

TU/e, 2MBA70

# **Solutions to problems for Measure and Probability Theory**



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## Chapter 2: Measurable spaces (sigma-algebras and measures)

### Problem 2.6

First note that if  $\mu(A \cap B) = \infty$  then by property 2 we have that also  $\mu(A)$ ,  $\mu(B)$  and  $\mu(A \cup B) = \infty$  and hence the result holds trivially. So assume now that  $\mu(A \cap B) < \infty$ . Since

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

it follows from property 1 that

$$\mu(A \cup B) = \mu(A \setminus (A \cap B)) + \mu(A \cap B) + \mu(B \setminus (A \cap B)).$$

Adding  $\mu(A \cap B) < \infty$  to both side we get

$$\begin{aligned} \mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus (A \cap B)) + \mu(A \cap B) + \mu(B \setminus (A \cap B)) + \mu(A \cap B) \\ &= \mu(A) + \mu(B), \end{aligned}$$

where the last line follows from applying property 3 twice.

### Problem 2.7

The idea is to construct a family of disjoint sets  $(E_i)_{i \in \mathbb{N}}$  with the following properties:

1.  $E_i \subset A_i$ , and
2.  $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} A_i$ .

If such a sequence exists then we have

$$\begin{aligned} \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) &= \mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) && \text{by 2} \\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{because } E_i \text{ are disjoint and } \mu \text{ is } \sigma\text{-additive} \\ &\leq \sum_{i=1}^{\infty} \mu(A_i) && \text{by 1 and monotone property of } \mu. \end{aligned}$$

So we are left to construct the required family of sets  $(E_i)_{i \in \mathbb{N}}$ . The following set will do:

$$E_1 = A_1 \quad E_i = A_i \setminus \bigcup_{k < i} A_k \text{ for all } i > 1.$$

Note that by definition the set  $E_i$  are pair-wise disjoint and property 1 holds. Finally, property 2 holds since  $\bigcup_{i=1}^k E_i = \bigcup_{i=1}^k A_i$  holds for all  $k \geq 1$ .

**Problem 2.9** (23 points) Let  $\mathcal{O}$  denote the open sets in  $\mathbb{R}$ .

1. (2 points) Note that the interval  $(a, b)$  is open for any  $a < b \in \mathbb{R}$ . Hence  $\mathcal{A}_1 \subset \mathcal{A}'_1 \subset \mathcal{O}$  and thus by Lemma 2.1.5 we have that  $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}'_1) \subset \sigma(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$ .

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2. (2 points) The inclusion  $\supset$  is trivial. So assume that  $x \in O$ . Then by definition there exist an  $r > 0$  such that the ball  $B_x(r) \subset O$ . But  $B_x(r) = (x - r, x + r) \in \mathcal{A}_1$  so  $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$ .
  3. (3 points) Take  $O \in \mathcal{O}$ . If we can show that  $O \in \sigma(\mathcal{A})$  then  $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$ . The result then follows from 1.  
 From 2 it follows that  $O$  is a union over a subset collection of interval  $(a, b)$  where  $a, b \in \mathbb{Q}$ . Since  $\mathbb{Q}$  is countable, the collection  $\{(a, b) : a < b \in \mathbb{Q}\}$  is also countable and hence  $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$ , from which it follows that  $\mathcal{B}_{\mathbb{R}} \subset \sigma(\mathcal{A})$ .
  4. (1 point) This follows immediately from 1 and 3 since these imply that  $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}'_1) \subset \mathcal{B}_{\mathbb{R}}$ .
  5. (3 points) The inclusion  $\subset$  is trivial, since  $(a, b] \subset (a, b + 1/j)$  for any  $j \in \mathbb{N}$ . For the other inclusion we argue by contradiction. Suppose that  $x \in \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$  but  $x \notin (a, b]$ . Then  $x > b$  and hence there exists a  $j \in \mathbb{N}$  such that  $(b - x) > 1/j$ . But this implies that  $x \notin (a, b + 1/j)$  which is a contradiction. So we conclude that  $(a, b] \supset \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$ .
  6. (3 points) This time the inclusion  $\supset$  is trivial since  $(a, b - 1/j] \subset (a, b)$  for every  $j \in \mathbb{N}$ . For the other inclusion suppose that  $x \in (a, b)$ . Then there exists a  $r > 0$  such that the interval  $(x - r, x + r) \subset (a, b)$ . In particular, this implies that  $b - (x + r) > 0$ . Now take any  $j \in \mathbb{N}$  such that  $j > 1/(b - (x + r))$ . Then  $b - x > r + 1/j$  which implies that  $(x - r, x + r) \subset (x - r, b - 1/j]$  and hence  $x \in \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$ .
  7. (4 points) It is clear that  $\mathcal{A}_2 \subset \mathcal{A}'_2$ . By 5 it follows that any interval  $(a, b]$  can be obtained as a countable intersection of intervals of the form  $(a, b + 1/j)$ . By 4  $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}'_1)$  which by Lemma 2.1.2 contains  $\bigcap_{j \in \mathbb{N}} (a, b + 1/j) = (a, b]$ . So we conclude that any interval  $(a, b] \in \mathcal{B}_{\mathbb{R}}$  from which it now follows that

$$\sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}'_2) \subset \sigma(\mathcal{A}'_1) = \mathcal{B}_{\mathbb{R}}.$$

For the other inclusion we consider a set  $(a, b)$  with  $a, b \in \mathbb{Q}$ . Then by 6 we have that  $(a, b) = \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$  where the later is a countable union of sets  $(c, d]$  with  $c, d \in \mathbb{Q}$  which must be in  $\sigma(\mathcal{A}_2)$  by definition of a  $\sigma$ -algebra. Hence, any interval  $(a, b) \in \sigma(\mathcal{A}_2)$  and we thus conclude, using 3, that

$$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}'_2) \subset \sigma(\mathcal{A}'_1) = \mathcal{B}_{\mathbb{R}},$$

which implies the result.

8. (2 points) Step 1 is to show that any interval  $[a, b)$  can be obtained as a countable intersection of intervals  $(a - 1/j, b)$ . From this we can conclude that any set  $[a, b)$  must be in  $\mathcal{B}_{\mathbb{R}}$  proving inclusions  $\subset$ .

For the other inclusions we have to show that any interval  $(a, b)$  can be obtained as a countable union of intervals  $[a + 1/j, b)$ , which implies that  $(a, b)$  must be in the  $\sigma$ -algebra generated by  $[a, b)$ .

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9. (3 points) The main tool is to show that each of the intervals  $(-\infty, a]$ ,  $(-\infty, a)$ ,  $(a, \infty)$  and  $[a, \infty)$  can be obtained by taking any allowed set operation for  $\sigma$ -algebras, i.e. countable unions/intersections and finite complements. This will help use prove the  $\subset$  inclusions.

Then we show that any set of the form  $(a, b)$ ,  $[a, b)$  or  $(a, b]$  can also be obtained through countable unions/intersections and finite complements of intervals of the forms  $(-\infty, a]$ ,  $(-\infty, a)$ ,  $(a, \infty)$  and  $[a, \infty)$ . These will then yield the  $\supset$  inclusions and finish the proof.

## Chapter 3: Measurable functions and stochastic objects

**Problem 3.2** “ $\subset$ ” By definition, the product  $\sigma$ -algebra  $\mathcal{F}_1 \otimes \mathcal{F}_2$  is defined as the  $\sigma$ -algebra generated by the collection

$$\mathcal{A} := \left\{ A \times B \subset \Omega_1 \times \Omega_2 : A \in \mathcal{F}_1, B \in \mathcal{F}_2 \right\}.$$

Since  $A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B)$ , we have that

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \sigma(\pi_1, \pi_2).$$

“ $\supset$ ” Let  $C \in \{\pi_i^{-1}(A) : i = 1, 2, A \in \mathcal{F}_1\}$ . Then there exist sets  $A \in \mathcal{F}_1$  or  $B \in \mathcal{F}_2$  such that  $C = \pi_1^{-1}(A) = A \times \Omega_2$  or  $C = \pi_2^{-1}(B) = \Omega_1 \times B$ . Either way, since  $\Omega_1 \in \mathcal{F}_1$  and  $\Omega_2 \in \mathcal{F}_2$ , we have that  $C \in \mathcal{A}$ .

**Problem 3.3** It is clear that  $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$ . Suppose a sequence of mutually disjoint sets  $B_i \in \mathcal{G}$ ,  $i \in \mathbb{N}$ , is given. Then,

$$f_{\#}\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) = \sum_{i=1}^{\infty} f_{\#}\mu(B_i).$$

### Problem 3.5

- (a) Some meaningful explanation would suffice.
- (b) By Proposition 2.1.8 and Problem 2.9, we know that  $\mathcal{B}_{\mathbb{R}}$  is generated by intervals of the form  $(-\infty, a]$  with  $a \in \mathbb{Q}$ . As a consequence,  $\mathcal{B}_{\mathbb{R}}$  is also generated by intervals of the form  $(a, +\infty)$  with  $a \in \mathbb{Q}$ . Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\}$$

is measurable for every  $a \in \mathbb{Q}$ . For brevity, we write  $\{f + g > a\}$ . The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \left( \{f > t\} \cap \{g > a - t\} \right).$$

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We first show the inclusion ‘ $\subset$ ’. If  $\omega \in \Omega$  is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some  $t \in \mathbb{Q}$  such that

$$f(\omega) > t > a - g(\omega),$$

and thus  $f(\omega) > t$  and  $g(\omega) > a - t$ . So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \left( \{f > t\} \cap \{g > a - t\} \right).$$

Now we will show the inclusion ‘ $\supset$ ’. Let  $\omega \in \Omega$  be such that  $f(\omega) > t$  and  $g(\omega) > a - t$ . Then, by adding the inequalities, we know that  $f(\omega) + g(\omega) > a$ .

(c) The constant function  $f(\omega) = a$  is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \quad \forall B \in \mathcal{B}_{\mathbb{R}}.$$

(d) Similar to the proof of Point (2) of Proposition 3.2.12.

(e) Let  $g(\omega) \neq 0$  for all  $\omega \in \Omega$ . Then, since  $g$  is measurable, we have that

$$\begin{aligned} \{1/g > a\} &= \{g < 1/a, g > 0\} \cup \{g > 1/a, g < 0\} \\ &= \left( \{g < 1/a\} \cap \{g > 0\} \right) \cup \left( \{g > 1/a\} \cap \{g < 0\} \right) \in \mathcal{F}, \end{aligned}$$

thus implying that  $1/g$  is measurable.

(f) Point (e) and Point (4) of Proposition 3.2.12 yields Point (5) of Proposition 3.2.12.

**Problem 3.6** From (3.6), we have for any  $a \in \mathbb{R}$ ,

$$\left\{ \sup_{n \geq 1} f_n > a \right\} = \bigcup_{n \geq 1} \{f_n > a\} \in \mathcal{F},$$

Since  $\mathcal{F}$  is a  $\sigma$ -algebra and  $f_n$  is measurable for all  $n \geq 1$ , i.e.,  $\{f_n > a\} \in \mathcal{F}$  for all  $n \geq 1$ .

**Problem 3.7**

(a) Note that

$$f_M = M \mathbf{1}_{\{f \geq M\}} + f \mathbf{1}_{\{|f| < M\}} - M \mathbf{1}_{\{f \leq -M\}}.$$

Since the sets

$$\{f \geq M\}, \quad \{f \leq -M\}, \quad \{|f| < M\} \quad \text{are } \mathcal{F}\text{-measurable,}$$

their corresponding indicator functions are  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable. Since  $f_M$  is the sum of products of  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable functions, we conclude that  $f_M$  is also  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable.

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- (b) It is easy to see that  $f_M$  converges pointwise to  $f$  as  $M \rightarrow \infty$ , i.e.,

$$\lim_{M \rightarrow \infty} f_M(\omega) = f(\omega) \quad \forall \omega \in \Omega.$$

Indeed, if  $\omega \in \Omega$  is such that  $f(\omega) = +\infty$ , then

$$\lim_{M \rightarrow \infty} f_M(\omega) = \lim_{M \rightarrow \infty} M = +\infty = f(\omega),$$

and similarly for  $\omega \in \Omega$  for which  $f(\omega) = -\infty$ . On the other hand, for any  $\omega \in \Omega$  with  $f(\omega) \in \mathbb{R}$ , there is some  $N_0(\omega) \in \mathbb{N}$  such that  $f_N(\omega) = f(\omega)$  for all  $N \geq N_0(\omega)$ , and hence,

$$\lim_{M \rightarrow \infty} f_M(\omega) = f(\omega).$$

Since  $f$  is the limit of a sequence of  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable functions, we conclude from Lemma 3.2.13 that  $f$  is  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable.

### Problem 3.9

- (a) For the probability space, take  $\Omega = [0, 1]$ ,  $\mathcal{F} = \mathcal{B}_{[0,1]}$  and  $\mathbb{P} = \lambda$  the Lebesgue measure restricted to  $[0, 1]$ .

Observe that the function  $H_{\gamma}(z)$  is continuous and hence has an inverse  $g_{\gamma}(y) = \gamma \tan(\pi(y - 1/2))$  on  $[0, 1]$ .

So the function  $Y[0, 1] \rightarrow \mathbb{R}$  defined by  $Y(x) = g_{\gamma}(x)$  has the correct distribution as

$$\mathbb{P}(Y^{-1}((-\infty, t])) = \mathbb{P}(g_{\gamma}^{-1}((-\infty, t])) = \lambda(H_{\gamma}((-\infty, t])) = H_{\gamma}(t).$$

- (b) Note that  $g_{\gamma}$  is continuous on  $[0, 1]$  and hence measurable.  
(c) For any  $t \geq 0$ , the cdf of the Poisson random variable is given by

$$F_{\lambda}(t) = \sum_{n=0}^{\lceil t \rceil} f_{\lambda}(n),$$

where  $\lceil t \rceil$  is the ceiling of  $t$ , i.e. the smallest integer  $k \geq t$ .

- (d) For the probability space, we again take  $\Omega = [0, 1]$ ,  $\mathcal{F} = \mathcal{B}_{[0,1]}$  and  $\mathbb{P} = \lambda$  the Lebesgue measure restricted to  $[0, 1]$ .

Now for any  $y \in [0, 1]$  let  $k := k(y)$  be such that

$$\sum_{n=1}^k f_{\lambda}(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_{\lambda}(n) < y,$$

where the last sum is interpreted as  $-1$  if  $k = 0$ .

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Now define  $X(y) = k(y) : [0, 1] \rightarrow \mathbb{R}$ . Then  $k(y) \leq t$  if and only if  $y \leq F_\lambda(t)$  and hence

$$X^{-1}((-\infty, t]) = \{y \in [0, 1] : k(y) \in (-\infty, t]\} = \{y \in [0, 1] : y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

- (e) It follows from the above computation that  $X^{-1}((-\infty, t]) = \{y \in [0, 1] : y \in (0, F_\lambda(t)]\}$ . Since the latter is a measurable set we conclude that  $X^{-1}((-\infty, t])$  is measurable for all  $t$  and since these generate the Borel  $\sigma$ -algebra  $X$  is measurable.
- (f) for any  $\ell \in \mathbb{N}$  define the sets  $A_\ell = (n - 1 - 1/\ell, n - 1 + 1/\ell]$ . Then  $A_\ell$  is a decreasing set with  $\lim_{\ell \rightarrow \infty} A_\ell = \{n\}$ . Moreover,  $A_\ell = (-\infty, n - 1 + 1/\ell] \setminus (-\infty, n - 1 - 1/\ell]$  and  $\mathbb{P}(A_1) < \infty$ . It now follows from continuity from above and (d) that

$$\begin{aligned} X_\# \mathbb{P}(\{n\}) &= \lim_{\ell \rightarrow \infty} X_\# \mathbb{P}(A_\ell) \\ &= \lim_{\ell \rightarrow \infty} X_\# \mathbb{P}((-\infty, n - 1 + 1/\ell]) - X_\# \mathbb{P}((-\infty, n - 1 - 1/\ell]) \\ &= F_\lambda(n - 1 + 1/\ell) - F_\lambda(n - 1 - 1/\ell) \\ &= \sum_{k=0}^n f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n). \end{aligned}$$

## Chapter 4: The Lebesgue Integral

### Problem 4.6

- (a) Let  $t \in \mathbb{R}$  and consider the set  $A_t = (-\infty, t]$ . Then by definition of the probability density function

$$\nu(A_t) = \int_{-\infty}^t \rho \, d\lambda = (X_\# \mathbb{P})((-\infty, t]).$$

We thus conclude that  $\nu$  and  $X_\# \mathbb{P}$  coincide on the family of set  $A_t$  and since these generate  $\mathcal{B}$  Theorem 2.2.17 implies that  $\nu = X_\# \mathbb{P}$ .

- (b) Since  $g$  is a simple function, there exist an  $N \in \mathbb{N}$ , constants  $(a_n)_{1 \leq n \leq N}$  and measurable sets  $(A_n)_{1 \leq n \leq N}$  such that

$$g = \sum_{n=1}^N a_n \mathbf{1}_{A_n}.$$

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Now, by first applying Proposition 4.8.11 and then part (a), we get that

$$\begin{aligned}
\mathbb{E}[g(X)] &= \int_{\Omega} g(X) \, d\mathbb{P} = \int_{\Omega} g \, dX_{\#}\mathbb{P} = \int_{\Omega} g \, d\nu \\
&= \int_{\Omega} \sum_{n=1}^N a_n \mathbf{1}_{A_n} \, d\nu = \sum_{n=1}^N a_n \nu(A_n) = \sum_{n=1}^N a_n \int_{A_n} \rho \, d\lambda \\
&= \int_{\mathbb{R}} \sum_{n=1}^N a_n \mathbf{1}_{A_n} \rho \, d\lambda = \int_{\mathbb{R}} g \rho \, d\lambda
\end{aligned}$$

(c) First note that by part (b) we have that

$$\int_{\Omega} [h]_n(X) \, d\mathbb{P} = \int_{\mathbb{R}} [h_n] \rho \, d\lambda.$$

Now we split the function  $[h_n]\rho$  into its positive and negative part and note that

$$([h_n]\rho)^+ = [h_n]^+ \rho^+ + [h_n]^- \rho^- \quad \text{and} \quad ([h_n]\rho)^- = [h_n]^+ \rho^- + [h_n]^- \rho^+,$$

where  $[h_n]^{\pm}$  and  $\rho^{\pm}$  denote the positive and negative parts of  $[h_n]$  and  $\rho$ .

We will show that

$$\int_{\Omega} h^+(X) \, d\mathbb{P} = \int_{\mathbb{R}} h^+ \rho \, d\lambda.$$

The proof for the negative part is similar.

$$\begin{aligned}
\int_{\mathbb{R}} h^+ \, d\nu &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h_n]^+ \, d\nu && \text{by Theorem 4.3.4} \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h_n]^+ \rho \, d\lambda && \text{by part (b)} \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h_n]^+ \rho^+ \, d\lambda - \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h_n]^+ \rho^- \, d\lambda && \text{by linearity of integration} \\
&= \int_{\mathbb{R}} h^+ + \rho^+ \, d\lambda - \int_{\mathbb{R}} h^+ + \rho^- \, d\lambda && \text{by Theorem 4.3.4} \\
&= \int_{\mathbb{R}} h^+ \rho \, d\lambda && \text{by linearity of integration}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbb{E}[h(X)] &= \int_{\Omega} h(X) \, d\mathbb{P} \\
&= \int_{\mathbb{R}} h \, dX_{\#}\mathbb{P} && \text{by Proposition 4.8.11} \\
&= \int_{\mathbb{R}} h \, d\nu && \text{by part (a)} \\
&= \int_{\mathbb{R}} h \rho \, d\lambda && \text{by part (c).}
\end{aligned}$$



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## **Chapter 5: Convergence of integrals and functions**