TU/e, 2MBA70

Solutions to problems for Measure and Probability Theory

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Version 0.2 October 3, 2024

Chapter 2: Measurable spaces (sigma-algebras and measures)

Problem 2.6
First note that if $\mu(A \cap B) = \infty$ then by property 2 we have that also $\mu(A)$, $\mu(B)$ and $\mu(A \cup B) = \infty$ and hence the result holds trivially. So assume now that $\mu(A \cap B) < \infty$. Since

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

it follows from property 1 that

$$\mu(A \cup B) = \mu(A \setminus (A \cap B))) + \mu(A \cap B) + \mu(B \setminus (A \cap B)).$$

Adding $\mu(A \cap B) < \infty$ to both side we get

$$\begin{aligned}\mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus (A \cap B))) + \mu(A \cap B) + \mu(B \setminus (A \cap B)) + \mu(A \cap B)\\ &= \mu(A) + \mu(B),\end{aligned}$$

where the last line follows from applying property 3 twice.

Problem 2.7
The idea is to construct a family of disjoint sets $(E_i)_{i \in \mathbb{N}}$ with the following properties:

1. $E_i \subset A_i$, and
2. $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} A_i$.

If such a sequence exists then we have

$$\begin{aligned}\mu(\bigcup_{i \in \mathbb{N}} A_i) &= \mu(\bigcup_{i \in \mathbb{N}} E_i) && \text{by 2}\\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{because } E_i \text{ are disjoint and } \mu \text{ is } \sigma\text{-additive}\\ &\leq \sum_{i=1}^{\infty} \mu(A_i) && \text{by 1 and monotone property of } \mu.\end{aligned}$$

So we are left to construct the required family of sets $(E_i)_{i \in \mathbb{N}}$. The following set will do:

$$E_1 = A_1 \quad E_i = A_i \setminus \bigcup_{k<i}^{i} A_k \text{ for all } i > 1.$$

Note that by definition the set E_i are pair-wise disjoint and property1 holds. Finally, property 2 holds since $\bigcup_{i=1}^{k} E_i = \bigcup_{i=1}^{k} A_i$ holds for all $k \geq 1$.

Problem 2.9 (23 points) Let $\mathcal{O}$ denote the open sets in $\mathbb{R}$.

1. (2 points) Note that the interval (a, b) is open for any $a < b \in \mathbb{R}$. Hence $\mathcal{A}_1 \subset \mathcal{A}_1' \subset \mathcal{O}$ and thus by Lemma 2.1.5 we have that $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \sigma(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$.

2. (2 points) The inclusion $\supset$ is trivial. So assume that $x \in O$. Then by definition there exist an $r > 0$ such that the ball $B_x(r) \subset O$. But $B_x(r) = (x - r, x + r) \in \mathcal{A}_1$ so $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$.

3. (3 points) Take $O \in \mathcal{O}$. If we can show that $O \in \sigma(\mathcal{A})$ then $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. The result then follows from 1.

   From 2 it follows that O is a union over a subset collection of interval (a, b) where $a, b \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, the collection $\{(a, b) \,:\, a < b \in \mathbb{Q}\}$ is also countable and hence $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$, from which it follows that $\mathcal{B}_{\mathbb{R}} \subset \sigma(\mathcal{A})$.

4. (1 point) This follows immediately from 1 and 3 since these imply that $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \mathcal{B}_{\mathbb{R}}$.

5. (3 points) The inclusion $\subset$ is trivial, since $(a, b] \subset (a + b + 1/j)$ for any $j \in \mathbb{N}$. For the other inclusion we argue by contradiction. Suppose that $x \in \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$ but $x \notin (a, b]$. Then $x > b$ and hence there exists a $j \in \mathbb{N}$ such that $(b - x) > 1/j$. But this implies that $x \notin (a, b + 1/j)$ which is a contradiction. So we conclude that $(a, b] \supset \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$.

6. (3 points) This time the inclusion $\supset$ is trivial since $(a, b - 1/j] \subset (a, b)$ for every $j \in \mathbb{N}$. For the other inclusion suppose that $x \in (a, b)$. Then there exists a $r > 0$ such that the interval $(x - r, x + r) \subset (a, b)$. In particular, this implies that $b - (x + r) > 0$. Now take any $j \in \mathbb{N}$ such that $j > 1/(b - (x + r))$. Then $b - x > r + 1/j$ which implies that $(x - r, x + r) \subset (x - r, b - 1/j]$ and hence $x \in \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$.

7. (4 points) It is clear that $\mathcal{A}_2 \subset \mathcal{A}_2'$. By 5 it follows that any interval $(a, b]$ can be obtained as a countable intersection of intervals of the form $(a, b + 1/j)$. By 4 $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1')$ which by Lemma 2.1.2 contains $\bigcap_{j \in \mathbb{N}} (a, b + 1/j) = (a, b]$. So we conclude that any interval $(a, b] \in \mathcal{B}_{\mathbb{R}}$ from which it now follows that

   $$\sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}_2') \subset \sigma(\mathcal{A}_1') = \mathcal{B}_{\mathbb{R}}.$$

   For the other inclusion we consider a set (a, b) with $a, b \in \mathbb{Q}$. Then by 6 we have that $(a, b) = \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$ where the later is a countable union of sets $(c, d]$ with $c, d \in \mathbb{Q}$ which must be in $\sigma(\mathcal{A}_2)$ by definition of a σ-algebra. Hence, any interval $(a, b) \in \sigma(\mathcal{A}_2)$ and we thus conclude, using 3, that

   $$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}_2') \subset \sigma(\mathcal{A}_1') = \mathcal{B}_{\mathbb{R}},$$

   which implies the result.

8. (2 points) Step 1 is to show that any interval $[a, b)$ can be obtained as a countable intersection of intervals $(a - 1/j, b)$. From this we can conclude that any set $[a, b)$ must be in $\mathcal{B}_{\mathbb{R}}$ proving inclusions $\subset$.

   For the other inclusions we have to show that any interval (a, b) can be obtained as a countable union of intervals $[a + 1/j, b)$, which implies that (a, b) must be in the σ-algebra generated by $[a, b)$.

9. (3 points) The main tool is to show that each of the intervals $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$ can be obtained by taking any allowed set operation for σ-algebras, i.e. countable unions/intersections and finite complements. This will help use prove the $\subset$ inclusions.

   Then we show that any set of the form (a, b), $[a, b)$ or $(a, b]$ can also be obtained through countable unions/intersections and finite complements of intervals of the forms $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$. These will then yield the $\supset$ inclusions and finish the proof.

Chapter 3: Measurable functions and stochastic objects

Problem 3.2 "$\subset$" By definition, the product σ-algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$ is defined as the σ-algebra generated by the collection

$$\mathcal{A} := \Big\{ A \times B \subset \Omega_1 \times \Omega_2 : A \in \mathcal{F}_1,\ B \in \mathcal{F}_2 \Big\}.$$

Since $A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B)$, we have that

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \sigma(\pi_1, \pi_2).$$

"$\supset$" Let $C \in \{\pi_i^{-1}(A) : i = 1, 2,\ A \in \mathcal{F}_1\}$. Then there exist sets $A \in \mathcal{F}_1$ or $B \in \mathcal{F}_2$ such that $C = \pi_1^{-1}(A) = A \times \Omega_2$ or $C = \pi_2^{-1}(B) = \Omega_1 \times B$. Either way, since $\Omega_1 \in \mathcal{F}_1$ and $\Omega_2 \in \mathcal{F}_2$, we have that $C \in \mathcal{A}$.

Problem 3.3 It is clear that $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Suppose a sequence of mutually disjoint sets $B_i \in \mathcal{G}$, $i \in \mathbb{N}$, is given. Then,

$$f_{\#}\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) = \sum_{i=1}^{\infty} f_{\#}\mu(B_i).$$

Problem 3.5

(a) Some meaningful explanation would suffice.

(b) By Proposition 2.1.8 and Problem 2.9, we know that $\mathcal{B}_{\mathbb{R}}$ is generated by intervals of the form $(-\infty, a]$ with $a \in \mathbb{Q}$. As a consequence, $\mathcal{B}_{\mathbb{R}}$ is also generated by intervals of the form $(a, +\infty)$ with $a \in \mathbb{Q}$. Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\big\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\big\}$$

is measurable for every $a \in \mathbb{Q}$. For brevity, we write $\{f + g > a\}$. The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \Big(\{f > t\} \cap \{g > a - t\}\Big).$$

We first show the inclusion '$\subset$'. If $\omega \in \Omega$ is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some $t \in \mathbb{Q}$ such that

$$f(\omega) > t > a - g(\omega),$$

and thus $f(\omega) > t$ and $g(\omega) > a - t$. So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \Big(\{f > t\} \cap \{g > a - t\} \Big).$$

Now we will show the inclusion '$\supset$'. Let $\omega \in \Omega$ be such that $f(\omega) > t$ and $g(\omega) > a - t$. Then, by adding the inequalities, we know that $f(\omega) + g(\omega) > a$.

(c) The constant function $f(\omega) = a$ is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \qquad \forall\, B \in \mathcal{B}_{\mathbb{R}}.$$

(d) Similar to the proof of Point (2) of Proposition 3.2.12.

(e) Let $g(\omega) \neq 0$ for all $\omega \in \Omega$. Then, since g is measurable, we have that

$$\begin{aligned}\{1/g > a\} &= \{g < 1/a,\ g > 0\} \cup \{g > 1/a,\ g < 0\} \\ &= \Big(\{g < 1/a\} \cap \{g > 0\}\Big) \cup \Big(\{g > 1/a\} \cap \{g < 0\}\Big) \in \mathcal{F},\end{aligned}$$

thus implying that $1/g$ is measurable.

(f) Point (e) and Point (4) of Proposition 3.2.12 yields Point (5) of Proposition 3.2.12.

Problem 3.6 From (3.6), we have for any $a \in \mathbb{R}$,

$$\Big\{\sup_{n \geq 1} f_n > a\Big\} = \bigcup_{n \geq 1} \{f_n > a\} \in \mathcal{F},$$

Since $\mathcal{F}$ is a σ-algebra and f_n is measurable for all $n \geq 1$, i.e., $\{f_n > a\} \in \mathcal{F}$ for all $n \geq 1$.

Problem 3.7

(a) Note that

$$f_M = M\mathbf{1}_{\{f \geq M\}} + f\mathbf{1}_{\{|f| < M\}} - M\mathbf{1}_{\{f \leq -M\}}.$$

Since the sets

$$\{f \geq M\}, \quad \{f \leq -M\}, \quad \{|f| < M\} \qquad \text{are } \mathcal{F}\text{-measurable},$$

their corresponding indicator functions are $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable. Since f_M is the sum of products of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable functions, we conclude that f_M is also $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

(b) It is easy to see that f_M converges pointwise to f as $M \to \infty$, i.e.,

$$\lim_{M\to\infty} f_M(\omega) = f(\omega) \qquad \forall\, \omega \in \Omega.$$

Indeed, if $\omega\Omega$ is such that $f(\omega) = +\infty$, then

$$\lim_{M\to\infty} f_M(\omega) = \lim_{M\to\infty} M = +\infty = f(\omega),$$

and similarly for $\omega \in \Omega$ for which $f(\omega) = -\infty$. On the other hand, for any $\omega \in \Omega$ with $f(\omega) \in \mathbb{R}$, there is some $N_0(\omega) \in \mathbb{N}$ such that $f_N(\omega) = f(\omega)$ for all $N \geq N_0(\omega)$, and hence,

$$\lim_{M\to\infty} f_M(\omega) = f(\omega).$$

Since f is the limit of a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable functions, we conclude from Lemma 3.2.13 that f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

Problem 3.9

(a) For the probability space, take $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0, 1]$.

Observe that the function $H_\gamma(z)$ is continuous and hence has an inverse $g_\gamma(y) = \gamma \tan(\pi(y - 1/2))$ on $[0, 1]$.

So the function $Y[0, 1] \to \mathbb{R}$ defined by $Y(x) = g_\gamma(x)$ has the correct distribution as

$$\mathbb{P}(Y^{-1}((-\infty, t])) = \mathbb{P}(g_\gamma^{-1}((-\infty, t])) = \lambda(H_\gamma((-\infty, t])) = H_\gamma(t).$$

(b) Note that g_γ is continuous on $[0, 1]$ and hence measurable.

(c) For any $t \geq 0$, the cdf of the Poisson random variable is given by

$$F_\lambda(t) = \sum_{n=0}^{\lceil t \rceil} f_\lambda(n),$$

where $\lceil t \rceil$ is the ceiling of t, i.e. the smallest integer $k \geq t$.

(d) For the probability space, we again take $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0, 1]$.

Now for any $y \in [0, 1]$ let $k := k(y)$ be such that

$$\sum_{n=1}^{k} f_\lambda(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_\lambda(n) < y,$$

where the last sum is interpreted as -1 if $k = 0$.

Now define $X(y) = k(y) : [0,1] \to \mathbb{R}$. Then $k(y) \leq t$ if and only if $y \leq F_\lambda(t)$ and hence

$$X^{-1}((-\infty, t]) = \{y \in [0,1] \,:\, k(y) \in (0,t]\} = \{y \in [0,1] \,:\, y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

(e) It follows from the above computation that $X^{-1}((-\infty,t]) = \{y \in [0,1] : y \in (0, F_\lambda(t)]\}$. Since the latter is a measurable set we conclude that $X^{-1}((-\infty,t])$ is measurable for all t and since these generate the Borel σ-algebra X is measurable.

(f) for any $\ell \in \mathbb{N}$ define the sets $A_\ell = (n-1-1/\ell), n-1+1/\ell]$. Then A_ℓ is a decreasing set with $\lim_{\ell\to\infty} A_\ell = \{n\}$. Moreover, $A_\ell = (-\infty, n-1+1/\ell] \setminus (-\infty, n-1-1/\ell]$ and $\mathbb{P}(A_1) < \infty$. It now follows from continuity from above and (d) that

$$\begin{aligned}
X_\#\mathbb{P}(\{n\}) &= \lim_{\ell\to\infty} X_\#\mathbb{P}(A_\ell) \\
&= \lim_{\ell\to\infty} X_\#\mathbb{P}((-\infty, n-1+1/\ell]) - X_\#\mathbb{P}((-\infty, n-1-1/\ell]) \\
&= F_\lambda(n-1+1/\ell) - F_\lambda(n-1-1/\ell) \\
&= \sum_{k=0}^{n} f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n).
\end{aligned}$$

Chapter 4: The Lebesgue Integral

Problem 4.2

The idea is to apply the monotone convergence theorem (Theorem 4.3.4). To this end we first note that

$$\|f_n(\omega) - f(\omega)\| \leq 2^{-n} \quad \text{for all } n \in \mathbb{N},\, \omega \in \Omega.$$

From this it follows that $f_n(\omega) \leq 2^{-n} + f(\omega)$ and hence

$$\begin{aligned}
\|[f_n](\omega) - f(\omega)\| &= \|2^n - f(\omega)\|\mathbf{1}_{2^n \leq f_n} + \|f_n(\omega) - f(\omega)\|\mathbf{1}_{f_n < 2^n} \\
&\leq 2^{-n} + 2^{-n}
\end{aligned}$$

form which we conclude that $[f_n] \to f$.

The final part is to show that $[f_n] \leq [f_{n+1}]$ which follows if we can show that $f_n \leq f_{n+1}$. For this we first note that for all $k \geq 1$ $(k+1)2^{-(n+1)} \leq k2^{-n}$. We also note that $2^n \leq 2n+1$. Now suppose that there exist an $n \geq 1$ and ω such that $f_n(\omega) > f_{n+1}(\omega)$. Then it must hold that $f_n(\omega) > 0$ and hence $f_n(\omega) = k2^{-n}$ for some $k \geq 1$. This then implies that $f_{n+1}(\omega) = \ell 2^{-n}$ for some $\ell \geq k+1$. But this cannot be the case as $[\ell 2^{-n}, (\ell+1)2^{-n}) \cap [k2^{-n}, (k+1)2^{-n}) = \emptyset$ while $f(\omega)$ should be in both sets.

Problem 4.3

(a) By definition, we have that $\nu_f(\Omega) = \int_\Omega f \,\mathrm{d}\mu = 1$. Now let $(A_n)_{n\in\mathbb{N}}$ be a family of mutually disjoint measurable sets. Then we have that the sequence

$$g_n := \sum_{i=1}^{n} f\,\mathbf{1}_{A_i} = f\,\mathbf{1}_{\bigcup_{i=1}^{n} A_i} \;\longrightarrow\; g := f\,\mathbf{1}_{\bigcup_{i\in\mathbb{N}} A_i} \qquad \text{pointwise monotonically.}$$

By MCT, we then have that

$$\nu_f\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \int_{\bigcup_{i\in\mathbb{N}} A_i} f\,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\bigcup_{i=1}^{n} A_i} f\,\mathrm{d}\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} f\,\mathrm{d}\mu = \sum_{i\in\mathbb{N}} \nu_f(A_i),$$

thus showing that ν_f is a probability measure on $(\Omega, \mathcal{F})$.

(b) Following the hint, we start by considering nonnegative simple functions g. Suppose $g = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}$ for $a_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$ mutually disjoint. Then,

$$\int_\Omega g\, d\nu_f = \sum_{i=1}^{n} a_i \;=\; \nu_f(A_i) = \sum_{i=1}^{n} a_i \int_{A_i} f\,\mathrm{d}\mu = \int_\Omega gf\,\mathrm{d}\mu.$$

Now let g be a nonnegative measurable function and $[g]_n$ be a sequence of nonnegative simple functions that converge pointwise monotonically to g. Then MCT yields

$$\int_\Omega g\,\mathrm{d}\nu_f = \lim_{n\to\infty} \int_\Omega [g]_n\,\mathrm{d}\nu_f = \lim_{n\to\infty} \int_\Omega [g]_n f\,\mathrm{d}\mu = \int_\Omega gf\,\mathrm{d}\mu,$$

where we used the fact that $[g]_n f$ converges pointwise monotonically to gf.

(c) Let g be measurable. Then $g = g^+ - g^-$, where $g^\pm$ are nonnegative measurable functions. Since f is nonnegative, we have that $(fg)^\pm = fg^\pm$. Due to (b), we deduce

$$\int_\Omega g^\pm\,\mathrm{d}\nu_f = \int_\Omega g^\pm f\,\mathrm{d}\mu = \int_\Omega (gf)^\pm\,\mathrm{d}\mu.$$

Hence, $g^\pm$ is ν_f-integrable if and only if $(gf)^\pm$ is μ-integrable. Consequently, g is ν_f-integrable if and only if gf is μ-integrable, since

$$\int_\Omega |g|\,\mathrm{d}\nu_f = \int_\Omega g^+\,\mathrm{d}\nu_f + \int_\Omega g^-\,\mathrm{d}\nu_f = \int_\Omega g^+ f\,\mathrm{d}\mu + \int_\Omega g^- f\,\mathrm{d}\mu = \int_\Omega |gf|\,\mathrm{d}\mu.$$

Problem 4.4

(⇒) Let f be μ-integrable. Then both $|f|\mathbf{1}_{\{|f|<n\}}$ and $|f|\mathbf{1}_{\{|f|\geq n\}}$ are integrable, due to the monotonicity of the integral. By linearity of the integral,

$$\int_\Omega |f|\mathbf{1}_{\{|f|\geq n\}}\,\mathrm{d}\mu = \int_\Omega |f|\,\mathrm{d}\mu - \int_\Omega |f|\mathbf{1}_{\{|f|<n\}}\,\mathrm{d}\mu.$$

Since the sequence $g_n := |f|\mathbf{1}_{\{|f|<n\}} \geq 0$ converges pointwise monotonically to $|f|$, we can apply MCT to obtain

$$\lim_{n\to\infty} \int_\Omega |f|\mathbf{1}_{\{|f|<n\}}\, dd\mu = \int_\Omega |f| \,\mathrm{d}\mu.$$

Hence,

$$\lim_{n\to\infty} \int_\Omega |f|\mathbf{1}_{\{|f|\geq n\}} \,\mathrm{d}\mu = \int_\Omega |f| \,\mathrm{d}\mu - \lim_{n\to\infty} \int_\Omega |f|\mathbf{1}_{\{|f|<n\}} \,\mathrm{d}\mu = 0.$$

(⇐) By assumption, there is some $N \geq 1$ such that

$$\int_\Omega |f|\mathbf{1}_{\{|f|\geq N\}} \,\mathrm{d}\mu \leq 1.$$

By linearity of the integral,

$$\int_\Omega |f| \,\mathrm{d}\mu = \int_\Omega |f|\mathbf{1}_{\{|f|<N\}} \,\mathrm{d}\mu + \int_\Omega |f|\mathbf{1}_{\{|f|\geq N\}} \,\mathrm{d}\mu \leq N\mu\big(\{|f| < N\}\big) + 1.$$

Since μ is a finite measure, the right-hand side is finite, implying that f is μ-integrable.

Problem 4.5
Observe that $\Omega = \bigcup_{n\in\mathbb{N}}\{|f| > n\}$.
We then get that

$$\sum_{n=1}^{\infty} \int_{\{|f|>n\}} |f| \,\mathrm{d}\mu = \int_\Omega |f| \,\mathrm{d}\mu < \infty.$$

This implies that for some N and all $n \geq N$: $\int_{\{|f|>n\}} |f| \,\mathrm{d}\mu < 1/n$ or else the sum cannot be finite.
Now let $\varepsilon > 0$, take $M > \max\{N, 2/\varepsilon\}$ and $\delta = \varepsilon/(2M)$. Then

$$\begin{aligned}\int_A |f| \,\mathrm{d}\mu &= \int_A |f|\mathbf{1}_{|f|\leq M} \,\mathrm{d}\mu + \int_A |f|\mathbf{1}_{|f|>M} \,\mathrm{d}\mu \\ &\leq M\mu(A) + \frac{1}{M} \leq M\delta + \frac{1}{M} < \varepsilon.\end{aligned}$$

Problem 4.6

(a) Let $t \in \mathbb{R}$ and consider the set $A_t = (-\infty, t]$. Then by definition of the probability density function

$$\nu(A_t) = \int_{-\infty}^{t} \rho \,\mathrm{d}\lambda = (X_\#\mathbb{P})((-\infty, t]).$$

We thus conclude that ν and $X_\#\mathbb{P}$ coincide on the family of set A_t and since these generate $\mathcal{B}$ Theorem 2.2.17 implies that $\nu = X_\#\mathbb{P}$.

(b) Since g is a simple function, there exist an $N \in \mathbb{N}$, constants $(a_n)_{1\leq n\leq N}$ and measurable sets $(A_n)_{1\leq n\leq N}$ such that

$$g = \sum_{n=1}^{N} a_n \mathbf{1}_{A_n}.$$

Now, by first applying Proposition 4.8.11 and then part (a), we get that

$$\begin{aligned}
\mathbb{E}[g(X)] &= \int_\Omega g(X)\,\mathrm{d}\mathbb{P} = \int_\Omega g\,\mathrm{d}X_{\#}\mathbb{P} = \int_\Omega g\,\mathrm{d}\nu \\
&= \int_\Omega \sum_{n=1}^{N} a_n \mathbf{1}_{A_n}\,\mathrm{d}\nu = \sum_{n=1}^{N} a_n \nu(A_n) = \sum_{n=1}^{N} a_n \int_{A_n} \rho\,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{n=1}^{N} a_n \mathbf{1}_{A_n} \rho\,\mathrm{d}\lambda = \int_{\mathbb{R}} g\rho\,\mathrm{d}\lambda
\end{aligned}$$

(c) First note that by part (b) we have that

$$\int_\Omega [h]_n(X)\,\mathrm{d}\mathbb{P} = \int_{\mathbb{R}} [h_n]\rho\,\mathrm{d}\lambda.$$

Now we split the function $[h_n]\rho$ into its positive and negative part and note that

$$([h_n]\rho)^+ = [h]_n^+\rho^+ + [h]_n^-\rho^- \quad \text{and} \quad ([h_n]\rho)^- = [h]_n^+\rho^- + [h]_n^-\rho^+,$$

where $[h]_n^\pm$ and $\rho^\pm$ denote the positive and negative parts of $[h]_n$ and ρ.

We will show that

$$\int_\Omega h^+(X)\,\mathrm{d}\mathbb{P} = \int_{\mathbb{R}} h^+\rho\,\mathrm{d}\lambda.$$

The proof for the negative part is similar.

$$\begin{aligned}
\int_{\mathbb{R}} h^+\,\mathrm{d}\nu &= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n^+\,\mathrm{d}\nu && \text{by Theorem 4.3.4} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n^+\rho\,\mathrm{d}\lambda && \text{by part (b)} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n^+\rho^+\,\mathrm{d}\lambda - \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n^+\rho^-\,\mathrm{d}\lambda && \text{by linearity of integration} \\
&= \int_{\mathbb{R}} h+\rho^+\,\mathrm{d}\lambda - \int_{\mathbb{R}} h+\rho^-\,\mathrm{d}\lambda && \text{by Theorem 4.3.4} \\
&= \int_{\mathbb{R}} h^+\rho\,\mathrm{d}\lambda && \text{by linearity of integration}
\end{aligned}$$

(d)

$$
\begin{aligned}
\mathbb{E}[h(X)] &= \int_\Omega h(X)\,\mathrm{d}\mathbb{P} \\
&= \int_{\mathbb{R}} h\,\mathrm{d}X_{\#}\mathbb{P} && \text{by Proposition 4.8.11} \\
&= \int_{\mathbb{R}} h\,\mathrm{d}\nu && \text{by part (a)} \\
&= \int_{\mathbb{R}} h\rho\,\mathrm{d}\lambda && \text{by part (c).}
\end{aligned}
$$

Problem 4.7 This follows from the following inequalities:

$$\int_{\mathbb{R}} |f|^p\,\mathrm{d}\mu \geq \int_{\{|f|\geq t\}} |f|^p\,\mathrm{d}\mu \geq t^p \mu(\{|f|\geq t\}).$$

Chapter 5: Convergence of integrals and functions

Problem 5.2

(a) Let $t_0 \in (a,b)$ be fixed. It suffices to check the continuity result for arbitrary sequences $(t_n)_{n\geq 1} \subset (a,b)$ such that $t_n \to t_0$ as $n\to\infty$. Fix such a sequence and define $g_n(\omega) := f(\omega, t_n)$ for all $\omega \in \Omega$ and $n \geq 1$. Since $\lim_{t\to t_0} f(\omega,t) = f(\omega,t_0)$ for all $\omega\in\Omega$, we deduce that $\lim_{n\to\infty} g_n(\omega) = f(\omega,t_0)$ for every $\omega\in\Omega$. Moreover, by assumption $|g_n| \leq g$ for all $n\geq 1$ and g is integrable. By the Dominated Convergence Theorem

$$\lim_{n\to\infty}\int_\Omega g_n(\omega)\,\mu(\mathrm{d}\omega) = \int_\Omega f(\omega,t_0)\,\mu(\mathrm{d}\omega).$$

As the chosen sequence was arbitrary, we deduce that $\lim_{t\to t_0} F(t) = F(t_0)$.

(b) If $t\mapsto f(\omega,t)$ is continuous on (a,b) for all $\omega\in\Omega$ then $\lim_{t\to t_0} f(\omega,t) = f(\omega,t_0)$ at every $t_0\in(a,b)$ for all $\omega\in\Omega$. In particular, (a) applies, showing that $\lim_{t\to t_0} F(t) = F(t_0)$ for every $t_0 \in (a,b)$, i.e., F is continuous on (a,b).

Problem 5.3

(1) We start by showing that $(\partial f/\partial t)(\cdot,t)$ is measurable. Let $(t_n)_{n\geq 1}\subset(a,b)$ be an arbitrary sequence with $t_n \neq t$ and $t_n\to t$ for $n\to\infty$. We set

$$g_n(\omega) = \frac{f(\omega,t_n)-f(\omega,t)}{t_n - t}.$$

Clearly, g_n is measurable for every $n\geq 1$. Moreover, $\lim_{n\to\infty} g_n(\omega) = (\partial f/\partial t)(\omega,t)$ by the definition of the derivative. Since $(\partial f/\partial t)(\cdot,t)$ is the pointwise limit of a sequence of measurable functions, it is also measurable. Clearly, $(\partial f/\partial t)(\cdot,t)$ is integrable since

$$\int_\Omega |(\partial f/\partial t)(\omega,t)|\,\mu(\mathrm{d}\omega) \leq \int_\Omega g\,\mathrm{d}\mu < +\infty.$$

(2) Let $t_0 \in (a,b)$ and suppose w.l.o.g. $t_0 < t$. Since $t \mapsto f(\omega, t)$ is differentiable on (a,b) for all $\omega \in \Omega$, the Mean Value Theorem gives

$$\frac{f(\omega,t) - f(\omega,t_0)}{t - t_0} = (\partial f/\partial t)(\omega, \tau) \qquad \text{for some } \tau \in (t_0, t).$$

Taking the modulus on both sides, we obtain

$$\left| \frac{f(\omega,t) - f(\omega,t_0)}{t - t_0} \right| \leq |(\partial f/\partial t)(\omega, \tau)| \leq g(\omega) \qquad \text{for all } \omega \in \Omega.$$

(3) We now have all the ingredients needed to apply the DCT, which yields

$$\lim_{n\to\infty} \frac{F(t_n) - F(t)}{t_n - t} = \lim_{n\to\infty} \int_\Omega g_n \, \mathrm{d}\mu = \int_\Omega (\partial f/\partial t)(\omega, t) \, \mu(\mathrm{d}\omega).$$

Since $t \in (a,b)$ and the sequence $(t_n)_{n\geq 1}$ was arbitrary, we conclude that F is differentiable on (a,b) with

$$F'(t) = \int_\Omega (\partial f/\partial t)(\omega, t) \, \mu(\mathrm{d}\omega).$$

Problem 5.3

(a) Note that the integrand $f_n(x) = \frac{1+nx^2}{(1+x^2)^n}$ is continuous on $[0,1]$ and non-negative. Hence, the Riemann integral and Lebesgue integral coincide, i.e.,

$$\int_0^1 f_n(x) \, \mathrm{d}x = \int_{[0,1]} f_n \, \mathrm{d}\lambda.$$

Observe that we have the following pointwise limit

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0,1], \end{cases}$$

i.e., $\lim_{n\to\infty} f_n = 0$ λ-almost everywhere. Moreover, $f_n(x) \leq 1$ for every $x \in [0,1]$ and $n \geq 1$. Since the constant function $g \equiv 1$ is λ-integrable on $[0,1]$, it is a valid dominator. Hence, the DCT gives

$$\lim_{n\to\infty} \int_0^1 f_n(x) \, \mathrm{d}x = \lim_{n\to\infty} \int_{[0,1]} f_n \, \mathrm{d}\lambda = \int_{[0,1]} \lim_{n\to\infty} f_n \, \mathrm{d}\lambda = 0$$

(b) For the purpose of convergence, we consider $n \geq 3$. Note that the integrand $f_n(x) = \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right)$ is continuous on $(0, +\infty)$ with pointwise limit

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ 1/2 & \text{if } x = 1, \\ 1/x^2 & \text{if } x > 1, \end{cases}$$

Setting the function

$$g(x) = \begin{cases} 1 & \text{for } x \in (0,1), \\ \frac{1}{x^2} & \text{for } x \geq 1, \end{cases}$$

we see that $f_n \leq g$ λ-almost everywhere in $(0, +\infty)$ and for all $n \geq 3$. Indeed, for $x \geq 1$, we obtain

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq \frac{x^{n-2}}{x^n} = \frac{1}{x^2},$$

while for $x \in (0,1)$, we have

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq 1.$$

Notice that g is non-negative and λ-integrable on $(0, +\infty)$. Indeed, using the MCT,

$$\begin{aligned} \int_{(0,+\infty)} g \, \mathrm{d}\lambda &= \int_{(0,1)} g \, \mathrm{d}\lambda + \int_{(1,+\infty)} g \, \mathrm{d}\lambda = 1 + \lim_{n\to\infty} \int_{(1,n)} g \, \mathrm{d}\lambda \\ &= 1 + \lim_{n\to\infty} \int_1^n \frac{1}{x^2} \, \mathrm{d}x = 1 + \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 2 < +\infty. \end{aligned}$$

To conclude, we apply DCT to deduce that the limit is 1.

Problem 5.4

The proof follows verbatim to the proof of the Dominated Convergence Theorem.

Problem 5.7

Let F_n denote the cdf of $Y_n = \|X_n - X\|$ and F_0 denote the cdf of 0. By Definition 5.2.9 and Lemma 5.2.8 we have that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $F_n(t) \to F_0(t)$ for all continuity points t of F_0. This is equivalent to showing that $1 - F_n(t) \to 1 - F_0(t)$, where

$$1 - F_0(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ 1 & \text{else.} \end{cases}$$

Now note that the only discontinuity point of F_0 is 0. Moreover, $1 - F_n(t) = 0 = F_0(t)$ for all $t < 0$. Hence it follows that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $1 - F_n(t) \to 0$ for all $t > 0$, which is what we needed to show.

Problem 5.8

(a) For this let $h_t(x) = \mathbf{1}_{(-\infty,t]}$ and note that

$$F_n(t) = (X_n)_\# \mathbb{P}_n((-\infty, x]) = \int_{\mathbb{R}} \mathbb{1}_{(-\infty,t]} \, \mathrm{d}(X_n)_\# \mathbb{P}_n = \int_{\mathbb{R}} h_t \, \mathrm{d}\mu_n.$$

and similarly $F(t) = \int_{\mathbb{R}} h_t \, \mathrm{d}\mu$

(b) The function h is discontinuous only at t, i.e. $\mathcal{C}_h = \mathbb{R} \setminus \{t\}$. Moreover, for any $\varepsilon > 0$

$$\mu((t-\varepsilon, t+\varepsilon)) = \mu((t-\varepsilon, t]) + \mu((t, t+\varepsilon)) = F(t) - F(t-\varepsilon) + F(t+\varepsilon) - F(t).$$

Since F is continuous at t, the right hand side goes to zero as $\varepsilon \to 0$. Therefore

$$\mu(\{t\}) = \lim_{\varepsilon \to 0} \mu((t-\varepsilon, t+\varepsilon)) = 0,$$

which implies that $\mu(\mathcal{C}_h) = 1$.

(c) The result follows by applying condition (2) in Theorem 5.2.7.

(d) Let $\varepsilon > 0$, pick such a δ and partition the interval $[-K, K]$ into $L_\delta := \lceil \frac{4K}{\delta} \rceil$ intervals $I_\ell = (a_\ell, b_\ell]$ of equal length, which is $\leq \delta/2 < \delta$. Now we define the simple function

$$\hat{g} := \sum_{\ell=1}^{L} h(b_\ell) \mathbb{1}_{I_\ell},$$

(e) Let $M = L$, $\beta_\ell = \sum_{t=1}^{\ell} h(b_t)$ and $t_\ell = b_\ell$. Then

$$\hat{g} := \sum_{\ell=1}^{L} \beta_\ell \mathbf{1}_{(-\infty, b_\ell]}.$$

(f) Using the representation in (e) we get

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[\hat{g}(X_n)] &= \lim_{n\to\infty} \sum_{\ell=1}^{L} \beta_\ell \int_{\mathbb{R}} \mathbf{1}_{X_n^{-1}((-\infty, b_\ell])} \, \mathrm{d}\mathbb{P} \\
&= \lim_{n\to\infty} \sum_{\ell=1}^{L} \beta_\ell (X_n)_\# \mathbb{P}((-\infty, b_\ell]) \\
&= \lim_{n\to\infty} \sum_{\ell=1}^{L} \beta_\ell F_n(b_\ell) \\
&= \sum_{\ell=1}^{L} F(b_\ell) = \mathbb{E}[\hat{g}(X)].
\end{aligned}$$

(g) Using the representation of $\hat{g}$ in (d) we note that $\|x - y\| < \varepsilon$ for all $x, y \in I_\ell$. This then implies that $\|g(x) - \hat{g}(y)\| \leq \varepsilon$ from which it follows that

$$\begin{aligned}
\|\mathbb{E}[g(X_n)] - \mathbb{E}[g(X)]\| &\leq \|\mathbb{E}[g(X_n)] - \mathbb{E}[\hat{g}(X_n)]\| + \|\mathbb{E}[g(X)] - \mathbb{E}[\hat{g}(X)]\| \\
&\quad + \|\mathbb{E}[\hat{g}(X_n)] - \mathbb{E}[\hat{g}(X)]\| \\
&\leq 2\varepsilon + \|\mathbb{E}[\hat{g}(X_n)] - \mathbb{E}[\hat{g}(X)]\|.
\end{aligned}$$

We have shown in (f) that the last term goes to zero as $n \to \infty$. Since ε was arbitrary we conclude that (**??**) holds.

(h) This now follows from Theorem 5.2.7 (3).

Problem 5.9 Suppose that $X_n \overset{\text{a.s.}}{\to} X$. Then by Lemma 5.2.16 this is equivalent to $\mathbb{P}(\|X_n - X\| > \varepsilon \text{ i.o.}) = 0$ for all $\varepsilon > 0$.
For now fix an $\varepsilon > 0$ and write $A_n := \{\|X_n - X\| > \varepsilon\}$. Recall that

$$\{A_n \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \bigcup_{k \geq n} A_n$$

and note two things:

(a) The sets $B_k := \bigcup_{n \geq k} A_n$ are non-increasing, i.e. $B_k \supset B_{k+1}$, and

(b) $\mathbb{P}(A_k) \leq \mathbb{P}(\bigcup_{n \geq k} A_n) = \mathbb{P}(B_k)$.

We then have that:

$$\begin{aligned}
0 &= \mathbb{P}(\{A_n \text{ i.o.}\}) && \text{by assumption} \\
&= \mathbb{P}(\bigcap_{k=1}^{\infty} B_k) && \text{by Lemma 5.2.16} \\
&= \lim_{k \to \infty} \mathbb{P}(B_k) && \text{by continuity form above (Proposition 2.2.15)} \\
&\geq \lim_{k \to \infty} \mathbb{P}(A_k) && \text{by (b).}
\end{aligned}$$

Chapter 6: L^p-spaces

Problem 6.2

Problem 6.4
Let $E_n := \{\omega \in \mathbb{R}^d \,:\, |f(\omega)| \geq n\}$. Since $\mathbf{1}_{E_n} f \to 0$ as $n \to \infty$, and $\mathbf{1}_{E_n}|f| \leq |f|$ for every $n \geq 1$, we can apply DCT to conclude that

$$\int_{E_n} |f| \,\mathrm{d}\mu = \int_{\mathbb{R}^d} \mathbf{1}_{E_n} |f| \,\mathrm{d}\mu \;\longrightarrow\; 0 \quad \text{as } n \to \infty.$$

Now pick some $n \geq 1$ such that $\int_{E_n} |f| \,\mathrm{d}\mu < \varepsilon/3$ and define

$$f_n(\omega) := \max\{-n, \min\{f(\omega), n\}\}, \qquad \omega \in \mathbb{R}^d,$$

i.e., f_n is a truncation of f. From Lusin's theorem, we find a continuous function g such that $f_n \equiv g$ on a compact set $K \subset \mathbb{R}^d$ with $\mu(\mathbb{R}^d \backslash K) < (2\varepsilon)/(3n)$. We assume w.l.o.g. that $|g| \leq n$, since otherwise, we can consider a truncation of g. Altogether, this yields

$$\begin{aligned}
\int_{\mathbb{R}^d} |f - g| \,\mathrm{d}\mu &= \int_{\mathbb{R}^d} |f - f_n| \,\mathrm{d}\mu + \int_{\mathbb{R}^d} |f_n - g| \,\mathrm{d}\mu \\
&= \int_{E_n} |f| \,\mathrm{d}\mu + \int_{\mathbb{R}^d \backslash K} |f_n - g| \,\mathrm{d}\mu \\
&\leq \frac{\varepsilon}{3} + 2n\,\mu(\mathbb{R}^d \backslash K) \leq \varepsilon.
\end{aligned}$$

Finally, $g \in L^1(\mathbb{R}^d, \mu)$ holds simply due to the triangle inequality.