathcal{V}^{\rightarrow}$, i.e., one that factors through $BO(\kappa) \hookrightarrow \mathcal{V}^{\sim} \hookrightarrow \mathcal{V}^{\rightarrow}$ for some k.

Example 3.2. Let $\mathfrak{b}$ be a classical tangential structure given by a map $B \rightarrow BO(\kappa)$ of spaces, e.g., induced by a map $G \rightarrow O(\kappa)$ of topological groups, or a rank-κ bundle $X \rightarrow BO(\kappa)$ on a space X. Then,

$$B_{(n,m)} = \begin{cases} \mathcal{EX}\left(\emptyset \leftarrow \emptyset \rightarrow B\right) = B, & n + m = \kappa, \\ \emptyset, & \text{else,} \end{cases}$$

where we abbreviated $\mathrm{Sing}_\bullet(B)$ to B in its last occurence.

Example 3.3. Consider $\mathbb{N} = (\mathbb{N}, \leq)$ with the standard order. Variframings ([AFR18b]) are given by $\mathrm{vfr} \colon \mathbb{N} \rightarrow \mathcal{V}^{\rightarrow}$, $k \mapsto \boldsymbol{k}$, $(k \leq K) \mapsto (\boldsymbol{k} \xrightarrow{-\oplus 0} \boldsymbol{K})$. We read $\mathrm{vfr}(k \leq K)$ as the standard[1] isomorphism $\boldsymbol{k} \boxplus (\boldsymbol{K} - \boldsymbol{k}) \cong \boldsymbol{K}$. Let us restrict vfr to depth 1 by choosing a pair $n \leq N$, i.e., consider $\mathrm{vfr}|_{n \leq N} \colon \{n \leq N\} \rightarrow \mathcal{V}^{\rightarrow}$. Then, for $m = N - n$, we have

$$\mathbb{U}^*(\mathrm{vfr}|_{n \leq N}) \simeq \mathcal{EX}(* \leftarrow * \rightarrow *) \simeq \Delta[1],$$

the exit path ∞-category of the nontrivially-linked point. Moreover, $\mathbb{U}^*(\mathrm{vfr}|_{n \leq N}) \rightarrow \mathcal{EX}(BO(n,m))$ is $\mathcal{EX}$ of

Thus, a *variframing* on $\mathfrak{S} = (M \leftarrow L \hookrightarrow N)$, i.e., a lift of $T\mathfrak{S}$ to this $\Delta[1]$, is a framing on M, a framing on N, and a framing on $\mathrm{N}_N M$ (see Remark 2.11).

Example 3.4 (point defects). The choice of a point p in a smooth manifold N of dimension n and a coordinate neighbourhood around it induce a linked space

$$\mathfrak{N}_p := \left(\{p\} \leftarrow S^{n-1} \xrightarrow{\iota_p = \iota} N \smallsetminus \{p\}\right)$$

---

[1]Up to, of course, the choice of a pairing function $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$

where the sphere is the unit sphere in coordinates. The link map of $T\mathfrak{N}_p$ reads

$$\varepsilon^0 \times \left(TS^{n-1} \oplus N(\iota) \right) : S^{n-1} \to * \times BO(n),$$

i.e., $\iota^*T\left(N \smallsetminus \{p\} \right) : S^{n-1} \to BO(n)$. A $\mathrm{vfr}_{0 \le n}$-structure on $\mathfrak{N}_p$ is a framing on N together with a framing on the normal bundle of ι. In this example, the latter always exists, which is why we will call such a configuration a trivial point defect.

Two relaxations of the tangential structure $\boldsymbol{\kappa}$ (or of any classical structure) are crucial for our purposes. They are termed, in increasing order of generality, the *stable* and *solid* replacements.

3.2. Linked morphism spaces.
Before proceeding to solid/stable structures in the linked setting, we need to identify the novel morphism spaces in the linked setting.

Notation. Given an embedding $\iota \colon \Sigma \hookrightarrow N$ and a point $q \in N$, we let $P(N)_{\Sigma,q} = P_{\Sigma,q}$ denote the space of paths in N that start in $\iota(\Sigma)$ and end in the point q, equipped with the compact-open topology. We use analogous notation when we work with a cofibration ι of simplicial sets.

The following result formalises and confirms the intuition that the link represents an infinitesimal expansion of the lower stratum into the higher stratum.

Theorem 3.5. *Let $\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N} \right)$ be a linked space, and $p \in \mathcal{M}$ and $q \in \mathcal{N}$ points in the two strata. We then have an equivalence*

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p,q) \simeq \mathcal{P}_{\mathcal{L}_p,q}$$

between the morphism space in $\mathcal{EX}$ from p to q and that of paths in $\mathcal{N}$ that start in $\iota(\mathcal{L}_p)$ (with ordinary fibre $\mathcal{L}_p = \{p\} \times_{\mathcal{M}} \mathcal{L}$) and end in q.

Proof. We will work with a model for morphism spaces that makes the proof particularly simple: by [Lur23, 01L5], the morphism space in $\mathcal{EX}$ is equivalent to the *right-pinched* morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{EX}}(p,q) \coloneqq \{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$. We will observe directly that $\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$ is in fact isomorphic to $\mathcal{P}_{\mathcal{L}_p,q} = \mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q)$. Indeed, at vertex level, the bijection

$$\left(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q) \right)_0 \cong \left(\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q) \right)_0$$

is clear: recalling that non-invertible 1-paths in $\mathcal{EX}$ are elements of $\mathcal{P}_0^{\Delta} \subset \mathcal{N}_1$ (as the exit index is necessarily 1 in this degree), let $(\gamma, 1) \in \mathcal{P}_0^{\Delta}$. For $p = d_1^{\mathcal{EX}}(\gamma, 1) \stackrel{\mathrm{def}}{=} \pi(d_1^{\mathcal{N}}(\gamma))$ to hold, we must have $d_1^{\mathcal{N}}(\gamma) \in \iota(\mathcal{L}_p)$. Similarly, $d_0^{\mathcal{EX}}(\gamma, 1) \stackrel{\mathrm{def}}{=} \iota(d_0^{\mathcal{N}}(\gamma))$, which yields the bijection.

Let now $k > 0$ and consider an exit $(k+1)$-path $(\gamma \colon \Delta[k] \star \Delta[0] \to \mathcal{N}, j)$ in $(\mathcal{EX}/q)_k \subset \mathcal{EX}_{k+1}$. Asking that (γ, j) be in $\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$ is equivalent to asking that

   (1) its $\mathcal{N}$-face

$$\Delta[k] \hookrightarrow \Delta[k] \star \Delta[0] \xrightarrow{\gamma} \mathcal{N},$$

   which is $d_{k+1}^{\mathcal{N}}(\gamma)$ under the standard identification $\Delta[k] \star \Delta[0] \simeq \Delta[k+1]$, is bottom, as by construction only then can the corresponding $\mathcal{EX}$-face be in $\mathcal{M}_k \subset \mathcal{EX}_k$;
   (2) and that it lies in particular in $\iota(\mathcal{L}_p)$.

Condition (1) implies moreover that $d_\ell^\mathcal{N}(\gamma) \in \mathcal{N}_k$ is vertical for all $\ell < k + 1$, since all other faces include the tip $\Delta[0] \hookrightarrow \Delta[k] \star \Delta[0]$ given by q, whence they are necessarily not bottom; and if some $d_\ell(\gamma)$ was top, that would contradict the bottomness of its (unique) common $(k-1)$-face with $d_{k+1}^\mathcal{N}(\gamma)$. In fact, (γ, j) has no n-face that is top once $n > 0$: given $\Delta[n] \hookrightarrow \Delta[k+1]$, there is necessarily a vertex in $d_{d+1}^\mathcal{N}(\gamma)$ that is hit by it.

But then the exit index j must be maximal: $j = k + 1$. For if not, then there would exist at least one top n-face for $n > 0$, the largest such, with $n = k + 1 - j$, for instance, being specified by $[n] \hookrightarrow [k+1]$, $\alpha \mapsto \ell + \alpha$. We thus obtain a bijection

$$(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q))_k \cong (\mathcal{L}_p \times_\mathcal{N} (\mathcal{N}/q))_k$$

in a fashion similar to the bijection of vertices: we have reduced exit paths (γ, j) in question on the LHS to those of index $k + 1$, and so to only a subset of $\mathcal{N}_{k+1}$, and specifically those such that $d_{k+1}^\mathcal{N}(\gamma) \in \mathcal{L}_p$. These are exactly the elements of the RHS. Finally, it is a direct check that $(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q))_* \xrightarrow{\cong} (\mathcal{L}_p \times_\mathcal{N} (\mathcal{N}/q))_*$ is functorial; for instance, any vertical face of such a $(\gamma, k+1)$ is again of maximal index: we have $\flat_{k+1,i}^{k+1} = k$ and, and as for degeneracies, $\sharp_{k+1,i}^{k+1} = k + 2$, for all $i < k + 1$ ([Tet23, §2]). $\qquad\square$

We will explore some immediate and useful consequences of this result. In the proofs, we will use Theorem 3.5 without mention.

Corollary 3.6. *Let $\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ be a linked space with $\mathcal{M}$ and $\mathcal{N}$ connected, and $p \in \mathcal{M}$, $q \in \mathcal{N}$. Then, π is an equivalence if and only if*

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q) \simeq \Omega\mathcal{N}.$$

Here, $\Omega\mathcal{N}$ denotes the based loop space of $\mathcal{N}$.

Proof. The fibre at any point of the source evaluation $\mathcal{P}_{\mathcal{L}_p, q} \to \mathcal{L}_p$ is equivalent to $\Omega\mathcal{N}$. Thus, the homotopy long exact sequence of this fibration implies that the fibre inclusion induces isomorphisms $\pi_*(\Omega\mathcal{N}) \cong \pi_*(\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q))$ iff $\pi_*(\mathcal{L}_p) \cong *$. Thus, $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q) \simeq \Omega\mathcal{N}$ iff π is a trivial Kan fibration. But π's being a trivial Kan fibration is tantamount to its being an equivalence ([Lur23, 00X2]). $\qquad\square$

We interpret this as saying that the space(s) of non-invertible paths in a linked space is (are) at its (their) largest when π is an equivalence; there are just as many as there are paths in the higher stratum. This is the case, for instance, when $\mathfrak{S}$ is induced by a manifold with boundary, as in Example 2.13. We have a maximal simplification in the other extreme, namely when π is trivial.

Corollary 3.7. *Consider a linked space of type $\mathfrak{S} = \left(* \leftarrow \mathcal{N} \xrightarrow{\mathrm{id}} \mathcal{N}\right)$.*

(1) *We have $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(*, q) \simeq *$.*
(2) *When $\mathcal{N} = \mathrm{Sing}_\bullet(N)$ for N a smooth manifold, we have*

$$\mathrm{Exit}(C(N)) \simeq \mathcal{EX}\left(* \leftarrow N \xrightarrow{\mathrm{id}} N\right),$$

where the LHS is the exit path ∞-category à la Lurie/MacPherson/AFR of the conically-smooth open cone $C(M) = * \amalg_{\{0\}\times N} ([0,1) \times N)$ on N with its canonical stratification over $\{0 < 1\}$.
(3) *The tangent bundle of the linked space $\mathfrak{N}_p$ given by a point defect $p \in N$ (Example 3.4) is determined, up to a contractible choice, by $\mathrm{T}N$.*

Proof.   (1) We have $\mathcal{L}_* = \mathcal{N}_* = \mathcal{N}$ and so $\mathcal{P}_{\mathcal{L}_*,q} \simeq \mathcal{N}/q \simeq *$ ([Lur23, 018Y]).

    (2) Statement (1) implies $\mathcal{EX}\left(* \leftarrow N \overset{\mathrm{id}}{\to} N\right) \simeq \mathrm{Sing}_\bullet(N)^\triangleleft$. The latter agrees with the LHS by [AFR18a, Proposition 3.3.8], since the cone locus $*$ is a (the) deepest stratum of $C(N)$.[2]

    (3) Let $p \neq q \in N$. In the corresponding locally-Kan categories, even though $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{N}_p)}(p,q) \simeq P(N)_{\iota_p(S^{n-1}),q}$ may have nontrivial homotopy type, the non-invertible paths are mapped per

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{N}_p)}(p,q) \to \mathrm{Hom}_{BO(0,n)}(*, T_q(N \smallsetminus \{p\})) \simeq *.$$

since $\mathcal{EX}(BO(0,n)) \simeq BO(n)^\triangleleft$ by (the proof of) statement (2), which is to say that the adjoined point $* \in BO(0) \subset BO(0,n)$ is initial.

$\hfill\square$

Remark 3.8. Item 3 of Corollary 3.7 is, in a sense, redundant, as the statement already follows from the requirement that $T\mathfrak{N}_p$ be a map of spans. It should be read as saying that there is no choice in its restriction to the link S^{n-1}.

3.3. **Replacements, I.** For $J \in \mathbb{N}$, a stably-J-framed smooth manifold M of dimension n is one with a framing on $TM \oplus \varepsilon^j$, $J = n + j$. In other words, this amounts to an injection $TM \hookrightarrow \varepsilon^J$ of bundles over M whose normal bundle, defined either using a metric on ε^J or as the quotient ε^J/TM, is parallelised. More generally, a solid J-framing on M is merely an injection $TM \hookrightarrow \varepsilon^J$. First of all, we notice that in order to impose such a condition on said normal bundle *in terms of tangential structures*, we must first separate it from the solid datum.

Let X be a smooth manifold equipped with a vector bundle $E \to X$ of rank r, and let $F \to X$ be another bundle, of rank R. Choosing a bundle embedding $E \hookrightarrow F$ is a 'reduction [or extension] of gauge group' on E in the following way. There is naturally a normal bundle N to E such that the embedding amounts to an isomorphism $\Phi \colon E \oplus N \cong F$. This Φ provides a filler for the diagram

$$
\begin{array}{ccc}
& & BO(r) \times BO(R-r) \\
& \overset{E \times N}{\nearrow} & \Big\downarrow \boxplus \\
X & \xrightarrow{\quad F \quad} & BO(R).
\end{array}
$$

Changing our point of view slightly, consider the limit space[3]

$$
(15) \qquad
\begin{array}{ccc}
X \times_{BO(R)} (BO(r) \times BO(R-r)) & \dashrightarrow & BO(r) \times BO(R-r) \\
\Big\downarrow & & \Big\downarrow \boxplus \\
X & \xrightarrow{\qquad\qquad F \qquad\qquad} & BO(R)
\end{array}
$$

which also admits a 'source evaluation' by projecting to the first factor:

$$\mathrm{ev}_0 \colon (BO(r) \times BO(R-r))|_F \to BO(r).$$

---

[2] More precisely, this is an equivalence of quasi-categories for Lurie's model from [Lur17], or, after translating to the complete Segal space model and using [AFR18a, Lemma 3.3.9], with that of Ayala et al.

[3] For the moment, we disregard the appropriate homotopy versions of such limits in order to ease notation; in terms of the tangential structure, this amounts to disregarding the choice of bundle isomorphism. Homotopy limits will be reincorporated into this account of their own accord below.

Now, writing

$$(16) \qquad (BO(r) \times BO(R - r))|_F := X \times_{BO(R)} (BO(r) \times BO(R - r)),$$

the choice of Φ can be expressed as follows:

Definition 3.9. A *solid F-structure* or *-reduction* on E (or on X when $E = \mathrm{T}X$) of is a lift of the form

$$
\begin{array}{ccc}
 & & (BO(r) \times BO(R-r))|_F \\
 & \nearrow & \downarrow {\scriptstyle \mathrm{ev}_0} \\
X & \xrightarrow{\quad E \quad} & BO(r)
\end{array}
\quad .
$$

We call R the *total rank* of the solid structure.

The normal bundle itself can be recovered from such a lift by projecting it to the second factor:

$$N \colon X \to (BO(r) \times BO(R - r))|_F \to BO(R - r).$$

Thus, a further, simultaneous reduction on N can be implemented using this projection: if N is to have $(B \to BO(R - r))$-structure, then we may consider the iterated fibre product

$$
(17) \quad
\begin{array}{ccc}
(BO(r) \times BO(R-r))|_F \times_{BO(R-r)} B & \dashrightarrow & (BO(r) \times BO(R-r))|_F \\
\downarrow & & \downarrow \\
B & \xrightarrow{\hspace{4cm}} & BO(R-r)
\end{array}
$$

and, writing

$$(18) \qquad (BO(r) \times BO(R - r))|_{(F,B)} := (BO(r) \times BO(R - r))|_F \times_{BO(R-r)} B,$$

ask for reductions of the following form:

Definition 3.10. A *solid (F, B)-structure* on E (or on X when $E = \mathrm{T}X$) is a lift of the form

$$
\begin{array}{ccc}
 & & (BO(r) \times BO(R-r))|_{(F,B)} \\
 & \nearrow & \downarrow {\scriptstyle \mathrm{ev}_0} \\
X & \xrightarrow{\quad E \quad} & BO(r)
\end{array}
\quad .
$$

When $B = \boldsymbol{R - r}$, this is a *stable F-structure*. We call $B \to BO(R - r)$, or B, the *normal structure*.

It is incidental that F is given as a bundle over X. More generally, when $F \colon Y \to BO(R)$ is any classical tangential structure with rank $R \geq r = \mathrm{rk}(E)$, the limit (15), and so also (17), still make sense. Then, a *solid Y- or (Y, B)-structure* is defined analogously, as is a *stable Y-structure*.

Solid replacements in the stratified context have been considered in [AFR18b]. Categorically speaking, they are cartesian fibration replacements. Namely, the assignment in the following Definition 3.11 extends (by a main result of [GHN17]) to a left-adjoint

$$\mathcal{C}\mathrm{at}_\infty / \mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{cart}} / \mathcal{V}^{\hookrightarrow}$$

to the forgetful functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{cart}} / \mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty / \mathcal{V}^{\hookrightarrow}$ from *solid/cartesian* tangential structures (i.e., cartesian fibrations over $\mathcal{V}^{\hookrightarrow}$) to tangential structures.

Definition 3.11. Given a tangential structure $\mathfrak{b} \colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, its *solid/cartesian replacement* is

$$\overline{\mathfrak{b}} \colon \overline{\mathcal{B}} = \overline{(\mathcal{B}, \mathfrak{b})} = (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})\{1\}} \mathcal{B} \to (\mathcal{V}^{\hookrightarrow})^{\{0\}},$$

the source evaluation from the fibre product along the target evaluation.

Note the direct correspondence with Definition 3.9 (and (15)), in view of $BO(r) \times BO(R-r)$'s being the link of the $(r, R-r)$-Grassmannian, viewing the target evaluation as the embedding $\boxplus$ off the link. We will make this precise.

A solid $(F : Y \to BO(R))$-structure on a rank-r bundle ought to be (a lift to) the restriction to $BO(r)$ of the solid replacement of F:

$$
\begin{array}{ccc}
\overline{Y}|_r := BO(r) \times_{(\mathcal{V}^{\hookrightarrow})\{0\}} \overline{Y} & \dashrightarrow & \overline{Y} \\
\Big\downarrow & \ulcorner & \Big\downarrow \overline{F} \\
BO(r) & \longrightarrow & \mathcal{V}^{\hookrightarrow}
\end{array} \; .
$$

This is the space of morphisms in $\mathcal{V}^{\hookrightarrow}$ that start in $BO(r)$ and end in the image of F inside $BO(R)$.

Lemma 3.12. $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q) \simeq \mathrm{Hom}_{\mathcal{EX}(BO(r, R-r))}(p, q)$, where $p \in BO(r)$ and $q \in BO(R)$.

Proof. $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q)$ is equivalent to the homotopy-fibre of

$$p^* \colon \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(\mathcal{B} \boxplus \mathrm{O})}(*, *) \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(\mathcal{B} \boxplus \mathrm{O})}(*, *),$$

i.e., using that morphism spaces in the homotopy-coherent nerve are equivalent to those in the original topological category ([HK20]), the homotopy-fibre of

$$(- \boxplus p) \colon BO_{\amalg}^{\infty} \to BO_{\amalg}^{\infty},$$

at q. The connected component of q in $BO_{\amalg}^{\infty}$ is $BO(R)$, and p^* maps only $BO(R-r)$ into it, so we have

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q) &= (BO(R-r) \boxplus p) \times_{BO(R)\{0\}} BO(R)^{\Delta[1]} \times_{BO(R)\{1\}} \{q\} \\
&= \pi^{-1}(p) \times_{BO(R)\{0\}} BO(R)^{\Delta[1]} \times_{BO(R)\{1\}} \{q\} \\
&= P(BO(R))_{\pi^{-1}(p), q} \\
&\simeq \mathrm{Hom}_{\mathcal{EX}(BO(r, R-r))}(p, q)
\end{aligned}
$$

by Theorem 3.5. By $\pi = \mathrm{pr}_1$ we denoted the link projection in $BO(r, R-r)$. $\square$

Remark 3.13. We should note that $\overline{Y}|_r$ is *not*, in general, the ∞-categorical homotopy fibre product $BO(r) \times_{\mathcal{V}^{\hookrightarrow}}^{\mathrm{h}} Y = BO(r) \times_{(\mathcal{V}^{\hookrightarrow})\{0\}} \mathrm{Isom}(\mathcal{V}^{\hookrightarrow}) \times_{(\mathcal{V}^{\hookrightarrow})\{1\}} Y$ in the sense of [Lur23, §01DE]: isomorphisms in $\mathcal{V}^{\hookrightarrow}$ from $BO(r)$ to $F(Y)$ exist iff $r = R$.

Proposition 3.14. $\overline{Y}|_r \simeq (BO(r) \times BO(R-r)) \times_{BO(R)}^{\mathrm{h}} Y$.

Proof. Written in full, the statement reads

$$BO(r) \times_{(\mathcal{V}^{\hookrightarrow})\{0\}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})\{1\}} Y$$
$$\overset{\simeq}{}$$
$$(BO(r) \times BO(R-r)) \times_{BO(R)\{0\}} BO(R)^{\Delta[1]} \times_{BO(R)\{1\}} Y.$$

By direct on inspection of $\mathbb{U}$ on exit paths of maximal index (see the proof of Theorem 3.5) and by Lemma 3.12 we have that $\mathbb{U} \colon \mathcal{E}\mathfrak{X} = \mathcal{E}\mathfrak{X}(BO(r, R-r)) \to \mathcal{V}^{\hookrightarrow}$ is fully-faithful, whence it is an equivalence onto its image $\mathcal{V}^{\hookrightarrow}(r, R)$, the full sub-$\infty$-category generated by $BO(r) \amalg BO(R) \subset \mathcal{V}^{\hookrightarrow}$. Thus, we observe that in the diagram

$$
\begin{array}{ccccccc}
\mathcal{F} \times_{BO(R)} Y & \dashrightarrow & \mathcal{F} & \overset{\ulcorner}{\hookleftarrow\!-\!-\!-\!\to} & \mathcal{E}\mathfrak{X}^{\Delta[1]} & \overset{\sim}{\longrightarrow} & \mathcal{V}^{\hookrightarrow}(r, R)^{\Delta[1]} \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_0 \times \mathrm{ev}_1} & & \big\downarrow{\scriptstyle \mathrm{ev}_0 \times \mathrm{ev}_1} \\
& & BO(r) \times BO(R) & \longhookrightarrow & \mathcal{E}\mathfrak{X}^{\times 2} & \overset{\sim}{\longrightarrow} & \mathcal{V}^{\hookrightarrow}(r, R)^{\times 2} \\
\big\downarrow & & \big\downarrow & & & & \\
Y & \overset{F}{\longrightarrow} & BO(R) & & & &
\end{array}
\quad,
$$

$\mathcal{F}$ coincides with $BO(r) \times_{(\mathcal{V}^{\hookrightarrow})\{0\}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})\{1\}} BO(R)$, and, by the pasting lemma and again by Theorem 3.5, also with

$$
\mathcal{E}\mathfrak{X}^{\Delta[1]} \times_{\mathcal{E}\mathfrak{X}^{\times 2}} (BO(r) \times BO(R)) = BO(r) \times_{\mathcal{E}\mathfrak{X}\{0\}} \mathcal{E}\mathfrak{X}^{\Delta[1]} \times_{\mathcal{E}\mathfrak{X}\{1\}} BO(R)
$$
$$
\simeq (BO(r) \times BO(R-r)) \times_{BO(R)\{0\}} BO(R)^{\Delta[1]}.
$$

Thus, both sides in the statement are equivalent to $\mathcal{F} \times_{BO(R)} Y$. $\qquad\square$

In terms of lifts, Proposition 3.14 reads:

Corollary 3.15. *A solid F-structure, in the sense of Definition 3.9, on a smooth manifold X of dimension r, is a cartesian F-structure, in the sense of Definition 3.11, on the (trivially-linked) manifold X.*

3.4. **Replacements, II.**

3.5. **A Grothendieck construction.**

3.6. **The linked universal frame and tautological bundles.**

4. Duals

Let $\mathcal{B} = Y \xrightarrow{F} BO(n) \subset \mathcal{V}^{\hookrightarrow}$ be a classical tangential structure. In this section, we introduce, for a smooth manifold or linked space X with solid $\mathcal{B}$-structure, the notion of a (quadratic) dual[4] and a collar, both of which depend on the solid structure in question. The two notions are closely related – the collar $\overline{X}$ is to be an extension of X that possesses on-the-nose $\mathcal{B}$-structure, and the dual $X^!$ is a space that parametrises the difference between X and $\overline{X}$. That is, the solid structure on X gives rise to data resembling that of a fibration with typical fibre X:

$$
\begin{array}{ccc}
X & \longhookrightarrow & \overline{X} \\
& & \big\downarrow{\scriptstyle p} \\
& & X^!
\end{array}
\quad .
$$

A $\mathcal{B}$-structured disk algebra A (necessarily n-dimensional) gives rise to a locally-constant factorisation algebra (*lcfa*) $\int_- A$ – which we sometimes abusively refer to

---

[4](so named as the idea resembles that of [GK94, §2.1.9])

as 'A' – not on X but on $\overline{X}$, which can be pushed forward down to $X^!$ along the constructible bundle map p, giving a lcfa thereon ([AFT17, §2.5]). These pushforwards are to factorisation homology as integration along fibres is to integration.

When X is a bordism, and if we rigidify from solid to stable, then $X^!$ is a particulary simple stratified/linked space, and this procedure produces a functorial field theory (FFT) of top dimension n landing in a Morita category (that of 1-disk algebras and 0-disk algebra bimodules) with values in the target symmetric-monoidal category of A. In this article, we restrict ourselves to codimension-1 field theories, so linked spaces of depth 1 will suffice.

Similar ideas have appeared in [Sch14], which expanded on an idea in [Lur08, §4]; see also [CS19] and the first half of [CHS21]. We should note that our approach differs from these in a number of important aspects:

(1) Once the background on linked spaces is established, the construction is much simpler; so much so that at this stage we can already start considering examples (see below).

(2) It works for any stable tangential structure with which one chooses to endow bordisms, not just for framings. We will also briefly discuss a possible extension to bordisms with solid tangential structure.

(3) In particular, we do not use Dunn–Lurie additivity ([Dun88]; [Lur17]), nor assume any additivity properties of our input disk algebras.

(4) We do not use the Lurie–Calaque–Scheimbauer (∞, n)-category model for bordisms, where a bordism is defined, roughly speaking, by cut functions on a manifold of full dimension. Instead, we deal with bordisms in the more familiar sense of the word, and so the term 'TQFT' carries for us the more practical (and older) Atiyah–Segal–Witten meaning.

(5) Our treatment of cutting-and-gluing is quite on-the-nose and conforms to that in more physically-minded literature, such as e.g. in [CMR18].

4.1. **Exposition, I: Examples.** We will consider a variety of examples before defining p formally.

Notation 4.1. In this section, we will resume here the notation $\boldsymbol{n} \coloneqq \mathbb{R}^n$. This will help elucidate the crucially different roles of some $\mathbb{R}$-factors that appear below.

Example 4.2 (codimension-0 bordisms, $\partial = \emptyset$). Say $X = M$ is a smooth n-manifold (which, in our convention, means $\partial M = \emptyset$), which needn't be compact. A solid $\mathcal{B}$-structure on M is necessarily an on-the-nose $\mathcal{B}$-structure. Then $\overline{X} = M \times \boldsymbol{0} = M$, and $M^! = \boldsymbol{0}$. Pushing A forward along the trivial M-bundle

$$
\begin{array}{c}
M \\
\big\downarrow \\
\boldsymbol{0}
\end{array}
$$

produces a 0-disk algebra with underlying object $\int_M A$. That is, top-dimensional bordisms without boundary are evaluated simply by factorisation homology. The pointing, i.e., the structure of a 0-disk algebra, is given by the functoriality of $\int_{\_} A$ along the inclusion of the empty set.

Example 4.3 (codimension-1 bordisms, $\partial = \emptyset$). Consider now a smooth manifold N with $\partial N = \emptyset$ of dimension $n - 1$. A stable $\mathcal{B}$-structure on N is a $\mathcal{B}$-structure on

$\overline{N} = N \times \mathbf{1}$, and we have $N^! = \mathbf{1}$. Pushing A forward along the product N-bundle

$$
\begin{array}{c}
N \times \mathbf{1} \\
\downarrow \\
\mathbf{1}
\end{array}
$$

produces a 1-disk algebra. Its underlying object is simply $\int_{N \times \mathbb{R}} A$.

Example 4.4 (codimension-0 bordisms, $\partial \neq \emptyset$ and connected). Let now M be a top-dimensional bordism with *connected* non-empty boundary $\partial = \partial M$. We consider it as a linked space $\mathfrak{M}$ of type

$$
\partial \overset{=}{\Leftarrow} \partial \overset{\iota_+}{\longrightarrow} M^\circ,
$$

as in Example 2.13. Recall that the restriction of $T\mathfrak{M}$ to the link ∂ is given, up to isomorphism, by the bundle $T\partial \oplus \varepsilon^1$. A stable $\mathcal{B}$-structure on $\mathfrak{M}$ (Section 3.4) amounts to a $\mathcal{B}$-structure on M° and a $\mathcal{B}$-structure on $\partial \times \mathbf{1}$ together with an equivalence of $\mathcal{B}$-structures on $\partial \times \mathbf{1}$ between the one mentioned and the one induced from the one on M° along $\mathrm{pr}_1^* \iota_+^* TM^\circ \simeq \mathrm{pr}_1(T\partial \oplus \varepsilon^1) \simeq \mathrm{pr}_1^* T(\partial \times \mathbf{1})$, where $\mathrm{pr}_1 \colon \partial \times \mathbf{1} \to \partial$ denotes the coordinate projection. E.g., when $\mathcal{B}$ classifies n-framings, this means that the framing splits along the boundary in the usual sense.

In order to make the notation suggestive, let us assume we are in an oriented context and ∂ is the incoming boundary. (This has no bearing on what follows except in notation.) This datum determines a $\mathcal{B}$-structure on

$$
\tag{19} \overline{M} = \partial \times \mathbb{R}_{\leq 0} \coprod_{\partial \times \{0\}} M.
$$

We now consider this collar as a linked space $\overline{\mathfrak{M}}$, still of depth one but with three strata:

$$
\tag{20}
$$

or

$$
\tag{21} \partial \times \mathbf{1} \xleftarrow{\mathrm{id} \times \{+\}} \partial \xrightarrow{\iota_+} M^\circ
$$

for short (see Figure 1). The stable $\mathcal{B}$-structure on $\mathfrak{M}$ is tantamount to $\mathcal{B}$-structures on the two higher strata $\partial \times \mathbf{1}$, M° of $\overline{\mathfrak{M}}$, and to an equivalence between their pullbacks onto the common link ∂.

We read off the quadratic dual directly from $\mathfrak{M}$:

$$
\mathfrak{M}^! = \left(\mathbf{1} \xleftarrow{+} \{0\} = \mathbf{0} \xrightarrow{=} \mathbf{0} \right),
$$

namely the linked half-line $\mathbb{R}_{\leq 0}$. The projection

$$
\begin{array}{c}
\overline{\mathfrak{M}} \\
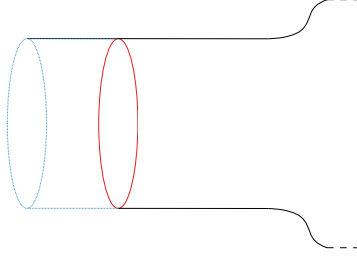\downarrow{\scriptstyle p} \\
\mathfrak{M}^!
\end{array}
$$

FIGURE 1. The three strata of the linked stable collar of a manifold with connected boundary.

off the collar is given by

(22)

$$
\begin{array}{c}
\partial \times \mathbf{1} \xleftarrow{\mathrm{id}\times\{+\}} \partial \xrightarrow{\iota_+} M^\circ \\
\end{array}
$$

for which we also write

(23)
$$
\begin{array}{ccc}
\partial \times \mathbf{1} & \xleftarrow{\mathrm{id}\times\{+\}} \partial \xrightarrow{\iota_+} & M^\circ \\
\downarrow & \downarrow & \downarrow \\
\mathbf{1} & \xleftarrow[+]{} \mathbf{0} \xrightarrow[=]{} & \mathbf{0}
\end{array},
$$

with the above meaning understood.[5] in The 'typical fibre' inclusion $\mathfrak{M} \hookrightarrow \overline{\mathfrak{M}}$ – a description which here begins to lose its familiar meaning – is given in the in the obvious manner. The linked projection p is the systematised version of the obvious projection $\overline{M} \to \mathbb{R}_{\leq 0}$ which remembers every stratum's history. (A disk algebra on this target, the pushforward algebra, is determined by a 1-disk algebra and a pointed left-module.) Because of this, it lends itself to generalisation, as we show below.

Remark 4.5. The examples above show clearly that $X^!$ parametrises the normal bundle of the stable $\mathcal{B}$-structure on X. In the non-trivially stratified Example 4.4 we see, however, that the classifying map thereof is stratification-reversing, which is why we only speak out loud about the stratification-preserving projections $\overline{X} \to X^!$. That oughtn't suggest that it is impossible to express classifiers of stratified normal bundles within an appropriate framework. When $\mathcal{B}$ is not a classical tangential

---

[5]We introduce better notation in Notation 4.13 which however obscures the link maps. For this reason, we keep this more verbose version until then.

structure, however, both the notion of collar and that of quadratic dual become very problematic, as the rank of the normal bundle may both increase or decrase as one moves along strata. This makes a general description of $X^!$ very hard, if not impossible. We restrict ourselves to (relaxations of) classical $\mathcal{B}$.

Example 4.6 (codimension-0 bordisms, $\partial = \partial_L \amalg \partial_R$). Let M be an n-dimensional bordism and $\partial M = \partial_L \amalg \partial_R$ with ∂_L, ∂_R connected. We see M as a linked space $\mathfrak{M}$ of depth 1 but now with three strata:

$$
\begin{array}{ccccc}
& \partial_L & & \partial_R & \\
\stackrel{=}{\swarrow} & & \stackrel{\iota_{+L}}{\searrow} \quad \stackrel{\iota_{+R}}{\swarrow} & & \stackrel{=}{\searrow} \\
\partial_L & & M^\circ & & \partial_R
\end{array} \quad .
$$

The meaning of a stable $\mathcal{B}$-structure on $\mathfrak{M}$ can be expressed analogously to Example 4.4.[6] We see that the collar $\overline{\mathfrak{M}}$, in short and suggestive notation (cf. (21)), is

$$
\partial_L \times \mathbf{1} = \partial_L \times \mathbb{R}_{<0} \xleftarrow{\mathrm{id} \times \{+_L\}} \partial_L \xrightarrow{\iota_{+L}} M^\circ \xleftarrow{\iota_{+R}} \partial_R \xrightarrow{\mathrm{id} \times \{+_R\}} \partial_R \times \mathbb{R}_{>0} = \partial_R \times \mathbf{1},
$$

the linked version of $\overline{M} = \partial_L \times \mathbb{R}_{\leq 0} \amalg_{\partial_L \times \{0\}} M \amalg_{\partial_R \times \{0\}} \partial_R \times \mathbb{R}_{\geq 0}$, and the dual $\mathfrak{M}^!$ is

$$
\begin{array}{ccccc}
& \mathbf{0} & & \mathbf{0} & \\
\stackrel{+_L}{\swarrow} & & \searrow \quad \swarrow & & \stackrel{+_R}{\searrow} \\
\mathbf{1} & & \mathbf{0} & & \mathbf{1}
\end{array} \quad ,
$$

the linked version of $M^! = \mathbb{R}_{\{0\}}$, the real line with three-fold stratification given by (say) the defect $\{0\} \subset \mathbb{R}$ the two components of its complement. We write $\mathfrak{M}^! = \left(\mathbf{1} \xleftarrow{+_L} \mathbf{0} \xrightarrow{+_R} \mathbf{1} \right)$. We observe the projection $p \colon \overline{\mathfrak{M}} \to \mathfrak{M}^!$ to be given, in short notation (cf. (23)), by

$$
\begin{array}{ccccccc}
\partial_L \times \mathbf{1} & \xleftarrow{\mathrm{id} \times \{+_L\}} & \partial_L & \xrightarrow{\iota_{+L}} & M^\circ & \xleftarrow{\iota_{+R}} & \partial_R & \xrightarrow{\mathrm{id} \times \{+_R\}} & \partial_R \times \mathbf{1} \\
\downarrow & & & & \downarrow & & & & \downarrow \\
\mathbf{1} & & \xleftarrow{\quad +_L \quad} & & \mathbf{0} & & \xrightarrow{\quad +_R \quad} & & \mathbf{1}
\end{array}
$$

where this time we omitted the induced maps on links.

Example 4.7 (codimension-1 bordisms, $\partial \neq \emptyset$ and connected). Let M be a bordism of dimension $n-1$ with connected boundary $\partial = \partial M$. As an ordinary space, the collar will be given simply by the product of (19) with $\mathbf{1}$. For us, it has again three strata, so $\overline{\mathfrak{M}}$, the analogue of (20), is given by

$$
\begin{array}{ccccc}
& \partial \times \mathbf{1} & & \partial \times \mathbf{1} & \\
\stackrel{(\mathrm{id}_\partial \times \{+\}) \times \mathrm{id}_{\mathbf{1}}}{\nearrow} & & \searrow \stackrel{=}{} \quad \stackrel{=}{} \nwarrow & & \stackrel{\iota_+ \times \mathrm{id}_{\mathbf{1}}}{\searrow} \\
\partial \times \mathbf{2} = \partial \times \mathbf{1} \times \mathbf{1} & & \partial \times \mathbf{1} & & M^\circ \times \mathbf{1}
\end{array}
$$

and the collar by

$$
\mathfrak{M}^! = \left(\mathbf{2} \hookleftarrow \mathbf{1} \xrightarrow{=} \mathbf{1} \right),
$$

---

[6]As the constructions of the stable collar and stable dual are the same for any classical $\mathcal{B}$, we will no longer mention this.

the linked half-plane $\mathbb{H}$. The projection $\overline{\mathfrak{M}} \to \mathfrak{M}^!$ is given exactly as in (22) up to a factor of $\mathbf{1}$.

Example 4.8 (codimension-1 bordisms, $\partial = \partial_L \amalg \partial_R$). This is, similarly to the discussion of codimension-0 bordisms, a small modification of Example 4.7. The dual is given, in short notation (cf. Example 4.6), by

$$\mathbf{2} \hookleftarrow \mathbf{1} \hookrightarrow \mathbf{2},$$

the linked version $\mathbb{R}^2_{\mathbb{R}}$, the real plane with three-fold stratification induced by (say) the defect $\{x = 0\} \subset \mathbb{R}^2$.

A distinct advantage of this approach is that it treats defects on the same footing as boundaries.

Example 4.9 (closed submanifold defects). Let M be a smooth n-manifold and $\Sigma \subset M$ a smooth submanifold of codimension m, which yields a linked manifold $\mathfrak{M}$ with link $\mathbb{S} := L = \mathbb{S}(\mathrm{N}_M\Sigma)$ (see Example 2.12). Assume $\mathbb{S}$ is connected, so Σ has codimension at least 2. (Otherwise, we speak of a 'cut locus' – to be treated in Section 4.2.) If $\mathfrak{M}$ is constructible (in that the normal bundle of $\mathbb{S}$ inside $M \smallsetminus \Sigma$ is trivialisable), then the collar $\overline{\mathfrak{M}}$ is again three-fold stratified, given by

$$
\begin{array}{ccccccc}
 & & \mathbb{S} & & & \mathbb{S} & \\
 & \swarrow & & \searrow & \nearrow & & \searrow \\
 & & & \stackrel{=}{\searrow} & \stackrel{=}{\swarrow} & & \\
\mathbb{S} \times \mathbf{1} & & & & \mathbb{S} & & M \smallsetminus \Sigma
\end{array}
$$

and the dual $\mathfrak{M}^!$ by the half-line $\left(\mathbf{1} \hookleftarrow \mathbf{0} \stackrel{=}{\to} \mathbf{0}\right)$ together with the obvious projection $\overline{\mathfrak{M}} \to \mathfrak{M}^!$. Thus, closed submanifold defects are treated exactly the same way as bordisms. Analogous considerations apply when M has dimension strictly smaller than n (see Example 4.7), or when there are multiple non-intersecting defects, or both (cf. Examples 4.6 and 4.8).

Remark 4.10. Note that the 'right side' of the collar in Example 4.9, $\mathbb{S} \stackrel{=}{\leftarrow} \mathbb{S} \hookrightarrow M \smallsetminus \Sigma$ is simply the linked version of the blow-up (a.k.a. unzip in the conically-smooth literature) of M at Σ. In light of the previous examples, we can identify the linked spaces induced by bordisms as those whose blow-ups coincide with themselves. The 'left side', in contrast is simply the collar as in the simple codimension-1 case of Example 4.3.

Remark 4.11. There is a different approach to closed submanifold defects than Example 4.9 which reflects the geometry around the defect submanifold better. We will explore this later. The version above should be read as the bare-bones, minimalist approach.

4.2. **Exposition, II: Cutting and gluing.** Finally, we will discuss cutting-and-gluing.
    Let M be (for simplicity) a bordism of dimension n, $\partial = \partial M = \partial_L \amalg \partial R$ the components of its boundary, and let $\iota \colon \Sigma \subset M^\circ$, a *cut locus*, be a closed codimension-1 submanifold in the interior, such that $M = M_L \cup_{\Sigma \times \mathbb{R}} M_R$ is a collar-gluing with $\partial_L \subset M_L$, $\partial_R \subset M_R$. These data are organised as a linked space $\mathfrak{M}_\Sigma$ in the obvious manner by mixing Examples 2.12 and 2.13. Writing $M_L^\circ = M_L \smallsetminus \partial_L$

and $M_R^\circ = M_R \smallsetminus \partial_R$, we have that the stable collar $\overline{\mathfrak{M}_\Sigma}$ is

$$
\begin{array}{ccccccccccccccccc}
& \partial_L & & \partial_L & & \Sigma & & \Sigma & & \Sigma & & \Sigma & & \partial_R & & \partial_R & \\
& \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\
\partial_L \times \mathbf{1} & & \partial_L & & M_L^\circ & & \Sigma & & \Sigma \times \mathbf{1} & & \Sigma & & M_R^\circ & & \partial_R & & \partial_R \times \mathbf{1}
\end{array}
$$

$,$

the linked version of

$$\overline{M_\Sigma} = \partial_L \times \mathbb{R}_{\leq 0} \cup_{\partial_L \times \{0\}} M_L \cup_{\Sigma \times \mathbb{R}_{<1}} \Sigma \times \mathbb{R} \cup_{\Sigma \times \mathbb{R}_{>-1}} M_R \cup_{\partial_R \times \{0\}} \partial_R \times \mathbb{R}_{\geq 0}.$$

The dual $\mathfrak{M}_\Sigma^!$ is

$$
\begin{array}{ccccccccc}
& \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & \\
& \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\
\mathbf{1} & & \mathbf{0} & & \mathbf{1} & & \mathbf{0} & & \mathbf{1}
\end{array}
$$

$,$

the linked version of $\mathbb{R}_{\{\pm 1\}}$, the real line with defects $\{-1\}$, $\{1\}$. The projection $p_\Sigma \colon \overline{\mathfrak{M}_\Sigma} \to \mathfrak{M}_\Sigma^!$ is, in short notation, as follows:

$$
\begin{array}{ccccccccccccccc}
\partial_L \times \mathbf{1} & \hookleftarrow & \partial_L & \hookrightarrow & M_L^\circ & \hookleftarrow & \Sigma & \hookrightarrow & \Sigma \times \mathbf{1} & \hookleftarrow & \Sigma & \hookrightarrow & M_R^\circ & \hookleftarrow & \partial_R & \hookrightarrow & \partial_R \times \mathbf{1} \\
\downarrow & & & & \downarrow\downarrow\downarrow & & & & \downarrow & & & & \downarrow\downarrow\downarrow & & & & \downarrow \\
\mathbf{1} & \longleftarrow & & & \mathbf{0} & & \longrightarrow & & \mathbf{1} & & \longleftarrow & & \mathbf{0} & & \longrightarrow & & \mathbf{1}
\end{array}
$$

By *cutting-and-gluing* (*of M along Σ*) we mean the *refinement* map (a map whose restrictions to the strata are all isomorphisms – a barbarian notion of recollement that will suffice for our purposes)

$$r \colon \mathfrak{M}_\Sigma \to \mathfrak{M},$$

with target as in Example 4.6, given, after decorating the outward copies of the cut locus with ℓ/r, by

$$
\begin{array}{ccccccccccccccccc}
\partial_L & \twoheadleftarrow & \partial_L & \hookrightarrow & M_L^\circ & \hookleftarrow & \Sigma_\ell & \twoheadrightarrow & \Sigma_\ell & \twoheadleftarrow & \Sigma_\ell & \hookrightarrow & \Sigma \times \mathbf{1} & \hookleftarrow & \Sigma_r & \twoheadrightarrow & \Sigma_r & \twoheadleftarrow & \Sigma_r & \hookrightarrow & M_R^\circ & \hookleftarrow & \partial_R & \twoheadrightarrow & \partial_R \\
\downarrow & & & & & & & & & & & & \downarrow & & & & & & & & & & & & \downarrow \\
\partial_L & \twoheadleftarrow & & & \partial_L & & \hookrightarrow & & & & & & M^\circ & & & & & & & & \partial_R & & & & \partial_R
\end{array}
$$

where the vertical maps are all (restricted) identities. For better legibility, we rendered the links in red.

Remark 4.12 (Coda). Even though we have postponed a systematic treatment of maps of linked spaces until Section 4.4, we should note here that, in order to conform to Definition 4.16, it remains to specify

$$
\begin{array}{ccc}
\Sigma_{\ell/r} & \dashrightarrow & (M^\circ)^I \\
\downarrow & & \downarrow \\
\Sigma_{\ell/r} \times M_{L/R}^\circ & \longrightarrow & M^\circ \times M^\circ
\end{array}
\quad,\quad
\begin{array}{ccc}
\Sigma_{\ell/r} & \dashrightarrow & (M^\circ)^I \\
\downarrow & & \downarrow \\
\Sigma_{\ell/r} \times (\Sigma \times \mathbf{1}) & \longrightarrow & M^\circ \times M^\circ
\end{array}
\quad,
$$

which are all given analogously. For instance, for the map $\Sigma_\ell \to (M^\circ)^I$ on the left, first reparametrise the gluing line as $\mathbb{R} \cong (-1,1)$ such that $\Sigma \times \mathbf{1} = \Sigma \times (-1,1)$ is the middle embedding. Extend now the tubular neighbourhood of Σ to times $(-2,2) \supset (-1,1)$. The link embedding $\Sigma_\ell \hookrightarrow M_L^\circ$ is given by following for a nonzero time τ with $-1 > \tau > -2$ the outward nonvanishing vector field X_Σ along Σ that

frames the rank-1 normal bundle of $\Sigma \subset M$ (whereas $\Sigma_r \hookrightarrow M_R^\circ$ is determined by following it for time $-\tau$), while the stratum map $\Sigma_\ell \to M^\circ$ hits the copy at time $-1 \in (-2, 2)$. Now, the map $\Sigma_\ell \to (M^\circ)^I$ at $p \in \Sigma_\ell$ can be chosen to be the path from p at time -1 to p at time τ along X_Σ. See Remark 4.21 for a more general version of this construction.

Now, as we show in Section 4.9, the operation $\overline{(-)} \twoheadrightarrow (-)^!$ is compatible with cutting and gluing, in the sense that it is covariant along refinements. Using Example 4.6 to get $p : \overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$, we have commuting square

(24)
$$
\begin{array}{ccc}
\overline{\mathfrak{M}_\Sigma} & \xrightarrow{\ \overline{r}\ } & \overline{\mathfrak{M}} \\
{\scriptstyle p_\Sigma}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
\mathfrak{M}^!_\Sigma & \xrightarrow[\ r^!\ ]{} & \mathfrak{M}^!
\end{array} \ .
$$

which we call a *cutting-and-gluing square*. In fact, both $\overline{r}$ and $r^!$ will be (linked versions of) constructible bundles, so that pushing forward an algebra along either path in the diagram will give equivalent results.

Before proceeding, let us lighten the notation even further by setting the following convention:

Notation 4.13. We will write

$$ M \rightarrowtail^{\ L\ } N $$

for a linked space of type $M \xleftarrow{\pi} L \xhookrightarrow{\iota} N$.

Now, $\overline{r} : \overline{\mathfrak{M}_\Sigma} \to \overline{\mathfrak{M}}$ is

$$
\begin{array}{ccccccccccccc}
\partial_L \times \mathbf{1} & \rightarrowtail & \partial_L & \leftarrowtail & M_L^\circ & \rightarrowtail & \Sigma & \rightarrowtail & \Sigma \times \mathbf{1} & \rightarrowtail & \Sigma & \rightarrowtail & M_R^\circ \rightarrowtail \partial_R \rightarrowtail \partial_R \times \mathbf{1} \\
\downarrow & & \downarrow & & & & & & & & & & \\
\partial_L \times \mathbf{1} & \rightarrowtail & \partial_L & \rightarrowtail & & & M^\circ & & & & & & \partial_R \rightarrowtail \partial_R \times \mathbf{1}
\end{array}
$$

and $r^! : \mathfrak{M}^!_\Sigma \to \mathfrak{M}^!$ is

$$
\begin{array}{ccccccccc}
\mathbf{1} & \rightarrowtail & \mathbf{0} & \leftarrowtail & \mathbf{1} & \rightarrowtail & \mathbf{0} & \leftarrowtail & \mathbf{1} \\
\downarrow & & & \searrow & \downarrow & \swarrow & & & \downarrow \\
\mathbf{1} & & & \rightarrowtail & \mathbf{0} & \rightarrowtail & & & \mathbf{1}
\end{array} \ ,
$$

the linked version of the map $\mathbb{R}_{\{\pm 1\}} \twoheadrightarrow \mathbb{R}_{\{0\}}$ that collapses $[-1, 1]$ onto $\{0\}$ and scales up the two sides of the former to $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$. We see by direct inspection that (24) commutes. See Figure 2.

4.3. **Exposition, III: Corners.** We will conclude the expository sections by giving an informal treatment of corners in the linked context, as well as the extension of $\overline{(-)} \twoheadrightarrow (-)^!$ to depth 2. In future work, this will be formalised, and extended to arbitrary depth, though the idea should become clear already here.

Figure 3 gives an essentially full picture of $\overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ induced by a manifold M (for simplicity of full dimension n – otherwise see Examples 4.7 and 4.8) with *a* corner – the passage to multiple corners is analogous to the passage to multiple boundary components (cf. Example 4.6).

Here, $\overline{\mathfrak{M}}$ is a depth-2 linked space with 9 strata: those of full dimension are (isomorphic to) $\partial^2 \times \mathbf{2}$, $\partial_R \times \mathbf{1}$, $\partial_L \times \mathbf{1}$ and M^2; then there is a cross' worth of
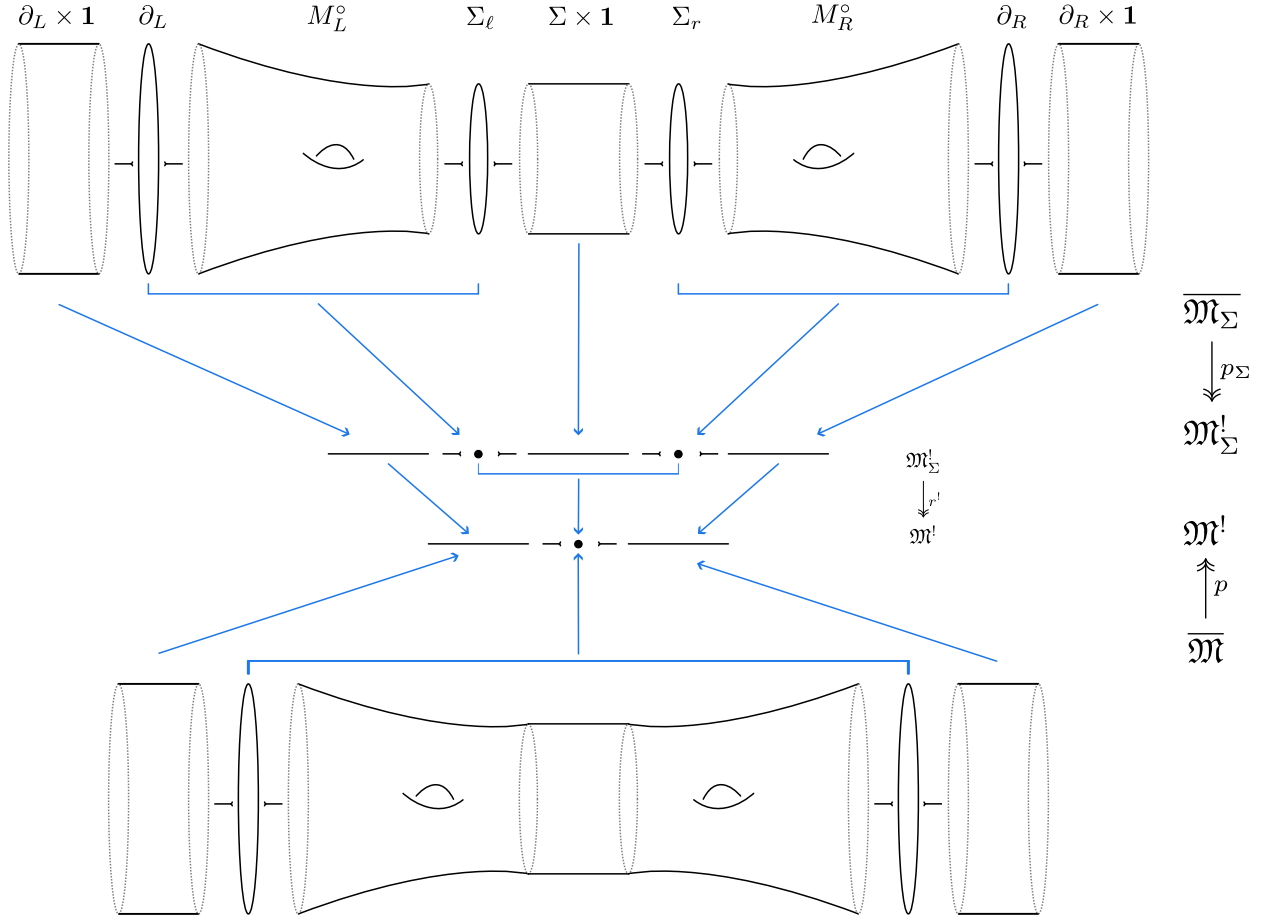
FIGURE 2. The cutting-and-gluing square (24) using Notation 4.13; $\overline{r} \colon \overline{\mathfrak{M}_\Sigma} \to \overline{\mathfrak{M}}$ omitted.

codimension-1 strata, depicted by solid lines in Figure 3, and finally their codimension-2 intersection, depicted by the solid point.

One can essentially see how the depth-2 dual

$$\mathfrak{M}^! = \begin{pmatrix} \partial^2 \rightarrowtail\!\!\!\!\!- \partial_R \\ \curlyvee \quad\diagdown\quad \curlyvee \\ \partial_L \rightarrowtail\!\!\!\!\!- M^\circ \end{pmatrix}$$

must be assembled by considering the adjacent depth-1 pairs. This is illustrated in Figure 4

Remark 4.14 (Coda). (This remark assumes familiarity with Definition 4.15.) Indeed, $\mathfrak{M}^!$ is, as a stratified space, the quarter-plane $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\leq 0}$. Let us introduce it as a *product linked space*, the simplest type of linked space of depth 2. . Each
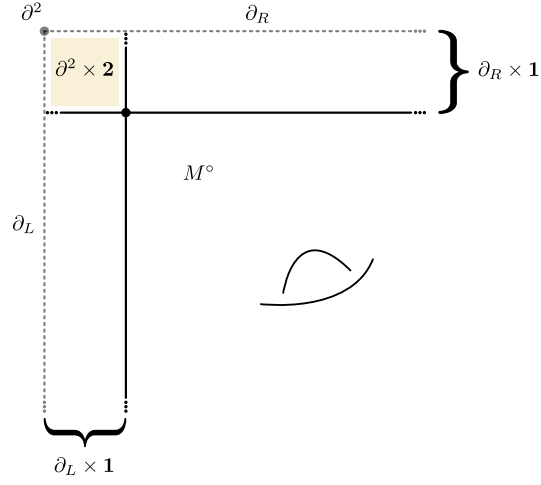
FIGURE 3. Construction of $\overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ induced by M with connected boundary $\partial = \partial_L \cup \partial^2 \cup \partial_R$, with ∂^2 denoting the corner.
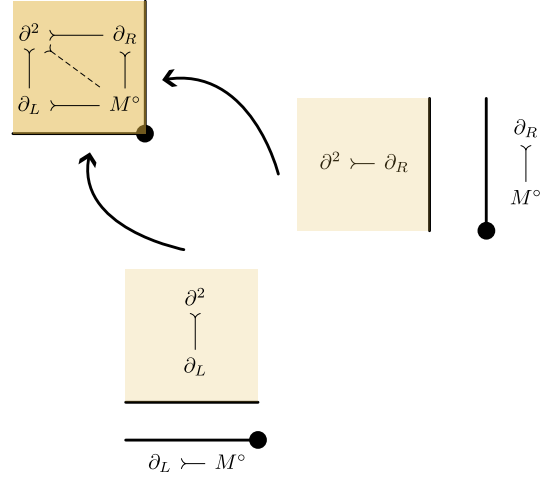

FIGURE 4. The assembly of a corner: $\mathfrak{M}^!$ from depth-1 data.

factor is $\mathbf{0} \stackrel{\mathbf{0}}{\rightarrowtail} \mathbf{1}$ stratified over $\mathfrak{P} = \{0 < 1\}$, so the product is of type

$$
\begin{array}{ccc}
\mathbf{0} \times \mathbf{0} & \stackrel{\mathbf{0}}{\rightarrowtail} & \mathbf{1} \times \mathbf{0} \\
{\scriptstyle \mathbf{0}} \big\uparrow & {\scriptstyle \mathbf{0} \times \mathbf{0}} & \big\uparrow {\scriptstyle \mathbf{0}} \\
\mathbf{1} \times \mathbf{0} & \stackrel{}{\underset{\mathbf{0}}{\rightarrowtail}} & \mathbf{1} \times \mathbf{1}
\end{array}
$$

and is stratified over

$$
\mathfrak{P} \times \mathfrak{P} = \left\{
\begin{array}{ccc}
(0,0) & < & (1,0) \\
\wedge & \diagdown & \wedge \\
(1,0) & < & (1,1)
\end{array}
\right\}.
$$

In a general product of depth-1 linked spaces, the new links will be given, for $(\mathfrak{p}, \mathfrak{q}) \leq (\mathfrak{p}', \mathfrak{q}')$ in $\mathfrak{P} \times \mathfrak{Q}$, by

$$L_{(\mathfrak{p},\mathfrak{q}),(\mathfrak{p}',\mathfrak{q}')} = L_{\mathfrak{p},\mathfrak{p}'} \times L_{\mathfrak{q},\mathfrak{q}'}.$$

In a general linked space of depth 2 (or indeed in any depth) indexed over a poset $\mathfrak{P}$ of depth 2, we ask that, for each concatenation $\mathfrak{p} < \mathfrak{q} < \mathfrak{r}$, commuting diagrams

$$L_{\mathfrak{p}\mathfrak{q}} \times_{M_\mathfrak{q}} L_{\mathfrak{q}\mathfrak{r}} \dashrightarrow L_{\mathfrak{p}\mathfrak{r}}$$
$$M_\mathfrak{p} \times M_\mathfrak{r}$$

We call the maps $L_{\mathfrak{p}\mathfrak{q}} \times_{M_\mathfrak{q}} L_{\mathfrak{q}\mathfrak{r}} \to L_{\mathfrak{p}\mathfrak{r}}$ *concatenation* maps and the condition that they cover $M_\mathfrak{p} \times M_\mathfrak{r}$ that concatenation be *rel endpoints*. The terminology is justified in view Theorem 3.5; see also Remark 4.21.

Finally, the projection $p \colon \overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ is visible in Figure 3: $\partial_{R/L} \times \mathbf{1}$ project to the two coordinates of the quarter plane, $\partial^2 \times \mathbf{2}$ projects to $\mathbf{2}$, and M° and its closure are collapsed to the corner point of the quarter plane.

4.4. **Maps of linked spaces.** Contrary to best practice, we have so far carefully worked around the issue of defining maps of linked spaces formally, as it wasn't strictly necessary, and have made do with span maps as they clearly induce maps between the respective exit path ∞-categories. The technical reason is that a simple approach to linked maps had to wait until after Section 3.2. The examples above, and the central interest of the projections $\overline{X} \to X^!$, show, however, that a more systematic treatment is inavoidable.

In this section, we propose a notion of a map of linked spaces. Any such notion should be well-behaved in two respects:

- maps should compose, and
- they should induce ∞-functors on exit paths.

Our proposal fulfills both criteria (Remark 4.17, Proposition 4.20) and enjoys a generous amount of simplicity.

Definition 4.15. A *linked space*

$$M \to \mathfrak{P}$$

of *depth* 1 is a collection of spaces indexed over a poset $\mathfrak{P}$ of depth 1: we have a space $M_\mathfrak{p}$ for each $\mathfrak{p} \in \mathfrak{P}$, and for each arrow $\mathfrak{p} \leq \mathfrak{q}$ a space $L_{\mathfrak{p}\mathfrak{q}}$, called a *link*, sitting in a span $M_p \xleftarrow{\pi} L_{pq} \xrightarrow{\iota} M_q$, such that

- if $\mathfrak{p} \leq \mathfrak{q}$ but $\mathfrak{p} \neq \mathfrak{q}$, then π is a proper fibre bundle and ι an embedding;
- if $\mathfrak{p} = \mathfrak{q}$, then

$$L_{\mathfrak{p}\mathfrak{p}} = M_\mathfrak{p}^I,$$

   the unbased path space of $M_\mathfrak{p}$ ($I = [0,1]$). The exponential is endowed with the compact-open topology, and

$$\pi = \mathrm{ev}_0, \quad \iota = \mathrm{ev}_1$$

   are, respectively, the source and target evaluations.

We call $M \to \mathfrak{P}$ a *linked manifold* (of *depth* 1) if all $M_\mathfrak{p}$, as well as $L_{\mathfrak{p}\mathfrak{q}}$ whenever $\mathfrak{p} \neq \mathfrak{q}$, are smooth riemannian manifolds. *Al coda: Remark 4.14.*

Definition 4.16. A *map* f, written

(25)
$$
\begin{array}{ccc}
M & \longrightarrow & N \\
\downarrow & & \downarrow \\
\mathfrak{P} & \longrightarrow & \mathfrak{Q}
\end{array}
,
$$

consists of a map of posets $\mathfrak{f}\colon \mathfrak{P} \to \mathfrak{Q}$ together with maps of spans

$$
f_{\mathfrak{p}\mathfrak{q}}\colon (M_{\mathfrak{p}} \leftarrow L_{\mathfrak{p}\mathfrak{q}} \to M_{\mathfrak{q}}) \to (N_{\mathfrak{f}(\mathfrak{q})} \leftarrow L'_{\mathfrak{f}(\mathfrak{p})\mathfrak{f}(\mathfrak{q})} \to N_{\mathfrak{f}(\mathfrak{q})})
$$

for each pair $\mathfrak{p} \leq \mathfrak{q}$ in $\mathfrak{P}$. This means that $f_{\mathfrak{p}\mathfrak{q}}$, with a slight abuse of notation, is a commutative diagram of the form

$$
\begin{array}{ccc}
L_{\mathfrak{p}\mathfrak{q}} & \xrightarrow{\;f_{\mathfrak{p}\mathfrak{q}}\;} & L'_{\mathfrak{f}(\mathfrak{p})\mathfrak{f}(\mathfrak{q})} \\
{\scriptstyle \pi \times \iota}\downarrow & & \downarrow{\scriptstyle \pi' \times \iota'} \\
M_{\mathfrak{p}} \times M_{\mathfrak{q}} & \xrightarrow{\;f_{\mathfrak{p}} \times f_{\mathfrak{q}}\;} & N_{\mathfrak{f}(\mathfrak{p})} \times N_{\mathfrak{f}(\mathfrak{q})}
\end{array}
$$

Remark 4.17. Since both poset maps and span maps compose, maps of linked spaces compose.

Warning 4.18. Diagrams of type (25), while literal in classical stratified geometry, are only figurative in the linked context!

We have thus obtained a notion of mapping from a non-trivially linked space $(M_0 \leftarrow L = L_{01} \to M_1) = (M \to \{0 < 1\})$ to a smooth space X. The latter is naturally a linked space indexed over the trivial poset $*$, so that the its full expression is $X \xleftarrow{\mathrm{ev}_0} X^I \xrightarrow{\mathrm{ev}_1} X$. Thus, a map

(26)
$$
\begin{array}{ccc}
M & \longrightarrow & X \\
\downarrow & & \downarrow \\
\{0 < 1\} & \longrightarrow & *
\end{array}
$$

is a commuting square

(27)
$$
\begin{array}{ccc}
L & \xrightarrow{\;f_L := f_{01}\;} & X^I \\
\downarrow & & \downarrow \\
M_0 \times M_1 & \xrightarrow{\;f_0 \times f_1\;} & X \times X
\end{array}
.
$$

The sector of (26) over the two identities in $\{0 < 1\}$,

$$
\begin{array}{ccc}
M_0^I & \longrightarrow & X^I \\
\downarrow & & \downarrow \\
M_0 \times M_0 & \longrightarrow & X \times X
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M_1^I & \longrightarrow & X^I \\
\downarrow & & \downarrow \\
M_1 \times M_1 & \longrightarrow & X \times X
\end{array}
,
$$

is determined by f_0 and f_1. *Al coda: Remark 4.12.*

What one might object to is that if the linked spaces in question are induced by depth-1 stratified spaces, then maps in our sense are more relaxed than stratified maps of the corresponding spaces: the latter give only a subset of the former. While this relaxation poses no problem for our purposes, let us illustrate it with the simplest example where it is detectable.

Example 4.19 (linked paths in a smooth space). Let $M \to \mathfrak{P}$ be induced by $\mathbb{R}_{\geq 0}$ with its standard stratification, i.e., $M_0 = \{0\}$, $M_1 = \mathbb{R}_{>0}$, $L = L_{01} = *$, with $\iota = \iota_+ \colon L = * \hookrightarrow \mathbb{R}_{>0}$ is given by the choice of some point $+ \in \mathbb{R}_{>0}$. (We keep to our implicit notation from Example 4.4.) Let also $X \to *$ be induced by a smooth manifold X as above. Then

$$
\begin{array}{ccc}
* & \xrightarrow{\gamma := f_{01}} & X^I \\
{\scriptstyle \mathrm{id} \times \iota_+} \downarrow & & \downarrow \\
\{0\} \times \mathbb{R}_{>0} & \xrightarrow{f_0 \times f_1} & X \times X
\end{array}
$$

is determined, besides the stratum-wise maps, by the choice of a single path

$$\gamma \colon f_0(0) \to f_1(+)$$

in X. Similarly, if $[0,1] \to \{0 < i > 1\}$ (i for *interior*) is the linked space of depth 1 with three strata induced by the standard stratification on $[0,1]$, with both links given again by $*$ and the embeddings into $(0,1)$ determined by a pair of points $\varepsilon < \delta$ in $(0,1)$, then a linked map

(28)
$$
\begin{array}{ccc}
[0,1] & \longrightarrow & X^I \\
\downarrow & & \downarrow \\
\{0 < i > 1\} & \longrightarrow & *
\end{array}
$$

is determined, besides the stratum-wise maps, by two paths

$$
\begin{aligned}
\gamma_\varepsilon &\colon f_0(0) \to f_i(\varepsilon), \\
\gamma_\delta &\colon f_1(1) \to f_i(\delta)
\end{aligned}
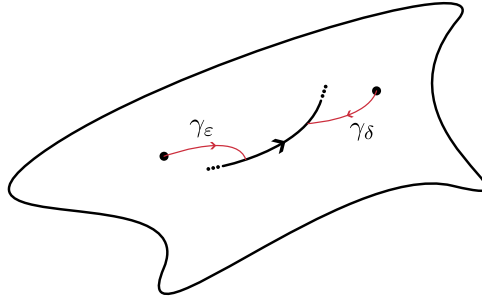$$

in X (see Figure 5).

FIGURE 5. A general linked path as in (28).

Proposition 4.20. *Given a map f of type*

$$
\begin{array}{ccc}
M & \longrightarrow & N \\
\downarrow & & \downarrow \\
\mathfrak{P} & \longrightarrow & \mathfrak{Q}
\end{array}
$$

in the sense of Definition 4.16, there is an induced map

$$f = f_* \colon \mathcal{EX}(M \to \mathfrak{P}) \to \mathcal{EX}(N \to \mathfrak{Q})$$

of exit path ∞-categories. In fact, this defines a(n ordinary) functor

$$\mathcal{LS} \to \mathcal{C}at_\infty$$

from linked spaces to ∞-categories and ∞-functors.

Proof. Without loss of generality, assume $|\mathfrak{P}| = 2$, since $\mathcal{EX}(M \to \mathfrak{P})$ otherwise splits dimension-wise into disjoint unions according to pairs of neighbouring strata of differing depth. The only non-obvious case is when $\mathfrak{Q} = *$, $X := N_*$. Let then $\mathfrak{p} \neq \mathfrak{q}$ in $\mathfrak{P}$, and $p \in M_\mathfrak{p}$, $q \in M_\mathfrak{q}$. We observe f_* in terms of the corresponding locally-Kan categories: on non-invertible paths, it should induce maps

(29) $\mathrm{Hom}_{\mathcal{EX}(M)}(p,q) \overset{\mathrm{Thm}}{\underset{3.5}{\simeq}} \mathcal{P}(M_\mathfrak{q})_{(L_{\mathfrak{p}\mathfrak{q}})_p,q} \to \mathcal{P}(X)_{f(p),f(q)} \simeq \mathrm{Hom}_{\mathrm{Sing}X}(f(p),f(q))$

while given is the commuting square (27). The latter, however, induces, after pulling back onto $p \in \mathfrak{M}_\mathfrak{p}$ (and writing $L := L_{\mathfrak{p}\mathfrak{q}}$), the square

$$
\begin{array}{ccc}
L_p & \xrightarrow{\ f_L\ } & X^I \\
\downarrow & & \downarrow \\
\{p\} \times M_\mathfrak{q} & \longrightarrow & \{f(p)\} \times X
\end{array}
\quad .
$$

Thus, the initial point γ_0 of a path $\gamma \in \mathcal{P}(M_\mathfrak{q})_{L_p,q}$ yields the path $f_L(\gamma_0)$ in X from $f(p)$ to γ_0. Thus,

$$f_* \colon \gamma \mapsto f_L(\gamma_0) * \gamma$$

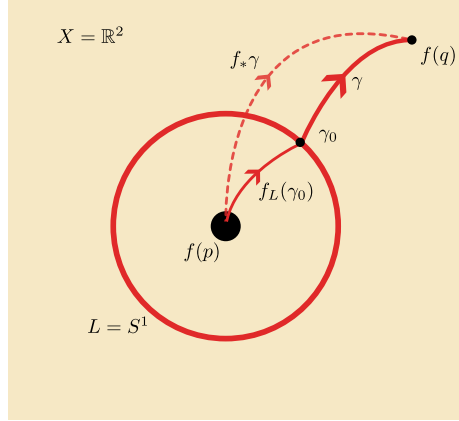provides (29) (see Figure 6). (We have not distinguished L and $\iota(L) \subseteq M_\mathfrak{q}$ in

FIGURE 6. The proof of Proposition 4.20 for $(M \to \mathfrak{P}) = (\{0\} \leftarrow S^1 \hookrightarrow \mathbb{R}^2 \smallsetminus \{0\})$ and $N = X = \mathbb{R}^2$, with f 'the identity'; cf. Remark 4.21.

notation.) The second statement is left for the reader. □

Remark 4.21 (linked maps induced by stratified maps). Consider a link projection $\pi \colon L \twoheadrightarrow M$. The *fibrewise open cone* on π is defined to be

$$C(\pi) = L \times \mathbb{R}_{\geq 0} \amalg_{L \times \{0\}} M.$$

Suppose that a constructible linked space $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xhookrightarrow{\iota} N \right)$ (recall Definition 2.14) is *induced by* a (conically-smooth) stratified space X over $\{0 < 1\}$ with strata $X_0 = M$ and $X_1 = X \smallsetminus X_0 = N$ in the sense that

$$X = C(\pi) \amalg_{L \times \mathbb{R}_{>0}} N.$$

and ι by the implicitly used open embedding $L \times \mathbb{R}_{>0} \hookrightarrow N$ at time 1. This construction appeared in [AFR18a, the proof of Lemma 6.1.7].

Now, let Y be smooth and $f\colon X \to Y$ a stratified map. There is an induced map of linked spaces $\mathfrak{S} \to Y$ whose non-obvious component $f_L\colon L \to Y^I$ covering $f|_0 \times f|_1\colon M \times N \to Y \times Y$ can be given, at $\gamma_0 \in L$, by simply following along the time coordinate in $C(\pi)$ from 0 to 1, from $f|_0(\pi(\gamma_0))$ to $\iota(\gamma_0)$.

4.5. **Duals in depth** 0**.**

4.6. **Cartesian duals.**

4.7. **Stable duals.**

4.8. **Duals of bordisms.**

4.9. **Covariance along refinements.**

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