Problem 11.2

(a) By definition we have that

$$\int_B \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} = \int_B X\,\mathrm{d}\mathbb{P},$$

holds for all $B \in \mathcal{H}$. Since by assumption both $\mathbb{E}[X|\mathcal{H}]$ and X are $\mathcal{H}$-measurable, the result follows from problem 8.2.

(b) Note that $a\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable. Moreover,

$$\int_B a\mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} = a\int_B \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} = a\int_B X\,\mathrm{d}\mathbb{P} = \int_B aX\,\mathrm{d}\mathbb{P}.$$

This proves the claim.

(c) Similarly to the previous point, we first note that since $\mathbb{E}[X|\mathcal{H}]$ and $\mathbb{E}[Y|\mathcal{H}]$ are $\mathcal{H}$-measurable so is $\mathbb{E}[X|\mathcal{H}] + \mathbb{E}[Y|\mathcal{H}]$. The result then follows because

$$\begin{aligned}\int_B \mathbb{E}[X|\mathcal{H}] + \mathbb{E}[Y|\mathcal{H}]\,\mathrm{d}\mathbb{P} &= \int_B \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} + \int_B \mathbb{E}[Y|\mathcal{H}]\,\mathrm{d}\mathbb{P} \\ &= \int_B X\,\mathrm{d}\mathbb{P} + \int_B Y\,\mathrm{d}\mathbb{P} = \int_B X + Y\,\mathrm{d}\mathbb{P}.\end{aligned}$$

(d) First we observe that for any $B \in \mathcal{H}$

$$\int_B \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} = \int_B X\,\mathrm{d}\mathbb{P} \le \int_B Y\,\mathrm{d}\mathbb{P} = \int_B \mathbb{E}[Y|\mathcal{H}]\,\mathrm{d}\mathbb{P}.$$

Now consider the event $A := \{\mathbb{E}[X|\mathcal{H}] > \mathbb{E}[Y|\mathcal{H}]\} \in \mathcal{H}$. If this event has non-zero measure then it would follow that

$$\int_A \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} > \int_A \mathbb{E}[Y|\mathcal{H}]\,\mathrm{d}\mathbb{P},$$

which is a contradiction. Hence we conclude that $\mathbb{E}[X|\mathcal{H}] \le \mathbb{E}[Y|\mathcal{H}]$ holds $\mathbb{P}$-almost everywhere.

Problem 11.3

(a) This follows by repeating the step for the solution to Problem 4.8 a).

(b) We start by observing that the result will directly follow from Theorem 2.15 if we can show that the σ-algebra $\sigma(X)$ satisfies the two properties.

The first one is immediate, from the fact that $X^{-1}(A) \cap X^{-1}(B) = X^{-1}(A\ capB)$. For the second one consider the intervals $I_n = (-n, n)$ and define $A_n := X^{-1}(I_n) \in \sigma(X)$. Since $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{R}$ it follows that $\bigcup_{n\in\mathbb{N}} A_n = \Omega$. Moreover since $|X| \le n$ on the set A_n it holds that

$$\mu(A_n) = \nu_X(A_n) = \int_{A_n} X\,\mathrm{d}\mathbb{P} \le n\int_{A_n} \mathrm{d}\mathbb{P} = n\mathbb{P}(A_n) \le n < \infty.$$

Thus the second condition of Theorem 2.15 is also satisfied and the result now follows.