Problem 11.4

The solution to this problem will closely follow the proof of Lemma 11.7. To this end let $g(x,y)$ denote the conditional density of X given $Y = y$ and define the function $\phi(y) := \int_{\mathbb{R}} h(x)g(x,y)\,\lambda(\mathrm{d}x)$. We now need to show that for any $A \in \mathcal{B}_{\mathbb{R}}$ it holds that

$$\int_{Y^{-1}(A)} \phi(Y)\,\mathrm{d}\mathbb{P} = \mathbb{E}[\mathbb{1}_A(Y)h(X)].$$

Using the change of variables formula and the definition of ϕ and g we get

$$\begin{aligned}
\int_{Y^{-1}(A)} \phi(Y)\,\mathrm{d}\mathbb{P} &= \int_{\Omega} \mathbb{1}_{Y^{-1}(A)}\phi(Y)\,\mathrm{d}\mathbb{P} \\
&= \int_{\mathbb{R}}\int_{A} (y)\phi(y)f_Y(y)\,\lambda(\mathrm{d}y) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_A(y)h(x)g(x,y)f_Y(y)\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y) \\
&= \int_{\mathbb{R}^2} \mathbb{1}_A(y)h(x)f(x,y)\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y) \\
&= \mathbb{E}[\mathbb{1}_A(Y)h(X)].
\end{aligned}$$

Problem 11.5

(a) First note that for any $\ell \geq 0$

$$\mathbb{P}(X > \ell) = \int_{\ell} \nu e^{-\nu x}\,\lambda(dx) = e^{-\nu\ell}.$$

By Lemma 11.5 we have that

$$\mathbb{P}(X > t+s|X > t) = \frac{\mathbb{P}(X > t+s, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > t)} = \frac{e^{-\nu(t+s)}}{e^{-\nu t}} = e^{-\nu s} = \mathbb{P}(X > s).$$

(b) Define the function $f(x,y) = \mathbb{1}_{x+y\leq t}$. By Lemma 11.8 we then have that

$$\mathbb{P}(X + Y \leq t|Y = y) = \mathbb{E}[f(X,Y)|Y = y] = \mathbb{E}[f(X,y)].$$

Using the probability density function we then get

$$\mathbb{E}[f(X,y)] = \int_{\mathbb{R}} f(x,y)\rho(x)\,\lambda(dx) = \mathbb{1}_{y<t}\int_0^{t-y} \nu e^{-\nu x}\,\lambda(dx) = \mathbb{1}_{y<t}(1 - e^{-\nu(t-y)}).$$

(c) We have that

$$\begin{aligned}
\mathbb{P}(X+Y\le t) &= \mathbb{E}[\mathbb{P}(X+Y\le t|Y)] \\
&= \int_{\mathbb{R}} \rho(y)\mathbb{P}(X+Y\le t|Y=y)\,\lambda(dx) \\
&= \int_{\mathbb{R}} \mathbb{1}_{0\le y<t}(1-e^{-\nu(t-y)})\nu e^{-\nu y}\,\lambda(dy) \\
&= \int_0^t \nu e^{-\nu y} - e^{-\nu t}\,\lambda(dy) \\
&= 1-e^{-\nu t}-te^{-\nu t}.
\end{aligned}$$