QUADRATIC DUALS OF LINKED SPACES
V0.5

ÖDÜL TETİK

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1. Introduction

> Of quantities some are discrete, others continuous... Discrete are number and language; continuous are lines, surfaces, bodies, and also, besides these, time and place. For the parts of a number have no common boundary at which they join together... A line, on the other hand, is a continuous quantity. For it is possible to find a common boundary at which its parts join together, a point. And for a surface, a line; for the parts of a plane join together at some common boundary. Similarly in the case of a body one could find a common boundary—a line or a surface—at which the parts of the body join together.
>
> ---
>
> *Categories*
> Aristotle [Ari84]

1.1. **Summary of results in this version.**

Section 2.1. Passage from the simplicial nerve version of the stratified Grassmannian (written $\mathcal{V}^{\hookrightarrow}$) to the homotopy coherent nerve version.

Section 2.2. Given $n, m \in \mathbb{N}$, the exit path ∞-category $\mathcal{EX}(B\mathrm{O}(n,m))$, in the sense of [Tet23], of the (n,m)-*Grassmannian*

$$B\mathrm{O}(n,m) = \left(B\mathrm{O}(n) \xleftarrow{\mathrm{pr}_1} B\mathrm{O}(n) \times B\mathrm{O}(m) \overset{\boxplus}{\hookrightarrow} B\mathrm{O}(n+m)\right)$$

has objects $B\mathrm{O}(n)_0 \amalg B\mathrm{O}(n+m)_0$, which canonically map to $\mathcal{V}_0^{\hookrightarrow}$ (cf. [Tet22]). Moreover, a non-invertible path $(\gamma, 1) \in \mathcal{P}_0^{\Delta} \subset \mathcal{EX}(B\mathrm{O}(n,m))_1$ corresponds to a path γ in $B\mathrm{O}(n+m)$ of type

$$V_{12} \boxplus V_{01} \to V_{02},$$

where $V_{01} \in B\mathrm{O}(n)$ and $V_{12} \in B\mathrm{O}(m)$. This datum arranges as the morphism

$$\begin{tikzcd}
& 1 \arrow[dr, "V_{12}"] & \\
0 \arrow[ur, "V_{01}"] \arrow[rr, "V_{02}"] \arrow[rr, dashed, bend right=60, "V_{12}\boxplus V_{01}"'] \arrow[rr, phantom, shift right=3ex, "\Uparrow"] & & 2
\end{tikzcd}$$

in $\mathcal{V}_1^{\hookrightarrow}$. Thus, we have an assignment

$$\mathcal{EX}_{\leq 1}(B\mathrm{O}(n,m)) \to \mathcal{V}_{\leq 1}^{\hookrightarrow}$$

Theorem (2.2)**.** *This assignment extends to an ∞-functor*

$$\mathbb{U} \colon \mathcal{EX}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow}.$$

Section 2.3. Given a *linked manifold*

$$\mathfrak{S} = \left(M \overset{\pi}{\twoheadleftarrow} L \overset{\iota}{\hookrightarrow} N \right),$$

that is, a triple of smooth manifolds where π is a fibre bundle and ι is an embedding, we construct (Construction 2.12) its tangent bundle in terms of purely smooth data:

$$\mathrm{T}\mathfrak{S} \colon \mathfrak{S} \to B\mathrm{O}(n, m).$$

This is informed by an informal discussion present in the conically-smooth setting ([AFR18b]) which becomes completely natural in the linked setting. Namely, we give it as the following span map:

(1)
$$\begin{tikzcd}
& L \arrow[rr, "\pi^*\mathrm{T}M \times \mathrm{N}_N M"] \arrow[ddl, two heads, "\pi"'] \arrow[dr, hook, "\iota"] & & B\mathrm{O}(n) \times B\mathrm{O}(m) \arrow[ddl, "\mathrm{pr}"'] \arrow[dr, hook, "\boxplus"] & \\
& & N \arrow[rr, "\mathrm{T}N"] & & B\mathrm{O}(n+m) \\
M \arrow[rr, "\mathrm{T}M"] & & B\mathrm{O}(n) & &
\end{tikzcd}$$

where $\mathrm{N}_M N$ is the 'normal bundle of M inside N', built over the link. That a span map like this can be given depends on the observation that the classifier $\mathrm{T}N$ can be chosen in such a way that the back square of (1) commutes honestly; see Lemma 2.10. This yields an ∞-functor $\mathcal{E}\mathcal{X}(\mathfrak{S}) \to \mathcal{V}^{\hookrightarrow}$ which we also call the tangent bundle of $\mathfrak{S}$.

Section 3.1. Tangential structures in the stratified setting, which are ∞-categories over $\mathcal{V}^{\hookrightarrow}$, pull back to the linked setting:

$$\mathbb{U}^* \colon \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{E}\mathcal{X}(B\mathrm{O}(n,m)).$$

We introduce the problem of finding a linked space $\mathfrak{B}$ such that

(2)
$$\mathbb{U}^*\mathcal{B} = \mathcal{E}\mathcal{X}(\mathfrak{B})$$

for a stratified tangential structure $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, and solve it in some simple cases: framings of fixed rank (Example 3.1); classical tangential structures given by space maps of type $B \to B\mathrm{O}(\kappa) \subset \mathcal{V}^{\hookrightarrow}$ (Example 3.2); and variframings, in depth 1 but of arbitrary codimension (Example 3.3). We also discuss variframed point defects (Example 3.4).

Sections 3.2 and 3.4. Given a tangential structure $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, the main problem of performing factorisation homology, given a $\mathcal{B}$-structured disk algebra, on bordisms is that bordisms do not in general have $\mathcal{B}$-structure, but may be asked instead to have a relaxation of it. The most rigid relaxation is the *stable* one (for fixed maximal rank, namely that of the dimension of the disk algebra), used implicitly in [Lur08]. A further relaxation, the *solid* one, was considered in the stratified setting in [AFR18b]. For instance, if a certain structure is determined by some bundle F over X, a solid F-structure is a bundle injection

$$\mathrm{T}X \hookrightarrow F,$$

while a stable F-structure is a solid one together with a trivialisation of the normal bundle of this injection. We first discuss both relaxations in the smooth setting, and express them as tangential structures over a single Grassmannian: Definition 3.16 gives the solid version, and

Definition 3.17 a further-restricted one, the simpliest special case of which recovers the stable version.

This justifies the way in which we transfer such structures to the linked setting. We see stratified solid structures as Cartesian fibration replacements of the original $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ à la [GHN17] (see Definition 3.18). Cartesian fibration replacements of 'classical' tangential structures, i.e., those of the form $Y \xrightarrow{F} BO(R) \hookrightarrow \mathcal{V}^{\sim} \subset \mathcal{V}^{\hookrightarrow}$ for Y an ∞-groupoid are for us of particular interest. Typically, F classifes a rank-R vector bundle on Y, or Y is of type BG for a group G over $O(R)$. We denote the replacement of such an (Y, F) by $\overline{Y}$, and set, for $r \leq R$,

$$\overline{Y}|_r := BO(r) \times_{\mathcal{V}^{\hookrightarrow}} \overline{Y},$$

At this point, determining the morphism spaces in $\mathcal{EX}(\mathfrak{S})$, for $\mathfrak{S}$ a linked space, becomes compulsory. We prove:

Theorem (3.5)**.** *If $p \in M$ and $q \in N$, then* $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p, q) \simeq \mathcal{P}_{\mathcal{L}_p, q}$.

Here, $\mathcal{P}_{\mathcal{L}_p,q}$ is the space of paths in N that start in the (the image of the) fibre of π at p (under ι) and end at q. We then explore some immediate consequences of this result in Corollaries 3.6 and 3.7, concerning linked spaces of two certain types. In particular, we recover the stratified homotopy type of the conically-smooth open cone on a smooth manifold N as $\mathrm{Exit}(C(N)) \simeq \mathcal{EX}\left(* \leftarrow N \xrightarrow{\mathrm{id}} N\right)$.

We are then in a position to identify the novel morphism spaces in the stratified Grassmannian. For $p \in BO(r)$ and $q \in BO(R)$, we obtain:

Proposition (3.19)**.** $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q) \simeq \mathrm{Hom}_{\mathcal{EX}(BO(r,R-r))}(p, q)$.

This leads to the following description of the restricted Cartesian replacement $\overline{Y}|_r$ as a classical homotopy pullback:

Proposition (3.21, 3.22)**.** *We have* $\overline{Y}|_r \simeq (BO(r) \times BO(R-r)) \times^{\mathrm{h}}_{BO(R)} Y$*. Consequently, over a smooth manfiold, solid F-structures and Cartesian F-structures coincide.*

Section 3.3. We give here a belated account of the category of linked spaces, as a coherent treatment had to wait until after Section 3.2. We propose a way of seeing a smooth space as a linked space in a natural way as well as that of a map of linked spaces in terms of smooth data only, in such a way that maps of linked spaces naturally induce maps ∞-functors on exit paths.

Section 3.5. For the Cartesian fibration replacement of a classical tangential structure $Y \to BO(R) \subset \mathcal{V}^{\hookrightarrow}$ of rank R we give here a partial solution to problem (2) in the linked setting, extending the results mentioned above in the smooth setting. Namely, for $\mathfrak{S}$ with M of dimension n and N of dimension $n+m$, assume for simplicity $R = n+m$. Then:

Proposition (3.26, 3.29)**.** *A Y-structure on N and a solid Y-structure on M, such that $\pi^*W \cong \mathrm{N}_N M$ over L, induce a Cartesian Y-structure on $\mathfrak{S}$.*

Here, $W \to M$ is the rank-m *normal bundle* $M \to BO(R-r) = BO(m)$ of the solid structure $M \to \overline{Y}|_r \simeq (BO(r) \times BO(R-r)) \times^{\mathrm{h}}_{BO(R)} Y$ given by projecting $\overline{Y}|_r \to BO(R-r)$, and $\mathrm{N}_N M$ refers to (1).

Section 4. Here we introduce the second main topic of the present work: *collars* and *quadratic duals* of linked spaces with a solid or stable $\mathcal{B}$-structure for $\mathcal{B}$ a classical tangential structure. The picture we present is that this datum induces an 'extension'

$$X \hookrightarrow \overline{X} \overset{p}{\twoheadrightarrow} X^!$$

where the middle term is the collar and the last the dual. The relevance to field theory consists in p being the map along which one pushes forward the disk algebra A of observables with $\mathcal{B}$-structure. In sloganic form:

$$\mathcal{Z}_A = \left(\overline{(-)} \twoheadrightarrow (-)^!\right)_* A.$$

This systematically relaxes factorisation homology in the sense that when the argument bordism X is of top dimension and without boundary, then $\overline{X} = X$ and $X^! = *$, so that the pushforward algebra is simply (the 0-disk algebra whose underlying object is) $\int_X A$.

The history of this idea and the differences of our approach to previous ones are explained very briefly at the beginning of this section. The relation to previous sections consists in the fact that the dual $X^!$ parametrises the normal bundle of the solid/stable $\mathcal{B}$-structure on X, while the collar $\overline{X}$ is in fact the 'interior' X° given a new stratification induced by that on X and its solid/stable $\mathcal{B}$-structure.

In Sections 4.1 to 4.3 we immediately discuss many examples with numerous illustrations: mostly bordisms of different codimensions, as well as cutting and gluing – all in the linked setting, which simplifies the theory substantially.

Cutting and gluing in this approach means the choice of a codimension-1 smooth submanifold $\Sigma \subset X^\circ$ in the interior, with trivial normal bundle. This induces a new linked space X_Σ (just as it induces a new stratification in the ordinary sense) and a *refinement*

$$r\colon X_\Sigma \to X,$$

i.e., a map that is stratum-wise an isomorphism onto its image. One of our results is that $\mathcal{Z} = \left(\overline{(-)} \twoheadrightarrow (-)^!\right)$ is covariant along refinements, so that r induces what we call a commuting *cutting-and-gluing* square (see (48))

$$\mathcal{Z}(r) = \left(\begin{tikzcd} \overline{X_\Sigma} \arrow[r, "\overline{r}"] \arrow[d, two heads, "p_\Sigma"'] & \overline{X} \arrow[d, two heads, "p"] \\ X^!_\Sigma \arrow[r, "r^!"'] & X^! \end{tikzcd}\right).$$

Figure 4 illustrates this. Now, applying a fundamental result of [AFT17], we have that $\mathcal{Z}_A(r)$ also commutes, which expresses the functoriality of $\mathcal{Z}_A$.

Although we restrict ourselves to depth 1 in this work, we also give a teaser on depth 2 by giving an explicit discussion of $\overline{X}$ and $X^!$ in the case of bordisms with corners; see Section 4.3.

1.2. **Prose.** We give a procedure to evaluate a non-singular disk algebra on singular spaces. This suggests a solution to some extent of a problem posed by Ayala–Francis–Tanaka, extending a construction, due to Lurie, Calaque and Scheimbauer, of framed functorial field theories given bulk/point datum in the form of an $\mathbb{E}_n$-algebra. The core novelty consists in the decomposition of a nice-enough stratified space into its strata and their links, organised as a linked space, which uses only smooth manifold data. We transfer some ideas from the theory of tangential structures in stratified geometry, due to Ayala–Francis–Rozenblyum to this setting, by constructing comparison maps from 'linked' Grassmannians to the stratified Grassmannian. This results in a much more accessible theory of bundles and tangential structures in the linked setting.

The main practical advantage of this approach is that it bypasses a good portion of the complicated higher-categorical treatment of bordism categories in terms of iterated Segal spaces. Concretely, this is because in the Lurie–Calaque–Scheimbauer approach, boundaries and corners are specified by cut functions on a manifold (without boundary) of full dimension, embedded, essentially, in a (colimit of) euclidean space(s). One justication for this is the relatively straightforward manner in which one may then introduce tangential structures on boundaries and corners, by simply introducing one on the containing manifold. On the other hand, this approach introduces many technical issues and impracticalities that are not of immediate interest from the point of view of the field theory itself. In our approach, we reverse the places where the difficulty and ease lie: we adopt the more complicated theory of tangential structures on stratified spaces à la Ayala–Francis–Rozenblyum, which we trade for an easier, more intuitive approach to bordisms, in tune with the Atiyah–Segal–Witten paradigm. In fact, we go further and transfer, as mentioned, the theory of stratified tangential structures to the simpler linked setting, which only uses non-singular data. Even though general linked geometry should be at least as complicated as conically-smooth stratified geometry, the linked spaces induced by bordisms are especially simple, whence the difficult part of this approach is also greatly simplified. It seems, therefore, that for the purpose of connecting BV-type field theory with FFT, the linked approach is 'correct'.

The idea, due to Lurie, of completing a bordism into an honestly framed collar and evaluating the disk algebra on the latter in a particular way in order to construct field theory, is generalised to the linked setting by defining factorisation algebras/homology therein in terms of algebraic coupling data, in such a way that it reproduces some well-known classifying statements in the stratified setting.

Our procedure does not depend intrinsically on the chosen non-singular tangential structure being that of a framing. In particular, it does not rely on Dunn–Lurie additivity, as did Lurie's original construction, and so suggests an extension to any tangential structure. In view of a result of Ginot et al. that generalises Lurie's correspondence between locally-constant factorisation algebras on $\mathbb{R}^n$ and $\mathbb{E}_n$-algebras to one between such on a smooth manifold M and $(M, \mathrm{T}M)$-structured disk algebras, we present, as a special case, a way to construct functorial field theories given observables algebras (and correlators) on *any* smooth manifold, not just on the local $\mathbb{R}^n$. This is achieved at least in the case when bordisms are 'stably-$(M, \mathrm{T}M)$-structured'.

The 'evaluation' of a structured disk algebra on a structured collar of a singular space (such as a bordism), when performed à la Lurie, naturally gives rise to representation-theoretic data, as was further systematised by Calaque–Scheimbauer. The datum of compatible actions is parametrised by especially simple spaces when one evaluates bordisms, namely, euclidean spaces with flag-like stratifications. We obtain such spaces directly from the stable tangential structure on bordisms, and call them their *quadratic duals*. They parametrise the Morita categories which provide the coefficents of the functorial field theories thus obtained. We show that they have the same shape regardless of tangential structure, but when bordisms are relaxed to be solidly-structured rather than stably, the quadratic duals become more complicated. We pose their construction problem, which is a certain stratified surgery problem, and present ad hoc solutions in some simple cases in low dimensions.

Acknowledgments.

Conventions. We say *smooth manifold* to refer always to one without boundary. In the linked context, it is understood moreover to be connected.

Definition 1.1. A *linked manifold* (of *depth* 1) is a span

$$\begin{tikzcd} & L \arrow[dl, two heads, "\pi"'] \arrow[dr, hook, "\iota"] & \\ M & & N \end{tikzcd}$$

of smooth manifolds, where π is a proper fibre bundle, and ι a closed embedding.

2. Linked tangent bundles

For technical reasons, we need to modify our convention for the homotopy-coherent nerve from that in [Lur09] to the one in [Lur23, 00KM]. The former was adopted in [Tet22].

2.1. **The stratified Grassmannian via the homotopy-coherent nerve.**

Recall the treatment of the stratified Grassmannian of [AFR18b] given in [Tet22]:

$$\mathcal{V}^{\hookrightarrow} := */\mathcal{B}^{\boxplus}\mathrm{O},$$

where

- $\mathcal{B}^{\boxplus}\mathrm{O} := \mathrm{N}^{\mathbf{\Delta}}(B^{\boxplus}\mathrm{O})$, where
  - $B^{\boxplus}\mathrm{O}$ is the simplicial category given by delooping the topological monoid $(B\mathrm{O}^{\infty}_{\amalg}, \boxplus)$ and taking the singular complex,
  - $\mathrm{N}^{\mathbf{\Delta}}$ takes the simplicial nerve as in [Lur09] (see below),
- $*/-$ takes the under ∞-category under the single object $*$ of $\mathcal{B}^{\boxplus}\mathrm{O}$.

Specifically, the simplicial set that is the simplicial nerve of a simplicially-enriched category $\mathcal{A}$ is defined levelwise by setting $\mathrm{N}^{\mathbf{\Delta}}(\mathcal{A})_k := \mathrm{Fun}(\mathfrak{C}[k], \mathcal{A})$, the set of functors of simplicially-enriched categories from $\mathfrak{C}[k]$ to $\mathcal{A}$, where

- the objects of $\mathfrak{C}[k]$ are the same as those of $[k]$,
- the simplicial hom-set from i to j, if $i \leq j$, is $\mathrm{Hom}_{\mathfrak{C}[k]}(i, j) := \mathrm{N}(P_{i,j})$, the nerve of the poset of subposets of k with least element i and greatest element j, ordered by inclusion (empty if $i > j$),
- composition is given by taking unions.

We will use N^{hc} instead of $\mathrm{N}^{\mathbf{\Delta}}$, which is defined levelwise by $\mathrm{N}^{\mathrm{hc}}(\mathcal{A})_k := \mathrm{Fun}(\mathrm{Path}[k], \mathcal{A})$, where $\mathrm{Path}[k] := \mathfrak{C}[k]^{\mathrm{op}}$. (This can be read as $\mathrm{Path}[k] = \mathrm{N}(P^{\mathrm{op}}_{i,j})$.) We will call functors $\mathrm{Path}[k] \to \mathcal{A}$ *k-paths* in $\mathcal{A}$. To illustrate, let us abuse notation and write

$$\mathcal{V}^{\hookrightarrow} := */\mathrm{N}^{\mathrm{hc}}(B^{\boxplus}\mathrm{O}).$$

We then have the bijection $\mathcal{V}^{\hookrightarrow}_1 \cong \mathrm{Fun}\left(\mathrm{Path}[2], B^{\boxplus}\mathrm{O}\right)$, and so a morphism in $\mathcal{V}^{\hookrightarrow}$ in this version is uniquely determined by data that can be summed up in the diagram

(3)
$$\begin{tikzcd}
 & 1 \arrow[dr, "V_{12}"] & \\
0 \arrow[ur, "V_{01}"] \arrow[rr, "V_{02}"] \arrow[rr, dashed, bend right, "V_{12}\boxplus V_{01}"'] \arrow[rr, phantom, shift right=3ex, "\Uparrow"] & & 2
\end{tikzcd}$$

where $\Rightarrow$ denotes a path from $V_{12} \boxplus V_{01}$ to V_{02} in $B\mathrm{O}^{\infty}_{\amalg}$. In the other version of $\mathcal{V}^{\hookrightarrow}$, in contrast, the path is to go in the reverse direction.

Since the two versions are clearly equivalent, we will keep this notation for $\mathcal{V}^{\hookrightarrow}$ to refer to the homotopy-coherent version. In particular, its maximal sub-∞-groupoid is still $B\mathrm{O}^{\infty}_{\amalg}$ ([Tet22, Proposition 2.7]). It will become clear momentarily why this is convenient.

2.2. **Grassmannians: linked to stratified.**

Definition 2.1 ([Tet23, Example 2.7])**.** Let $n, m \in \mathbb{N}$. We call the linked space

$$\begin{tikzcd}
 & B\mathrm{O}(n) \times B\mathrm{O}(m) \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\boxplus"] & \\
B\mathrm{O}(n) & & B\mathrm{O}(n+m)
\end{tikzcd}$$

the (n, m)-*Grassmannian* and denote it by $B\mathrm{O}(n, m)$.

The goal of this section is to construct a map

$$\mathbb{U} : \mathcal{E}\mathcal{X}(B\mathrm{O}(n, m)) \to \mathcal{V}^{\hookrightarrow}$$

from the exit path ∞-category of the (n, m)-Grassmannian to the stratified Grassmannian. The theory of tangential structures on linked spaces will then be able to interface with the conically-smooth variant, which we expand upon in Section 2.3. We will be completely explicit excepting specification of some trivial choices.

The map restricted to $B\mathrm{O}(n)_{\bullet}$ and $B\mathrm{O}(n+m)_{\bullet}$ inside $\mathcal{E}\mathcal{X}$ is defined to be inclusion into the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$. It remains to define the restriction

$$\mathcal{E}\mathcal{X}_{k+1} \supset \mathcal{P}^{\Delta}_k \to \mathcal{V}^{\hookrightarrow}_{k+1} \cong \mathrm{Fun}\left(\mathrm{Path}[k+2], B^{\boxplus}\mathrm{O}\right),$$

for $k \geq 0$. We will explain dimensions 1 and 2 verbosely before giving the full definition without further explanation.

2.2.1. 1-*morphisms.* An element $(\gamma, 1)$ of $\mathcal{P}^{\Delta}_0$ – the exit index in this dimension is necessarily 1 – corresponds to a path γ in $B\mathrm{O}(n+m)$ whose starting point is a direct sum $V_{12} \boxplus V_{01}$ with $V_{01} \in B\mathrm{O}(n)$, $V_{12} \in B\mathrm{O}(m)$. Denoting the endpoint by V_{02}, γ determines a 2-path by arranging the data exactly as in (3). We have thus defined

(4)
$$\mathcal{E}\mathcal{X}_{\leq 1} \to \mathcal{V}^{\hookrightarrow}_{\leq 1}.$$

Observe that this is compatible with face maps: the source of the image of γ is V_{01}, and the target is V_{02} ([Tet22, §2.2.2]), which are, by construction, the images of the source and target of γ in $\mathcal{E}\mathcal{X}$, respectively:

$$d_1^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}(\gamma, 1)) = V_{01} = \mathbb{U}\left(\mathrm{pr}\left(d_1^{B\mathrm{O}(n+m)}\gamma\right)\right) = \mathbb{U}\left(d_1^{\mathcal{E}\mathcal{X}}(\gamma, 1)\right),$$

$$d_0^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}\gamma) = V_{02} = \mathbb{U}\left(d_0^{BO(n+m)}\gamma\right) = \mathbb{U}\left(d_0^{\mathcal{E}\mathcal{X}}(\gamma,1)\right).$$

Compatibility with degeneracies is also clear.

Theorem 2.2. *For any pair of natural numbers $n, m \in \mathbb{N}$, the assignment* (4) *extends to an ∞-functor*

$$\mathbb{U}\colon \mathcal{E}\mathcal{X}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow},$$

which we call the unpacking map.

First, we will discuss what $\mathbb{U}$ has to do with 2-morphisms at a phenomenological level so as to elucidate the essential issues to be overcome.

2.2.2. *Intermezzo: 2-morphisms.* Exit paths in $\mathcal{P}_k^{\Delta}$ come in $k+1$ classes according to their exit indices, which need to be mapped to $\mathcal{V}^{\hookrightarrow}$ in different ways.

First, in order to uniquely determine a 3-path in $B^{\boxplus}\mathrm{O}$, it is enough to map out of the sets $N_{\leq 2}(P_{i,j}^{\mathrm{op}})$ into $B\mathrm{O}_{\leq 2} := (B\mathrm{O}_{\amalg}^{\infty})_{\leq 2}$ (and in general, a κ-path in $B^{\boxplus}\mathrm{O}$ is determined in dimensions $\leq \kappa - 1$), since higher dimensions are degenerate. The simple (non-decomposable) morphisms in Path$[3]$ are of type $\mathrm{N}_0(P_{\alpha\beta}^{\mathrm{op}}) \ni \underline{\alpha\beta} := \{\alpha < \beta\} \subset [3]$, which by a 3-path are mapped to $V_{\alpha\beta} \in B\mathrm{O}_0$. When $\beta = \alpha + 2$ (of which type there are two pairs), there are arrows $\underline{\alpha, \alpha+1, \beta} = \underline{\alpha+1, \beta} \cup \underline{\alpha, \alpha+1} > \underline{\alpha\beta}$ in $N_1(P_{\alpha\beta}^{\mathrm{op}})$, which determine paths

$$V_{\alpha+1,\beta} \boxplus V_{\alpha,\alpha+1} \to V_{\alpha\beta},$$

i.e., $V_{12} \boxplus V_{01} \to V_{02}$ and $V_{23} \boxplus V_{12} \to V_{13}$ in $B\mathrm{O}_1$, namely two of the face 2-paths. The remaining two faces are supplied analogously by considering $(\alpha, \beta) = (0,3)$ and the compositions $\underline{013} = \underline{13} \cup \underline{01}$ and $\underline{023} = \underline{23} \cup \underline{02}$. Finally, again for $(\alpha, \beta) = (0,3)$, consider $\underline{0123} = \underline{23} \cup \underline{12} \cup \underline{01}$, which is to be mapped to $V_{0123} = V_{23} \boxplus V_{12} \boxplus V_{01}$. Out of $N_1(P_{0,3}^{\mathrm{op}})$ we receive paths $V_{0123} \to V_{03}$, $V_{0123} \to V_{013}$, $V_{0123} \to V_{023}$. The two non-degenerate elements $(\underline{0123} > \underline{023} > \underline{03})$ and $(\underline{0123} > \underline{013} > \underline{03})$ in $N_2(P_{0,3}^{\mathrm{op}})$ are to map in $B\mathrm{O}_2$ to

$$\begin{tikzcd} & V_{023} \arrow[dr] & \\ V_{0123} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} = \begin{tikzcd} & V_{23} \boxplus V_{02} \arrow[dr] & \\ V_{23} \boxplus V_{12} \boxplus V_{01} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} \tag{5}$$

and

$$\begin{tikzcd} & V_{013} \arrow[dr] & \\ V_{0123} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} = \begin{tikzcd} & V_{13} \boxplus V_{01} \arrow[dr] & \\ V_{23} \boxplus V_{12} \boxplus V_{01} \arrow[ur] \arrow[rr] & & V_{03} \end{tikzcd} \tag{6}$$

We have thus summed up the data needed to provide a functor Path$[3] \to B^{\boxplus}\mathrm{O}$.

Now, let us start with paths of exit index $2 \in \{1,2\}$. Such an exit path $(\gamma, 2)$ (in $\mathcal{P}_1^{\Delta} \subset \mathcal{E}\mathcal{X}_2$) consists of a 2-simplex $\gamma \in B\mathrm{O}(n+m)_2$ of type

$$\begin{tikzcd} & K & \\ W \boxplus V \arrow[ur, "\gamma_{\boxplus}"] \arrow[r, "\gamma_W \boxplus \gamma_V"'] & W' \boxplus V' \arrow[u, "\gamma_{\boxplus'}"'] \end{tikzcd} \tag{7}$$

where the bottom edge comes from $BO(n) \times BO(m)$ (whence it is $\boxplus$ of two paths). The natural choice for the image, visualised as a 3-simplex of $\mathcal{B}^{\boxplus}O$, is

$$\mathbb{U}(\gamma, 2) = \begin{tikzcd} & 2 \arrow[dr, "W'"] & \\ 1 \arrow[ur, "0"] \arrow[rr, "W"] & & 3 \\ & 0 \arrow[ul, "V"] \arrow[uu, dashed, "V'"] \arrow[ur, "K"'] & \end{tikzcd} .$$

Indeed, the edges in (7) supply the face triangles – in fact, the fact that the bottom edge is of type $\gamma_W \boxplus \gamma_V$ is crucial, since the summand paths supply the triangles adjacent to the edge decorated by the zero vector space. The only wrinkle is that the upper face requires a path $W' \to W$, which can be taken to be the (standard) inverse of γ_W, which we will denote by γ_W^{-1}. As for (5), i.e.,

$$\begin{tikzcd} & W' \boxplus V' \arrow[dr, "\gamma_{\boxplus'}"] & \\ W' \boxplus V \arrow[ur, "\text{(i)}"] \arrow[rr, "\text{(ii)}"'] & & K \end{tikzcd} , \tag{8}$$

note that we are still free to (and have to) choose the paths $W' \boxplus V \to W' \boxplus V'$ and $W' \boxplus V \to K$ (corresponding to the arrows $\underline{0123} > \underline{023}$ and $\underline{0123} > \underline{03}$). To this end, consider the diagram

$$\begin{tikzcd} W' \boxplus V' \arrow[r, "\gamma_{\boxplus'}"] & K \\ W' \boxplus V \arrow[u, "\mathrm{id}_{W'} \boxplus \gamma_V"] \arrow[r, "\gamma_W^{-1} \boxplus \mathrm{id}_V"'] & W \boxplus V \arrow[u, "\gamma_{\boxplus}"'] \arrow[ul, "\gamma_W \boxplus \gamma_V"'] \end{tikzcd}$$

and choose the obvious fillers. (By id_A we mean the constant loop at A.) This suggests using (i) = $\mathrm{id}_{W'} \boxplus \gamma_V$, (ii) = $(\gamma_W^{-1} \boxplus \mathrm{id}_V) * \gamma_{\boxplus}$ (we denote concatenation from left to right) whereupon the obvious filler can be chosen. Similarly, for (6), i.e.,

$$\begin{tikzcd} & W \boxplus V \arrow[dr, "\gamma_{\boxplus}"] & \\ W' \boxplus V \arrow[ur, "\text{(i)}'"] \arrow[rr, "\text{(ii)}'"'] & & K \end{tikzcd} , \tag{9}$$

consider

$$\begin{tikzcd} W \boxplus V \arrow[r, "\gamma_{\boxplus}"] & K \\ W' \boxplus V \arrow[u, "\gamma_W^{-1} \boxplus \mathrm{id}_V"] \arrow[r, "\mathrm{id}_W \boxplus \gamma_V"'] & W' \boxplus V' \arrow[u, "\gamma_{\boxplus'}"'] \arrow[ul, "\gamma_W^{-1} \boxplus \gamma_V^{-1}"'] \end{tikzcd}$$

and proceed similarly. (We give a systematic account in Section 2.2.3.) This completes the construction of $N_{\leq 2}(P_{0,3}^{\mathrm{op}}) \to BO_{\leq 2}$ and so in toto of the 3-path $\mathrm{Path}[3] \to B^{\boxplus}O$ associated with the exit path $(\gamma, 2)$. The image of an index-1 exit path

$$
\begin{tikzcd}
K \arrow[r, "\gamma_K"] & K' \\
W \boxplus V \arrow[u, "\gamma_{\boxplus}"] \arrow[ur, "\gamma_{\boxplus'}"'] &
\end{tikzcd}
\tag{10}
$$

is constructed analogously, with its picture as a 3-simplex of $\mathcal{B}^{\boxplus}\mathrm{O}$ given by

$$
\begin{tikzcd}
& 2 \arrow[dr, "0"] & \\
1 \arrow[ur, "W"] \arrow[rr, "W"] & & 3 \\
& 0 \arrow[ul, "V"] \arrow[uu, "K"] \arrow[ur, "K'"'] &
\end{tikzcd}
\cdot
$$

We have thus defined

$$\mathcal{E}\mathcal{X}_{\leq 2} \to \mathcal{V}^{\hookrightarrow}_{\leq 2}.$$

As for simpliciality, consider again an index-2 exit path $(\gamma, 2)$ as in (7). Its source edge is the path $d_2^{\mathcal{E}\mathcal{X}}(\gamma, 2) = (\gamma_V \colon V \to V') \in B\mathrm{O}_1 \subset \mathcal{E}\mathcal{X}_1$. Since its two remaining edges are vertical, they are the elements of $\mathcal{P}_0^{\Delta}$ induced by $\gamma_{\boxplus'}$ and $\gamma_{\boxplus}$. Now, recall that $\mathbb{U}(\gamma, 2)$ is identified with an element of $\mathcal{B}^{\boxplus}\mathrm{O}_3 \cong \mathcal{V}_2^{\hookrightarrow}$ due to $\mathcal{V}^{\hookrightarrow}$'s being an under-$\infty$-category, via $\Delta[0] \star \Delta[2] \simeq \Delta[3]$. Accordingly, face maps apply on the factor $\Delta[2]$ (i.e., ∂ acts as $\mathrm{id}_{\Delta[0]} \star \partial$). In the picture in $\mathcal{B}^{\boxplus}\mathrm{O}_3$, this means that when pulling back along a face map $\partial \colon \Delta[1] \hookrightarrow \Delta[2]$, we restrict to the triangle whose top edge is specified by ∂; e.g., $\partial_2 \colon \Delta[1] \hookrightarrow \Delta[2]$, which skips 2, applies to give

$$
d_2^{\mathcal{V}^{\hookrightarrow}} \mathbb{U}(\gamma, 2) =
\begin{tikzcd}
& 1 \arrow[dr, "0"] & \\
0 \arrow[ur, "V"] \arrow[rr, "V'"] & & 2
\end{tikzcd}
,
$$

which is precisely $\mathbb{U}(\gamma_V)$. So as to avoid confusion, note in particular that the top face

$$
\begin{tikzcd}
& 2 \arrow[dr, "W'"] & \\
1 \arrow[ur, "0"] \arrow[rr, "W"] & & 3
\end{tikzcd}
,
$$

where we inverted γ_W, is *not* a face in $\mathcal{V}^{\hookrightarrow}$, nor is $\gamma_W \in B\mathrm{O}(m)_1 \subset \mathcal{E}\mathcal{X}_1$ of $(\gamma, 2)$ in $\mathcal{E}\mathcal{X}$.

We leave the analogous treatment of the remaining two faces and of the index-1 case to the reader. Now, we proceed to finally give the general construction.

2.2.3. *Proof of Theorem 2.2.* We will first give a systematic account of $\mathcal{P}_1^{\Delta} \to \left[\mathrm{Path}[3], B^{\boxplus}\mathrm{O}\right]$ in such a way that the ideas generalise to all dimensions. At some places, it will conventient to slightly rearrange the visual representation of exit paths. For $(\gamma, 1) \in \mathcal{P}_0^{\Delta}$, the diagram $W \boxplus V \to K$ depicts $\gamma \in B\mathrm{O}_1$. Instead, $V \xrightarrow{(W,\gamma)} K$ or $V \xrightarrow{W} K$ for short, depicts $(\gamma, 1)$ more properly. Similarly, we also depict $(\gamma, 2) \in \mathcal{P}_1^{\Delta}$ by

$$\begin{tikzcd} & K \\ V \arrow[ur, "W"] \arrow[r] & V' \arrow[u, "W'"'] \end{tikzcd},$$

etc.[1]

Notation 2.3. $[A, B] \coloneqq \mathrm{Fun}(A, B)$.

Definition 2.4. We use $\mathbf{F}\colon B\mathrm{O}(n) \times B\mathrm{O}(m) \to B\mathrm{O}(m)$, for *f*ibre, to denote the second coordinate projection. When we apply $\mathbf{F}$ to a bottom face of an exit path (γ, j), we mean that first the corresponding face of γ is to be taken, which is then (unambiguously) to be identified with a simplex of the link, and then $\mathbf{F}$ is to be applied. Namely, we have, by abuse of notation, a map

$$\mathbf{F}\colon \mathcal{P}_*^{\Delta} \to B\mathrm{O}(m)_*$$

for each $* \geq 0$, given by the composition

$$\begin{tikzcd} \mathcal{P}_*^{\Delta} \arrow[r, "(\gamma,j)\mapsto\Gamma_j=\gamma\circ C_j"] \arrow[rrr, dashed, bend right, "\mathbf{F}"'] & \mathcal{P}_* \arrow[r, two heads] & \mathcal{L}_* \arrow[r, "\mathbf{F}"] & B\mathrm{O}(m)_* \end{tikzcd}$$

where we have not omitted $*$ since $\mathcal{P}^{\Delta}$ is not a simplicial set.

Notation 2.5. For X a space, we denote by Op the canonical isomorphism $X \simeq X^{\mathrm{op}}$ of Kan complexes, by which we mean $\mathrm{Sing}_{\bullet}(X) \simeq \mathrm{Sing}_{\bullet}(X)^{\mathrm{op}}$ ([Lur23, 003R]). This inverts simplices of all dimensions in a compatible fashion.

Notation 2.6. Let $\alpha_0, \ldots, \alpha_\ell \in [k]$. By $\mathrm{Path}[\alpha_0, \ldots, \alpha_\ell] \cong \mathrm{Path}[\ell]$ we denote the simplicial subcategory of $\mathrm{Path}[k]$ generated by the objects $\alpha_0, \ldots, \alpha_\ell$.

Lemma 2.7. $d_i^{\mathcal{V}^{\hookrightarrow}} = d_{i+1}^{\mathcal{B}^{\boxplus}\mathrm{O}}$.

Proof. $(\mathrm{id}_0 \star \partial_i\colon \Delta[0] \star \Delta[1] \hookrightarrow \Delta[0] \star \Delta[2]) = (\partial_{1+i}\colon \Delta[2] \hookrightarrow \Delta[3])$. □

The exitence of the restriction of $\mathbb{U}\colon \mathcal{EX} = \mathcal{EX}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow}$ to non-invertible paths will be shown inductively. It is defined on $B\mathrm{O}(n), B\mathrm{O}(n+m) \subset \mathcal{EX}$ by inclusion into the maximal sub-∞-groupoid of $\mathcal{V}^{\hookrightarrow}$, and on $\mathcal{P}_*^{\Delta} \subset \mathcal{EX}_{*+1}$, using $\mathcal{V}_k^{\hookrightarrow} \cong \left[\mathrm{Path}[k+2], B^{\boxplus}\mathrm{O}\right]$ and that a $(k+2)$-path is determined on hom-ssets in dimensions $\leq k-1$, as follows:

$\underline{\mathcal{P}_0^{\Delta} \to \left[\mathrm{Path}[2], B^{\boxplus}\mathrm{O}\right]:}$

$$\mathbb{U}|\mathcal{P}_{\mathbf{1},0}^{\Delta} : (\gamma, 1) = \begin{array}{c} K \\ {\scriptstyle\gamma}\uparrow \\ W \boxplus V \end{array} \mapsto \begin{cases} \mathrm{N}_0(P_{0,a}^{\mathrm{op}}) \to B\mathrm{O}_0, & \underline{0,a} \mapsto \{a-1\}^*(\gamma, 1) = \begin{cases} V, & a = 1 \\ K, & a = 2 \end{cases} \\ \mathrm{N}_0(P_{1,2}^{\mathrm{op}}) \to B\mathrm{O}_0, & \underline{12} \mapsto \mathbf{F}(\gamma, j) = W \\ \mathrm{N}_1(P_{0,2}) \to B\mathrm{O}_1, & (\underline{012} > \underline{02}) \mapsto \gamma \end{cases}$$

$\underline{\mathcal{P}_1^{\Delta} \to \left[\mathrm{Path}[3], B^{\boxplus}\mathrm{O}\right]:}$

[1]This is essentially a variant of Notation 4.12.

Induced faces. Let $(\gamma, j) \in \mathcal{P}_1^{\Delta}$. We first define the faces of $\mathbb{U}(\gamma, j)$. The three faces $d_{0,1,2}^{\mathcal{V}^{\hookrightarrow}}\mathbb{U}(\gamma, 2)$ are defined by $\mathbb{U}|\mathcal{E}\mathcal{X}_0$ via simpliciality, i.e., by

$$d_i^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}(\gamma, j)) := \mathbb{U}(d_i^{\mathcal{E}\mathcal{X}}(\gamma, j)). \tag{11}$$

This fixes $\mathbb{U}(\gamma, j)$, by Lemma 2.7, on the subcategories $\mathrm{Path}[0, k, l] \cong \mathrm{Path}[2]$ of $\mathrm{Path}[3]$, for $1 \le k < l \le 3$ (Notation 2.5).

The top face. The remaining $\mathcal{B}^{\boxplus}\mathrm{O}$-face $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}(\mathbb{U}(\gamma, j))$ is the restriction to $\mathrm{Path}[1, 2, 3]$. The edges of $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}(\mathbb{U}(\gamma, j)) \in \mathcal{B}^{\boxplus}\mathrm{O}_2 \cong \mathcal{V}_1^{\hookrightarrow}$ are already specified by the $d_i^{\mathcal{V}^{\hookrightarrow}}(\mathbb{U}(\gamma, j))$, so only

$$\begin{aligned} \mathrm{N}_1(P_{1,3}^{\mathrm{op}}) &\to B\mathrm{O}_1 \\ (\underline{23} \cup \underline{12} > \underline{13}) &\mapsto \big(\mathbb{U}_{(\gamma,j)}(\underline{23}) \boxplus \mathbb{U}_{(\gamma,j)}(\underline{12}) \to \mathbb{U}_{(\gamma,j)}(\underline{13})\big) \\ &= (\mathbb{U}(d_0 d_0(\gamma, j)) \boxplus \mathbb{U}(d_0 d_2(\gamma, j)) \to \mathbb{U}(d_0 d_1(\gamma, j))) \end{aligned}$$

remains. This is determined by $\mathbf{F}$:

$$\mathbb{U}_{(\gamma,j)}|\mathrm{N}_1(P_{1,3}^{\mathrm{op}}) := \mathrm{Op}\mathbf{F}(\gamma, j). \tag{12}$$

Remark 2.8 (interrupting the proof). We should note that it is immaterial that (12) is 'not functorial' (although $\mathbb{U}$ will be). As noted in Lemma 2.9, the direct sums appearing in the $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}$-face are trivial in that all summands but one are zero, the non-zero one being determined by the exit index j. We use $\mathrm{Op}\mathbf{F}$ to supply *only the path* in $B\mathrm{O}(m)$. We have $\mathbb{U}_{\gamma,j}(\underline{23}) = 0$ if $j = 1$, and $\mathbb{U}_{\gamma,j}(\underline{12}) = 0$ if $j = 2$. In the former case, the edges of $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}$ are specified by simpliciality as in

$$\begin{tikzcd} & 2 \arrow[dr, "0"] & \\ 1 \arrow[ur, "W"] \arrow[rr, "W"] & & 3 \end{tikzcd},$$

and $\mathrm{Op}\mathbf{F}(\gamma, 1)$ is $\mathrm{Op}(\mathrm{id}_W) = \mathrm{id}_W \colon W = W \boxplus 0 \to W$. Here, (12) happens to be functorial as $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}$ happens to lie in $\mathcal{V}^{\sim}$. In the latter case, the filler of

$$\begin{tikzcd} & 2 \arrow[dr, "W'"] & \\ 1 \arrow[ur, "0"] \arrow[rr, "W"] & & 3 \end{tikzcd}$$

is supplied by $\mathrm{Op}\mathbf{F}(\gamma, 2) = \mathrm{Op}(\gamma_W) = \gamma_W^{-1} \colon W' = 0 \boxplus W' \to W$. This breaks functoriality, since $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}$ is *not* invertible. Still, the path is as desired.

1**-paths induced by functoriality.** Some 1-paths in the image of $\mathbb{U}(\gamma, j)$ are determined by the data provided thus far and by imposing functoriality. Namely, we have the following decompositions:

$$
\begin{aligned}
(\mathrm{i}) = (\underline{0123} > \underline{023}) &= (\underline{23} \cup \underline{012} > \underline{23} \cup \underline{02}) \\
&= \mathrm{id}_{\underline{23}} \cup [\underline{012} > \underline{02}] \\
&\in \mathrm{Im}\left(\cup : \mathrm{N}_1(P^{\mathrm{op}}_{2,3}) \times \mathrm{N}_1(P^{\mathrm{op}}_{0,2}) \to \mathrm{N}_1(P^{\mathrm{op}}_{0,3})\right), \\
(\mathrm{i})' = (\underline{0123} > \underline{013}) &= (\underline{123} \cup \underline{01} > \underline{13} \cup \underline{01}) \\
&= [\underline{123} > \underline{13}] \cup \mathrm{id}_{\underline{01}} \\
&\in \mathrm{Im}\left(\cup : \mathrm{N}_1(P^{\mathrm{op}}_{1,3}) \times \mathrm{N}_1(P^{\mathrm{op}}_{0,1}) \to \mathrm{N}_1(P^{\mathrm{op}}_{0,3})\right).
\end{aligned}
$$

Thus, functoriality imposes

$$
\begin{aligned}
\mathbb{U}_{(\gamma,j)}(\mathrm{i}) &= \mathrm{id}_{\mathbb{U}_{(\gamma,j)}(\underline{23})} \boxplus \mathbb{U}_{(\gamma,j)}(\underline{012} > \underline{02}) \\
\mathbb{U}_{(\gamma,j)}(\mathrm{i})' &= \mathbb{U}_{(\gamma,j)}(\underline{123} > \underline{13}) \boxplus \mathrm{id}_{\mathbb{U}_{(\gamma,j)}(\underline{01})}
\end{aligned}
$$

where the first non-constant summand is determined by (11) and the second by (12). In particular, the ad hoc assignments in (8) and (9) were correct.

Remining 1**-path and** 2**-paths.** It remains to specify $\mathbb{U}(\gamma, j)$ on the 'long path' in $\mathrm{N}_1(P^{\mathrm{op}}_{0,3})$ and on $\mathrm{N}_2(P^{\mathrm{op}}_{0,3})$. In contrast to the paths induced by functoriality, $\underline{03}$ is $\cup$-simple (not a composition), so

$$(\mathrm{ii}) = (\underline{0123} > \underline{03})$$

presents a genuine choice, and was not handled systematically in Section 2.2.2. It is considered most naturally in conjunction with the two non-degenerate elements in $N_2(P^{\mathrm{op}}_{0,3})$ to be mapped, as it is their (necessarily-)common composition:

$$
\begin{tikzcd}
& \underline{0123} \arrow[dl, "\mathrm{id}_{\underline{23}}\cup[\underline{012}>\underline{02}]"'] \arrow[dr, "{[\underline{123}>\underline{13}]\cup\mathrm{id}_{\underline{01}}}"] \arrow[dd, dotted, "(\mathrm{ii})" description] & \\
\underline{023} \arrow[dr, "(11)"'] & & \underline{013} \arrow[dl, "(11)"] \\
& \underline{03} &
\end{tikzcd}
$$

First, note that, regardless of exit index, this square decomposes into two triangles:

$$
\begin{tikzcd}
& \underline{0123} \arrow[dl, "\mathrm{id}_{\underline{23}}\cup[\underline{012}>\underline{02}]"'] \arrow[dr, "{[\underline{123}>\underline{13}]\cup\mathrm{id}_{\underline{01}}}"] & \\
\underline{023} \arrow[dr, "(11)"'] & & \underline{013} \arrow[ll, "d_2\gamma" description] \arrow[dl, "(11)"] \\
& \underline{03} &
\end{tikzcd}
$$

For $j = 2$ (using the labels in (7)), this reads

$$\begin{tikzcd}
& W' \boxplus 0 \boxplus V = W' \boxplus V \arrow[dl, "\mathrm{id}_{W'} \boxplus \gamma_V"'] \arrow[dr, "\gamma_W^{-1} \boxplus \mathrm{id}_V"] & \\
W' \boxplus V' \arrow[dr, "\gamma_{\boxplus'}"'] & & W \boxplus V \arrow[ll, "\gamma_W \boxplus \gamma_V"'] \arrow[dl, "\gamma_{\boxplus}"] \\
& K &
\end{tikzcd},$$

and for $j = 1$ (using the labels in (10)),

$$\begin{tikzcd}
& 0 \boxplus W \boxplus V = W \boxplus V \arrow[dl, "\mathrm{id}_0 \boxplus \gamma_{\boxplus} = \gamma_{\boxplus}"'] \arrow[dr, "\mathrm{id}_{0 \boxplus W} \boxplus \mathrm{id}_V = \mathrm{id}_W \boxplus \mathrm{id}_V"] & \\
0 \boxplus K = K \arrow[dr, "\gamma_K"'] & & W \boxplus V \arrow[ll, "\gamma_{\boxplus}"'] \arrow[dl, "\gamma_{\boxplus'}"] \\
& K' &
\end{tikzcd}.$$

For both indices, the bottom triangle is filled by γ itself, and the top one has a canonical filler. This suggest assigning to (ii) the outer-left concatenation, $\mathbb{U}_{(\gamma,j)}(\mathrm{ii}) = \mathbb{U}_{(\gamma,j)}(\underline{0123} > \underline{023}) * \mathbb{U}_{(\gamma,j)}(\underline{023} > \underline{03})$. Accordingly, $\mathbb{U}(\gamma, j)|\mathrm{N}_2(P^{\mathrm{op}}_{0,3})$ is determined by said fillers. This concludes the construction of $\mathbb{U}|\mathcal{P}^{\Delta}_1$.

The induction step. Assume now that the $\mathbb{U}|\mathcal{P}^{\Delta}_{<k}$ have been constructed simplicially.

$\underline{\mathcal{P}^{\Delta}_k \to \left[\mathrm{Path}[k+2], B^{\boxplus}\mathrm{O}\right]}$: We have constructed $\mathbb{U}|\mathcal{P}^{\Delta}_1$ as independently of exit indices as possible, and the same ideas apply here mutatis mutandis.

Let $(\gamma, j) \in \mathcal{P}^{\Delta}_k$.

Induced faces. The restriction of $\mathbb{U}(\gamma, j)$ to the subcategories $\mathrm{Path}[0, k_1, \dots, k_{k+1}] \cong \mathrm{Path}[k+1]$ of $\mathrm{Path}[k+2]$, $1 \le k_\ell < k_{\ell'} \le k+2 \ \forall 1 \le \ell < \ell' \le k+1$, are determined by setting

$$d_i^{\mathcal{Y}^{\hookrightarrow}}(\mathbb{U}(\gamma, j)) = \mathbb{U}(d_i^{\mathcal{E}\mathcal{X}}(\gamma, j)). \tag{13}$$

The top face. The restriction $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}(\mathbb{U}(\gamma, j))$ to $\mathrm{Path}[1, \dots, k+2]$ is given, in view of Lemma 2.9, by $\mathrm{Op}\mathbf{F}(\gamma, j)$.

The rest of $P^{\mathrm{op}}_{0,k+2}$. All arrows in $P^{\mathrm{op}}_{0,k+2}$ out of the initial $\underline{0, \dots, k+2} = [k+2]$ except for $\underline{0, \dots, k+2} > \underline{0, k+2}$ are clearly $\cup$-composite and are thus determined by (13) and functoriality. This is well-defined due to the inductive assumption that so is $\mathbb{U}|\mathcal{P}^{\Delta}_{<k}$.

We set

$$\begin{aligned}
\mathbb{U}_{(\gamma,j)}\left([k+2] > \underline{0, k+2}\right) &= \mathbb{U}_{(\gamma,j)}\left([k+2] > [k+2] \setminus \{1\}\right) \\
&\quad * \mathbb{U}_{(\gamma,j)}\left([k+2] \setminus \{1\} > [k+2] \setminus \{1, 2\}\right) \\
&\quad * \cdots \\
&\quad * \mathbb{U}_{(\gamma,j)}\left(\underline{0, k+1, k+2} > \underline{0, k+2}\right).
\end{aligned}$$

Among the $\cup$-simple ones, there are $k+1$ arrows $\underline{0, i, k+2} > \underline{0, k+2}$. In the $(k+1)$-cube $P_{0,k+2}$, the $(k+1)$-simplex with vertices the domains $\underline{0, i, k+2}$, and finally $\underline{0, k+2}$, is filled by $\gamma \in B\mathrm{O}(n+m)_{k+1}$ according to the rule $\alpha \mapsto \underline{0, \alpha+1, k+2}$ for $k+2 > \alpha \in [k+2]$, and

$k+2 \mapsto \underline{0, k+2}$. Thus, in any chain in $\mathrm{N}_{k+1}(P^{\mathrm{op}}_{0,k+2})$ from $[k+2]$ to $\underline{0, k+2}$, the concatenation of the images of the short arrows is homotopic to $\mathbb{U}_{(\gamma,j)}([k+2] > \underline{0, k+2})$, which provides $\mathbb{U}(\gamma, j)|\mathrm{N}_{k+1}(P^{\mathrm{op}}_{0,k+2})$. This concludes the construction of $\mathbb{U}$.

Admittedly, the construction above is more brute-force than elightening. It does not rely on universal properties, which might be explained by the low-level (in the programming sense) status it has in linked space theory, and, as such, does not benefit from higher-level universal-property arguments. Essentially, we force the construction inductively from the lowest non-trivial assignment $\mathcal{P}^{\Delta}_0 \to \left[\mathrm{Path}[2], B^{\boxplus}\mathrm{O}\right]$, guided by an explicit systematisation of the first 'induction step', $\mathcal{P}^{\Delta}_1 \to \left[\mathrm{Path}[3], B^{\boxplus}\mathrm{O}\right]$ – whose redressed presentation was, strictly speaking, not necessary – which illustrates the fundamental idea quite well already.

The reason why the construction then works out in all dimensions remains slightly mysterious: the core phenomenon leveraged is buried in the argument that we use to construct the restriction $\mathrm{N}_\bullet(P^{\mathrm{op}}_{0,k+2}) \to B\mathrm{O}_\bullet$ of $\mathbb{U}(\gamma, j)$, $(\gamma, j) \in \mathcal{P}^{\Delta}_k$, once all $\mathcal{B}^{\boxplus}\mathrm{O}$-faces have been forced by lower dimensions and simpliciality. Here, recognising $P^{\mathrm{op}}_{0,k+2}$ to be a cube of dimension $k+1$ (as was also noted in [Lur09, §1.1.5], [Lur23, 00LM]), we observe that $\gamma \in B\mathrm{O}(n+m)_{k+1}$ itself provides a filler for a particular $(k+1)$-simplex within this cube (the one with vertices $\underline{0, i, k+2}$, $1 < i < k+2$, and $\underline{0, k+2}$). The rest of the cube has a canonical filler, which has, crucially, to do with the inversion of the paths originating in the pr-fibre $B\mathrm{O}(m)$ (paths like $\gamma^{-1}_W : W' \to W$), necessitated above by the edge arrangements in $\mathcal{B}^{\boxplus}\mathrm{O}$. So, as disconcerting as this might have seemed above, it proves absolutely essential.

As a final illustration, consider an exit 3-path of index 3:

$$\mathcal{P}^{\Delta}_2 \ni (\gamma, 3) = \begin{tikzcd} & & K \\ & W_1 \boxplus V_1 \arrow[ur, dotted, tail] \arrow[dr, dotted, tail] & \\ W_0 \boxplus V_0 \arrow[rr] \arrow[ur, dotted, tail] \arrow[uurr, bend left] & & W_2 \boxplus V_2 \arrow[uu] \end{tikzcd} .$$

The (1d) edges of $\mathbb{U}(\gamma, 3) \in \mathcal{B}^{\boxplus}\mathrm{O}_4$ are given, due to (13), as follows:

$$\begin{cases} \underline{01} \mapsto V_0,\ \underline{02} \mapsto V_1,\ \underline{03} \mapsto V_2,\ \underline{04} \mapsto K, \\ \underline{14} \mapsto W_0,\ \underline{24} \mapsto W_1,\ \underline{34} \mapsto W_2 \end{cases}$$

and the remaining edges are zero. Now, the 3-cube $P_{0,4}^{\mathrm{op}}$ and its image under $\mathbb{U}(\gamma, j)$ look as follows:

$$
\begin{tikzcd}[column sep=small]
& \underline{01234} \arrow[rr] \arrow[dl] \arrow[dd] & & \underline{0124} \arrow[dl] \arrow[dd] \\
\underline{0234} \arrow[rr, crossing over] \arrow[dd] & & \underline{024} & \\
& \underline{0134} \arrow[rr] \arrow[dl] & & \underline{014} \arrow[dl] \\
\underline{034} \arrow[rr] & & \underline{04} \arrow[from=uu, crossing over] &
\end{tikzcd}
\mapsto
\begin{tikzcd}[column sep=tiny]
& W_2 \boxplus V_0 \arrow[rr, "\gamma^{-1}_{W12}\boxplus\mathrm{id}"] \arrow[dl, "\mathrm{id}\boxplus\gamma_{V01}"'] \arrow[dd, equals] & & W_1 \boxplus V_0 \arrow[dl, "\mathrm{id}\boxplus\gamma_{V01}"'] \arrow[dd, "\gamma^{-1}_{W01}\boxplus\mathrm{id}"] \\
W_2 \boxplus V_1 \arrow[rr, crossing over, "\mathrm{id}^{-1}_{W12}\boxplus\mathrm{id}"] \arrow[dd, "\mathrm{id}\boxplus\gamma_{V12}"'] & & W_1 \boxplus V_1 & \\
& W_2 \boxplus V_0 \arrow[rr, "\gamma^{-1}_{W}\boxplus\mathrm{id}"] \arrow[dl, "\mathrm{id}\boxplus\gamma_{V02}"'] & & W_0 \boxplus V_0 \\
W_2 \boxplus V_2 \arrow[rr, "\gamma_{\boxplus 2}"] & & K \arrow[from=uu, crossing over, "\mathrm{id}_{\boxplus 1}"] \arrow[ur, "\gamma_{\boxplus 0}"'] &
\end{tikzcd}
.
$$

Painting in the outer-left concatenation chosen to be the image of $\underline{01234} > \underline{04}$ (in green) and the (edges of the) 'lower' tetrahedron given by $\gamma \in BO_3$ itself (in blue), we see that

$$
\begin{tikzcd}[column sep=small]
& W_2 \boxplus V_0 \arrow[rr] \arrow[dl, green] \arrow[dd, equals] & & W_1 \boxplus V_0 \arrow[dl] \arrow[dd] \\
W_2 \boxplus V_1 \arrow[rr, crossing over] \arrow[dd, green] & & W_1 \boxplus V_1 \arrow[dd, blue] \arrow[ddll, blue, "\gamma_{W12}\boxplus\gamma_{V12}"'] & \\
& W_2 \boxplus V_0 \arrow[rr] \arrow[dl] & & W_0 \boxplus V_0, \arrow[ul, blue, "\gamma_{W01}\boxplus\gamma_{V01}"'] \arrow[dl, blue] \arrow[dlll, blue, "\gamma_{W02}\boxplus\gamma_{V02}"] \\
W_2 \boxplus V_2 \arrow[rr, cyan] & & K &
\end{tikzcd}
$$

homotopy-commutes by inspection.

Lemma 2.9 (ad $d_0^{\mathcal{B}^{\boxplus}\mathrm{O}}\mathbb{U}(\gamma, j)$)**.**

Proof. □

2.3. **Classifying maps of linked tangent bundles.** We start with a technical observation that lets us give the tangent bundle of a linked by means of unstratified data, which formalises an informal discussion present in [AFR18b, §2.1.4] in the conically-smooth setting. From now on, we assume all smooth manifolds Hausdorff and paracompact.

Let $\iota\colon L \hookrightarrow N$ be a clossed embedding of smooth manifolds, and $E \to N$ a rank-$(n+m)$ vector bundle classified by $E\colon N \to BO(n+m)$, equipped with the inner product induced by that on the separable Hilbert space $H \cong \mathbb{R}^\infty$ used to construct the Grassmannians $BO(k) = \mathrm{Gr}_k(H)$. Let further E_0 be a rank-n vector sub-bundle of ι^*E, classified by $E_0\colon L \to BO(n)$. The pullback bundle itself is classified by $\iota^*E\colon L \hookrightarrow N \to BO(n+m)$.

The normal bundle $E_0^\perp \subset \iota^*E$, classified by $E_0^\perp\colon L \to BO(m)$, satisfies $E_0 \oplus E_0^\perp \cong \iota^*E$. It is classical that the Whitney sum is classified as follows: Consider the isomorphism

$$\Phi\colon H \oplus H \cong H$$

given by sending, with respect to a fixed basis of H indexed over $\mathbb{N}$, the first copy to odd coordinates and the second copy to even coordinates. The (abstract) direct sum precomposes with this isomorphism to give a map

$$E_0^\perp \widetilde{\boxplus} E_0 \colon L \xrightarrow{E_0 \times E_0^\perp} \mathrm{Gr}_n(H) \times \mathrm{Gr}_m(H) \xrightarrow{\boxplus} \mathrm{Gr}_{n+m}(H \oplus H) \overset{\Phi}{\cong} \mathrm{Gr}_{n+m}(H) = B\mathrm{O}(n+m).$$

The classifier $\oplus_W \colon L \to B\mathrm{O}(n+m)$ of the Whitney sum $E_0 \oplus E_0^\perp$ is then homotopic to $E_0^\perp \widetilde{\boxplus} E_0$.

Let us concatenate this homotopy with the standard one from $\iota^* E$ to $\oplus_W$ (classifying the inverse of the bundle isomorphism $E_0 \oplus E_0^\perp \cong \iota^* E$ given fibrewise by $(v, w) \mapsto v + w$) to obtain a homotopy

$$h \colon \iota^* E \to \oplus_W \to E_0^\perp \widetilde{\boxplus} E_0,$$

of maps $L \to B\mathrm{O}(n+m)$, which sits in the commutative diagram

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, hook, "\iota"'] & B\mathrm{O}(n+m)^I \arrow[d, "\mathrm{ev}_0"] \\ N \arrow[r, "E"'] \arrow[ur, dotted, "H"] & B\mathrm{O}(n+m) \end{tikzcd} .$$

As ι, being a closed embedding, is a cofibration, there exists a homotopy extension $H \colon N \to B\mathrm{O}(n+m)^I$ as depicted. We may now consider

$$E' \coloneqq H_1 \colon N \to B\mathrm{O}(n+m)$$

and apply the inverse isomorphism $\Phi^{-1} \colon H \cong H \oplus H$ to obtain

$$\Phi^{-1} E' \colon N \to \mathrm{Gr}_n(H \oplus H).$$

On the other hand, applying Φ^{-1} to $E_0^\perp \widetilde{\boxplus} E_0$ recovers $E_0^\perp \boxplus E_0 = \boxplus \circ E_0 \times E_0^\perp$. Therefore the two classifiers

$$E_0^\perp \boxplus E_0, \iota^* E' \colon L \to \mathrm{Gr}_{n+m}(H \oplus H)$$

coincide on-the-nose. Below, we will denote E' simply by E since the two are isomorphic/homotopic. We have proved:

Lemma 2.10. *Let*

- $\iota \colon L \hookrightarrow N$ *be a closed embedding of smooth manifolds,*
- $E \to N$ *a rank-*$(n+m)$ *vector bundle equipped with an inner product,*
- *and* $E_0 \hookrightarrow \iota^* E$ *a rank-*n *vector sub-bundle.*

Then there exists a classifier $E \colon N \to \mathrm{Gr}_{n+m}(H \oplus H)$ *of the isomorphism class of* $E \to N$ *such that the diagram*

$$\begin{tikzcd} L \arrow[r, "E_0 \times E_0^\perp"] \arrow[d, hook, "\iota"'] & \mathrm{Gr}_n(H) \times \mathrm{Gr}_m(H) \arrow[d, "\boxplus"] \\ N \arrow[r, "E"'] & \mathrm{Gr}_{n+m}(H \oplus H) \end{tikzcd} \tag{14}$$

commutes. [2]

Notation 2.11. We will sometimes write simply $B\mathrm{O}(k)$ for $\mathrm{Gr}_k(H^{\oplus -}) \cong B\mathrm{O}(k)$ for any countable number of copies of H, and therefore abuse notation in diagrams of type (14).

[2] The point being that it doesn't just homotopy-commute.

Let now $\mathfrak{S} = \left(M \xleftarrow{\pi} L \xrightarrow{\iota} N\right)$ be a linked manifold, with each manifold riemannian. As above, they are all assumed Hausdorff and paracompact. Given this contractible choice of metrics, we will show that there is a 'canonical' map

$$\mathrm{T}\mathfrak{S}\colon \mathfrak{S} \to BO(n, m) \tag{15}$$

of linked spaces, which we call the *tangent bundle* of $\mathfrak{S}$.

Construction 2.12. Let $\mathfrak{S}$ be as above. Since $\mathrm{d}\pi$ surjects, the induced linear dual map $(\pi^*\mathrm{T}M)^\vee \hookrightarrow (\mathrm{T}L)^\vee$, of bundles over L, injects. Using the metrics, this gives an injection $\pi^*\mathrm{T}M \hookrightarrow \mathrm{T}L$. Composing with $\mathrm{d}\iota$, we have a bundle injection

$$\pi^*\mathrm{T}M \hookrightarrow \mathrm{T}L \hookrightarrow \iota^*\mathrm{T}N$$

over L. Let us denote the normal bundle of this injection by

$$\mathrm{N} := \mathrm{N}_N M := (\pi^*\mathrm{T}M)^\perp \subset \iota^*\mathrm{T}N.$$

Now, in the diagram

$$\begin{tikzcd}
{} & L \arrow[rr, "\pi^*\mathrm{T}M\times\mathrm{N}"] \arrow[ddl, two heads, "\pi"'] \arrow[dr, hook, "\iota"] & & BO(n)\times BO(m) \arrow[ddl, two heads, "\mathrm{pr}"'] \arrow[dr, hook, "\boxplus"] & \\
& & N \arrow[rr, "\mathrm{T}N"] & & BO(n+m) \\
M \arrow[rr, "\mathrm{T}M"] & & BO(n) & &
\end{tikzcd} \tag{16}$$

the back square

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & BO(n)\times BO(m) \arrow[d] \\
N \arrow[r] & BO(n+m)
\end{tikzcd}$$

commutes using Lemma 2.10 and Notation 2.11, and the front square commutes trivially. This yields the span map (15).

Remark 2.13. Writing $\mathrm{N}_L M := (\pi^*\mathrm{T}M)^\perp \subset \mathrm{T}L$, we have a splitting $\mathrm{T}L \cong \pi^*\mathrm{T}M \oplus \mathrm{N}_L M$. Similarly, writing $\mathrm{N}_N L := (\mathrm{T}L)^\perp \subset \iota^*\mathrm{T}N$, we have a splitting $\iota^*\mathrm{T}N \cong \mathrm{T}L \oplus \mathrm{N}_N L \cong \pi^*\mathrm{T}M \oplus \mathrm{N}_L M \oplus \mathrm{N}_N L$. Thus

$$\mathrm{N}_N M \cong \mathrm{N}_L M \oplus \mathrm{N}_N L.$$

In practice, the bundle N is best determined in two steps via this decomposition.

Applying $\mathcal{E}\mathcal{X}$ and post-composing with $\mathbb{U}$, we have the induced map

$$\mathcal{E}\mathcal{X}(\mathfrak{S}) \to \mathcal{V}^{\hookrightarrow},$$

which is the linked version of the (conically-smooth) constructible tangent bundle.

Example 2.14. If L is induced by a closed submanifold inclusion $M \subset \overline{N}$ as $L = \mathbb{S}(\mathrm{N}_N M)$, the sphere bundle of the normal bundle ([Tet23, Example 2.6]), then L has dimension $n + m - 1$, $\mathrm{N}_L M$ has rank $m - 1$, and $\mathrm{N}_L N$ has rank 1. More specifically, in the conically-smooth context, the link (of a pair of strata) comes with an open embedding $L \times \mathbb{R} \hookrightarrow N$, which is tantamount to the triviality of the latter normal bundle, i.e., $\mathrm{N}_N L \simeq \varepsilon^1$, or, equivalently, to a diffeomorphism

$L \times \mathbb{R} \simeq \mathbb{S}(\mathrm{N}_N M) \times \mathbb{R}$. This $\mathbb{R}$-factor incarnates the extra $\mathbb{E}_1$-structure featuring in the classification of stratified locally-constant factorisation algebras on stratified spaces of type $M \subset \overline{N}$.

Example 2.15. An even simpler situation arises when L (and $\mathfrak{S}$) is induced by a boundary $M = \partial\overline{N} \subset \overline{N}$ as $L \cong M$, the boundary pushed diffeomorphically into the interior $N = \overline{N} \setminus M$ by following the flow of a nowhere-vanishing inward pointing vector field along the boundary (which always exists) for a chosen non-zero time ([Tet23, Example 2.5]); we will denote this closed link embedding later by ι_+. Then $\mathrm{N}_L M = 0$ and $\mathrm{N}_N L \simeq \varepsilon^1$ again. The type of classification statement mentioned in Example 2.14 reduces here to Kontsevich's Swiss-Cheese conjecture (e.g., [Tho16]).

Definition 2.16. We call a linked manifold with M of dimension n and N of dimension $n+m$ *constructible* if L is of dimension $n+m-1$, and its normal bundle in N is trivialisable.

Making the linked tangent bundle (16) an on-the-nose span map may be justified by the fact that the only real homotopy involved in (the proof of) Lemma 2.10 is the classical one between $\iota^* E$ and $E_0^{\perp} \widetilde{\boxplus} E_0$ over L, which is canonical in the sense that it does not depend on E or E_0. This choice contains no geometric information, so it would be unwise to change the (n,m)-Grassmannian by taking a replacement just to remain agnostic about it. Besides, from a more practical point of view, the map $\mathbb{U} \colon \mathcal{E}\mathcal{X}(B\mathrm{O}(n,m)) \to \mathcal{V}^{\hookrightarrow}$ is natural only for this span $B\mathrm{O}(n,m)$.

3. Linked geometry

3.1. **Adapting AFR-type structures.** The ∞-category of *tangential structures* is the over-∞-category $\mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow}$ ([AFR18b][3]). Via

$$\mathbb{U}^* \colon \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{E}\mathcal{X}(B\mathrm{O}(n,m)),$$

these transfer to tangential structures on linked manifolds: given $\mathfrak{S}$, and writing $\mathcal{B}_{(n,m)} := \mathbb{U}^*\mathcal{B}$, we may define the space (homotopy type) of $\mathcal{B}$-structures on $\mathfrak{S}$ to be

$$\mathcal{B}\text{-red}(\mathfrak{S}) := \mathrm{Map}_{/B\mathrm{O}(n,m)}\left(\mathcal{E}\mathcal{X}(\mathfrak{S}), \mathcal{B}_{(n,m)}\right),$$

the mapping space in $\mathcal{C}\mathrm{at}_\infty/\mathcal{E}\mathcal{X}(B\mathrm{O}(n,m))$, where the first argument uses $\mathrm{T}\mathfrak{S}$ (Construction 2.12). Equivalently,

$$\mathcal{B}\text{-red}(\mathfrak{S}) = \Gamma\left((\mathrm{T}\mathfrak{S})^* \mathcal{B}_{(n,m)}\right),$$

the homotopy-sections of $(\mathrm{T}\mathfrak{S})^* \mathcal{B}_{(n,m)} \to \mathcal{E}\mathcal{X}(\mathfrak{S})$.

Given $\mathcal{B}$ and (n,m), the natural question is whether

$$\mathcal{B}_{(n,m)} = \mathcal{E}\mathcal{X}(\mathfrak{B}) \tag{17}$$

for a linked space $\mathfrak{B}$, which would enable us to discuss stratified tangential structures without having to refer to exit paths. We will restrict ourselves in this paper to the case where $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ is induced by a classical tangential structure by a Cartesian fibration replacement, defined in Section 3.4.

The reason we consider this problem at all is that such tangential structures are central to our considerations in Section 4, where we consider linked spaces induced by bordisms and defects,

[3]– see [Lur09, §3] for $\mathcal{C}\mathrm{at}_\infty$

equipped with Cartesian fibration replacements of classical tangential structures. These correspond essentially to bordisms with solid structures, as will become clear. Understanding these structures is crucial for an understanding of what we call quadratic duals, whose definition relies on normal bundles with respect to Cartesian structures. Besides applications, this is the main theoretical reason for our restriction to Cartesian structures: for arbitrary stratified tangential structures, it is not clear how to define normal bundles, if at all possible; cf. Remark 4.5.

With this restriction, we give in Section 3.4 a solution to problem (17) for $\mathfrak{S}$ smooth, i.e., consisting of a single stratum, and then in Section 3.5 a partial solution for arbitrary $\mathfrak{S}$.

We will first discuss the simplest example imaginable. To begin with, recall that for $\kappa \in \mathbb{N}$, rank-κ framings (*κ-framings*) are expressed by the tangential structure $\boldsymbol{\kappa}\colon * \to \mathcal{V}^{\hookrightarrow}$ that sends the point to $\boldsymbol{\kappa} := \mathbb{R}^{\kappa}$.

Example 3.1 (framings). We have

$$\boldsymbol{\kappa}_{(n,m)} = \begin{cases} \mathcal{E}\mathcal{X}\,(\emptyset \leftarrow \emptyset \to *) = *, & n+m=\kappa, \\ \emptyset, & \text{else} \end{cases}$$

with

$$\left(\boldsymbol{\kappa}_{(n,m)} \to \mathcal{E}\mathcal{X}\,(B\mathrm{O}(n,m))\right) = \mathcal{E}\mathcal{X}\left((\emptyset \leftarrow \emptyset \to *) \to B\mathrm{O}(n,m), * \xrightarrow{\boldsymbol{\kappa}} B\mathrm{O}(n+m)\right).$$

This reflects the fact that a nontrivially stratified space does not admit a κ-framing: the else-statement implies that for a linked space to admit a κ-framing its bulk must be κ-dimensional. The first statement implies moreover that for a lift of $\mathrm{T}\mathfrak{S}$ to $(\emptyset \leftarrow \emptyset \to *)$ to exist, the space must be of type $\mathfrak{S} = (\emptyset \leftarrow \emptyset \to N)$ (if non-empty), and $\dim N = \kappa$.

Similar considerations apply to any *classical tangential structure* $\mathfrak{b}\colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, i.e., one that factors through $B\mathrm{O}(\kappa) \hookrightarrow \mathcal{V}^{\sim} \hookrightarrow \mathcal{V}^{\hookrightarrow}$ for some k.

Example 3.2. Let $\mathfrak{b}$ be a classical tangential structure given by a map $B \to B\mathrm{O}(\kappa)$ of spaces, e.g., induced by a map $G \to \mathrm{O}(\kappa)$ of topological groups, or a rank-κ bundle $X \to B\mathrm{O}(\kappa)$ on a space X. Then,

$$B_{(n,m)} = \begin{cases} \mathcal{E}\mathcal{X}\,(\emptyset \leftarrow \emptyset \to B) = B, & n+m=\kappa, \\ \emptyset, & \text{else,} \end{cases}$$

where we abbreviated $\mathrm{Sing}_{\bullet}(B)$ to B in its last occurence.

Example 3.3. Consider $\mathbb{N} = (\mathbb{N}, \leq)$ with the standard order. Variframings ([AFR18b]) are given by $\mathrm{vfr}\colon \mathbb{N} \to \mathcal{V}^{\hookrightarrow}$, $k \mapsto \boldsymbol{k}$, $(k \leq K) \mapsto (\boldsymbol{k} \xhookrightarrow{-\oplus 0} \boldsymbol{K})$. We read $\mathrm{vfr}(k \leq K)$ as the standard[4] isomorphism $\boldsymbol{k} \boxplus (\boldsymbol{K} - \boldsymbol{k}) \cong \boldsymbol{K}$. Let us restrict vfr to depth 1 by choosing a pair $n \leq N$, i.e., consider $\mathrm{vfr}|_{n \leq N}\colon \{n \leq N\} \to \mathcal{V}^{\hookrightarrow}$. Then, for $m = N - n$, we have

$$\mathbb{U}^*(\mathrm{vfr}|_{n \leq N}) \simeq \mathcal{E}\mathcal{X}(* \leftarrow * \to *) \simeq \Delta[1],$$

[4]Up to, of course, the choice of a pairing function $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$

the exit path ∞-category of the nontrivially-linked point. Moreover, $\mathbb{U}^*(\mathrm{vfr}|_{n\le N}) \to \mathcal{EX}(B\mathrm{O}(n,m))$ is $\mathcal{EX}$ of

$$\begin{tikzcd}
 & & * \arrow[dll] \arrow[d, "\boldsymbol{n\times m}"] \arrow[drr] & & \\
* \arrow[d, "\boldsymbol{n}"] & & B\mathrm{O}(n)\times B\mathrm{O}(m) \arrow[dll, "\mathrm{pr}"'] \arrow[drr, "\boxplus"] & & * \arrow[d, "\boldsymbol{N}"] \\
B\mathrm{O}(n) & & & & B\mathrm{O}(N)
\end{tikzcd} .$$

Thus, a *variframing* on $\mathfrak{S} = (M \leftarrow L \hookrightarrow N)$, i.e., a lift of $\mathrm{T}\mathfrak{S}$ to this $\Delta[1]$, is a framing on M, a framing on N, and a framing on $\mathrm{N}_N M$.

Example 3.4 (point defects). The choice of a point p in a smooth manifold N of dimension n and a coordinate neighbourhood around it induce a linked space

$$\mathfrak{N}_p := \Big(\{p\} \leftarrow S^{n-1} \xhookrightarrow{\iota_p=\iota} N \smallsetminus \{p\}\Big)$$

where the sphere is the unit sphere in coordinates. The link map of $\mathrm{T}\mathfrak{N}_p$ reads

$$\varepsilon^0 \times \Big(\mathrm{T}S^{n-1} \oplus \mathrm{N}(\iota)\Big) : S^{n-1} \to * \times B\mathrm{O}(n),$$

i.e., $\iota^*\mathrm{T}\left(N \smallsetminus \{p\}\right) : S^{n-1} \to B\mathrm{O}(n)$. A $\mathrm{vfr}_{0\le n}$-structure on $\mathfrak{N}_p$ is a framing on N together with a framing on the normal bundle of ι. In this example, the latter always exists, which is why we will call such a configuration a trivial point defect.

Two relaxations of the tangential structure $\boldsymbol{\kappa}$ (or of any classical structure) are crucial for our purposes. They are termed, in increasing order of generality, the *stable* and *solid* replacements.

3.2. **Linked morphism spaces.** Before proceeding to solid/stable structures in the linked setting, we need to identify the novel morphism spaces in the linked setting.

Notation. Given an embedding $\iota\colon \Sigma \hookrightarrow N$ and a point $q \in N$, we let $P(N)_{\Sigma,q} = P_{\Sigma,q}$ denote the space of paths in N that start in $\iota(\Sigma)$ and end in the point q, equipped with the compact-open topology. We use analogous notation when we work with a cofibration ι of simplicial sets.

The following result formalises and confirms the intuition that the link represents an infinitesimal expansion of the lower stratum into the higher stratum.

Theorem 3.5. *Let* $\mathfrak{S} = \Big(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\Big)$ *be a linked space, and* $p \in \mathcal{M}$ *and* $q \in \mathcal{N}$ *points in the two strata. We then have an equivalence*

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(p,q) \simeq \mathcal{P}_{\mathcal{L}_p,q}$$

between the morphism space in $\mathcal{EX}$ *from* p *to* q *and that of paths in* $\mathcal{N}$ *that start in* $\iota(\mathcal{L}_p)$ *(with ordinary fibre* $\mathcal{L}_p = \{p\} \times_{\mathcal{M}} \mathcal{L}$*) and end in* q*.*

Proof. We will work with a model for morphism spaces that makes the proof particularly simple: by [Lur23, 01L5], the morphism space in $\mathcal{EX}$ is equivalent to the *right-pinched* morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{EX}}(p,q) := \{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$. We will observe directly that $\{p\} \times_{\mathcal{EX}} (\mathcal{EX}/q)$ is in fact isomorphic to $\mathcal{P}_{\mathcal{L}_p,q} = \mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q)$. The proof of this is essentially contained in the proof of [Tet23, Theorem 2.3], but we will extract and rewrite it here in order to make an independent reading possible.

Indeed, at vertex level, the bijection

$$(\{p\} \times_{\mathcal{E}\mathcal{X}} (\mathcal{E}\mathcal{X}/q))_0 \cong \left(\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q)\right)_0$$

is clear: recalling that non-invertible 1-paths in $\mathcal{E}\mathcal{X}$ are elements of $\mathcal{P}_0^{\Delta} \subset \mathcal{N}_1$ (as the exit index is necessarily 1 in this degree), let $(\gamma, 1) \in \mathcal{P}_0^{\Delta}$. For $p = d_1^{\mathcal{E}\mathcal{X}}(\gamma, 1) \overset{\text{def}}{=} \pi(d_1^{\mathcal{N}}(\gamma))$ to hold, we must have $d_1^{\mathcal{N}}(\gamma) \in \iota(\mathcal{L}_p)$. Similarly, $d_0^{\mathcal{E}\mathcal{X}}(\gamma, 1) \overset{\text{def}}{=} \iota(d_0^{\mathcal{N}}(\gamma))$, which yields the bijection.

Let now $k > 0$ and consider an exit $(k+1)$-path $(\gamma\colon \Delta[k] \star \Delta[0] \to \mathcal{N}, j)$ in $(\mathcal{E}\mathcal{X}/q)_k \subset \mathcal{E}\mathcal{X}_{k+1}$. Asking that (γ, j) be in $\{p\} \times_{\mathcal{E}\mathcal{X}} (\mathcal{E}\mathcal{X}/q)$ is equivalent to asking that

(1) its $\mathcal{N}$-face

$$\Delta[k] \hookrightarrow \Delta[k] \star \Delta[0] \xrightarrow{\gamma} \mathcal{N},$$

which is $d_{k+1}^{\mathcal{N}}(\gamma)$ under the standard identification $\Delta[k] \star \Delta[0] \simeq \Delta[k+1]$, is bottom, as by construction only then can the corresponding $\mathcal{E}\mathcal{X}$-face be in $\mathcal{M}_k \subset \mathcal{E}\mathcal{X}_k$;

(2) and that it lies in particular in $\iota(\mathcal{L}_p)$.

Condition (1) implies moreover that $d_\ell^{\mathcal{N}}(\gamma) \in \mathcal{N}_k$ is vertical for all $\ell < k+1$, since all other faces include the tip $\Delta[0] \hookrightarrow \Delta[k] \star \Delta[0]$ given by q, whence they are necessarily not bottom; and if some $d_\ell(\gamma)$ was top, that would contradict the bottomness of its (unique) common $(k-1)$-face with $d_{k+1}^{\mathcal{N}}(\gamma)$. In fact, (γ, j) has no n-face that is top once $n > 0$: given $\Delta[n] \hookrightarrow \Delta[k+1]$, there is necessarily a vertex in $d_{d+1}^{\mathcal{N}}(\gamma)$ that is hit by it.

But then the exit index j must be maximal: $j = k+1$. For if not, then there would exist at least one top n-face for $n > 0$, the largest such, with $n = k+1-j$, for instance, being specified by $[n] \hookrightarrow [k+1]$, $\alpha \mapsto \ell + \alpha$. We thus obtain a bijection

$$(\{p\} \times_{\mathcal{E}\mathcal{X}} (\mathcal{E}\mathcal{X}/q))_k \cong \left(\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q)\right)_k$$

in a fashion similar to the bijection of vertices: we have reduced exit paths (γ, j) in question on the LHS to those of index $k+1$, and so to only a subset of $\mathcal{N}_{k+1}$, and specifically those such that $d_{k+1}^{\mathcal{N}}(\gamma) \in \mathcal{L}_p$. These are exactly the elements of the RHS. Finally, it is a direct check that $(\{p\} \times_{\mathcal{E}\mathcal{X}} (\mathcal{E}\mathcal{X}/q))_* \xrightarrow{\cong} \left(\mathcal{L}_p \times_{\mathcal{N}} (\mathcal{N}/q)\right)_*$ is functorial; for instance, any vertical face of such a $(\gamma, k+1)$ is again of maximal index: we have $\flat_{k+1,i}^{k+1} = k$ and, and as for degeneracies, $\sharp_{k+1,i}^{k+1} = k+2$, for all $i < k+1$ ([Tet23, §2]). □

We will explore some immediate and useful consequences of this result. In the proofs, we will use Theorem 3.5 without mention.

Corollary 3.6. *Let $\mathfrak{S} = \left(\mathcal{M} \xleftarrow{\pi} \mathcal{L} \xrightarrow{\iota} \mathcal{N}\right)$ be a linked space with $\mathcal{M}$ and $\mathcal{N}$ connected, and $p \in \mathcal{M}$, $q \in \mathcal{N}$. Then, π is an equivalence if and only if*

$$\mathrm{Hom}_{\mathcal{E}\mathcal{X}(\mathfrak{S})}(p, q) \simeq \Omega\mathcal{N}.$$

Here, $\Omega\mathcal{N}$ denotes the based loop space of $\mathcal{N}$.

Proof. The fibre at any point of the source evaluation $\mathcal{P}_{\mathcal{L}_p, q} \to \mathcal{L}_p$ is equivalent to $\Omega\mathcal{N}$. Thus, the homotopy long exact sequence of this fibration implies that the fibre inclusion induces isomorphisms $\pi_*(\Omega\mathcal{N}) \cong \pi_*(\mathrm{Hom}_{\mathcal{E}\mathcal{X}(\mathfrak{S})}(p, q))$ iff $\pi_*(\mathcal{L}_p) \cong *$. Thus, $\mathrm{Hom}_{\mathcal{E}\mathcal{X}(\mathfrak{S})}(p, q) \simeq \Omega\mathcal{N}$ iff π is a trivial Kan fibration. But π's being a trivial Kan fibration is tantamount to its being an equivalence ([Lur23, 00X2]). □

We interpret this as saying that the space(s) of non-invertible paths in a linked space is (are) at its (their) largest when π is an equivalence; there are just as many as there are paths in the higher stratum. This is the case, for instance, when $\mathfrak{S}$ is induced by a manifold with boundary, as in Example 2.15. We have a maximal simplification in the other extreme, namely when π is trivial.

Corollary 3.7. *Consider a linked space of type* $\mathfrak{S} = \left(* \leftarrow \mathcal{N} \xrightarrow{\mathrm{id}} \mathcal{N}\right)$.

(1) *We have* $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{S})}(*, q) \simeq *$.
(2) *When* $\mathcal{N} = \mathrm{Sing}_\bullet(N)$ *for* N *a smooth manifold, we have*

$$\mathrm{Exit}(C(N)) \simeq \mathcal{EX}\left(* \leftarrow N \xrightarrow{\mathrm{id}} N\right),$$

where the LHS is the exit path ∞*-category à la Lurie/MacPherson/AFR of the conically-smooth open cone* $C(M) = * \amalg_{\{0\}\times N} ([0,1) \times N)$ *on* N *with its canonical stratification over* $\{0 < 1\}$.
(3) *The tangent bundle of the linked space* $\mathfrak{N}_p$ *given by a point defect* $p \in N$ *(Example 3.4) is determined, up to a contractible choice, by* $\mathrm{T}N$.

Proof. (1) We have $\mathcal{L}_* = \mathcal{N}_* = \mathcal{N}$ and so $\mathcal{P}_{\mathcal{L}_*,q} \simeq \mathcal{N}/q \simeq *$ ([Lur23, 018Y]).

(2) Statement (1) implies $\mathcal{EX}\left(* \leftarrow N \xrightarrow{\mathrm{id}} N\right) \simeq \mathrm{Sing}_\bullet(N)^{\triangleleft}$. The latter agrees with the LHS by [AFR18a, Proposition 3.3.8], since the cone locus $*$ is a (the) deepest stratum of $C(N)$.[5]

(3) Let $p \neq q \in N$. In the corresponding locally-Kan categories, even though $\mathrm{Hom}_{\mathcal{EX}(\mathfrak{N}_p)}(p, q) \simeq P(N)_{\iota_p(S^{n-1}),q}$ may have nontrivial homotopy type, the non-invertible paths are mapped per

$$\mathrm{Hom}_{\mathcal{EX}(\mathfrak{N}_p)}(p, q) \to \mathrm{Hom}_{B\mathrm{O}(0,n)}(*, T_q(N \setminus \{p\})) \simeq *.$$

since $\mathcal{EX}(B\mathrm{O}(0, n)) \simeq B\mathrm{O}(n)^{\triangleleft}$ by (the proof of) statement (2), which is to say that the adjoined point $* \in B\mathrm{O}(0) \subset B\mathrm{O}(0, n)$ is initial.

□

Remark 3.8. Item 3 of Corollary 3.7 is, in a sense, redundant, as the statement already follows from the requirement that $\mathrm{T}\mathfrak{N}_p$ be a map of spans. It should be read as saying that there is no choice in its restriction to the link S^{n-1}.

3.3. **Intermezzo: Maps of linked spaces.** Contrary to best practice, we have so far carefully worked around the issue of defining maps of linked spaces formally, as it wasn't strictly necessary, and have made do with span maps as they clearly induce maps between the respective exit path ∞-categories. The technical reason is that a simple approach to linked maps had to wait until after Section 3.2. The examples above, and the central interest of the projections $\overline{X} \to X^{!}$ in Section 4, show, however, that a more systematic treatment is inavoidable.

In this section, we propose a notion of a map of linked spaces. Any such notion should be well-behaved in two respects:

- maps should compose, and
- they should induce ∞-functors on exit paths.

[5]More precisely, this is an equivalence of quasi-categories for Lurie's model from [Lur17], or, after translating to the complete Segal space model and using [AFR18a, Lemma 3.3.9], with that of Ayala et al.

Our proposal fulfills both criteria (Remark 3.11, Proposition 3.14) and enjoys a generous amount of simplicity.

Definition 3.9. A *linked space*, denoted by

$$M \to \mathfrak{P},$$

of *depth* 1 is a collection of spaces indexed over a poset $\mathfrak{P}$ of depth 1: we have a space $M_\mathfrak{p}$ for each $\mathfrak{p} \in \mathfrak{P}$, and for each arrow $\mathfrak{p} \leq \mathfrak{q}$ a space $L_{\mathfrak{pq}}$, called a *link*, sitting in a span $M_p \xleftarrow{\pi} L_{pq} \xrightarrow{\iota} M_q$, such that

- if $\mathfrak{p} \leq \mathfrak{q}$ but $\mathfrak{p} \neq \mathfrak{q}$, then π is a proper fibre bundle and ι an embedding;
- if $\mathfrak{p} = \mathfrak{q}$, then
$$L_{\mathfrak{pp}} = M_\mathfrak{p}^I,$$
the unbased path space of $M_\mathfrak{p}$ ($I = [0,1]$). The exponential is endowed with the compact-open topology, and
$$\pi = \mathrm{ev}_0, \quad \iota = \mathrm{ev}_1$$
are, respectively, the source and target evaluations.

We call $M \to \mathfrak{P}$ a *linked manifold* (of *depth* 1) if all $M_\mathfrak{p}$, as well as $L_{\mathfrak{pq}}$ whenever $\mathfrak{p} \neq \mathfrak{q}$, are smooth riemannian manifolds.

Definition 3.10. A *map* f, written

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ \mathfrak{P} \arrow[r] & \mathfrak{Q} \end{tikzcd}, \tag{18}$$

consists of a map of posets $\mathfrak{f}\colon \mathfrak{P} \to \mathfrak{Q}$ together with maps of spans

$$f_{\mathfrak{pq}}\colon (M_\mathfrak{p} \leftarrow L_{\mathfrak{pq}} \to M_\mathfrak{q}) \to (N_{\mathfrak{f}(\mathfrak{q})} \leftarrow L'_{\mathfrak{f}(\mathfrak{p})\mathfrak{f}(\mathfrak{q})} \to N_{\mathfrak{f}(\mathfrak{q})})$$

for each pair $\mathfrak{p} \leq \mathfrak{q}$ in $\mathfrak{P}$. This means that $f_{\mathfrak{pq}}$, with a slight abuse of notation, is a commutative diagram of the form

$$\begin{tikzcd} L_{\mathfrak{pq}} \arrow[r, "f_{\mathfrak{pq}}"] \arrow[d, "\pi\times\iota"'] & L'_{\mathfrak{f}(\mathfrak{p})\mathfrak{f}(\mathfrak{q})} \arrow[d, "\pi'\times\iota'"] \\ M_\mathfrak{p} \times M_\mathfrak{q} \arrow[r, "f_\mathfrak{p}\times f_\mathfrak{q}"] & N_{\mathfrak{f}(\mathfrak{p})} \times N_{\mathfrak{f}(\mathfrak{q})} \end{tikzcd}$$

Remark 3.11. Since both poset maps and span maps compose, maps of linked spaces compose.

Warning 3.12. Diagrams of type (18), while literal in classical stratified geometry, are only figurative in the linked context!

We have thus obtained a notion of mapping from a non-trivially linked space ($M_0 \leftarrow L = L_{01} \to M_1$) $= (M \to \{0 < 1\})$ to a smooth space X. The latter is naturally a linked space indexed over the trivial poset $*$, so that the its full expression is $X \xleftarrow{\mathrm{ev}_0} X^I \xrightarrow{\mathrm{ev}_1} X$. Thus, a map

$$\begin{tikzcd} M \arrow[r] \arrow[d] & X \arrow[d] \\ \{0 < 1\} \arrow[r] & * \end{tikzcd} \tag{19}$$

is a commuting square

$$
\begin{tikzcd}
L \arrow[r, "f_L := f_{01}"] \arrow[d] & X^I \arrow[d] \\
M_0 \times M_1 \arrow[r, "f_0 \times f_1"] & X \times X
\end{tikzcd}
\quad . \tag{20}
$$

The sector of (19) over the two identities in $\{0 < 1\}$,

$$
\begin{tikzcd}
M_0^I \arrow[r] \arrow[d] & X^I \arrow[d] \\
M_0 \times M_0 \arrow[r] & X \times X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
M_1^I \arrow[r] \arrow[d] & X^I \arrow[d] \\
M_1 \times M_1 \arrow[r] & X \times X
\end{tikzcd}
\quad ,
$$

is determined by f_0 and f_1.

What one might object to is that if the linked spaces in question are induced by depth-1 stratified spaces, then maps in our sense are more relaxed than stratified maps of the corresponding spaces: the latter give only a subset of the former. While this relaxation poses no problem for our purposes, let us illustrate it with the simplest example where it is detectable.

Example 3.13 (linked paths in a smooth space). Let $M \to \mathfrak{P}$ be induced by $\mathbb{R}_{\geq 0}$ with its standard stratification, i.e., $M_0 = \{0\}$, $M_1 = \mathbb{R}_{>0}$, $L = L_{01} = *$, with $\iota = \iota_+ \colon L = * \hookrightarrow \mathbb{R}_{>0}$ is given by the choice of some point $+ \in \mathbb{R}_{>0}$. (We keep to our implicit notation from Example 4.4.) Let also $X \to *$ be induced by a smooth manifold X as above. Then

$$
\begin{tikzcd}
* \arrow[r, "\gamma := f_{01}"] \arrow[d, "\mathrm{id} \times \iota_+"'] & X^I \arrow[d] \\
\{0\} \times \mathbb{R}_{>0} \arrow[r, "f_0 \times f_1"] & X \times X
\end{tikzcd}
$$

is determined, besides the stratum-wise maps, by the choice of a single path

$$\gamma \colon f_0(0) \to f_1(+)$$

in X. Similarly, if $[0,1] \to \{0 < i > 1\}$ (i for *i*nterior) is the linked space of depth 1 with three strata induced by the standard stratification on $[0,1]$, with both links given again by $*$ and the embeddings into $(0,1)$ determined by a pair of points $\varepsilon < \delta$ in $(0,1)$, then a linked map

$$
\begin{tikzcd}
{[0,1]} \arrow[r] \arrow[d] & X^I \arrow[d] \\
\{0 < i > 1\} \arrow[r] & *
\end{tikzcd}
\tag{21}
$$

is determined, besides the stratum-wise maps, by two paths

$$
\begin{aligned}
\gamma_\varepsilon &\colon f_0(0) \to f_i(\varepsilon), \\
\gamma_\delta &\colon f_1(1) \to f_i(\delta)
\end{aligned}
$$

in X (see Figure 1).

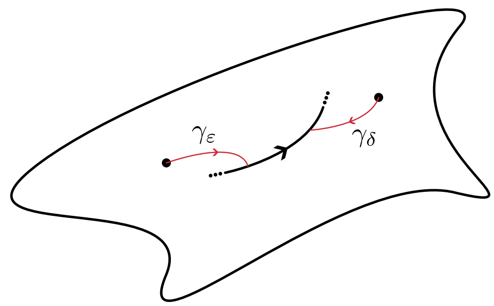

FIGURE 1. A general linked path as in (21).

Proposition 3.14. *Given a map f of type*

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ \mathfrak{P} \arrow[r] & \mathfrak{Q} \end{tikzcd}$$

in the sense of Definition 3.10, there is an induced map

$$f = f_* \colon \mathcal{EX}(M \to \mathfrak{P}) \to \mathcal{EX}(N \to \mathfrak{Q})$$

of exit path ∞-categories. In fact, this defines a(n ordinary) functor

$$\mathcal{LS} \to \mathfrak{C}\mathrm{at}_\infty$$

from linked spaces to ∞-categories and ∞-functors.

Proof. Without loss of generality, assume $|\mathfrak{P}| = 2$, since $\mathcal{EX}(M \to \mathfrak{P})$ otherwise splits dimension-wise into disjoint unions according to pairs of neighbouring strata of differing depth. The only non-obvious case is when $\mathfrak{Q} = *$, $X := N_*$. Let then $\mathfrak{p} \neq \mathfrak{q}$ in $\mathfrak{P}$, and $p \in M_\mathfrak{p}$, $q \in M_\mathfrak{q}$. We observe f_* in terms of the corresponding locally-Kan categories: on non-invertible paths, it should induce maps

$$\mathrm{Hom}_{\mathcal{EX}(M)}(p,q) \overset{\mathrm{Thm}}{\underset{3.5}{\simeq}} \mathcal{P}(M_\mathfrak{q})_{(L_{\mathfrak{pq}})_p, q} \to \mathcal{P}(X)_{f(p),f(q)} \simeq \mathrm{Hom}_{\mathrm{Sing}X}(f(p), f(q)) \tag{22}$$

while given is the commuting square (20). The latter, however, induces, after pulling back onto $p \in \mathfrak{M}_\mathfrak{p}$ (and writing $L := L_{\mathfrak{pq}}$), the square

$$\begin{tikzcd} L_p \arrow[r, "f_L"] \arrow[d] & X^I \arrow[d] \\ \{p\} \times M_\mathfrak{q} \arrow[r] & \{f(p)\} \times X \end{tikzcd} .$$

Thus, the initial point γ_0 of a path $\gamma \in \mathcal{P}(M_\mathfrak{q})_{L_p, q}$ yields the path $f_L(\gamma_0)$ in X from $f(p)$ to γ_0. Thus,

$$f_* \colon \gamma \mapsto f_L(\gamma_0) * \gamma$$

provides (22) (see Figure 2). (We have not distinguished L and $\iota(L) \subseteq M_\mathfrak{q}$ in notation.) The second statement is left for the reader. $\square$

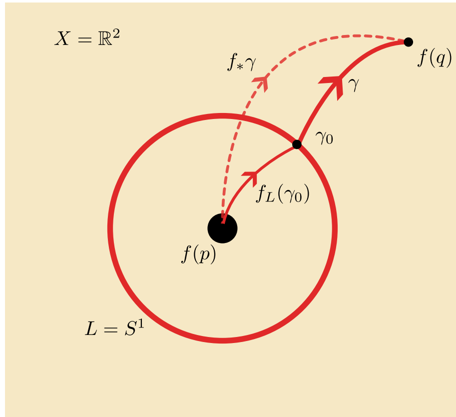

FIGURE 2. The proof of Proposition 3.14 for $(M \to \mathfrak{P}) = (\{0\} \twoheadleftarrow S^1 \hookrightarrow \mathbb{R}^2 \smallsetminus \{0\})$ and $N = X = \mathbb{R}^2$, with f 'the identity'; cf. Remark 3.15.

Remark 3.15 (linked maps induced by stratified maps). Consider a link projection $\pi\colon L \twoheadrightarrow M$. The *fibrewise open cone* on π is defined to be

$$C(\pi) = L \times \mathbb{R}_{\geq 0} \amalg_{L \times \{0\}} M.$$

Suppose that a constructible (Definition 2.16) linked space $\mathfrak{S} = \left(M \overset{\pi}{\twoheadleftarrow} L \overset{\iota}{\hookrightarrow} N\right)$ is *induced by* a (conically-smooth) stratified space X over $\{0 < 1\}$ with strata $X_0 = M$ and $X_1 = X \smallsetminus X_0 = N$ in the sense that

$$X \cong C(\pi) \amalg_{L \times \mathbb{R}_{>0}} N.$$

and ι by the implicitly used open embedding $L \times \mathbb{R}_{>0} \hookrightarrow N$ at time 1. This construction appeared in [AFR18a, the proof of Lemma 6.1.7].

Now, let Y be smooth and $f\colon X \to Y$ a stratified map. There is an induced map of linked spaces $\mathfrak{S} \to Y$ whose non-obvious component $f_L\colon L \to Y^I$ covering $f|_0 \times f|_1\colon M \times N \to Y \times Y$ can be given, at $\gamma_0 \in L$ (notation from the proof of Proposition 3.14), by simply following along the time coordinate in $C(\pi)$ from 0 to 1, from $f|_0(\pi(\gamma_0))$ to $\iota(\gamma_0)$.

3.4. **Cartesian replacements, I: The smooth case.** For $J \in \mathbb{N}$, a stably-J-framed smooth manifold M of dimension n is one with a framing on $\mathrm{T}M \oplus \varepsilon^j$, $J = n + j$. In other words, this amounts to an injection $\mathrm{T}M \hookrightarrow \varepsilon^J$ of bundles over M whose normal bundle, defined either using a metric on ε^J or as the quotient $\varepsilon^J/\mathrm{T}M$, is parallelised. More generally, a solid J-framing on M is merely an injection $\mathrm{T}M \hookrightarrow \varepsilon^J$. First of all, we notice that in order to impose such a condition on said normal bundle *in terms of tangential structures*, we must first separate it from the solid datum.

Let X be a smooth manifold equipped with a vector bundle $E \to X$ of rank r, and let $F \to X$ be another bundle, of rank R. Choosing a bundle embedding $E \hookrightarrow F$ is a 'reduction [or extension] of gauge group' on E in the following way. There is naturally a normal bundle N to E such that the embedding amounts to an isomorphism $\Phi\colon E \oplus N \cong F$. This Φ provides a filler for the diagram

$$\begin{tikzcd} & BO(r) \times BO(R-r) \arrow[d, "\boxplus"] \\ X \arrow[ur, dotted, "E\times N"] \arrow[r, "F"] & BO(R). \end{tikzcd}$$

Changing our point of view slightly, consider the limit space[6]

$$\begin{tikzcd} X \times_{BO(R)} (BO(r) \times BO(R-r)) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & BO(r) \times BO(R-r) \arrow[d, "\boxplus"] \\ X \arrow[r, "F"] & BO(R) \end{tikzcd} \tag{23}$$

which also admits a 'source evaluation' by projecting to the first factor:

$$\mathrm{ev}_0 \colon (BO(r) \times BO(R-r))|_F \to BO(r).$$

Now, writing

$$(BO(r) \times BO(R-r))|_F \coloneqq X \times_{BO(R)} (BO(r) \times BO(R-r)), \tag{24}$$

the choice of Φ can be expressed as follows:

Definition 3.16. A *solid F-structure* or *-reduction* on E (or on X when $E = TX$) of is a lift of the form

$$\begin{tikzcd} & (BO(r) \times BO(R-r))|_F \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[ur, dotted] \arrow[r, "E"] & BO(r) \end{tikzcd} .$$

We call R the *total rank* of the solid structure.

The normal bundle itself can be recovered from such a lift by projecting it to the second factor:

$$N \colon X \to (BO(r) \times BO(R-r))|_F \to BO(R-r).$$

Thus, a further, simultaneous reduction on N can be implemented using this projection: if N is to have $(B \to BO(R-r))$-structure, then we may consider the iterated fibre product

$$\begin{tikzcd} (BO(r) \times BO(R-r))|_F \times_{BO(R-r)} B \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & (BO(r) \times BO(R-r))|_F \arrow[d] \\ B \arrow[r] & BO(R-r) \end{tikzcd} \tag{25}$$

and, writing

$$(BO(r) \times BO(R-r))|_{(F,B)} \coloneqq (BO(r) \times BO(R-r))|_F \times_{BO(R-r)} B, \tag{26}$$

ask for reductions of the following form:

[6]For the moment, we disregard the appropriate homotopy versions of such limits in order to ease notation; in terms of the tangential structure, this amounts to disregarding the choice of bundle isomorphism. Homotopy limits will be reincorporated into this account of their own accord below.

Definition 3.17. A *solid (F, B)-structure* on E (or on X when $E = \mathrm{T}X$) is a lift of the form

$$\begin{tikzcd} & (B\mathrm{O}(r) \times B\mathrm{O}(R-r))|_{(F,B)} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "E"] \arrow[ur, dotted] & B\mathrm{O}(r) \end{tikzcd} .$$

When $B = \boldsymbol{R-r}$, this is a *stable F-structure.* We call $B \to B\mathrm{O}(R-r)$, or B, the *normal structure*.

It is incidental that F is given as a bundle over X. More generally, when $F : Y \to B\mathrm{O}(R)$ is any classical tangential structure with rank $R \geq r = \mathrm{rk}(E)$, the limit (23), and so also (25), still make sense. Then, a *solid Y- or (Y, B)-structure* is defined analogously, as is a *stable Y-structure*.

Solid replacements in the stratified context have been considered in [AFR18b]. Categorically speaking, they are Cartesian fibration replacements. Namely, the assignment in the following Definition 3.18 extends (by a main result of [GHN17]) to a left-adjoint

$$\mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{cart}}/\mathcal{V}^{\hookrightarrow}$$

to the forgetful functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{cart}}/\mathcal{V}^{\hookrightarrow} \to \mathcal{C}\mathrm{at}_\infty/\mathcal{V}^{\hookrightarrow}$ from *solid/Cartesian* tangential structures (i.e., Cartesian fibrations over $\mathcal{V}^{\hookrightarrow}$) to tangential structures.

Definition 3.18. Given a tangential structure $\mathfrak{b} : \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, its *solid/Cartesian replacement* is

$$\overline{\mathfrak{b}} : \overline{\mathcal{B}} = \overline{(\mathcal{B}, \mathfrak{b})} = (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} \mathcal{B} \to (\mathcal{V}^{\hookrightarrow})^{\{0\}},$$

the source evaluation from the fibre product along the target evaluation.

Note the direct correspondence with Definition 3.16 (and (23)), in view of $B\mathrm{O}(r) \times B\mathrm{O}(R-r)$'s being the link of the $(r, R-r)$-Grassmannian, viewing the target evaluation as the embedding $\boxplus$ off the link. We will make this precise.

A solid $(F : Y \to B\mathrm{O}(R))$-structure on a rank-$r$ bundle ought to be (a lift to) the restriction to $B\mathrm{O}(r)$ of the solid replacement of F:

$$(27) \qquad \begin{tikzcd} \overline{Y}|_r \coloneqq B\mathrm{O}(r) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} \overline{Y} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & \overline{Y} \arrow[d, "\overline{F}"] \\ B\mathrm{O}(r) \arrow[r] & \mathcal{V}^{\hookrightarrow} \end{tikzcd} .$$

This is the space of morphisms in $\mathcal{V}^{\hookrightarrow}$ that start in $B\mathrm{O}(r)$ and end in the image of F inside $B\mathrm{O}(R)$.

Lemma 3.19. $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q) \simeq \mathrm{Hom}_{\mathcal{E}\mathcal{X}(B\mathrm{O}(r,R-r))}(p, q)$*, where* $p \in B\mathrm{O}(r)$ *and* $q \in B\mathrm{O}(R)$.

Proof. $\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p, q)$ is equivalent to the homotopy-fibre of

$$p^* : \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(\mathcal{B}^{\boxplus}\mathrm{O})}(*, *) \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(\mathcal{B}^{\boxplus}\mathrm{O})}(*, *),$$

i.e., using that morphism spaces in the homotopy-coherent nerve are equivalent to those in the original topological category ([HK20]), the homotopy-fibre of

$$(- \boxplus p) : B\mathrm{O}^\infty_{\amalg} \to B\mathrm{O}^\infty_{\amalg},$$

at q. The connected component of q in $BO^{\infty}_{\amalg}$ is $BO(R)$, and p^* maps only $BO(R-r)$ into it, so we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{V}^{\hookrightarrow}}(p,q) &= (BO(R-r) \boxplus p) \times_{BO(R)^{\{0\}}} BO(R)^{\Delta[1]} \times_{BO(R)^{\{1\}}} \{q\} \\
&= \pi^{-1}(p) \times_{BO(R)^{\{0\}}} BO(R)^{\Delta[1]} \times_{BO(R)^{\{1\}}} \{q\} \\
&= P(BO(R))_{\pi^{-1}(p),q} \\
&\simeq \mathrm{Hom}_{\mathcal{E}\mathcal{X}(BO(r,R-r))}(p,q)
\end{aligned}$$

by Theorem 3.5. By $\pi = \mathrm{pr}_1$ we denoted the link projection in $BO(r, R-r)$. □

Remark 3.20. We should note that $\overline{Y}|_r$ is *not*, in general, the ∞-categorical homotopy fibre product $BO(r) \times^{\mathrm{h}}_{\mathcal{V}^{\hookrightarrow}} Y = BO(r) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} \mathrm{Isom}(\mathcal{V}^{\hookrightarrow}) \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} Y$ in the sense of [Lur23, §01DE]: isomorphisms in $\mathcal{V}^{\hookrightarrow}$ from $BO(r)$ to $F(Y)$ exist iff $r = R$. A *Kan* fibration replacement rather than a Cartesian one would employ $BO(r) \times^{\mathrm{h}}_{\mathcal{V}^{\hookrightarrow}} Y$.

Proposition 3.21. $\overline{Y}|_r \simeq (BO(r) \times BO(R-r)) \times^{\mathrm{h}}_{BO(R)} Y$.

Proof. Written in full, the statement reads

$$\begin{gathered}
BO(r) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} Y \\
\simeq \\
(BO(r) \times BO(R-r)) \times_{BO(R)^{\{0\}}} BO(R)^{\Delta[1]} \times_{BO(R)^{\{1\}}} Y.
\end{gathered}$$

By direct on inspection of $\mathbb{U}$ on exit paths of maximal index (see the proof of Theorem 3.5) and by Lemma 3.19 we have that $\mathbb{U}\colon \mathcal{E}\mathcal{X} = \mathcal{E}\mathcal{X}(BO(r, R-r)) \to \mathcal{V}^{\hookrightarrow}$ is fully-faithful, whence it is an equivalence onto its image $\mathcal{V}^{\hookrightarrow}(r,R)$, the full sub-$\infty$-category generated by $BO(r) \amalg BO(R) \subset \mathcal{V}^{\hookrightarrow}$. Thus, we observe that in the diagram

$$\begin{tikzcd}
\mathcal{F} \times_{BO(R)} Y \arrow[r, dashed] \arrow[dd, dashed] & \mathcal{F} \arrow[r, hook, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}\mathcal{X}^{\Delta[1]} \arrow[r, "\sim"] \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_1"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{V}^{\hookrightarrow}(r,R)^{\Delta[1]} \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_1"] \\
 & BO(r) \times BO(R) \arrow[r, hook] \arrow[d] & \mathcal{E}\mathcal{X}^{\times 2} \arrow[r, "\sim"] & \mathcal{V}^{\hookrightarrow}(r,R)^{\times 2} \\
Y \arrow[r, "F"] & BO(R) & &
\end{tikzcd},$$

$\mathcal{F}$ coincides with $BO(r) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} BO(R)$, and, by the pasting lemma and again by Theorem 3.5, also with

$$\begin{aligned}
\mathcal{E}\mathcal{X}^{\Delta[1]} \times_{\mathcal{E}\mathcal{X}^{\times 2}} (BO(r) \times BO(R)) &= BO(r) \times_{\mathcal{E}\mathcal{X}^{\{0\}}} \mathcal{E}\mathcal{X}^{\Delta[1]} \times_{\mathcal{E}\mathcal{X}^{\{1\}}} BO(R) \\
&\simeq (BO(r) \times BO(R-r)) \times_{BO(R)^{\{0\}}} BO(R)^{\Delta[1]}.
\end{aligned}$$

Thus, both sides in the statement are equivalent to $\mathcal{F} \times_{BO(R)} Y$. □

In terms of lifts, Proposition 3.21 reads:

Corollary 3.22. *A solid F-structure, in the sense of Definition 3.16, on a smooth manifold X of dimension r, is a Cartesian F-structure, in the sense of Definition 3.18, on the (trivially-linked) manifold X.*

We note the following consequence for future reference.

Corollary 3.23. *There is a cofibration $\overline{Y}|_r \hookrightarrow \overline{Y}|_R$.*

Proof. Using Proposition 3.21, this is induced simply by the simplicial subset inclusion

$$\begin{aligned}&(B\mathrm{O}(r) \times B\mathrm{O}(R-r)) \times_{B\mathrm{O}(R)^{\{0\}}} B\mathrm{O}(R)^{\Delta[1]} \times_{B\mathrm{O}(R)^{\{1\}}} Y \\ &\hookrightarrow B\mathrm{O}(R)^{\Delta[1]} \times_{B\mathrm{O}(R)^{\{1\}}} Y = B\mathrm{O}(R) \times_{B\mathrm{O}(R)^{\{0\}}} B\mathrm{O}(R)^{\Delta[1]} \times_{B\mathrm{O}(R)^{\{1\}}} Y.\end{aligned}$$

□

3.5. **Cartesian replacements, II: The linked case.** We now generalise the discussion from Cartesian structures on smooth spaces to those on linked spaces. Throughout this section, let $F\colon Y \to B\mathrm{O}(R) \subset \mathcal{V}^{\hookrightarrow}$ be a classical tangential structure, and $\mathfrak{S} = \left(M \overset{\pi}{\twoheadleftarrow} L \overset{\iota}{\hookrightarrow} N\right)$ be a linked manifold with $\dim M = n$, $\dim N = n+m$. For simplicity, let us assume $R = n+m$, leaving the obvious modifications for the case $R > n+m$ to the reader. We will sometimes abuse notation by not distinguishing $B\mathrm{O}(n,m)$ from $\mathcal{EX}(B\mathrm{O}(n,m))$, or $\mathfrak{S}$ from $\mathcal{EX}(\mathfrak{S})$, etc., and also keep to Notation 2.11 throughout.

Since $\mathbb{U}\colon B\mathrm{O}(n,m) \hookrightarrow \mathcal{V}^{\hookrightarrow}$ is fully faithful, the (iterated) fibre product

$$\overline{Y}_{(n,m)} \coloneqq B\mathrm{O}(n,m) \times_{\mathcal{V}^{\hookrightarrow}} \overline{Y}$$

as in Section 3.1 is simply a restriction, and so, in light of the results of Section 3.4, we are in a position to understand the meaning of a Cartesian Y-structure in the linked setting.

Definition 3.24. A *Cartesian Y-structure* on $\mathfrak{S}$ is a lift of type

$$\begin{tikzcd} & \overline{Y}_{(n,m)} \arrow[d] \\ \mathfrak{S} \arrow[r, "\mathrm{T}\mathfrak{S}"'] \arrow[ur, dotted, "t"] & B\mathrm{O}(n,m) \end{tikzcd}.[7]$$

The main goal of this section is to identify a linked space $\mathfrak{B}$ over $B\mathrm{O}(n,m)$ such that span maps $\mathfrak{S} \to \mathfrak{B}$ that lift $\mathrm{T}\mathfrak{S}$ induce Cartesian Y-structures on $\mathfrak{S}$. This is achieved in Proposition 3.26. This constitutes a partial solution to the problem (17) in depth 1.

At a point $p \in M$, let us write $\mathrm{T}_p \coloneqq \mathrm{T}\mathfrak{S}(p) \in B\mathrm{O}(n)$. A point-wise lift

$$t_p \in B\mathrm{O}(n,m) \times_{(\mathcal{V}^{\hookrightarrow})^{\{0\}}} (\mathcal{V}^{\hookrightarrow})^{\Delta[1]} \times_{(\mathcal{V}^{\hookrightarrow})^{\{1\}}} Y$$

is determined by a path of type

$$W_p \boxplus \mathrm{T}_p \xrightarrow{t_p} F_p^M$$

in $B\mathrm{O}(n+m)$ where

$$\begin{gathered} F_p^M \in F(Y) \subset B\mathrm{O}(n+m), \\ W_p \in B\mathrm{O}(m). \end{gathered}$$

Since

$$\mathrm{T}\mathfrak{S}|_M\colon M \hookrightarrow \mathfrak{S} \to B\mathrm{O}(n,m)$$

factors through

$$M \xrightarrow{\mathrm{T}M} B\mathrm{O}(n) \hookrightarrow B\mathrm{O}(n,m)^{\sim} \hookrightarrow B\mathrm{O}(n,m)$$

[7]We consider honest lifts rather than homotopy lifts in view of Proposition 3.21.

(note $B\mathrm{O}(n,m)^{\sim} \simeq B\mathrm{O}(n) \amalg B\mathrm{O}(n+m)$ exactly like $\mathcal{V}^{\sim} \simeq B\mathrm{O}^{\infty}_{\amalg}$),

$$t|_M : M \to \overline{Y}_{(n,m)}$$

factors through

$$B\mathrm{O}(n) \times_{B\mathrm{O}(n,m)} B\mathrm{O}(n,m) \times_{\mathcal{V}^{\hookrightarrow}} \overline{Y} = \overline{Y}|_n$$

(recall (27)), we identify W_-, via Proposition 3.21, as the 'normal bundle'

$$W\colon M \xrightarrow{t|_M} (B\mathrm{O}(n) \times B\mathrm{O}(m)) \times^{\mathrm{h}}_{B\mathrm{O}(n+m)} Y \xrightarrow{\mathrm{pr}_2 \circ \mathrm{ev}_0} B\mathrm{O}(m) \ .$$

of the induced solid F-structure $t|_M$ on M. Of course, $\mathrm{T}M$ can be similarly identified to be $\mathrm{pr}_1 \circ \mathrm{ev}_0 \circ t|_M$ and F^M_- to be $\mathrm{ev}_1 \circ t|_M$. Similarly, a lift t_q at $q \in N$ is a path of type

$$\mathrm{T}_q \xrightarrow{t_q} F^N_q .$$

That is, $t|_N$ factors, again as a special case of Proposition 3.21, through

$$B\mathrm{O}(n+m)^{\Delta[1]} \times_{B\mathrm{O}(n+m)^{\{1\}}} Y$$

which projects to $B\mathrm{O}(n+m)$ via ev_0 so that $\mathrm{T}N = \mathrm{ev}_0 \circ t|_N$ on the nose.

Let us consider now a non-invertible exit 1-path $\gamma = (\widehat{\gamma}, 1)$ in $\mathcal{E}\mathcal{X}(\mathfrak{S})$ and explicate the application of $\mathrm{T}\mathfrak{S}$ thereon: γ is determined by a path $\widehat{\gamma}\colon \widehat{p} \to q$ in N, and so $\mathrm{T}\mathfrak{S}(\gamma)$ will be determined by a path in $B\mathrm{O}(n+m)_1$ with source

$$\boxplus\, ((\pi^*\mathrm{T}M \times \mathrm{N}_N M)\,(\widehat{p})) = \mathrm{N}_{\widehat{p}} \boxplus \mathrm{T}_p M,$$

where $\mathrm{N} = \mathrm{N}_N M$, and actual path

$$\mathrm{T}_{\widehat{p}} N \xrightarrow{\mathrm{T}N(\widehat{\gamma})} \mathrm{T}_q N$$

which is consistent as $\mathcal{E}\mathcal{X}(\mathfrak{S}) \to \mathcal{E}\mathcal{X}(B\mathrm{O}(n,m))$'s being induced by a span map $\mathfrak{S} \to B\mathrm{O}(n,m)$ implies

$$\mathrm{N}_{\widehat{p}} \boxplus \mathrm{T}_p M = \mathrm{T}_{\widehat{p}} N. \tag{28}$$

Let us place all players involved in a diagram:

$$\begin{tikzcd}
N_1 & B\mathrm{O}(n+m)_1 \\
\left(\widehat{p} \xrightarrow{\widehat{\gamma}} q\right) \arrow[u, phantom, "\Cup"] \arrow[r, "\mathrm{d}\iota"] \arrow[d, no head] & \left(\mathrm{N}_{\widehat{p}} \boxplus \mathrm{T}_p M \xrightarrow{\mathrm{T}_{\widehat{\gamma}} N} \mathrm{T}_q N\right) \arrow[u, phantom, "\Cup"] \arrow[d, no head] \\
\left(p \underset{\gamma}{\to} q\right) \arrow[r, "\mathrm{T}_\gamma \mathfrak{S}"'] \arrow[d, phantom, "\Cap"] & \left(\mathrm{T}_p M \underset{\mathrm{T}_L \mathfrak{S}(\gamma)}{\longrightarrow} \mathrm{T}_q N\right) \arrow[d, phantom, "\Cap"] \\
\mathcal{E}\mathcal{X}(\mathfrak{S})_1 & B\mathrm{O}(n,m)_1
\end{tikzcd} \tag{29}$$

Now, a lift t_γ necessarily factors as

$$t_\gamma\colon \Delta[1] \times \Delta[1] \to B\mathrm{O}(n+m) \hookrightarrow \mathcal{V}^{\hookrightarrow}$$

and, resuming the notation from the beginning of Section 2.2.3, we have

$$\left(p \xrightarrow{\gamma} q\right) \overset{t}{\mapsto} \left(\begin{tikzcd} F^M_p \arrow[r, "\rho"] & F^N_q \\ \mathrm{T}_p M \arrow[u, "{(W_p, t_p)}"] \arrow[r, "{(\mathrm{N}_{\widehat{p}}, \mathrm{T}_{\widehat{\gamma}} N)}"'] & \mathrm{T}_q N \arrow[u, "{(0, t_q)}"'] \end{tikzcd}\right) \tag{30}$$

in view of (29). First of all, this implies[8]

$$W_p = \mathrm{N}_{\widehat{p}}. \tag{31}$$

Further, let us take $\ell \in L$ and see it, via the constant-loop inclusion

$$L \hookrightarrow P(L) \overset{\iota_*}{\hookrightarrow} P(N), \tag{32}$$

as an exit path $\pi(\ell) \to \iota(\ell)$ in $\mathfrak{S}$. Taking $p = \pi(\ell)$ and $q = \iota(\ell)$, (30) reduces to the triangle

$$t_\gamma = \left(\begin{tikzcd} F^M_p \arrow[rr, "\rho"] & & F^N_{\iota(\ell)} \\ & \mathrm{N}_\ell \boxplus \mathrm{T}_p M \arrow[ul, "t_p"] \arrow[ur, "t_q"'] & \end{tikzcd}\right)$$

Varying ℓ and rewriting, we see that this provides a point-wise filler for the triangle

$$\begin{tikzcd} \pi^* F^M \arrow[rr, "\rho"] \arrow[dr, dotted, bend right] & & \iota^* F^N \arrow[dl, dotted, bend left] \\ & \mathrm{N} \oplus \pi^* \mathrm{T} M \arrow[ul, "\pi^*(t|_M)"] \arrow[ur, "\iota^*(t|_N)"'] \arrow[d, dotted] & \\ & L & \end{tikzcd} \tag{33}$$

of bundle isomorphisms over L. The bundles $F^{(-)}$ are

$$F^{(-)} \colon (-) \xrightarrow{t|_{(-)}} \overline{Y}_{(n,m)} \xrightarrow{\mathrm{ev}_1} Y \xrightarrow{F} BO(n+m)$$

But (33) can always be achieved, in fact up to contractible choice: To be given is a morphism in the space of bundles under $\mathrm{T}_L \mathfrak{S} = \mathrm{N} \oplus \pi^* \mathrm{T} M$, that is,

$$\rho \in \mathrm{Hom}_{\mathrm{T}_L \mathfrak{S}/\mathrm{Map}(L, BO(n+m))}(\pi^* t|_M, \iota^* t|_N).$$

This space is contractible (in particular non-empty) for so is $\mathrm{T}_L \mathfrak{S}/\mathrm{Map}(L, BO(n+m))$ already.[9] We thus see that the only compatibility condition of substance between the structures on M and N induced by t is given by (31), which we may rephrase along $L \hookrightarrow P(N)$ (32) as

$$\pi^* W = \mathrm{N}_N M, \tag{34}$$

[8]This is necessary for the phantom diagonal in (30) to exist. Perhaps surprisingly, for t_γ to be functorial and to satisfy its defining lifting property, this must be an honest equality, just as (28) must. Alternatively, modify and apply Lemma 2.10.

[9]The reader unhappy with our sudden move to bundles under $\mathrm{T}_L \mathfrak{S}$ can instead invert (contractibly-uniquely) one of the legs of the horn to transform it into an inner horn, and apply the essential uniqueness of composition.

and not by (33). In particular, we may rewrite

$$\mathrm{N} \oplus \pi^*\mathrm{T}M = \pi^*(W \oplus \mathrm{T}M)$$

above.

In brief, a Cartesian Y-structure on $\mathfrak{S}$ entails

- a solid Y-structure on M determined by a bundle isomorphism $W \oplus \mathrm{T}M \simeq F^M$
- a Y-structure on N determined by a bundle isomorphism $\mathrm{T}N \simeq F^N$

such that $\pi^*W = \mathrm{N}_N M$ over L.[10] These data, together with the condition, are summed up in a lift of $\mathrm{T}\mathfrak{S}$ in

$$\begin{tikzcd}
 & & & & \overline{Y}|_n \arrow[dl, equal] \arrow[dr, hook, "\text{Cor 3.23}"] \arrow[dd, "\mathrm{ev}_0"] & \\
 & & & \overline{Y}|_n \arrow[ddd, "\mathrm{pr}_1\circ\mathrm{ev}_0"] & & \overline{Y}|_{n+m} \arrow[dd, "\mathrm{ev}_0"] \\
 & L \arrow[rrr, "\mathrm{T}_L\mathfrak{S}=(\pi^*\mathrm{T}M\times\mathrm{N})"] \arrow[dr, hook, "\iota"] \arrow[ddl, two heads, "\pi"'] & & & BO(n)\times BO(m) \arrow[ddl, two heads, "\mathrm{pr}_1"'] \arrow[dr, hook, "\boxplus"] & \\
 & & N \arrow[rrr, "\mathrm{T}N"'] & & & BO(n+m) \\
M \arrow[rrr, "\mathrm{T}M"] & & & BO(n) & &
\end{tikzcd} \tag{35}$$

along the indicated span map

$$\overline{Y}|_{(n,m)} \coloneqq \left(\overline{Y}|_n \overset{=}{\twoheadleftarrow} \overline{Y}|_n \hookrightarrow \overline{Y}|_{n+m}\right) \to BO(n,m),$$

namely the one given by the solid structure on M, the structure on Y, and

$$\left(L \to \overline{Y}|_n\right) = \left(L \xrightarrow{\pi} M \to \overline{Y}|_n\right), \tag{36}$$

which is implied by the lift's being a span map. That with this definition of the link lift we obtain (34) is clear since the two sides are both classified by

$$\mathrm{pr}_2 \circ \mathrm{ev}_0 \circ t|_M \circ \pi \colon L \to BO(m).$$

Conversely, the structures on M and N couple to a Cartesian Y-structure in the manner above iff setting (36) extends them to a span map lift as in (35).

Definition 3.25. A $\overline{Y}|_{(n,m)}$*-structure on* $\mathfrak{S}$ is a lift of span maps of $\mathrm{T}\mathfrak{S}$ to $\overline{Y}|_{(n,m)}$: a solid Y-structure $t_M \colon M \to \overline{Y}|_n$ on M, a Y-structure $t_N \colon \overline{Y}|_{n+m}$ on N, such that, defining $t_L \colon L \to \overline{Y}|_n$ by (36), the triple $(t_M, t_L, t_N) \colon \mathfrak{S} \to \overline{Y}|_{(n,m)}$ is a span map that lifts $\mathrm{T}\mathfrak{S}$.

Proposition 3.26. *A* $\overline{Y}|_{(n,m)}$*-structure*

$$(t_M, t_L, t_N) \colon \mathfrak{S} \to \overline{Y}|_{(n,m)}$$

induces a Cartesian Y-structure

$$t \colon \mathfrak{S} \to \overline{Y}_{(n,m)}$$

[10]We mean of course that their classifiers are equal.

on $\mathfrak{S}$, such that

$$t|_M = t_M, \quad t|_N = t_N.$$

The abused notation should not obfuscate the point of this proposition. The $\overline{Y}|_{(n,m)}$-structure is given in terms of *actual spaces and maps thereof*, so the notation for it is literal, whereas the $\overline{Y}_{(n,m)}$-structure claimed to be induced thereby is given in terms of an ∞-functor. We will relax the equality $\pi^* W = \mathrm{N}$ of classifiers momentarily in Corollary 3.29.

Proof of Proposition 3.26. We leverage Theorem 3.5 in order to work at the level of simplicial categories, much like in the proof of Proposition 3.14. Of course, $t_M \amalg t_N$ defines the restriction of t to $\mathfrak{S}^{\sim} = M \amalg N$. Let now $p \in M$, $q \in N$. By Theorem 3.5, it suffices to provide a map

$$P_{L_p,q} \to \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_p, t_q) = \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(q)).$$

As discussed above, the $\overline{Y}|_{(n,m)}$-structure yields up to contractible choice a filler

$$\rho\colon \pi^* t_M \to \iota^* t_N \text{ in } \mathrm{T}_L\mathfrak{S}/\mathrm{Map}(L, B\mathrm{O}(n+m)), \tag{37}$$

which by restriction gives a map

$$\rho|_p\colon P_{L_p,q} \xrightarrow{\mathrm{ev}_0} L_p \xrightarrow{\rho|} \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(\mathrm{ev}_0(-))), \tag{38}$$

where $\mathrm{ev}_0(-)$ in the second argument takes $\widehat{\gamma} \in P_{L_p,q}$ to its initial point.[11] Similarly, the restriction of t_N by restriction gives a map

$$t_N|\colon P_{L_p,q} \to \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_N(\mathrm{ev}_0(-)), t_N(q)). \tag{39}$$

Remark 3.27 (interrupting the proof). The expressions (38), (39) make literal sense if we allow dependent pair types, or Σ-types, in quasicategories. These are common in type theory but perhaps not so much in mainstream mathematics (see e.g. [Uni13, §1.6] for an exposition). Instead, one could also replace the dependent targets (still formally) by $\coprod_{\widehat{\gamma}\in P_{L_p,q}} \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(\mathrm{ev}_0(\widehat{\gamma})))$ and $\coprod_{\widehat{\gamma}\in P_{L_p,q}} \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_N(\mathrm{ev}_0(\widehat{\gamma})), t_N(q))$, or equally well (now literally) by the iterated fibre products

$$\{t_M(p)\} \times_{\left((\mathcal{V}^{\hookrightarrow})^{\Delta[1]}\right)^{\{0\}}} \left((\mathcal{V}^{\hookrightarrow})^{\Delta[1]}\right)^{\Delta[1]} \times_{\left((\mathcal{V}^{\hookrightarrow})^{\Delta[1]}\right)^{\{1\}}} P_{L_p,q}$$

for (38), and

$$P_{L_p,q} \times_{\left((\mathcal{V}^{\hookrightarrow})^{\Delta[1]}\right)^{\{0\}}} \left((\mathcal{V}^{\hookrightarrow})^{\Delta[1]}\right)^{\Delta[1]} \times_{\left((\mathcal{V}^{\hookrightarrow})^{\Delta[1]}\right)^{\{1\}}} \{t_N(q)\}$$

for (39), using

$$t_N \circ \mathrm{ev}_0\colon P_{L_p,q} \to (\mathcal{V}^{\hookrightarrow})^{\Delta[1]}$$

in both. The rest of the proof accepts either taste.

We may now compose to obtain the desired map

$$\begin{aligned} P_{L_p,q} &\xrightarrow{\rho|\times t_N|} \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(\mathrm{ev}_0(-))) \times \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_N(\mathrm{ev}_0(-)), t_N(q)) \\ &\xrightarrow{\circ} \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(q)). \end{aligned}$$

[11] As usual, we do not distinguish L from $\iota(L)$ in notation.

This defines t functorially:
(40)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}\mathcal{X}}(p,p') \times \mathrm{Hom}_{\mathcal{E}\mathcal{X}}(p',q) \arrow[r] \arrow[d, dashed, "\sim"] & \mathrm{Hom}_{\mathcal{E}\mathcal{X}}(p,q) \arrow[d, dashed, "\sim"] \\
P(M)_{p,p'} \times P(N)_{L_{p'},q} \arrow[r] \arrow[d] & P(N)_{L_p,q} \arrow[d] \\
\mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_M(p')) \times \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p'), t_N(q)) \arrow[r] & \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(q))
\end{tikzcd}$$

where $p, p' \in M$ and $q \in N$, homotopy-commutes, as does
(41)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}\mathcal{X}}(p,q) \times \mathrm{Hom}_{\mathcal{E}\mathcal{X}}(q,q') \arrow[r] \arrow[d, dashed, "\sim"] & \mathrm{Hom}_{\mathcal{E}\mathcal{X}}(p,q') \arrow[d, dashed, "\sim"] \\
P(N)_{L_p,q} \times P(N)_{q,q'} \arrow[r] \arrow[d] & P(N)_{L_p,q'} \arrow[d] \\
\mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(q)) \times \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_N(q), t_N(q')) \arrow[r] & \mathrm{Hom}_{(\mathcal{V}^{\hookrightarrow})^{\Delta[1]}}(t_M(p), t_N(q'))
\end{tikzcd}$$

where $p \in M$ and $q, q' \in N$. Let us consider (40) first. For the sake of readability, we will sometimes write $[-,-]_{(-)} := \mathrm{Hom}_{(-)}(-,-)$. Unpacking the construction, the lower square is

$$\begin{tikzcd}
P(M)_{p,p'} \times P(N)_{L_{p'},q} \arrow[rr] \arrow[d, "t_M \times (\rho|_{p'} \times t_N|)"] & & P(N)_{L_p,q} \arrow[d, "\rho|_p \times t_N|"] \\
{[t_M(p), t_M(p')] \times ([t_M(p'), t_N(\mathrm{ev}_0(-))] \times [t_N(\mathrm{ev}_0(-)), t_N(q)])} \arrow[d, "\mathrm{id} \times \circ"] & & {[t_M(p), t_N(\mathrm{ev}_0(-))] \times [t_N(\mathrm{ev}_0(-)), t_N(q)]} \arrow[d, "\circ"] \\
{[t_M(p), t_M(p')] \times [t_M(p'), t_N(q)]} \arrow[rr, "\circ"] & & {[t_M(p), t_N(q)]}
\end{tikzcd}$$

Let now $\delta \in P(M)_{p,p'}$, $\widetilde{\gamma} \in P(N)_{L_{p'},q}$, and let $\widetilde{\epsilon} \in P(N)_{L_p,q}$ a choice of composition, with composition 2-path $\Gamma = (\widetilde{\Gamma}, 2) \in \mathcal{P}_1^{\Delta} \subset \mathcal{E}\mathcal{X}_2$ necessarily of maximal exit index. This is to say that

$$d_2\Gamma = \pi(d_2\widetilde{\Gamma}) = \delta,\ d_0\Gamma = (d_0\widetilde{\Gamma}, \flat_{2,0}) = (d_0\widehat{\Gamma}, 1) = (\widetilde{\epsilon}, 1),\ d_1\Gamma = (d_0\widetilde{\Gamma}, \flat_{2,1}) = (\widetilde{\gamma}, 1) \tag{42}$$

where $\widetilde{\delta} := d_2\widetilde{\Gamma}$ is a path in L covering δ. Say $\ell \in L_p$ is the source of $\widehat{\delta}$ and $\ell' \in L_{p'}$ its target, so that the two compositions in the diagram evaluate to

$$\begin{tikzcd}
(\delta, \widetilde{\gamma}) \arrow[r, mapsto] \arrow[d, mapsto] & \widetilde{\epsilon} \arrow[d, mapsto] \\
\left(t_M(p) \xrightarrow{t_M(\delta)} t_M(p'), t_M(p') \xrightarrow{\rho(\ell')} t_N(\ell'), t_N(\ell') \xrightarrow{t_N(\widetilde{\gamma})} t_N(q)\right) \arrow[d, mapsto] & \left(t_M(p) \xrightarrow{\rho(\ell)} t_N(\ell), t_N(\ell) \xrightarrow{t_N(\widetilde{\epsilon})} t_N(q)\right) \arrow[d, mapsto] \\
\left(t_M(p) \xrightarrow{t_M(\delta)} t_M(p'), t_M(p') \xrightarrow{\rho(\ell') * t_N(\widetilde{\gamma})} t_N(q)\right) \arrow[d, mapsto] & \left(t_M(p) \xrightarrow{\rho(\ell) * t_N(\widetilde{\epsilon})} t_N(q)\right) \\
\left(t_M(p) \xrightarrow{t_M(\delta) * \rho(\ell') * t_N(\widetilde{\gamma})} t_N(q)\right) &
\end{tikzcd}$$

but Γ provides two fillers, namely ρ applied to $\widetilde{\delta}$ which underlies its low edge, and t_N applied to $\widetilde{\Gamma}$ itself, that fill

$$\begin{tikzcd}
t_M(p) \arrow[rr, "t_M(\delta)"] \arrow[d, "\rho(\ell)"'] \arrow[drr, phantom, "\rho(\widetilde{\delta})"] & & t_M(p') \arrow[d, "\rho(\ell')"] \\
t_N(\ell) \arrow[rr, "t_N(\widetilde{\delta})" description] \arrow[dr, "t_N(\widetilde{\epsilon})"'] \arrow[drr, phantom, "t_N(\widetilde{\Gamma})"] & & t_N(\ell') \arrow[dl, "t_N(\widetilde{\gamma})"] \\
& t_N(q) &
\end{tikzcd}$$

as depicted. This exactly provides the desired homotopy $t_M(\delta) * \rho(\ell') * t_N(\widetilde{\gamma}) \sim \rho(\ell) * t_N(\widetilde{\epsilon})$.

As for (41), let $\widetilde{\delta} \in P(N)_{L_p,q}$, $\gamma \in P(N)_{q,q'}$ and let $\widetilde{\epsilon} \in P(N)_{L_p,q'}$ be a choice of composition with filler $\Gamma = (\widetilde{\Gamma}, q) \in \mathcal{E}\mathcal{X}_1$ necessarily of minimal exit index satisfying the index-1 analogue of (42). Say now $\ell \in L_p$ is the source of $\widetilde{\delta}$ and of $\widetilde{\varepsilon}$. The compositions in the lower square evaluate to

$$\begin{tikzcd}
(\widetilde{\delta}, \gamma) \arrow[r, mapsto] \arrow[d, mapsto] & \widetilde{\varepsilon} \arrow[d, mapsto] \\
\left(t_M(p) \xrightarrow{\rho(\ell) * t_N(\widetilde{\delta}) * t_N(\gamma)} t_N(q')\right) & \left(t_M(p) \xrightarrow{\rho(\ell) * t_N(\widetilde{\epsilon})} t_N(q')\right)
\end{tikzcd}$$

but now it suffices to observe that the filler

$$t_N(\widetilde{\Gamma}) = \left(\begin{tikzcd} t_N(q) \arrow[r, "t_N(\gamma)"] & t_N(q') \\ t_N(\ell) \arrow[u, "t_N(\widetilde{\delta})"] \arrow[ur, "t_N(\widetilde{\epsilon})"'] & \end{tikzcd} \right)$$

induces the desired homotopy $\rho(\ell)*t_N(\widetilde{\delta})*t_N(\gamma) \sim \rho(\ell)*t_N(\widetilde{\epsilon})$. This concludes the proof. $\square$

Remark 3.28. In the simplicial incarnation of $\mathcal{EX}(\mathfrak{S})$, non-trivial compositions are given, by virtue of Theorem 3.5, in terms of maps of the form

$$P(M)_{p,p'} \times P(N)_{L_{p'},q} \to P(N)_{L_p,q}$$

and

$$P(N)_{L_p,q} \times P(N)_{q,q'} \to P(N)_{L_p,q'}$$

as featured in (40), (41). While the essential uniqueness of the latter concatenation is clear, that of the former is a consequence of [Tet23, Theorem 2.3], [Lur23, 01L5], and the essential uniqueness of composition.

As noted in Footnote 8, we may work with any bundle isomorphism $\pi^*W \cong \mathrm{N}_N M$ by modifying and applying Lemma 2.10, and so still obtain a ρ as in (37). The rest of the proof of Proposition 3.26 goes through verbatim, so we obtain the following more practical result:

Corollary 3.29. *A Y-structure on N and a solid Y-structure on M, such that $\pi^*W \cong \mathrm{N}_N M$ over L, induce a Cartesian Y-structure on $\mathfrak{S}$.*

3.6. **Cartesian replacements, III: Normal bundles.**

3.7. **A Grothendieck construction.**

3.8. **The linked universal frame and tautological bundles.**

4. Duals

Let $\mathcal{B} = Y \xrightarrow{F} BO(n) \subset \mathcal{V}^{\hookrightarrow}$ be a classical tangential structure. In this section, we introduce, for a smooth manifold or linked space X with solid $\mathcal{B}$-structure, the notion of a (quadratic) dual[12] and a collar, both of which depend on the solid structure in question. The two notions are closely related – the collar $\overline{X}$ is to be an extension of X that possesses on-the-nose $\mathcal{B}$-structure, and the dual $X^!$ is a space that parametrises the difference between X and $\overline{X}$. That is, the solid structure on X gives rise to data resembling that of a fibration with typical fibre X:

$$\begin{tikzcd} X \arrow[r, hook] & \overline{X} \arrow[d, two heads, "p"] \\ & X^! \end{tikzcd} .$$

A $\mathcal{B}$-structured disk algebra A (necessarily n-dimensional) gives rise to a locally-constant factorisation algebra (*lcfa*) $\int_{-} A$ – which we sometimes abusively refer to as 'A' – not on X but on $\overline{X}$, which can be pushed forward down to $X^!$ along the constructible bundle map p, giving a

[12](so named as the idea resembles that of [GK94, §2.1.9])

lcfa thereon ([AFT17, §2.5]). These pushforwards are to factorisation homology as integration along fibres is to integration.

When X is a bordism, and if we rigidify from solid to stable, then $X^!$ is a particulary simple stratified/linked space, and this procedure produces a functorial field theory (*FFT*) of top dimension n landing in a Morita category (that of 1-disk algebras and 0-disk algebra bimodules) with values in the target symmetric-monoidal category of A. In this article, we restrict ourselves to codimension-1 field theories, so linked spaces of depth 1 will suffice.

Similar ideas have appeared in [Sch14], which expanded on an idea in [Lur08, §4]; see also [CS19] and the first half of [CHS21]. We should note that our approach differs from these in a number of important aspects:

(1) Once the background on linked spaces is established, the construction is much simpler; so much so that at this stage we can already start considering examples (see below).
(2) It works for any stable tangential structure with which one chooses to endow bordisms, not just for (stable) framings. We will also briefly discuss a possible extension to bordisms with solid tangential structure.
(3) In particular, we do not use Dunn–Lurie additivity ([Dun88]; [Lur17]), nor assume any additivity properties of our input disk algebras.
(4) We do not use the Lurie–Calaque–Scheimbauer (∞, n)-category model for bordisms, where a bordism is defined, roughly speaking, by cut functions on a manifold of full dimension. Instead, we deal with bordisms in the more familiar sense of the word, and so the term 'TQFT' carries for us the more practical (and older) Atiyah–Segal–Witten meaning.
(5) The field theory needn't be fully extended.
(6) Defects are first-class citizens.
(7) Our treatment of cutting-and-gluing is quite on-the-nose and conforms to that in more physically-minded literature, such as e.g. in [CMR18].

4.1. **Exposition, I: Examples.** We will consider a variety of examples before defining p formally. In this section, the link embeddings $\iota\colon L \hookrightarrow N$ are *closed*.

Notation 4.1. In this section, we will resume the notation $\boldsymbol{n} \coloneqq \mathbb{R}^n$. This will help clarify the crucially different roles of some $\mathbb{R}$-factors that appear below.

Example 4.2 (codimension-0 bordisms, $\partial = \emptyset$). Say $X = M$ is a smooth n-manifold (which, in our convention, means $\partial M = \emptyset$), which needn't be compact. A solid $\mathcal{B}$-structure on M is necessarily an on-the-nose $\mathcal{B}$-structure. Then $\overline{X} = M \times \mathbf{0} = M$, and $M^! = \mathbf{0}$. Pushing A forward along the trivial M-bundle

$$\begin{array}{c} M \\ \big\downarrow\!\!\!\downarrow \\ \mathbf{0} \end{array}$$

produces a 0-disk algebra with underlying object $\int_M A$. That is, top-dimensional bordisms without boundary are evaluated simply by factorisation homology. The pointing, i.e., the structure of a 0-disk algebra, is given by the functoriality of $\int_{\_} A$ along the inclusion of the empty set.

Example 4.3 (codimension-1 bordisms, $\partial = \emptyset$). Consider now a smooth manifold N with $\partial N = \emptyset$ of dimension $n-1$. A stable $\mathcal{B}$-structure on N is a $\mathcal{B}$-structure on $\overline{N} = N \times \mathbf{1}$, and we have $N^! = \mathbf{1}$. Pushing A forward along the product N-bundle

$$\begin{tikzcd} N \times \mathbf{1} \arrow[d, two heads] \\ \mathbf{1} \end{tikzcd}$$

produces a 1-disk algebra. Its underlying object is simply $\int_{N\times\mathbb{R}} A$.

Example 4.4 (codimension-0 bordisms, $\partial \neq \emptyset$ and connected). Let now M be a top-dimensional bordism with *connected* non-empty boundary $\partial = \partial M$. We consider it as a linked space $\mathfrak{M}$ of type

$$\partial \xleftarrow{=} \partial \xhookrightarrow{\iota_+} M^\circ,$$

as in Example 2.15. Recall that the restriction of $\mathrm{T}\mathfrak{M}$ to the link ∂ is given, up to isomorphism, by the bundle $\mathrm{T}\partial \oplus \varepsilon^1$. A stable $\mathcal{B}$-structure on $\mathfrak{M}$ (Section 3.5) amounts to a $\mathcal{B}$-structure on M° and a $\mathcal{B}$-structure on $\partial \times \mathbf{1}$ together with an equivalence of $\mathcal{B}$-structures on $\partial \times \mathbf{1}$ between the one mentioned and the one induced from the one on M° along $\mathrm{pr}_1^*\iota_+^*\mathrm{T}M^\circ \simeq \mathrm{pr}_1(\mathrm{T}\partial \oplus \varepsilon^1) \simeq {\mathrm{pr}_1}^*\mathrm{T}(\partial \times \mathbf{1})$, where $\mathrm{pr}_1 : \partial \times \mathbf{1} \to \partial$ denotes the coordinate projection. E.g., when $\mathcal{B}$ classifies n-framings, this means that the framing splits along the boundary in the usual sense.

In order to make the notation suggestive, let us assume we are in an oriented context and ∂ is the incoming boundary. (This has no bearing on what follows except in notation.) This datum determines a $\mathcal{B}$-structure on

$$\overline{M} = \partial \times \mathbb{R}_{\leq 0} \coprod_{\partial \times \{0\}} M. \tag{43}$$

We now consider this collar as a linked space $\overline{\mathfrak{M}}$, still of depth one but with three strata:

$$\begin{tikzcd}[column sep=small] & \partial \arrow[dl, hook', "\mathrm{id}\times\{+\}"'] \arrow[dr, two heads, "="] & & \partial \arrow[dl, two heads, "="'] \arrow[dr, hook, "\iota_+"] & \\ \partial \times \mathbb{R}_{<0} = \partial \times \mathbf{1} & & \partial & & M^\circ \end{tikzcd} \tag{44}$$

or

$$\partial \times \mathbf{1} \xleftarrow{\mathrm{id}\times\{+\}} \partial \xhookrightarrow{\iota_+} M^\circ \tag{45}$$

for short (see Figure 3). The stable $\mathcal{B}$-structure on $\overline{\mathfrak{M}}$ is tantamount to $\mathcal{B}$-structures on the two higher strata $\partial \times \mathbf{1}$, M° of $\overline{\mathfrak{M}}$, and to an equivalence between their pullbacks onto the common link ∂.

We read off the quadratic dual directly from $\mathfrak{M}$:

$$\mathfrak{M}^! = \left(\mathbf{1} \xhookleftarrow{+} \{0\} = \mathbf{0} \xrightarrow{=} \mathbf{0}\right),$$

namely the linked half-line $\mathbb{R}_{\leq 0}$. The projection

$$\begin{tikzcd} \overline{\mathfrak{M}} \arrow[d, two heads, "p"] \\ \mathfrak{M}^! \end{tikzcd}$$

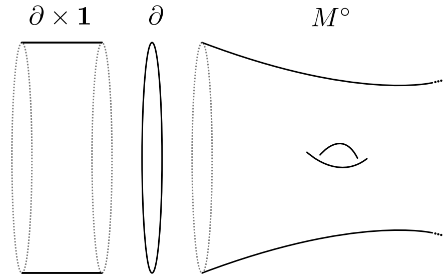

FIGURE 3. The three strata of the linked stable collar of a manifold with connected boundary.

off the collar is given by

$$
\begin{tikzcd}
 & \partial \arrow[dl, hook', "\mathrm{id}\times\{+\}"'] \arrow[dd, dashed, two heads] \arrow[dr, two heads, "="] & & \partial \arrow[dl, two heads, "="'] \arrow[dddl, dashed, two heads] \arrow[dr, hook, "\iota_+"] & \\
\partial\times\mathbf{1} \arrow[dd, two heads, "\mathrm{pr}_2"] & & \partial \arrow[dd, two heads] & & M^\circ \arrow[ddll, two heads] \\
 & \mathbf{0} \arrow[dl, hook', "+"] \arrow[dr, two heads, "="] & & & \\
\mathbf{1} & & \mathbf{0} & &
\end{tikzcd}
\tag{46}
$$
,

for which we also write

$$
\begin{tikzcd}
\partial\times\mathbf{1} \arrow[d, two heads] & \partial \arrow[l, hook', "\mathrm{id}\times\{+\}"'] \arrow[r, hook, "\iota_+"] \arrow[d, two heads] & M^\circ \arrow[d, two heads] \\
\mathbf{1} & \mathbf{0} \arrow[l, hook', "+"] \arrow[r, two heads, "="'] & \mathbf{0}
\end{tikzcd}
\tag{47}
$$
,

with the above meaning understood.[13] in The 'typical fibre' inclusion $\mathfrak{M} \hookrightarrow \overline{\mathfrak{M}}$ – a description which here begins to lose its familiar meaning – is given in the in the obvious manner. The linked projection p is the systematised version of the obvious projection $\overline{M} \to \mathbb{R}_{\leq 0}$ which remembers every stratum's history. (A disk algebra on this target, the pushforward algebra, is determined by a 1-disk algebra and a pointed left-module.) Because of this, it lends itself to generalisation, as we show below.

Remark 4.5. The examples above show clearly that $X^!$ parametrises the normal bundle of the stable $\mathcal{B}$-structure on X. In the non-trivially stratified Example 4.4 we see, however, that the classifying map thereof is stratification-reversing, which is why we only speak out loud about the stratification-preserving projections $\overline{X} \to X^!$. That oughtn't suggest that it is impossible to express classifiers of stratified normal bundles within an appropriate framework. When $\mathcal{B}$ is not a classical tangential structure, however, both the notion of collar and that of quadratic dual become very problematic, as the rank of the normal bundle may both increase or decrase

[13]We introduce better notation in Notation 4.12 which however obscures the link maps. For this reason, we keep this more verbose version until then.

as one moves along strata. This makes a general description of $X^!$ very hard, if not impossible. We restrict ourselves to (relaxations of) classical $\mathcal{B}$.

Example 4.6 (codimension-0 bordisms, $\partial = \partial_L \amalg \partial_R$). Let M be an n-dimensional bordism and $\partial M = \partial_L \amalg \partial_R$ with ∂_L, ∂_R connected. We see M as a linked space $\mathfrak{M}$ of depth 1 but now with three strata:

$$\begin{tikzcd} & \partial_L \arrow[dl, two heads, "="'] \arrow[dr, hook, "\iota_{+L}"] & & \partial_R \arrow[dl, hook', "\iota_{+R}"'] \arrow[dr, two heads, "="] & \\ \partial_L & & M^\circ & & \partial_R \end{tikzcd} .$$

The meaning of a stable $\mathcal{B}$-structure on $\mathfrak{M}$ can be expressed analogously to Example 4.4.[14] We see that the collar $\overline{\mathfrak{M}}$, in short and suggestive notation (cf. (45)), is

$$\partial_L \times \mathbf{1} = \partial_L \times \mathbb{R}_{<0} \xhookleftarrow{\mathrm{id}\times\{+_L\}} \partial_L \xhookrightarrow{\iota_{+L}} M^\circ \xhookleftarrow{\iota_{+R}} \partial_R \xhookrightarrow{\mathrm{id}\times\{+_R\}} \partial_R \times \mathbb{R}_{>0} = \partial_R \times \mathbf{1},$$

the linked version of $\overline{M} = \partial_L \times \mathbb{R}_{\leq 0} \coprod_{\partial_L \times \{0\}} M \coprod_{\partial_R \times \{0\}} \partial_R \times \mathbb{R}_{\geq 0}$, and the dual $\mathfrak{M}^!$ is

$$\begin{tikzcd} & \mathbf{0} \arrow[dl, hook', "+_L"'] \arrow[dr, two heads] & & \mathbf{0} \arrow[dl, two heads] \arrow[dr, hook, "+_R"] & \\ \mathbf{1} & & \mathbf{0} & & \mathbf{1} \end{tikzcd} ,$$

the linked version of $M^! = \mathbb{R}_{\{0\}}$, the real line with three-fold stratification given by (say) the defect $\{0\} \subset \mathbb{R}$ the two components of its complement. We write $\mathfrak{M}^! = \left(\mathbf{1} \xhookleftarrow{+_L} \mathbf{0} \xhookrightarrow{+_R} \mathbf{1}\right)$. We observe the projection $p\colon \overline{\mathfrak{M}} \to \mathfrak{M}^!$ to be given, in short notation (cf. (47)), by

$$\begin{tikzcd} \partial_L \times \mathbf{1} \arrow[d, two heads] & \partial_L \arrow[l, hook', "\mathrm{id}\times\{+_L\}"'] \arrow[r, hook, "\iota_{+L}"] & M^\circ \arrow[d, two heads] & \partial_R \arrow[l, hook', "\iota_{+R}"'] \arrow[r, hook, "\mathrm{id}\times\{+_R\}"] & \partial_R \times \mathbf{1} \arrow[d, two heads] \\ \mathbf{1} & & \mathbf{0} \arrow[ll, hook', "+_L"'] \arrow[rr, hook, "+_R"] & & \mathbf{1} \end{tikzcd}$$

where this time we omitted the induced maps on links.

Example 4.7 (codimension-1 bordisms, $\partial \neq \emptyset$ and connected). Let M be a bordism of dimension $n-1$ with connected boundary $\partial = \partial M$. As an ordinary space, the collar will be given simply by the product of (43) with $\mathbf{1}$. For us, it has again three strata, so $\overline{\mathfrak{M}}$, the analogue of (44), is given by

$$\begin{tikzcd} & \partial \times \mathbf{1} \arrow[dl, hook', "(\mathrm{id}_\partial\times\{+\})\times\mathrm{id}_\mathbf{1}"'] \arrow[dr, two heads, "="] & & \partial \times \mathbf{1} \arrow[dl, two heads, "="'] \arrow[dr, hook, "\iota_+\times\mathrm{id}_\mathbf{1}"] & \\ \partial \times \mathbf{2} = \partial \times \mathbf{1} \times \mathbf{1} & & \partial \times \mathbf{1} & & M^\circ \times \mathbf{1} \end{tikzcd}$$

and the collar by

$$\mathfrak{M}^! = \left(\mathbf{2} \hookleftarrow \mathbf{1} \xrightarrow{=} \mathbf{1}\right),$$

the linked half-plane $\mathbb{H}$. The projection $\overline{\mathfrak{M}} \to \mathfrak{M}^!$ is given exactly as in (46) up to a factor of $\mathbf{1}$.

[14] As the constructions of the stable collar and stable dual are the same for any classical $\mathcal{B}$, we will no longer mention this.

Example 4.8 (codimension-1 bordisms, $\partial = \partial_L \amalg \partial_R$). This is, similarly to the discussion of codimension-0 bordisms, a small modification of Example 4.7. The dual is given, in short notation (cf. Example 4.6), by

$$\mathbf{2} \hookleftarrow \mathbf{1} \hookrightarrow \mathbf{2},$$

the linked version $\mathbb{R}^2_{\mathbb{R}}$, the real plane with three-fold stratification induced by (say) the defect $\{x = 0\} \subset \mathbb{R}^2$.

A distinct advantage of this approach is that it treats defects on the same footing as boundaries.

Example 4.9 (closed submanifold defects). Let M be a smooth n-manifold and $\Sigma \subset M$ a smooth submanifold of codimension m, which yields a linked manifold $\mathfrak{M}$ with link $\mathbb{S} := L = \mathbb{S}(\mathrm{N}_M \Sigma)$ (see Example 2.14). Assume $\mathbb{S}$ is connected, so Σ has codimension at least 2. (Otherwise, we speak of a 'cut locus' – to be treated in Section 4.2.) If $\mathfrak{M}$ is constructible (in that the normal bundle of $\mathbb{S}$ inside $M \setminus \Sigma$ is trivialisable), then the collar $\overline{\mathfrak{M}}$ is again three-fold stratified, given by

$$\begin{tikzcd}
& \mathbb{S} \arrow[dl, hook'] \arrow[dr, two heads, "="'] & & \mathbb{S} \arrow[dl, two heads, "="] \arrow[dr, hook] & \\
\mathbb{S} \times \mathbf{1} & & \mathbb{S} & & M \setminus \Sigma
\end{tikzcd}$$

and the dual $\mathfrak{M}^!$ by the half-line $\left(\mathbf{1} \hookleftarrow \mathbf{0} \xrightarrow{=} \mathbf{0}\right)$ together with the obvious projection $\overline{\mathfrak{M}} \to \mathfrak{M}^!$. Thus, closed submanifold defects are treated exactly the same way as bordisms. Analogous considerations apply when M has dimension strictly smaller than n (see Example 4.7), or when there are multiple non-intersecting defects, or both (cf. Examples 4.6 and 4.8).

Remark 4.10. Note that the 'right side' of the collar in Example 4.9, $\mathbb{S} \overset{=}{\twoheadleftarrow} \mathbb{S} \hookrightarrow M \setminus \Sigma$ is simply the linked version of the blow-up (a.k.a. unzip in the conically-smooth literature) of M at Σ. In light of the previous examples, we can identify the linked spaces induced by bordisms as those whose blow-ups coincide with themselves. The 'left side', in contrast is simply the collar as in the simple codimension-1 case of Example 4.3.

Remark 4.11. There is a different approach to closed submanifold defects than Example 4.9 which reflects the geometry around the defect submanifold better. We will explore this later. The version above should be read as the bare-bones, minimalist approach.

4.2. **Exposition, II: Cutting and gluing.** Finally, we will discuss cutting-and-gluing.

Let M be (for simplicity) a bordism of dimension n, $\partial = \partial M = \partial_L \amalg \partial R$ the components of its boundary, and let $\iota\colon \Sigma \subset M^\circ$, a *cut locus*, be a closed codimension-1 submanifold in the interior, such that $M = M_L \cup_{\Sigma\times\mathbb{R}} M_R$ is a collar-gluing with $\partial_L \subset M_L$, $\partial_R \subset M_R$. These data are organised as a linked space $\mathfrak{M}_\Sigma$ in the obvious manner by mixing Examples 2.14 and 2.15. Writing $M_L^\circ = M_L \setminus \partial_L$ and $M_R^\circ = M_R \setminus \partial_R$, we have that the stable collar $\overline{\mathfrak{M}_\Sigma}$ is

$$\begin{tikzcd}[column sep=tiny]
& \partial_L \arrow[dl, hook'] \arrow[dr, two heads] & & \partial_L \arrow[dl, two heads] \arrow[dr, hook] & & \Sigma \arrow[dl, hook'] \arrow[dr, two heads] & & \Sigma \arrow[dl, two heads] \arrow[dr, hook] & & \Sigma \arrow[dl, hook'] \arrow[dr, two heads] & & \Sigma \arrow[dl, two heads] \arrow[dr, hook] & & \partial_R \arrow[dl, hook'] \arrow[dr, two heads] & & \partial_R \arrow[dl, two heads] \arrow[dr, hook] & \\
\partial_L \times \mathbf{1} & & \partial_L & & M_L^\circ & & \Sigma & & \Sigma \times \mathbf{1} & & \Sigma & & M_R^\circ & & \partial_R & & \partial_R \times \mathbf{1}
\end{tikzcd},$$

the linked version of

$$\overline{M_\Sigma} = \partial_L \times \mathbb{R}_{\leq 0} \cup_{\partial_L \times \{0\}} M_L \cup_{\Sigma\times\mathbb{R}_{<1}} \Sigma \times \mathbb{R} \cup_{\Sigma\times\mathbb{R}_{>-1}} M_R \cup_{\partial_R\times\{0\}} \partial_R \times \mathbb{R}_{\geq 0}.$$

The dual $\mathfrak{M}^!_\Sigma$ is

$$\begin{tikzcd}[column sep=small]
& \mathbf{0} \arrow[dl, hook'] \arrow[dr, two heads] & & \mathbf{0} \arrow[dl, two heads] \arrow[dr, hook] & & \mathbf{0} \arrow[dl, hook'] \arrow[dr, two heads] & & \mathbf{0} \arrow[dl, two heads] \arrow[dr, hook] & \\
\mathbf{1} & & \mathbf{0} & & \mathbf{1} & & \mathbf{0} & & \mathbf{1}
\end{tikzcd},$$

the linked version of $\mathbb{R}_{\{\pm 1\}}$, the real line with defects $\{-1\}, \{1\}$. The projection $p_\Sigma \colon \overline{\mathfrak{M}_\Sigma} \to \mathfrak{M}^!_\Sigma$ is, in short notation, as follows:

$$\begin{tikzcd}[column sep=small]
\partial_L \times \mathbf{1} \arrow[d, two heads] & \partial_L \arrow[l, hook'] \arrow[r, hook] \arrow[dr, two heads] & M^\circ_L \arrow[d, two heads] & \Sigma \arrow[l, hook'] \arrow[r, hook] \arrow[dl, two heads] & \Sigma \times \mathbf{1} \arrow[d, two heads] & \Sigma \arrow[l, hook'] \arrow[r, hook] \arrow[dr, two heads] & M^\circ_R \arrow[d, two heads] & \partial_R \arrow[l, hook'] \arrow[r, hook] \arrow[dl, two heads] & \partial_R \times \mathbf{1} \arrow[d, two heads] \\
\mathbf{1} & & \mathbf{0} \arrow[ll, hook'] \arrow[rr, hook] & & \mathbf{1} & & \mathbf{0} \arrow[ll, hook'] \arrow[rr, hook] & & \mathbf{1}
\end{tikzcd}$$

By *cutting-and-gluing* (*of M along* Σ) we mean the *refinement* map (a map whose restrictions to the strata are all isomorphisms – a barbarian notion of recollement that will suffice for our purposes)

$$r \colon \mathfrak{M}_\Sigma \to \mathfrak{M},$$

with target as in Example 4.6, given, after decorating the outward copies of the cut locus with ℓ/r, by

$$\begin{tikzcd}[column sep=small]
\partial_L \arrow[d] & {\color{red}\partial_L} \arrow[l, two heads] \arrow[r, hook] \arrow[drr] & M^\circ_L & {\color{red}\Sigma_\ell} \arrow[l, hook'] \arrow[r, two heads] \arrow[drrr, two heads] & \Sigma_\ell & {\color{red}\Sigma_\ell} \arrow[l, two heads] \arrow[r, hook] \arrow[dr, two heads] & \Sigma \times \mathbf{1} \arrow[d] & {\color{red}\Sigma_r} \arrow[l, hook'] \arrow[r, two heads] \arrow[dl, two heads] & \Sigma_r & {\color{red}\Sigma_r} \arrow[l, two heads] \arrow[r, hook] \arrow[dlll, two heads] & M^\circ_R & {\color{red}\partial_R} \arrow[l, hook'] \arrow[r, two heads] \arrow[dll] & \partial_R \arrow[d] \\
\partial_L & & & {\color{red}\partial_L} \arrow[lll, two heads] \arrow[rrr, hook, two heads] & & & M^\circ & & & {\color{red}\partial_R} \arrow[lll, hook', two heads] \arrow[rrr, two heads] & & & \partial_R
\end{tikzcd}$$

where the vertical maps are all (restricted) identities. For better legibility, we rendered the links in red.

In order to conform to Definition 3.10, it remains to specify

$$\begin{tikzcd}
{\color{red}\Sigma_{\ell/r}} \arrow[r, dotted] \arrow[d] & (M^\circ)^I \arrow[d] \\
\Sigma_{\ell/r} \times M^\circ_{L/R} \arrow[r] & M^\circ \times M^\circ
\end{tikzcd}, \qquad
\begin{tikzcd}
{\color{red}\Sigma_{\ell/r}} \arrow[r, dotted] \arrow[d] & (M^\circ)^I \arrow[d] \\
\Sigma_{\ell/r} \times (\Sigma \times \mathbf{1}) \arrow[r] & M^\circ \times M^\circ
\end{tikzcd},$$

which are all given analogously. For instance, for the map ${\color{red}\Sigma_\ell} \to (M^\circ)^I$ on the left, first reparametrise the gluing line as $\mathbb{R} \cong (-1, 1)$ such that $\Sigma \times \mathbf{1} = \Sigma \times (-1, 1)$ is the middle embedding. Extend now the tubular neighbourhood of Σ to times $(-2, 2) \supset (-1, 1)$. The link embedding ${\color{red}\Sigma_\ell} \hookrightarrow M^\circ_L$ is given by following for a nonzero time τ with $-1 > \tau > -2$ the outward nonvanishing vector field X_Σ along Σ that frames the rank-1 normal bundle of $\Sigma \subset M$ (whereas $\Sigma_r \hookrightarrow M^\circ_R$ is determined by following it for time $-\tau$), while the stratum map $\Sigma_\ell \to M^\circ$ hits the copy at time $-1 \in (-2, 2)$. Now, the map ${\color{red}\Sigma_\ell} \to (M^\circ)^I$ at $p \in {\color{red}\Sigma_\ell}$ can be chosen to be the path from p at time -1 to p at time τ along X_Σ. See Remark 3.15 for a more general version of this construction.

Now, as we show in Section 4.8, the operation $\overline{(-)} \twoheadrightarrow (-)^!$ is compatible with cutting and gluing, in the sense that it is covariant along refinements. Using Example 4.6 to get

$p : \overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$, we have commuting square

(48)
$$\begin{tikzcd}
\overline{\mathfrak{M}_\Sigma} \arrow[r, "\overline{r}"] \arrow[d, two heads, "p_\Sigma"'] & \overline{\mathfrak{M}} \arrow[d, two heads, "p"] \\
\mathfrak{M}^!_\Sigma \arrow[r, "r^!"'] & \mathfrak{M}^!
\end{tikzcd} .$$

which we call a *cutting-and-gluing square*. In fact, both $\overline{r}$ and $r^!$ will be (linked versions of) constructible bundles, so that pushing forward an algebra along either path in the diagram will give equivalent results.

Before proceeding, let us lighten the notation even further by setting the following convention:

Notation 4.12. We will write

$$\begin{tikzcd} M \arrow[r, tail, no head, "L"] & N \end{tikzcd}$$

for a linked space of type $M \overset{\pi}{\twoheadleftarrow} L \overset{\iota}{\hookrightarrow} N$.

Now, $\overline{r} : \overline{\mathfrak{M}_\Sigma} \to \overline{\mathfrak{M}}$ is

$$\begin{tikzcd}
\partial_L \times \mathbf{1} \arrow[d] & \partial_L \arrow[l, tail, no head] \arrow[r, tail, no head] \arrow[d] & M_L^\circ \arrow[drr] & \Sigma \arrow[l, tail, no head] \arrow[r, tail, no head] \arrow[dr] & \Sigma \times \mathbf{1} \arrow[d] & \Sigma \arrow[l, tail, no head] \arrow[r, tail, no head] \arrow[dl] & M_R^\circ \arrow[dll] & \partial_R \arrow[l, tail, no head] \arrow[r, tail, no head] \arrow[d] & \partial_R \times \mathbf{1} \arrow[d] \\
\partial_L \times \mathbf{1} & \partial_L \arrow[l, tail, no head] \arrow[rrr, tail, no head] & & & M^\circ & & & \partial_R \arrow[lll, tail, no head] \arrow[r, tail, no head] & \partial_R \times \mathbf{1}
\end{tikzcd}$$

and $r^! : \mathfrak{M}^!_\Sigma \to \mathfrak{M}^!$ is

$$\begin{tikzcd}
\mathbf{1} \arrow[d] & \mathbf{0} \arrow[l, tail, no head] \arrow[r, tail, no head] \arrow[dr] & \mathbf{1} \arrow[d] & \mathbf{0} \arrow[l, tail, no head] \arrow[r, tail, no head] \arrow[dl] & \mathbf{1} \arrow[d] \\
\mathbf{1} & & \mathbf{0} \arrow[ll, tail, no head] \arrow[rr, tail, no head] & & \mathbf{1}
\end{tikzcd},$$

the linked version of the map $\mathbb{R}_{\{\pm 1\}} \twoheadrightarrow \mathbb{R}_{\{0\}}$ that collapses $[-1, 1]$ onto $\{0\}$ and scales up the two sides of the former to $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$. We see by direct inspection that (48) commutes. See Figure 4.

4.3. **Exposition, III: Corners.** We will conclude the expository sections by giving an informal treatment of corners in the linked context, as well as the extension of $\overline{(-)} \twoheadrightarrow (-)^!$ to depth 2. In future work, this will be formalised, and extended to arbitrary depth, though the idea should become clear already here.

Figure 5 gives an essentially full picture of $\overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ induced by a manifold M (for simplicity of full dimension n – otherwise see Examples 4.7 and 4.8) with *a* corner – the passage to multiple corners is analogous to the passage to multiple boundary components (cf. Example 4.6).

Here, $\overline{\mathfrak{M}}$ is a depth-2 linked space with 9 strata: those of full dimension are (isomorphic to) $\partial^2 \times \mathbf{2}$, $\partial_R \times \mathbf{1}$, $\partial_L \times \mathbf{1}$ and M^2; then there is a cross' worth of codimension-1 strata, depicted by solid lines in Figure 5, and finally their codimension-2 intersection, depicted by the solid point.

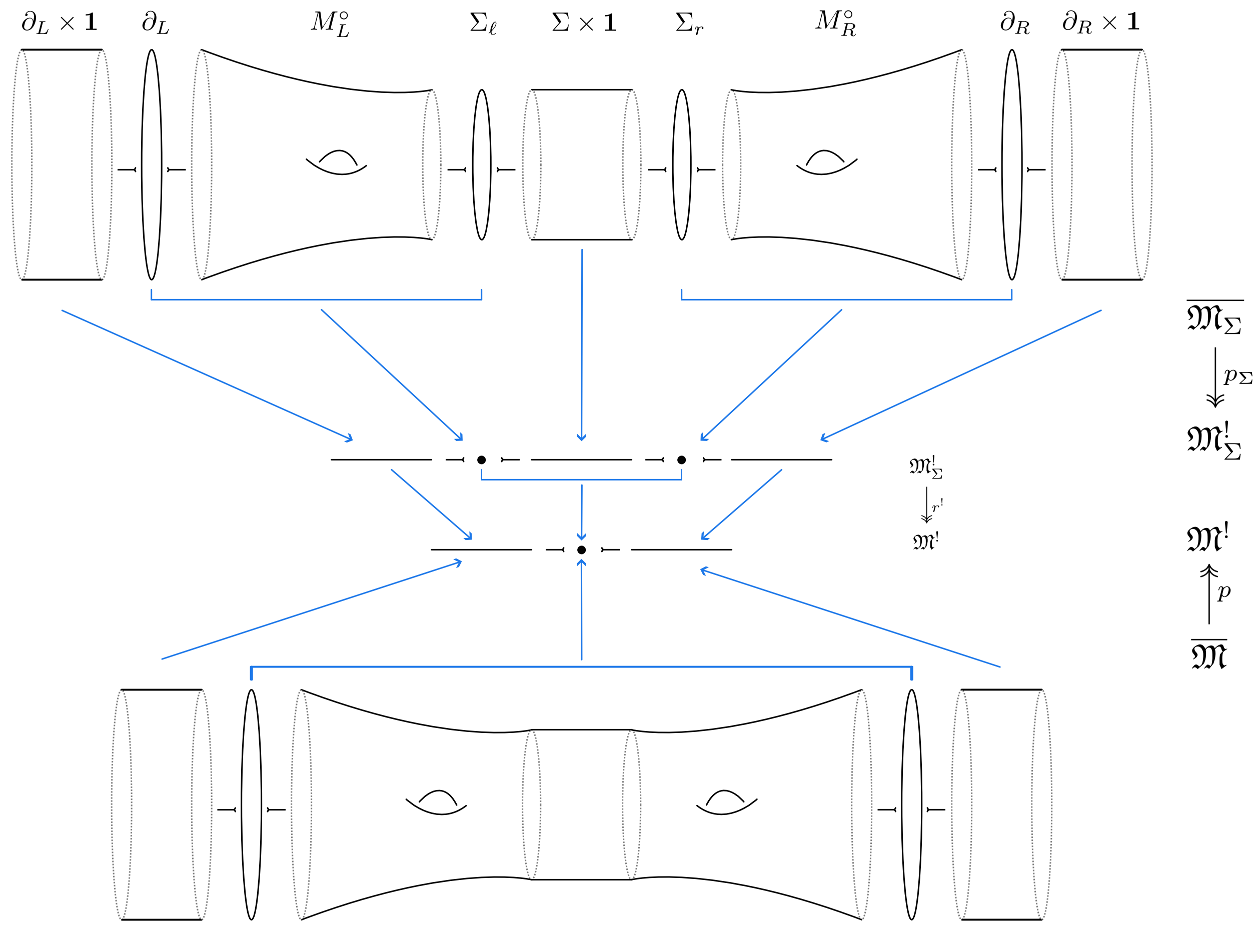

FIGURE 4. The cutting-and-gluing square (48) using Notation 4.12; $\overline{r}\colon \overline{\mathfrak{M}_\Sigma} \to \overline{\mathfrak{M}}$ omitted.

One can essentially see how the depth-2 dual

$$\mathfrak{M}^! = \left(\begin{tikzcd} \partial^2 \arrow[r, tail, no head] \arrow[d, tail, no head] \arrow[dr, tail, no head, dashed] & \partial_R \arrow[d, tail, no head] \\ \partial_L \arrow[r, tail, no head] & M^\circ \end{tikzcd} \right)$$

must be assembled by considering the adjacent depth-1 pairs. This is illustrated in Figure 6.

Now, recall Definition 3.9. Indeed, $\mathfrak{M}^!$ is, as a stratified space, the quarter-plane $\mathbb{R}_{\leq 0} \times \mathbb{R}_{\leq 0}$. Let us introduce it as a *product linked space*, the simplest type of linked space of depth 2. .

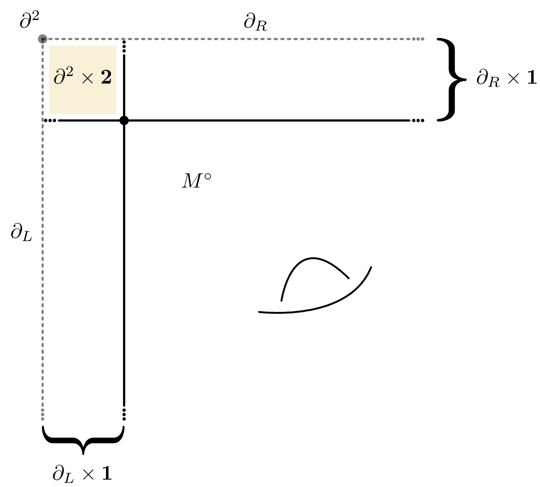

FIGURE 5. Construction of $\overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ induced by M with connected boundary $\partial = \partial_L \cup \partial^2 \cup \partial_R$, with ∂^2 denoting the corner.

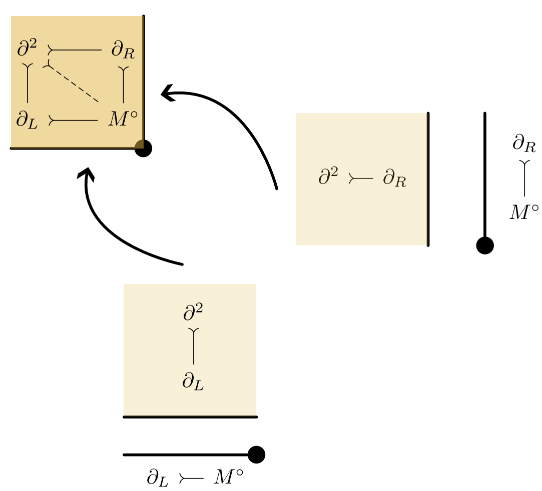

FIGURE 6. The assembly of a corner: $\mathfrak{M}^!$ from depth-1 data.

Each factor is $\mathbf{0} \overset{\mathbf{0}}{\rightarrowtail} \mathbf{1}$ stratified over $\mathfrak{P} = \{0 < 1\}$, so the product is

$$\begin{tikzcd}
\mathbf{0}\times\mathbf{0} \arrow[r, tail, "\mathbf{0}"] \arrow[d, tail, "\mathbf{0}"'] \arrow[dr, tail, "\mathbf{0}\times\mathbf{0}"] & \mathbf{1}\times\mathbf{0} \arrow[d, tail, "\mathbf{0}"] \\
\mathbf{1}\times\mathbf{0} \arrow[r, tail, "\mathbf{0}"'] & \mathbf{1}\times\mathbf{1}
\end{tikzcd}$$

stratified over

$$\mathfrak{P} \times \mathfrak{P} = \left\{ \begin{array}{ccc} (0,0) & < & (1,0) \\ \wedge & \nwarrow & \wedge \\ (1,0) & < & (1,1) \end{array} \right\}.$$

In a general product of depth-1 linked spaces, the new links will be given, for $(\mathfrak{p},\mathfrak{q}) \leq (\mathfrak{p}',\mathfrak{q}')$ in $\mathfrak{P} \times \mathfrak{Q}$, by

$$L_{(\mathfrak{p},\mathfrak{q}),(\mathfrak{p}',\mathfrak{q}')} = L_{\mathfrak{p},\mathfrak{p}'} \times L_{\mathfrak{q},\mathfrak{q}'}.$$

In a general linked space of depth 2 (or indeed in any depth) indexed over a poset $\mathfrak{P}$ of depth 2, we ask that, for each concantenation $\mathfrak{p} < \mathfrak{q} < \mathfrak{r}$, commuting diagrams

$$\begin{tikzcd} L_{\mathfrak{p}\mathfrak{q}} \times_{M_{\mathfrak{q}}} L_{\mathfrak{q}\mathfrak{r}} \arrow[rr, dotted] \arrow[dr] & & L_{\mathfrak{p}\mathfrak{r}} \arrow[dl] \\ & M_{\mathfrak{p}} \times M_{\mathfrak{r}} & \end{tikzcd}.$$

We call the maps $L_{\mathfrak{p}\mathfrak{q}} \times_{M_{\mathfrak{q}}} L_{\mathfrak{q}\mathfrak{r}} \to L_{\mathfrak{p}\mathfrak{r}}$ *concatenation* maps and the condition that they cover $M_{\mathfrak{p}} \times M_{\mathfrak{r}}$ that concatenation be *rel endpoints*. The terminology is justified in view Theorem 3.5; see also Remark 3.15.

Finally, the projection $p\colon \overline{\mathfrak{M}} \twoheadrightarrow \mathfrak{M}^!$ is visible in Figure 5: $\partial_{R/L} \times \mathbf{1}$ project to the two coordinates of the quarter plane, $\partial^2 \times \mathbf{2}$ projects to $\mathbf{2}$, and M° and its closure are collapsed to the corner point of the quarter plane.

4.4. **Duals in depth** 0**.**

4.5. **Cartesian duals.**

4.6. **Stable duals.**

4.7. **Duals of bordisms.**

4.8. **Covariance along refinements.**

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