KOSZUL DUALS OF LINKED SPACES
V0.1

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Abstract.

Contents

1. Introduction

1.1. Summary of results written down so far.

Section 4.1. Passage from the simplicial nerve version of the stratified Grassmannian (written $\mathcal{V}^{\hookrightarrow}$) to the homotopy coherent nerve version. No new results.

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Section 4.2. Given $n, m \in \mathbb{N}$, the exit path ∞-category $\mathcal{EX}(B\mathrm{O}(n,m))$, in the sense of [9], of the (n,m)-*Grassmannian*

$$B\mathrm{O}(n,m) = \left(B\mathrm{O}(n) \xleftarrow{\mathrm{pr}_1} B\mathrm{O}(n) \times B\mathrm{O}(m) \overset{\boxplus}{\hookrightarrow} B\mathrm{O}(n+m) \right)$$

has objects $B\mathrm{O}(n)_0 \amalg B\mathrm{O}(n+m)_0$, which canonically map to $\mathcal{V}_0^{\hookrightarrow}$ (cf. ([10])). Moreover, a non-invertible path $(\gamma, 1) \in \mathcal{P}_0^{\Delta} \subset \mathcal{EX}(B\mathrm{O}(n,m))_1$ corresponds to a path γ in $B\mathrm{O}(n+m)$ of type

$$V_{12} \boxplus V_{01} \to V_{02},$$

where $V_{01} \in B\mathrm{O}(n)$ and $V_{12} \in B\mathrm{O}(m)$. This datum arranges as the morphism

$$\begin{tikzcd} & 1 \arrow[dr, "V_{12}"] & \\ 0 \arrow[ur, "V_{01}"] \arrow[rr, "V_{02}"] \arrow[rr, dashed, bend right=60, "V_{12}\boxplus V_{01}"'] \arrow[rr, phantom, bend right=30, "\Uparrow"] & & 2 \end{tikzcd}$$

in $\mathcal{V}_1^{\hookrightarrow}$. Thus, we have an assignment

$$\mathcal{EX}_{\leq 1}(B\mathrm{O}(n,m)) \to \mathcal{V}_{\leq 1}^{\hookrightarrow} \tag{1}$$

Theorem (Theorem 4.2)**.** *The assignment* (1) *extends to an ∞-functor*

$$\mathbb{U}\colon \mathcal{EX}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow}.$$

Section 4.3. Given a linked manifold $\mathfrak{S} = \left(M \xrightarrow{\pi} L \overset{\iota}{\hookrightarrow} N\right)$, we construct (Construction 4.10) its tangent bundle in terms of purely smooth data:

$$\mathrm{T}\mathfrak{S}\colon \mathfrak{S} \to B\mathrm{O}(n,m).$$

This is informed by an informal discussion present in the conically-smooth setting ([2]) which becomes completely natural in the linked setting. Namely, we give it as the following span map (cf. Remark 4.11):

(2)

$$\begin{tikzcd} L \arrow[rr, "\pi^*\mathrm{T}M \times \mathrm{N}_M N"] \arrow[dr, hook, "\iota"] \arrow[dd, "\pi"'] & & B\mathrm{O}(n) \times B\mathrm{O}(m) \arrow[dr, hook, "\boxplus"] \arrow[dd, "\mathrm{pr}"] & \\ & N \arrow[rr, "\mathrm{T}N", crossing over] & & B\mathrm{O}(n+m) \\ M \arrow[rr, "\mathrm{T}M"] & & B\mathrm{O}(n) & \end{tikzcd} \cdot$$

This yields an ∞-functor $\mathcal{EX}(\mathfrak{S}) \to \mathcal{V}^{\hookrightarrow}$ which we also call the tangent bundle of the linked manifold $\mathfrak{S}$.

Section 5.1. Tangential structures in the stratified setting, which are ∞-categories over $\mathcal{V}^{\hookrightarrow}$, pull back to the linked setting:

$$\mathbb{U}^*\colon \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{EX}(B\mathrm{O}(n,m)).$$

We introduce the problem of finding a linked space $\mathfrak{B}$ such that

$$\mathfrak{U}^*\mathcal{B} = \mathcal{EX}(\mathfrak{B})$$

for a stratified tangential structure $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, and solve it in some simple cases: framings of fixed rank (Example 5.1); classical tangential structures given by space maps of type $B \to B\mathrm{O}(\kappa) \subset \mathcal{V}^{\hookrightarrow}$ (Example 5.2); and variframings, in depth 1 but of arbitrary codimension (Example 5.3). We also discuss variframed point defects (Example 5.4).

Section 5.2. Given a tangential structure $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, the main problem of performing factorisation homology, given a $\mathcal{B}$-structured disk-algebra, on bordisms is that bordisms do not in general have $\mathcal{B}$-structure, but may be asked instead to have a relaxation of it. The most rigid relaxation is the *stable* one (for fixed maximal rank, namely that of the dimension of the disk-algebra), used implicitly in [8]. A further relaxation, the *solid* one, was considered in [2]. First, we discuss both relaxations in the smooth setting, and express them as tangential structures over a single Grassmannian: Definition 5.5 gives the solid version, and Definition 5.6 a further-restricted one, a special case of which recovers the stable structure.

This justifies the way in which we transfer of such structures to the linked setting: we see stratified solid structures as cartesian fibration replacements of the original $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ à la [3] (see Definition 5.7). At this point, determining the morphism spaces in $\mathcal{EX}(\mathfrak{S})$, for $\mathfrak{S}$ a linked space, becomes compulsory. We obtain:

Theorem (Theorem 5.9)**.** *If $p \in M$ and $q \in N$, then* $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q) \simeq \mathcal{P}_{\mathcal{L}_p,q}$.

Here, $\mathcal{P}_{\mathcal{L}_p,q}$ is the space of paths in N that start in the (the image of the) fibre of π at p (under ι) and end at q. We then explore some immediate consequences of this result in Corollaries 5.10 and 5.11, concerning linked spaces of two certain types. In particular, we recover the stratified homotopy type of the conically-smooth open cone on a smooth manifold N as

$$\mathrm{Exit}(C(N)) \simeq \mathcal{EX}\left(* \leftarrow N \xrightarrow{\mathrm{id}} N\right).$$

1.2. **Prose.** We give a procedure to evaluate a non-singular disk-algebra on singular spaces. This suggests a solution to some extent of a problem posed by Ayala–Francis–Tanaka, extending a construction, due to Lurie, Calaque and Scheimbauer, of framed functorial field theories given bulk/point datum in the form of an $\mathbb{E}_n$-algebra. The core novelty consists in the decomposition of a nice-enough stratified space into its strata and their links, organised as a linked space, which uses only smooth manifold data. We transfer some ideas from the theory of tangential structures in stratified geometry, due to Ayala–Francis–Rozenblyum to this setting, by constructing comparison maps from 'linked' Grassmannians to the stratified Grassmannian. This results in a much more accessible theory of bundles and tangential structures in the linked setting.

The main practical advantage of this approach is that it bypasses a good portion of the complicated higher-categorical treatment of bordism categories in terms of iterated Segal spaces. Concretely, this is because in the Lurie–Calaque–Scheimbauer approach, boundaries and corners are specified by cut functions on a manifold (without boundary) of full dimension, embedded, essentially, in a (colimit of) euclidean space(s). One justication for this is the relatively straightforward manner in which one may then introduce tangential structures on boundaries and corners, by simply introducing one on the containing manifold. On the other hand, this approach introduces many technical issues and impracticalities that are not of immediate interest from the point of view of the field theory itself. In our approach, we reverse the places where the difficulty and ease lie: we adopt the more complicated theory of tangential structures on stratified spaces à la Ayala–Francis–Rozenblyum, which we trade for an easier, more intuitive approach to bordisms, in tune with the Atiyah–Segal–Witten paradigm. In fact, we go further and transfer, as mentioned, the theory of stratified tangential structures to the simpler linked setting, which only uses non-singular data. Even though general linked geometry should be at least as complicated as conically-smooth stratified geometry, the linked spaces induced by bordisms are especially simple, whence the difficult part of this approach is also greatly simplified. It seems, therefore, that for the purpose of connecting BV-type field theory with FFT, the linked approach is 'correct'.

The idea, due to Lurie, of completing a bordism into an honestly framed collar and evaluating the disk-algebra on the latter in a particular way in order to construct field theory, is generalised to the linked setting by defining factorisation algebras/homology therein in terms of algebraic coupling data, in such a way that it reproduces some well-known classifying statements in the stratified setting.

Our procedure does not depend intrinsically on the chosen non-singular tangential structure being that of a framing. In particular, it does not rely on Dunn–Lurie additivity, as did Lurie's original construction, and so suggests an extension to any tangential structure. In view of a result of Ginot et al. that generalises Lurie's correspondence between locally-constant factorisation algebras on $\mathbb{R}^n$ and $\mathbb{E}_n$-algebras to one between such on a smooth manifold M and $(M, \mathrm{T}M)$-structured disk-algebras, we present, as a special case, a way to construct functorial field theories given observables algebras (and correlators) on *any* smooth manifold, not just on the local $\mathbb{R}^n$. This is achieved at least in the case when bordisms are 'stably-$(M, \mathrm{T}M)$-structured'.

The 'evaluation' of a structured disk-algebra on a structured collar of a singular space (such as a bordism), when performed à la Lurie, naturally gives rise to representation-theoretic data, as was further systematised by Calaque–Scheimbauer. The datum of compatible actions is parametrised by especially simple spaces when one evaluates bordisms, namely, euclidean spaces with flag-like stratifications. We obtain such spaces directly from the stable tangential structure on bordisms, and call them their *Koszul duals*. They parametrise the Morita categories which provide the coefficents of the functorial field theories thus obtained. We show that they have the same shape regardless of tangential structure, but when bordisms are relaxed to be solidly-structured rather than stably, the Koszul duals become more complicated. We pose their construction problem, which is a certain stratified surgery problem, and present ad hoc solutions in some simple cases in low dimensions.

Acknowledgments.

Conventions. We say *smooth manifold* to refer always to one without boundary.

2. Linking in higher depth

3. Linked manifolds and their exit paths

3.1. **Linked manifolds.**

Definition 3.1. A *linked manifold* (of *depth* 1) is a span

$$\begin{tikzcd} & L \arrow[dl, two heads, "\pi"'] \arrow[dr, hook, "\iota"] & \\ M & & N \end{tikzcd}$$

of smooth manifolds, where π is a fibre bundle, and ι a closed embedding.

3.2. **Examples of depth one.**

3.3. **Short exit paths.**

4. Stratified tangent bundles

For technical reasons, we need to modify our convention for the homotopy-coherent nerve from that in [6] to the one in [7, 00KM]. The former was adopted in [10].

4.1. **The stratified Grassmannian via the homotopy-coherent nerve.** Recall the treatment of the stratified Grassmannian of [2] given in [10]:

$$\mathcal{V}^{\hookrightarrow} := */\mathcal{B}^{\boxplus}\mathrm{O},$$

where

- $\mathcal{B}^{\boxplus}\mathrm{O} := \mathrm{N}^{\boldsymbol{\Delta}}(B^{\boxplus}\mathrm{O})$, where
  - $B^{\boxplus}\mathrm{O}$ is the simplicial category given by delooping the topological monoid $(B\mathrm{O}^{\infty}_{\mathrm{II}}, \boxplus)$ and taking the singular complex,
  - $\mathrm{N}^{\boldsymbol{\Delta}}$ takes the simplicial nerve as in [6] (see below),
- $*/-$ takes the under ∞-category under the single object $*$ of $\mathcal{B}^{\boxplus}\mathrm{O}$.

Specifically, the simplicial set that is the simplicial nerve of a simplicially-enriched category $\mathcal{A}$ is defined levelwise by setting $\mathrm{N}^{\boldsymbol{\Delta}}(\mathcal{A})_k := \mathrm{Fun}(\mathfrak{C}[k], \mathcal{A})$, the set of functors of simplicially-enriched categories from $\mathfrak{C}[k]$ to $\mathcal{A}$, where

- the objects of $\mathfrak{C}[k]$ are the same as those of $[k]$,
- the simplicial hom-set from i to j, if $i \le j$, is $\mathrm{Hom}_{\mathfrak{C}[k]}(i,j) := \mathrm{N}(P_{i,j})$, the nerve of the poset of subposets of k with least element i and greatest element j, ordered by inclusion (empty if $i > j$),
- composition is given by taking unions.

We will use N^{hc} instead of $\mathrm{N}^{\boldsymbol{\Delta}}$, which is defined levelwise by $\mathrm{N}^{\mathrm{hc}}(\mathcal{A})_k := \mathrm{Fun}(\mathrm{Path}[k], \mathcal{A})$, where $\mathrm{Path}[k] := \mathfrak{C}[k]^{\mathrm{op}}$. (This can be read as $\mathrm{Path}[k] = \mathrm{N}(P^{\mathrm{op}}_{i,j})$.) We will call functors $\mathrm{Path}[k] \to \mathcal{A}$ *k-paths* in $\mathcal{A}$. To illustrate, let us abuse notation and write

$$\mathcal{V}^{\hookrightarrow} := */\mathrm{N}^{\mathrm{hc}}(B^{\boxplus}\mathrm{O}).$$

We then have the bijection $\mathcal{V}^{\hookrightarrow}_1 \cong \mathrm{Fun}\left(\mathrm{Path}[2], B^{\boxplus}\mathrm{O}\right)$, and so a morphism in $\mathcal{V}^{\hookrightarrow}$ in this version is uniquely determined by data that can be summed up in the diagram

(3)
$$\begin{tikzcd} & 1 \arrow[dr, "V_{12}"] & \\ 0 \arrow[ur, "V_{01}"] \arrow[rr, "V_{02}"] \arrow[rr, dashed, bend right=60, "V_{12}\boxplus V_{01}"'] & \arrow[phantom, d, "\Uparrow"] & 2 \\ & & \end{tikzcd}$$

where $\Rightarrow$ denotes a path from $V_{12} \boxplus V_{01}$ to V_{02} in $B\mathrm{O}^{\infty}_{\mathrm{II}}$. In the other version of $\mathcal{V}^{\hookrightarrow}$, in contrast, the path is to go in the reverse direction.

Since the two versions are clearly equivalent, we will keep this notation for $\mathcal{V}^{\hookrightarrow}$ to refer to the homotopy-coherent version. In particular, its maximal sub-∞-groupoid is still $B\mathrm{O}^{\infty}_{\amalg}$ ([10, Proposition 2.7]). It will become clear momentarily why this is convenient.

4.2. **Grassmannians: linked to stratified.**

Definition 4.1 ([9, Example 2.7])**.** Let $n, m \in \mathbb{N}$. We call the linked space

$$\begin{tikzcd} & B\mathrm{O}(n) \times B\mathrm{O}(m) \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\boxplus"] & \\ B\mathrm{O}(n) & & B\mathrm{O}(n+m) \end{tikzcd}$$

the (n,m)-*Grassmannian* and denote it by $B\mathrm{O}(n,m)$.

The goal of this section is to construct a map

$$\mathbb{U} : \mathcal{EX}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow}$$

from the exit path ∞-category of the (n,m)-Grassmannian to the stratified Grassmannian. The theory of tangential structures on linked spaces will then be able to interface with the conically-smooth variant, which we expand upon in Section 4.3. We will be completely explicit excepting specification of some trivial choices.

The map restricted to $B\mathrm{O}(n)_{\bullet}$ and $B\mathrm{O}(n+m)_{\bullet}$ inside $\mathcal{EX}$ is defined to be inclusion into the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$. It remains to define the restriction

$$\mathcal{EX}_{k+1} \supset \mathcal{P}^{\Delta}_{k} \to \mathcal{V}^{\hookrightarrow}_{k+1} \cong \mathrm{Fun}\left(\mathrm{Path}[k+2], B^{\boxplus}\mathrm{O}\right),$$

for $k \geq 0$. We will explain dimensions 1 and 2 verbosely before giving the full definition without further explanation.

4.2.1. 1-*morphisms.* An element $(\gamma, 1)$ of $\mathcal{P}^{\Delta}_{0}$ – the exit index in this dimension is necessarily 1 – corresponds to a path γ in $B\mathrm{O}(n+m)$ whose starting point is a direct sum $V_{12} \boxplus V_{01}$ with $V_{01} \in B\mathrm{O}(n)$, $V_{12} \in B\mathrm{O}(m)$. Denoting the endpoint by V_{02}, γ determines a 2-path by arranging the data exactly as in (3). We have thus defined

$$\mathcal{EX}_{\leq 1} \to \mathcal{V}^{\hookrightarrow}_{\leq 1}. \tag{4}$$

Observe that this is compatible with face maps: the source of the image of γ is V_{01}, and the target is V_{02} ([10, §2.2.2]), which are, by construction, the images of the source and target of γ in $\mathcal{EX}$, respectively:

$$d_1^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}(\gamma,1)) = V_{01} = \mathbb{U}\left(\mathrm{pr}\left(d_1^{B\mathrm{O}(n+m)}\gamma\right)\right) = \mathbb{U}\left(d_1^{\mathcal{EX}}(\gamma,1)\right),$$
$$d_0^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}\gamma) = V_{02} = \mathbb{U}\left(d_0^{B\mathrm{O}(n+m)}\gamma\right) = \mathbb{U}\left(d_0^{\mathcal{EX}}(\gamma,1)\right).$$

Compatibility with degeneracies is also clear.

Theorem 4.2. *For any pair of natural numbers $n, m \in \mathbb{N}$, the assignment* (4) *extends to an ∞-functor*

$$\mathbb{U} \colon \mathcal{EX}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow},$$

which we call the unpacking map.

First, we will discuss what $\mathbb{U}$ has to do with 2-morphisms at a phenomenological level so as to elucidate the essential issues to be overcome.

4.2.2. *Intermezzo: 2-morphisms.* Exit paths in $\mathcal{P}_k^{\Delta}$ come in $k+1$ classes according to their exit indices, which need to be mapped to $\mathcal{V}^{\hookrightarrow}$ in different ways.

First, in order to uniquely determine a 3-path in $B^{\boxplus}\mathrm{O}$, it is enough to map out of the sets $N_{\leq 2}(P^{\mathrm{op}}_{i,j})$ into $B\mathrm{O}_{\leq 2} \coloneqq (B\mathrm{O}^{\infty}_{\amalg})_{\leq 2}$ (and in general, a κ-path in $B^{\boxplus}\mathrm{O}$ is determined in dimensions $\leq \kappa - 1$), since higher dimensions are degenerate. The simple (non-decomposable) morphisms in Path[3] are of type $\mathrm{N}_0(P^{\mathrm{op}}_{\alpha\beta}) \ni \underline{\alpha\beta} \coloneqq \{\alpha < \beta\} \subset [3]$, which by a 3-path are mapped to $V_{\alpha\beta} \in B\mathrm{O}_0$. When $\beta = \alpha + 2$ (of which type there are two pairs), there are arrows $\underline{\alpha, \alpha+1, \beta} = \underline{\alpha+1, \beta} \cup \underline{\alpha, \alpha+1} > \underline{\alpha\beta}$ in $N_1(P^{\mathrm{op}}_{\alpha\beta})$, which determine paths

$$V_{\alpha+1,\beta} \boxplus V_{\alpha,\alpha+1} \to V_{\alpha\beta},$$

i.e., $V_{12} \boxplus V_{01} \to V_{02}$ and $V_{23} \boxplus V_{12} \to V_{13}$ in $B\mathrm{O}_1$, namely two of the face 2-paths. The remaining two faces are supplied analogously by considering $(\alpha, \beta) = (0, 3)$ and the compositions $\underline{013} = \underline{13} \cup \underline{01}$ and $\underline{023} = \underline{23} \cup \underline{02}$. Finally, again for $(\alpha, \beta) = (0, 3)$, consider $\underline{0123} = \underline{23} \cup \underline{12} \cup \underline{01}$, which is to be mapped to $V_{0123} = V_{23} \boxplus V_{12} \boxplus V_{01}$. Out of $N_1(P^{\mathrm{op}}_{0,3})$ we receive paths $V_{0123} \to V_{03}$, $V_{0123} \to V_{013}$, $V_{0123} \to V_{023}$. The two non-degenerate elements $(\underline{0123} > \underline{023} > \underline{03})$ and $(\underline{0123} > \underline{013} > \underline{03})$ in $N_2(P^{\mathrm{op}}_{0,3})$ are to map in $B\mathrm{O}_2$ to

$$\begin{tikzcd} & V_{023} \arrow[dr] & \\ V_{0123} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} \quad = \quad \begin{tikzcd} & V_{23} \boxplus V_{02} \arrow[dr] & \\ V_{23} \boxplus V_{12} \boxplus V_{01} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} \tag{5}$$

and

$$\begin{tikzcd} & V_{013} \arrow[dr] & \\ V_{0123} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} \quad = \quad \begin{tikzcd} & V_{13} \boxplus V_{01} \arrow[dr] & \\ V_{23} \boxplus V_{12} \boxplus V_{01} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} \tag{6}$$

We have thus summed up the data needed to provide a functor $\mathrm{Path}[3] \to B^{\boxplus}\mathrm{O}$.

Now, let us start with paths of exit index $2 \in \{1, 2\}$. Such an exit path $(\gamma, 2)$ (in $\mathcal{P}_1^{\Delta} \subset \mathcal{EX}_2$) consists of a 2-simplex $\gamma \in B\mathrm{O}(n+m)_2$ of type

$$\begin{tikzcd} & K \\ W \boxplus V \arrow[ur, "\gamma_{\boxplus}"] \arrow[r, "\gamma_W \boxplus \gamma_V"'] & W' \boxplus V' \arrow[u, "\gamma_{\boxplus'}"'] \end{tikzcd} \tag{7}$$

where the bottom edge comes from $B\mathrm{O}(n) \times B\mathrm{O}(m)$ (whence it is $\boxplus$ of two paths). The natural choice for the image, visualised as a 3-simplex of $\mathcal{B}^{\boxplus}\mathrm{O}$, is

$$\mathbb{U}(\gamma, 2) = \begin{tikzcd} & 2 \arrow[dr, "W'"] & \\ 1 \arrow[ur, "0"] \arrow[rr, "W"] & & 3 \\ & 0 \arrow[ul, "V"] \arrow[uu, "V'"] \arrow[ur, "K"'] & \end{tikzcd} .$$

Indeed, the edges in (7) supply the face triangles – in fact, the fact that the bottom edge is of type $\gamma_W \boxplus \gamma_V$ is crucial, since the summand paths supply the triangles adjacent to the edge decorated by the zero vector space. The only wrinkle is that the upper face requires a path $W' \to W$, which can be taken to be the (standard) inverse of γ_W, which we will denote by γ_W^{-1}. As for (5), i.e.,

$$
\begin{tikzcd}
 & W' \boxplus V' \arrow[dr, "\gamma_{\boxplus'}"] & \\
W' \boxplus V \arrow[ur, "\text{(i)}"] \arrow[rr, "\text{(ii)}"'] & & K
\end{tikzcd}, \tag{8}
$$

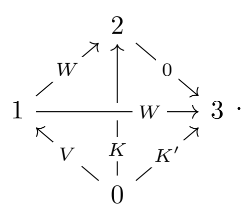
note that we are still free to (and have to) choose the paths $W' \boxplus V \to W' \boxplus V'$ and $W' \boxplus V \to K$ (corresponding to the arrows $\underline{0123} > \underline{023}$ and $\underline{0123} > \underline{03}$). To this end, consider the diagram $\begin{tikzcd} W' \boxplus V' \arrow[r, "\gamma_{\boxplus'}"] & K \\ W' \boxplus V \arrow[u, "\mathrm{id}_{W'} \boxplus \gamma_V"] \arrow[r, "\gamma_W^{-1} \boxplus \mathrm{id}_V"'] & W \boxplus V \arrow[u, "\gamma_{\boxplus}"'] \arrow[ul, "\gamma_W \boxplus \gamma_V"'] \end{tikzcd}$ and choose the obvious fillers. (By id_A we mean the constant loop at A.) This suggests using (i) $= \mathrm{id}_{W'} \boxplus \gamma_V$, (ii) $= (\gamma_W^{-1} \boxplus \mathrm{id}_V) * \gamma_{\boxplus}$ (we denote concatenation from left to right) whereupon the obvious filler can be chosen. Similarly, for (6), i.e.,

$$
\begin{tikzcd}
 & W \boxplus V \arrow[dr, "\gamma_{\boxplus}"] & \\
W' \boxplus V \arrow[ur, "\text{(i)}'"] \arrow[rr, "\text{(ii)}'"'] & & K
\end{tikzcd}, \tag{9}
$$

consider $\begin{tikzcd} W \boxplus V \arrow[r, "\gamma_{\boxplus}"] & K \\ W' \boxplus V \arrow[u, "\gamma_W^{-1} \boxplus \mathrm{id}_V"] \arrow[r, "\mathrm{id}_W \boxplus \gamma_V"'] & W' \boxplus V' \arrow[u, "\gamma_{\boxplus'}"'] \arrow[ul, "\gamma_W^{-1} \boxplus \gamma_V^{-1}"'] \end{tikzcd}$ and proceed similarly. (We give a systematic account in Section 4.2.3.) This completes the construction of $N_{\leq 2}(P_{0,3}^{\mathrm{op}}) \to B\mathrm{O}_{\leq 2}$ and so in toto of the 3-path $\mathrm{Path}[3] \to B^{\boxplus}\mathrm{O}$ associated with the exit path $(\gamma, 2)$. The image of an index-1 exit path

$$
\begin{tikzcd}
K \arrow[r, "\gamma_K"] & K' \\
W \boxplus V \arrow[u, "\gamma_{\boxplus}"] \arrow[ur, "\gamma_{\boxplus'}"'] &
\end{tikzcd} \tag{10}
$$

is constructed analogously, with its picture as a 3-simplex of $\mathcal{B}^{\boxplus}\mathrm{O}$ given by



We have thus defined

$$\mathcal{E}\mathcal{X}_{\leq 2} \to \mathcal{V}_{\leq 2}^{\hookrightarrow}.$$

As for simpliciality, consider again an index-2 exit path $(\gamma, 2)$ as in (7). Its source edge is the path $d_2^{\mathcal{E}\mathcal{X}}(\gamma, 2) = (\gamma_V : V \to V') \in B\mathrm{O}_1 \subset \mathcal{E}\mathcal{X}_1$. Since its two remaining edges are vertical, they are the elements of $\mathcal{P}_0^{\Delta}$ induced by $\gamma_{\boxplus'}$ and $\gamma_{\boxplus}$. Now, recall that $\mathbb{U}(\gamma, 2)$ is identified with an element of $\mathcal{B}^{\boxplus}\mathrm{O}_3 \cong \mathcal{V}_2^{\hookrightarrow}$ due to $\mathcal{V}^{\hookrightarrow}$'s being an under-$\infty$-category, via $\Delta[0] \star \Delta[2] \simeq \Delta[3]$. Accordingly, face maps apply on the factor $\Delta[2]$ (i.e., ∂ acts as $\mathrm{id}_{\Delta[0]} \star \partial$). In the picture in $\mathcal{B}^{\boxplus}\mathrm{O}_3$, this means that when pulling back along a face map $\partial\colon \Delta[1] \hookrightarrow \Delta[2]$, we restrict to the triangle whose top edge is specified by ∂; e.g., $\partial_2\colon \Delta[1] \hookrightarrow \Delta[2]$, which skips 2, applies to give $d_2^{\mathcal{V}^{\hookrightarrow}}\mathbb{U}(\gamma,2) = \begin{tikzcd}[sep=small] & 1 \arrow[dr, "0"] & \\ 0 \arrow[ur, "V"] \arrow[rr, "V'"'] & & 2 \end{tikzcd}$, which is precisely $\mathbb{U}(\gamma_V)$. So as to avoid confusion, note in particular that the top face $\begin{tikzcd}[sep=small] & 2 \arrow[dr, "W'"] & \\ 1 \arrow[ur, "0"] \arrow[rr, "W"'] & & 3 \end{tikzcd}$, where we inverted γ_W, is *not* a face in $\mathcal{V}^{\hookrightarrow}$, nor is $\gamma_W \in B\mathrm{O}(m)_1 \subset \mathcal{E}\mathcal{X}_1$ of $(\gamma,2)$ in $\mathcal{E}\mathcal{X}$.

We leave the analogous treatment of the remaining two faces and of the index-1 case to the reader. Now, we proceed to finally give the general construction.

4.2.3. *Proof of Theorem 4.2.* We will first give a systematic account of $\mathcal{P}_1^{\Delta} \to \big[\mathrm{Path}[3], B^{\boxplus}\mathrm{O}\big]$ in such a way that the ideas generalise to all dimensions. At some places, it will conventient to slightly rearrange the visual representation of exit paths. For $(\gamma,1) \in \mathcal{P}_0^{\Delta}$, the diagram $W \boxplus V \to K$ depicts $\gamma \in B\mathrm{O}_1$. Instead, $V \xrightarrow{(W,\gamma)} K$ or $V \xrightarrow{W} K$ for short, depicts $(\gamma,1)$ more properly. Similarly, we also depict $(\gamma,2) \in \mathcal{P}_1^{\Delta}$ by $\begin{tikzcd}[sep=small] & K \\ V \arrow[ur, "W"] \arrow[r] & V' \arrow[u, "W'"'] \end{tikzcd}$, etc.

Notation 4.3. $[A,B] \coloneqq \mathrm{Fun}(A,B)$.

Definition 4.4. We use $\mathbf{F}\colon B\mathrm{O}(n) \times B\mathrm{O}(m) \to B\mathrm{O}(m)$, for *f*ibre, to denote the second coordinate projection. When we apply $\mathbf{F}$ to a bottom face of an exit path (γ,j), we mean that first the corresponding face of γ is to be taken, which is then (unambiguously) to be identified with a simplex of the link, and then $\mathbf{F}$ is to be applied. Namely, we have, by abuse of notation, a map

$$\mathbf{F}\colon \mathcal{P}_*^{\Delta} \to B\mathrm{O}(m)_*$$

for each $* \geq 0$, given by the composition

$$\begin{tikzcd} \mathcal{P}_*^{\Delta} \arrow[r, "(\gamma,j)\mapsto\Gamma_j=\gamma\circ\mathcal{C}_j"] \arrow[rrr, dashed, bend right, "\mathbf{F}"'] & \mathcal{P}_* \arrow[r, two heads] & \mathcal{L}_* \arrow[r, "\mathbf{F}"] & B\mathrm{O}(m)_* \end{tikzcd}$$

where we have not omitted $*$ since $\mathcal{P}^{\Delta}$ is not a simplicial set.

Notation 4.5. For X a space, we denote by Op the canonical isomorphism $X \simeq X^{\mathrm{op}}$ of Kan complexes, by which we mean $\mathrm{Sing}_\bullet(X) \simeq \mathrm{Sing}_\bullet(X)^{\mathrm{op}}$ ([7, 003R]). This inverts simplices of all dimensions in a compatible fashion.

Notation 4.6. Let $\alpha_0,\ldots,\alpha_\ell \in [k]$. By $\mathrm{Path}[\alpha_0,\ldots,\alpha_\ell] \cong \mathrm{Path}[\ell]$ we denote the simplicial subcategory of $\mathrm{Path}[k]$ generated by the objects $\alpha_0,\ldots,\alpha_\ell$.

Lemma 4.7. $d_i^{\mathcal{V}^{\hookrightarrow}} = d_{i+1}^{\mathcal{B}^{\boxplus}\mathrm{O}}$.

Proof. $(\mathrm{id}_0 \star \partial_i\colon \Delta[0]\star\Delta[1] \hookrightarrow \Delta[0]\star\Delta[2]) = (\partial_{1+i}\colon \Delta[2] \hookrightarrow \Delta[3])$. □

The exitence of the restriction of $\mathbb{U}\colon \mathcal{E}\mathcal{X} = \mathcal{E}\mathcal{X}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow}$ to non-invertible paths will be shown inductively. It is defined on $B\mathrm{O}(n), B\mathrm{O}(n+m) \subset \mathcal{E}\mathcal{X}$ by inclusion into the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$, and on $\mathcal{P}_*^{\Delta} \subset \mathcal{E}\mathcal{X}_{*+1}$, using $\mathcal{V}_k^{\hookrightarrow} \cong \big[\mathrm{Path}[k+2], B^{\boxplus}\mathrm{O}\big]$ and that a $(k+2)$-path is determined on hom-ssets in dimensions $\leq k-1$, as follows:

$\underline{\mathcal{P}^{\Delta}_0 \to \big[\mathrm{Path}[2], B^{\boxplus}\mathrm{O}\big] :}$

$$\mathbb{U}|\mathcal{P}^{\Delta}_{\mathbf{1},0} : (\gamma, 1) = \begin{array}{c} K \\ \gamma\big\uparrow \\ W \boxplus V \end{array} \mapsto \begin{cases} \mathrm{N}_0(P^{\mathrm{op}}_{0,a}) \to BO_0, & \underline{0,a} \mapsto \{a-1\}^*(\gamma, 1) = \begin{cases} V, & a = 1 \\ K, & a = 2 \end{cases} \\ \mathrm{N}_0(P^{\mathrm{op}}_{1,2}) \to BO_0, & \underline{12} \mapsto \mathbf{F}(\gamma, j) = W \\ \mathrm{N}_1(P_{0,2}) \to BO_1, & (\underline{012} > \underline{02}) \mapsto \gamma \end{cases}$$

$\underline{\mathcal{P}^{\Delta}_1 \to \big[\mathrm{Path}[3], B^{\boxplus}\mathrm{O}\big] :}$

Induced faces. Let $(\gamma, j) \in \mathcal{P}^{\Delta}_1$. We first define the faces of $\mathbb{U}(\gamma, j)$. The three faces $d^{\mathcal{V}^{\hookleftarrow}}_{0,1,2}\mathbb{U}(\gamma, 2)$ are defined by $\mathbb{U}|\mathcal{E}\mathcal{X}_0$ via simpliciality, i.e., by

$$d^{\mathcal{V}^{\hookleftarrow}}_i(\mathbb{U}(\gamma, j)) \coloneqq \mathbb{U}(d^{\mathcal{E}\mathcal{X}}_i(\gamma, j)). \tag{11}$$

This fixes $\mathbb{U}(\gamma, j)$, by Lemma 4.7, on the subcategories $\mathrm{Path}[0, k, l] \cong \mathrm{Path}[2]$ of $\mathrm{Path}[3]$, for $1 \leq k < l \leq 3$ (Notation 4.5).

The top face. The remaining $\mathcal{B}^{\boxplus}\mathrm{O}$-face $d^{\mathcal{B}^{\boxplus}\mathrm{O}}_0(\mathbb{U}(\gamma, j))$ is the restriction to $\mathrm{Path}[1, 2, 3]$. The edges of $d^{\mathcal{B}^{\boxplus}\mathrm{O}}_0(\mathbb{U}(\gamma, j)) \in \mathcal{B}^{\boxplus}\mathrm{O}_2 \cong \mathcal{V}^{\hookleftarrow}_1$ are already specified by the $d^{\mathcal{V}^{\hookleftarrow}}_i(\mathbb{U}(\gamma, j))$, so only

$$\begin{aligned} &\mathrm{N}_1(P^{\mathrm{op}}_{1,3}) \to BO_1 \\ (\underline{23} \cup \underline{12} > \underline{13}) &\mapsto \big(\mathbb{U}_{(\gamma,j)}(\underline{23}) \boxplus \mathbb{U}_{(\gamma,j)}(\underline{12}) \to \mathbb{U}_{(\gamma,j)}(\underline{13})\big) \\ &= (\mathbb{U}(d_0 d_0(\gamma, j)) \boxplus \mathbb{U}(d_0 d_2(\gamma, j)) \to \mathbb{U}(d_0 d_1(\gamma, j))) \end{aligned}$$

remains. This is determined by $\mathbf{F}$:

$$\mathbb{U}_{(\gamma,j)}|\mathrm{N}_1(P^{\mathrm{op}}_{1,3}) \coloneqq \mathrm{Op}\mathbf{F}(\gamma, j). \tag{12}$$

Remark 4.8 (interrupting the proof). We should note that it is immaterial that (12) is 'not functorial' (although $\mathbb{U}$ will be). As noted in Lemma 4.9, the direct sums appearing in the $d^{\mathcal{B}^{\boxplus}\mathrm{O}}_0$-face are trivial in that all summands but one are zero, the non-zero one being determined by the exit index j. We use $\mathrm{Op}\mathbf{F}$ to supply *only the path* in $BO(m)$. We have $\mathbb{U}_{\gamma,j}(\underline{23}) = 0$ if $j = 1$, and $\mathbb{U}_{\gamma,j}(\underline{12}) = 0$ if $j = 2$. In the former case, the edges of $d^{\mathcal{B}^{\boxplus}\mathrm{O}}_0$ are specified by simpliciality as in $\begin{tikzcd}[column sep=small, row sep=small] & 2 \arrow[dr, "0"] & \\ 1 \arrow[ur, "W"] \arrow[rr, "W"'] & & 3 \end{tikzcd}$, and $\mathrm{Op}\mathbf{F}(\gamma, 1)$ is $\mathrm{Op}(\mathrm{id}_W) = \mathrm{id}_W \colon W = W \boxplus 0 \to W$. Here, (12) happens to be functorial as $d^{\mathcal{B}^{\boxplus}\mathrm{O}}_0$ happens to lie in $\mathcal{V}^{\sim}$. In the latter case, the filler of $\begin{tikzcd}[column sep=small, row sep=small] & 2 \arrow[dr, "W'"] & \\ 1 \arrow[ur, "0"] \arrow[rr, "W"'] & & 3 \end{tikzcd}$ is supplied by $\mathrm{Op}\mathbf{F}(\gamma, 2) = \mathrm{Op}(\gamma_W) = \gamma^{-1}_W \colon W' = 0 \boxplus W' \to W$. This breaks functoriality, since $d^{\mathcal{B}^{\boxplus}\mathrm{O}}_0$ is *not* invertible. Still, the path is as desired.

(The proof cont.)

1-paths induced by functoriality. Some 1-paths in the image of $\mathbb{U}(\gamma, j)$ are determined by the data provided thus far and by imposing functoriality. Namely, we have the following decompositions:

$$\begin{aligned}
\text{(i)} = (\underline{0123} > \underline{023}) &= (\underline{23} \cup \underline{012} > \underline{23} \cup \underline{02}) \\
&= \mathrm{id}_{\underline{23}} \cup [\underline{012} > \underline{02}] \\
&\in \mathrm{Im}\left(\cup : \mathrm{N}_1(P^{\mathrm{op}}_{2,3}) \times \mathrm{N}_1(P^{\mathrm{op}}_{0,2}) \to \mathrm{N}_1(P^{\mathrm{op}}_{0,3})\right), \\
\text{(i)}' = (\underline{0123} > \underline{013}) &= (\underline{123} \cup \underline{01} > \underline{13} \cup \underline{01}) \\
&= [\underline{123} > \underline{13}] \cup \mathrm{id}_{\underline{01}} \\
&\in \mathrm{Im}\left(\cup : \mathrm{N}_1(P^{\mathrm{op}}_{1,3}) \times \mathrm{N}_1(P^{\mathrm{op}}_{0,1}) \to \mathrm{N}_1(P^{\mathrm{op}}_{0,3})\right).
\end{aligned}$$

Thus, functoriality imposes

$$\begin{aligned}
\mathbb{U}_{(\gamma,j)}(\text{i}) &= \mathrm{id}_{\mathbb{U}_{(\gamma,j)}(\underline{23})} \boxplus \mathbb{U}_{(\gamma,j)}(\underline{012} > \underline{02}) \\
\mathbb{U}_{(\gamma,j)}(\text{i})' &= \mathbb{U}_{(\gamma,j)}(\underline{123} > \underline{13}) \boxplus \mathrm{id}_{\mathbb{U}_{(\gamma,j)}(\underline{01})}
\end{aligned}$$

where the first non-constant summand is determined by (11) and the second by (12). In particular, the ad hoc assignments in (8) and (9) were correct.

Remining 1-path and 2-paths. It remains to specify $\mathbb{U}(\gamma, j)$ on the 'long path' in $\mathrm{N}_1(P^{\mathrm{op}}_{0,3})$ and on $\mathrm{N}_2(P^{\mathrm{op}}_{0,3})$. In contrast to the paths induced by functoriality, $\underline{03}$ is $\cup$-simple (not a composition), so

$$\text{(ii)} = (\underline{0123} > \underline{03})$$

presents a genuine choice, and was not handled systematically in Section 4.2.2. It is considered most naturally in conjunction with the two non-degenerate elements in $N_2(P^{\mathrm{op}}_{0,3})$ to be mapped, as it is their (necessarily-)common composition:

$$\begin{tikzcd}
& \underline{0123} \arrow[dl, "\mathrm{id}_{\underline{23}}\cup{[\underline{012}>\underline{02}]}"'] \arrow[dr, "{[\underline{123}>\underline{13}]\cup\mathrm{id}_{\underline{01}}}"] \arrow[dd, dotted, "\text{(ii)}" description] & \\
\underline{023} \arrow[dr, "(11)"'] & & \underline{013} \arrow[dl, "(11)"] \\
& \underline{03} &
\end{tikzcd}$$

First, note that, regardless of exit index, this square decomposes into two triangles:

$$\begin{tikzcd}
& \underline{0123} \arrow[dl, "\mathrm{id}_{\underline{23}}\cup{[\underline{012}>\underline{02}]}"'] \arrow[dr, "{[\underline{123}>\underline{13}]\cup\mathrm{id}_{\underline{01}}}"] & \\
\underline{023} \arrow[dr, "(11)"'] & & \underline{013} \arrow[ll, "d_2\gamma" description] \arrow[dl, "(11)"] \\
& \underline{03} &
\end{tikzcd}$$

For $j = 2$ (using the labels in (7)), this reads

$$\begin{tikzcd}
& W' \boxplus 0 \boxplus V = W' \boxplus V \arrow[dl, "\mathrm{id}_{W'}\boxplus\gamma_V"'] \arrow[dr, "\gamma_W^{-1}\boxplus\mathrm{id}_V"] & \\
W' \boxplus V' \arrow[dr, "\gamma_{\boxplus'}"'] & & W \boxplus V \arrow[ll, "\gamma_W\boxplus\gamma_V" description] \arrow[dl, "\gamma_{\boxplus}"] \\
& K &
\end{tikzcd},$$

and for $j = 1$ (using the labels in (10)),

$$\begin{tikzcd} & 0 \boxplus W \boxplus V = W \boxplus V \arrow[dl, "\mathrm{id}_0 \boxplus \gamma_\boxplus = \gamma_\boxplus"'] \arrow[dr, "\mathrm{id}_{0 \boxplus W} \boxplus \mathrm{id}_V = \mathrm{id}_W \boxplus \mathrm{id}_V"] & \\ 0 \boxplus K = K \arrow[dr, "\gamma_K"'] & & W \boxplus V \arrow[ll, "\gamma_\boxplus"'] \arrow[dl, "\gamma_{\boxplus'}"] \\ & K' & \end{tikzcd} .$$

For both indices, the bottom triangle is filled by γ itself, and the top one has a canonical filler. This suggest assigning to (ii) the outer-left concatenation, $\mathbb{U}_{(\gamma,j)}(\mathrm{ii}) = \mathbb{U}_{(\gamma,j)}(\underline{0123} > \underline{023}) * \mathbb{U}_{(\gamma,j)}(\underline{023} > \underline{03})$. Accordingly, $\mathbb{U}(\gamma, j)|\mathrm{N}_2(P^{\mathrm{op}}_{0,3})$ is determined by said fillers. This concludes the construction of $\mathbb{U}|\mathcal{P}^{\Delta}_1$.

The induction step. Assume now that the $\mathbb{U}|\mathcal{P}^{\Delta}_{<k}$ have been constructed simplicially.

$\underline{\mathcal{P}^{\Delta}_k \to \left[\mathrm{Path}[k+2], B^{\boxplus}\mathrm{O}\right]}$: We have constructed $\mathbb{U}|\mathcal{P}^{\Delta}_1$ as independently of exit indices as possible, and the same ideas apply here mutatis mutandis.

Let $(\gamma, j) \in \mathcal{P}^{\Delta}_k$.

Induced faces. The restriction of $\mathbb{U}(\gamma, j)$ to the subcategories $\mathrm{Path}[0, k_1, \ldots, k_{k+1}] \cong \mathrm{Path}[k+1]$ of $\mathrm{Path}[k+2]$, $1 \leq k_\ell < k_{\ell'} \leq k+2$ $\forall 1 \leq \ell < \ell' \leq k+1$, are determined by setting

$$d_i^{\mathcal{V}^{\leftrightarrow}}(\mathbb{U}(\gamma, j)) = \mathbb{U}(d_i^{\mathcal{E}\mathcal{X}}(\gamma, j)). \tag{13}$$

The top face. The restriction $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}(\mathbb{U}(\gamma, j))$ to $\mathrm{Path}[1, \ldots, k+2]$ is given, in view of Lemma 4.9, by $\mathrm{Op}\mathbf{F}(\gamma, j)$.

The rest of $P^{\mathrm{op}}_{0,k+2}$. All arrows in $P^{\mathrm{op}}_{0,k+2}$ out of the initial $\underline{0, \ldots, k+2} = [k+2]$ except for $\underline{0, \ldots, k+2} > \underline{0, k+2}$ are clearly $\cup$-composite and are thus determined by (13) and functoriality. This is well-defined due to the inductive assumption that so is $\mathbb{U}|\mathcal{P}^{\Delta}_{<k}$.

We set

$$\begin{aligned} \mathbb{U}_{(\gamma,j)}\left([k+2] > \underline{0, k+2}\right) = {} & \mathbb{U}_{(\gamma,j)}\left([k+2] > [k+2] \smallsetminus \{1\}\right) \\ & * \mathbb{U}_{(\gamma,j)}\left([k+2] \smallsetminus \{1\} > [k+2] \smallsetminus \{1, 2\}\right) \\ & * \cdots \\ & * \mathbb{U}_{(\gamma,j)}\left(\underline{0, k+1, k+2} > \underline{0, k+2}\right). \end{aligned}$$

Among the $\cup$-simple ones, there are $k+1$ arrows $\underline{0, i, k+2} > \underline{0, k+2}$. In the $(k+1)$-cube $P_{0,k+2}$, the $(k+1)$-simplex with vertices the domains $\underline{0, i, k+2}$, and finally $\underline{0, k+2}$, is filled by $\gamma \in B\mathrm{O}(n+m)_{k+1}$ according to the rule $\alpha \mapsto \underline{0, \alpha+1, k+2}$ for $k+2 > \alpha \in [k+2]$, and $k+2 \mapsto \underline{0, k+2}$. Thus, in any chain in $\mathrm{N}_{k+1}(P^{\mathrm{op}}_{0,k+2})$ from $[k+2]$ to $\underline{0, k+2}$, the concatenation of the images of the short arrows is homotopic to $\mathbb{U}_{(\gamma,j)}([k+2] > \underline{0, k+2})$, which provides $\mathbb{U}(\gamma, j)|\mathrm{N}_{k+1}(P^{\mathrm{op}}_{0,k+2})$.

This concludes the construction of $\mathbb{U}$.

Admittedly, the construction above is more brute-force than elightening. It does not rely on universal properties, which might be explained by the low-level (in the programming sense) status it has in linked space theory, and, as such, does not benefit from higher-level universal-property arguments. Essentially, we force the construction inductively from the lowest non-trivial assignment $\mathcal{P}^{\Delta}_0 \to \left[\mathrm{Path}[2], B^{\boxplus}\mathrm{O}\right]$, guided by an explicit systematisation of the first 'induction step',

$\mathcal{P}^{\Delta}_1 \to \big[\mathrm{Path}[3], B^{\boxplus}\mathrm{O}\big]$ – whose redressed presentation was, strictly speaking, not necessary – which illustrates the fundamental idea quite well already.

The reason why the construction then works out in all dimensions remains slightly mysterious: the core phenomenon leveraged is buried in the argument that we use to construct the restriction $\mathrm{N}_\bullet(P^{\mathrm{op}}_{0,k+2}) \to B\mathrm{O}_\bullet$ of $\mathbb{U}(\gamma, j)$, $(\gamma, j) \in \mathcal{P}^{\Delta}_k$, once all $\mathcal{B}^{\boxplus}\mathrm{O}$-faces have been forced by lower dimensions and simpliciality. Here, recognising $P^{\mathrm{op}}_{0,k+2}$ to be a cube of dimension $k+1$ (as was also noted in [6, §1.1.5], [7, 00LM]), we observe that $\gamma \in B\mathrm{O}(n+m)_{k+1}$ itself provides a filler for a particular $(k+1)$-simplex within this cube (the one with vertices $\underline{0, i, k+2}$, $1 < i < k+2$, and $\underline{0, k+2}$). The rest of the cube has a canonical filler, which has, crucially, to do with the inversion of the paths originating in the pr-fibre $B\mathrm{O}(m)$ (paths like $\gamma^{-1}_W \colon W' \to W$), necessitated above by the edge arrangements in $\mathcal{B}^{\boxplus}\mathrm{O}$. So, as disconcerting as this might have seemed above, it proves absolutely essential.

As a final illustration, consider an exit 3-path of index 3:

$$\mathcal{P}^{\Delta}_2 \ni (\gamma, 3) = \begin{tikzcd} & & K \\ & W_1 \boxplus V_1 \arrow[ur, dotted] \arrow[dr, dotted] & \\ W_0 \boxplus V_0 \arrow[ur, dotted] \arrow[rr] \arrow[uurr, bend left] & & W_2 \boxplus V_2 \arrow[uu] \end{tikzcd} .$$

The (1d) edges of $\mathbb{U}(\gamma, 3) \in \mathcal{B}^{\boxplus}\mathrm{O}_4$ are given, due to (13), as follows:

$$\begin{cases} \underline{01} \mapsto V_0,\ \underline{02} \mapsto V_1,\ \underline{03} \mapsto V_2,\ \underline{04} \mapsto K, \\ \underline{14} \mapsto W_0,\ \underline{24} \mapsto W_1,\ \underline{34} \mapsto W_2 \end{cases}$$

and the remaining edges are zero. Now, the 3-cube $P^{\mathrm{op}}_{0,4}$ and its image under $\mathbb{U}(\gamma, j)$ look as follows:

$$\begin{tikzcd} & \underline{01234} \arrow[rr] \arrow[dl] \arrow[dd] & & \underline{0124} \arrow[dl] \arrow[dd] \\ \underline{0234} \arrow[rr] \arrow[dd] & & \underline{024} \arrow[dd] & \\ & \underline{0134} \arrow[rr] \arrow[dl] & & \underline{014} \arrow[dl] \\ \underline{034} \arrow[rr] & & \underline{04} & \end{tikzcd} \mapsto \begin{tikzcd} & W_2 \boxplus V_0 \arrow[rr, "\gamma^{-1}_{W12} \boxplus \mathrm{id}"] \arrow[dl, "\mathrm{id} \boxplus \gamma_{V01}"'] \arrow[dd, equal] & & W_1 \boxplus V_0 \arrow[dl, "\mathrm{id} \boxplus \gamma_{V01}"'] \arrow[dd, "\gamma^{-1}_{W01} \boxplus \mathrm{id}"] \\ W_2 \boxplus V_1 \arrow[rr, "\mathrm{id}^{-1}_{W12} \boxplus \mathrm{id}"] \arrow[dd, "\mathrm{id} \boxplus \gamma_{V12}"'] & & W_1 \boxplus V_1 \arrow[dd, "\mathrm{id}_{\boxplus 1}"] & \\ & W_2 \boxplus V_0 \arrow[rr, "\gamma^{-1}_{W02} \boxplus \mathrm{id}"] \arrow[dl, "\mathrm{id} \boxplus \gamma_{V02}"'] & & W_0 \boxplus V_0 \arrow[dl, "\gamma_{\boxplus 0}"] \\ W_2 \boxplus V_2 \arrow[rr, "\gamma_{\boxplus 2}"] & & K & \end{tikzcd} .$$

Painting in the outer-left concatenation chosen to be the image of $\underline{01234} > \underline{04}$ (in green) and the (edges of the) ‘lower’ tetrahedron given by $\gamma \in B\mathrm{O}_3$ itself (in blue), we see that

$$\begin{tikzcd}
& W_2 \boxplus V_0 \arrow[rr] \arrow[dl] \arrow[dd, equal] & & W_1 \boxplus V_0 \arrow[dl] \arrow[dd] \\
W_2 \boxplus V_1 \arrow[rr] \arrow[dd] & & W_1 \boxplus V_1 \arrow[dd] \arrow[dll, "\gamma_{W12} \boxplus \gamma_{V12}"] & \\
& W_2 \boxplus V_0 \arrow[rr] \arrow[dl] & & W_0 \boxplus V_0, \arrow[ul, "\gamma_{W01} \boxplus \gamma_{V01}"'] \arrow[dlll, "\gamma_{W02} \boxplus \gamma_{V02}"] \arrow[dl] \\
W_2 \boxplus V_2 \arrow[rr] & & K &
\end{tikzcd}$$

homotopy-commutes by inspection.

Lemma 4.9 (ad $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}\mathbb{U}(\gamma, j)$)**.**

Proof. □

4.3. **Classifying maps of linked tangent bundles.**

Construction 4.10 (linked tangent bundle)**.** Let $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xrightarrow{\iota} N\right)$ be a linked manifold, with each manifold riemannian. Given this contractible choice, we will show that there is a 'canonical' map

$$\mathrm{T}\mathfrak{S} \colon \mathfrak{S} \to B\mathrm{O}(n, m) \tag{14}$$

of linked spaces, which we call the *tangent bundle* of $\mathfrak{S}$.

Since $\mathrm{d}\pi$ surjects, the induced linear dual map $(\pi^*\mathrm{T}M)^\vee \hookrightarrow (\mathrm{T}L)^\vee$, of bundles over L, injects. Using the metrics, this gives an injection $\pi^*\mathrm{T}M \hookrightarrow \mathrm{T}L$. Let us assume all manifolds paracompact, so that vector subbundles split. Then, writing $\mathrm{N}_L M \coloneqq (\pi^*\mathrm{T}M)^\perp \subset \mathrm{T}L$, we have a splitting $\mathrm{T}L \simeq \pi^*\mathrm{T}M \oplus \mathrm{N}_L M$. Similarly, writing $\mathrm{N}_N L \coloneqq (\iota_*\mathrm{T}L)^\perp \subset \iota^*\mathrm{T}N$, we have a splitting $\iota^*\mathrm{T}N \simeq \mathrm{T}L \oplus \mathrm{N}_N L \simeq \pi^*\mathrm{T}M \oplus \mathrm{N}_L M \oplus \mathrm{N}_N L$. This implies precisely that in the diagram

(15)
$$\begin{tikzcd}
& L \arrow[rr, "\pi^*\mathrm{T}M \times (\mathrm{N}_L M \oplus \mathrm{N}_N L)"] \arrow[dr, hook, "\iota"] \arrow[dl, two heads, "\pi"'] & & B\mathrm{O}(n) \times B\mathrm{O}(m) \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\boxplus"] & \\
& & N \arrow[rr, "\mathrm{T}N"] & & B\mathrm{O}(n+m) \ , \\
M \arrow[rr, "\mathrm{T}M"] & & B\mathrm{O}(n) & &
\end{tikzcd}$$

the back square

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & B\mathrm{O}(n) \times B\mathrm{O}(m) \arrow[d] \\
N \arrow[r] & B\mathrm{O}(n+m)
\end{tikzcd}$$

commutes, and the front square commutes trivially. This yields (14).

Remark 4.11. $\mathrm{N}_L M \oplus \mathrm{N}_N L \simeq \mathrm{N}_N M \coloneqq (\pi^*\mathrm{T}M)^\perp \subset \iota^*\mathrm{T}N$ with respect to the pulled back metric.

Applying $\mathcal{EX}$ and post-composing with $\mathfrak{U}$, we have the induced map

$$\mathcal{EX}(\mathfrak{S}) \to \mathcal{V}^{\hookrightarrow},$$

which is the linked version of the (conically-smooth) constructible tangent bundle.

Example 4.12. If L is induced by a closed submanifold inclusion $M \subset \overline{N}$ as $L = \mathbb{S}(\mathrm{N}_N M)$, the sphere bundle of the normal bundle ([9, Example 2.6]), then L has dimension $n+m-1$, $\mathrm{N}_L M$ has rank $m-1$, and $\mathrm{N}_L N$ has rank 1. More specifically, in the conically-smooth context, the link (of a pair of strata) comes with an open embedding $L \times \mathbb{R} \hookrightarrow N$, which is tantamount to the triviality of the latter normal bundle, i.e., $\mathrm{N}_N L \simeq \varepsilon^1$, or, equivalently, to a diffeomorphism $L \times \mathbb{R} \simeq \mathbb{S}\left(\mathrm{N}_N M\right) \times \mathbb{R}$. This $\mathbb{R}$-factor incarnates the extra $\mathbb{E}_1$-structure featuring in the classification of stratified locally-constant factorisation algebras on stratified spaces of type $M \subset \overline{N}$.

Example 4.13. An even simpler situation arises when L (and $\mathfrak{S}$) is induced by a boundary $M = \partial\overline{N} \subset \overline{N}$ as $L \cong M$, the boundary pushed diffeomorphically into the interior $N = \overline{N} \smallsetminus M$ by following the flow of a nowhere-vanishing inward pointing vector field along the boundary (which always exists) for a chosen time ([9, Example 2.5]). Then $\mathrm{N}_L M = 0$ and $\mathrm{N}_N L \simeq \varepsilon^1$ again. The type of classification statement mentioned in Example 4.12 reduces here to Kontsevich's Swiss-Cheese conjecture (e.g., [11]).

Definition 4.14. We call a linked manifold with M of dimension n and N of dimension $n+m$ *constructible* if L is of dimension $n+m-1$, and its normal bundle in N is trivialisable.

5. Linked geometry

5.1. **Adapting AFR-type structures.** The ∞-category of *tangential structures* is the over-∞-category $\mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow}$ ([2]; see [6, §3] for $\mathcal{C}\mathrm{at}_\infty$). Via

$$\mathbb{U}^* \colon \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{EX}(B\mathrm{O}(n,m)),$$

these transfer to tangential structures on linked manifolds: given $\mathfrak{S}$, and writing $\mathcal{B}_{(n,m)} := \mathbb{U}^*\mathcal{B}$, we may define the space (homotopy type) of $\mathcal{B}$-structures on $\mathfrak{S}$ to be

$$\mathcal{B}\text{-red}(\mathfrak{S}) := \mathrm{Map}_{/B\mathrm{O}(n,m)}\left(\mathcal{EX}(\mathfrak{S}), \mathcal{B}_{(n,m)}\right),$$

the mapping space in $\mathcal{C}\mathrm{at}_\infty/\mathcal{EX}(B\mathrm{O}(n,m))$, where the first argument uses $\mathrm{T}\mathfrak{S}$ (Construction 4.10). Equivalently, $\mathcal{B}\text{-red}(\mathfrak{S}) = \Gamma\left((\mathrm{T}\mathfrak{S})^*\,\mathcal{B}_{(n,m)}\right)$, the homotopy-sections of $(\mathrm{T}\mathfrak{S})^*\,\mathcal{B}_{(n,m)} \to \mathcal{EX}(\mathfrak{S})$.

Given $\mathcal{B}$ and (n,m), the natural question is whether

$$\mathcal{B}_{(n,m)} = \mathcal{EX}(\mathfrak{B})$$

for a linked space $\mathfrak{B}$, which would enable us to discuss stratified tangential structures without having to refer to exit paths.

We will first discuss the simplest example imaginable. To begin with, recall that for $\kappa \in \mathbb{N}$, rank-κ framings (*κ-framings*) are expressed by the tangential structure $\boldsymbol{\kappa} \colon * \to \mathcal{V}^{\hookrightarrow}$ that sends the point to $\boldsymbol{\kappa} := \mathbb{R}^\kappa$.

Example 5.1 (framings). We have

$$\boldsymbol{\kappa}_{(n,m)} = \begin{cases} \mathcal{EX}\left(\emptyset \leftarrow \emptyset \to *\right) = *, & n+m = \kappa, \\ \emptyset, & \text{else} \end{cases}$$

with

$$\left(\boldsymbol{\kappa}_{(n,m)} \to \mathcal{EX}\left(BO(n,m)\right)\right) = \mathcal{EX}\left((\emptyset \leftarrow \emptyset \to *) \to BO(n,m), * \xrightarrow{\boldsymbol{\kappa}} BO(n+m)\right).$$

This reflects the fact that a nontrivially stratified space does not admit a κ-framing: the else-statement implies that for a linked space to admit a κ-framing its bulk must be κ-dimensional. The first statement implies moreover that for a lift of $\mathrm{T}\mathfrak{S}$ to $(\emptyset \leftarrow \emptyset \to *)$ to exist, the space must be of type $\mathfrak{S} = (\emptyset \leftarrow \emptyset \to N)$ (if non-empty), and $\dim N = \kappa$.

Similar considerations apply to any *classical tangential structure* $\mathfrak{b}\colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, i.e., one that factors through $BO(\kappa) \hookrightarrow \mathcal{V}^{\sim} \hookrightarrow \mathcal{V}^{\hookrightarrow}$ for some k.

Example 5.2. Let $\mathfrak{b}$ be a classical tangential structure given by a map $B \to BO(\kappa)$ of spaces, e.g., induced by a map $G \to O(\kappa)$ of topological groups, or a rank-κ bundle $X \to BO(\kappa)$ on a space X. Then,

$$B_{(n,m)} = \begin{cases} \mathcal{EX}\left(\emptyset \leftarrow \emptyset \to B\right) = B, & n+m = \kappa, \\ \emptyset, & \text{else}, \end{cases}$$

where we abbreviated $\mathrm{Sing}_\bullet(B)$ to B in its last occurence.

Example 5.3. Consider $\mathbb{N} = (\mathbb{N}, \leq)$ with the standard order. Variframings ([2]) are given by $\mathrm{vfr}\colon \mathbb{N} \to \mathcal{V}^{\hookrightarrow}$, $k \mapsto \boldsymbol{k}$, $(k \leq K) \mapsto (\boldsymbol{k} \xhookrightarrow{-\oplus 0} \boldsymbol{K})$. We read $\mathrm{vfr}(k \leq K)$ as the standard[1] isomorphism $\boldsymbol{k} \boxplus (\boldsymbol{K} - \boldsymbol{k}) \cong \boldsymbol{K}$. Let us restrict vfr to depth 1 by choosing a pair $n \leq N$, i.e., consider $\mathrm{vfr}|_{n\leq N}\colon \{n \leq N\} \to \mathcal{V}^{\hookrightarrow}$. Then, for $m = N - n$, we have

$$\mathbb{U}^*(\mathrm{vfr}|_{n\leq N}) \simeq \mathcal{EX}(* \leftarrow * \to *) \simeq \Delta[1],$$

the exit path ∞-category of the nontrivially-linked point. Moreover, $\mathbb{U}^*(\mathrm{vfr}|_{n\leq N}) \to \mathcal{EX}(BO(n,m))$ is $\mathcal{EX}$ of

$$\begin{tikzcd}
 & * \arrow[dl] \arrow[dr] \arrow[d, "\boldsymbol{n}\times\boldsymbol{m}"] & \\
* \arrow[d, "\boldsymbol{n}"] & BO(n) \times BO(m) \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\boxplus"] & * \arrow[d, "\boldsymbol{N}"] \\
BO(n) & & BO(N)
\end{tikzcd}.$$

Thus, a *variframing* on $\mathfrak{S} = (M \leftarrow L \hookrightarrow N)$, i.e., a lift of $\mathrm{T}\mathfrak{S}$ to this $\Delta[1]$, is a framing on M, a framing on N, and a framing on $\mathrm{N}_N M$ (see Remark 4.11).

Example 5.4 (point defects). The choice of a point p in a smooth manifold N of dimension n and a coordinate neighbourhood around it induce a linked space

$$\mathfrak{N}_p := \left(\{p\} \leftarrow S^{n-1} \xhookrightarrow{\iota_p = \iota} N \setminus \{p\}\right)$$

where the sphere is the unit sphere in coordinates. The link map of $\mathrm{T}\mathfrak{N}_p$ reads

$$\varepsilon^0 \times \left(\mathrm{T}S^{n-1} \oplus \mathrm{N}(\iota)\right) : S^{n-1} \to * \times BO(n),$$

i.e., $\iota^*\mathrm{T}\left(N \setminus \{p\}\right) : S^{n-1} \to BO(n)$. A $\mathrm{vfr}_{0\leq n}$-structure on $\mathfrak{N}_p$ is a framing on N together with a framing on the normal bundle of ι. In this example, the latter always exists, which is why we will call such a configuration a trivial point defect.

[1]Up to, of course, the choice of a pairing function $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$

Two relaxations of the tangential structure $\boldsymbol{\kappa}$ (or of any classical structure) are crucial for our purposes. They are termed, in increasing order of generality, the *stable* and *solid* replacements.

5.2. **Replacements and normal bundles.** For $J \in \mathbb{N}$, a stably-J-framed smooth manifold M of dimension n is one with a framing on $\mathrm{T}M \oplus \varepsilon^j$, $J = n + j$. In other words, this amounts to an injection $\mathrm{T}M \hookrightarrow \varepsilon^J$ of bundles over M whose normal bundle, defined either using a metric on ε^J or as the quotient $\varepsilon^J/\mathrm{T}M$, is parallelised. More generally, a solid J-framing on M is merely an injection $\mathrm{T}M \hookrightarrow \varepsilon^J$. First of all, we notice that in order to impose such a condition on said normal bundle *in terms of tangential structures*, we must first separate it from the solid datum.

Let X be a smooth manifold equipped with a vector bundle $E \to X$ of rank r, and let $F \to X$ be another bundle, of rank R. Choosing a bundle embedding $E \hookrightarrow F$ is a 'reduction [or extension] of gauge group' on E in the following way. There is naturally a normal bundle N to E such that the embedding amounts to an isomorphism $\Phi\colon E \oplus N \cong F$. This Φ provides a filler for the diagram

$$\begin{tikzcd} & B\mathrm{O}(r) \times B\mathrm{O}(R-r) \arrow[d, "\boxplus"] \\ X \arrow[r, "F"'] \arrow[ur, dotted, "E\times N"] & B\mathrm{O}(R). \end{tikzcd}$$

Changing our point of view slightly, consider the limit space

$$\begin{tikzcd} X \times_{B\mathrm{O}(R)} (B\mathrm{O}(r) \times B\mathrm{O}(R-r)) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & B\mathrm{O}(r) \times B\mathrm{O}(R-r) \arrow[d, "\boxplus"] \\ X \arrow[r, "F"] & B\mathrm{O}(R) \end{tikzcd} \tag{16}$$

which also admits a 'source evaluation' by projecting to the first factor:

$$\mathrm{ev}_0\colon (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \to B\mathrm{O}(r).$$

Now, writing

$$(B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \coloneqq X \times_{B\mathrm{O}(R)} (B\mathrm{O}(r) \times B\mathrm{O}(R-r)), \tag{17}$$

the choice of Φ can be expressed as follows:

Definition 5.5. A *solid F-structure* or *-reduction* on E (or on X when $E = \mathrm{T}X$) of is a lift of the form

$$\begin{tikzcd} & (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "E"'] \arrow[ur, dotted] & B\mathrm{O}(r) \end{tikzcd} .$$

We call R the *total rank* of the solid structure.

The normal bundle itself can be recovered from such a lift by projecting it to the second factor:

$$N\colon X \to (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_F \to B\mathrm{O}(R-r).$$

Thus, a further, simultaneous reduction on N can be implemented using this projection: if N is to have $(B \to B\mathrm{O}(R-r))$-structure, then we may consider the iterated fibre product

$$\begin{tikzcd} (BO(r) \times BO(R-r))|_F \times_{BO(R-r)} B \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & (BO(r) \times BO(R-r))|_F \arrow[d] \\ B \arrow[r] & BO(R-r) \end{tikzcd} \tag{18}$$

and, writing

$$(BO(r) \times BO(R-r))|_{(F,B)} \coloneqq (BO(r) \times BO(R-r))|_F \times_{BO(R-r)} B, \tag{19}$$

ask for reductions of the following form:

Definition 5.6. A *solid* (F, B)*-structure* on E (or on X when $E = \mathrm{T}X$) is a lift of the form

$$\begin{tikzcd} & (BO(r) \times BO(R-r))|_{(F,B)} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[ur, dotted] \arrow[r, "E"] & BO(r) \end{tikzcd} .$$

When $B = \boldsymbol{R-r}$, this is a *stable* F*-structure.* We call $B \to BO(R-r)$, or B, the *normal structure.*

It is incidental that F is given as a bundle over X. More generally, when $F\colon Y \to BO(R)$ is any classical tangential structure with rank $R \geq r = \mathrm{rk}(E)$, the limit (16), and so also (18), still make sense. Then, a *solid* Y*- or* (Y, B)*-structure* is defined analogously, as is a *stable* Y*-structure.*

Solid replacements in the stratified context have been considered in [2]. Categorically speaking, they are cartesian fibration replacements. Namely, the assignment in the following Definition 5.7 extends (by a main result of [3]) to a left-adjoint $\mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{cart}}/\mathcal{V}^{\hookrightarrow}$ to the forgetful functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{cart}}/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow}$ from *solid/cartesian* tangential structures (i.e., cartesian fibrations on $\mathcal{V}^{\hookrightarrow}$) to tangential structures.

Definition 5.7. Given a tangential structure $\mathfrak{b}\colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, its *solid replacement* is

$$\overline{\mathfrak{b}}\colon \overline{\mathcal{B}} = \overline{(\mathcal{B}, \mathfrak{b})} = (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} \mathcal{B} \to (\mathcal{V}^{\hookrightarrow})^{\{0\}},$$

the source evaluation from the fibre product along the target evaluation.

Note the direct correspondence with Definition 5.5 (and (16)), in view of $BO(r) \times BO(R-r)$'s being the link of the $(r, R-r)$-Grassmannian, viewing the target evaluation as the embedding $\boxplus$ off the link. A simple observation will make this precise.

Notation 5.8. Given an embedding $\iota\colon \Sigma \hookrightarrow N$ and a point $q \in N$, we let $P(N)_{\Sigma,q} = P_{\Sigma,q}$ denote the space of paths in N that start in $\iota(\Sigma)$ and end in the point q, equipped with the compact-open topology. We use analogous notation when we work with a cofibration ι of simplicial sets.

The following result formalises and confirms the intuition the link represents an infinitesimal expansion of the lower stratum into the higher stratum.

Theorem 5.9. *Let* $\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ *be a linked space, and* $p \in \mathcal{M}$ *and* $q \in \mathcal{N}$ *points in the two strata. We then have an equivalence*

$$\mathrm{Hom}_{\mathcal{E}\mathcal{X}(\mathfrak{S})}(p, q) \simeq \mathcal{P}_{\mathcal{L}_p, q}$$

between the morphism space in $\mathcal{EX}$ from p to q and that of paths in $\mathcal{N}$ that start in $\iota(\mathcal{L}_p)$ (with ordinary fibre $\mathcal{L}_p = \{p\} \times_{\mathcal{M}} \mathcal{L}$) and end in q.

Proof. We will work with a model for morphism spaces that makes the proof particularly simple: by [7, 01L5], the morphism space in $\mathcal{EX}$ is equivalent to the *right-pinched* morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{EX}}(p,q) \coloneqq \{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$. We will observe directly that $\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$ is in fact isomorphic to $\mathcal{P}_{\mathcal{L}_p,q} = \mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q)$. Indeed, at vertex level, the bijection

$$(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q))_0 \cong (\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q))_0$$

is clear: recalling that non-invertible 1-paths in $\mathcal{EX}$ are elements of $\mathcal{P}^{\Delta}_0 \subset \mathcal{N}_1$ (as the exit index is necessarily 1 in this degree), let $(\gamma,1) \in \mathcal{P}^{\Delta}_0$. For $p = d^{\mathcal{EX}}_1(\gamma,1) \overset{\mathrm{def}}{=} \pi(d^{\mathcal{N}}_1(\gamma))$ to hold, we must have $d^{\mathcal{N}}_1(\gamma) \in \iota(\mathcal{L}_p)$. Similarly, $d^{\mathcal{EX}}_0(\gamma,1) \overset{\mathrm{def}}{=} \iota(d^{\mathcal{N}}_0(\gamma))$, which yields the bijection.

Let now $k > 0$ and consider an exit $(k+1)$-path $(\gamma\colon \Delta[k] \star \Delta[0] \to \mathcal{N}, j)$ in $(\mathcal{EX}/q)_k \subset \mathcal{EX}_{k+1}$. Asking that (γ,j) be in $\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$ is equivalent to asking that

(1) its $\mathcal{N}$-face
$$\Delta[k] \hookrightarrow \Delta[k] \star \Delta[0] \xrightarrow{\gamma} \mathcal{N},$$
which is $d^{\mathcal{N}}_{k+1}(\gamma)$ under the standard identification $\Delta[k] \star \Delta[0] \simeq \Delta[k+1]$, is bottom, as by construction only then can the corresponding $\mathcal{EX}$-face be in $\mathcal{M}_k \subset \mathcal{EX}_k$;

(2) and that it lies in particular in $\iota(\mathcal{L}_p)$.

Condition (1) implies moreover that $d^{\mathcal{N}}_{\ell}(\gamma) \in \mathcal{N}_k$ is vertical for all $\ell < k+1$, since all other faces include the tip $\Delta[0] \hookrightarrow \Delta[k] \star \Delta[0]$ given by q, whence they are necessarily not bottom; and if some $d_\ell(\gamma)$ was top, that would contradict the bottomness of its (unique) common $(k-1)$-face with $d^{\mathcal{N}}_{k+1}(\gamma)$. In fact, (γ,j) has no n-face that is top once $n > 0$: given $\Delta[n] \hookrightarrow \Delta[k+1]$, there is necessarily a vertex in $d^{\mathcal{N}}_{d+1}(\gamma)$ that is hit by it.

But then the exit index j must be maximal: $j = k+1$. For if not, then there would exist at least one top n-face for $n > 0$, the largest such, with $n = k+1-j$, for instance, being specified by $[n] \hookrightarrow [k+1]$, $\alpha \mapsto \ell + \alpha$. We thus obtain a bijection

$$(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q))_k \cong (\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q))_k$$

in a fashion similar to the bijection of vertices: we have reduced exit paths (γ,j) in question on the LHS to those of index $k+1$, and so to only a subset of $\mathcal{N}_{k+1}$, and specifically those such that $d^{\mathcal{N}}_{k+1}(\gamma) \in \mathcal{L}_p$. These are exactly the elements of the RHS. Finally, it is a direct check that $(\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q))_* \xrightarrow{\cong} (\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q))_*$ is functorial; for instance, any vertical face of such a $(\gamma, k+1)$ is again of maximal index: we have $\flat^{k+1}_{k+1,i} = k$ and, and as for degeneracies, $\sharp^{k+1}_{k+1,i} = k+2$, for all $i < k+1$ ([9, §2]). □

Before proceeding to solid/stable structures in the linked setting, we will explore some immediate and useful consequences of this result. In the proofs, we will use Theorem 5.9 without mention.

Corollary 5.10. *Let* $\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ *be a linked space with* $\mathcal{M}$ *and* $\mathcal{N}$ *connected, and* $p \in \mathcal{M}$, $q \in \mathcal{N}$. *Then,* π *is an equivalence if and only if*

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p,q) \simeq \Omega\mathcal{N}.$$

Here, $\Omega\mathcal{N}$ *denotes the based loop space of* $\mathcal{N}$.

Proof. The fibre at any point of the source evaluation $\mathcal{P}_{\mathcal{L}_p,q} \to \mathcal{L}_p$ is equivalent to $\Omega\mathcal{N}$. Thus, the homotopy long exact sequence of this fibration implies that the fibre inclusion induces isomorphisms $\pi_*(\Omega\mathcal{N}) \cong \pi_*(\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p,q))$ iff $\pi_*(\mathcal{L}_p) \cong *$. Thus, $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p,q) \simeq \Omega\mathcal{N}$ iff π is a trivial Kan fibration. But π's being a trivial Kan fibration is tantamount to its being an equivalence ([7, 00X2]). □

We interpret this as saying that the space(s) of non-invertible paths in a linked space is (are) at its (their) largest when π is an equivalence; there are just as many as there are paths in the higher stratum. This is the case, for instance, when $\mathfrak{S}$ is induced by a manifold with boundary, as in Example 4.13. We have a maximal simplification in the other extreme, namely when π is trivial.

Corollary 5.11. *Consider a linked space of type* $\mathfrak{S} = \left(* \leftarrow \mathcal{N} \xrightarrow{\mathrm{id}} \mathcal{N}\right)$.

1. *We have* $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(*,q) \simeq *$.
2. *When* $\mathcal{N} = \mathrm{Sing}_\bullet(N)$ *for* N *a smooth manifold, we have*

   $$\mathrm{Exit}(C(N)) \simeq \mathcal{EX}\left(* \leftarrow N \xrightarrow{\mathrm{id}} N\right),$$

   where the LHS is the exit path ∞*-category à la Lurie/MacPherson/AFR of the conically-smooth open cone* $C(M) = * \amalg_{\{0\}\times N} ([0,1) \times N)$ *on* N *with its canonical stratification over* $\{0 < 1\}$.
3. *The tangent bundle of the linked space* $\mathfrak{N}_p$ *given by a point defect* $p \in N$ *(Example 5.4) is determined, up to a contractible choice, by* $\mathrm{T}N$.

Proof. (1) We have $\mathcal{L}_* = \mathcal{N}_* = \mathcal{N}$ and so $\mathcal{P}_{\mathcal{L}_*,q} \simeq \mathcal{N}/q \simeq *$ ([7, 018Y]).

(2) Statement (1) implies $\mathcal{EX}\left(* \leftarrow N \xrightarrow{\mathrm{id}} N\right) \simeq \mathrm{Sing}_\bullet(N)^{\triangleleft}$. The latter agrees with the LHS by [1, Proposition 3.3.8], since the cone locus $*$ is a (the) deepest stratum of $C(N)$.[2]

(3) Let $p \neq q \in N$. In the corresponding locally-Kan categories, even though $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{N}_p)}(p,q) \simeq P(N)_{\iota_p(S^{n-1}),q}$ may have nontrivial homotopy type, the non-invertible paths are mapped per

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{N}_p)}(p,q) \to \mathrm{Hom}_{B\mathrm{O}(0,n)}(*, T_q(N \setminus \{p\})) \simeq *.$$

since $\mathcal{EX}(B\mathrm{O}(0,n)) \simeq B\mathrm{O}(n)^{\triangleleft}$ by (the proof of) statement (2), which is to say that the adjoined point $* \in B\mathrm{O}(0) \subset B\mathrm{O}(0,n)$ is initial.

□

Remark 5.12. Item 3 of Corollary 5.11 is, in a sense, redundant, as the statement already follows from the requirement that $\mathrm{T}\mathfrak{N}_p$ be a map of spans. It should be read as saying that there is no choice in its restriction to the link S^{n-1}.

[2] More precisely, this is an equivalence of quasi-categories for Lurie's model from [5], or, after translating to the complete Segal space model and using [1, Lemma 3.3.9], with that of Ayala et al.

A solid $(F : Y \to B\mathrm{O}(R))$-structure on a rank-$r$ bundle ought to be (a lift to) the restriction to $B\mathrm{O}(r)$ of the solid replacement of F:

$$\begin{tikzcd} \overline{Y}|_r \coloneqq B\mathrm{O}(r) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} \overline{Y} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & \overline{Y} \arrow[d, "\overline{F}"] \\ B\mathrm{O}(r) \arrow[r] & \mathcal{V}^{\hookrightarrow} \end{tikzcd} .$$

This is the space of morphisms in $\mathcal{V}^{\hookrightarrow}$ that start in $B\mathrm{O}(r)$ and end in the image of F inside $B\mathrm{O}(R)$, and $\overline{Y}|_r \to B\mathrm{O}(r)$ is the source evaluation:

$$\left(\overline{Y}|_r \to B\mathrm{O}(r)\right) = \Big((B\mathrm{O}(r) \times B\mathrm{O}(R-r)|_F) \xrightarrow{\mathrm{ev}_0} B\mathrm{O}(r)\Big).$$

Proposition 5.13. $\overline{Y}|_r \simeq (B\mathrm{O}(r) \times B\mathrm{O}(R-r)) \times^{\mathrm{h}}_{B\mathrm{O}(R)} (Y)$.

Lemma 5.14. $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q) \simeq \mathrm{Hom}_{\mathcal{E}\mathcal{X}(B\mathrm{O}(r,R-r))}(p, q)$, *where* $p \in B\mathrm{O}(r)$ *and* $q \in B\mathrm{O}(R)$.

Proof. $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q)$ is equivalent to the homotopy-fibre of

$$p^* \colon \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(\mathcal{B}^{\boxplus}\mathrm{O})}(*, *) \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(\mathcal{B}^{\boxplus}\mathrm{O})}(*, *),$$

i.e., using that morphism spaces in the homotopy-coherent nerve are equivalent to those in the original topological category ([4]), the homotopy-fibre of

$$(- \boxplus p) \colon B\mathrm{O}^{\infty}_{\mathrm{II}} \to B\mathrm{O}^{\infty}_{\mathrm{II}},$$

at q. The connected component of q in $B\mathrm{O}^{\infty}_{\mathrm{II}}$ is $B\mathrm{O}(R)$, and p^* maps only $B\mathrm{O}(R-r)$ into it, so we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q) &= (B\mathrm{O}(R-r) \boxplus p) \times_{B\mathrm{O}(R)^{\{0\}}} B\mathrm{O}(R)^{\Delta[1]} \times_{B\mathrm{O}(R)^{\{1\}}} \{q\} \\ &= \pi^{-1}(p) \times_{B\mathrm{O}(R)^{\{0\}}} B\mathrm{O}(R)^{\Delta[1]} \times_{B\mathrm{O}(R)^{\{1\}}} \{q\} \\ &= P(B\mathrm{O}(R))_{\pi^{-1}(p),q} \\ &\simeq \mathrm{Hom}_{\mathcal{E}\mathcal{X}(B\mathrm{O}(r,R-r))}(p, q) \end{aligned}$$

by Theorem 5.9. By $\pi = \mathrm{pr}_1$ we denoted the link projection in $B\mathrm{O}(r, R-r)$. □

Corollary 5.15. $\mathbb{U}$ *is fully faithful.*

Proof. That $\mathbb{U} \colon \mathrm{Hom}_{\mathcal{E}\mathcal{X}}(p, q) \to \mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q)$ is an equivalence when $p, q \in B\mathrm{O}(r)$ or $p, q \in B\mathrm{O}(R)$ is clear, since $B\mathrm{O}^{\infty}_{\mathrm{II}}$ is the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$. Let now $p \in B\mathrm{O}(r)$, $q \in B\mathrm{O}(R)$. □

Proof of Proposition 5.13. □

In other words:

Corollary 5.16. *A solid F-structure, in the sense of Definition 5.5, on a smooth manifold X of dimension r, is a solid F-structure, in the sense of Definition 5.7, on the trivially-linked manifold X.*

This leads to a natural elaboration on the idea of Example 5.3 in the context of replacements pulled back to the nontrivially-linked setting:

5.3. **A Grothendieck construction.**

5.4. **The linked universal frame and tautological bundles.**

6. Duals

6.1. Stable duals.

6.2. Solid duals.

6.3. Duals of framed bordisms.

References

[1] D. Ayala, J. Francis and N. Rozenblyum. 'A stratified homotopy hypothesis'. *Journal of the European Mathematical Society* 21.4 (2018), 1071–1178. arXiv: 1502.01713 [math.AT] (cit. on p. 20).

[2] D. Ayala, J. Francis and N. Rozenblyum. 'Factorization homology I: Higher categories'. *Advances in Mathematics* 333 (2018), 1042–1177. arXiv: 1504.04007 [math.AT] (cit. on pp. 2, 3, 5, 15, 16, 18).

[3] D. Gepner, R. Haugseng and T. Nikolaus. 'Lax colimits and free fibrations in ∞-categories'. *Documenta Mathematica* 22 (2017), 1225–1266 (cit. on pp. 3, 18).

[4] F. Hebestreit and A. Krause. 'Mapping spaces in homotopy coherent nerves' (2020). arXiv: 2011.09345 [math.AT] (cit. on p. 21).

[5] J. Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf (cit. on p. 20).

[6] J. Lurie. *Higher Topos Theory*. Princeton University Press, 2009 (cit. on pp. 5, 13, 15).

[7] J. Lurie. *Kerodon*. 2023. URL: https://kerodon.net (cit. on pp. 5, 9, 13, 19, 20).

[8] J. Lurie. 'On the classification of topological field theories'. *Current developments in mathematics* 2008.1 (2008), 129–280 (cit. on p. 3).

[9] Ö. Tetik. 'Linked spaces and exit paths' (2023). arXiv: 2301.02063 [math.AT] (cit. on pp. 2, 6, 15, 19).

[10] Ö. Tetik. 'The stratified Grassmannian' (2022). arXiv: 2211.13824 [math.AT] (cit. on pp. 2, 5, 6).

[11] J. Thomas. 'Kontsevich's Swiss cheese conjecture'. *Geometry & Topology* 20.1 (2016), 1–48. DOI: 10.2140/gt.2016.20.1 (cit. on p. 15).

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