Problem 7.6

(a)

(b) By definition

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}} f\,\mathrm{d}\mu_n - \int_{\mathbb{R}} f\,\mathrm{d}\mu\right| \le \varepsilon,$$

implies that for any $\delta > 0$

$$\left|\int_{\mathbb{R}} f\,\mathrm{d}\mu_n - \int_{\mathbb{R}} f\,\mathrm{d}\mu\right| < \varepsilon + \delta,$$

holds for large enough n. Note that this holds for any $\varepsilon, \delta > 0$.

Now pick $\eta > 0$ and set $\varepsilon = \eta/2 = \delta$, then the above inequality implies that

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}} f\,\mathrm{d}\mu_n - \int_{\mathbb{R}} f\,\mathrm{d}\mu\right| = 0.$$

(c) Consider the sequence of sets $A_n = \mathbb{R} \setminus [-n, n]$. Then $A_n \supset A_{n+1}$ and $A_n \downarrow \emptyset$. Hence, it follows from Proposition 2.12 2) that $\lim_{n\to\infty} \mu(A_n) = 0$. Thus, there exists a N such that $\mu(A_n) < \varepsilon/(2M)$ holds for all $n \ge N$. We can then take any $\alpha > N$.

(d) The function

$$g(x) = \mathbb{1}_{[-\alpha,\alpha]}(x) + \mathbb{1}_{(-(\alpha+1),-\alpha)}(x)\,(x + (\alpha+1)) + \mathbb{1}_{(\alpha,\alpha+1)}(x)\,(-x + \alpha + 1)$$

does the trick. This is simply a linear increase from zero to one from $-(\alpha+1)$ to $-\alpha$ and from $\alpha + 1$ to α.

(e) Observe that g is a non-negative continuous bounded function that is zero outside the interval $[-(\alpha+1), \alpha+1]$, and thus we can apply (3). Using linearity of the integral, the fact that $|f| \le M$ and the definition of g, we get

$$\begin{aligned}
\left|\int_{\mathbb{R}} f\,\mathrm{d}\mu - \int_{\mathbb{R}} fg\,\mathrm{d}\mu\right| = \left|\int_{\mathbb{R}} f(1-g)\,\mathrm{d}\mu\right| &\le M\int_{\mathbb{R}}(1-g)\,\mathrm{d}\mu \\
&\le M\int_{\mathbb{R}}(1-g)\,\mathrm{d}\mu \\
&= M\left(1 - \int_{\mathbb{R}} g\,\mathrm{d}\mu\right) \\
&\le M\mu(\mathbb{R}\backslash[-\alpha,\alpha]) < \frac{\varepsilon}{2}.
\end{aligned}$$

(f) Again, using linearity of the integral and the fact that $|f| \le M$ we get

$$\begin{aligned}
\left|\int_{\mathbb{R}} f\,\mathrm{d}\mu_n - \int_{\mathbb{R}} fg\,\mathrm{d}\mu_n\right| &= \left|\int_{\mathbb{R}} f(1-g)\,\mathrm{d}\mu_n\right| \le M\int_{\mathbb{R}}(1-g)\,\mathrm{d}\mu_n \\
&\le M\int_{\mathbb{R}}(1-g)\,\mathrm{d}\mu_n = M\left(1 - \int_{\mathbb{R}} g\,\mathrm{d}\mu_n\right)
\end{aligned}$$

Now observe that the integral in the last term converges to $\int_{\mathbb{R}} g \, \mathrm{d}\mu$ by (3). Thus, we obtain

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}} f \, \mathrm{d}\mu_n - \int_{\mathbb{R}} fg \, \mathrm{d}\mu_n \right| \leq M \int_{\mathbb{R}} (1-g) \, \mathrm{d}\mu \leq M\mu(\mathbb{R}\backslash[-\alpha,\alpha]) < \frac{\varepsilon}{2}.$$

(g) Recall that

$$\begin{aligned} \left| \int_{\mathbb{R}} f \, \mathrm{d}\mu_n - \int_{\mathbb{R}} f \, \mathrm{d}\mu \right| \leq \left| \int_{\mathbb{R}} f \, \mathrm{d}\mu_n - \int_{\mathbb{R}} fg \, \mathrm{d}\mu_n \right| + \left| \int_{\mathbb{R}} f \, \mathrm{d}\mu - \int_{\mathbb{R}} fg \, \mathrm{d}\mu \right| \\ + \left| \int_{\mathbb{R}} fg \, \mathrm{d}\mu_n - \int_{\mathbb{R}} fg \, \mathrm{d}\mu \right|. \end{aligned}$$

For the first two terms, the (e) and (f) imply that the $\limsup_{n\to\infty}$ is bounded by $\varepsilon/2$. For the third term we not that fg is a continuous bounded function and hence this term converges to zero by our assumption that (3) holds.

Together we then have that

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}} f \, \mathrm{d}\mu_n - \int_{\mathbb{R}} f \, \mathrm{d}\mu \right| < \varepsilon,$$

which implies the result.