Problem 6.2

(a) The implication from right to left is by definition of $\overleftarrow{F}$ and the fact that F is non-decreasing. The implication from left to right is because F is right continuous.

(b) Consider the preimage of $(-\infty, t]$ under X. Then, using the above observation, we have

$$X^{-1}((-\infty,t]) = \{\omega \in \Omega \,:\, \overleftarrow{F}(U(\omega)) \in (-\infty,t]\}$$
$$= \{\omega \in \Omega \,:, U(\omega) \in (-\infty, F(t)]\} = U^{-1}((-\infty, F(t)]) \in \mathcal{B}_{[0,1]}.$$

Hence, X is measurable. Finally, the above computation, together with Lemma 6.5, also implies that

$$\mathbb{P}\left(X^{-1}((-\infty,t])\right) = \mathbb{P}\left(U^{-1}((-\infty,F(t)])\right) = F(t).$$

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and U a standard normal random variable. We will show that $X = \overleftarrow{F} \circ U$ is a random variable with the right probability measure. Since we can construct a standard uniform random variable on the probability $([0,1], \mathcal{B}_{[0,1]}, \lambda|_{[0,1]})$ this also implies the last part.
which finished the proof.

Problem 6.3

(a) For the probability space, take $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0,1]$.

Observe that the function $H_\gamma(z)$ is continuous and hence has an inverse $g_\gamma(y) = \gamma \tan(\pi(y - 1/2))$ on $[0,1]$.

So the function $Y[0,1] \to \mathbb{R}$ defined by $Y(x) = g_\gamma(x)$ has the correct distribution as

$$\mathbb{P}(Y^{-1}((-\infty,t])) = \mathbb{P}(g_\gamma^{-1}((-\infty,t])) = \lambda(H_\gamma((-\infty,t])) = H_\gamma(t).$$

(b) Note that g_γ is continuous on $[0,1]$ and hence measurable.

(c) For any $t \geq 0$, the cdf of the Poisson random variable is given by

$$F_\lambda(t) = \sum_{n=0}^{\lceil t \rceil} f_\lambda(n),$$

where $\lceil t \rceil$ is the ceiling of t, i.e. the smallest integer $k \geq t$.

(d) For the probability space, we again take $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0,1]$.

Now for any $y \in [0,1]$ let $k := k(y)$ be such that

$$\sum_{n=1}^{k} f_\lambda(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_\lambda(n) < y,$$

where the last sum is interpreted as -1 if $k = 0$.

Now define $X(y) = k(y) : [0,1] \to \mathbb{R}$. Then $k(y) \le t$ if and only if $y \le F_\lambda(t)$ and hence

$$X^{-1}((-\infty, t]) = \{y \in [0,1] \,:\, k(y) \in (0,t]\} = \{y \in [0,1] \,:\, y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

(e) It follows from the above computation that $X^{-1}((-\infty,t]) = \{y \in [0,1] \,:\, y \in (0, F_\lambda(t)]\}$. Since the latter is a measurable set we conclude that $X^{-1}((-\infty,t])$ is measurable for all t and since these generate the Borel σ-algebra X is measurable.

(f) for any $\ell \in \mathbb{N}$ define the sets $A_\ell = (n-1-1/\ell), n-1+1/\ell]$. Then A_ℓ is a decreasing set with $\lim_{\ell\to\infty} A_\ell = \{n\}$. Moreover, $A_\ell = (-\infty, n-1+1/\ell] \setminus (-\infty, n-1-1/\ell]$ and $\mathbb{P}(A_1) < \infty$. It now follows from continuity from above and (d) that

$$\begin{aligned}
X_\#\mathbb{P}(\{n\}) &= \lim_{\ell\to\infty} X_\#\mathbb{P}(A_\ell) \\
&= \lim_{\ell\to\infty} X_\#\mathbb{P}((-\infty, n-1+1/\ell]) - X_\#\mathbb{P}((-\infty, n-1-1/\ell]) \\
&= F_\lambda(n-1+1/\ell) - F_\lambda(n-1-1/\ell) \\
&= \sum_{k=0}^{n} f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n).
\end{aligned}$$