TU/e, 2MBA70

# Solutions to problems for Measure and Probability Theory

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## Chapter 2: Measurable spaces (sigma-algebras and measures)

**Problem 2.6** (23 points) Let $\mathcal{O}$ denote the open sets in $\mathbb{R}$.

(a) (2 points) Note that the interval $(a, b)$ is open for any $a < b \in \mathbb{R}$. Hence $\mathcal{A}_1 \subset \mathcal{A}_1' \subset \mathcal{O}$ and thus by Lemma 2.1.5 we have that $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \sigma(\mathcal{O}) = \mathcal{B}_\mathbb{R}$.

(b) (2 points) The inclusion $\supset$ is trivial. So assume that $x \in O$. Then by definition there exist an $r > 0$ such that the ball $B_x(r) \subset O$. But $B_x(r) = (x - r, x + r) \in \mathcal{A}_1$ so $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$.

(c) (3 points) Take $O \in \mathcal{O}$. If we can show that $O \in \sigma(\mathcal{A})$ then $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. The result then follows from 1.

From 2 it follows that $O$ is a union over a subset collection of interval $(a, b)$ where $a, b \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, the collection $\{(a, b) : a < b \in \mathbb{Q}\}$ is also countable and hence $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$, from which it follows that $\mathcal{B}_\mathbb{R} \subset \sigma(\mathcal{A})$.

(d) (1 point) This follows immediately from 1 and 3 since these imply that $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \mathcal{B}_\mathbb{R}$.

(e) (3 points) The inclusion $\subset$ is trivial, since $(a, b] \subset (a + b + 1/j)$ for any $j \in \mathbb{N}$. For the other inclusion we argue by contradiction. Suppose that $x \in \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$ but $x \notin (a, b]$. Then $x > b$ and hence there exists a $j \in \mathbb{N}$ such that $(b - x) > 1/j$. But this implies that $x \notin (a, b + 1/j)$ which is a contradiction. So we conclude that $(a, b] \supset \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$.

(f) (3 points) This time the inclusion $\supset$ is trivial since $(a, b - 1/j] \subset (a, b)$ for every $j \in \mathbb{N}$. For the other inclusion suppose that $x \in (a, b)$. Then there exists a $r > 0$ such that the interval $(x - r, x + r) \subset (a, b)$. In particular, this implies that $b - (x + r) > 0$. Now take any $j \in \mathbb{N}$ such that $j > 1/(b - (x + r))$. Then $b - x > r + 1/j$ which implies that $(x - r, x + r) \subset (x - r, b - 1/j]$ and hence $x \in \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$.

(g) (4 points) It is clear that $\mathcal{A}_2 \subset \mathcal{A}_2'$. By 5 it follows that any interval $(a, b]$ can be obtained as a countable intersection of intervals of the form $(a, b + 1/j)$. By 4 $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{A}_1')$ which by Lemma 2.1.2 contains $\bigcap_{j \in \mathbb{N}} (a, b + 1/j) = (a, b]$. So we conclude that any interval $(a, b] \in \mathcal{B}_\mathbb{R}$ from which it now follows that

$$\sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}_2') \subset \sigma(\mathcal{A}_1') = \mathcal{B}_\mathbb{R}.$$

For the other inclusion we consider a set $(a, b)$ with $a, b \in \mathbb{Q}$. Then by 6 we have that $(a, b) = \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$ where the later is a countable union of sets $(c, d]$ with $c, d \in \mathbb{Q}$ which must be in $\sigma(\mathcal{A}_2)$ by definition of a $\sigma$-algebra. Hence, any interval $(a, b) \in \sigma(\mathcal{A}_2)$ and we thus conclude, using 3, that

$$\mathcal{B}_\mathbb{R} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}_2') \subset \sigma(\mathcal{A}_1') = \mathcal{B}_\mathbb{R},$$

which implies the result.

(h) (2 points) Step 1 is to show that any interval $[a, b)$ can be obtained as a countable intersection of intervals $(a - 1/j, b)$. From this we can conclude that any set $[a, b)$ must be in $\mathcal{B}_{\mathbb{R}}$ proving inclusions $\subset$.

For the other inclusions we have to show that any interval $(a, b)$ can be obtained as a countable union of intervals $[a+1/j, b)$, which implies that $(a, b)$ must be in the $\sigma$-algebra generated by $[a, b)$.

(i) (3 points) The main tool is to show that each of the intervals $(-\infty, a]$, $(-\infty, a)$, $(a, \infty)$ and $[a, \infty)$ can be obtained by taking any allowed set operation for $\sigma$-algebras, i.e. countable unions/intersections and finite complements. This will help use prove the $\subset$ inclusions.

Then we show that any set of the form $(a, b)$, $[a, b)$ or $(a, b]$ can also be obtained through countable unions/intersections and finite complements of intervals of the forms $(-\infty, a]$, $(-\infty, a)$, $(a, \infty)$ and $[a, \infty)$. These will then yield the $\supset$ inclusions and finish the proof.

**Problem 2.9**

First note that if $\mu(A \cap B) = \infty$ then by property 2 we have that also $\mu(A)$, $\mu(B)$ and $\mu(A \cup B) = \infty$ and hence the result holds trivially. So assume now that $\mu(A \cap B) < \infty$. Since

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

it follows from property 1 that

$$\mu(A \cup B) = \mu(A \setminus (A \cap B))) + \mu(A \cap B) + \mu(B \setminus (A \cap B)).$$

Adding $\mu(A \cap B) < \infty$ to both side we get

$$\begin{aligned}\mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus (A \cap B))) + \mu(A \cap B) + \mu(B \setminus (A \cap B)) + \mu(A \cap B) \\ &= \mu(A) + \mu(B),\end{aligned}$$

where the last line follows from applying property 3 twice.

**Problem 2.11**

The idea is to construct a family of disjoint sets $(E_i)_{i \in \mathbb{N}}$ with the following properties:

1. $E_i \subset A_i$, and
2. $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} A_i$.

If such a sequence exists then we have

$$\begin{aligned}\mu(\bigcup_{i \in \mathbb{N}} A_i) &= \mu(\bigcup_{i \in \mathbb{N}} E_i) && \text{by 2} \\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{because } E_i \text{ are disjoint and } \mu \text{ is } \sigma\text{-additive} \\ &\leq \sum_{i=1}^{\infty} \mu(A_i) && \text{by 1 and monotone property of } \mu.\end{aligned}$$

So we are left to construct the required family of sets $(E_i)_{i\in\mathbb{N}}$. The following set will do:

$$E_1 = A_1 \quad E_i = A_i \setminus \bigcup_{k<i}^{i} A_k \text{ for all } i > 1.$$

Note that by definition the set $E_i$ are pair-wise disjoint and property1 holds. Finally, property 2 holds since $\bigcup_{i=1}^k E_i = \bigcup_{i=1}^k A_i$ holds for all $k \geq 1$.

**Problem 2.12**

(a) We first make the following observations about $\mathcal{N}$:

- because $\mu(\emptyset) = 0$ it holds that $\emptyset \in \mathcal{N}$,
- if $N, M \in \mathcal{N}$ then $N \setminus M \in \mathcal{N}$ since $N \setminus M \subset N$, and
- if $(N_i)_{i\geq 1}$ is a family of sets in $\mathcal{N}$ then so is $\bigcup_{i\geq 1} N_i$.

From the first point it follows that $\emptyset = \emptyset \cup \emptyset \in \overline{\mathcal{F}}$ and $\Omega = \Omega \cup \emptyset \in \overline{\mathcal{F}}$.

Furthermore, if $A, B \in \mathcal{F}$ and $N, M \in \mathcal{N}$, then by the second point and because $A \setminus B \in \mathcal{F}$,

$$(A \cup N) \setminus (B \cup M) = (A \setminus B) \cup (N \setminus M) \in \overline{\mathcal{F}}.$$

Finally, let $(A_i \cup N_i)_{i\geq 1}$ be a collection of sets in $\mathcal{N}$. Then using the third point we get

$$\bigcup_{i\geq 1} A_i \cup N_i = \bigcup_{i\geq 1} A_i \cup \bigcup_{i\geq 1} N_i \in \overline{\mathcal{F}}.$$

(b) From the definition we immediately get that $\mu(\emptyset) = 0$. Now, let $(A_i \cup N_i)_{i\geq 1}$ be a collection of disjoint sets in $\mathcal{N}$. Then

$$\bar{\mu}(\bigcup_{i\geq 1} A_i \cup N_i) = \bar{\mu}(\bigcup_{i\geq 1} A_i \cup \bigcup_{i\geq 1} N_i) = \mu(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} \mu(A_i) = \sum_{i\geq 1} \bar{\mu}(A_i \cup N_i).$$

(c) This follows from the fact that $\bar{\mu}|_{\mathcal{F}}(A) = \bar{\mu}(A \cup \emptyset) = \mu(A)$.

(d) Suppose that $N \subset \Omega$ is a null set for $\overline{\mathcal{F}}$. Then there exists an $A \cup M \in \overline{\mathcal{F}}$ such that $N \subset A \cup M$ and $\bar{\mu}(A \cup M) = \mu(A) = 0$. However, since $M \in \mathcal{N}$, there must also exist a $B \in \mathcal{F}$ with $M \subset B$ and $\mu(B) = 0$. But this implies that $N \subset A \cup B \in \mathcal{F}$ which implies that $N \in \mathcal{N}$. Therefore, since $N = \emptyset \cup N$ it follows that $N \in \overline{\mathcal{F}}$ and hence every null set is part of the $\sigma$-algebra.

## Chapter 3: Measurable functions

**Problem 3.2**

(a) First we note that $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}$ and $f^{-1}(E) = \Omega \in \mathcal{F}$. So $\emptyset, E \in \mathcal{H}$.

Next, let $B \in \mathcal{H}$. Then

$$f^{-1}(E \setminus B) = \Omega \setminus f^{-1}(B) \in \mathcal{F},$$

since by definition $f^{-1}(B) \in \mathcal{F}$. So $E \setminus B \in \mathcal{H}$.

Finally, if $(B_i)_{i\in\mathbb{N}}$ is a sequence of sets in $\mathcal{H}$, then

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) \in \mathcal{F},$$

which shows that $\bigcup_{i=1}^{\infty} B_i \in \mathcal{H}$, completing the proof that $\mathcal{H}$ is a $\sigma$-algebra.

(b) By construction $\mathcal{A} \subseteq \mathcal{H}$. It therefore follows from Lemma 2.5 that $\mathcal{G} = \sigma(\mathcal{A}) \subseteq \mathcal{H}$. But this then implies that $f^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{G}$ which means that $f$ is $(\mathcal{F}, \mathcal{G})$-measurable.

**Problem 3.3** "$\subset$" By definition, the product $\sigma$-algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$ is defined as the $\sigma$-algebra generated by the collection

$$\mathcal{A} := \Big\{A \times B \subset \Omega_1 \times \Omega_2 : A \in \mathcal{F}_1,\ B \in \mathcal{F}_2\Big\}.$$

Since $A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B)$, we have that

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \sigma(\pi_1, \pi_2).$$

"$\supset$" Let $C \in \{\pi_i^{-1}(A) : i = 1, 2,\ A \in \mathcal{F}_1\}$. Then there exist sets $A \in \mathcal{F}_1$ or $B \in \mathcal{F}_2$ such that $C = \pi_1^{-1}(A) = A \times \Omega_2$ or $C = \pi_2^{-1}(B) = \Omega_1 \times B$. Either way, since $\Omega_1 \in \mathcal{F}_1$ and $\Omega_2 \in \mathcal{F}_2$, we have that $C \in \mathcal{A}$.

**Problem 3.4** It is clear that $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Suppose a sequence of mutually disjoint sets $B_i \in \mathcal{G}$, $i \in \mathbb{N}$, is given. Then,

$$f_{\#}\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) = \sum_{i=1}^{\infty} f_{\#}\mu(B_i).$$

**Problem 3.6**

(a) By Proposition 2.8, we know that $\mathcal{B}_{\mathbb{R}}$ is generated by intervals of the form $(-\infty, a]$ with $a \in \mathbb{Q}$. As a consequence, $\mathcal{B}_{\mathbb{R}}$ is also generated by intervals of the form $(a, +\infty)$ with $a \in \mathbb{Q}$. Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\big\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\big\}$$

is measurable for every $a \in \mathbb{Q}$. For brevity, we write $\{f + g > a\}$. The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \Big( \{f > t\} \cap \{g > a - t\} \Big).$$

We first show the inclusion '$\subset$'. If $\omega \in \Omega$ is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some $t \in \mathbb{Q}$ such that

$$f(\omega) > t > a - g(\omega),$$

and thus $f(\omega) > t$ and $g(\omega) > a - t$. So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \Big( \{f > t\} \cap \{g > a - t\} \Big).$$

Now we will show the inclusion '$\supset$'. Let $\omega \in \Omega$ be such that $f(\omega) > t$ and $g(\omega) > a - t$. Then, by adding the inequalities, we know that $f(\omega) + g(\omega) > a$.

(b) The constant function $f(\omega) = a$ is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \qquad \forall\, B \in \mathcal{B}_{\mathbb{R}}.$$

(c) Similar to the proof of Point (2) of Proposition 3.2.12.

(d) Let $g(\omega) \neq 0$ for all $\omega \in \Omega$. Then, since $g$ is measurable, we have that

$$\begin{aligned}\{1/g > a\} &= \{g < 1/a,\ g > 0\} \cup \{g > 1/a,\ g < 0\} \\ &= \Big(\{g < 1/a\} \cap \{g > 0\}\Big) \cup \Big(\{g > 1/a\} \cap \{g < 0\}\Big) \in \mathcal{F},\end{aligned}$$

thus implying that $1/g$ is measurable.

(e) The previous part of this exercise together with point (4) of Proposition 3.12 yields Point (5) of Proposition 3.12.

**Problem 3.7** From (3.6), we have for any $a \in \mathbb{R}$,

$$\Big\{\sup_{n \geq 1} f_n > a\Big\} = \bigcup_{n \geq 1} \{f_n > a\} \in \mathcal{F},$$

Since $\mathcal{F}$ is a $\sigma$-algebra and $f_n$ is measurable for all $n \geq 1$, i.e., $\{f_n > a\} \in \mathcal{F}$ for all $n \geq 1$.

**Problem 3.8**

(a) Note that
$$f_M = M\mathbf{1}_{\{f\geq M\}} + f\mathbf{1}_{\{|f|<M\}} - M\mathbf{1}_{\{f\leq -M\}}.$$
Since the sets
$$\{f \geq M\}, \quad \{f \leq -M\}, \quad \{|f| < M\} \qquad \text{are } \mathcal{F}\text{-measurable},$$
their corresponding indicator functions are $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable. Since $f_M$ is the sum of products of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable functions, we conclude that $f_M$ is also $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

(b) It is easy to see that $f_M$ converges pointwise to $f$ as $M \to \infty$, i.e.,
$$\lim_{M\to\infty} f_M(\omega) = f(\omega) \qquad \forall\, \omega \in \Omega.$$
Indeed, if $\omega\Omega$ is such that $f(\omega) = +\infty$, then
$$\lim_{M\to\infty} f_M(\omega) = \lim_{M\to\infty} M = +\infty = f(\omega),$$
and similarly for $\omega \in \Omega$ for which $f(\omega) = -\infty$. On the other hand, for any $\omega \in \Omega$ with $f(\omega) \in \mathbb{R}$, there is some $N_0(\omega) \in \mathbb{N}$ such that $f_N(\omega) = f(\omega)$ for all $N \geq N_0(\omega)$, and hence,
$$\lim_{M\to\infty} f_M(\omega) = f(\omega).$$
Since $f$ is the limit of a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable functions, we conclude from Lemma 3.2.13 that $f$ is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

## Chapter 4: The Lebesgue Integral

**Problem 4.2**

**Problem 4.3**

(a) The fact that the sets are disjoint is immediate from the definition. Measurability follows from Lemma 3.11

(b) Let us fix a $\omega \in \Omega$. Then if $f(\omega) = +\infty$ we get that $f_n(\omega) = 2^n$ holds for all $n \geq 1$ and hence $\lim_{n\to\infty} f_n(\omega) = +\infty = f(\omega)$. So assume that $f(\omega) < +\infty$. Then there exists an $M \in \mathbb{N}$ such that $f(\omega) < M$. Hence, for all $n \geq M$ we have that
$$\|f_n(\omega) - f(\omega)\| = f(\omega) - f_n(\omega) \leq 2^{-n},$$
which implies that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

(c) Fix $n \geq 1$ and $\omega \in \Omega$. Clearly, if $f(\omega) = +\infty$ then $f_n(\omega) = 2^n < +\infty = f(\omega)$.

(d) Fix $\omega \in \Omega$ such that $f(\omega) < +\infty$ and $\omega \in A_k^n$ for some $0 \le k < N_n = n2^n$.

Note that $k2^{-n} \le f(\omega) < (k+1)2^{-n}$ holds and this interval can be split into two intervals as follows:

$$[k2^{-n}, (k+1)2^{-n}) = [(2k)2^{-(n+1)}, (2k+1)2^{-(n+1)}) \cup [(2k+1)2^{-(n+1)}, (2k+2)2^{-(n+1)}).$$

Hence, we conclude that either $\omega \in A_{2k}^{n+1}$ or $\omega \in A_{2k+1}^{n+1}$. In both case we get that

$$f_n(\omega) = k2^{-n} = 2kn^{-(n+1)} \le f_{n+1}(\omega).$$

(e) Now let us consider the case where $\omega \in A_k^n$ with $k = n2^n$, so that $n \le f(\omega) < +\infty$. Then, if $f(\omega) \ge n+1$ it follows that $f_n(\omega) = n < n+1 = f_{n+1}(\omega)$. If, on the other hand, $n \le f(\omega) < n+1$ there exists an $2n\,2^n \le \ell \le (2n+2)\,2^n$ such that $\omega \in A_\ell^{n+1}$, which then implies that

$$f_n(\omega) = n = (2n2^n)\,2^{-(n+1)} \le f_{n+1}(\omega).$$

**Problem 4.5**

(a) First suppose $f = \sum_{i=1}^N a_i \mathbb{1}_{A_i}$ is a simple function. Then $f\mathbb{1}_B = \sum_{i=1}^N a_i \mathbb{1}_{A_i \cap B}$ is also a simple function and thus

$$\int_B f \,\mathrm{d}\mu = \int_\Omega f \mathbb{1}_B \,\mathrm{d}\mu = \sum_{i=1}^N a_i \mu(A_i \cap B) \le \mu(B) \sum_{i=1}^N a_i \mu(A_i) = 0.$$

Now let $f$ be a non-negative function and $g \le f$ be a simple function. Then $g\mathbb{1}_B \le f\mathbb{1}_B$ and thus by Definition 4.7

$$\int_B f \,\mathrm{d}\mu = \int_\Omega f \mathbb{1}_B \,\mathrm{d}\mu \ge \int_\Omega g \mathbb{1}_B \,\mathrm{d}\mu = 0,$$

which implies the result.

(b) Suppose $f \le g$ are non-negative functions and observe that if $h$ is a simple function such that $h \le f$ then also $h \le g$. Therefore we get

$$\int_\Omega f \,\mathrm{d}\mu = \sup_{h \le f} \left\{ \int_\Omega h \,\mathrm{d}\mu \right\} \le \sup_{h \le g} \left\{ \int_\Omega h \,\mathrm{d}\mu \right\} = \int_\Omega g \,\mathrm{d}\mu.$$

(c) Suppose that $h$ is a simple function. Then $\alpha h$ is also simple and it immediately follows that $\int_\Omega (\alpha h) \,\mathrm{d}\mu = \alpha \int_\Omega h \,\mathrm{d}\mu$. Now let $f$ be non-negative. Then $h \le f \iff \alpha h \le \alpha f$

and $h \le \alpha f \iff \alpha^{-1} h \le f$. Thus by Definition 4.7 we have

$$\begin{aligned}
\alpha \int_\Omega f \,\mathrm{d}\mu &= \alpha \sup_{h \le f} \left\{ \int_\Omega h \,\mathrm{d}\mu \right\} \\
&= \sup_{h \le f} \alpha \left\{ \int_\Omega h \,\mathrm{d}\mu \right\} \\
&= \sup_{h \le f} \left\{ \int_\Omega (\alpha h) \,\mathrm{d}\mu \right\} \\
&= \sup_{\alpha^{-1} h \le f} \left\{ \int_\Omega h) \,\mathrm{d}\mu \right\} \\
&= \sup_{h \le \alpha f} \left\{ \int_\Omega (\alpha h) \,\mathrm{d}\mu \right\} = \int_\Omega (\alpha f) \,\mathrm{d}\mu.
\end{aligned}$$

**Problem 4.8**

(a) By definition, we have that $\nu_f(\Omega) = \int_\Omega f \,\mathrm{d}\mu = 1$. Now let $(A_n)_{n\in\mathbb{N}}$ be a family of mutually disjoint measurable sets. Then we have that the sequence

$$g_n := \sum_{i=1}^n f\, \mathbf{1}_{A_i} = f\, \mathbf{1}_{\bigcup_{i=1}^n A_i} \longrightarrow g := f\, \mathbf{1}_{\bigcup_{i\in\mathbb{N}} A_i} \qquad \text{pointwise monotonically.}$$

By MCT, we then have that

$$\nu_f\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \int_{\bigcup_{i\in\mathbb{N}} A_i} f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\bigcup_{i=1}^n A_i} f \,\mathrm{d}\mu = \lim_{n\to\infty} \sum_{i=1}^n \int_{A_i} f \,\mathrm{d}\mu = \sum_{i\in\mathbb{N}} \nu_f(A_i),$$

thus showing that $\nu_f$ is a probability measure on $(\Omega, \mathcal{F})$.

(b) Following the hint, we start by considering nonnegative simple functions $g$. Suppose $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ for $a_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$ mutually disjoint. Then,

$$\int_\Omega g\, d\nu_f = \sum_{i=1}^n a_i = \nu_f(A_i) = \sum_{i=1}^n a_i \int_{A_i} f \,\mathrm{d}\mu = \int_\Omega gf \,\mathrm{d}\mu.$$

Now let $g$ be a nonnegative measurable function and $[g]_n$ be a sequence of nonnegative simple functions that converge pointwise monotonically to $g$. Then MCT yields

$$\int_\Omega g \,\mathrm{d}\nu_f = \lim_{n\to\infty} \int_\Omega [g]_n \,\mathrm{d}\nu_f = \lim_{n\to\infty} \int_\Omega [g]_n f \,\mathrm{d}\mu = \int_\Omega gf \,\mathrm{d}\mu,$$

where we used the fact that $[g]_n f$ converges pointwise monotonically to $gf$.

(c) Let $g$ be measurable. Then $g = g^+ - g^-$, where $g^\pm$ are nonnegative measurable functions. Since $f$ is nonnegative, we have that $(fg)^\pm = fg^\pm$. Due to (b), we deduce

$$\int_\Omega g^\pm \,\mathrm{d}\nu_f = \int_\Omega g^\pm f \,\mathrm{d}\mu = \int_\Omega (gf)^\pm \,\mathrm{d}\mu.$$

Hence, $g^\pm$ is $\nu_f$-integrable if and only if $(gf)^\pm$ is $\mu$-integrable. Consequently, $g$ is $\nu_f$-integrable if and only if $gf$ is $\mu$-integrable, since

$$\int_\Omega |g| \,\mathrm{d}\nu_f = \int_\Omega g^+ \,\mathrm{d}\nu_f + \int_\Omega g^- \,\mathrm{d}\nu_f = \int_\Omega g^+ f \,\mathrm{d}\mu + \int_\Omega g^- f \,\mathrm{d}\mu = \int_\Omega |gf| \,\mathrm{d}\mu.$$

**Problem 4.9**

**($\Rightarrow$)** Let $f$ be $\mu$-integrable. Then both $|f|\mathbf{1}_{\{|f|<n\}}$ and $|f|\mathbf{1}_{\{|f|\geq n\}}$ are integrable, due to the monotonicity of the integral. By linearity of the integral,

$$\int_\Omega |f|\mathbf{1}_{\{|f|\geq n\}} \,\mathrm{d}\mu = \int_\Omega |f| \,\mathrm{d}\mu - \int_\Omega |f|\mathbf{1}_{\{|f|<n\}} \,\mathrm{d}\mu.$$

Since the sequence $g_n := |f|\mathbf{1}_{\{|f|<n\}} \geq 0$ converges pointwise monotonically to $|f|$, we can apply MCT to obtain

$$\lim_{n\to\infty} \int_\Omega |f|\mathbf{1}_{\{|f|<n\}} \, dd\mu = \int_\Omega |f| \,\mathrm{d}\mu.$$

Hence,

$$\lim_{n\to\infty} \int_\Omega |f|\mathbf{1}_{\{|f|\geq n\}} \,\mathrm{d}\mu = \int_\Omega |f| \,\mathrm{d}\mu - \lim_{n\to\infty} \int_\Omega |f|\mathbf{1}_{\{|f|<n\}} \,\mathrm{d}\mu = 0.$$

**($\Leftarrow$)** By assumption, there is some $N \geq 1$ such that

$$\int_\Omega |f|\mathbf{1}_{\{|f|\geq N\}} \,\mathrm{d}\mu \leq 1.$$

By linearity of the integral,

$$\int_\Omega |f| \,\mathrm{d}\mu = \int_\Omega |f|\mathbf{1}_{\{|f|<N\}} \,\mathrm{d}\mu + \int_\Omega |f|\mathbf{1}_{\{|f|\geq N\}} \,\mathrm{d}\mu \leq N\mu\big(\{|f| < N\}\big) + 1.$$

Since $\mu$ is a finite measure, the right-hand side is finite, implying that $f$ is $\mu$-integrable.

**Problem 4.10**
Observe that $\Omega = \bigcup_{n\in\mathbb{N}}\{|f| > n\}$.
We then get that

$$\sum_{n=1}^{\infty} \int_{\{|f|>n\}} |f| \,\mathrm{d}\mu = \int_\Omega |f| \,\mathrm{d}\mu < \infty.$$

This implies that for some $N$ and all $n \geq N$: $\int_{\{|f|>n\}} |f| \, \mathrm{d}\mu < 1/n$ or else the sum cannot be finite.
Now let $\varepsilon > 0$, take $M > \max\{N, 2/\varepsilon\}$ and $\delta = \varepsilon/(2M)$. Then

$$\begin{aligned} \int_A |f| \, \mathrm{d}\mu &= \int_A |f| \mathbf{1}_{|f| \leq M} \, \mathrm{d}\mu + \int_A |f| \mathbf{1}_{|f|>M} \, \mathrm{d}\mu \\ &\leq M\mu(A) + \frac{1}{M} \leq M\delta + \frac{1}{M} < \varepsilon. \end{aligned}$$

## Chapter 5: Product spaces and Lebesgue integration

**Problem 5.2**

(a) Note that $\mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{F}_1 \times \mathcal{F}_2$, and hence

$$\sigma(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{F}_1 \otimes \mathcal{F}_2.$$

(b) Let $B \in \mathcal{A}_2$. Then we have that

$$\Omega_1 \times B = \bigcup_{n \geq 1} A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$$

since $A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ for all $n \geq 1$. So $\Omega_1 \in \Sigma$

For the second property, let $C \in \Sigma$ and note that $C^c \times B = (\Omega_1 \times B) \setminus (C \times B)$. Since both these sets are in $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ it follows that $C^c \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ and hence $C^c \in \Sigma$.

Finally consider a countable sequence $(C_n)_{n \geq 1}$ of sets in $\Sigma$. Then for any $B \in \mathcal{A}_2$

$$\left( \bigcup_{n \geq 1} C_n \right) \times B = \bigcup_{n \geq 1} (C_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2),$$

since each $C_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(c) Note that $\mathcal{A}_1 \subset \Sigma_1 \subset \mathcal{F}_1$. From which it follows that $\Sigma_1 = \mathcal{F}_1$. But then, from the definition of $\Sigma_1$ we have that $\mathcal{F}_1 \times \mathcal{A}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(d) We can show in a similar fashion that

$$\Sigma_2 := \{C \in \mathcal{F}_2 \, : \, B \times C \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2) \, \forall B \in \mathcal{A}_1\}.$$

is a $\sigma$-algebra on $\Omega_2$, from which we conclude that $\mathcal{A}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(e) take any $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Then

$$A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B) = \bigcup_{n,m \geq 1} (A \times B_m) \cap (A_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2).$$

From this we conclude that $\mathcal{F}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$, which finishes the proof.

## Problem 5.2

(a) Let $t_0 \in (a, b)$ be fixed. It suffices to check the continuity result for arbitrary sequences $(t_n)_{n\geq 1} \subset (a, b)$ such that $t_n \to t_0$ as $n \to \infty$. Fix such a sequence and define $g_n(\omega) := f(\omega, t_n)$ for all $\omega \in \Omega$ and $n \geq 1$. Since $\lim_{t\to t_0} f(\omega, t) = f(\omega, t_0)$ for all $\omega \in \Omega$, we deduce that $\lim_{n\to\infty} g_n(\omega) = f(\omega, t_0)$ for every $\omega \in \Omega$. Moreover, by assumption $|g_n| \leq g$ for all $n \geq 1$ and $g$ is integrable. By the Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_\Omega g_n(\omega)\, \mu(\mathrm{d}\omega) = \int_\Omega f(\omega, t_0)\, \mu(\mathrm{d}\omega).$$

As the chosen sequence was arbitrary, we deduce that $\lim_{t\to t_0} F(t) = F(t_0)$.

(b) If $t \mapsto f(\omega, t)$ is continuous on $(a, b)$ for all $\omega \in \Omega$ then $\lim_{t\to t_0} f(\omega, t) = f(\omega, t_0)$ at every $t_0 \in (a, b)$ for all $\omega \in \Omega$. In particular, (a) applies, showing that $\lim_{t\to t_0} F(t) = F(t_0)$ for every $t_0 \in (a, b)$, i.e., $F$ is continuous on $(a, b)$.

## Problem 5.3

(1) We start by showing that $(\partial f/\partial t)(\cdot, t)$ is measurable. Let $(t_n)_{n\geq 1} \subset (a, b)$ be an arbitrary sequence with $t_n \neq t$ and $t_n \to t$ for $n \to \infty$. We set

$$g_n(\omega) = \frac{f(\omega, t_n) - f(\omega, t)}{t_n - t}.$$

Clearly, $g_n$ is measurable for every $n \geq 1$. Moreover, $\lim_{n\to\infty} g_n(\omega) = (\partial f/\partial t)(\omega, t)$ by the definition of the derivative. Since $(\partial f/\partial t)(\cdot, t)$ is the pointwise limit of a sequence of measurable functions, it is also measurable. Clearly, $(\partial f/\partial t)(\cdot, t)$ is integrable since

$$\int_\Omega |(\partial f/\partial t)(\omega, t)|\, \mu(\mathrm{d}\omega) \leq \int_\Omega g\, \mathrm{d}\mu < +\infty.$$

(2) Let $t_0 \in (a, b)$ and suppose w.l.o.g. $t_0 < t$. Since $t \mapsto f(\omega, t)$ is differentiable on $(a, b)$ for all $\omega \in \Omega$, the Mean Value Theorem gives

$$\frac{f(\omega, t) - f(\omega, t_0)}{t - t_0} = (\partial f/\partial t)(\omega, \tau) \qquad \text{for some } \tau \in (t_0, t).$$

Taking the modulus on both sides, we obtain

$$\left|\frac{f(\omega, t) - f(\omega, t_0)}{t - t_0}\right| \leq |(\partial f/\partial t)(\omega, \tau)| \leq g(\omega) \qquad \text{for all } \omega \in \Omega.$$

(3) We now have all the ingredients needed to apply the DCT, which yields

$$\lim_{n\to\infty} \frac{F(t_n) - F(t)}{t_n - t} = \lim_{n\to\infty} \int_\Omega g_n\, \mathrm{d}\mu = \int_\Omega (\partial f/\partial t)(\omega, t)\, \mu(\mathrm{d}\omega).$$

Since $t \in (a, b)$ and the sequence $(t_n)_{n\geq 1}$ was arbitrary, we conclude that $F$ is differentiable on $(a, b)$ with

$$F'(t) = \int_\Omega (\partial f/\partial t)(\omega, t)\, \mu(\mathrm{d}\omega).$$

### Problem 5.3

(a) Note that the integrand $f_n(x) = \frac{1+nx^2}{(1+x^2)^n}$ is continuous on $[0,1]$ and non-negative. Hence, the Riemann integral and Lebesgue integral coincide, i.e.,

$$\int_0^1 f_n(x)\,\mathrm{d}x = \int_{[0,1]} f_n \,\mathrm{d}\lambda.$$

Observe that we have the following pointwise limit

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0,1], \end{cases}$$

i.e., $\lim_{n\to\infty} f_n = 0$ $\lambda$-almost everywhere. Moreover, $f_n(x) \leq 1$ for every $x \in [0,1]$ and $n \geq 1$. Since the constant function $g \equiv 1$ is $\lambda$-integrable on $[0,1]$, it is a valid dominator. Hence, the DCT gives

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{[0,1]} f_n\,\mathrm{d}\lambda = \int_{[0,1]} \lim_{n\to\infty} f_n\,\mathrm{d}\lambda = 0$$

(b) For the purpose of convergence, we consider $n \geq 3$. Note that the integrand $f_n(x) = \frac{x^{n-2}}{1+x^n}\cos\left(\frac{\pi x}{n}\right)$ is continuous on $(0,+\infty)$ with pointwise limit

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ 1/2 & \text{if } x = 1, \\ 1/x^2 & \text{if } x > 1, \end{cases}$$

Setting the function

$$g(x) = \begin{cases} 1 & \text{for } x \in (0,1), \\ \frac{1}{x^2} & \text{for } x \geq 1, \end{cases}$$

we see that $f_n \leq g$ $\lambda$-almost everywhere in $(0,+\infty)$ and for all $n \geq 3$. Indeed, for $x \geq 1$, we obtain

$$|f_n(x)| \leq \left|\frac{x^{n-2}}{1+x^n}\cos\left(\frac{\pi x}{n}\right)\right| \leq \frac{x^{n-2}}{1+x^n} \leq \frac{x^{n-2}}{x^n} = \frac{1}{x^2},$$

while for $x \in (0,1)$, we have

$$|f_n(x)| \leq \left|\frac{x^{n-2}}{1+x^n}\cos\left(\frac{\pi x}{n}\right)\right| \leq \frac{x^{n-2}}{1+x^n} \leq 1.$$

Notice that $g$ is non-negative and $\lambda$-integrable on $(0,+\infty)$. Indeed, using the MCT,

$$\begin{aligned}\int_{(0,+\infty)} g\,\mathrm{d}\lambda &= \int_{(0,1)} g\,\mathrm{d}\lambda + \int_{(1,+\infty)} g\,\mathrm{d}\lambda = 1 + \lim_{n\to\infty}\int_{(1,n)} g\,\mathrm{d}\lambda \\ &= 1 + \lim_{n\to\infty}\int_1^n \frac{1}{x^2}\,\mathrm{d}x = 1 + \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 2 < +\infty.\end{aligned}$$

To conclude, we apply DCT to deduce that the limit is 1.

**Problem 5.4**
The proof follows verbatim to the proof of the Dominated Convergence Theorem.

**Problem 5.7**
Let $F_n$ denote the cdf of $Y_n = \|X_n - X\|$ and $F_0$ denote the cdf of $0$. By Definition 5.2.9 and Lemma 5.2.8 we have that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $F_n(t) \to F_0(t)$ for all continuity points $t$ of $F_0$. This is equivalent to showing that $1 - F_n(t) \to 1 - F_0(t)$, where

$$1 - F_0(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ 1 & \text{else.} \end{cases}$$

Now note that the only discontinuity point of $F_0$ is 0. Moreover, $1 - F_n(t) = 0 = F_0(t)$ for all $t < 0$. Hence it follows that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $1 - F_n(t) \to 0$ for all $t > 0$, which is what we needed to show.

**Problem 5.8**

(a) For this let $h_t(x) = \mathbf{1}_{(-\infty,t]}$ and note that

$$F_n(t) = (X_n)_\# \mathbb{P}_n((-\infty, x]) = \int_{\mathbb{R}} \mathbb{1}_{(-\infty,t]} \, \mathrm{d}(X_n)_\# \mathbb{P}_n = \int_{\mathbb{R}} h_t \, \mathrm{d}\mu_n.$$

and similarly $F(t) = \int_{\mathbb{R}} h_t \, \mathrm{d}\mu$

(b) The function $h$ is discontinuous only at $t$, i.e. $\mathcal{C}_h = \mathbb{R} \setminus \{t\}$. Moreover, for any $\varepsilon > 0$

$$\mu((t - \varepsilon, t + \varepsilon)) = \mu((t - \varepsilon, t]) + \mu((t, t + \varepsilon)) = F(t) - F(t - \varepsilon) + F(t + \varepsilon) - F(t).$$

Since $F$ is continuous at $t$, the right hand side goes to zero as $\varepsilon \to 0$. Therefore

$$\mu(\{t\}) = \lim_{\varepsilon \to 0} \mu((t - \varepsilon, t + \varepsilon)) = 0,$$

which implies that $\mu(\mathcal{C}_h) = 1$.

(c) The result follows by applying condition (2) in Theorem 5.2.7.

(d) Let $\varepsilon > 0$, pick such a $\delta$ and partition the interval $[-K, K]$ into $L_\delta := \left\lceil \frac{4K}{\delta} \right\rceil$ intervals $I_\ell = (a_\ell, b_\ell]$ of equal length, which is $\leq \delta/2 < \delta$. Now we define the simple function

$$\hat{g} := \sum_{\ell=1}^{L} h(b_\ell) \mathbb{1}_{I_\ell},$$

(e) Let $M = L$, $\beta_\ell = \sum_{t=1}^{\ell} h(b_t)$ and $t_\ell = b_\ell$. Then

$$\hat{g} := \sum_{\ell=1}^{L} \beta_\ell \mathbf{1}_{(-\infty, b_\ell]}.$$

(f) Using the representation in (e) we get

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}[\hat{g}(X_n)] &= \lim_{n\to\infty}\sum_{\ell=1}^{L}\beta_\ell\int_{\mathbb{R}}\mathbf{1}_{X_n^{-1}((-\infty,b_\ell])}\,\mathrm{d}\mathbb{P}\\ &= \lim_{n\to\infty}\sum_{\ell=1}^{L}\beta_\ell(X_n)_{\#}\mathbb{P}((-\infty,b_\ell])\\ &= \lim_{n\to\infty}\sum_{\ell=1}^{L}\beta_\ell F_n(b_\ell)\\ &= \sum_{\ell=1}^{L}F(b_\ell) = \mathbb{E}[\hat{g}(X)].\end{aligned}$$

(g) Using the representation of $\hat{g}$ in (d) we note that $\|x-y\|<\varepsilon$ for all $x,y\in I_\ell$. This then implies that $\|g(x)-\hat{g}(y)\|\le\varepsilon$ from which it follows that

$$\begin{aligned}\|\mathbb{E}[g(X_n)]-\mathbb{E}[g(X)]\| &\le \|\mathbb{E}[g(X_n)]-\mathbb{E}[\hat{g}(X_n)]\|+\|\mathbb{E}[g(X)]-\mathbb{E}[\hat{g}(X)]\|\\ &\quad+\|\mathbb{E}[\hat{g}(X_n)]-\mathbb{E}[\hat{g}(X)]\|\\ &\le 2\varepsilon+\|\mathbb{E}[\hat{g}(X_n)]-\mathbb{E}[\hat{g}(X)]\|.\end{aligned}$$

We have shown in (f) that the last term goes to zero as $n\to\infty$. Since $\varepsilon$ was arbitrary we conclude that (**??**) holds.

(h) This now follows from Theorem 5.2.7 (3).

**Problem 5.9** Suppose that $X_n \overset{\text{a.s.}}{\to} X$. Then by Lemma 5.2.16 this is equivalent to $\mathbb{P}(\|X_n-X\|>\varepsilon\text{ i.o.})=0$ for all $\varepsilon>0$.
For now fix an $\varepsilon>0$ and write $A_n:=\{\|X_n-X\|>\varepsilon\}$. Recall that

$$\{A_n\text{ i.o.}\}=\bigcap_{k=1}^{\infty}\bigcup_{k\ge n}A_n$$

and note two things:

(a) The sets $B_k:=\bigcup_{n\ge k}A_n$ are non-increasing, i.e. $B_k\supset B_{k+1}$, and

(b) $\mathbb{P}(A_k)\le\mathbb{P}(\bigcup_{n\ge k}A_n)=\mathbb{P}(B_k)$.

We then have that:

$$\begin{aligned}0 &= \mathbb{P}(\{A_n\text{ i.o.}\}) && \text{by assumption}\\ &= \mathbb{P}(\bigcap_{k=1}^{\infty}B_k) && \text{by Lemma 5.2.16}\\ &= \lim_{k\to\infty}\mathbb{P}(B_k) && \text{by continuity form above (Proposition 2.2.15)}\\ &\ge \lim_{k\to\infty}\mathbb{P}(A_k) && \text{by (b).}\end{aligned}$$

# Probability I: The basics

**Problem 6.2**

(a) The implication from right to left is by definition of $\overleftarrow{F}$ and the fact that $F$ is non-decreasing. The implication from left to right is because $F$ is right continuous.

(b) Consider the preimage of $(-\infty, t]$ under $X$. Then, using the above observation, we have

$$
\begin{aligned}
X^{-1}((-\infty,t]) &= \{\omega \in \Omega \,:\, \overleftarrow{F}(U(\omega)) \in (-\infty,t]\} \\
&= \{\omega \in \Omega \,:\,, U(\omega) \in (-\infty, F(t)]\} = U^{-1}((-\infty, F(t)]) \in \mathcal{B}_{[0,1]}.
\end{aligned}
$$

Hence, $X$ is measurable. Finally, the above computation, together with Lemma 6.5, also implies that

$$
\mathbb{P}\left(X^{-1}((-\infty,t])\right) = \mathbb{P}\left(U^{-1}((-\infty,F(t)])\right) = F(t).
$$

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $U$ a standard normal random variable. We will show that $X = \overleftarrow{F} \circ U$ is a random variable with the right probability measure. Since we can construct a standard uniform random variable on the probability $([0,1], \mathcal{B}_{[0,1]}, \lambda|_{[0,1]})$ this also implies the last part.
which finished the proof.

**Problem 6.3**

(a) For the probability space, take $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0,1]$.

Observe that the function $H_\gamma(z)$ is continuous and hence has an inverse $g_\gamma(y) = \gamma \tan(\pi(y - 1/2))$ on $[0,1]$.

So the function $Y[0,1] \to \mathbb{R}$ defined by $Y(x) = g_\gamma(x)$ has the correct distribution as

$$
\mathbb{P}(Y^{-1}((-\infty,t])) = \mathbb{P}(g_\gamma^{-1}((-\infty,t])) = \lambda(H_\gamma((-\infty,t])) = H_\gamma(t).
$$

(b) Note that $g_\gamma$ is continuous on $[0,1]$ and hence measurable.

(c) For any $t \geq 0$, the cdf of the Poisson random variable is given by

$$
F_\lambda(t) = \sum_{n=0}^{\lceil t \rceil} f_\lambda(n),
$$

where $\lceil t \rceil$ is the ceiling of $t$, i.e. the smallest integer $k \geq t$.

(d) For the probability space, we again take $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0,1]$.

Now for any $y \in [0,1]$ let $k := k(y)$ be such that

$$\sum_{n=1}^{k} f_\lambda(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_\lambda(n) < y,$$

where the last sum is interpreted as $-1$ if $k = 0$.

Now define $X(y) = k(y) : [0,1] \to \mathbb{R}$. Then $k(y) \leq t$ if and only if $y \leq F_\lambda(t)$ and hence

$$X^{-1}((-\infty, t]) = \{y \in [0,1] \,:\, k(y) \in (0,t]\} = \{y \in [0,1] \,:\, y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

(e) It follows from the above computation that $X^{-1}((-\infty,t]) = \{y \in [0,1] : y \in (0, F_\lambda(t)]\}$. Since the latter is a measurable set we conclude that $X^{-1}((-\infty,t])$ is measurable for all $t$ and since these generate the Borel $\sigma$-algebra $X$ is measurable.

(f) for any $\ell \in \mathbb{N}$ define the sets $A_\ell = (n-1-1/\ell), n-1+1/\ell]$. Then $A_\ell$ is a decreasing set with $\lim_{\ell\to\infty} A_\ell = \{n\}$. Moreover, $A_\ell = (-\infty, n-1+1/\ell] \setminus (-\infty, n-1-1/\ell]$ and $\mathbb{P}(A_1) < \infty$. It now follows from continuity from above and (d) that

$$\begin{aligned}
X_\#\mathbb{P}(\{n\}) &= \lim_{\ell\to\infty} X_\#\mathbb{P}(A_\ell) \\
&= \lim_{\ell\to\infty} X_\#\mathbb{P}((-\infty, n-1+1/\ell]) - X_\#\mathbb{P}((-\infty, n-1-1/\ell]) \\
&= F_\lambda(n-1+1/\ell) - F_\lambda(n-1-1/\ell) \\
&= \sum_{k=0}^{n} f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n).
\end{aligned}$$

**Problem 6.5**

Define for any $j \in \mathbb{Z}$, $p_j := \mathbb{P}(X^{-1}(\{j\}))$. Then, since $(X^{-1}(j))_{j\in\mathbb{Z}}$ is a family of disjoint sets and $\mathbb{P}$ is a probability measure we get that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\bigcup_{j\in\mathbb{Z}} X^{-1}(j)) = \sum_{j\in\mathbb{Z}} p_j.$$

Now let $A \subset \mathbb{R}$ be a measurable set and note that

$$X^{-1}(A) = \bigcup_{j\in\mathbb{Z}\cap A} X^{-1}(j).$$

Then it follows that

$$\mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}\left(\bigcup_{j\in\mathbb{Z}\cap A} X^{-1}(j)\right) = \sum_{j\in\mathbb{Z}\cap A} p_j = \sum_{j\in\mathbb{Z}} \delta_j(A) p_j.$$

**Problem 6.7**

(a) We first observe that

$$\nu((-\infty,t]) = X_{\#}\mathbb{P}((-\infty,t])$$

holds by definition of the pdf. The idea is to extend this to general sets of the $\sigma$-algebra using Theorem 2.15, since the family $\mathcal{A}_1 := \{(-\infty,t] \,:\, t \in \mathbb{R}\}$ generates $\mathcal{B}$. The problem is that this family does not satisfy the first requirement $A \cap B \in \mathcal{A}_1$ for all $A, B \in \mathcal{A}_1$. However, we also know that $\mathcal{B}$ is generated by the family $\mathcal{A}_2 := \{(a,b] \,:\, a < b \in \mathbb{R}\}$, and that this family does satisfy the requirement $A \cap B \in \mathcal{A}_2$ for all $A, B \in \mathcal{A}_2$. So lets show that $\nu$ agrees with $X_{\#}\mathbb{P}$ on $\mathcal{A}_2$.

We first note that $(a,b] = (-\infty,b] \setminus (-\infty,a]$. Hence

$$\begin{aligned}
X_{\#}\mathbb{P}((a,b]) &= X_{\#}\mathbb{P}((-\infty,b]) - X_{\#}\mathbb{P}((-\infty,a]) \\
&= \int_{(-\infty,b]} \mathrm{d}\mathbb{P} - \int_{(-\infty,a]} \mathrm{d}\mathbb{P} \\
&= \int_{(a,b]} \mathrm{d}\mathbb{P} = \nu((a,b]).
\end{aligned}$$

Finally, we note that $\mathbb{R} = \bigcup_{n\in\mathbb{N}}(-n,n]$ and that

$$\nu((-n,n]) = X_{\#}\mathbb{P}((-n,n]) < \infty.$$

Thus, by Theorem 2.15 we conclude that $\nu = X_{\#}\mathbb{P}$ on $\sigma(\mathcal{A}_2) = \mathcal{B}$.

(b) Let $g = \sum_{i=1}^N a_i \mathbb{1}_{A_i}$ be a simple function. Then by definition of $\nu$ and linearity of the integral we get

$$\begin{aligned}
\int_{\mathbb{R}} g \,\mathrm{d}\nu &= \sum_{i=1}^N a_i \int_{A_i} \mathrm{d}\nu \\
&= \sum_{i=1}^N a_i \nu(A_i) \\
&= \sum_{i=1}^N a_i \int_{A_i} \rho \,\mathrm{d}\lambda \\
&= \int_{\Omega} \sum_{i=1}^N a_i \mathbb{1}_{A_i} \rho \,\mathrm{d}\lambda = \int_{\mathbb{R}} g\rho \,\mathrm{d}\lambda.
\end{aligned}$$

(c) By using (b) and monotone convergence twice we get

$$\begin{aligned}
\int_{\mathbb{R}} h \,\mathrm{d}\nu &= \int_{\mathbb{R}} \lim_{n\to\infty} [h]_n \,\mathrm{d}\nu \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n \,\mathrm{d}\nu \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n \rho \,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} [h]_n \rho \,\mathrm{d}\lambda = \int_{\mathbb{R}} h\rho \,\mathrm{d}\lambda.
\end{aligned}$$

(d) Using the change of variables result (Proposition 4.14) (a) and (c) we get

$$\mathbb{E}[h(X)] = \int_\Omega h \circ X \,\mathrm{d}\mathbb{P} = \int_\mathbb{R} h \,\mathrm{d}X_\# \mathbb{P} = \int_\mathbb{R} h \,\mathrm{d}\nu = \int_\mathbb{R} h\rho \,\mathrm{d}\lambda.$$

**Problem 6.8**

(a) This follows from the following computation

$$\int_\Omega |f|^p \,\mathrm{d}\mu \geq \int_\Omega |f|^p \mathbb{1}_{|f|\geq t} \,\mathrm{d}\mu \geq t^p \int_\Omega \mathbb{1}_{|f|\geq t} \,\mathrm{d}\mu = t^p \mu(\{\omega \in \Omega \,:\, |f| \geq t\}).$$

(b) Using the result for $p = 1$ we get

$$\mathbb{P}(|X| \geq t) = \mu(\omega \in \Omega \,:\, |X(\omega) \geq t\}) \leq \frac{1}{t}\int_\Omega X \,\mathrm{d}\mathbb{P} = \frac{1}{t}\mathbb{E}[X].$$

(c) Take $f(\omega) = X(\omega) - \mathbb{E}[X]$, which is measurable. Then using the first result with $p = 2$ gives

$$\begin{aligned}\mathbb{P}(|X - \mathbb{E}[X]| \geq t) &= \mathbb{P}(|X - \mathbb{E}[X]|^2 \geq t^2) \\ &\leq \frac{1}{t^2}\mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \frac{1}{t^2}(\mathbb{E}[X^2] - \mathbb{E}[X]^2) = \frac{\mathrm{Var}(X)}{t^2}.\end{aligned}$$

**Problem 6.4** From Theorem 6.5.9, we find for any $\varepsilon > 0$ a continuous and bounded function $g \in L^1(\Omega, \mu)$ such that

$$\|f - g\|_1 < \frac{\varepsilon}{2}.$$

Let $M > 0$ and set $g_M := \varphi_M g$, where is a continuous function with compact support satisfying $0 \leq \varphi_M \leq 1$, $\varphi_M \equiv 1$ on $\overline{B_M}$ and $\varphi_M \equiv 0$ on $B^c_{M+1}$. Notice that

$$\int_{\mathbb{R}^d} |g - g_M| \,\mathrm{d}\mu = \int_{B^c_M} g \,\mathrm{d}\mu \leq \|g\|_{\sup} \mu(B^c_M).$$

Since $\mu$ is finite, the continuity from above of $\mu$ gives $\lim_{M\to\infty} \mu(B^c_M) = 0$. Hence, we find some $M = M_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^d} |g - g_M| \,\mathrm{d}\mu < \frac{\varepsilon}{2}.$$

Altogether, we've found some $g_M$ such that

$$\|f - g_M\|_1 \leq \|f - g\|_1 + \|g - g_M\| < \varepsilon.$$

## Chapter 5: Convergence of integrals and functions

## Chapter 6: $L^p$-spaces

## Chapter 7: Fubini-Tonelli

**Problem 7.4**
One direction is easy. Assume that $X_1$ and $X_2$ are independent according to Definition 7.1.4. Now take any $a, b \in \mathbb{R}$ and note that $A_1 := X_1^{-1}((-\infty, a]) \in \sigma(X_1)$ and $A_2 := X_2^{-1}((-\infty, b]) \in \sigma(X_2)$. Then by the definition of independence we have that

$$\mathbb{P}(X_1 \leq a, X_2 \leq b) = \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \leq b).$$

So let us focus now on the other direction. Assume that for all $a, b \in \mathbb{R}$

$$\mathbb{P}(X_1 \leq a, X_2 \leq b) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \leq b).$$

We now have to show that $X_1$ and $X_2$ are independent according to Definition 7.1.4.
First note that since the family $(-\infty, a] \times (-\infty, b]$ generate the 2-dimensional Borel $\sigma$-algebra we have, using Theorem 2.2.17, that

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2) = \mathbb{P}(X_1 \in B_1)\mathbb{P}(X_2 \in B_2)$$

for all $B_1, B_2 \in \mathcal{B}_{\mathbb{R}}$.
Now fix a set $B_2 \in \mathcal{B}_{\mathbb{R}}$, set $A_2 := X_2^{-1}(B_2) \in \sigma(X_2)$, and define the following two measures on the space $(\Omega, \sigma(X_1))$

$$\mu_1(A) = \mathbb{P}(A \cap A_2) \quad \text{and} \quad \mu_2(A) = \mathbb{P}(A)\mathbb{P}(A_2).$$

Let $a \in \mathbb{R}$ and consider the set $A_1 := X_1^{-1}((-\infty, a]) \in \sigma(X_1)$. Then, by our assumption we have that

$$\mu_1(A_1) = \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(X_1 \leq a, X_2 \in B_2) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \in B_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = \mu_2(A_1).$$

In other words, the measures $\mu_1, \mu_2$ coincide on the set $\{X_1^{-1}((-\infty, a]) \,:\, a \in \mathbb{R}\}$.
Since the set $(-\infty, a]$ generate $\mathcal{B}_{\mathbb{R}}$ it follows that

$$\sigma(X_1) = \sigma(\{X_1^{-1}((-\infty, a]) \,:\, a \in \mathbb{R}\}).$$

In addition, this set satisfies the conditions of Theorem 2.2.17 and hence we conclude that $\mu_1(A) = \mu_2(A)$ for all $A \in \sigma(X_1)$.
We can repeat this argument for the two measures on $(\Omega, \sigma(X_2))$

$$\nu_1(A) = \mathbb{P}(A_1 \cap A) \quad \text{and} \quad \nu_2(A) = \mathbb{P}(A_1)\mathbb{P}(A),$$

where $A_1 \in \{X_1^{-1}((-\infty, a]) \,:\, a \in \mathbb{R}\}$ is fixed.
From this we conclude that for any $A_1 \in \sigma(X_1)$ and $A_2 \in \sigma(X_2)$

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$$

and hence $X_1$ and $X_2$ are independent.

## Chapter 8: Radon-Nikodym

**Problem 8.2** First note that since $f-g$ is $\mathcal{H}$-measurable, we have that $\{f \geq g\}, \{f < g\} \in \mathcal{H}$. We then write

$$\int_\Omega \|f-g\| \,\mathrm{d}\mathbb{P} = \int_{f \geq g} (f-g) \,\mathrm{d}\mathbb{P} - \int_{f<g} (f-g) \,\mathrm{d}\mathbb{P}.$$

Since $\int_B f \,\mathrm{d}\mathbb{P} = \int_B g \,\mathrm{d}\mathbb{P}$ holds for all $B \in \mathcal{H}$ both integrals on the right hand side are zero.

**Problem 8.4**

(a) By definition we have that

$$\int_B \mathbb{E}[X|\mathcal{H}] \,\mathrm{d}\mathbb{P} = \int_B X \,\mathrm{d}\mathbb{P},$$

holds for all $B \in \mathcal{H}$. Since by assumption both $\mathbb{E}[X|\mathcal{H}]$ and $X$ are $\mathcal{H}$-measurable, the result follows from problem 8.2.

(b) Note that $a\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable. Moreover,

$$\int_B a\mathbb{E}[X|\mathcal{H}] \,\mathrm{d}\mathbb{P} = a \int_B \mathbb{E}[X|\mathcal{H}] \,\mathrm{d}\mathbb{P} = a \int_B X \,\mathrm{d}\mathbb{P} = \int_B aX \,\mathrm{d}\mathbb{P}.$$

This proves the claim.

(c) Similarly to the previous point, we first note that since $\mathbb{E}[X|\mathcal{H}]$ and $\mathbb{E}[Y|\mathcal{H}]$ are $\mathcal{H}$-measurable so is $\mathbb{E}[X|\mathcal{H}] + \mathbb{E}[Y|\mathcal{H}]$. The result then follows because

$$\begin{aligned} \int_B \mathbb{E}[X|\mathcal{H}] + \mathbb{E}[Y|\mathcal{H}] \,\mathrm{d}\mathbb{P} &= \int_B \mathbb{E}[X|\mathcal{H}] \,\mathrm{d}\mathbb{P} + \int_B \mathbb{E}[Y|\mathcal{H}] \,\mathrm{d}\mathbb{P} \\ &= \int_B X \,\mathrm{d}\mathbb{P} + \int_B Y \,\mathrm{d}\mathbb{P} = \int_B X + Y \,\mathrm{d}\mathbb{P}. \end{aligned}$$

(d) First we observe that for any $B \in \mathcal{H}$

$$\int_B \mathbb{E}[X|\mathcal{H}] \,\mathrm{d}\mathbb{P} = \int_B X \,\mathrm{d}\mathbb{P} \leq \int_B Y \,\mathrm{d}\mathbb{P} = \int_B \mathbb{E}[Y|\mathcal{H}] \,\mathrm{d}\mathbb{P}.$$

Now consider the event $A := \{\mathbb{E}[X|\mathcal{H}] > \mathbb{E}[Y|\mathcal{H}]\} \in \mathcal{H}$. If this event has non-zero measure then it would follow that

$$\int_A \mathbb{E}[X|\mathcal{H}] \,\mathrm{d}\mathbb{P} > \int_A \mathbb{E}[Y|\mathcal{H}] \,\mathrm{d}\mathbb{P},$$

which is a contradiction. Hence we conclude that $\mathbb{E}[X|\mathcal{H}] \leq \mathbb{E}[Y|\mathcal{H}]$ holds $\mathbb{P}$-almost everywhere.