STRUCTURED POISSON–MORITA THEORY

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Abstract.

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1. Introduction

2. Preliminaries

We recall some material needed to construct classifying maps out of stratified spaces.

2.1. **Striation sheaves and ∞-categories.** Let $\mathbf{\Delta}$ denote the simplex category, and $\mathsf{pSh}(\mathbf{\Delta})$ denote $\mathcal{S}$p-valued preheaves on $\mathbf{\Delta}$.[1] It is well-known that

$$\mathcal{C}\mathrm{at}_\infty \simeq \mathsf{pSh}^{\mathrm{cSegal}}(\mathbf{\Delta})$$

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[1]Unless otherwise specified, all (pre)sheaves are $\mathcal{S}$p-valued, i.e. valued in the ∞-category of spaces. Script initials indicate ∞-categories. We use the term '∞-category' in the sense of Joyal or Lurie ([Lur09]; [Lur]), i.e. to mean a quasi-category, a simplicial set that satisfies the Kan condition for inner horns. For $(\infty, 0)$-categories (by which we mean Kan complexes) we reserve the term '∞-groupoid'. When we work with a complete Segal space, we call it an '∞-category' also.

([JT07]; [Toë05]) where the right hand side denotes complete Segal spaces. It was shown in [AFR18a] that
$$\mathcal{S}\mathrm{tri} \simeq \mathsf{pSh}^{\mathrm{cSegal}}(\mathbf{\Delta}),$$
where $\mathcal{S}\mathrm{tri} \subset \mathsf{pSh}(\mathrm{Strat})$ denotes striation sheaves on the ordinary category of stratified spaces. These are presheaves which satisfy a set of properties that generalise the complete-Segal axioms from $\mathbf{\Delta}$ to Strat, and are also locally-constant when restricted to smooth manifolds along $\mathrm{Mnf} \xrightarrow{K\times -} \mathrm{Strat}$ for any stratified space K [AFR18a, §4]. The latter condition is called *constructibility*, and is equivalent to being stratified-homotopy invariant; see [AFR18a, §2] and below. The map
$$\mathrm{st}^*\colon \mathcal{S}\mathrm{tri} \xrightarrow{\sim} \mathsf{pSh}^{\mathrm{cSegal}}(\mathbf{\Delta})$$
is given by restriction along the opposite of the standard cosimplicial space functor $\mathrm{st}\colon \mathbf{\Delta} \to \mathrm{Strat}$ that to $[p]$ attaches $\Delta^p = (\overline{C}^p(*) \to [p])$, the p-fold closed cone on the point with its natural iterative stratification. We wrote st^* instead of $(\mathrm{st}^{\mathrm{op}})^*$.

2.2. **Exit paths.** The right-adjoint st_* to st^* at $\mathcal{F}$ is given by $\mathrm{Ran}_{\mathrm{st}^{\mathrm{op}}}\mathcal{F}$. As $\mathcal{S}\mathrm{p}$ is complete, $\mathrm{st}_*\mathcal{F}$ at a stratified space X can be computed as a limit over $\mathrm{const}_X \downarrow \mathcal{F}$. Expressed more economically, we have
$$\mathrm{st}_*\mathcal{F}(X) \simeq \mathcal{S}\mathrm{p}(\mathbf{Ex}(X), \mathcal{F}),$$
where the exit-path functor
$$\mathbf{Ex}\colon \mathrm{Strat} \to \mathsf{pSh}(\mathbf{\Delta})$$
is the composition
$$\mathrm{Strat} \xrightarrow{\text{Yoneda}} \mathsf{pSh}(\mathrm{Strat}) \xrightarrow{\mathrm{st}^*} \mathsf{pSh}(\mathbf{\Delta}).$$
Note that the naïve Yoneda embedding does not give $\mathcal{S}\mathrm{p}$-valued presheaves; 'Yoneda' here takes X to $(Y \mapsto \mathcal{S}\mathrm{trat}(Y, X))$, after taking X, Y canonically from Strat to $\mathcal{S}\mathrm{trat}$, where $\mathcal{S}\mathrm{trat} \simeq \mathrm{Strat}[H^{-1}]$ with H denoting stratified homotopy equivalences.[2] In fact, $\mathbf{Ex}$ factors through striation sheaves and lands therefore in complete Segal spaces, and so induces after localisation a functor $\mathbf{Ex}\colon \mathcal{S}\mathrm{trat} \to \mathsf{pSh}^{\mathrm{cSegal}}(\mathbf{\Delta}) \simeq \mathcal{C}\mathrm{at}_\infty$.

2.3. **Extended simplices, stratified homotopies.** Let us define the cosimplicial space $\Delta_e^\bullet\colon \mathbf{\Delta} \to \mathrm{Mnf} \hookrightarrow \mathrm{Strat}$, where $\mathrm{Mnf} \hookrightarrow \mathrm{Strat}$ includes smooth manifolds by stratifying by connected components, i.e. with $\pi_0(M)$ as the (discrete) stratifying poset, $M \to \pi_0(M)$ the projection. One sets $\Delta_e^p = \{(x_0, \ldots, x_p) \in \mathbb{R}^{p+1} : \sum x_k = 1\}$. To $\delta_i\colon [n] \hookrightarrow [n+1]$, which skips the i'th entry in $[n+1]$, one assigns
$$\big((x_k)_{k\le n} \mapsto (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)\big).$$
To $\sigma_i\colon [n+1] \twoheadrightarrow [n]$, which sends $i-1, i \mapsto i-1$, one assigns
$$\big((x_k)_{k\le n+1} \mapsto (x_0, \ldots, x_{i-2}, x_{i-1} + x_i, x_{i+1}, \ldots, x_{n+1})\big).$$
Hence, in general, the j'th image-coordinate is the sum of those source-coordinates whose indices hit j, with the empty sum understood to be zero.

Given maps $f, g\colon X \to Y$, a stratified homotopy $H\colon X \times \mathbb{R} \to Y$ is easily defined using a time coordinate in $\Delta_e^1 = (\mathbb{R} \to *)$, rather than $[0,1] = (\overline{C}(*) \to [1])$,

[2]See [AFR18a] for an independent definition of $\mathcal{S}\mathrm{trat}$.

by asking $H|_{X\times\{0\}} = f$, $H|_{X\times\{1\}} = g$.[3] Higher homotopies, similarly, have time coordinates in $\mathbb{R}^p \simeq \Delta^p_e$. From now on, we will write $\boldsymbol{\delta}^*_i$, $\boldsymbol{\sigma}^*_i$ for pullbacks in the $\Delta^\bullet_e$ direction, and instead equivalently ask, for notational ease, that $\boldsymbol{\delta}^*_0 H = f$, $\boldsymbol{\delta}^*_1 H = g$, and similarly for higher homotopies. This corresponds to simply exchanging 0 and 1 for a 1-homotopy, etc.

2.4. **Vector bundles.** The notion of a (conically-smooth) *vector bundle* $V \to X$ over X was defined in [AFR18a]. For simplicity, all our vector bundles will have finite rank. We now consider the complete Segal space $\mathcal{V}\mathrm{ect}^{\hookrightarrow}\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ which sends $[p]$ to the space of length-$(p+1)$ fibrewise injective sequences $(V_0 \to \cdots \to V_p)$ of vector bundles over $\Delta^\bullet_e$, in the following sense. Denoting by $\boldsymbol{V}^p_k$ a length-$(p+1)$ sequence over Δ^k_e, and omitting p if it is clear or fixed, we define $\mathcal{V}\mathrm{ect}^{\hookrightarrow}$ as follows:

Definition 2.1.

- Points of $\mathcal{V}\mathrm{ect}^{\hookrightarrow}[p]$ are sequences $\boldsymbol{V}_0$; a path from $\boldsymbol{V}_0$ to $\boldsymbol{W}_0$ is a sequence $\boldsymbol{P}_1$ with $\boldsymbol{\delta}^*_0\boldsymbol{P}_1 = \boldsymbol{V}_0$, $\boldsymbol{\delta}^*_1\boldsymbol{P}_1 = \boldsymbol{W}_0$; a path $\boldsymbol{P}_1 \to \boldsymbol{Q}_1$ between paths $\boldsymbol{P}, \boldsymbol{Q}\colon \boldsymbol{V}_0 \to \boldsymbol{W}_0$ is a sequence $\boldsymbol{L}_2$ with $\boldsymbol{\delta}^*_0\boldsymbol{L} = \boldsymbol{P}$, $\boldsymbol{\delta}^*_1\boldsymbol{L} = \boldsymbol{Q}$; and similarly for higher paths.
- On $\delta_k\colon [n] \hookrightarrow [n+1]$, we ask that $\delta^*_k := \mathcal{V}\mathrm{ect}^{\hookrightarrow}(\delta_k)\colon \mathcal{V}\mathrm{ect}^{\hookrightarrow}[n+1] \to \mathcal{V}\mathrm{ect}^{\hookrightarrow}[n]$ send a point $(V_0 \to \cdots \to V_{n+1})$ to the point
$$(V_0 \to \cdots \to \widehat{V_k} \to \cdots V_{n+1})$$
with the arrows around $\widehat{V_k}$ composed; a path $\boldsymbol{P}_1\colon \boldsymbol{V}_0 \to \boldsymbol{V}'_0$ to the path
$$(P_0 \to \cdots \to \widehat{P_k} \to \cdots \to P_{n+1})\colon \delta^*_k\boldsymbol{V}_0 \to \delta^*_k\boldsymbol{V}'_0;$$
and so on. On $\sigma_i\colon [n+1] \twoheadrightarrow [n]$, $\sigma^*_i := \mathcal{V}\mathrm{ect}^{\hookrightarrow}(\sigma_i)$ sends $(P_0 \to \cdots \to P_n)$ to
$$(P_0 \to \cdots \to P_{i-2} \to P_{i-1} \xrightarrow{\mathrm{id}} P_{i-1} \to P_i \to \cdots \to P_n).$$

Remark. Technically, we haven't defined what the 'pullbacks' in the first point of 2.1 are along. We may say that $\mathcal{V}\mathrm{ect}^{\hookrightarrow}[p]$ is of type $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathrm{Set}$, sending $[k]$ to the set of length-$(p+1)$ fibrewise injective sequences of vector bundles over Δ^k_e, and $[l] \to [k]$ to the (ordinary) pullback of the along $\Delta^l_e \to \Delta^k_e$. In this sense, $\boldsymbol{\delta}^*_0\boldsymbol{P}_1$, for instance, is indeed the restriction of $\boldsymbol{P}_1$ to the singleton $\{1\} \subset \Delta^1_e$, seen as a bundle over Δ^0_e; similarly for higher paths.

Definition 2.2. We call
$$\mathrm{st}_*\mathcal{V}\mathrm{ect}^{\hookrightarrow}(X) = \mathcal{S}\mathrm{p}(\mathbf{Ex}(X), \mathcal{V}\mathrm{ect}^{\hookrightarrow})$$
the space of ***($\hookrightarrow$-)constructible vector bundles*** on X.

Notation 1. We will sometimes write $\mathcal{V}$ (resp. V) for $\mathcal{V}\mathrm{ect}$ (resp. Vect).

[3] One may still think of the smooth $\mathbb{R}$-coordinate as a smooth $[0,1]$-coordinate, as a smooth map out of $[0,1]$ by definition comes with the requirement that there be a (non-specified) smooth open extension of the map around $[0,1] \subset \mathbb{R}$; hence 'extended'.

As is easily seen, we have an equivalence $\mathrm{B}O \simeq \mathcal{V}^{\sim}$ of ∞-groupoids, so, on smooth manifolds, $\sim$-constructible vector bundles correspond exactly to ordinary vector bundles.

Definition 2.3. We call $\mathcal{B}\mathrm{un}[p] := |\mathcal{B}\mathrm{un}[p][\bullet]|_{\bullet} := |\{X \xrightarrow{\mathrm{cbl}} \Delta^p \times \Delta_e^{\bullet}\}|_{\bullet}$ the space of ***(constructible) bundles*** over Δ^p. A point is a constructible bundle $X \to \Delta^p$; a path $X_0 \to X_1$ is a bundle $X_{01} \to \Delta^p \times \Delta_e^1$ with $\boldsymbol{\delta}_0^* X_{01} = X_0$, $\boldsymbol{\delta}_1^* X_{01} = X_1$ as bundles over $\Delta^p = \Delta^p \times \Delta_e^0$, and similarly for higher paths. Pullbacks in the first argument are given by restriction.

Construction 2.4. Each constructible bundle $f\colon X \xrightarrow{\mathrm{cbl}} B$ gives a functor $\widehat{f}\colon \mathbf{Ex}(B) \to \mathcal{B}\mathrm{un}$ in $\mathsf{pSh}^{\mathrm{cSegal}}(\boldsymbol{\Delta})$, called the B***-point*** corresponding to f, by restrictions, as follows. At $[p]$, we define $\widehat{f}[p]\colon \mathcal{S}\mathrm{trat}(\Delta^p, B) \to \mathcal{B}\mathrm{un}[p]$, on $\gamma\colon \Delta^p \times \Delta_e^l \to B$, by

$$\widehat{f}[p]\gamma := (\gamma^*(X \xrightarrow{f} B) \xrightarrow{\mathrm{cbl}} \Delta^p \times \Delta_e^l),$$

using the fact that constructible bundles pull back to constructible bundles.

2.5. **Stratified paths.** Before we proceed, we need to collect some facts on paths. Let $X = (X \to P)$ be a stratified space. We will use the generic stratum indices $p, q \in P$. There is an explicit description of $\mathbf{Ex}(X)$ at the level of objects and morphisms. Invertible and noninvertible morphisms therein differ in a fundamental way. We start with the former.

Observation 2.5. *The maximal ∞-subgroupoid of $\mathbf{Ex}(X)$ is the ∞-groupoid of the space $\coprod_p X_p$, i.e. $\coprod_p \mathbf{Ex}(X_p)$.*

Proof. For a stratified path $\gamma\colon \Delta^1 = (\overline{C}(*) \to [1]) \to X$, we will have $\gamma(0) \in X_p$ and $\gamma(1) \in X_q$ for some p and q. Since $0 \in \overline{C}(*)_0$ and $1 \in (0,1] = \overline{C}(*)_1$, we must have $p \preceq q$. If $p \neq q$, then any inverse path γ^{-1} must satisfy $\gamma^{-1}(0) \in X_q$ and $\gamma^{-1}(1) \in X_p$, but γ^{-1} being stratified implies $q \prec p$, which is impossible. So if γ has an inverse, then $p = q$, which is tantamount to the statement. □

The space $(X_p \to X_q)$ of paths starting in X_p and ending in X_q is equivalent to $\mathrm{Link}_{X_p}(X)_q = \mathrm{Link}_{X_p}(X) \cap X_q$, as was observed by Ayala et al. We will prove it briefly, circumventing some machinery developed by these authors.

If A, B are pointed stratified spaces, one writes $\mathcal{S}\mathrm{trat}_*(A, B)$ for the pointed sector of $\mathcal{S}\mathrm{trat}(A, B)$. For any pointed stratified space (X, x) and a basic $(B, 0) := (\mathbb{R}^i \times C(Z), (0, 0))$ with Z compact stratified, we have:

Observation 2.6. *For any neighbourhood $U \subset X$ of x, inclusion induces an equivalence*

$$\mathcal{S}\mathrm{trat}_*(B, U) \simeq \mathcal{S}\mathrm{trat}_*(B, X).$$

Proof. We may assume $i = 0$, since localising at stratified homotopies is equivalent to localising at projections off $- \times \mathbb{R}^i$. Let now $f\colon C(Z) \to X$ with $f(0) = x$ be given, and consider $f^{-1}U \supset \{0\}$. Pick a rescaling $\phi\colon C(Z) \times \mathbb{R} \to C(Z)$ such that $\phi(0, -) \equiv x$, together with some $t \geq 0$ such that $\phi_{t'}(C(Z)) \subset f^{-1}U$ for all

$t' > t$ (any rescaling that is strictly monotonous with respect to inclusion after some nonnegative time will do), pick some such pair $t' > t$, and smoothly reparametrise the $\mathbb{R}$ component such that $t' = 1$, $t = 0$. Now, $H = f \circ \phi$ is a homotopy from f to a map that factors through U.[4] □

Let $i_0 \colon [0] \hookrightarrow [p]$ denote the inclusion at 0.

Corollary 2.7. *We have* $(\{x\} \to X) \simeq \mathrm{Link}_x(X)$.

Proof. The Observation still applies with $\overline{C}(Z)$ as B. Consider the case $Z = *$, i.e. $\mathcal{S}\mathrm{trat}_*(\overline{C}(Z), X)$. Fixing some neighbourhood of x of type $U = \mathbb{R}^j \times C(K)$, we have $\mathcal{S}\mathrm{trat}_*(\overline{C}(*), X) \simeq \mathcal{S}\mathrm{trat}_*(\overline{C}(*), \mathbb{R}^j \times C(K))$. Now, we are interested in the latter space with the constant path removed, which leaves, after cancelling the $\mathbb{R}^j$-component and identifying a path $\gamma\colon [0,1] \to C(K)$, $\gamma(0) = 0$, with its endpoint $\gamma(1) \in K = \{t\} \times K$ (which holds for some $t \neq 0$, its exact value being immaterial up to homotopy): $\mathcal{S}\mathrm{trat}(*, K) \simeq K \simeq \mathrm{Link}_0(C(K)) \simeq \mathrm{Link}_x(X)$. The last two equivalences are standard. □

Passing from $\{x\}$ to X_p and finally taking q-components gives the claim. We record it for future reference:

Corollary 2.8. *We have* $(X_p \to X_q) \simeq \mathrm{Link}_{X_p}(X)_q$.

Example 2.9. Consider $X = \mathbb{R}^2$ with distinguished subspaces $\mathbb{R} = \{y = 0\} \subset \mathbb{R}^2$, $\{0\} \subset \mathbb{R}$. This describes the stratified space $s\colon X \to P$ with $P = \{0 \leq 1 \leq 2\}$[5] and

$$X_0 = \{0\}, \quad X_1 = \mathbb{R}, \quad X_2 = \mathbb{R}^2 \setminus \mathbb{R}.$$

Then $\mathrm{Link}_{X_0}(X) \simeq S^1$ and $\mathrm{Link}_{X_1}(X) \simeq \mathbb{R} \amalg \mathbb{R}$, in particular:

$$(X_0 \to X_1) \simeq S^0, (X_0 \to X_2) \simeq \mathbb{R} \amalg \mathbb{R}, (X_1 \to X_2) \simeq \mathbb{R} \amalg \mathbb{R}.$$

Remark 2.10. Observation 2.6 is not true on closed cones whose *both* ends are required to be sent to x; e.g. for $B = \overline{C}(*) = [0,1]$, and $X = M$ a smooth manifold, a loop in $\mathcal{S}\mathrm{trat}_*(B, M)$ of course need not be homotopable to one within an arbitrary neighbourhood of its base point, as this would be tantamount to contractibility. Still, if $\overline{C}(Z)|_1$ is not required to be sent to x, the statement for $B = \overline{C}(Z)$ remains true by a similar argument.

2.6. **Absolute exit paths and the constructible tangent bundle.** Given $p\colon X \xrightarrow{\text{cbl}} B$, consider the corresponding B-point $\widehat{p}\colon \mathbf{Ex}(B) \to \mathcal{B}\mathrm{un}$ from 2.4, and the pushforward $p_* \coloneqq (p \circ -)\colon \mathbf{Ex}(X) \to \mathbf{Ex}(B)$.

Lemma 2.11. *The fibre of p_* at some $\gamma\colon \Delta^k \to B$ is equivalent to the space of sections $\Delta^k \to \gamma^* X$ of $\widehat{p}(\gamma)$.*

[4] In order to conform to the conventions of 2.3 perfectly, one can again reparametrise $\mathbb{R}$ such that t and t' switch places.

[5] Stratifying posets are equipped with the Alexandrov topology, i.e. downward-closed subsets are closed.

Proof. For simplicity, we consider only $\star = k$, $\bullet = 0$ in explicit detail. If the bundle $\widehat{p}(\gamma)\colon \gamma^* X \to \Delta^k$ has a section $s\colon \Delta^k \to \gamma^* X$, the composition

$$\Gamma\colon \Delta^k \xrightarrow{s} \gamma^* X \xrightarrow{\mathrm{pr}} X$$

defines a point in $\mathbf{Ex}(X)[k][0]$ with $p_*\Gamma = p \circ \mathrm{pr} \circ s = \gamma \circ \widehat{p}(\gamma) \circ s = \gamma$. Conversely, if $\Gamma\colon \Delta^k \to X$ in $\mathbf{Ex}(X)[k][0]$ is given, with pushforward $\gamma \coloneqq p_*\Gamma\colon \Delta^k \to B$ in $\mathbf{Ex}(B)[k][0]$, then the pullback of X along γ can be computed in two steps via the following commutative diagram:

$$\begin{tikzcd}
\Gamma^*(X \times_B X) \arrow[r] \arrow[d] & X \times_B X \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^k \arrow[r] & X \arrow[r] & B.
\end{tikzcd}$$

Thus $\Gamma^*(X \times_B X) \simeq \gamma^* X$ by the pasting law, as both little squares are pullback squares. Now, the diagonal section $X \to X \times_B X$ induces a map $s\colon \Delta^k \to \Gamma^*(X \times_B X) \simeq \gamma^* X$ via the universal property of $\Gamma^*(X \times_B X)$ by the map $\mathrm{id} \times (\mathrm{diag} \circ \Gamma)\colon \Delta^k \to \Delta^k \times X \times_B X$. That s is a section is precisely the commutativity of the lower triangle in the diagram

$$\begin{tikzcd}
\Delta^k \arrow[dr, dotted, "s"] \arrow[drr, bend left, "\mathrm{diag}\circ\Gamma"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \Gamma^*(X \times_B X) \arrow[r] \arrow[d] & X \times_B X \arrow[d, shift right] \\
& \Delta^k \arrow[r, "\Gamma"] & X. \arrow[u, bend right, "\mathrm{diag}"']
\end{tikzcd}$$

We have $\Gamma \simeq \mathrm{pr} \circ s\colon \Delta^k \to X$, as desired. We leave it to the reader the check that this correspondence extends in both the $\star$ and $\bullet$ directions. $\square$

We continue with a map $p\colon X \xrightarrow{\mathrm{cbl}} B$. Lemma 2.11 recovers [AFR18a, 0.0.3= 6.4.3], once we recall the so-called absolute exit-path category.

Definition 2.12. We call $\mathcal{E}\mathrm{xit} = |\mathcal{E}\mathrm{xit}[\star][\bullet]|_\bullet$ the ***absolute exit-path category***, defined exactly like $\mathcal{B}\mathrm{un}$ in 2.3, except that all bundle maps are required to be equipped with sections. Forgetting sections defines a functor $\mathcal{E}\mathrm{xit} \to \mathcal{B}\mathrm{un}$.

We will blackbox the fact that $\mathcal{B}\mathrm{un}$ and $\mathcal{E}\mathrm{xit}$ are indeed striation sheaves, hence ∞-categories.

Corollary 2.13. $\mathbf{Ex}(X)$ *is a pullback in* ∞*-categories of* $\mathcal{E}\mathrm{xit} \to \mathcal{B}\mathrm{un}$ *along* $\widehat{p}\colon \mathbf{Ex}(B) \to \mathcal{B}\mathrm{un}$.

As is clear from the proof of 2.11, the projection map $\mathbf{Ex}(X) \to \mathcal{E}\mathrm{xit}$ is given by the bundle $X \times_B X \rightleftarrows X$. We will write $\mathrm{Diag} = \mathrm{Diag}_X\colon \mathbf{Ex}(X) \to \mathcal{E}\mathrm{xit}$ for this map.

Any vector bundle has a section, the 0-section.

There is a symmetric monoidal functor

$$T\colon \mathcal{E}\mathrm{xit} \to \mathcal{V}\mathrm{ect}^{\hookrightarrow},$$

called the ***constructible tangent bundle***, that can be characterised uniquely in invariant language. Below, we will only recall a rather explicit description that will suffice for our purposes.

A stratified space X considered as a bundle $X \to *$ induces an X-point of $\mathcal{E}\mathrm{xit}$, i.e. a functor $\mathbf{Ex}(X) \to \mathcal{E}\mathrm{xit}$. We will denote the composition with T by

$$T_X\colon \mathbf{Ex}(X) \to \mathcal{V}\mathrm{ect}^{\hookrightarrow}.$$

In general, we will write $T_{(X\to B)}$ for the map $\mathbf{Ex}(X) \to \mathcal{V}\mathrm{ect}^{\hookrightarrow}$, since the intermediate map $\mathbf{Ex}(X) \to \mathcal{E}\mathrm{xit}$ depends on $(X \to B)$ and not just on X.

Recall that the strata of a stratified space are smooth manifolds. Hence, as $\mathrm{Link}_Y X$ is again a stratified space, each $(\mathrm{Link}_Y X)_p$ is a smooth manifold. It is enough to describe T_X on the objects X_p and, using 2.8, on the morphism spaces $L_{pq} :=(\mathrm{Link}_{X_p} X)_q$. We will now recall such a description, and use it below. Let X be a stratified space considered as a bundle $X \to *$.

Theorem 2.14 ([AFR18b])**.**

(1) *The functor T_X restricts on a stratum X_p to the classifying map of TX_p.*

(2) *It induces the following span of bundle maps over L_{pq}:*

$$\begin{tikzcd} & TL_{pq} \arrow[dl] \arrow[dr] & \\ {}^{*}TX_p & & {}^{*}TX_q \end{tikzcd}$$

where the left-superscript * indicates a pulled back bundle on L_{pq}. More precisely, the left map is the differential of the canonical projection $L_{pq} \to X_p$, and the right map is the differential of the (canonical-up-to-contractible-choice) composition

$$L_{pq} \hookrightarrow L_{pq} \times (-1,1) \hookrightarrow X_q,$$

the embedding of a tubular neighbourhood of X_p into X_q, restricted to the link.

(3) *Up to the contractible choice of a riemannian metric, T_X restricts on the morphism space L_{pq} to the following bundle map over L_{pq}:*

$${}^{*}TX_p \to TL_{pq} \to {}^{*}TX_q.$$

In the last point above, one first considers the map $({}^{*}TX_p)^{\vee} \to (TL_{pq})^{\vee}$ dual to the left leg of the span above. Then, a riemannian metric identifies this with a map ${}^{*}TX_p \to TL_{pq}$.

We will now work out some examples, and then move on to the first main topic of the present paper.

2.7. **Examples.** In the first example, we elaborate a little bit on a discussion already present in [AFR18b], bypassing the use of parallel transport. We then consider two typical defects in euclidean space.

2.7.1. *Hemispherically-stratified disks.* Consider the unit 1-disk $\mathbb{D}^1$ with 0d strata $\{-1\}$, $\{1\}$; and its interior $(-1,1)$ as the single 1d stratum. Similarly, consider $\mathbb{D}^2$ with the same 0d strata as $\mathbb{D}^1$; with two 1d strata, the interiors of the lower and upper hemispheres; and a single 2d stratum, the interior $\mathbb{D}^2 \setminus \partial\mathbb{D}^2$. We list these strata, in the given order, as follows:

$$\mathbb{D}^1_{(0,-)}, \quad \mathbb{D}^1_{(0,+)}, \quad \mathbb{D}^1_{\triangleright},$$

and

$$\mathbb{D}^2_{(0,-)}, \quad \mathbb{D}^2_{(0,+)}, \quad \mathbb{D}^2_{(1,-)}, \quad \mathbb{D}^2_{(1,+)}, \quad \mathbb{D}^2_{\triangleright}.$$

This describes the ***hemispherical stratifications*** on $\mathbb{D}^1$ and $\mathbb{D}^2$. The stratifying posets are respectively $P_1^{\mathrm{hem}} \coloneqq B_1^{\triangleright}$ and $P_2^{\mathrm{hem}} \coloneqq B_2^{\triangleright}$, the respective right-cones (obtained by adjoining a new maximal element) on the posets $B_1 \coloneqq [0] \times \{\pm\}$ and $B_2 \coloneqq [1] \times \{\pm\}$ ('B' for 'Boundary'). Here, B_n has the partial order given by declaring

$$(i,\sigma) \leq (j,\sigma') \text{ if } \begin{cases} i < j \text{ and irrespective of } \sigma, \sigma' \\ i = j \text{ and } \sigma = \sigma' \text{ simultaneously.} \end{cases}$$

One can define B_n, $P_n^{\mathrm{hem}} \coloneqq B_n^{\triangleright}$ analogously for larger n, and hemispherically-stratify $\mathbb{D}^n$ accordingly. For the adjoined maximal element we write $\triangleright \in P^{\mathrm{hem}}$. Finally, we will write P^{hem} for P_n^{hem} if n is clear. For the rest of this example, we will drop the 'hem's.

Set $\mathbb{D}^n \coloneqq (\mathbb{D}^n \to P)$. We have $T_{\mathbb{D}^n}|_p \simeq T\mathbb{D}^n_p$ for the restriction to the p-stratum for any $p \in P$. For simplicity, we now restrict to $n = 2$. With regard to morphisms, we note that since all strata are contractible, we have $L_{pq} \simeq *$ if $\mathbb{D}^n_p \subset \overline{\mathbb{D}^n_q}$, and $L_{pq} = \emptyset$ otherwise:

$$L_{p \neq \triangleright, \triangleright} \simeq *, \quad L_{(0,\pm),(0,\mp)} = \emptyset, \quad L_{(0,\pm),(1,\pm)} \simeq *.$$

Now, $L_{p\neq\triangleright,\triangleright} \to \mathrm{Ar}(\mathcal{V}^{\hookrightarrow}) = \mathcal{F}\mathrm{un}([1], \mathcal{V}^{\hookrightarrow})$ classifies the composite bundle map

$$\pi_p^* T\mathbb{D}^2_p \simeq (\pi_p^* T\mathbb{D}^2_p)^{\vee} \to (TL_{p,\triangleright})^{\vee} \simeq TL_{p,\triangleright} \hookrightarrow TL_{p,\triangleright} \oplus \varepsilon^1 \to \gamma_{\triangleright}^* T\mathbb{D}^2_{\triangleright} \simeq \mathbb{R}^2$$

over $L_{p,\triangleright}$, which is equivalent to

$$0 \hookrightarrow \mathbb{R}^2 \quad \text{if } p = (0,\pm),$$
$$\mathbb{R} \xhookrightarrow{-\oplus 0} \mathbb{R}^2 \quad \text{if } p = (1,\pm).$$

Similarly, $L_{(0,\pm),(1,\pm)}$ classifies $0 \hookrightarrow \mathbb{R}$.

2.7.2. 3*-space with a line defect.* Consider $X \coloneqq \mathbb{R}^3_{|} = (\mathbb{R}^3 \to [1])$ where X_0 is a line and $X_1 = X_0^c$, the complement. We have $L_{01} \simeq S^1 \times \mathbb{R}$. Using $T(S^1 \times \mathbb{R}) \simeq TS^1 \times T\mathbb{R}$, the map $\pi_0 \colon L_{01} \to S^1$ induces the coordinate projection $TS^1 \times T\mathbb{R} \to T\mathbb{R}$, which we will consider as the bundle map

$$TL_{01} \simeq TS^1 \oplus \varepsilon^1_{L_{01}} \to \varepsilon^1_{L_{01}} \simeq \pi_0^* T\mathbb{R}.$$

Employing the standard riemannian metric, this is equivalent to

$$\varepsilon^1 \xhookrightarrow{0\oplus -} TS^1 \oplus \varepsilon^1.$$

Next, let $\gamma_1 \colon L_{01} \xhookrightarrow{-\times 0} L_{01} \times \mathbb{R} \hookrightarrow X_1$ embed the link into a tubular neighbourhood of S^1, minus the core line X_0 itself. Postcomposing with the induced map $TS^1 \oplus \varepsilon^1 \simeq TL_{01} \overset{\sim}{\hookrightarrow} \gamma_1^* TX_1$, and observing $X_1 \simeq L_{01} \times \mathbb{R}$, we obtain

$$\varepsilon^1 \hookrightarrow TL_{01} \overset{\sim}{\hookrightarrow} \gamma_1^*(TS^1 \oplus \varepsilon^2) \simeq \gamma_1^* TS^1 \oplus \varepsilon^2,$$

which is the bundle map classified by L_{01}. Fibrewise, this map is equivalent to

$$0 \oplus \mathrm{id} \oplus 0.$$

This result extends mutatis mutandis to 3-spaces with any number of nonintersecting line defects.

2.7.3. *3-space with a circle defect.* Consider $X := \mathbb{R}^3_\circ = (\mathbb{R}^3 \to [1])$ where X_0 is a circle and $X_1 = X_0^c$, the complement. We have $L_{01} \simeq T^2$, and $\pi_0 \colon L_{01} \to S^1$ (say, the first coordinate projection) again induces a coordinate projection

$$TL_{01} \simeq TS^1 \oplus TS^1 \to \pi_0^* TS^1.$$

Note that $\pi_0^* TS^1 \simeq TS^1 \times \varepsilon^0_{S^1}$, which we will denote, by abuse of notation, by TS^1.[6] After dualising, $T\pi_0$ is equivalent to

$$TS^1 \times \varepsilon^0_{S^1} \xhookrightarrow{\mathrm{id} \oplus 0} TL_{01}.$$

Similarly to the previous examples, we have $\gamma_1^* TX_1 \simeq TL_{01} \oplus \varepsilon^1$, so L_{01} classifies the bundle morphism

$$TS^1 \xhookrightarrow{\mathrm{id} \oplus 0 \oplus 0} TL_{01} \oplus \varepsilon^1.$$

3. Structure groups

3.1. **Reductions and extensions.** Let X be a stratified space, and $v \colon \mathbf{Ex}(X) \to \mathcal{V}^{\hookrightarrow}$ classify a vector bundle on X. The following definition appears in [AFR18b].

Definition 3.1. An ∞-category over $\mathcal{V}^{\hookrightarrow}$, $\mathfrak{b} \colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$, is called a ***tangential structure***, and a lift $t_X^{\mathcal{B}} \colon \mathbf{Ex}(X) \to \mathcal{B}$ of v to $\mathcal{B}$, together with a homotopy $\mathfrak{t}_X^{\mathcal{B}}$ from the composition $\mathbf{Ex}(X) \to \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ to $T_X \colon \mathbf{Ex}(X) \to \mathcal{V}^{\hookrightarrow}$, is called a $\mathcal{B}$-***reduction*** on v. If $\mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ is in particular a right-fibration, it is called ***solid***. Diagrammatically:

$$\begin{tikzcd} & \mathcal{B} \arrow[d, "\mathfrak{b}"] \\ \mathbf{Ex}(X) \arrow[ur, dashed, "t_X^{\mathcal{B}}"] \arrow[r, "T_X"'] \arrow[ur, phantom, shift right=3ex, "\Rightarrow \mathfrak{t}_X^{\mathcal{B}}"] & \mathcal{V}^{\hookrightarrow} \end{tikzcd}$$

This data describes 'exactly', up to well-defined ∞-categorical ambiguity, an object in the functor-∞-category from $\mathbf{Ex}(X)$ to $\mathcal{B}$ in the over-∞-category over $\mathcal{V}^{\hookrightarrow}$. Thus,

$$\mathcal{B}\text{-red}(X) := \mathcal{M}\mathrm{ap}_{/\mathcal{V}^{\hookrightarrow}}(\mathbf{Ex}(X), \mathcal{B}).$$

For more detail, see **??** below. By

$$T_X^{\mathcal{B}} \in \mathcal{B}\text{-red}(X)$$

[6] This denotes now a bundle over L_{01}.

we therefore mean the data of a lift $t_X^{\mathcal{B}}$ as above as well as a homotopy $\mathfrak{b} \circ t_X^{\mathcal{B}} \simeq T_X$. We sometimes denote by $T_X^{\mathcal{B}}$ also only $\mathfrak{b} \circ t_X^{\mathcal{B}}$ if the homotopy is canonical or implicit or unimportant, such as when $\mathcal{B}$ is a product tangential structure (see e.g. 5.3 below).

Remark 3.2. Instead of reductions, one could also talk about extensions in the more familiar terms of principal bundles. Say X is a smooth n-manifold, so T_X factors through $\mathrm{B}O(n) \hookrightarrow \mathcal{V}^{\sim} \hookrightarrow \mathcal{V}^{\hookrightarrow}$ by virtue of $\mathbf{Ex}(X)$'s being an ∞-groupoid, and classifies the (orthonormal) frame bundle $O(M)$. Say $\mathcal{B} = \mathrm{B}H$ and $\mathfrak{b} = \mathrm{B}b$ for $b\colon H \to O(n)$ a map of Lie groups. Then, the lift to $\mathcal{B}$ defines a principal H-bundle Q on X, the b-reduction of $O(M)$, and the composition of the lift with $\mathfrak{b}\colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ is the b-extension $Q \times_b O(n)$, the associated principal $\mathrm{O}(n)$-bundle. A homotopy that commutes the lifting diagram above is then precisely an isomorphism $Q \times_b O(n) \xrightarrow{\sim} O(M)$. This need not be a contractible choice. For instance, there is a unique lift to $\mathrm{B}* = *$, so the associated bundle is also unique, but inequivalent framings correspond to inequivalent isomorphisms as above.

3.2. **Variframings.**

Definition 3.3. Recall the ***local dimension*** functor $\dim\colon \mathbf{Ex}(X) \to \mathbb{Z}_{\geq 0}$, which sends a point to the local (Lebesgue) dimension of the stratum in which the point lies, and is defined on morphisms in the obvious way.

Definition 3.4. The tangential structure $\varepsilon^{\bullet}\colon \mathbb{Z}_{\geq 0} \to \mathcal{V}^{\hookrightarrow}$, which sends $i \mapsto \mathbb{R}^i$ and $(i \leq j) \mapsto (- \oplus 0\colon \mathbb{R}^i \hookrightarrow \mathbb{R}^j)$, is called the ***variframing*** tangential structure, denoted by vfr.

Definition 3.5. Post-composition of $\varepsilon^{\bullet}$ with its left-inverse, the dimension functor $\mathcal{V}^{\hookrightarrow} \to \mathbb{Z}_{\geq 0}$, gives a vector bundle $\varepsilon_X^{\dim}\colon \mathbf{Ex}(X) \to \mathcal{V}^{\hookrightarrow}$, called the (***stratified***) ***trivial bundle***.

A variframing (that is, a vfr-reduction) on X (that is, on T_X) is tantamount to a natural equivalence $\varepsilon_X^{\dim} \simeq T_X$.

Besides giving a framing on each stratum, a variframing also provides compatibility between strata via the links. Namely, let f be a variframing $\varepsilon_X^{\dim} \xrightarrow{\sim} T_X$. Since $L_{pq} \simeq \mathrm{Ar}(\mathbf{Ex}(X))|_{X_p \times X_q}$, we will have an induced map

$$f|_{L_{pq}}\colon L_{pq} \to \mathrm{Equiv}(\varepsilon_X^{\dim}, T_X)|_{X_p \times X_q},$$

where the restriction in the target is meant in the obvious way. That is, f gives, after introducing a metric, a commutative square

$$\begin{tikzcd}
\varepsilon^{D_p}_{L_{pq}} \arrow[r, "\varepsilon^{\dim}|_{L_{pq}}"] \arrow[d, "\sim"'] & \varepsilon^{D_q}_{L_{pq}} \arrow[d, "\sim"] \\
\pi_p^* TX_p \arrow[r, "T_X|_{L_{pq}}"] & \gamma_q^* TX_q
\end{tikzcd}$$

of bundle maps over L_{pq}. Here, $D_k \coloneqq \dim(X_k)$. Since the top horizontal map includes as first coordinates, these squares expresses the requirement that the framing 'split', though not directly in the usual sense but rather over the link. Due to the existence of a nonvanishing inward pointing vector field on the boundary of a classical manifold with boundary $(X, \partial X)$, hence the fact, setting $\boldsymbol{X} \coloneqq (\boldsymbol{X} \to [1])$ with $\boldsymbol{X}_0 = \partial X$ and $\boldsymbol{X}_1 = X \setminus \partial X$, that $L_{01} \simeq \partial X$, we see that this recovers the usual notion of a framing split over the boundary. In fact, a variframing on (X, ∂) is the same as a framing on X, ∂X and $\mathbb{R}$ such that the restriction of the former to a tubular neighbourhood of ∂X is the product framing induced by the latter two.

3.3. General reductions.

3.4. Right-fibration replacements and solid reductions.

3.5. Exodromy.

4. Reductions

We base the shape of our Morita theory on the framed 'base case' given in [Sch14]. This was originally proposed and explained in [Lur08]. Scheimbauer provides the details of a definition of $\mathrm{Alg}_n = (\mathrm{Alg}_n)_{\bullet_1, \dots, \bullet_n}$, an n-fold (complete) Segal space, or Morita (∞, n)-category of $\mathbb{E}_n$-algebras and higher homotopy-modules, with coefficients in a suitable symmetric-monoidal ∞-category $\mathcal{T}$. It is a particular arrangement of locally-constant stratified factorisaton algebras, and we adopt this approach. First, therefore, we articulate the particular notions of factorisation algebra our context demands.

Let $\mathfrak{b} \colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ be a tangential structure. Let X be a stratified $\mathcal{B}$-space, i.e. with a $\mathcal{B}$-reduction $T_X^{\mathcal{B}}$ of its tangent bundle. Consider a stratified $\mathcal{B}$-space $U = (U, T_U^{\mathcal{B}})$, and an open embedding $U \overset{\iota}{\hookrightarrow} X$ such that the following diagram homotopy-commutes:

(1)
$$\begin{tikzcd}
\mathbf{Ex}(U) \arrow[d, "\iota_*"'] \arrow[dr] \arrow[drrr] & & & \\
& \mathcal{E}\mathrm{xit} \arrow[r] & \mathcal{V}^{\hookrightarrow} & \mathcal{B} \arrow[l] \\
\mathbf{Ex}(X) \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}$$

This amounts to asking $T_U^{\mathcal{B}} \simeq T_X^{\mathcal{B}} \circ \iota_*$, which we now make precise.

Remark 4.1. Here and below, we always use that a composition exists, and is unique up to homotopy, and employ the standard abuse of notation of writing '$\circ$' and 'the composition'. See e.g. [Lur22, Tag 0043]. In fact, by a result of Joyal, composition is unique up to a contractible choice [Lur22, Tag 007A]. Similarly, we implicitly use standard facts such as compositions respecting homotopies [Lur22, Tag 0048], and do not concern ourselves with the choice of composition.

Definition 4.2. Given a stratified open embedding $\iota \colon U \hookrightarrow X$, we define

$$\iota^* \colon \mathcal{B}\text{-red}(X) \to \mathcal{B}\text{-red}(U)$$

using the induced diagram

$$
\begin{tikzcd}
 & & \mathcal{B} \arrow[d, "\mathfrak{b}"] \\
\mathbf{Ex}(U) \arrow[r, "\iota_*"] \arrow[urr, dashed, bend left, "t_X^{\mathcal{B}} \circ \iota_*"] \arrow[rr, bend right, "T_U"'] \arrow[r, phantom, shift left=4ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & \mathbf{Ex}(X) \arrow[r, "T_X"] \arrow[ur, "t_X^{\mathcal{B}}"] \arrow[r, phantom, shift left=3ex, "\mathfrak{t}_X^{\mathcal{B}}\Downarrow"] & \mathcal{V}^{\hookrightarrow}
\end{tikzcd}
$$

Namely, $\iota^* T_X^{\mathcal{B}}$ is defined by the lift $\iota^* t_X^{\mathcal{B}} \coloneqq t_X^{\mathcal{B}} \circ \iota_*$ and the equivalence $\mathfrak{b}_* \iota^* t_X^{\mathcal{B}} \simeq T_U$ given by the composition of the equivalences in the diagram. We write $\iota^* \mathfrak{t}_X^{\mathcal{B}}$ for the latter. The bottom 2-equivalence classifies the (stratified) derivative $\mathrm{d}\iota \colon T_U \xrightarrow{\sim} \iota^* T_X$, which is stratum-wise the ordinary derivative, and is induced by the latter over links.

Definition 4.3. A stratified open embedding $\iota \colon U \hookrightarrow X$, together with an equivalence $e_\iota \colon T_U^{\mathcal{B}} \simeq \iota^* T_X^{\mathcal{B}}$ within $\mathcal{B}$-red(U) is called a $\mathcal{B}$***-open*** in X. The space $\mathcal{B}$-open$_X(\iota)$ of $\mathcal{B}$-open-structures on ι is defined to be

$$\mathcal{B}\text{-open}_X(\iota) \coloneqq \mathcal{M}\mathrm{ap}^{\sim}_{/\iota^* T_X^{\mathcal{B}}}(*, \mathcal{B}\text{-red}(U)).$$

We must justify Definition 4.3 by seeing that it reproduces the familiar notion of equivalence of reductions of principal bundles in the trivially-stratified setting. To this end, we must first understand the hom-space that e_ι inhabits. It can again be expressed in an over-∞-category, or in this case as an over-∞-groupoid.

Notation 4.4. We denote by $[-,-]$ the ∞-category of ∞-functors between the arguments. We denote by $[\![-,-]\!]$ the hom-space (∞-groupoid) between two objects in an ∞-category. Both notions, as already remarked, are well-defined up to equivalence, with contractible choice.

We should stress that this notation does not indicate taking connected components. We will use absolute value symbols for that; see below.

Lemma 4.5. *An equivalence*

$$e_\iota \colon T_U^{\mathcal{B}} \xrightarrow{\sim} \iota^* T_X^{\mathcal{B}} \quad \textit{in} \quad \mathcal{B}\text{-red}(U)$$

is presented by

(1) *an equivalence*

$$\epsilon_\iota \colon t_U^{\mathcal{B}} \xrightarrow{\sim} \iota^* t_X^{\mathcal{B}} \quad \textit{in} \quad [\mathbf{Ex}(U), \mathcal{B}],$$

(2) *together with a homotopy*

$$\mathfrak{e}_\iota \colon \iota^* \mathfrak{t}_X^{\mathcal{B}} \circ \mathfrak{b}_* \epsilon_\iota \xrightarrow{\sim} \mathfrak{t}_U^{\mathcal{B}} \quad \textit{in} \quad [\![\mathfrak{b}_* t_U^{\mathcal{B}}, T_U]\!]_{[\mathbf{Ex}(U), \mathcal{V}^{\hookrightarrow}]}.$$

Proof. By [Lur09, Prop. 5.5.5.12] dualised from under to over, $\mathcal{B}$-red(U) is the homotopy fibre that fits the ∞-pullback diagram

$$\begin{tikzcd} \mathcal{B}\text{-red}(U) \arrow[r, dashed] \arrow[d, dashed] & {[\mathbf{Ex}(U), \mathcal{B}]} \arrow[d, "\mathfrak{b}_*"] \\ * \arrow[r, "T_U"'] & {[\mathbf{Ex}(U), \mathcal{V}^{\hookrightarrow}]} \end{tikzcd}$$

The restriction to the relevant hom-space fits thus the ∞-pullback of ∞-groupoids

$$\begin{tikzcd} {[\![T_U^{\mathcal{B}}, \iota^* T_X^{\mathcal{B}}]\!]_{\mathcal{B}\text{-red}(U)}} \arrow[r, dashed] \arrow[d, dashed] & {[\![t_U^{\mathcal{B}}, \iota^* t_X^{\mathcal{B}}]\!]_{[\mathbf{Ex}(U), \mathcal{B}]}} \arrow[d, "\mathfrak{b}_*"] \\ * \simeq [\![*, *]\!]_* \arrow[r, "\mathrm{id}_{T_U}"'] & {[\![T_U, T_U]\!]_{[\mathbf{Ex}(U), \mathcal{V}^{\hookrightarrow}]}} \end{tikzcd}$$

where, given $\epsilon_\iota \colon t_U^{\mathcal{B}} \xrightarrow{\sim} \iota^* t_X^{\mathcal{B}}$, the induced equivalence

$$\mathfrak{b}_*\epsilon_\iota \colon \mathfrak{b}_* t_U^{\mathcal{B}} \xrightarrow{\sim} \mathfrak{b}_* \iota^* t_X^{\mathcal{B}} \quad \text{within} \quad [\mathbf{Ex}(U), \mathcal{V}^{\hookrightarrow}]$$

is understood to be in $[\![T_U, T_U]\!]$ using the equivalences $\mathfrak{t}_U^{\mathcal{B}} \colon \mathfrak{b}_* t_U^{\mathcal{B}} \xrightarrow{\sim} T_U$ and $\iota^* \mathfrak{t}_X^{\mathcal{B}} \colon \mathfrak{b}_* \iota^* t_X^{\mathcal{B}} \xrightarrow{\sim} T_U$ specified by the chosen reductions $T_U^{\mathcal{B}}$ and $\iota^* T_X^{\mathcal{B}}$, respectively. Thus, given ϵ_ι, we are considering, up to a contractible choice involved in inversion, the diagram

$$\begin{tikzcd} & \mathfrak{b}_* \iota^* t_X^{\mathcal{B}} \arrow[d, "\iota^* \mathfrak{t}_X^{\mathcal{B}}"] \\ \mathfrak{b}_* t_U^{\mathcal{B}} \arrow[ur, "\mathfrak{b}_* \epsilon_\iota"] & T_U \arrow[l, "(\mathfrak{t}_U^{\mathcal{B}})^{-1}"] \end{tikzcd}$$

of equivalences within $[\mathbf{Ex}(U), \mathcal{V}^{\hookrightarrow}]$ describing a loop at T_U.

By the above, an equivalence $e_\iota \colon T_U^{\mathcal{B}} \xrightarrow{\sim} \iota^* T_X^{\mathcal{B}}$ of reductions is characterised by the condition that this induced loop be homotopic to id_{T_U}. Such a homotopy is the same (again up to a contractible choice) as a 2-equivalence $\mathfrak{e}_\iota$ as in the diagram

$$\begin{tikzcd} & \mathfrak{b}_* \iota^* t_X^{\mathcal{B}} \arrow[d, "\iota^* \mathfrak{t}_X^{\mathcal{B}}"] \\ \mathfrak{b}_* t_U^{\mathcal{B}} \arrow[ur, "\mathfrak{b}_* \epsilon_\iota"] \arrow[r, "\mathfrak{t}_U^{\mathcal{B}}"'] \arrow[ur, phantom, shift right=3ex, "\Searrow \mathfrak{e}_\iota"] & T_U \end{tikzcd} \tag{2}$$

within $[\mathbf{Ex}(U), \mathcal{V}^{\hookrightarrow}]$. □

The precise data encoded by e_ι, which consists thus of two parts, is crucial for geometric purposes. These parts play complementary roles in opposite geometries. In terms of principal bundles:

- The equivalence ϵ_ι is an equivalence of the reduced $\mathcal{B}$-bundles. It encodes geometric information if $\mathfrak{b}$ has nontrivial fibres, such as with (s)pin, product, or solid tangential structures.
- The equivalence $\mathfrak{b}_*\epsilon_\iota \simeq \mathrm{id}_{T_U}$ is the equivariant commuting diagram of maps to the tangent (frame) bundle out of the two corresponding extended $\mathcal{V}^{\hookrightarrow}$-bundles. It encodes geometric information if $\mathcal{B}$ is 'trivial', i.e. a 0-type, such as with (vari)framings, in which case $t_U^{\mathcal{B}} \simeq \iota^* t_X^{\mathcal{B}}$ up to a contractible choice, making ϵ_ι trivial.

We explicate this translation in the following remark, as advertised.

Remark 4.6. In terms of principal bundles, an e_ι as above would, in the context of 3.2, amount to an isomorphism

$$Q \times_b O(n) \simeq (\iota^* Q') \times_b O(n)$$

over $O(U)$ of principal $O(n)$-bundles on U, where Q is the given b-reduction on U, and Q' the b-reduction on X. This is the same as an equivalence of reductions in the classical sense, i.e. a map

$$f\colon Q \to \iota^* Q'$$

of H-bundles such that

$$r_{\iota^* Q'} \circ f = r_Q$$

with $r_Q\colon Q \to O(U)$ the (b-equivariant) reduction on U, and

$$r_{\iota^* Q'} = \iota^* r_{Q'}\colon \iota^* Q' \to \iota^* O(X) \simeq O(U)$$

(the b-equivariant) reduction on X pulled back onto U. The last equivalence $O(U) \simeq \iota^* O(X)$ on the associated vector bundles is given by the derivative of ι. As this is canonical, we will denote this (and other such equivalences) by '$=$' instead of '$\simeq$'. This will cause no confusion, as we employ '$=$' otherwise only for definitional equality.

Stratum-wise, 4.6 exhausts the geometric information. The new information is along links. We illustrate this in an example.

Let X and U be variframed with variframings $T_X^{\mathcal{B}} \in \mathrm{vfr}(X)$, $T_U^{\mathcal{B}} \in \mathrm{vfr}(U)$, and let $\iota\colon U \hookrightarrow X$ be a vfr-open specified by an equivalence $e_\iota \in \mathrm{vfr}_U(T_U^{\mathrm{vfr}}, \iota^* T_X^{\mathrm{vfr}})$. Finally, let U_p, U_q be two strata of respective dimension D_p, D_q.

Observation 4.7. *Up to contractible choice, e_ι is equivalent to a 3-equivalence commuting the diagram*

$$\begin{tikzcd} \varepsilon^{D_p}_{L^U_{pq}} \arrow[r, bend left=50, ""{name=A, below}] \arrow[r, bend right=50, "0^{q-p}"', ""{name=B, above}] \arrow[Rightarrow, from=A, to=B, shift right=4, "\mathfrak{t}_U^{\mathrm{vfr}}"'] \arrow[Rightarrow, from=A, to=B, shift left=4, "\iota^*\mathfrak{t}_X^{\mathrm{vfr}}"] & \varepsilon^{D_q}_{L^U_{pq}} \end{tikzcd} \quad \mathfrak{e}_\iota\colon \mathfrak{t}_U^{\mathrm{vfr}} \Rrightarrow \iota^*\mathfrak{t}_X^{\mathrm{vfr}}.$$

Proof. As we discussed, $t_U^{\mathrm{vfr}} \simeq \iota^* t_X^{\mathrm{vfr}}$ up to a contractible choice due to the hom-spaces of $\mathbb{Z}_{\geq 0}$ being (-1)-types. This collapses the hypotenuse in 2, so e_ι is presented by an equivalence $\mathfrak{e}_\iota$ that fits[7]

$$\begin{tikzcd}
\varepsilon_U^{\dim} \arrow[r, bend left, "\iota^* \mathfrak{t}_X^{\mathrm{vfr}}"] \arrow[r, bend right, "\mathfrak{t}_U^{\mathrm{vfr}}"'] \arrow[r, phantom, "\mathfrak{e}_\iota \Uparrow"] & T_U
\end{tikzcd}$$

Let U_p, U_q be two strata and D_p, D_q their respective dimensions. We write again p, q for the strata indices of X hit by the underlying poset map of ι applied to p, q. Now, ι connects the two framings $\mathfrak{t}_U^{\mathrm{vfr}}$, $\iota\mathfrak{t}_X^{\mathrm{vfr}}$, restricted for the moment only to the strata, via $\mathrm{d}\iota$ as in the commutative diagram

$$\begin{tikzcd}
& \iota^*\pi_X^*\varepsilon_{X_p}^{D_p} \arrow[dl, "\sim"'] & & \iota^*\pi_X^*TX_p \arrow[ll, "\sim"'] & \iota^*TL_{pq}^X \arrow[l, "\mathrm{d}(\pi_X\circ\iota)"'] \arrow[r, hook] & {} \\
\varepsilon_{L_{pq}}^{D_p} & \pi^*\varepsilon_{U_p}^{D_p} \arrow[l, "\sim"] \arrow[u, "\sim"] & & \pi^*TU_p \arrow[ll, "\sim"] \arrow[u, "\sim"] & TL_{pq}^U \arrow[l, "\mathrm{d}\pi^U"] \arrow[u] \arrow[ul, "\mathrm{d}(\iota\circ\pi)"] \arrow[r, hook] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, hook] & \iota^*TL_{pq}^X \oplus \varepsilon^1 \arrow[r, hook, "\mathrm{d}(\gamma_X\circ\iota)"] & \iota^*\gamma_X^*TX_q \arrow[r, "\sim"] & \iota^*\gamma_X^*\varepsilon_{X_q}^{D_q} \arrow[dr, "\sim"] & \\
{} \arrow[r, hook] & TL_{pq}^U \oplus \varepsilon^1 \arrow[u] \arrow[ur, "\mathrm{d}(\iota\circ\gamma)"] \arrow[r, hook, "\mathrm{d}\gamma^U"'] & \gamma^*TU_q \arrow[u, "\sim"] \arrow[r, "\sim"'] & \gamma^*\varepsilon_{U_q}^{D_q} \arrow[u, "\sim"] \arrow[r, "\sim"'] & \varepsilon_{L_{pq}}^{D_q}
\end{tikzcd}$$

of bundle maps over L_{pq}^U. After composition, and using the projections $TL^{U/X} \oplus \varepsilon^1 \to TL^{U/X}$ instead of the sections $TL^{U/X} \hookrightarrow TL^{U/X} \oplus \varepsilon^1$, $\mathfrak{e}_\iota$ thus gives equivalences

$$\begin{tikzcd}
& TL_{pq}^U \oplus \varepsilon^1 \arrow[dl] \arrow[dr] \arrow[dd, "\mathrm{d}\iota\oplus\mathrm{id}"] & \\
\varepsilon_{L_{pq}^U}^{D_p} \arrow[r, phantom, "\Longleftarrow"] & {} \arrow[r, phantom, "\Longrightarrow"] & \varepsilon_{L_{pq}^U}^{D_q} \\
& \iota^*TL_{pq}^X \oplus \varepsilon^1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We read this diagram as depicting two spans between $\varepsilon_{L^U}^{D_p}$ and $\varepsilon_{L^U}^{D_q}$, both with roof $TL^U \oplus \varepsilon^1$, where the second span factors via $\mathrm{d}\iota \oplus \mathrm{id}$ through the intermediate bottom span with roof $\iota^*TL^X \oplus \varepsilon^1$. We now (1) choose a metric, (2) reverse the source arrow in each span while inverting $\mathrm{d}\iota \oplus \mathrm{id}$ (which is needed to reverse the source arrow of the second span), (3) compose through $TL^U \oplus \varepsilon^1$, (4) get induced equivalences of bundle maps to the inclusion map 0^{q-p} as first coordinates, provided

[7]This is in the reverse direction compared to said diagram, but this changes nothing up to a contractible choice.

by $\mathfrak{t}_U^{\mathcal{B}}$ and $\iota^*\mathfrak{t}_X^{\mathcal{B}}$, and finally (4) get a 3-equivalence of bundle maps over L_{pq}, '$\mathfrak{e}_\iota$' by abuse of notation,

$$\begin{tikzcd}[column sep=huge] \varepsilon^{D_p}_{L^U_{pq}} \arrow[r, bend left] \arrow[r, bend right, "0^{q-p}"'] \arrow[r, phantom, "\mathfrak{t}_U^{\mathrm{vfr}} \Downarrow \overset{\mathfrak{e}_\iota}{\Rrightarrow} \Downarrow \iota^*\mathfrak{t}_X^{\mathrm{vfr}}"] & \varepsilon^{D_q}_{L^U_{pq}} \end{tikzcd}$$

which is equivalent to the data of $\mathfrak{e}_\iota$ up to contractible choice (since every choice (1)–(4) is contractible). $\square$

5. Ambient-compatibility

We will occasionally identify a $\mathcal{B}$-open $(U, \alpha\colon U \to X)$ (with e_α implicit) with its image under α, and sometimes denote it by U_α or simply by α.

Consider two $\mathcal{B}$-opens α and β, such that $U_\alpha \subseteq U_\beta$, and denote this inclusion by $\iota_{\alpha\beta}$. Then, the two maps $(\alpha)_*, \beta_* \circ (\iota_{\alpha\beta})_*\colon \mathbf{Ex}(U_\alpha) \to \mathbf{Ex}(X)$ are canonically equivalent. They are not the same, since $\iota_{\alpha\beta}$ is short for the open embedding $U_\alpha \hookrightarrow U_\beta$ through their images in X, i.e. $\iota_{\alpha\beta}$ is the 'transition function' $\beta^{-1} \circ \alpha$ through $\mathrm{Im}(\alpha) \cap \mathrm{Im}(\beta)$.

Definition 5.1. We set

$$\mathfrak{e}_{\alpha\beta} := \iota_{\alpha\beta}^*(e_\beta^{-1}) \circ \mathfrak{id}_{\alpha\beta} \circ e_\alpha\colon T_\alpha^{\mathcal{B}} \simeq \iota_{\alpha\beta}^* T_\beta^{\mathcal{B}},$$

where

(1) e_γ, $\gamma = \alpha, \beta$, are the given $\mathcal{B}$-open structures $T_\gamma^{\mathcal{B}} := T_{U_\gamma}^{\mathcal{B}} \simeq \iota_\gamma^* T_X^{\mathcal{B}}$;

(2) the equivalence $\mathfrak{id}_{\alpha\beta} : \iota_\alpha^* T_X^{\mathcal{B}} \simeq \iota_{\alpha\beta}^* \iota_\beta^* T_X^{\mathcal{B}}$ is

$$(\mathbf{Ex}(U_\alpha) \xrightarrow{\alpha_*} \mathbf{Ex}(X) \to \mathcal{B}) \simeq (\mathbf{Ex}(U_\alpha) \xrightarrow{(\iota_{\alpha\beta})_*} \mathbf{Ex}(U_\beta) \to \mathbf{Ex}(X) \to \mathcal{B}),$$

which is $\mathfrak{b}_*$ applied to the canonical equivalence

$$(\mathbf{Ex}(U_\alpha) \to \mathbf{Ex}(X)) \simeq (\mathbf{Ex}(U_\alpha) \to \mathbf{Ex}(U_\beta) \to \mathbf{Ex}(X)).$$

The former is homotopy-over $\mathcal{V}^{\hookrightarrow}$ via the respective $\mathcal{B}$-open structures, where the intermediate homotopy that commutes

$$\begin{tikzcd} \mathbf{Ex}(U_\alpha) \arrow[rr] \arrow[dr] & & \mathbf{Ex}(U_\beta) \arrow[dl] \\ & \mathcal{V}^{\hookrightarrow} & \end{tikzcd}$$

classifies the derivative $\mathrm{d}\iota_{\alpha\beta}\colon T_\alpha \xrightarrow{\sim} \iota_{\alpha\beta}^* T_\beta$.

Definition 5.2. Let U_α, U_β be as above and let the inclusion $U_\alpha \subseteq U_\beta$ be accompanied by an equivalence

$$e_{\alpha\beta}\colon T_\alpha^{\mathcal{B}} \simeq \iota_{\alpha\beta}^* T_\beta^{\mathcal{B}} \quad \text{within} \quad \mathcal{B}\text{-red}(\alpha).$$

We call $(U_\alpha \subseteq U_\beta, e_{\alpha\beta})$, or $e_{\alpha\beta}$, ***ambient-compatible***, if there is a 3-equivalence

$$e_{\alpha\beta} \simeq \mathfrak{e}_{\alpha\beta} \quad \text{within} \quad [\![T_\alpha^{\mathcal{B}}, \iota_{\alpha\beta}^* T_\beta^{\mathcal{B}}]\!]_{\mathcal{B}\text{-red}_\alpha}.$$

The space of ambient-compatible $\mathcal{B}$-inclusions will be denoted by

$$X^{\mathcal{B}}(U_\alpha \,|\, U_\beta) := [\![T_\alpha^{\mathcal{B}}, \iota_{\alpha\beta}^* T_\beta^{\mathcal{B}}]\!]_{/\mathfrak{e}_{\alpha\beta}}$$

the over-∞-groupoid over $\mathfrak{e}_{\alpha\beta}$. This is understood to be empty when U_α is not a subset of U_β, and to be the point[8] when $U_\alpha = \emptyset$. Without this restriction, we write

$$\widehat{X}^{\mathcal{B}}(U_\alpha \,|\, U_\beta) \coloneqq [\![T^{\mathcal{B}}_\alpha, \iota^*_{\alpha\beta} T^{\mathcal{B}}_\beta]\!].$$

5.1. **Reductions on the point.** Consider the simplest simplest case, $U = X = *$, and $\mathcal{B} = \underline{\mathcal{S}} = \mathcal{V}^{\hookrightarrow} \times \mathcal{S} \xrightarrow{\mathrm{pr}} \mathcal{V}^{\hookrightarrow}$, where $\mathcal{S}$ is a space. (One can treat a general ∞-category $\mathcal{S}$ similarly.) Then, $\mathcal{B}$-structures on U are the same as points of S, 2-equivalences like e_α above are the same as paths in $\mathcal{S}$, and so on. If $\mathcal{S}$ is a 1-type, modding out the 3-equivalences $e_{\alpha\beta}$ may be justified. Otherwise, this corresponds to Postnikov-truncating $\mathcal{S}$ to a 1-type.

This observation generalises to the case $U = X = \mathbb{B}^n$ an open $\mathcal{B}$-ball whenever $\mathcal{B}$-red(U) is not a 1-type. For instance, when U is stratified trivially, $\mathcal{B}\text{-red} \simeq \mathcal{B}|_{\mathbb{R}^n}$, which is equivalent to $B\mathrm{O}(n)$, $B\mathrm{SO}(n)$, etc., for typical choices of $\mathcal{B}$.

Example 5.3 ($\underline{S^1}$-reductions on the $*$.). A further motivation for ambient-compatibility is that it excludes equivalences $e_{\alpha\beta}$ that are not equivalent to $\mathfrak{e}_{\alpha\beta}$. (This will make 'ambient $\mathcal{B}$-factorisation algebras' much more manageable; see below.) This is apparent already in the case $U_\alpha = U_\beta = X = *$ and $\mathcal{B} = \underline{\mathcal{S}}$ for e.g. $\mathcal{S} = S^1$. We have that $p_\gamma \coloneqq T^{\underline{S^1}}_\gamma$, $\gamma = \alpha, \beta, X$ are points in S^1, and the triviality of the inclusions simplify matters to make $e_{\alpha/\beta}$ paths $p_{\alpha/\beta} \to p_X$, $\mathfrak{id}_{\alpha\beta}$ the constant path at p_X, and so $\mathfrak{e}_{\alpha\beta}$ the path $p_\alpha \to p_\beta$ that is the (class of the) concatenation $e_\alpha * e_\beta^{-1}$. Now, if $e_{\alpha\beta}$ is $e_\alpha * e_\beta^{-1}$ concatenated with any nontrivial loop at p_β, then indeed $e_{\alpha\beta} \colon T^{\mathcal{B}}_\alpha \simeq \iota^*_{\alpha\beta} T^{\mathcal{B}}_\beta$, but $e_{\alpha\beta} \not\simeq \mathfrak{e}_{\alpha\beta}$, and so it does not descend to $X^{\mathcal{B}}(U_\alpha \,|\, U_\beta)$. In this example, $X^{\mathcal{B}}(U_\alpha \,|\, U_\beta)$ is always contractible (in particular, never empty), corresponding to the fact that S^1 is a connected 1-type. Considering *all* structured embeddings $* \to *$, i.e. $\widehat{X}^{\mathcal{B}}(U_\alpha \,|\, U_\beta)$, would here yield a $\mathbb{Z}$'s worth instead.

Next, take a self-embedding $*' \coloneqq * \to *$ of the ambient space with an equivalence $p_{X'} \to p_X$. Then, an $e_{\alpha X'}$ is ambient-compatible if and only if $e_{\alpha X'} \simeq e_{X'}^{-1} \circ e_\alpha$, or $e_\alpha \simeq e_{X'} \circ e_{\alpha X'} = e_{X'X}^{-1} \circ e_{\alpha X'}$, where $e_{X'X} \coloneqq e_{X'}^{-1}$ is the 'change of basis'. This readily lifts to the fact that $X^{\mathcal{B}}(U_\alpha \,|\, U_\beta) \simeq X'^{\mathcal{B}}(U'_\alpha \,|\, U'_\beta)$, where U'_γ is a $\mathcal{B}$-open of X' via $e'_\gamma \coloneqq e_{XX'} \circ e_\gamma$, where $e_{XX'} \coloneqq e_{X'X}^{-1} \simeq e_{X'}$. Slightly more generally, if $\mathcal{S}$ is a connected 1-type, the space of ambient $\underline{\mathcal{S}}$-factorisation algebras on $*$ (see below) is a homotopy-$\pi_1(\mathcal{S})$-torsor, realised by concatenation with $e_{XX'}$ (see **??**).

Example 5.4 (solid reductions on the point). Consider $B\mathrm{SO}(2) \to \mathcal{V}^{\sim} \to \mathcal{V}^{\hookrightarrow}$, with its right-fibration replacement

$$\mathbf{s}B\mathrm{SO}(2) \coloneqq \mathrm{Ar}(\mathcal{V}^{\hookrightarrow})|_{B\mathrm{SO}(2)} \xrightarrow{\text{source}} \mathcal{V}^{\hookrightarrow},$$

the induced solid tangential structure. A $\mathbf{s}B\mathrm{SO}(2)$-reduction on $*$ is the same as a rank-2 oriented bundle on $*$, which would be the full-dimensional oriented collar fed into factorisation homology in the induced fully-extended 2-FFT (which would just produce the coefficient oriented 2-disk-algebra itself). Incidentally, $\underline{S^1}$-reductions and $\mathbf{s}B\mathrm{SO}(2)$-reductions on $*$ are equivalent.

[8] The ∞-groupoid given by the constant complete Segal space at $*$.

5.2. **Factorisation property, I: general compositions.** Let $U_\alpha \hookrightarrow U_\beta \hookrightarrow U_\gamma$ be (stratified) open embeddings. We have a natural composition map

$$\widehat{\circledast} \colon [\![T^{\mathcal{B}}_\beta, \iota^*_{\beta\gamma} T^{\mathcal{B}}_\gamma]\!] \times [\![T^{\mathcal{B}}_\alpha, \iota^*_{\alpha\beta} T^{\mathcal{B}}_\beta]\!] \to [\![T^{\mathcal{B}}_\alpha, \iota^*_{\alpha\gamma} T^{\mathcal{B}}_\gamma]\!], \tag{3}$$

$$(e_{\beta\gamma}, e_{\alpha\beta}) \mapsto e_{\alpha\beta} \widehat{\circledast} e_{\beta\gamma} \coloneqq (\iota^*_{\alpha\beta} e_{\beta\gamma}) \circ e_{\alpha\beta},$$

where $\iota^*_{\alpha\beta} e_{\beta\gamma} \colon \iota^*_{\alpha\beta} T^{\mathcal{B}}_\beta \simeq \iota^*_{\alpha\beta}\iota^*_{\beta\gamma} T^{\mathcal{B}}_\gamma = \iota^*_{\alpha\gamma} T^{\mathcal{B}}_\gamma$ is the induced equivalence on the pulled back reductions.

When the U_i are not $\mathcal{B}$-opens in an ambient space, $\widehat{\circledast}$ is badly-behaved.As this case is still of some interest, since it is about plain structured nested open embeddings, let us briefly examine the behaviour of $\widehat{\circledast}$.

Example 5.5. (1) In order to present a composition that is not an equivalence, consider $\underline{S^1}$-reductions on the point, where the domain of $\widehat{\circledast}$ will be a $\mathbb{Z} \times \mathbb{Z}$'s worth while its target equivalent to $\mathbb{Z}$, and the fibres of $\widehat{\circledast}$ are non-contractible. Still, $\widehat{\circledast}$ is essentially surjective.

(2) (with disconnected $\mathcal{B}$) In order to present a composition that is not essentially surjective, pick any disconnected space $\mathcal{S} = \mathcal{S}_1 \amalg \mathcal{S}_2$. Using the notation of 5.3, take $p_\alpha, p_\gamma \in \mathcal{S}_1$ but $p_\beta \in \mathcal{S}_2$. Then, no path $e_{\alpha\gamma}$ factors through a path through p_β. In fact, the domain of $\widehat{\circledast}$ is empty.

On the other hand, when $\mathcal{B}$ is connected, such examples do no apply, but it is enough to pick a tangential structure that is a connected ∞-category but not an ∞-groupoid.

Example 5.6 (with connected $\mathcal{B}$). It is again enough to consider $U_\alpha = U_\beta = U_\gamma = X = *$ in order to present a minimal example. Namely, take $\mathcal{B} = \underline{\mathbb{R}_{\geq 0}}$, where $\mathbb{R}_{\geq 0}$ has its boundary stratification,[9] or any other $\underline{Y}$ where the stratifying poset of Y has a nontrivial arrow (i.e. a subcategory of shape $\bullet \lneq \bullet$). Let $T^{\underline{\mathbb{R}_{\geq 0}}}_\alpha = 0 = T^{\underline{\mathbb{R}_{\geq 0}}}_\gamma$, and $T^{\underline{\mathbb{R}_{\geq 0}}}_\beta = 1$. It is enough to write $T^{\underline{\mathbb{R}_{\geq 0}}}_\beta \simeq 1$, since the former will then never be stratified-homotopic to 0. It is also for this reason that $e_{\alpha\gamma} \in [\![0, 0]\!] \simeq *$ does not factor through 1 via $\widehat{\circledast}$. In fact, both factors in the domain of $\widehat{\circledast}$ are empty.

5.3. **Factorisation property, II: compositions in an ambient space.** Without further restriction, $\circledast$ on $\mathcal{B}$-opens might still have slightly suboptimal behaviour: in the context of the first point of 5.5, since $\mathcal{B}$ is already connected, it is immaterial that the U_i are specifically $\mathcal{B}$-opens in some X, so the behaviour is the same: the fibres of $\circledast$ are nontrivial. In view of 5.4, this is not an artificial example. The second point of 5.5 becomes a non-example, since all $T^{\mathcal{B}}_i$ are now necessarily in either $\mathcal{S}_1$ or $\mathcal{S}_2$.

We now observe that $\circledast$ is well-defined when restricted to ambient-compatible embeddings.

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[9]As is customary, we use 'X' and '$\mathbf{Ex}(X)$' interchangeably in such contexts; we will also write '$X \to Y$' while meaning '$\mathbf{Ex}(X) \to \mathbf{Ex}(Y)$'.

Lemma 5.7. *Let $U_\alpha \hookrightarrow U_\beta \hookrightarrow U_\gamma$ be nested $\mathcal{B}$-opens in X. The composition map of 3 descends to a map*

$$\circledast\colon X^{\mathcal{B}}(U_\alpha \,|\, U_\beta) \times X^{\mathcal{B}}(U_\beta \,|\, U_\gamma) \to X^{\mathcal{B}}(U_\alpha \,|\, U_\gamma).$$

Proof. First, we observe $\mathfrak{id}_{\alpha\beta} \widehat{\circledast} \mathfrak{id}_{\beta\gamma} = \iota_{\alpha\beta}^* \mathfrak{id}_{\beta\gamma} \circ \mathfrak{id}_{\alpha\beta} = \mathfrak{id}_{\alpha\gamma}$. Now, $\mathfrak{e}_{\alpha\beta} \widehat{\circledast} \mathfrak{e}_{\beta\gamma}$ is the upper right composition in the diagram

$$\begin{tikzcd}
T_\alpha^{\mathcal{B}} \arrow[r, "e_\alpha"] & \iota_\alpha^* T_X^{\mathcal{B}} \arrow[r, "\mathfrak{id}_{\alpha\beta}"] \arrow[dd, "\mathfrak{id}_{\alpha\gamma}"'] \arrow[ddrr, "\mathfrak{id}_{\alpha\beta}\widehat{\circledast}\mathfrak{id}_{\beta\gamma}"'] & \iota_{\alpha\beta}^* \iota_\beta^* T_X^{\mathcal{B}} \arrow[r, "\iota_{\alpha\beta}^* e_\beta^{-1}"] \arrow[dr, equal] & \iota_{\alpha\beta}^* T_\beta^{\mathcal{B}} \arrow[d, "\iota_{\alpha\beta}^* e_\beta"] \\
 & & & \iota_{\alpha\beta}^* \iota_\beta^* T_X^{\mathcal{B}} \arrow[d, "\iota_{\alpha\beta}^* \mathfrak{id}_{\beta\gamma}"] \\
 & \iota_{\alpha\gamma}^* \iota_\gamma^* T_X^{\mathcal{B}} \arrow[rr, equal] \arrow[drr, "\iota_{\alpha\gamma}^* e_\gamma^{-1}"'] & & \iota_{\alpha\beta}^* \iota_{\beta\gamma}^* \iota_\gamma^* T_X^{\mathcal{B}} \arrow[d, "\iota_{\alpha\beta}^* \iota_{\beta\gamma}^* e_\gamma^{-1}"] \\
 & & & \iota_{\alpha\gamma}^* T_\gamma^{\mathcal{B}} = \iota_{\alpha\beta}^* \iota_{\beta\gamma}^* T_\gamma^{\mathcal{B}}
\end{tikzcd}$$

whereas $\mathfrak{e}_{\alpha\gamma}$ is the lower right composition, thus $\mathfrak{e}_{\alpha\gamma} = \mathfrak{e}_{\alpha\beta} \widehat{\circledast} \mathfrak{e}_{\beta\gamma}$, since all the inner triangles and the square commute. The statement follows immediately: given $f\colon e_{\beta\gamma} \simeq \mathfrak{e}_{\beta\gamma}$, we have $\iota_{\alpha\beta}^* f\colon \iota_{\alpha\beta}^* e_{\beta\gamma} \simeq \iota_{\alpha\beta}^* \mathfrak{e}_{\beta\gamma}$, and given $g\colon e_{\alpha\beta} \simeq \mathfrak{e}_{\alpha\beta}$, we have

$$e_{\alpha\beta} \widehat{\circledast} e_{\beta\gamma} = \iota_{\alpha\beta}^* e_{\beta\gamma} \circ e_{\alpha\beta} \simeq_g \iota_{\alpha\beta}^* e_{\beta\gamma} \mathfrak{e}_{\alpha\beta} \simeq_{\iota_{\alpha\beta}^* f} \iota_{\alpha\beta}^* \mathfrak{e}_{\beta\gamma} \circ \mathfrak{e}_{\alpha\beta} = \mathfrak{e}_{\alpha\beta} \widehat{\circledast} \mathfrak{e}_{\beta\gamma} = \mathfrak{e}_{\alpha\gamma}.$$

Thus, $\widehat{\circledast}$ descends. □

In particular, $\circledast$ satisfies a weak factorisation property (not meant in the sense of Beilinson–Drinfeld; this will be discussed below), which we note for future reference.

Corollary 5.8. *Given an ambient-compatible embedding $e_{\alpha\gamma}$, we have $e_{\alpha\beta} \circledast e_{\beta\gamma} \simeq e_{\alpha\gamma}$ for any intermediate ambient-compatible embeddings $e_{\alpha\beta}$, $e_{\beta\gamma}$.*

We now define the operad whose algebras will be 'ambient' prefactorisation algebras with $\mathcal{B}$-structure.

Definition 5.9. Let $\mathbb{O}_X^{\mathcal{B}}$ denote the operad whose colours are $\mathcal{B}$-opens, i.e. whose space[10] of colours is given by

$$\coprod_\iota \mathcal{B}\text{-open}_X(\iota),$$

where the coproduct is taken over all stratified open embeddings into X, and whose k-multihoms are given by

$$\mathbb{O}_X^{\mathcal{B}}(\alpha_1, \ldots, \alpha_k \,|\, \beta) = \begin{cases} \prod_{i=1}^k X^{\mathcal{B}}(U_{\alpha_i} \,|\, U_\beta) & \text{if } \coprod_{i=1}^k U_\alpha \subseteq U_\beta \\ \emptyset & \text{else.} \end{cases}$$

[10]In that $\mathcal{B}$-open$_X(\iota)$ is an ∞-groupoid for each ι.

Let a finite disjoint collection $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ of embeddings be given, as well as collections $\boldsymbol{\beta}_i = (\beta_{i1}, \ldots, \beta_{il_i})$ for $i = 1, \ldots, k$ with intermediate embeddings $\boldsymbol{\iota}_{\boldsymbol{\beta}\boldsymbol{\alpha}} = (\iota_{\beta_{il}\alpha_i})_{1 \le i \le k, 1 \le l \le l_i}$, where $\iota_{ba} \colon U_{\beta_b} \hookrightarrow U_{\alpha_a}$, such that $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ \boldsymbol{\iota}_{\boldsymbol{\beta}\boldsymbol{\alpha}}$ coordinate-wise, in the obvious sense. We write $\mathbb{O}^{\mathcal{B}}_X(\boldsymbol{\delta} \,|\, \epsilon) \coloneqq \mathbb{O}^{\mathcal{B}}_X(\delta_1, \ldots, \delta_k \,|\, \epsilon)$.

We define the general composition

$$\circledast \colon \mathbb{O}^{\mathcal{B}}_X(\boldsymbol{\alpha} \,|\, \gamma) \times \mathbb{O}^{\mathcal{B}}_X(\boldsymbol{\beta}_1 \,|\, \alpha_1) \times \cdots \times \mathbb{O}^{\mathcal{B}}_X(\boldsymbol{\beta}_k \,|\, \alpha_k) \to \mathbb{O}^{\mathcal{B}}_X(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_k \,|\, \gamma)$$

to be the product of the compositions

$$\mathbb{O}^{\mathcal{B}}_X(\alpha_i, \gamma) \times \mathbb{O}^{\mathcal{B}}_X(\beta_{il}, \alpha_i) \to \mathbb{O}^{\mathcal{B}}_X(\beta_{il}, \gamma)$$

given by $\circledast$, using 5.7. Now, $\mathbb{O}^{\mathcal{B}}_X$ thus defined is a unital operad by asking

$$\eta_a \colon * \to \mathbb{O}^{\mathcal{B}}_X(a, a)$$

to hit $\mathfrak{e}_{aa} = \mathrm{id}_{T^{\mathcal{B}}_a}$. It is symmetric using the associator for $\times$.

We call an $\mathbb{O}^{\mathcal{B}}_X$-algebra in $\mathcal{T}$ an ***ambient $\mathcal{B}$-prefactorisation algebra***, or ***a-$\mathcal{B}$-pFA***, on X with values in $\mathcal{T}$.

Here, algebras in $\mathcal{T}$ over a coloured operad are meant in the usual sense: they are operad maps $\mathbb{O}^{\mathcal{B}}_X \to \underline{\mathcal{T}}$ with target the operad induced by the symmetric-monoidal structure on $\mathcal{T}$ by taking

$$\underline{\mathcal{T}}(t_1, \ldots, t_n \,|\, s) \coloneqq \mathcal{T}(t_1 \otimes \cdots \otimes t_n, s).$$

Definition 5.10. Let $\widehat{\mathbb{O}}^{\mathcal{B}}_X$ denote the operad with colours $\mathcal{B}$-opens in X and k-multihoms

$$\widehat{\mathbb{O}}^{\mathcal{B}}_X(\alpha_1, \ldots, \alpha_k \,|\, \beta) = \begin{cases} \prod_{i=1}^k \widehat{X}^{\mathcal{B}}(U_{\alpha_i} \,|\, U_\beta) & \text{if } \coprod_{i=1}^k U_\alpha \subseteq U_\beta \\ \emptyset & \text{else.} \end{cases}$$

The general composition maps are similarly defined the be the products of the compositions

$$\widehat{\circledast} \colon \widehat{\mathbb{O}}^{\mathcal{B}}_X(\alpha_i, \gamma) \times \widehat{\mathbb{O}}^{\mathcal{B}}_X(\beta_{il}, \alpha_i) \to \widehat{\mathbb{O}}^{\mathcal{B}}_X(\beta_{il}, \gamma)$$

given directly by 3. It has the same unit $\eta_a \colon * \to \widehat{\mathbb{O}}^{\mathcal{B}}_X(a, a)$ as above, and symmetric in the same way.

We call an $\widehat{\mathbb{O}}^{\mathcal{B}}_X$-algebra in $\mathcal{T}$ a ***$\mathcal{B}$-prefactorisation algebra***, or ***$\mathcal{B}$-pFA***, on X with values in $\mathcal{T}$.

Remark 5.11. The ambient/non-ambient distinction is akin to the small/big étale site distinction on an object. The underlying 'site' (we have not yet discussed a topology) would be $\mathcal{S}\mathrm{trat}^{\mathcal{B}\text{-emb}}$ of $\mathcal{B}$-spaces with $\mathcal{B}$-embeddings. A factorising precosheaf on the over-∞-category $\mathcal{S}\mathrm{trat}^{\mathcal{B}\text{-emb}}_{/(X, T^{\mathcal{B}}_X)}$ is precisely an ambient $\mathcal{B}$-pFA on $(X, T^{\mathcal{B}}_X)$. For a given topology on $\mathcal{S}\mathrm{trat}^{\mathcal{B}\text{-emb}}$, the induced 'small étale site' is precisely $\mathcal{S}\mathrm{trat}^{\mathcal{B}\text{-emb}}_{/(X, T^{\mathcal{B}}_X)}$ with the induced topology. If one ignores the compatibilities entailed by over-ness in the induced topology, one would have the 'big' version.

5.4. **Structured multiplications.**

Notation 5.12. Given $e_{\beta\alpha} \in \mathbb{O}_X^{\mathcal{B}}(\beta \,|\, \alpha)$ (resp. in $\widehat{\mathbb{O}}_X^{\mathcal{B}}(\beta \,|\, \alpha)$) and an a-$\mathcal{B}$-pFA (resp. a $\mathcal{B}$-pFA) F, we write

$$e_{\beta\alpha}^F \coloneqq F(e_{\beta\alpha})\colon F(\beta) \to F(\alpha).$$

We briefly note some behaviour familiar from ordinary prefactorisation algebras.

Remark 5.13. Due to 5.8, whenever we have, for F an a-$\mathcal{B}$-pFA, a system

$$\begin{tikzcd} F(b) \arrow[r, "e_{ba}^F"'] \arrow[rr, bend left, "e_{bc}^F"] & F(a) \arrow[r, "e_{ac}^F"'] & F(c), \end{tikzcd}$$

we have $e_{ba}^F \circledast e_{ac}^F \simeq e_{bc}^F$, witnessed by the image under F of the corresponding equivalence in $\mathbb{O}_X^{\mathcal{B}}(b \,|\, c)$. In particular, denoting by $\alpha \cap \beta$ the (embedding of the) intersection of α, β, we get $e_{\alpha\cap\beta,\gamma}^F \simeq e_{\alpha\cap\beta,\alpha}^F \circledast e_{\alpha\gamma}^F$ as well as $e_{\alpha\cap\beta,\gamma}^F \simeq e_{\alpha\cap\beta,\beta}^F \circledast e_{\beta\gamma}^F$, for any such e's, by setting $b = \alpha \cap \beta$, $c = \gamma$ and $a = \alpha/\beta$.

Example 5.14. If F is only a $\mathcal{B}$-pFA, this need not be true, as already illustrated by $\underline{S^1}$- or $\mathbf{s}B\mathrm{SO}(2)$-reductions on the point. Another simple but more geometric (though essentially the same) example is provided by $X = \mathbb{R}^3_{|}$, the euclidean 3-space with a line defect (see 2.7.2), and $\mathcal{B}$ stratified orientations, using the standard block inclusions $\mathrm{SO}(k) \hookrightarrow \mathrm{SO}(k+1)$ that extend by 1 in the last coordinate.[11] Let $\mathbb{R}^3_{|}$ have its standard orientation, and the line defect, say the z-axis, the induced orientation, so that we have a well-defined reduction $\mathbf{Ex}(\mathbb{R}^3_{|}) \to \mathcal{B}$. Consider now an open ball $\mathbb{B}_\alpha$ within an open tubular neighbourhood Z_β of the defect, both with the induced stratifications, such that $\mathbb{B}_\alpha$ does not intersect the defect. (So $\mathbb{B}_\alpha$ is trivially stratified and Z_β has two strata.) We may for some real ϕ, put the orientation $1 \times e^{i\phi}$ on $\mathbb{B}_\alpha$, and take e_α to be $1 \times e^{i(1-t)\phi}$ concatenated with a nontrivial loop at $1 \times 1 \times 1$ constant in the first coordinate. If we take $e_{\alpha\beta} = 1 \times e^{i(1-t)\phi}$ and $e_\beta = \mathrm{id}$, then $e_\alpha \not\simeq e_{\alpha\beta} \widehat{\circledast} e_\beta$. Thus, a non-ambient stratified oriented prefactorisation algebra on $\mathbb{R}^3_{|}$ need not have this weak factorisation property. (That is, $e_\alpha^F, (e_{\alpha\beta} \widehat{\circledast} e_\beta)^F\colon F(\mathbb{B}_\alpha) \to F(\mathbb{R}^3)$ need not be equivalent.)

Construction 5.15 (families of structure maps: the ambient-compatible case)**.** We have $\prod_{i=1}^k X^{\mathcal{B}}(\alpha_i \,|\, \beta) \simeq X^{\mathcal{B}}(\coprod_{i=1}^k \alpha_i \,|\, \beta)$ for formal reasons. Indeed, for given $e_{i\beta} \in X^{\mathcal{B}}(\alpha_i \,|\, \beta)$, we have $\coprod e_{i\beta} \simeq \mathfrak{e}_{\coprod \alpha_i, \beta}$ due to $\mathfrak{e}_{\coprod \alpha_i, \beta} = \coprod \mathfrak{e}_{i\beta}$, and the inverse map is obvious. This gives $\mathbb{O}_X^{\mathcal{B}}(\boldsymbol{\alpha} \,|\, \beta) \simeq \mathbb{O}_X^{\mathcal{B}}(\coprod \alpha_i \,|\, \beta)$. In particular,

$$\mathbb{O}_X^{\mathcal{B}}(\boldsymbol{\alpha} \,|\, \coprod \alpha_i) \simeq \mathbb{O}_X^{\mathcal{B}}(\coprod \alpha_i \,|\, \coprod \alpha_i).$$

Thus, for F an a-$\mathcal{B}$-pFA, we have a canonical map

$$\mathfrak{e}_{\coprod \alpha_i}^F\colon \bigotimes F(\alpha_i) \to F(\coprod \alpha_i)$$

[11] *Warning:* This does not describe induced Stokes orientations.

induced by the unit at $\coprod \alpha_i$. More precisely, $\mathfrak{e}_{\coprod \alpha_i, \coprod \alpha_i} \in \mathbb{O}_X^{\mathcal{B}}(\coprod \alpha_i \,|\, \coprod \alpha_i)$ gives a family $(\mathfrak{e}_{\alpha_j, \coprod \alpha_i})_j$, and thus a family $(\mathfrak{e}^F_{\alpha_j, \coprod \alpha_i} \colon F(\alpha_j) \to F(\coprod \alpha_i))_j$, which gives the $\mathfrak{e}^F_{\coprod \alpha_i}$ above after tensoring this family.

This need not be an equivalence: the operad does not prescribe a map in the inverse direction; we only know that every other $e^F_{\alpha_i}$ coming from $\mathbb{O}_X^{\mathcal{B}}(\boldsymbol{\alpha} \,|\, \coprod \alpha_i)$ will be equivalent to this one.

We note that the composition

$$\circledast \colon \mathbb{O}_X^{\mathcal{B}}(\coprod \alpha_i \,|\, \beta) \times \mathbb{O}_X^{\mathcal{B}}(\boldsymbol{\alpha} \,|\, \coprod \alpha_i) \to \mathbb{O}_X^{\mathcal{B}}(\boldsymbol{\alpha} \,|\, \beta)$$

prescribes, upon fixing the unit at $\coprod \alpha_i$ in the second argument, a family of maps

$$(\mathfrak{e}_{\coprod \alpha_i} \circledast -)^F \colon X^{\mathcal{B}}(\coprod \alpha_i \,|\, \beta) \to \mathcal{T}(\bigotimes F(\alpha_i), F(\beta)). \tag{4}$$

Note that any other $e_{\coprod \alpha_i}$ in place of $\mathfrak{e}_{\coprod \alpha_i}$ will give an equivalent such map between mapping spaces. These are the analogues here of the ordinary structure maps of a prefactorisation algebra. The above is a canonically-specified sector of a larger space of such maps.

Remark 5.16 (families of structure maps: non-ambient-compatible case). We also have $\prod \widehat{X}^{\mathcal{B}}(\alpha_i \,|\, \beta) \simeq \widehat{X}^{\mathcal{B}}(\coprod \alpha_i \,|\, \beta)$, and in the same way we get a map $\mathfrak{e}^F \colon \bigotimes F(\alpha_i) \to F(\coprod \alpha_i)$ for F a $\mathcal{B}$-pFA. In this case, F may give inequivalent such maps. Therefore, $\widehat{\mathbb{O}}_X^{\mathcal{B}}(\boldsymbol{\alpha} \,|\, \coprod \alpha_i) \simeq \widehat{\mathbb{O}}_X^{\mathcal{B}}(\coprod \alpha_i \,|\, \coprod \alpha_i)$ and we can compose $\widehat{\circledast} \colon \widehat{\mathbb{O}}_X^{\mathcal{B}}(\coprod \alpha_i \,|\, \beta) \times \widehat{\mathbb{O}}_X^{\mathcal{B}}(\boldsymbol{\alpha} \,|\, \coprod \alpha_i) \to \widehat{\mathbb{O}}_X^{\mathcal{B}}(\boldsymbol{\alpha}_i \,|\, \beta)$. Now, F may give maps $(e_{\coprod \alpha_i} \widehat{\circledast} -)^F \colon \widehat{X}^{\mathcal{B}}(\coprod \alpha_i \,|\, \beta) \to \mathcal{T}(\bigotimes F(\alpha_i), F(\beta))$ that may be inequivalent for inequivalent choices of $e_{\coprod \alpha_i}$.

6. Locality

We adapt the Čech cosheaf condition. First, by the 'intersection' of two $\mathcal{B}$-opens U, V in X we must mean the limit of the inclusions within $\mathcal{S}\mathrm{trat}^{\mathcal{B}\text{-emb}}$, or equivalently over $(X, T_X^{\mathcal{B}})$. It is easily seen that $(U \cap V, \iota_{UV}^* T_X^{\mathcal{B}})$, is such a limit,[12] and from now on we will suppress this $\mathcal{B}$-reduction in notation. We start with $\mathcal{B}$-versions of common cosheaf theory notation.

Definition 6.1. Let F be an a-$\mathcal{B}$-pFA on X and $\boldsymbol{U} = \{U_\alpha\}_\alpha$ be a cover of a $\mathcal{B}$-open U by $\mathcal{B}$-opens. We will call $\boldsymbol{U}$ a $\mathcal{B}$***-cover*** of U. Let $P\boldsymbol{U} \subset \coprod_n \boldsymbol{U}^n$ consist of families (of any finite size) that are pairwise disjoint.

When $\boldsymbol{U}$ is fixed, we will omit it in notation.

Notation 6.2. For $0 \le l \le k-1$, let $\boldsymbol{A}^{\widehat{l}} \in P\boldsymbol{U}^k$ denote $\boldsymbol{A} \in P\boldsymbol{U}^{k+1}$ with the l-component omitted, and similarly for $\boldsymbol{\alpha}^{\widehat{l}} \in \boldsymbol{A}^{\widehat{l}}$. We write $\bigcap \boldsymbol{\alpha} := \bigcap_{\alpha_i \in \boldsymbol{\alpha}} \alpha_i$.

Definition 6.3. For $k \ge 0$, let

$$\check{\mathcal{C}}_k^F(\boldsymbol{U}) := \check{\mathcal{C}}_k(\boldsymbol{U}; F) := \coprod_{\boldsymbol{A} \in P\boldsymbol{U}^{k+1}} \bigotimes_{\boldsymbol{\alpha} \in \boldsymbol{A}} F\left(\bigcap_{\alpha_i \in \boldsymbol{\alpha}} \alpha_i\right).$$

[12]One could also induce the $\mathcal{B}$-structure from U or V; the limiting $\mathcal{B}$-structure is taken in an ∞-groupoid, $\mathcal{B}\text{-open}(\iota_{UV})$, so this choice is immaterial.

Definition 6.4. We set

$$\mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_n \,|\, \check{\mathcal{C}}_m) \coloneqq \coprod_{\rho\in\boldsymbol{\Delta}([m],[n])} \prod_{\boldsymbol{A}\in P\boldsymbol{U}^{n+1}} \prod_{\boldsymbol{\alpha}\in\boldsymbol{A}} \mathbb{O}_X^{\mathcal{B}}(\cap\boldsymbol{\alpha} \,|\, \cap\rho^*\boldsymbol{\alpha}),$$

where $\rho^*(\boldsymbol{\alpha}) \in \rho^*(\boldsymbol{A}) \in P\boldsymbol{U}^{m+1}$ is $\boldsymbol{\alpha}$ whose indices are restricted along ρ in the obvious sense.

Definition 6.5. Consider the ∞-category whose objects are the same as those of $\boldsymbol{\Delta}$, and whose morphism space from $[n]$ to $[m]$ is given by $\mathbb{O}_X^{\mathcal{B}}\left(\check{\mathcal{C}}_m \,|\, \check{\mathcal{C}}_n\right)$, with compositions

$$\check{\circledast}\colon \mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_m \,|\, \check{\mathcal{C}}_n) \times \mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_k \,|\, \check{\mathcal{C}}_m) \to \mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_k \,|\, \check{\mathcal{C}}_n),$$

defined by

$$\begin{aligned}&(e_{\cap\alpha\,|\,\cap\rho^*\boldsymbol{\alpha}})_{\boldsymbol{A},\boldsymbol{\alpha}}, (e_{\cap\boldsymbol{\beta}\,|\,\cap\sigma^*\boldsymbol{\beta}})_{\boldsymbol{B},\boldsymbol{\beta}}\\ &\mapsto (e_{\cap\boldsymbol{\gamma}\,|\,\sigma^*\boldsymbol{\gamma}} \circledast e_{\sigma^*\boldsymbol{\gamma}\,|\,\rho^*\sigma^*\boldsymbol{\gamma}})_{\boldsymbol{\gamma},\boldsymbol{C}}\end{aligned}$$

from the (ρ,σ)-component to the $\sigma\circ\rho$-component, using, in the noted instance, the coordinates $\boldsymbol{B} = \boldsymbol{C}$, $\boldsymbol{A} = \sigma^*\boldsymbol{C}$, $\boldsymbol{\alpha} = \sigma^*\boldsymbol{\gamma}$.

We denote by $\boldsymbol{\Delta}_X^{\mathcal{B}}(\boldsymbol{U})$ the *opposite* ∞-category. We will call an object of $\mathsf{pSh}(\boldsymbol{\Delta}_X^{\mathcal{B}};\mathcal{T}) = [\boldsymbol{\Delta}_X^{B}(\boldsymbol{U})^{\mathrm{op}},\mathcal{T}]$ a $\mathcal{B}$-***simplicial object*** in $\mathcal{T}$ over $\boldsymbol{U}$.

This $\boldsymbol{\Delta}_X^{\mathcal{B}}(\boldsymbol{U})$ is a version of the ordinary simplex category $\boldsymbol{\Delta}$ that reflects the geometric information we carry, whence the dependence on $\mathcal{B}$ and X. We now arrive at

Definition 6.6. The diagram

$$\check{\mathcal{C}}_\bullet(\boldsymbol{U};F) \in \mathsf{pSh}(\boldsymbol{\Delta}_X^{\mathcal{B}};\mathcal{T}),$$

called the ***a-$\mathcal{B}$-Čech complex*** of F over $\boldsymbol{U}$, is defined on objects by

$$[n] \mapsto \check{\mathcal{C}}_n^F(\boldsymbol{U}),$$

and on morphism spaces by

$$(\boldsymbol{\Delta}_X^{\mathcal{B}})^{\mathrm{op}}([n],[m]) = \mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_n \,|\, \check{\mathcal{C}}_m) \to \mathcal{T}(\check{\mathcal{C}}_n^F, \check{\mathcal{C}}_m^F),$$

$$(e_{\cap\boldsymbol{\alpha}\,|\,\cap\rho^*\boldsymbol{\alpha}})_{\rho,\boldsymbol{A},\boldsymbol{\alpha}} \mapsto (\otimes_{\boldsymbol{\alpha}\in\boldsymbol{A}} e^F_{\cap\boldsymbol{\alpha}\,|\,\cap\rho^*\boldsymbol{\alpha}}\colon \otimes_{\boldsymbol{\alpha}\in\boldsymbol{A}} F(\cap\boldsymbol{\alpha}) \to \otimes_{\rho^*\boldsymbol{\alpha}\in\rho^*\boldsymbol{A}} F(\cap\rho^*\boldsymbol{\alpha}))_{\boldsymbol{A}},$$

where the extension by units

$$\otimes_{\rho^*\boldsymbol{\alpha}\in\rho^*\boldsymbol{A}} F(\cap\rho^*(\boldsymbol{\alpha})) \to \otimes_{\boldsymbol{\beta}\in\rho^*\boldsymbol{A}} F(\cap\boldsymbol{\beta})) \hookrightarrow \check{\mathcal{C}}_m^F,$$

into the $\rho^*\boldsymbol{A} \in P\boldsymbol{U}^{m+1}$ component of $\check{\mathcal{C}}_m^F$, is understood if necessary (when ρ is not injective). Functoriality is immediate by inspection, by virtue of the way $\check{\circledast}$ is defined in terms of $\circledast$ and of the functoriality of F.

In sum, $\check{\mathcal{C}}_\bullet$ replaces the ordinary Čech complex, which is a plain simplicial object. It is a homotopical generalisation in the sense that its '0-truncation' is the ordinary Čech complex, a statement that we will make precise (in 6.13). This is essentially trivial, but a necessary sanity check. We start by making precise how we truncate.

Definition 6.7. Let $\|-\|$ (resp. $|-|$) denote (-1)-truncation (resp. 0-truncation), that is, $\|\mathcal{C}\|$ (resp. $|\mathcal{C}|$) is the ∞-category whose hom-spaces are the (-1)-truncations (resp. 0-truncations) of those of $\mathcal{C}$, with composition defined by composing representatives and then taking equivalence classes. For a more general account of truncations, see [Lur09, §5.5.6].

We (-1)- or 0-truncate ∞-operads by truncating the underlying ∞-categories. The results are readily checked to be ordinary coloured operads. This is immediate since we are only (-1)- or 0-truncating.

Definition 6.8. Let $\mathcal{O}$, $\mathcal{P}$ be two ∞-operads, and $F\colon \mathcal{O} \to \mathcal{P}$ be a map. Let $\mathcal{O} \to |\mathcal{O}|$ be the 0-truncation. We call any pair

$$(|F|\colon |\mathcal{O}| \to \mathcal{P}, \mathfrak{f}\colon |\mathcal{O}| \to \mathcal{O}|),$$

with the second coordinate a section of truncation, such that

$$\begin{tikzcd} |\mathcal{O}| \arrow[r, "\mathfrak{f}"] \arrow[d, "|F|"'] & \mathcal{O} \arrow[dl, "F"] \\ \mathcal{P} & \end{tikzcd}$$

homotopy-commutes, a 0-***truncation*** of F. Given a section $\mathfrak{f}$, the canonical choice is $|F| = \mathfrak{f}^* F$. This choice, when made, will be clear from context.

We define a 0-truncation of a map between ∞-categories in exactly the same by removing operadicity conditions.

Recall the operad pFA_X of prefactorisation algebras on X. Its objects are (stratified) opens of X, and the multihom from $\boldsymbol{\alpha}$ to β is $*$ if $\coprod \boldsymbol{\alpha} \subseteq \beta$, and $\emptyset$ otherwise.

Definition 6.9. Let

$$\mathfrak{e}\colon \mathrm{pFA}_X \to \mathbb{O}_X^{\mathcal{B}}$$

denote the map given by sending an open β to the $\mathcal{B}$-open $(\beta, \beta^* T_X^{\mathcal{B}})$ with the identity equivalence to the same $\mathcal{B}$-structure giving the $\mathcal{B}$-open structure, and

$$\mathrm{pFA}_X(\boldsymbol{\alpha} \,|\, \beta) \ni * \mapsto (\mathfrak{e}_{\alpha_i \,|\, \beta})_{\boldsymbol{\alpha}} \in \mathbb{O}_X^{\mathcal{B}}((\boldsymbol{\alpha}, \boldsymbol{\alpha}^* T_X^{\mathcal{B}}) \,|\, (\beta, \beta^* T_X^{\mathcal{B}}))$$

if $\coprod \boldsymbol{\alpha} \subseteq \beta$.

Lemma 6.10. *The map $\mathfrak{e}\colon \mathrm{pFA}_X \to \mathbb{O}_X^{\mathcal{B}}$ defines a suboperad. Moreover, postcomposition with 0-truncation gives an equivalence*

$$|\mathfrak{e}|\colon \mathrm{pFA}_X \xrightarrow{\sim} |\mathbb{O}_X^{\mathcal{B}}|.$$

Proof. The observation, from the proof of 5.7, that $\mathfrak{e}_{\alpha \,|\, \beta} \circledast \mathfrak{e}_{\beta \,|\, \gamma} = \mathfrak{e}_{\alpha \,|\, \gamma}$, immediately implies that $\mathfrak{e}$ is a suboperad. Since the multihoms of pFA_X are (-1)-types and those of $\mathbb{O}_X^{\mathcal{B}}$ are connected, the equivalence follows. □

Definition 6.11. By

$$\mathfrak{e}^{\boldsymbol{\Delta}}\colon \boldsymbol{\Delta} \hookrightarrow \boldsymbol{\Delta}_X^{\mathcal{B}}(\boldsymbol{U})$$

we denote the map that sends $\rho\colon [n] \to [m]$ to $(\mathfrak{e}_{\cap \boldsymbol{\alpha} \,|\, \cap \rho^* \boldsymbol{\alpha}})_{\boldsymbol{A}, \boldsymbol{\alpha}}$ in the ρ-component of $\mathbb{O}_X^{\mathcal{B}}(\check{\mathcal{C}}_m, \check{\mathcal{C}}_n)$.

Lemma 6.12. *The postcomposition*

$$|\mathfrak{e}^{\boldsymbol{\Delta}}|\colon \boldsymbol{\Delta} \xrightarrow{\sim} |\boldsymbol{\Delta}^{\mathcal{B}}_X|$$

of $\mathfrak{e}^{\boldsymbol{\Delta}}$ *with* 0*-truncation is an equivalence.*

Proof. Since $\mathbb{O}^{\mathcal{B}}_X(\cap\boldsymbol{\alpha} \,|\, \cap\rho^*\boldsymbol{\alpha})$ is connected for any $\boldsymbol{A}$, $\boldsymbol{\alpha} \in \boldsymbol{A}$, the product remains connected, so that

$$|\mathbb{O}^{\mathcal{B}}_X(\check{\mathcal{C}}_m \,|\, \check{\mathcal{C}}_n)| \simeq \coprod_{\boldsymbol{\Delta}([n],[m])} * \simeq \boldsymbol{\Delta}([n],[m]).$$

This implies that $|\mathfrak{e}^{\boldsymbol{\Delta}}|\colon \boldsymbol{\Delta} \xrightarrow{\sim} |\boldsymbol{\Delta}^{\mathcal{B}}_X|$, is an equivalence of ordinary categories, whose functoriality is warranted, for $[n] \xrightarrow{\rho} [m] \xrightarrow{\sigma} [k]$ in $\boldsymbol{\Delta}$, by

$$\begin{aligned}
|\mathfrak{e}^{\boldsymbol{\Delta}}(\rho)\check{\circledast}\mathfrak{e}^{\boldsymbol{\Delta}}(\sigma)| &= |(\mathfrak{e}_{\cap\boldsymbol{\alpha}\,|\,\cap\rho^*\boldsymbol{\alpha}})_{\boldsymbol{A},\boldsymbol{\alpha}}\check{\circledast}(\mathfrak{e}_{\cap\boldsymbol{\beta}\,|\,\cap\sigma^*\boldsymbol{\beta}})_{\boldsymbol{B},\boldsymbol{\beta}}| \\
&= |(\mathfrak{e}_{\cap\boldsymbol{\gamma}\,|\,\sigma^*\boldsymbol{\gamma}} \circledast \mathfrak{e}_{\cap\sigma^*\boldsymbol{\gamma}\,|\,\cap\rho^*\sigma^*\boldsymbol{\gamma}})_{\boldsymbol{C},\boldsymbol{\gamma}}| = |(\mathfrak{e}_{\cap\boldsymbol{\gamma}\,|\,\cap(\sigma\circ\rho)^*\boldsymbol{\gamma}})_{\boldsymbol{C},\boldsymbol{\gamma}}| \\
&= |\mathfrak{e}^{\boldsymbol{\Delta}}|(\sigma\circ\rho).
\end{aligned}$$

□

Proposition 6.13. *For* F *an a-*$\mathcal{B}$*-pFA on* X *and* $\boldsymbol{U}$ *a fixed* $\mathcal{B}$*-cover of a fixed* $\mathcal{B}$*-open* U*, we have*

$$|\check{\mathcal{C}}^F_\bullet| \simeq \check{C}^{|F|}_\bullet$$

in $\mathcal{T}^{\boldsymbol{\Delta}^{\mathrm{op}}}$.

Proof. The LHS fits into

$$\begin{tikzcd}
{|\boldsymbol{\Delta}^{\mathcal{B}}_X|} \arrow[r, "|\mathfrak{e}^{\boldsymbol{\Delta}}|^{-1}", "\sim"'] \arrow[d, dashed] & \boldsymbol{\Delta} \arrow[r, "\mathfrak{e}^{\boldsymbol{\Delta}}"] & \boldsymbol{\Delta}^{\mathcal{B}}_X \arrow[dll, "(\check{\mathcal{C}}^F_\bullet)^{\mathrm{op}}"] \\
\mathcal{T}^{\mathrm{op}} & &
\end{tikzcd}$$

which defines it in $\mathcal{T}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ using 6.12. The truncation $|F|$ fits into

$$\begin{tikzcd}
{|\mathbb{O}^{\mathcal{B}}_X|} \arrow[r, "|\mathfrak{e}|^{-1}", "\sim"'] \arrow[d, dashed] & \mathrm{pFA}_X \arrow[r, "\mathfrak{e}"] & \mathbb{O}^{\mathcal{B}}_X \arrow[dll, "F"] \\
\underline{\mathcal{T}} & &
\end{tikzcd}$$

Up to equivalence, we may collapse $|\mathfrak{e}^{\boldsymbol{\Delta}}|^{-1}$ and $|\mathfrak{e}|^{-1}$ so that the statement becomes the homotopy-commutation of

$$\begin{tikzcd}
\boldsymbol{\Delta}^{\mathrm{op}} \arrow[r, "(\mathfrak{e}^{\boldsymbol{\Delta}})^{\mathrm{op}}"] \arrow[d, "\check{C}^{\mathfrak{e}^*F}_\bullet"'] & (\boldsymbol{\Delta}^{\mathcal{B}}_X)^{\mathrm{op}} \arrow[dl, "\check{\mathcal{C}}^F_\bullet"] \\
\mathcal{T} &
\end{tikzcd}$$

which is trivial by inspection. □

Remark 6.14. There are in general inequivalent suboperads $\mathrm{pFA}_X \to \widehat{\mathbb{O}}^{\mathcal{B}}_X$ (in other words, it need not be true that $\mathrm{pFA}_X \simeq |\widehat{\mathbb{O}}^{\mathcal{B}}_X|$ unless $\mathcal{B}$ is a 0-type such as $\mathbb{Z}_{\geq 0}$), so that $|F|$ need not be unique if F is a $\mathcal{B}$-pFA. Still, a non-ambient version $\widehat{\boldsymbol{\Delta}}^{\mathcal{B}}_X$ of $\boldsymbol{\Delta}^{\mathcal{B}}_X$ may still be defined using similarly defined hom-spaces $\widehat{\mathbb{O}}^{\mathcal{B}}_X(\check{\mathcal{C}}_n, \check{\mathcal{C}}_m)$, yielding a non-ambient $\widehat{\check{\mathcal{C}}}^F_\bullet$. Then, $|\mathfrak{e}^{\boldsymbol{\Delta}}|$ and $|\mathfrak{e}|$ need not be equivalences, but it is possible, given a truncation $|F|$, to find a corresponding Čech truncation such that $|\widehat{\check{\mathcal{C}}}^F_\bullet| \simeq \check{C}^{|F|}_\bullet$. Conversely, it is possible, given a truncation $|\widehat{\check{\mathcal{C}}}^F_\bullet|$, to find a corresponding truncation $|F|$ satisfying the same equivalence.

After this sanity check, we now come to the cosheaf condition.

By a colimit of a functor of ∞-categories, we will mean an initial object, unique up to contractible choice, in the under-∞-category over that diagram in the target. An equivalent definition with full detail is given in [Lur22, Tag 02VZ].

We could simply ask $\operatorname{colim} \check{\mathcal{C}}^F_\bullet$ to be equivalent to $F(U)$ for varying U and $\boldsymbol{U}$ as a cosheaf condition. But this would be useless on two accounts: 1) we would not have a generic description of a cocone $\check{\mathcal{C}}^F_\bullet \to F(U)$ to compare the two sides; and 2) we would not know how to use these colimits themselves to define a $\mathcal{B}$-factorisation algebra from local data, as is commonly done to construct ordinary locally-constant factorisation algebras. We therefore address this issue now.

Notation 6.15. For $k \geq 0$, we write

$$\mathfrak{e}^F_k \coloneqq (\otimes_{\boldsymbol{\alpha}\in\boldsymbol{A}} \mathfrak{e}_{\cap\boldsymbol{\alpha}\,|\,U})^F_{\boldsymbol{A}\in P\boldsymbol{U}^{k+1}} : \check{\mathcal{C}}^F_k \to F(U),$$

where the family over $\boldsymbol{A}$ is understood to give a map out of the outer coproduct over $\boldsymbol{A}$ defining $\check{\mathcal{C}}^F_k$.

Proposition 6.16. *The collection* $\mathfrak{e}^F_\bullet : \check{\mathcal{C}}^F_\bullet \to F(U)$ *lifts to an object of* $\mathcal{T}^{\check{\mathcal{C}}^F_\bullet/}$.

In other words, there is a cocone under $\check{\mathcal{C}}^F_\bullet$ with cotip $F(U)$ and edges $\mathfrak{e}^F_\bullet$.

Proof. We provide, for each $d^F_{n\,|\,m} \coloneqq (d^F_{\cap\boldsymbol{\alpha}\,|\,\cap\rho^*\boldsymbol{\alpha}})_{\rho,\boldsymbol{A},\boldsymbol{\alpha}} \in [\![\check{\mathcal{C}}^F_n, \check{\mathcal{C}}^F_m]\!]$ in the image of F out of $\mathbb{O}^{\mathcal{B}}_X(\check{\mathcal{C}}_n\,|\,\check{\mathcal{C}}_m)$, a 2-equivalence $\mathfrak{d}_{n\,|\,m}$ commuting

$$\begin{tikzcd} \check{\mathcal{C}}^F_n \arrow[rr, "d^F_{n\,|\,m}"] \arrow[dr, "\mathfrak{e}^F_n"'] & \arrow[d, phantom, "\Downarrow \mathfrak{d}_{n\,|\,m}"] & \check{\mathcal{C}}^F_m \arrow[dl, "\mathfrak{e}^F_m"] \\ & F(U) & \end{tikzcd} \tag{5}$$

such that for any other $f^F_{n\,|\,m} \in [\![\check{\mathcal{C}}^F_n, \check{\mathcal{C}}^F_m]\!]$, there are equivalences commuting

$$\begin{tikzcd} \mathfrak{e}^F_m f^F_{n\,|\,m} \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "\mathfrak{f}^F_{n\,|\,m}"'] & \arrow[d, Rightarrow] & \mathfrak{e}^F_m d^F_{n\,|\,m} \arrow[dl, Rightarrow, "\mathfrak{d}^F_{n\,|\,m}"] \\ & \mathfrak{e}^F_n & \end{tikzcd} \tag{6}$$

as well as, for each other $g^F_{m\,|\,k} \in [\![\check{\mathfrak{C}}^F_m, \check{\mathfrak{C}}^F_k]\!]$ fitting

$$
\begin{tikzcd}
\check{\mathfrak{C}}^F_n \arrow[rr, "d^F_{n\,|\,m}"] \arrow[rrrr, dashed, bend left, "(gd)^F_{n\,|\,k}"] \arrow[ddrr, bend right, "\mathfrak{e}^F_n"'] & & \check{\mathfrak{C}}^F_m \arrow[rr, "g^F_{m\,|\,k}"] \arrow[dd, "\mathfrak{e}^F_m"] & & \check{\mathfrak{C}}^F_k \arrow[ddll, bend left, "\mathfrak{e}^F_k"] \\
& \arrow[r, phantom, "\Swarrow\,\mathfrak{d}_{n\,|\,m}"] & \arrow[r, phantom, "\Leftarrow\,\mathfrak{dg}_{n\,|\,k}"] & \arrow[r, phantom, "\Nwarrow\,\mathfrak{g}_{m\,|\,k}"] & \\
& & F(U) & &
\end{tikzcd}
\tag{7}
$$

where $\mathfrak{dg}_{n\,|\,m}$ is the specified equivalence attached to $(gd)^F_{n\,|\,k} \coloneqq g^F_{m\,|\,k} \circ d^F_{n\,|\,m}$, a further 3-equivalence witnessing $\mathfrak{dg}_{n\,|\,k}$ as the corresponding composition of 2-equivalences, that is, one that commutes

$$
\begin{tikzcd}
\mathfrak{e}^F_k (gd)^F_{n\,|\,k} \arrow[rr, Rightarrow, "(d^F_{n\,|\,m})^* \mathfrak{g}_{m\,|\,k}"] \arrow[dr, Rightarrow, "\mathfrak{dg}_{n\,|\,k}"'] \arrow[r, phantom, shift right=4, "\Downarrow"] & & \mathfrak{e}^F_m d^F_{n\,|\,m} \arrow[dl, Rightarrow, "\mathfrak{d}_{n\,|\,m}"] \\
& \mathfrak{e}^F_n &
\end{tikzcd}
\tag{8}
$$

These two types of data, 6 and 8, must commute in both directions, in the sense that in each occurrence of maps of type

$$
\begin{tikzcd}
\check{\mathfrak{C}}^F_n \arrow[r, shift left, "f^F_{n\,|\,m}"] \arrow[r, shift right, "d^F_{n\,|\,m}"'] & \check{\mathfrak{C}}^F_m \arrow[r, shift left, "h^F_{n\,|\,m}"] \arrow[r, shift right, "g^F_{n\,|\,m}"'] & \check{\mathfrak{C}}^F_k ,
\end{tikzcd}
$$

there are commutations

$$
\begin{tikzcd}
\mathfrak{e}^F_k (gf)^F_{n\,|\,k} \arrow[rr, Rightarrow, "(\mathfrak{e}^F_k g^F_{m\,|\,k})_* -"] \arrow[dr, Rightarrow, "\mathfrak{fg}_{n\,|\,k}"'] \arrow[r, phantom, shift right=4, "\Downarrow"] & & \mathfrak{e}^F_k (gd)^F_{n\,|\,k} \arrow[dl, Rightarrow, "\mathfrak{dg}_{n\,|\,k}"] \\
& \mathfrak{e}^F_n &
\end{tikzcd}
\tag{9}
$$

where the horizontal pushforward is applied to the specified horizontal map in 6, and

$$
\begin{tikzcd}
\mathfrak{e}^F_k (hd)^F_{n\,|\,k} \arrow[rr, Rightarrow, "(\mathfrak{e}^F_k d^F_{n\,|\,m})^* -"] \arrow[dr, Rightarrow, "\mathfrak{dh}_{n\,|\,k}"'] \arrow[r, phantom, shift right=4, "\Downarrow"] & & \mathfrak{e}^F_k (gd)^F_{n\,|\,m} \arrow[dl, Rightarrow, "\mathfrak{dg}_{n\,|\,k}"] \\
& \mathfrak{e}^F_n &
\end{tikzcd}
\tag{10}
$$

where the horizontal pullback is applied to the specified horizontal map in 6 with f replaced by h and d replaced by g. This data will exhibit $F(U)$ as a cocone under $\check{\mathfrak{C}}^F_\bullet$. From the construction it will be clear that the higher morphisms necessitated by the existence of this cocone (the $(n+1)$-morphisms extending the n-morphisms of $\check{\mathfrak{C}}^F_\bullet$ into the cocone for higher n than already considered) will be trivial.

Now, we have homotopies $\widehat{d}_{\cap\boldsymbol{\alpha}\,|\,\cap\rho^*\boldsymbol{\alpha}}\colon d_{\cap\boldsymbol{\alpha}\,|\,\cap\rho^*\boldsymbol{\alpha}} \xrightarrow{\sim} \mathfrak{e}_{\cap\boldsymbol{\alpha}\,|\,\cap\rho^*\boldsymbol{\alpha}}$ specified by the definition of $X^{\mathcal{B}}(\cap\boldsymbol{\alpha}\,|\,\cap\rho^*\boldsymbol{\alpha})$ for each $\rho, \boldsymbol{A}, \boldsymbol{\alpha}$, which assemble into

$$\widehat{d}_{n\,|\,m} \in [\![d_{n\,|\,m}, \mathfrak{e}_{n\,|\,m}]\!]_{\mathbb{O}_X^{\mathcal{B}}(\check{\mathfrak{C}}_n\,|\,\check{\mathfrak{C}}_m)}$$

where (see 6.11)

$$\mathfrak{e}_{n\,|\,m} \coloneqq \prod_{\rho\in\boldsymbol{\Delta}([n],[m])} \mathfrak{e}^{\boldsymbol{\Delta}}(\rho).$$

Noting

$$\mathfrak{e}^F_m \mathfrak{e}^F_{n\,|\,m} = \mathfrak{e}^F_n \tag{11}$$

by inspection as in the proof of 5.7, we set

$$\mathfrak{d}_{n\,|\,m} \coloneqq (\mathfrak{e}^F_m)_* \widehat{d}^F_{n\,|\,m}\colon \mathfrak{e}^F_m d^F_{n\,|\,m} \xrightarrow{\sim} \mathfrak{e}^F_m \mathfrak{e}^F_{n\,|\,m} = \mathfrak{e}^F_n.$$

Any $\widehat{df} \in [\![f_{n\,|\,m}, d_{n\,|\,m}]\!]_{\mathbb{O}_X^{\mathcal{B}}(\check{\mathfrak{C}}_n\,|\,\check{\mathfrak{C}}_m)}$ covers $\mathfrak{e}_{n\,|\,m}$ along $\widehat{f}$ and $\widehat{d}$, i.e. $\widehat{d}\widehat{df} \simeq \widehat{f}$ witnessed by a specified $\underline{df}$, so we have

$$\begin{tikzcd} f^F_{n\,|\,m} \arrow[rr, Rightarrow, "\widehat{df}^F"] \arrow[dr, Rightarrow, "\widehat{f}^F"'] & \arrow[d, phantom, "\Downarrow \underline{df}^F"] & d^F_{n\,|\,m} \arrow[dl, Rightarrow, "\widehat{d}^F"] \\ & \mathfrak{e}^F_{n\,|\,m} & \end{tikzcd} \quad \text{in} \quad [\![\check{\mathfrak{C}}^F_n, \check{\mathfrak{C}}^F_m]\!],$$

applying $(\mathfrak{e}^F_m)_*$ to which, by the above and again using 11, gives

$$\begin{tikzcd} \mathfrak{e}^F_m f^F_{n\,|\,m} \arrow[rr, Rightarrow, "(\mathfrak{e}^F_m)_*\widehat{df}"] \arrow[dr, Rightarrow, "\mathfrak{f}_{n\,|\,m}"'] & \arrow[d, phantom, "\Downarrow (\mathfrak{e}^F_m)_*\underline{df}^F"] & \mathfrak{e}^F_m d^F_{n\,|\,m} \arrow[dl, Rightarrow, "\mathfrak{d}_{n\,|\,m}"] \\ & \mathfrak{e}^F_n & \end{tikzcd} \quad \text{in} \quad [\![\check{\mathfrak{C}}^F_n, F(U)]\!],$$

which provides 6. Next, $\widehat{gd}_{n\,|\,k}$ factors as

$$\begin{tikzcd} (g\circ d)_{n\,|\,k} \arrow[r, "\widehat{gd}"] \arrow[d, "d^*_{n\,|\,m}\widehat{g}_{m\,|\,k}"'] & \mathfrak{e}_{n\,|\,k} \arrow[d, equal] \\ \mathfrak{e}_{m\,|\,k} d_{n\,|\,m} \arrow[r, "(\mathfrak{e}_{m\,|\,k})_*\widehat{d}_{n\,|\,m}"'] & \mathfrak{e}_{m\,|\,k}\mathfrak{e}_{n\,|\,m} \end{tikzcd} \quad \text{in} \quad \mathbb{O}_X^{\mathcal{B}}(\check{\mathfrak{C}}_n\,|\,\check{\mathfrak{C}}_m),$$

where the right equality follows by inspection as in 6.12. This gives

$$\mathfrak{d}\mathfrak{g}_{n\,|\,k} = (\mathfrak{e}^F_k)_*(\mathfrak{e}^F_{m\,|\,k})_*\widehat{d}^F_{n\,|\,m}(d^F_{n\,|\,m})^*\widehat{g}^F_{m\,|\,k} = (\mathfrak{e}^F_m)_*\widehat{d}^F_{n\,|\,m}(d^F_{n\,|\,m})^*\widehat{g}^F_{m\,|\,k}.$$

where we used 11. Concurrently, we have, by construction,

$$\mathfrak{d}_{n\,|\,m}(d^F_{n\,|\,m})^*\mathfrak{g}_{m\,|\,k} = (\mathfrak{e}^F_m)_*\widehat{d}^F_{n\,|\,m}(d^F_{n\,|\,m})^*\widehat{g}^F_{m\,|\,k}$$

which provides 8 simply as a composition diagram (see 4.1 regarding the choice). The commutations 9 and 10 follow analogously as composition diagrams. This implies that the higher simplices in the cocone are trivial, as advertised. □

Remark 6.17. This construction is impossible without defining $X^{\mathcal{B}}(-,-)$ as an over-∞-category; if it was only a homotopy fibre, paths incompatible over the equivalences to the respective $\mathfrak{c}$ would have to be admitted, but then the vertex-wise $\mathfrak{c}_\bullet^F$ would not be liftable to a cocone in general, unless each reduction space $\mathcal{B}$-red is a 0-type. Working with good covers, one can relax (!) this condition to $\mathcal{B}$ being a 0-type.

A $\mathcal{B}$-cover is ***factorising*** if its underlying cover is factorising.

Definition 6.18. We call an a-$\mathcal{B}$-pFA F on a stratified $\mathcal{B}$-space X with coefficients in $\mathcal{T}$ an ***ambient*** $\mathcal{B}$***-factorisation algebra*** or ***a-***$\mathcal{B}$***-FA***, if, for all $\mathcal{B}$-opens U and every factorising $\mathcal{B}$-cover $\boldsymbol{U}$ of U, $F(U) \in \mathcal{T}^{\check{\mathcal{C}}_\bullet^F(\boldsymbol{U})/}$ as specified in Prop. 6.16 is initial.

We may restrict to '$\mathcal{B}$-Galois theory' by considering locally-constant a-$\mathcal{B}$-FAs, appropriately defined. The following definitions are direct adaptations from ordinary factorisation algebra theory, and are morally a substitute for the local constancy condition on the rank of a (co)sheaf of modules.

Definition 6.19. A $\mathcal{B}$***-disk of index*** k in a $\mathcal{B}$-space X is a $\mathcal{B}$-open of type $(\mathbb{R}^k \times C(L), T^{\mathcal{B}}) \xrightarrow{\iota, e_\iota} X$, where, for X of dimension n, L has dimension $n-k-1$.

It is understood that dimension -1 implies empty. For a nontrivial example, take X to be the halfplane $\mathbb{R} \times \mathbb{R}_{\geq 0}$ with boundary $\simeq \mathbb{R}$. Taking $k = 1$ and $L = *$ so that $C(L) \simeq [0,1)$, the map $\iota \colon \mathbb{R} \times C(L) \hookrightarrow X$, $(x,t) \mapsto (x,t)$ is a plain disk of index 1. Promoting this to a $\mathcal{B}$-open would give a $\mathcal{B}$-disk of index 1.

Definition 6.20.

7. Additivity - this will be expanded and put somewhere else

It was conjectured by Ginot [Gin15] and Ayala [private communication] that locally-constant factorisation algebras on a product stratified space should satisfy an additivity property, also known as 'exponential property', or 'Fubini law', similar to Dunn–Lurie additivity. Recall (see e.g. Ginot, op. cit.) that the latter reduces to such a Fubini law for such FAs on the trivially-stratified product space $\mathbb{R}^n$ via Lurie's theorem that the ∞-categories of $\mathbb{E}_k$-algebras and locally-constant factorisation algebras on $\mathbb{R}^k$ are equivalent.

This is not true, without further qualification, even with variframings which would be one direct generalisation of the ordinary setting since $\mathbb{E}_k$-algebras and framed k-disk algebras are equivalent.

Counterexample. Take $X = \mathbb{R}^3_{|}$, which is $\mathbb{R}^3$ with a line defect X_0, and $X_1 = \mathbb{R}^3 \setminus X_0$. (See 2.12.2 for a description of the stratified tangent bundle on this space.) In the context of example 4.10, take $U = X$, $\iota = \mathrm{id}$, and let T_U^{vfr}, T_X^{vfr} be two variframings on X. (Even though $U = X$, we consider ι as just giving a stratified open embedding.) Up to contractible choice, each variframing is determined by a homotopy between the bundle maps

$$f, 0^{3-1} \colon \varepsilon^1 \hookrightarrow \varepsilon^3 \quad \text{over} \quad S^1,$$

which is equivalent to the link, where f is part of the data of the tangent bundle, and 0^{3-1} embeds, fibrewise, as the first coordinate. Let $a, b\colon f \to 0^{3-1}$ be the homotopies determined by T_U^{vfr}, T_X^{vfr}, respectively. An equivalence $e_\iota\colon T_U^{\mathrm{vfr}} \to T_X^{\mathrm{vfr}}$ will be determined, again up to contractible choice, by a homotopy $H\colon a \to b$.

Now, since ε^1 has rank 1, f and 0^{3-1} are both determined by the point in $\mathbb{R}^3 \setminus \{0\}$ they send $1 \in \mathbb{R}$ to, at each point on the base S^1. Thus, they are maps

$$f, 0^{3-1}\colon S^1 \to S^2.$$

In fact, 0^{3-1} is the constant loop at $1 \in S^2$. We may assume, since S^2 is connected, that f hits 1, so it is another such loop. Now, H will give an equality $[a] = [b]$ in $\pi_2(S^2) \neq *$, and in fact, there may be inequivalent such H, since $\pi_3(S^2) \neq *$. (One has these nontrivialities already in [Cur69]: see the last sentence therein. Currently, it is known that $\pi_n(S^2) \neq *$ for $n \geq 2$; see [IMW16].

On the other hand, write $\mathbb{R}^3_| = \mathbb{R} \times \mathbb{R}^2_*$, where $\mathbb{R}^2_*$ is $\mathbb{R}^2$ with a point defect $*$. One can transfer a given variframing on either side to the other, so we can say $\mathbb{R}^3_|$ is isomorphic as a variframed stratified space to the RHS, with the transferred variframing. Additivity would state, with target symmetric-monoidal ∞-category $\mathcal{T}$, that

$$\text{vfr-FA}(\mathbb{R}^3_|; \mathcal{T}) \simeq \text{vfr-FA}(\mathbb{R}; \text{vfr-FA}(\mathbb{R}^2_*; \mathcal{T})),$$

ambient-compatible version or otherwise. In the outer category on the RHS, we have only two framings (=variframings), and having picked the induced one from $\mathbb{R}^3_|$ (say one that induces the canonical (rightward) framing on $\mathbb{R}$), all of the equivalences between them are trivial by virtue of $GL_+(1)$ being a (-1)-type (and so in particular a 0-type). In the inner category, similarly to the above, just the self-equivalences of $T^{\mathrm{vfr}}_{\mathbb{R}^2_*}$ are determined (even if one fixes the framings on the individual strata and their self-equivalences to be identities, like implicitly done above) by homotopies between homotopies between the maps $g, 0^{2-1} : S^1 \to S^1$,[13] with g again coming from the stratified tangent bundle. But the higher homotopy groups of S^1 are trivial.

[13]The link is again equivalent to S^1, but we have ε^2 as the target bundle instead.

8. Pushforwards

Factorisation homology of a disk algebra A on X, a stratified space of local type (geometry) corresponding to the type of the disk algebra, is the object of global sections of the factorisation algebra F_A associated to A, constructed locally from A. That is, $\int_X A \simeq F_A(X)$. For any constructible bundle $\rho\colon X \to Y$, one may pushforward A to a disk algebra $\rho_* A$ and indeed any factorisation algebra F to $\rho_* F$ such that $\rho_* F_A \simeq F_{\rho_* A}$. We have $F(X) = \rho_* F(Y)$ since $\rho^{-1} Y = X$. Since the singleton $*$ is final, we may uniformly remove the choice of Y and express factorisation homology by $F(X) = \rho_* F(*)$ for ρ the map to $*$.

In the case of stratified $\mathcal{B}$-spaces $\mathcal{S}\mathrm{trat}^{\mathcal{B}}$, we now generalise from open $\mathcal{B}$-embeddings to all stratified maps between $\mathcal{B}$-spaces that have stratified underlying maps, and are equipped with an equivalence $T_X \simeq f^* T_Y$ of maps $\mathbf{Ex}(X) \to \mathcal{V}^{\hookrightarrow}$ (which means $T_Y^{\mathcal{B}}$ does indeed induce a $\mathcal{B}$-reduction on X) such that $T_X^{\mathcal{B}} \simeq f^* T_Y^{\mathcal{B}}$. Here, an underlying singleton $*$, with some chosen $\mathcal{B}$-reduction, need not be final even if $\mathcal{B}$ is framings. (Solid framings, the most-well studied case due to direct extrapolatability to TQFTs, is rather special, as we will discuss below.) Indeed, since $T_X \simeq \rho^* T_*$ is required to compare the two $\mathcal{B}$-reductions on X in the first place, X would have to be 0-dimensional. Thus, to carry out collapse (or general factorisation homology in the above sense) in the structured context, we first need to find a final object in $\mathcal{S}\mathrm{trat}^{\mathcal{B}}$.

Definition 8.1. The ***classifying space*** of the tangential structure $\mathfrak{b}\colon \mathcal{B} \to \mathcal{V}^{\hookrightarrow}$ is a stratified space $\mathbf{B}$ together with equivalences

- $\mathbf{Ex}(\mathbf{B}) \simeq \mathcal{B}$,
- $T_{\mathbf{B}} \simeq \mathfrak{b}$.

It is a $\mathcal{B}$-space with $t_{\mathbf{B}}^{\mathcal{B}} \coloneqq \mathrm{id}_{\mathcal{B}}$, and with $\mathfrak{t}_{\mathbf{B}}^{\mathcal{B}}$ the equivalence in the second point.

Lemma 8.2. *The classifying space, if it exists, is final in* $\mathcal{S}\mathrm{trat}^{\mathcal{B}}$.

Proof. Any $\mathcal{B}$-space $(X, T_X^{\mathcal{B}})$ comes with a map $t_X^{\mathcal{B}}\colon \mathbf{Ex}(X) \to \mathbf{Ex}(\mathbf{B})$. Since $\mathbf{Ex}$ is fully faithful, there exists a unique (up to contractible choice) map $\mathbf{t}\colon X \to \mathbf{B}$ with $\mathbf{Ex}(\mathbf{t}) = t_X^{\mathcal{B}}$. In fact, $\mathbf{t}$ promotes to a $\mathcal{B}$-map via $e_{\mathbf{t}}$ presented as follows: $\mathbf{Ex}(\mathbf{t})^* T_{\mathbf{B}} \simeq (t_X^{\mathcal{B}})^* \mathfrak{b} \simeq_{\mathfrak{t}_X^{\mathcal{B}}} T_X$ provides $\mathfrak{e}_{\mathbf{t}}$, and $\epsilon_{\mathbf{t}} = \mathrm{id}$ works due to $\mathbf{Ex}(\mathbf{t})^* t_{\mathbf{B}}^{\mathcal{B}} = t_X^{\mathcal{B}}$. Thus, every $\mathcal{B}$-space admits a $\mathcal{B}$-map to $\mathbf{B}$.

Let now $(\mathbf{s}, e_{\mathbf{s}})\colon X \to \mathbf{B}$ be another $\mathcal{B}$-map. Then, $\epsilon_{\mathbf{s}}\colon t_X^{\mathcal{B}} \simeq \mathbf{Ex}(\mathbf{s})^* t_{\mathbf{B}}^{\mathcal{B}} = \mathbf{Ex}(\mathbf{s})$, but then $\mathbf{Ex}(\mathbf{s}) \simeq \mathbf{Ex}(\mathbf{t})$, so $\mathbf{s} \simeq \mathbf{t}$, since $\mathbf{Ex}$ is fully faithful. Now, to compare $\epsilon_{\mathbf{s}}$ and $\epsilon_{\mathbf{t}} = \mathrm{id}$, consider diagram 2 in the proof of 4.5. In this context it reads, after substituting $\mathbf{s} \simeq \mathbf{t}$,

$$\begin{tikzcd}
& (t_X^{\mathcal{B}})^*\mathfrak{b} \arrow[d, "(t_X^{\mathcal{B}})^*\mathfrak{t}_{\mathbf{B}}^{\mathcal{B}}"] \\
& (t_X^{\mathcal{B}})^* T_{\mathbf{B}} \arrow[d, "\mathfrak{e}_{\mathbf{t}}"] \\
\mathfrak{b}_* t_X^{\mathcal{B}} \arrow[uur, "\mathfrak{b}_*\mathfrak{e}_{\mathbf{s}}"] \arrow[r, "\mathfrak{t}_X^{\mathcal{B}}"'] \arrow[ur, phantom, "\Rightarrow \mathfrak{e}_{\mathbf{s}}"] & T_X
\end{tikzcd} \tag{12}$$

where $\mathfrak{t}_{\mathbf{B}}^{\mathcal{B}}$ is the inverse of the chosen equivalence $T_{\mathbf{B}} \simeq \mathfrak{b}$. Since $\mathfrak{e}_{\mathbf{t}}$ is given (by its construction above) $(t_X^{\mathcal{B}})^*(\mathfrak{t}_{\mathbf{B}}^{\mathcal{B}})^{-1}$, the vertical composition is in fact $\mathfrak{t}_X^{\mathcal{B}}$, the horizontal map. □

Example 8.3. Let $\mathcal{B} = \mathrm{B}* \simeq *$, $\mathfrak{b} \colon * \mapsto \mathbb{R}^n$, classify n-framings. Then $\mathbf{B} = \mathbb{R}^n$ satisfies $T_{\mathbf{B}} = \mathfrak{b}$, while $*$ itself would not.

Asking for solid n-framings already complicates matters immensely.

9. Structured collapse

10. Poisson structures

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