Problem 8.6

(a) Define the sets
$$B_j := \bigcup_{i \geq j} A_i, \qquad j \in \mathbb{N}.$$
Clearly the sequence $(B_j)_{j\in\mathbb{N}}$ is decreasing and $\{A_n \text{ i.o.}\} \subset B_j$ for every $j \in \mathbb{N}$.

By assumption, and the σ-subadditivity of $\mathbb{P}$,
$$\mathbb{P}(B_1) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i) < +\infty.$$
Moreover, the summability also gives
$$\lim_{j\to\infty} \mathbb{P}(B_j) \leq \limsup_{j\to\infty} \sum_{i=j}^{\infty} \mathbb{P}(A_i) = 0.$$
Hence, by the continuity from above of μ, we obtain
$$\mathbb{P}(\{A_n \text{ i.o.}\}) \leq \mathbb{P}\left(\bigcup_{j=1}^{\infty} B_j\right) = \lim_{j\to\infty} \mathbb{P}(B_j) = 0,$$
i.e., $\{A_n \text{ i.o.}\}$ is a null set. In other words, $\mathbb{P}$-almost every ω is in only finitely many A_n.

(b) We will prove that
$$\mathbb{P}(\Omega \setminus \{A_n \text{ i.o.}\}) = 0,$$
from which the result follows since $\mathbb{P}(\Omega) = 1$.

First note that
$$\Omega \setminus \{A_n \text{ i.o.}\} = \bigcup_{k\geq 1} \left(\bigcup_{n\geq k} A_n\right)^c = \bigcup_{k\geq 1} \bigcap_{n\geq k} A_n^c.$$
Next, since A_n are mutually exclusive, so are A_n^c. Thus, for any $k \geq 1$ we have that
$$\begin{aligned}\mathbb{P}\left(\bigcap_{n\geq k} A_n^c\right) &= \prod_{n\geq k} \mathbb{P}(A_n^c) = \prod_{n\geq k} (1 - \mathbb{P}(A_m)) \\ &\leq \prod_{n\geq k} e^{-\mathbb{P}(A_n)} = e^{-\sum_{n\geq k} \mathbb{P}(A_n)} = 0.\end{aligned}$$
Here we used that for any $0 \leq x \leq 1$ it holds that $1 - x \leq e^{-x}$.

Finally, using σ-subadditivity we conclude that
$$\mathbb{P}(\Omega \setminus \{A_n \text{ i.o.}\}) = \mathbb{P}\left(\bigcup_{k\geq 1} \bigcap_{n\geq k} A_n^c\right) \leq \sum_{k\geq 1} \mathbb{P}\left(\bigcap_{n\geq k} A_n^c\right) = 0.$$

Problem 8.7
Fix $\varepsilon > 0$ and define $A_n(\varepsilon) : \{|X_n - X| > \varepsilon\}$. Then the assumption translates to

$$\sum_{n\geq 1} \mathbb{P}(A_n(\varepsilon)) < \infty.$$

By Lemma 8.11 1) this then implies that $\mathbb{P}(A_n(\varepsilon) \text{ i.o.}) = 0$. Since $\varepsilon > 0$ was arbitrary, Lemma 8.9 now implies that $X_n \overset{\text{a.s.}}{\to} X$.