Seminar on loop quantum gravity
Talk 4
3d TQFTs from spherical fusion categories

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In the last talk, Pavel explained (among many other things) how one can discretise BF theory and write its partition function, using a triangulation, as a sum of weights depending on data assigned edges and vertices (elements of $\hat{G}$ and intertwiners), in such a way that it is independent of the choice of triangulation. The goal of this talk is to give a systematic discussion of such discrete theories in 3d.

Contents

For a complete list of references, see the seminar page at Pavel Safronov's website. The non-handdrawn figures are taken from [31] (Turaev—Viro, 1992), [7] (Balsam—Kirillov, 2010), and [22] (Kirillov, 2011).

1 Overview

The idea of a state-sum model is the following.

1. **Discretise** your spacetime (or something of higher codimension) by taking a triangulation, or some kind of dual triangulation.
2. **Colour** by attaching algebraic data to the simplicial bits. These can be spins, Hilbert spaces, algebras, categories, maps, numbers...
3. **Evaluate**, i.e. contract matching dual pieces, i.e. take a trace.
4. **Sum** over (a class of nice) triangulations.

The problem is that this will depend on the choice of triangulation. The main solution strategy is:

5. Find **local moves** that let you walk from any triangulation to any other.
6. Translate invariance under these moves into **algebraic conditions**, i.e. get a diagram for each move.

Physicists will do things in this order, because they'll start with plans to attach specific algebraic data to the simplicial bits in a specific way, so they'll have to check invariance later. Mathematicians will proceed in the reverse order, because they'll want to know what *can* be attached, given the constraints. In the latter case, the chain of determination will look like this:

the type of (T)QFT, i.e. dimension and flavour

$\downarrow$

type of discretisation such that a *finite* number of moves suffice
(also, find the moves!)

$\downarrow$

the moves become diagrams for the evaluation maps

$\downarrow$

the diagrams tell us what type of algebra/category we need

Functoriality. In general, it is not a given that what we get in the end will be functorial! In organic state-sums, however, the gluing property (multiplicativity) will come for free. Contributions from cylinders (which shouldn't contribute as they are the identities) may be a problem. Gluing only implies that cylinders will be projections in the sense that $p^2 = p$ (cylinder on a cylinder is still a cylinder). Still, one can mod out the cylinders in an appropriate way, so we need not worry.

We will cover the following constructions in 3d:

1. The Turaev–Viro state-sum TQFT, which is indeed functorial, and does not require, and is insensitive to, orientations.
   - A colouring will be given on a *dual* triangulation by attaching symbols to the *edges*. So there will only be one kind of vertex, and a tetrahedron will have a 6-tuple attached.
   - There are *3 moves* (as opposed to 2 in 2d) and a permutation symmetry.
   - It turns out that *quantum 6j-symbols* transform as required, so can be used to build this TQFT.
2. The quasi-generalisation by Barrett–Westbury, who redid TV more generally – except that this construction requires orientation. BW can accept a larger class of algebraic data as input.

- At each *vertex*, one edge is distinguished.
- Each *edge* gets an object and an orientation.
- Each *face* gets a 'Hilbert space' $H(a, b, c)$ specified by the incident objects a, b, c.
- Each *tetrahedron*, being itself oriented, has 2 incoming and 2 outgoing faces, and gives a certain map

$$\bigotimes_{\text{incoming}}^{2} H(\dots) \to \bigotimes_{\text{outgoing}}^{2} H(\dots).$$

  Such a map is specified by duality if we can specify a map

$$\bigotimes_{\text{all 4 faces}}^{4} H(...) \to \mathbb{k}$$

  to the base field, given only the information of the 6 objects on the edges. This is the incarnation of the 6j-symbols, and it is provided by a certain trace, given by translating the coloured tetrahedron to a closed trivalent graph.

The space $H(a, b, c)$ is defined to be $\mathrm{Hom}(a, b \otimes c)$ for a the distinguished object. We already see most of the structure the category must have: it must be *monoidal, additive, and with duals*. The precise relations are determined by formulating invariance under the 3 local moves. Basically, one wants a finite semisimple spherical category, also called a spherical fusion category. Spherical means that the left- and right-traces coincide.

Tannaka–Krein duality for fusion categories says that they are the representation categories of weak Hopf algebras.

When $\mathcal{C} = \mathrm{Rep}(U_q\mathfrak{sl}_2)$, BW coincides with TV.

For $\mathcal{C} = \mathcal{V}\mathrm{ec}^G$ the category of G-graded vector spaces for a finite group G, the theory is related to BF theory/Chern–Simons. The state space (the value on a 2-manifold) is the space of flat G-connections modulo gauge equivalence (which comes from 'modding out' the cylinders). Allegedly, the theory described is Dijkgraaf–Witten theory, which is supposed to be a version of Chern–Simons for finite G instead of a Lie group. It certainly looks the part.

3. The extension by Kirillov–Balsam of the TV theory to corners, i.e. closed codimension-2 submanifolds (circles). It turns out that a spherical fusion category as input is already sufficient to do this. The theory is still functorial.

   To recall, such an extension is a 1-2-3d (functorial) TQFT, whereas a usual 3d TQFT would be 2-3d. The 2-3d TQFT is a representation of the usual 2-3d cobordism category, whereas the extension is a representation of the 1-2-3d cobordism 2-category:

   - the objects are closed 1-manifolds,
   - the morphisms are 2-manifolds with (possible empty) boundary,
   - the 2-morphisms are 3-manifolds with (possibly empty) corners.

   Usually, such a TQFT would map objects to algebras, morphisms to representations, and 2-morphisms to intertwiners.

   Incidentally, the hom-sets of type $\mathrm{Hom}(a, b \otimes c)$ of the BW construction are modified to the more familiar version $\mathrm{Hom}(b \otimes c, a)$ (fusion!) in the KB construction.

   This theory assigns the Drinfeld centre $Z(\mathcal{C})$ to the objects S^1. The Drinfeld centre of a fusion category imitates the centre of an algebra. This assignment will come as no surprise to those familiar with 1-2d

TQFTs constructed from Frobenius algebras, where it is easy to see from Morse theory that the state space has to be the centre. This is the appropriate 2-category analogue of that situation.

There is an adjoint pair between $\mathcal{C}$ and $Z(\mathcal{C})$, where $Z(\mathcal{C}) \to \mathcal{C}$ forgets, and $\mathcal{C} \to Z(\mathcal{C})$ sends an object to the sum of its simple conjugates:

$$V \mapsto \bigoplus_{X \text{ simple}} X \otimes V \otimes X^\vee.$$

That the simple objects are finite in number is an assumption on the category. Similarly, there is a projection from hom-sets in $\mathcal{C}$ to hom-sets in $Z(\mathcal{C})$.

Note that this theory is not *fully* extended, that is, it is not 0-1-2-3d. One thing to always keep in mind is that a fully extended functorial TQFT is determined, by the cobordism hypothesis, by its value at a point.

2 The Turaev–Viro construction

Let M be a 3-manifold with possibly empty boundary ∂M. For convenience we work in the PL category: notably, if two PL manifolds are glued along boundary components, the result is a PL manifold without modification, and discretisation data glue as well. In any case, in these dimensions, the PL category is equivalent to the topological category.

2.1 Discretisation: first pass

This will fail a little bit.

Take a triangulation of M. Just as in the discretisation of BF theory, we would like to colour the simplices with symbols subject to some consistency relations.

Symbols. Let R be a commutative ring, R^* its invertibles, I a finite index set with a function $I \to R^*$, $i \mapsto w_i$.

- These i (and thus w_i) will be assigned to the edges. These are the 'spins'.
- Some $w \in R^*$ will be assigned to all vertices.
- Lastly, the weight of a tetrahedron will be determined by the weights on its edges: since there are 6 of those, this will be a weight of type $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \in R$ for the 6-tuple (i, j, k, l, m, n).

Constraints.

- At each face of a tetrahedron, 3 edges form a triangle. The Clebsch–Gordan fusion rule tells us which two spin representations we can fuse to give a specific spin: given spins (i, j, k) whether $i \otimes j$ projects to k. In the case i, j, k are numbers, Clebsch–Gordan sees whether these can become the edge lengths of a triangle. This is insensitive to order. Imitating this, we will only allow certain admissible unordered triples.
- An ordered 6-tuple (i, j, k, l, m, n), then, is admissible if its face triples $(i, j, k), (k, l, m), (m, n, i), (j, l, n)$ are admissible.
- The previous point is ambiguous: there are $\#S_4 = 24$ ways to attach such an admissible 6-tuple a tetrahedron so that the face triples are admissible: the symmetry group S_4 acting on the 4 vertices acts freely and transitively on the set of these possibilities. We must require that the symbols $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ are invariant under these permutations.

- We have to find the local moves that transform any triangulation into any other, then impose relations such that the state-sums are invariant.

Note that the symmetry requirement on the $6j$-symbols makes the weight insensitive to orientation.

The total weight without boundary. Assume M closed. Let $\phi\colon \{\text{edges}\} \to I$ be an admissible colouring, meaning each face is coloured by an admissible triple. The weight $|M|_\phi$ of the colouring is the product of the following contributions:

- Each vertex contributes $\frac{1}{w^2}$.
- Each edge E contributes $w^2_{\phi(E)}$
- Each tetrahedron contributes its symbol.

The state-sum without boundary. Set $|M| = \sum_{\phi \text{ admissible}} |M|_\phi$. (This is the famous TV invariant.)

The total weight with boundary. Let ϕ be as above, automatically inducing an admissible colouring on ∂M. Again, define $|M|_\phi$ by modifying the above contributions as follows:

- Take the square root of the contributions of boundary edges and vertices.

The reason is that we want things to be multiplicative under gluing, so 'multiplying two matching boundary vertices/edges should give the internal vertex/edge they get identified to' (see below).

The state-sum with boundary. Assume $\partial M = F_+ \amalg F_-$. To make an honest TQFT out of this construction, we should assign linear objects to surfaces, $F \mapsto H(F)$ and a linear map $H(F_+) \to H(F_-)$ to M.

The natural way to do this is to make the number $|M|_\phi$, which depends on ϕ, depend on data on ∂M. As always in such situations, the idea is to fix ϕ on ∂M and vary its extension to M.

Therefore, let $H(F)$ be the free R-module over admissible colourings of F, and let α and β be admissible colourings of F_+ and F_-, respectively. The map

$$H(M)\colon H(F_+) \to H(F_=)$$

has (α, β)-matrix element

$$H(M)_{\alpha\amalg\beta} = \sum_{\substack{\phi \text{ admissible,} \\ \phi|_{\partial M} = \alpha\amalg\beta}} |M|_\phi,$$

the state-sum with the state on the boundary fixed to be $\alpha \amalg \beta$.

The contributions are set such that multiplicativity of H under gluing boundary surfaces is obvious.

Alexander moves. There is a simple transformation that provides a universal method of refinement: the Alexander move. It works on general polyhedra. Let T be a triangulation, E a simplex, $b \in E^\circ$ a point in the interior of E, which we will take to be the barycentre of E.

The star $\mathrm{St}_T E$ of E is the union of all closed simplices containing the interior E°. The Alexander move at (E, b) transforms T by replacing $\mathrm{St}_T E$ by the cone over $\partial\mathrm{St}$ centred at b. With 'cone' here is meant the projection of the literal coneback onto the polyhedron/manifold as in the picture:

The Alexander theorem says that any two triangulations can be transformed into each other by Alexander moves and their inverses. Equivalently, given two triangulations, a common refinement can be found using Alexander moves.

The problem is that in general, checking invariance under Alexander moves will require infinitely many checks: the moves depend on where E lies in M and the combinatorics of $\mathrm{St}_T E$ within T. Turaev and Viro circumvented this problem by using a dual discretisation, where they could reduce Alexander moves to 3 (previously known) *universal local moves*. One has to know how to transform the dual discretisation, which is why we needed to look at Alexander moves.

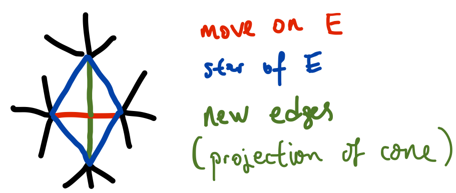
2.2 The 'right' discretisation

Any triangulation T induces a relative cell subdivision of $(M, \partial M)$, which will be the dual discretisation we will use. This is the union or collection $\{A^*\}_A$ of barycentric stars A^* of simplices A of T. Here,

$$A^* = \bigcup_{\substack{m \\ A_i}} [A, A_1, \ldots, A_m],$$

where $A \subset A_1 \subset \cdots A_m$ is an increasing sequence of simplices of T and $[A, A_1, \ldots, A_m]$ is the dual m-simplex with the barycentres of A, A_i as vertices. We will see shortly how a colourings of triangulations induce colourings on these cell subdivisions.

The figure depicts the situation where $\dim M = 2$, $\dim E = 1$, $A = E$. In general, we have $\dim A^* = \operatorname{codim} A$.



The effect of the Alexander move at E is a kind of thickening that is very predictable. The aforementioned 3 moves will allow us to perform this thickening. First, some pictures for $\dim E = 1$ in dimensions 2 and 3:

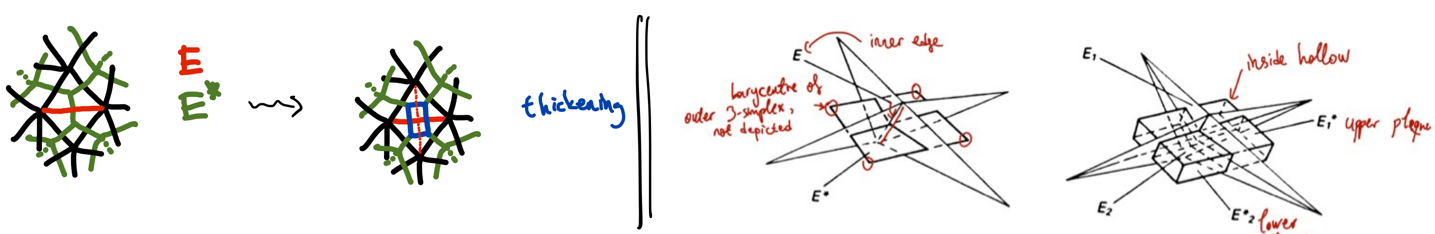


Fig. 5.1.

The 3 moves are stills from a film where a bubble is created on the barycentric star E^*, which is then stretched into the transformed star. Here they are:

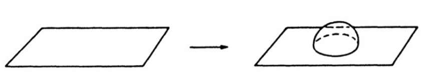


Fig. 7.

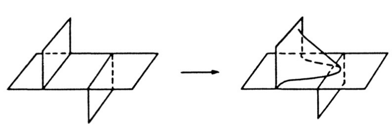


Fig. 8.

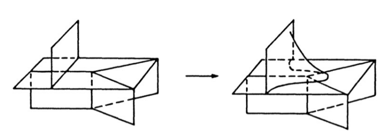


Fig. 9.

The problem is that these moves take us out of our geometric category. In particular, we don't know how to colour them. First, let us extend our category:

A fake surface (or simple 2-polyhedron) is a 2-polyhedron that can be obtained by gluing surfaces with boundary onto a closed surface along generic immersions of the boundary curves. Generic means here that the immersion has no triple points, and some garden variety transversality conditions hold. Cell subdivisions of 3-manfiolds obtained from triangulations as above are fake surfaces.

Thus, locally around a point, a fake surface can look like:

(1) $\mathbb{R}^2$

(2) $\mathbb{R}^2_+$

(3) 3 halfplanes meeting at a common line

(4) 3 quadrants meeting at a common line (or the last item cut in half)

(5) the last item, with 3 more quadrants attached along the meeting edge pairs

The boundary of a fake surface consists of points whose neihbourhoods are not of type (1), (3), (5). It is a finite trivalent graph (with loops).

Colouring and evaluating fake surfaces. Let T be a triangulation of M with colouring $\phi\colon \{\text{edges}\} \to I$,and T^* the induced (relative) cell subdivision, which is a fake surface. A colouring ϕ^* of T^* is induced by setting: $\phi^*(E^*) = \phi(E)$ where E^* is the 2-cell dual to the edge E of T, i.e. its barycentric star as above. One defines admissability of colourings of fake surfaces in such a way that ϕ^* is admissible if ϕ is.

The weight $|M^*|_{\phi^*} := |T^*|_{\phi^*}$ (we abuse notation because the weights will not depend on the choice of T) should be defined in such a way that

- $|M|_\phi = |M^*|_{\phi^*}$, and
- $|M^*|_{\phi^*} = |K^*|_{\phi|K^*} \cdot |L^*|_{\phi|L^*}$ if $M^* = T^*$ is obtained by gluing fake surfaces K^*, L^* along boundary components.

There are some complications. The strata of a fake surface consist of mixes of vertices, edges and faces glued together in complicated ways. We must pick exponents for weights w in such way that that the whole product is multiplicative, i.e. the exponents are additive.

Turaev and Viro use the additivity of the Euler characteristic. Let X be a fake surface, Γ_i its 2-strata, E_i its boudary edges, and x_i its interior vertices. A colouring $\psi\colon \{\Gamma_i\} \to I$ evaluates X via

$$|X|_\psi = w^{-2\chi(X)+\chi(\partial X)} \prod_{\Gamma_i} w_{\psi(\Gamma)}^{2\chi(\Gamma_i)} \prod_{E_i} w_{(\partial\psi)(E_i)}^{\chi(E_i)} \prod_{x_i} |\hat{T}^\psi_{x_i}|.$$

Here, $\partial\psi$ colours boundary edges by assigning them the colour of the 2-strata which contain them. For an internal vertex x, its adjacent 1- and 2-strata determine the vertices and edges of a tetrahedron T_x. Then, $\hat{T}_x$ is its dual tetrahedron: its edges correspond to 2-strata (and faces to 1-strata) of X at X, so ψ can colour its edges. (This is the reason for taking the dual tetrahedron.) Then, $|\hat{T}_x|$ is simply the symbol of this coloured tetrahedron.

For comparison, the weight of a colouring ϕ of M in symbols looks like this:

$$|M|_\phi = w^{-2a+e} \prod_{E_i \in \partial M} w_{\phi(E_i)} \prod_{E_i \notin \partial M} w^2_{\phi(E_i)} \prod_T |T^\phi|$$

with a the total number of vertices, e the number of boundary vertices.

This is exactly the same product if the fake surface is the dual of a triangulation of a 3-manifold. The general case incorporates fake surfaces obtained by performing the moves $\mathcal{B}, \mathcal{M}, \mathcal{L}$ without destroying the gluing property.

Let us mention that this defines a functorial 'polyhedral' TQFT in a similar way.

2.3 The algebraic constraints

Invariance under $\mathcal{B}$ translates for the weights w, w_i to the equation

$$w^2 = w_j^2 \sum_{\substack{(j,k,l) \\ \text{admissible}}} w_k^2 w_l^2$$

for all $j \in I$. In particular, one must require irreducibility, i.e. that for any $l_1, l_2 \in I$ there exist $l_3, \dots l_n$ such that all (l_i, l_{i+1}, l_{i+2}) are admissible.

The idea is that before the bubble, the local (partial) state-sum is

$$\frac{1}{w^2} w_j^2$$

for some colouring j and after the bubble it becomes

$$\sum_{\substack{(j,k,l) \\ \text{admissible, } j \text{ fixed}}} \frac{1}{w^4} w_k^2 w_l^2.$$

Invariance under $\mathcal{M}$ and $\mathcal{L}$ translate to similar equations with sums.

Turaev and Viro have shown that these equations hold for weights coming from $U_q\mathfrak{sl}_2$ at q a root of unity – see their paper for the calculations. In any case, we will see shortly what the proper context for these equations is.

3 The Barrett–Westbury construction

The input for the state-sum is a spherical fusion category $\mathcal{C}$. We will talk about the details of what this means in the appropriate places.

The discretisation is simply a triangulation (but we will also use some simple dual information). Deviating from BW's original construction we will allow polytope triangulations, since the construction is the same. Thus, for instance, the recipe for a torus by the identification of the opposing edges of a square is allowed here since it gives a polytope decomposition (though not a usual one).

Here is a summary that one should keep in mind: *The (dual graph of the) triangulation will be interpreted as a morphism. Its weight will be a trace. If M is closed, this will be an endomorphism, and its trace a number. If there is a boundary, the trace will contract only in the bulk, leaving uncontracted factors on the boundary. This is because morphisms come from tetrahedra.*

3.1 Discrete data

We will attach the following data to a triangulation. Data on the dual graph will be induced.

- Each edge e gets an object $l(e)$, such that $l(\bar{e}) = l(e)^*$. Physically speaking, this would be time-reversal.

- Each face C (can be any planar polygon) gets a state space determined by its edges:

$$H(C,l) = \langle l(e_1), \dots, l(e_n) \rangle.$$

We can have $H(C,l)$ simply collect spins, i.e. elements of $l(e_1) \otimes \cdots \otimes l(e_n)$. Since there is no notion of 'element' in our category, we have to put

$$\langle l(e_1), \dots, l(e_n) \rangle = \mathrm{Hom}(1, l(e_1) \otimes \cdots \otimes l(e_n))$$

  - This ought to depend only on the cyclic order of the edges, reflecting the nature of orientation.
  - This definition of $H(C,l)$ is nice also because

$$\langle l(e_1), \dots, l(e_n) \rangle = \left(\bigotimes_{\text{incoming}} l(e) \right) \otimes \left(\bigotimes_{\text{outgoing}} l(e) \right).$$

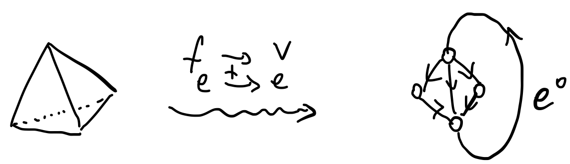
  - Reversing the orientations and the order should give the dual space: we'd like to have

$$\langle l(e_1), \dots, l(e_n) \rangle^* = \langle l(e_n)^*, \dots, l(e_1)^* \rangle$$

For a surface Σ, we can put these spaces together and set

$$H(\Sigma, l) = \bigotimes_{\text{2-cells } C \text{ of } \Sigma} H(C,l)$$

- A 3-cell T (which does not have to be a tetrahedron) should evaluate into the ground ring, or $\mathrm{End}(1_{\mathcal{C}})$. The dual graph of a 3-cell is closed – in fact it will be a *loop for one of the edges* (let's call it e°) up to some local tangle, as in the picture:



Thus the graph – let's call it $T^\vee$ describes a morphism in $\mathrm{End}\, l(e^\circ)$. We'd like to take a trace

$$\mathrm{tr} \colon \mathrm{End}(l(e^\circ)) \to \mathrm{End}(1) = \Bbbk$$

This number depends on the morphisms at the vertices. On the original triangulation, these are attached to faces, so let's call them $\phi_{C,l}$. So we can think about the trace above as the evaluation of $T^\vee$ under $\bigotimes_{C \in \partial T} \phi_{C,l}$, i.e.

$$\mathrm{tr}\left(T^\vee, \bigotimes \phi_{C,l} \right) = \left(\bigotimes \phi_{C,l} \right) \left(l(e_1) \otimes \cdots \otimes l(e_n) \right).$$

In particular, these $\phi_{C,l}$ thus make up a basis $H(C,l)^*$.

- (Yes, the graph $T^\vee$, which is *planar*, is not uniquely determined by T: one can also close the loop on the other side of the local tangle. We will return to this.)
- Since the theory Z should assign to T the state space of its boundary, i.e. since we wish to define

$$Z(T,l) \in H(\partial T, l)$$

, we can use this pairing to define $Z(T,l)$ by requiring

$$\left(\bigotimes \phi_{C,l} \right) (Z(T,l)) = \left(\bigotimes \phi_{C,l} \right) (T^\vee).$$

In other words, we identify $Z(T,l)$ with $T^\vee$, without morphisms put in at the vertices.

3.2 Complete definition

Let us assume invariance under the choice of triangulation for the moment. We will force it anyway, since we have not settled on an input category yet.

We have *almost* the seen the theory on objects (2-manifolds) N:

$$Z(N,l) = H(N,l) = \bigotimes_{\text{2-cells C of N}} H(C,l)$$

'Elements' – morphisms into this out of 1 – are basically collections of spins on the edges (from representations determined by the colouring l) , put together in a tensor product. We will define $Z(N)$ in a moment.

We have all the ingredients we need to define Z on 3-manifolds. We will also give the state-sum, and discuss it afterwards.

We will have

$$Z(M,l) \in Z(\partial M,l).$$

This will be a morphism if ∂M has to components with opposing orientations, since $H(\overline{N}) = H(N)^*$ because we have already required this for the individual edges. Set

$$Z(M,l) = \mathrm{ev}\left(\bigotimes_{\text{3-cells T of M}} Z(T,l)\right),$$

where ev contracts bulk 2-cells, which come up twice, once with each orientation.

The state-sum is defined by

$$Z(M) = \frac{1}{\mathcal{D}^{2\#(\text{bulk } v)} \cdot \mathcal{D}^{\#(\text{bdy v})}} \sum_{\text{simple } l} \left(Z(M,l) \prod_{\text{bulk } e} d_{l(e)} \prod_{\text{bdy } e} \sqrt{d_{l(e)}} \right)$$

Here:

- d_X is the categorical ('quantum' is also used) dimension of the object X, i.e. the trace of its identitiy morphism.
- $\mathcal{D}$ should be called the size or length of the category, but is called its quantum dimension:

$$\mathcal{D} = \sqrt{\sum_{\text{simple } X} d_X}.$$

- Simple means $\mathrm{End}(X) = \Bbbk$. This is because we'd like to work with simple representations and are emulating Schur's lemma.

Note that the edge weights $d_{l(e)}$, $\sqrt{d_{l(e)}}$ work just as in the previous section. The exponents are chosen such that we have functoriality. Clearly, the same goes for the vertex weights $\frac{1}{\mathcal{D}^2}$, $\frac{1}{\mathcal{D}}$; the value of $\mathcal{D}$ is simply for normalisation: it ensures $Z(S^2) = \mathbb{1}$. The distinction between bulk and boundary vertices is for functoriality just like with the edges.

Functoriality and Z on surfaces. Let's set

$$H(N) \coloneqq \bigoplus_l H(N,l).$$

Clearly, if ∂M has incoming component N_1 and outgoing component N_2, then $Z(M)\colon H(N_1) \to H(N_2)$.

We have here the familiar problem that identitiy morphisms (cylinders) are nontrivial, but they are projections, i.e.

$$p_N := Z(N \times I) \colon H(N) \to H(N)$$

satisfies $p^2 = p$. Thus we can mod out the cylinders by setting

$$Z(N) := \mathrm{Im}(p_N).$$

This works and makes Z a functorial TQFT, because we have the following gluing property: let M have (oriented) boundary components $N_0, N, \overline{N}$ and M' obtained from M by gluing $N, \overline{N}$. Then

$$Z(M') = \mathrm{ev}_N Z(M)$$

where $\mathrm{ev}_N \colon H(N) \otimes H(\overline{N}) \to \Bbbk$ is the pairing.

3.3 Moves and constraints

Without even looking at the moves that transform polytope triangulations (which are really simple!), we can already see most of the structure that we have to impose on our category $\mathcal{C}$. It must:

- be additive and monoidal: *tensoring spin reps.*
- (not necessarily have a braiding: *does not have to be symmetric/ribbon/modular: this is more general than Reshetikhin–Turaev: in fact supposedly a kind of square root of it.*)
- have strict duals with neutral object 1: *orientation-sensitivity, morphisms.* (Strictness is no real restriction.)
- have a trace, for which mutually dual evaluation and coevaluation maps: $\mathrm{coev}_X \colon 1 \to X \otimes X^\vee$, $\mathrm{ev}_X \colon X^\vee \otimes X \to 1$ with the usual relations.
- ev and coev allow the definition of two distinct traces: 'left' and 'right'. They must coincide: *given a* 3*-cell, essentially 2 distinct planar graphs can be obtained from it: the graph as seen on the* 3*-cell is essentially on* S^2*: the planar one will be a stereographic projection. The two versions differ by pulling the loop around the pole of the projection.* This is called the spherical property.
- be nondegenerate, i.e. $(f, g) \mapsto \mathrm{tr}(fg)$ should always be nondegenerate: *for sensible pairings.*
- finite, i.e. it must have finitely many simples. *Among other things, this is so that* $\mathcal{D}$ *is finite.*
- be semisimple: *for cancelling certain sums in diagrams.* Semisimplicity is about factoring maps through simples: let $j \in J$ be representatives of isomorphism classes of simples. For all objects X, Y, we require that the composition-and-addition map

  $$\bigoplus_{j \in J} \mathrm{Hom}(X, j) \otimes \mathrm{Hom}(j, Y) \to \mathrm{Hom}(X, Y)$$

  is an isomorphism. *This also ensures that the TV invariant isn't trivial: if* $\mathcal{C}$ *is the representation category of a certain type of Hopf algebra which is not semisimple, then all projective reps have zero categorical dimension. See BW's paper, last 2 pages.*
- $\Bbbk$ must be algebraically closed and of characteristic 0. *We are taking square roots. We choose those once and for all. There is one more reason for this that we'll see.*

Such categories are called spherical fusion categories. In particular, we have identified essentially all the conditions needed without even looking at the local moves. The weights with the $\mathcal{D}$ and $d_{l(e)}$ are picked so that for $\mathcal{C} = \mathrm{Rep}(U_q\mathfrak{sl}_2)$ at q a root of unity. Without this restriction, one could probably set up an even simpler state-sum for a functorial TQFT.

The moves for polytope triangulations. They consist in removing vertices, edges, and faces (and the inverse operations) as in the pictures:

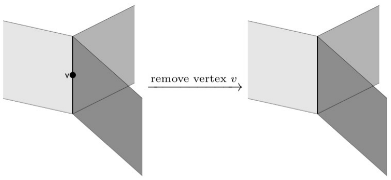


FIGURE 8. Move M1

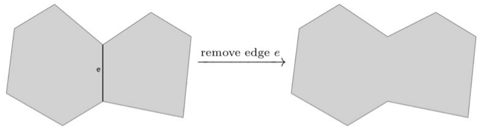


FIGURE 9. Move M2

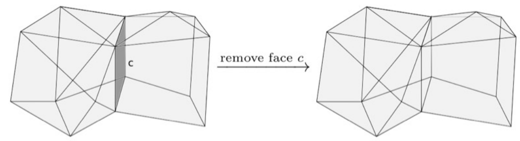


FIGURE 10. Move M3

It is very straightforward to check invariance, using semisimplicity to reduce sums, and freely evaluating the graphs thanks to our many assumptions.

4 The Balsam–Kirillov construction

We have already covered a lot of material from Balsam–Kirillov's paper minus the extension to codimension 2. That is the aim of this section. Because all this has to fit into one talk, I will be very brief.

The circle is assigned the Drinfeld centre $Z(\mathcal{C})$. I will try to explain why this is a good idea. The category $Z(\mathcal{C})$ has objects (z, ϕ_z) with $z \in \mathrm{Ob}(\mathcal{C})$ and ϕ_z a natural isomorphism

$$z \otimes - \to - \otimes z$$

called half-braiding, satisfying some identities, so that $Z(\mathcal{C})$ is not just semisimple, finite, and pivotal just like $\mathcal{C}$, but also braided (in fact modular). There is an left-adjoint $I \colon \mathcal{C} \to Z(\mathcal{C})$ to the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$ defined by

$$I(X) = \bigoplus_{j \text{ simple}} j \otimes X \otimes j^{\vee}$$

so that $\mathrm{Hom}_{Z(\mathcal{C})}(I(X), z) = \mathrm{Hom}_{\mathcal{C}}(X, z)$. In pictures this replaces the one-strand X by a sum over 3-strand tangles. The result is central (there is half-braiding) follows from simple acrobatics.

Following BK, let us consider surfaces Σ with boundary as closed surfaces with embedded disks. We denote the standard PL-disk by $D^2 \coloneqq I^{\times 2}$. Let $P_0 \in D^2$ be a distinguished point, giving a distinguished point on each such embedded disk. We will carry this information around. A surface with finitely many disjoint embedded disks is called an extended surface. A colouring of an extended surface is the choice of an object of $Z(\mathcal{C})$ for each embedded disk.

Analogously, we will think of gluing of embedded disks as embedded tubes connecting them. Let M be a 3-manifold. An embedded tube is an embedding of $I \times D^2$ with boundary two embedded disks with distinguished points, say, P_1, P_2, together with the choice of an oriented arc

$$\gamma \colon P_1 \to P_2$$

in M. A closed embedded tube is one where the two embedded boundary disks coincide (so γ is a loop). An extended 3-manifold with boundary is one with finitely many disjoint embedded tubes. A colouring is the choice of an object z_T of $Z(\mathcal{C})$ for each embedded tube T. The idea is that this induces a colouring of ∂M when we assign z_T (resp. z_T^*) to the outgoing (resp. incoming) embedded disk.

The state-sum. Let us attach a state space to an extended 2-manifold Σ with l a colouring of its (polytope triangulation's) edges.

If C is a 2-cell that is an embedded disk D_α with colouring $z_\alpha \in \mathrm{Ob}(Z(\mathcal{C}))$ attached, we set

$$H(C,l) = \langle z_\alpha, l(e_1), \ldots, l(e_n)\rangle$$

where the orientation e_1 is at the distinguished point. Otherwise, $H(C,l)$ is defined as in the previous section. In particular, we require that the (interior of the) embedded disks are 2-cells of the polytope triangulation. Now, as before, we may put, for a coloured (in both senses) extended surface,

$$H(\Sigma, \{z_\alpha\}, l) := \bigotimes_C H(C,l)$$

and

$$H(\Sigma, \{z_\alpha\}) := \bigoplus_l H(\Sigma, \{z_\alpha\}, l)$$

This clearly satisfies $H(\Sigma, \{z_\alpha\})^* = H(\overline{\Sigma}, \{z_\alpha^*\})$ and is thus functorial in the usual sense.

The basic idea of the proof of the following gluing property is why attaching $Z(\mathcal{C})$ to S^1 is a good idea:

Gluing property. Let D_a, D_b be two of the embedded disks of Σ, and obtain Σ' by removing their interiors and attaching a cylinder connecting their boundaries. Say D_a is outgoing. Then we have an isomorphism

$$H(\Sigma', \{z_\alpha\}_{D'}) = \bigoplus_{\substack{z\in Z(\mathcal{C})\\ \text{simple}}} H(\Sigma, \{z_\alpha\}_{D'}, z, z^*)$$

where D' is the collection of the embedded disks of Σ' (i.e. those of Σ minus D_a and D_b), and where z and z^* are attached to D_a and D_b, respectively.

In words, this means that attaching a cylinder to cancel a pair of boundary circles (embedded disks) is equivalent to summing over all mutually dual *central* states that one can put on those circles.

Proof. Let's fix notation away from the two disks:

$$H_0(l) = \bigotimes_{C\neq D_a, D_b} H(C,l).$$

Each summand on the RHS looks by definition like

$$H_0(l) \otimes \langle z, l(\partial D_a)\rangle \otimes \langle z^*, l(\partial D_b)\rangle.$$

Similarly, each summand on the LHS looks, for $l = l'|_\Sigma$, like

$$H(\Sigma', \{z_\alpha\}_{D'}, l') = H_0(l) \otimes \langle l(\partial D_a), l'(\gamma), l(\partial D_b), l'(\gamma)^*\rangle$$

where $\gamma\colon P_a \to P_b$ is the arc connecting the distinguished points. Here we are using the polytope decomposition of the cylinder with a single 1-cell, γ, and a single 2-cell.

Now we can forget about $H_0(l)$ and compare the rest of both sides. Starting with the LHS, we immediately see

$$\begin{aligned}
\textstyle\bigoplus_{\substack{j\in\mathcal{C}\\ \text{simple}}} \langle l(\partial D_a), j, l(\partial D_b), j^*\rangle &= \bigoplus \mathrm{Hom}_{\mathcal{C}}(l(\partial D_a)^*, j\otimes l(\partial D_b)\otimes j^*)\\
&= \mathrm{Hom}_{\mathcal{C}}(l(\partial D_a)^*, I(l(\partial D_B)))\\
&= \mathrm{Hom}_{Z(\mathcal{C}}(I(l(\partial D_a))^*, I(l(\partial D_b)))\\
&= \bigoplus_{\substack{z\in Z(\mathcal{C})\\ \text{simple}}} \langle z, l(\partial D_a)\rangle \otimes \langle z^*, l(\partial D_b)\rangle
\end{aligned}$$

□

One can now continue setting up the TQFT as in the previous section, with the difference that the dual graphs will include arcs, coming from the arcs connecting distinguished points, connecting two vertices. In particular, one must require that the polytope triangulation contain the distinguished points as 0-cells. The evaluation works otherwise in the same way. There no new local moves.

5 String nets

Let us briefly look at another description of the state spaces. For a surface Σ, there is an isomorphism

$$Z(\Sigma) = H^{\text{string}}(\Sigma)$$

where the RHS is the space of string nets (spin networks) on Σ. We will look at closed Σ, but there is an analogous description in codimension 2.

String nets are essentially just coloured dual graphs. Let T be any triangulation; then the map

$$\pi\colon H(\Sigma, T) \to H^{\text{string}}(\Sigma)$$

is defined by sending a colouring $\phi \in \bigotimes_C H(C, l)$ (for an edge-colouring l) to the dual graph $T^\vee$ with the induced colouring given by l on the edges and the induced colouring ϕ on the vertices, multiplied (for functoriality) by the factor $\sqrt{d_l(T)}$ were $d_l(T)$ is the product of $d_{l(e)}$ over all the edges in T:

$$\phi \mapsto (T^\vee, \sqrt{d_l(T)}\phi).$$

Here, the edges are unoriented in the sense that internal ones come up twice, once with each orientation.

It is easy to carry over invariance under the local moves to string nets. More interestingly:

Proposition. *Applying the cylinder morphism modifies $H^{\text{string}}(\Sigma)$ to $H^{\text{string}}(\Sigma - T^0)$, where T^0 stands for the vertices in T.*

This is shown by using the actual definition:

$$H^{\text{string}}(\Sigma) := \mathrm{VGraph}(\Sigma)/N(\Sigma)$$

where $\mathrm{VGraph}(\Sigma)$ consists of linear combinations of coloured graphs, and N consists of null graphs: linear combinations Γ such that there is an embedded disk $D \subset \Sigma$, such that:

- Γ intersects ∂D transversally;
- the terms of Γ coincide outside D;
- $\Gamma \cap D$ evaluates to 0.

Here, graphs are evaluated in the usual way. In particular, if $e_1, \dots e_n$ are the edges of a (single) graph $\Gamma \subset \Sigma$ meeting the boundary $\partial\Sigma$, then we interpret the evaluation as giving a state in the boundary:

$$\langle \Gamma \rangle_\Sigma \in \langle l(e_1), \dots l(e_n) \rangle = H(\Sigma, l).$$

Recall that in our notation, evaluation on a 2-cell C lands that cell's state space: $\langle \Gamma \rangle_C \in H(C, l)$.

Using this proposition (which can be shown by a straightforward hands-on semisimplicity calculation), the inverse map to π is defined, as one would hope, as follows:

1. Replace Γ, using semisimplicity, by a graph Γ' that intersects each edge of T at most once, through an edge that is coloured by l:

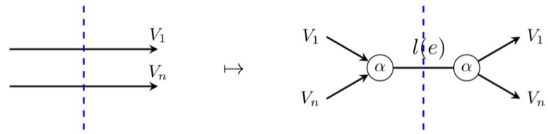


FIGURE 10. Transformation $\Gamma \mapsto \Gamma^l$.

2. Tensor over 2-cells, normalise by $\sqrt{d_l}$;
3. Sum over colourings l:

$$\Gamma \mapsto \sum_l \sqrt{d_l} \bigotimes_{\text{2-cells } C} \langle \Gamma' \rangle_C \in H(\Sigma, T).$$