THE STRATIFIED GRASSMANNIAN

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Abstract. We give two full definitions of the stratified Grassmannian of Ayala, Francis and Rozenblyum that bypass the theory of conically-smooth stratified spaces: first as an ∞-category, and then as a simplicially-enriched category. We show that the homotopy-coherent nerve of the latter is equivalent to the former. We also prove that the maximal sub-∞-groupoid of the stratified Grassmannian is equivalent to the disjoint union $\coprod_{k\geq 0} B\mathrm{O}(k)$ of the standard infinite Grassmannians.

Introduction and summary

For the purpose of constructing a theory of factorisation homology that can take as input any (∞, n)-category and evaluate it on appropriate variframed stratified spaces, Ayala–Francis–Rozenblyum defined in [AFR18b] the 'fibrewise constructible tangent bundle' T^{fib} of a (conically-smooth) stratified space which intrinsically depended on their earlier work, in part with Tanaka, on the general theory of conically-smooth stratified spaces ([AFT17]; [AFR18a]). The functor T^{fib} on a stratified space X (i.e., its nonrelative special case to which we restrict ourselves) is given in the form of a classifying map with domain $\mathrm{Exit}(X) \to \mathcal{V}^{\mathrm{inj}}$, which is their version of the exit path ∞-category of Lurie–MacPherson ([Lur]; the two notions are equivalent by a result of [AFR18a]), and with target the 'stratified Grassmannian'. By exodromy, such a functor classifies a constructible sheaf on X, which may be interpreted as the sheaf of sections of the 'tangent bundle'; to our knowledge no étalé space of this sheaf has been discussed in the literature.

There are numerous senses in which T^{fib} is the 'correct' notion of tangent bundle in stratified space theory. One is the ease with which one may define and work with tangential structures, and spaces of tangential structures, on stratified spaces using T^{fib}; this has been already exploited in op.s cit. Another is our work in preparation that employs it more explicitly in functorial field theory, in order to generalise a suggestion of Lurie from [Lur08] (taken up in [Sch14]; see also [CS19]) to construct fully-extended functorial field theories given $\mathbb{E}_n$-algebras. Our work in preparation shows that this the 'framed base case' of a more general construction that yields field theories given disk-algebras with any tangential structure. Scrutinising bordisms using T^{fib} is the starting point of that construction.

Indeed, this text was intended to be a section in an upcoming paper, but we separated it as it might be of independent interest. Our purpose is to give a direct construction of a variant of $\mathcal{V}^{\mathrm{inj}}$, that is suggested by and claimed in [AFR18b] to be equivalent to it. Our upcoming work will in particular give a corresponding

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variant of T^{fib} (in the nonrelative case) that completely circumvents the theory of conically-smooth stratified spaces, though it is directly informed by it.

We now summarise the contents.

Definition 1.8, suggested by [AFR18b, Remark 2.7], gives a full definition of the stratified Grassmannian as an ∞-category, in a way that bypasses the theory of conically-smooth stratified spaces and vector bundles. Since we do not show that our version is equivalent to $\mathcal{V}^{\mathrm{inj}}$, we denote it by $\mathcal{V}^{\hookrightarrow}$.

Proposition 1.13 proves that the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$ is the disjoint union of the ordinary infinite Grassmannians. The proof is a direct comparison; it in fact yields an isomorphism, contrary, we believe, to the situation with $\mathcal{V}^{\mathrm{inj}}$.

Definition 2.7 gives a simplicial version of $\mathcal{V}^{\hookrightarrow}$, and Corollary 2.6 and Remark 2.8 prove that its homotopy-coherent (i.e., simplicial) nerve is equivalent to $\mathcal{V}^{\hookrightarrow}$.This is preceded by a necessary interlude, Section 2.1, where we construct enriched undercategories.

Some standard constructions are gathered in Appendix A for the sake of self-containedness and in order to set some of the notation used in the main text.

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Conventions

We say ∞-category to mean a quasicategory, that is, a simplicial set that satisfies the weak Kan condition ([BV73]; [Joy08]; [Lur08]; [JT07]). We reserve the term ∞*-groupoid* to mean a Kan complex. We freely specify ∞-categories by presenting topologically- or Kan-enriched categories; the simplicial nerve construction involved is recalled in Appendix A.1. Topological spaces are convenient, i.e., compactly generated and weakly Hausdorff. When a topological space X appears in place of an ∞-category, we mean the ∞-groupoid $\mathrm{Sing}_\bullet(X)$ of its singular chains. By a simplicial category we mean a simplicially-enriched category, and by a Kan-enriched category we mean a simplicial category whose hom-spaces are Kan complexes. The set $\mathbb{N}$ of natural numbers includes 0. When we use it as a category, we mean the poset with the standard order $\leq$. We write $\boldsymbol{n} \coloneqq \mathbb{R}^n$.

1. The stratified Grassmannian

1.1. **Direct sums on infinite Grassmannians.** Let $\mathrm{Gr}_k(\boldsymbol{n})$ denote the Grassmannian of k-planes inside $\boldsymbol{n}$.

For $n \leq m$ in $\mathbb{N}$, inclusion as first coordinates, $\boldsymbol{n} \hookrightarrow \boldsymbol{m}$, gives, for each $k \in \mathbb{N}$, an $\mathbb{N}$-shaped diagram $n \mapsto \mathrm{Gr}_k(\boldsymbol{n})$, $[n \leq m] \mapsto [(-,0)\colon \mathrm{Gr}_k(\boldsymbol{n}) \hookrightarrow \mathrm{Gr}_k(\boldsymbol{m})]$ of spaces, with maps closed subspace inclusions.

Notation. $B\mathrm{O}(k) \coloneqq \mathrm{Gr}_k(\mathbb{R}^\infty) \coloneqq \operatorname{colim} \mathrm{Gr}_k(-)$.

Notation. $B\mathrm{O}_{\amalg} \coloneqq \coprod_{k\geq 0} B\mathrm{O}(k)$.

Remark 1.1. This is not to be confused with the (zeroeth space of the connected component of the real K-theory) spectrum $B\mathrm{O}$, where a different colimit is taken.

Direct-summing gives maps

$$\oplus : \mathrm{Gr}_k(\boldsymbol{n}) \times \mathrm{Gr}_l(\boldsymbol{m}) \to \mathrm{Gr}_{k+l}(\boldsymbol{n}+\boldsymbol{m}),$$

using specifically the isomorphisms $\boldsymbol{n} \oplus \boldsymbol{m} \cong \boldsymbol{n}+\boldsymbol{m}$, $v \oplus w \mapsto (v, w)$. These isomorphism will be denoted by $\equiv$ below.

After a slight modification, this induces a product operation on $B\mathrm{O}_{\amalg}$, as we now explain.

Definition 1.2. A representative $(V \subseteq \boldsymbol{n}) \in [V \subseteq \boldsymbol{n}] \in B\mathrm{O}(k)$ is *minimal* if n is such that the restriction of V to $\boldsymbol{n-1} \hookrightarrow \boldsymbol{n}$ has dimension $k-1$.

All other representatives $V \subseteq \boldsymbol{m}$, of such a class necessarily satisfy $m \geq n$ and are of type $(V \subseteq \boldsymbol{m}) = (-, 0)(V \subseteq \boldsymbol{n})$. In particular, the minimal representative is uniquely determined. All written representatives below are minimal.

Construction 1.3. The map

$$\begin{aligned} \boxplus : B\mathrm{O}(k) \times B\mathrm{O}(l) &\to B\mathrm{O}(k+l) \\ ([V \subseteq \boldsymbol{n}], [W \subseteq \boldsymbol{m}]) &\mapsto [V \oplus W \subseteq \boldsymbol{n} \oplus \boldsymbol{m} \equiv \boldsymbol{n}+\boldsymbol{m}] \end{aligned}$$

is well-defined due to minimality. Thus, we have a map

$$\boxplus \colon B\mathrm{O}_{\amalg} \times B\mathrm{O}_{\amalg} \to B\mathrm{O}_{\amalg},$$

which is easily seen to be associative.

1.2. **Its delooping.** We now apply the bar construction to $\boxplus$.

Definition 1.4. Let $B^{\boxplus}\mathrm{O}$ denote the Kan-enriched category with a single object $*$, endomorphism space $B\mathrm{O}_{\amalg}$, and composition $\boxplus$.[1]

Definition 1.5. $\mathfrak{B}^{\boxplus}\mathrm{O} \coloneqq \mathrm{N}^{\boldsymbol{\Delta}}(B^{\boxplus}\mathrm{O})$.

Using the notation from Appendix A.1, we will discuss explicitly the 1-, 2- and 3-simplices of $\mathfrak{B}^{\boxplus}\mathrm{O}$ for future reference, and leave higher vertices to the interested reader.

Warning 1.6. In the following, when we say ‘vector space’, we mean a point of $B\mathrm{O}_{\amalg}$, i.e., the class of a subspace of some $\mathbb{R}^n$.

1.2.1. 1-*morphisms.* Let $F \colon \mathfrak{C}[1] \to B^{\boxplus}\mathrm{O}$ be a map of simplicial categories, i.e., a 1-simplex of $\mathfrak{B}^{\boxplus}\mathrm{O}$. Both objects $0, 1 \in [1]$ are sent to $*$. The mapping poset $P_{0,1}$ has the sole nontrivial element $\underline{01} \coloneqq \{0,1\} \in N_0(P_{0,1})$, the image of which determines F. Write $F(\underline{01}) = V_{01} \in \mathrm{Sing}_0 = \mathrm{Sing}_0 B\mathrm{O}_{\amalg}$; so V_{01} is a vector space.

[1]By our convention, $B\mathrm{O}_{\amalg}$ here means the Kan complex $\mathrm{Sing}_\bullet B\mathrm{O}_{\amalg}$ of its singular chains, $\boxplus$ means $\mathrm{Sing}_\bullet(\boxplus)$.

1.2.2. 2-*morphisms.* Let $F\colon \mathfrak{C}[2] \to B^{\boxplus}\mathrm{O}$ be a 2-simplex of $\mathfrak{B}^{\boxplus}\mathrm{O}$. Let $\iota_{ab}^{*}F\colon \mathfrak{C}[1] \to B^{\boxplus}\mathrm{O}$ be the three nontrivial faces, $\iota_{ab}\colon [1] \hookrightarrow [2]$ given by $0 \mapsto a$, $1 \mapsto b$ for $a < b$ in $[2]$, so that they are determined by vector spaces $V_{ab} = F(\underline{ab}) = \mathrm{Sing}_0 B\mathrm{O}_{\mathrm{II}}$ as above. The mapping poset

$$P_{0,2} = \{\underline{02} \prec \underline{012}\}$$

includes two new pieces of information: a vector space $V_{012} = F(\underline{012}) \in \mathrm{Sing}_0$, and, seeing $\prec \in N_1(P_{0,2})$, a path $\gamma = F(\prec) \in \mathrm{Sing}_1$ with source V_{02} and target V_{012}. Notice now that $\underline{012}$ is in the image of

$$P_{1,2} \times P_{0,1} \to P_{0,2},$$

namely $\underline{012} = \underline{12} \cup \underline{01}$. As F is functorial, we have $V_{012} = V_{12} \boxplus V_{01}$. Thus, F is determined by three spaces V_{01}, V_{12} and V_{02}, together with a path $\gamma\colon V_{02} \to V_{12} \boxplus V_{01}$ in $B\mathrm{O}_{\mathrm{II}}$. Pictorially:

(1)
$$\begin{tikzcd} & * \arrow[dr, "V_{12}"] & \\ * \arrow[ur, "V_{01}"] \arrow[rr, "V_{02}"] \arrow[rr, dashed, bend right, "V_{12}\boxplus V_{01}"'] \arrow[rr, phantom, bend right=20, "\Downarrow F(\prec)"] & & * \end{tikzcd}$$

If V_{12} is the identity, i.e. $V_{12} = 0$, then this is just a path from V_{02} to V_{12}.

1.2.3. 3-*morphisms.* Let $F\colon \mathfrak{C}[3] \to B^{\boxplus}\mathrm{O}$ be a 3-simplex of $\mathfrak{B}^{\boxplus}\mathrm{O}$. The six non-degenerate edges $V_{ab} = F(\underline{ab})$, $0 \le a < b \le 3$, are vector spaces. The four non-degenerate faces

$$\iota_{abc}\colon \mathfrak{C}[2] \hookrightarrow \mathfrak{C}[3] \to B^{\boxplus}\mathrm{O}$$

are of the form of 1, which specifically is the face $(a, b, c) = (0, 1, 2)$. Explicitly, we have four paths $V_{ac} \xrightarrow{\sim} V_{abc} = V_{bc} \boxplus V_{ab}$:

(2) $\quad V_{02} \simeq V_{12} \boxplus V_{01}, \quad V_{03} \simeq V_{13} \boxplus V_{01}, \quad V_{13} \simeq V_{23} \boxplus V_{12}, \quad V_{03} \simeq V_{23} \boxplus V_{02}.$

The mapping poset $P_{0,3}$ is as follows:

$$\begin{tikzcd} & \underline{03} \arrow[dl, "\prec_1"'] \arrow[dr, "\prec_2"] \arrow[dd, "\prec_{12}"] & \\ \underline{013} \arrow[dr, "\prec_2"'] & & \underline{023} \arrow[dl, "\prec_1"] \\ & \underline{0123} & \end{tikzcd}$$

The left and right triangles therein depict the two non-degenerate elements of $N_2(P_{0,3})$, which F maps to $\mathrm{Sing}_2(B\mathrm{O}_{\mathrm{II}})$. That is, writing $\gamma_{-} = F(\prec_{-})$, F gives homotopies filling the triangles in

(3)
$$\begin{tikzcd} & V_{03} \arrow[dl, "\gamma_1"'] \arrow[dr, "\gamma_2"] \arrow[dd, "\gamma_{12}"] & \\ V_{13} \boxplus V_{01} \arrow[r, phantom, "\Longrightarrow"] \arrow[dr, "\gamma_2"'] & {} & V_{23} \boxplus V_{02} \arrow[l, phantom, "\Longleftarrow"] \arrow[dl, "\gamma_1"] \\ & V_{0123} & \end{tikzcd}$$

The first and third paths of 2 give further decompositions of the sums on the left and right. For future reference, note the decomposition

$$V_{0123} = V_{23} \boxplus V_{12} \boxplus V_{01}. \tag{4}$$

Remark 1.7. If all but V_{01}, V_{02}, V_{03} are non-zero, then the right triangle of 3 reduces to

$$\begin{tikzcd}
V_{03} \arrow[dd, "\gamma_{12}"'] \arrow[dr, "\gamma_2"] & \\
\arrow[r, phantom, "\Leftarrow"] & V_{02} \arrow[dl, "\gamma_1"] \\
V_{01} &
\end{tikzcd}$$

In globular terms, this describes vertical composition.

1.3. **The coslice of the delooping.** The advan e of the following definition is two-fold: it bypasses the theory of conically-smooth stratified spaces, and it is more amenable to related bundle-theoretic constructions than the original definition. See Appendix A.2 for the meaning of the notation.

Definition 1.8. $\mathcal{V}^{\hookrightarrow} := */\mathfrak{B}^{\boxplus}\mathrm{O}$.

We call this the *stratified Grassmannian*, and elucidate now its morphisms up to dimension 2.

Remark 1.9. Note that $\mathfrak{B}^{\boxplus}\mathrm{O}$ is far from being an ∞-groupoid: only the zero vector space is invertible.

Via the identification $\Delta^0 \star \Delta^n \simeq \Delta^{n+1}$ as in Remark A.10, n-simplicies of $\mathcal{V}^{\hookrightarrow}$ are $(n+1)$-simplices $\mathfrak{B}^{\boxplus}\mathrm{O}$ with no qualification, since $\mathfrak{B}^{\boxplus}\mathrm{O}$ has a unique 0-simplex. Thus:

1.3.1. 0-*simplices.* A 0-simplex is a vector space V (see Warning 1.6).

1.3.2. 1-*morphisms.* A 1-simplex is as in Equation (1), with source V_{01} and target V_{02} in the sense, or convention, of A.11, together with a map $V_{01} \hookrightarrow V_{12} \boxplus V_{01} \xrightarrow{\gamma^{-1}} V_{02}$. In this sense, morphisms of $\mathcal{V}^{\hookrightarrow}$ can be said to be 'injections of vector spaces'.

1.3.3. 2-*morphisms.* A 2-simplex is a map

$$\gamma\colon \Delta^0 \star \Delta^2 \to \mathfrak{B}^{\boxplus}\mathrm{O}$$

whose edges may be described as follows:

$$\begin{tikzcd}
& 2 \arrow[dr, "W_{23}"] & \\
1 \arrow[ur, "W_{12}"] \arrow[rr, "W_{13}"] & & 3 \\
& 0 \arrow[ul, "V_{01}"] \arrow[uu, "V_{02}"] \arrow[ur, "V_{03}"'] &
\end{tikzcd} \tag{5}$$

We have the following three induced faces of γ:

$$\begin{tikzcd}
\Delta^0 \star \Delta^1 \arrow[r, hook, "\mathrm{id}\times f_i"] \arrow[d, "\sim"] & \Delta^0 \star \Delta^2 \arrow[r, "\gamma"] \arrow[d, "\sim"] & \mathfrak{B}^{\boxplus}\mathrm{O} \\
\Delta^2 \arrow[r, dashed, "\mathrm{id}\times f_i"'] \arrow[rru, bend right, "\gamma_i"'] & \Delta^3 \arrow[ru, dashed, "\gamma"'] &
\end{tikzcd}$$

where $i = 0, 1, 2$ is skipped by $f_i\colon [1] \hookrightarrow [2]$.

We call the $\{0,1\}$-edge of Δ^2 its *source edge*, and the $\{0,2\}$-edge of Δ^2 its *target edge*. In globular terms, this is justified by the special case where $V_{12} = 0$ is the identity. Say, therefore, the induced f_2-face of γ is its *source face*, and the f_1-face its *target face*. These two faces share their respective source edges:

$$\begin{tikzcd}
\Delta^0 \star \Delta^0 \arrow[r, hook, "\mathrm{id}\times f_1"] \arrow[d, "\sim"] & \Delta^0 \star \Delta^1 \simeq \Delta^2 \arrow[r, "\gamma_2{,}\gamma_1"] & \mathfrak{B}^{\boxplus}\mathrm{O} \\
\Delta^1 \arrow[rru, dashed, "\gamma_{01}"'] & &
\end{tikzcd}$$

where $f_1\colon [0] \hookrightarrow [1]$ skips 1.

This common source edge γ_{01} is determined by the vector space V_{01} (5). Then the source face of γ is of type $\gamma_2 = (V_{01} \hookrightarrow W_{12} \boxplus V_{01} \simeq V_{02})$, and its target face is of type $\gamma_1 = (V_{01} \hookrightarrow W_{13} \boxplus V_{01} \simeq V_{03})$. The intermediate face is of type $\gamma_0 = (V_{02} \hookrightarrow W_{23} \boxplus V_{02} \simeq V_{03})$, yielding the diagram

$$\begin{tikzcd}
V_{01} \arrow[r, hook] \arrow[rrd, hook, bend right] & W_{12} \boxplus V_{01} \arrow[r, "\gamma_2", "\sim"'] & V_{02} \arrow[r, hook] & W_{23} \boxplus V_{02} \arrow[r, "\gamma_0", "\sim"'] & V_{03} \\
 & & W_{13} \boxplus V_{01} \arrow[rru, bend right, "\sim", "\gamma_1"'] & &
\end{tikzcd} \tag{6}$$

Consider now the final face of γ, given as follows:

$$\begin{tikzcd}
\Delta^2 \arrow[r, hook, "\iota_1", "\text{A.6}"'] \arrow[rr, dashed, bend right, "\Gamma"'] & \Delta^0 \star \Delta^2 \arrow[r, "\gamma"] & \mathfrak{B}^{\boxplus}\mathrm{O}
\end{tikzcd}$$

It is of type $\Gamma = (W_{12} \hookrightarrow W_{23} \boxplus W_{12} \simeq W_{13})$. Composing (concatenating) the upper maps in 6 and inserting Γ gives

$$\begin{tikzcd}
V_{01} \arrow[r, hook] \arrow[rd] & W_{23} \boxplus W_{12} \boxplus V_{01} \arrow[r, "(\mathrm{id}\boxplus\gamma_2)*\gamma_0", "\sim"'] \arrow[d, "\Gamma\boxplus\mathrm{id}", "\sim"'] & V_{03} \\
 & W_{13} \boxplus V_{01} \arrow[ru, "\gamma_1{,}\sim"'] &
\end{tikzcd} \tag{7}$$

The left triangle clearly commutes. The homotopy in the left triangle of 3 (with all paths inverted) commutes the right triangle of 7 in view of 4.

Remark 1.10. For γ a path in $B\mathrm{O}_{\mathrm{II}}$ from V to W, and $K \in B\mathrm{O}_{\mathrm{II}}$ another vector space, we mean by $\mathrm{id} \boxplus \gamma$ the path from $K \boxplus V$ to $K \boxplus W$ that is given, in terms of minimal representatives (Definition 1.2), by $\mathrm{id}_{\dim K} \oplus \gamma$. The path $\gamma \boxplus \mathrm{id}$ is defined similarly.

Remark 1.11. If all but V_{01}, V_{02}, V_{03} are non-zero, then 7 reduces to

$$\begin{tikzcd} V_{01} \arrow[r, "\gamma_2", "\sim"'] \arrow[rr, bend right, "\sim", "\gamma_1"'] & V_{02} \arrow[r, "\gamma_0", "\sim"'] & V_{03} \end{tikzcd}$$

together with a homotopy commuting it, i.e., a homotopy $\gamma_1 \simeq \gamma_2 * \gamma_0$ in $B\mathrm{O}_{\mathrm{II}}$ between paths from V_{01} to V_{03}.

Notation 1.12. For $\mathcal{C}$ an ∞-category, let $\mathcal{C}^{\sim}$ denote its maximal sub-∞-groupoid, i.e. the ∞-groupoid obtained by discarding the non-invertible morphisms of $\mathcal{C}$. That of $\mathcal{V}^{\hookrightarrow}$ is denoted by $\mathcal{V}^{\sim}$.

Proposition 1.13. $\mathcal{V}^{\sim} \simeq B\mathrm{O}_{\mathrm{II}}$.

Perhaps surprisingly, one may in fact promote this to an isomorphism of simplicial sets. We give the correspondences at the levels of vertices and morphisms whence the stated categorical (in Lurie's terminology), i.e., weak (in Joyal's terminology) equivalence will be clear.

Proof. The one-to-one correspondence at the level of vertices is given in Section 1.3.1. We first consider the identity morphism of a point of $\mathcal{V}^{\hookrightarrow}$. Let $d\colon [1] \to [0]$ be the trivial degeneracy map. The identity 1-morphism of V_{01} in $\mathcal{V}^{\hookrightarrow}$ is by definition the pullback

$$\mathrm{id}_{V_{01}} = \left(\Delta^0 \star \Delta^1 \overset{\mathrm{id}\star d}{\to} \Delta^0 \star \Delta^0 \overset{V}{\to} \mathfrak{B}^{\boxplus}\mathrm{O}\right).$$

In terms of 1, where now $V_{01} = V$, the edge $W_{12} = V_{12}$ is the further pullback

$$\begin{tikzcd} \Delta^1 \arrow[r, hook, "\iota_1", "A.6"'] \arrow[d] \arrow[rrr, dashed, bend left, "W_{12}"] & \Delta^0 \star \Delta^1 \arrow[r, "\mathrm{id}\star d"] & \Delta^0 \star \Delta^0 \arrow[r, "V"] \arrow[d, "\sim"] & \mathfrak{B}^{\boxplus}\mathrm{O} \\ \Delta^0 \arrow[rr, hook, "f_0"] & & \Delta^1 \arrow[ur, dashed] & \end{tikzcd}$$

which factors as the diagram indicates. (Here, f_0 skips $0 \in [1]$.) The corresponding map $\mathfrak{C}[1] \to B^{\boxplus}\mathrm{O}$ of simplicial categories factors then as

$$\mathfrak{C}[1] \to \mathfrak{C}[0] \to \mathfrak{C}[1] \to B^{\boxplus}\mathrm{O},$$

which picks out the identity morphism of $B^{\boxplus}\mathrm{O}$, i.e., $W_{12} = 0$, and, moreover, the path $V_{01} \overset{\sim}{\to} V_{01}$ is constant.

Now, a 1-morphism $\gamma_2 = \left(V_{01} \hookrightarrow W_{12} \boxplus V_{01} \overset{\sim}{\to} V_{02}\right)$ in the sense of Section 1.3.2 is invertible if there is a 2-morphism γ with source face γ_2 and target face $\gamma_1 = \mathrm{id}_{V_{01}}$. In terms of 5, this means $V_{03} = V_{01}$, $W_{13} = 0$, and so $W_{12} = 0 = W_{23}$, whence we are in the situation of Remark 1.11 with γ_1 constant. Thus, invertible morphisms correspond exactly to paths in $B\mathrm{O}_{\mathrm{II}}$. $\square$

2. The simplicial Grassmannian

We will give a useful description of the stratified Grassmannian. Recall that $\mathcal{V}^{\hookrightarrow} = */\mathfrak{B}^{\boxplus}\mathrm{O} = */(\mathrm{N}^{\boldsymbol{\Delta}}(B^{\boxplus}\mathrm{O}))$ where $B^{\boxplus}\mathrm{O}$ is a simplicial category.

We give a definition of a simplicial under-category that is compatible with taking simplicial nerves, in the sense that there will be an equivalence of ∞-categories

$$*/(\mathrm{N}^{\boldsymbol{\Delta}}(B^{\boxplus}\mathrm{O})) \overset{2.5}{\simeq} \mathrm{N}^{\boldsymbol{\Delta}}(*/B^{\boxplus}\mathrm{O}).$$

Thus, in order to define an ∞-functor to/from $\mathcal{V}^{\hookrightarrow}$, it suffices to define a functor to/from $*/B^{\boxplus}\mathrm{O}$ from/to a simplicial category.

The counit of the adjuntion $\mathfrak{C} \dashv \mathrm{N}^{\boldsymbol{\Delta}}$ will give an equivalence

$$\mathfrak{C}\mathcal{V}^{\hookrightarrow} \overset{2.6}{\simeq} */B^{\boxplus}\mathrm{O}$$

between the simplicial category associated to $\mathcal{V}^{\hookrightarrow}$ (Definition A.4) and $*/B^{\boxplus}\mathrm{O}$.

2.1. **(Co)slices of enriched categories.** Recall that the notion of simplicial enrichment makes use of the monoidal structure on sSet given by the cartesian product, with monoidal unit the constant singleton simplicial set pt. Moreover, sSet, being bicomplete, admits pullbacks, by means of which we may define simplicial (co)slices.

We will only define under-categories, and leave the generalisation to the reader.

Definition 2.1. Let $\mathcal{C}$ be a simplicial category, and x an object. The *simplical under-category* $x/\mathcal{C}$ at x is the simplicial category with objects morphisms in $\mathcal{C}$ with source x, and whose hom-space from $f\colon x \to y$ to $g\colon x \to y'$ is given as the pullback

$$\begin{tikzcd}
\mathrm{Hom}_{x/\mathcal{C}}(f,g) \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Hom}_{\mathcal{C}}(y,y') \arrow[d, "f^*"] \\
\mathrm{pt} \arrow[r, "g"'] & \mathrm{Hom}_{\mathcal{C}}(x,y')
\end{tikzcd}$$

in sSet. Let $f\colon x \to y$, $g\colon x \to y'$, $h\colon x \to y''$ be objects of $x/\mathcal{C}$. In order to give a composition map $\mathrm{Hom}_{x/\mathcal{C}}(f,g) \times \mathrm{Hom}_{x/\mathcal{C}}(g,h) \to \mathrm{Hom}_{x/\mathcal{C}}(f,h)$, consider the projection

$$\pi \times \pi\colon \mathrm{Hom}_{x/\mathcal{C}}(f,g) \times \mathrm{Hom}_{x/\mathcal{C}}(g,h) \to \mathrm{Hom}_{\mathcal{C}}(y,y') \times \mathrm{Hom}_{\mathcal{C}}(y',y'')$$

and postcompose it with composition in $\mathcal{C}$ to define a map

$$\Pi\colon \mathrm{Hom}_{x/\mathcal{C}}(f,g) \times \mathrm{Hom}_{x/\mathcal{C}}(g,h) \to \mathrm{Hom}_{\mathcal{C}}(y,y'').$$

Lemma 2.3 below implies that there is a unique induced map

$$\mathrm{Hom}_{x/\mathcal{C}}(f,g) \times \mathrm{Hom}_{x/\mathcal{C}}(g,h) \to \mathrm{Hom}_{x/\mathcal{C}}(f,h),$$

which we take to define composition in $x/\mathcal{C}$. We leave its associativity to the reader.

Remark 2.2. If $\mathcal{C}$ is Kan-enriched, then, by a result of Joyal ([Lur09, Proposition 2.1.2.1]) combined with [Lur09, Lemma 2.1.3.3], precomposition with f, i.e., the right vertical map f^* of Definition 2.1, is a Kan fibration. Therefore, the homotopy fibre of f^* at h, i.e., the homotopy pullback of $\mathrm{pt} \xrightarrow{h} \mathrm{Hom}_{\mathcal{C}}(x,y') \xleftarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(y,y')$ coincides with the ordinary pullback $\mathrm{Hom}_{x/\mathcal{C}}(f,g)$.

Lemma 2.3. *In the context of Definition 2.1, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Hom}_{x/\mathcal{C}}(f,g) \times \mathrm{Hom}_{x/\mathcal{C}}(g,h) \arrow[r, "\Pi"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(y,y'') \arrow[d, "f^*"] \\
\mathrm{pt} \arrow[r, "h"'] & \mathrm{Hom}_{\mathcal{C}}(x,y'')
\end{tikzcd}$$

The moral proof (i.e., for an enrichment in Set rather than in sSet) reads '$f^*(\phi'\phi) = (\phi'\phi)f = \phi'(\phi f) = \phi' f^*(\phi) = \phi' g = g^*\phi = h$', where $\phi\colon y \to y'$ and $\phi'\colon y' \to y''$. We give a general argument that will be valid for $\mathcal{C}$ enriched in any monoidal category with pullbacks.

Proof. The composition $f^*\Pi$ factors as

$$\begin{tikzcd}
\mathrm{Hom}_{x/\mathcal{C}}(f,g) \times \mathrm{Hom}_{x/\mathcal{C}}(g,h) \arrow[r] \arrow[d, "\pi\times\pi"] & \mathrm{pt} \times \mathrm{Hom}_{x/\mathcal{C}}(g,h) \arrow[r] \arrow[d, "g\times\pi"] & \mathrm{pt} \arrow[dddll, bend left, "h"] \\
\mathrm{Hom}_{\mathcal{C}}(y,y') \times \mathrm{Hom}_{\mathcal{C}}(y',y'') \arrow[r, "f^*\times\mathrm{id}"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(x,y') \times \mathrm{Hom}_{\mathcal{C}}(y',y'') \arrow[ddl] & \\
\mathrm{Hom}_{\mathcal{C}}(y,y'') \arrow[d] & & \\
\mathrm{Hom}_{\mathcal{C}}(x,y'') & &
\end{tikzcd}$$

where the inner triangle commutes by the associativity of the composition in $\mathcal{C}$, the upper left square by the universal property of $\mathrm{Hom}_{x/\mathcal{C}}(f,g)$, and finally the outer right square by the universal property of $\mathrm{Hom}_{x/\mathcal{C}}(g,h)$ and the definition of g^*; that is, $g^*\colon \mathrm{Hom}_{\mathcal{C}}(y',y'') \times \mathrm{Hom}_{\mathcal{C}}(x,y'')$ is the map $\mathrm{Hom}_{\mathcal{C}}(y',y'') = \mathrm{pt} \times \mathrm{Hom}_{\mathcal{C}}(y',y'') \xrightarrow{g\times\mathrm{id}} \mathrm{Hom}_{\mathcal{C}}(x,y') \times \mathrm{Hom}_{\mathcal{C}}(y',y'') \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(x,y'')$. □

2.2. The simplician Grassmannian.

Notation 2.4. Let $\mathcal{D}$ be an ∞-category, and x, x' objects. Unless stated otherwise, the homotopy category $\mathrm{h}\mathcal{D}$ is regarded as $\mathcal{H}$-enriched, with $\mathcal{H}$ the homotopy category of the topological category of CW complexes, or, equivalently, the category obtained from sSet by localising at weak homotopy equivalences of simplicial sets (which are defined in the sense of classical topology after taking geometric realisations). (For details, see [Lur09, §1.1.4].) We will denote the hom-space from x to x' in the homotopy category $\mathrm{h}\mathcal{D}$ by $\mathrm{Hom}_{\mathcal{D}}(x,x') \coloneqq \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(x,x')$. We reserve the expression $\mathrm{Map}_{\mathcal{D}}(x,x') \in \mathrm{sSet}$ for a representative of $\mathrm{Hom}_{\mathcal{D}}(x,x')$, such as $\{x\} \times_{\mathcal{D}^{\Delta\{0\}}} \mathcal{D}^{\Delta^1} \times_{\mathcal{D}^{\Delta\{1\}}} \{x'\}$. This deviates compatibly from the notation of [Lur09], in that the notation of op. cit. is coarser.

In the following, $\mathcal{C}$ is Kan-enriched, x is an object, and $x/\mathcal{C}$ is the simplicial undercategory.

Proposition 2.5. $x/(\mathrm{N}^{\boldsymbol{\Delta}}\mathcal{C}) \simeq \mathrm{N}^{\boldsymbol{\Delta}}(x/\mathcal{C})$.

Proof. We will write $\widehat{\mathcal{D}} := \mathrm{N}^{\mathbf{\Delta}}\mathcal{D}$. By their respective definitions, there is an obvious bijection between the objects of the two sides of the stated equivalence. Let $f\colon x \to y$, $g\colon x \to y'$ be objects of $\widehat{x/\mathcal{C}}$. By [Lur22, 01LE], there is a canonical equivalence

$$\mathrm{Hom}_{x/\mathcal{C}}(f,g) \simeq \mathrm{Map}_{\widehat{x/\mathcal{C}}}(f,g)$$

of Kan complexes. Let now $f\colon x \to y$, $g\colon x \to y'$ be objects of $x/\widehat{\mathcal{C}}$, so that there is an induced map $f^*\colon \mathrm{Hom}_{\widehat{\mathcal{C}}}(y,y') \to \mathrm{Hom}_{\widehat{\mathcal{C}}}(x,y')$. By [Lur09, Lemma 5.5.5.12], there is a homotopy fibre sequence

$$\mathrm{Hom}_{x/\widehat{\mathcal{C}}}(f,g) \to \mathrm{Hom}_{\widehat{\mathcal{C}}}(y,y') \xrightarrow{f^*} \mathrm{Hom}_{\widehat{\mathcal{C}}}(x,y')$$

at g, which, moreover, is a Kan fibration (see Remark 2.2 or the proof of the cited result). Again by [Lur22, 01LE], this Kan fibration can be replaced by

$$\mathrm{Hom}_{\mathcal{C}}(y,y') \to \mathrm{Hom}_{\mathcal{C}}(x,y'),$$

whose ordinary fibre at g is $\mathrm{Hom}_{x/\mathcal{C}}(f,g)$ by definition. We thus have a canonical equivalence

$$\mathrm{Map}_{\widehat{x/\mathcal{C}}}(f,g) \simeq \mathrm{Map}_{x/\widehat{\mathcal{C}}}(f,g)$$

of Kan complexes. □

Corollary 2.6. $\mathfrak{C}\mathcal{V}^{\hookrightarrow} \simeq */B^{\boxplus}\mathrm{O}$.

Proof. We have $\mathfrak{C}\mathcal{V}^{\hookrightarrow} = \mathfrak{C}(*/\mathrm{N}^{\mathbf{\Delta}}B^{\boxplus}\mathrm{O}) \simeq \mathfrak{C}(\mathrm{N}^{\mathbf{\Delta}}(*/B^{\boxplus}\mathrm{O})) \simeq */B^{\boxplus}\mathrm{O}$, where we used Proposition 2.5 and that the counit of the adjunction $\mathfrak{C} \dashv \mathrm{N}^{\mathbf{\Delta}}$ is an equivalence by the Comparison Theorem ([Lur09, Theorem 1.1.5.13]). □

Definition 2.7. The simplicial category $*/B^{\boxplus}\mathrm{O}$ is called the *simplicial Grassmannian.*

Remark 2.8. Corollary 2.6 can be interpreted backwards to motivate this definition: the simplicial Grassmannian is the 'underlying' simplicial category of the ∞-category $\mathcal{V}^{\hookrightarrow}$, in the sense that its simplicial nerve recovers $\mathcal{V}^{\hookrightarrow}$; i.e., $\mathrm{N}^{\mathbf{\Delta}}(*/B^{\boxplus}\mathrm{O}) \simeq \mathrm{N}^{\mathbf{\Delta}}\mathfrak{C}\mathcal{V}^{\hookrightarrow} \simeq \mathcal{V}^{\hookrightarrow}$ via the equivalence given by the unit of the adjunction $\mathfrak{C} \dashv \mathrm{N}^{\mathbf{\Delta}}$.

Appendix A. Nerves, slices

Let $\mathbf{\Delta}$ denote the simplex category, and $\mathrm{sSet} = \mathrm{pSh}(\mathbf{\Delta})$ that of simplicial sets, i.e. set-valued presheaves on $\mathbf{\Delta}$, and $\mathcal{C}\mathrm{at}_{\mathbf{\Delta}}$ that of simplicial categories, that is, the category of categories enriched over sSet. We assume the reader is familiar with the nerve $\mathrm{N}(C) = \mathrm{N}_\bullet(C) \in \mathrm{sSet}$ of an ordinary category C.

In this section, $\Delta^k = \mathrm{Hom}_{\mathbf{\Delta}}(-,[k]) \in \mathrm{sSet}$ denotes the standard k-simplex. For $X = X_\bullet$ a simplicial set, the Yoneda lemma identifies elements of X_k with maps $\Delta^k \to X$; a fact we will use without mention.

A.1. **Simplicial nerves.** We recall the simplicial nerve construction ([Cor82], though see [Lur22, 00KT]) that featured crucially in the delooping $\mathfrak{B}^{\boxplus}\mathrm{O}$ of $(B\mathrm{O}_{\mathrm{II}}, \boxplus)$, following [Lur09, §1.1.5]. Some authors call the simplicial nerve the homotopy-coherent nerve.

Similarly to the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathrm{sSet}$, $[k] \mapsto \Delta^k$, which gives a simplicial set for each $k \in \mathbb{N}$, there exists functor

$$\mathfrak{C}\colon \mathbf{\Delta} \to \mathcal{C}\mathrm{at}_{\mathbf{\Delta}}.$$

Definition A.1. We first define $\mathfrak{C}$ on objects, then on morphisms.

(1) The simplicial category $\mathfrak{C}[k]$ has the same objects as those of $[k]$, and the simplicial sets of morphisms in each $\mathfrak{C}[k]$ are given by

$$\mathrm{Hom}_{\mathfrak{C}[k]}(i, j) = \mathrm{N}(P_{i,j}),$$

where $P_{i,j}$, $0 \leq i, j \leq k$ is empty if $i > j$, and

$$P_{i,j} = \{I \subseteq \{i \leq a + 1 \leq \cdots \leq j\} \subseteq [k] : a, b \in I\}$$

if $i \leq j$. Its nerve is taken with respect to its poset structure, with partial order $\preceq$ given by inclusions of the I. For each triple $i \leq j \leq p$ in $[k]$, there is a map $P_{j,p} \times P_{i,j} \to P_{i,p}$ defined by taking unions. The nerve functor applied to these maps defines maps $\mathrm{Hom}_{\mathfrak{C}[k]}(j, p) \times \mathrm{Hom}_{\mathfrak{C}[k]}(i, j) \to \mathrm{Hom}_{\mathfrak{C}[k]}(i, p)$ of simplicial sets, which is associative since so is taking unions.

(2) A map $f\colon [l] \to [k]$ in $\mathbf{\Delta}$ induces a map $\mathfrak{C}[l] \to \mathfrak{C}[k]$ as follows: on objects, it is given by $[l] \ni i \mapsto f(i) \in [k]$, and on the mapping posets it is given by $P_{i,j} \ni I \mapsto f(I) \in P_{f(i),f(j)}$, applying N to which defines the map $f = \mathfrak{C}f\colon \mathfrak{C}[l] \to \mathfrak{C}[k]$.

Definition A.2. We call the $P_{i,j}$ *mapping posets*, and their nerves *mapping spaces*.

Definition A.3. The *simplicial nerve* $\mathrm{N}^{\mathbf{\Delta}}(\mathcal{D}) = \mathrm{N}^{\mathbf{\Delta}}_{\bullet}(\mathcal{D})$ of a simplicial category $\mathcal{D}$ is the simplicial set whose set of k-simplices is defined by

$$\mathrm{N}^{\mathbf{\Delta}}_k(\mathcal{D}) = \mathrm{Hom}_{\mathcal{C}\mathrm{at}_{\mathbf{\Delta}}}(\mathfrak{C}[k], \mathcal{D}).$$

This is contravariant in $[k]$ via the covariance of $\mathfrak{C}$.

In other words, $\mathrm{N}^{\mathbf{\Delta}}$ is the restriction of the Yoneda embedding $\mathcal{C}\mathrm{at}_{\mathbf{\Delta}} \to \mathrm{pSh}(\mathcal{C}\mathrm{at}_{\mathbf{\Delta}})$ along $\mathfrak{C}\colon \mathbf{\Delta} \to \mathcal{C}\mathrm{at}_{\mathbf{\Delta}}$.

If $\mathcal{D}$ is Kan-enriched, then $\mathrm{N}^{\mathbf{\Delta}}(\mathcal{D})$ is an ∞-category ([CP86], [Lur09, Proposition 1.1.5.10], [Lur22, 00LJ]).

Definition A.4. Since $\mathcal{C}\mathrm{at}_{\mathbf{\Delta}}$ is cocomplete, $\mathfrak{C}$ admits a left Kan extension

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, "\mathfrak{C}"] \arrow[d, hook, "\mathbf{y}"'] & \mathcal{C}\mathrm{at}_{\mathbf{\Delta}} \\ \mathrm{sSet} \arrow[ur, dashed] & \end{tikzcd}$$

where $\mathbf{y}$ is the Yoneda embedding. Given a simplicial set $\mathcal{C}$, we denote the resulting simplicial category by $\mathfrak{C}\mathcal{C}$, and call it its *associated simplicial category*.

A.2. **Joins and (co)slices.** For $f\colon K \to \mathcal{C}$ a functor from a simplicial set to an ∞-category, there is ([Joy02], [Lur22, 01GP]) a right fibration $\mathcal{C}/f \to \mathcal{C}$ and a left fibration $f/\mathcal{C} \to \mathcal{C}$, whose domains are respectively called the *slice* and *coslice* of $\mathcal{C}$ at f. We will recall their definitions, but refer the reader to the op. cit. for the named lifting properties.

In the following, our convention is that $X_{-1} = \emptyset$ for $X_\bullet$ a simplicial set, and a product is empty if one of its factors is $\emptyset$. The following is equivalent to the more standard definition; see [Lur22, 0234]. It is a simplicial version of Milnor's general topological construction from [Mil56].

Definition A.5. The *join* $X \star Y = (X \star Y)_\bullet$ of two simplicial sets $X = X_\bullet$, $Y = Y_\bullet$ is defined by

$$(X \star Y)_k = \{(\pi, f_-, f_+) : \pi\colon \Delta^k \to \Delta^1, f_-\colon \Delta^k|_0 \to X, f_+\colon \Delta^k|_1 \to Y\},$$

where π, f_-, f_+ are maps of simplicial sets, and $\Delta^k|_i = \{i\} \times_{\Delta^1} \Delta^k$, $i = 0, 1$, is defined using π. Given $\phi\colon [l] \to [k]$ in $\mathbf{\Delta}$, the corresponding $\phi\colon \Delta^l \to \Delta^k$ defines a map $(X \star Y)_k \to (X \star Y)_l$ by restrictions.

Remark A.6. We have injections $\iota_0\colon X \hookrightarrow X \star Y$, $\iota_1\colon Y \hookrightarrow X \star Y$. For the former, let $f\colon \Delta^k \to X$ be a k-simplex of X. Defining $\pi\colon \Delta^k \to \{0\} \hookrightarrow \Delta^1$ and setting $f_- = f$, and necessarily $f_+\colon \emptyset \to Y$, gives a map $X_k \to (X \star Y)_k$. In the inclusion of Y into $X \star Y$, π is defined by factoring through the projection to 1 and setting f_- empty instead.

Remark A.7. The join construction is functorial in both arguments. Given $\phi\colon X \to X'$, $\psi\colon Y \to Y'$, we write $\phi \star \psi$ for the induced map $X \star Y \to X' \star Y'$.

Definition A.8. Let K be a simplicial set, $\mathcal{C}$ an ∞-category, and $f\colon K \to \mathcal{C}$ a map. The *slice* $\mathcal{C}/f$ of $\mathcal{C}$ at f is the simplicial set defined by

$$(\mathcal{C}/f)_n = (\mathrm{Hom}_{\mathrm{sSet}})_K(\Delta^n \star K, \mathcal{C}),$$

where the subscript K indicates that the set in question consists of maps $\phi\colon \Delta^n \star K \to \mathcal{C}$ whose precomposition $K \overset{\iota_1}{\hookrightarrow} \Delta^n \star K \overset{\phi}{\to} \mathcal{C}$ is f.

The face and degeneracy maps are given by precomposition and functoriality: a map $\psi\colon \Delta^m \to \Delta^n$ induces a map $\Delta^m \star K \overset{\psi\star\mathrm{id}}{\to} \Delta^n \star K \overset{\phi}{\to} \mathcal{C}$, which is clearly in $(\mathcal{C}/f)_m$, i.e., $(\phi \circ (\psi \star \mathrm{id}))|_K = f$. The slice is again an ∞-category.

The projection $\mathcal{C}/f \to \mathcal{C}$ is given by precomposing $\phi\colon \Delta^n \star K \to \mathcal{C}$ with $\Delta^n \overset{\iota_0}{\hookrightarrow} \Delta^n \star K$.

The *coslice* $f/\mathcal{C}$ is defined analogously, with $\Delta^n \star K$ replaced by $K \star \Delta^n$, ι_1 by ι_0 and vice versa, throughout. It is again an ∞-category.

Notation A.9. Let $\iota_x\colon \Delta^0 \to \mathcal{C}$ be given by a vertex $x \in \mathcal{C}_0$. We write $\mathcal{C}/x \coloneqq \mathcal{C}/\iota_x$, $x/\mathcal{C} \coloneqq \iota_x/\mathcal{C}$. They are respectively called the *over-* and *under-∞-category* at x.

Remark A.10. There are canonical isomorphisms $\Delta^k \star \Delta^l \simeq \Delta^{k+1+l}$, such that the composition $\Delta^k \overset{\iota_0}{\hookrightarrow} \Delta^k \star \Delta^l \overset{\sim}{\to} \Delta^{k+1+l}$ is given by $[k] \hookrightarrow [k+1+l]$, $i \mapsto i$, and such that the composition $\Delta^l \overset{\iota_1}{\hookrightarrow} \Delta^k \star \Delta^l \overset{\sim}{\to} \Delta^{k+1+l}$ is given by $[l] \hookrightarrow [k+1+l]$, $i \mapsto k+1+i$.

Remark A.11. We should explicate the degeneracies in an under-∞-category $x/\mathcal{C}$. Via Remark A.10, a 0-simplex of $x/\mathcal{C}$ is a 1-simplex of $\mathcal{C}$ with source x. Given a 1-simplex of $x/\mathcal{C}$, written $\sigma\colon \Delta^0 \star \Delta^1 \to \mathcal{C}$, the source and target σ_0, σ_1, are given, according to Definition A.8, by $\sigma_0\colon \Delta^0 \star \Delta^0 \overset{\mathrm{id}\star 0}{\hookrightarrow} \Delta^0 \star \Delta^1 \overset{\sigma}{\to} \mathcal{C}$, and similarly with $\mathrm{id} \star 1$ for σ_1. The faces of higher simplices are to be understood analogously.

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