TU/e, 2MBA70

Solutions to problems for Measure and Probability Theory

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Chapter 2: Measurable spaces (sigma-algebras and measures)

Problem 2.6 (23 points) Let $\mathcal{O}$ denote the open sets in $\mathbb{R}$.

(a) (2 points) Note that the interval (a, b) is open for any $a < b \in \mathbb{R}$. Hence $\mathcal{A}_1 \subset \mathcal{A}_1' \subset \mathcal{O}$ and thus by Lemma 2.1.5 we have that $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \sigma(\mathcal{O}) = \mathcal{B}_\mathbb{R}$.

(b) (2 points) The inclusion $\supset$ is trivial. So assume that $x \in O$. Then by definition there exist an $r > 0$ such that the ball $B_x(r) \subset O$. But $B_x(r) = (x - r, x + r) \in \mathcal{A}_1$ so $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$.

(c) (3 points) Take $O \in \mathcal{O}$. If we can show that $O \in \sigma(\mathcal{A})$ then $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. The result then follows from 1.

From 2 it follows that O is a union over a subset collection of interval (a, b) where $a, b \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, the collection $\{(a, b) : a < b \in \mathbb{Q}\}$ is also countable and hence $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$, from which it follows that $\mathcal{B}_\mathbb{R} \subset \sigma(\mathcal{A})$.

(d) (1 point) This follows immediately from 1 and 3 since these imply that $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \mathcal{B}_\mathbb{R}$.

(e) (3 points) The inclusion $\subset$ is trivial, since $(a, b] \subset (a + b + 1/j)$ for any $j \in \mathbb{N}$. For the other inclusion we argue by contradiction. Suppose that $x \in \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$ but $x \notin (a, b]$. Then $x > b$ and hence there exists a $j \in \mathbb{N}$ such that $(b - x) > 1/j$. But this implies that $x \notin (a, b + 1/j)$ which is a contradiction. So we conclude that $(a, b] \supset \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$.

(f) (3 points) This time the inclusion $\supset$ is trivial since $(a, b - 1/j] \subset (a, b)$ for every $j \in \mathbb{N}$. For the other inclusion suppose that $x \in (a, b)$. Then there exists a $r > 0$ such that the interval $(x - r, x + r) \subset (a, b)$. In particular, this implies that $b - (x + r) > 0$. Now take any $j \in \mathbb{N}$ such that $j > 1/(b - (x + r))$. Then $b - x > r + 1/j$ which implies that $(x - r, x + r) \subset (x - r, b - 1/j]$ and hence $x \in \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$.

(g) (4 points) It is clear that $\mathcal{A}_2 \subset \mathcal{A}_2'$. By 5 it follows that any interval $(a, b]$ can be obtained as a countable intersection of intervals of the form $(a, b + 1/j)$. By 4 $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{A}_1')$ which by Lemma 2.1.2 contains $\bigcap_{j \in \mathbb{N}} (a, b + 1/j) = (a, b]$. So we conclude that any interval $(a, b] \in \mathcal{B}_\mathbb{R}$ from which it now follows that

$$\sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}_2') \subset \sigma(\mathcal{A}_1') = \mathcal{B}_\mathbb{R}.$$

For the other inclusion we consider a set (a, b) with $a, b \in \mathbb{Q}$. Then by 6 we have that $(a, b) = \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$ where the later is a countable union of sets $(c, d]$ with $c, d \in \mathbb{Q}$ which must be in $\sigma(\mathcal{A}_2)$ by definition of a σ-algebra. Hence, any interval $(a, b) \in \sigma(\mathcal{A}_2)$ and we thus conclude, using 3, that

$$\mathcal{B}_\mathbb{R} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}_2') \subset \sigma(\mathcal{A}_1') = \mathcal{B}_\mathbb{R},$$

which implies the result.

(h) (2 points) Step 1 is to show that any interval $[a, b)$ can be obtained as a countable intersection of intervals $(a - 1/j, b)$. From this we can conclude that any set $[a, b)$ must be in $\mathcal{B}_{\mathbb{R}}$ proving inclusions $\subset$.

For the other inclusions we have to show that any interval (a, b) can be obtained as a countable union of intervals $[a+1/j, b)$, which implies that (a, b) must be in the σ-algebra generated by $[a, b)$.

(i) (3 points) The main tool is to show that each of the intervals $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$ can be obtained by taking any allowed set operation for σ-algebras, i.e. countable unions/intersections and finite complements. This will help use prove the $\subset$ inclusions.

Then we show that any set of the form (a, b), $[a, b)$ or $(a, b]$ can also be obtained through countable unions/intersections and finite complements of intervals of the forms $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$. These will then yield the $\supset$ inclusions and finish the proof.

Problem 2.9

First note that if $\mu(A \cap B) = \infty$ then by property 2 we have that also $\mu(A)$, $\mu(B)$ and $\mu(A \cup B) = \infty$ and hence the result holds trivially. So assume now that $\mu(A \cap B) < \infty$. Since

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

it follows from property 1 that

$$\mu(A \cup B) = \mu(A \setminus (A \cap B))) + \mu(A \cap B) + \mu(B \setminus (A \cap B)).$$

Adding $\mu(A \cap B) < \infty$ to both side we get

$$\begin{aligned}\mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus (A \cap B))) + \mu(A \cap B) + \mu(B \setminus (A \cap B)) + \mu(A \cap B) \\ &= \mu(A) + \mu(B),\end{aligned}$$

where the last line follows from applying property 3 twice.

Problem 2.11

The idea is to construct a family of disjoint sets $(E_i)_{i \in \mathbb{N}}$ with the following properties:

1. $E_i \subset A_i$, and
2. $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} A_i$.

If such a sequence exists then we have

$$\begin{aligned}\mu(\bigcup_{i \in \mathbb{N}} A_i) &= \mu(\bigcup_{i \in \mathbb{N}} E_i) && \text{by 2} \\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{because } E_i \text{ are disjoint and } \mu \text{ is } \sigma\text{-additive} \\ &\leq \sum_{i=1}^{\infty} \mu(A_i) && \text{by 1 and monotone property of } \mu.\end{aligned}$$

So we are left to construct the required family of sets $(E_i)_{i\in\mathbb{N}}$. The following set will do:

$$E_1 = A_1 \quad E_i = A_i \setminus \bigcup_{k<i}^{i} A_k \text{ for all } i > 1.$$

Note that by definition the set E_i are pair-wise disjoint and property1 holds. Finally, property 2 holds since $\bigcup_{i=1}^k E_i = \bigcup_{i=1}^k A_i$ holds for all $k \geq 1$.

Problem 2.12

(a) (4 pt)

1 pt We first make the following observations about $\mathcal{N}$:

- because $\mu(\emptyset) = 0$ it holds that $\emptyset \in \mathcal{N}$,
- if $N, M \in \mathcal{N}$ then $N \setminus M \in \mathcal{N}$ since $N \setminus M \subset N$, and
- if $(N_i)_{i\geq 1}$ is a family of sets in $\mathcal{N}$ then so is $\bigcup_{i\geq 1} N_i$.

1 pt From the first point it follows that $\emptyset = \emptyset \cup \emptyset \in \overline{\mathcal{F}}$ and $\Omega = \Omega \cup \emptyset \in \overline{\mathcal{F}}$.

1 pt Furthermore, if $A, B \in \mathcal{F}$ and $N, M \in \mathcal{N}$, then by the second point and because $A \setminus B \in \mathcal{F}$,

$$(A \cup N) \setminus (B \cup M) = (A \setminus B) \cup (N \setminus M) \in \overline{\mathcal{F}}.$$

1 pt Finally, let $(A_i \cup N_i)_{i\geq 1}$ be a collection of sets in $\mathcal{N}$. Then using the third point we get

$$\bigcup_{i\geq 1} A_i \cup N_i = \bigcup_{i\geq 1} A_i \cup \bigcup_{i\geq 1} N_i \in \overline{\mathcal{F}}.$$

(b) (1 pt) From the definition we immediately get that $\mu(\emptyset) = 0$. Now, let $(A_i \cup N_i)_{i\geq 1}$ be a collection of disjoint sets in $\mathcal{N}$. Then

$$\bar{\mu}(\bigcup_{i\geq 1} A_i \cup N_i) = \bar{\mu}(\bigcup_{i\geq 1} A_i \cup \bigcup_{i\geq 1} N_i) = \mu(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} \mu(A_i) = \sum_{i\geq 1} \bar{\mu}(A_i \cup N_i).$$

(c) (1 pt) This follows from the fact that $\bar{\mu}|_{\mathcal{F}}(A) = \bar{\mu}(A \cup \emptyset) = \mu(A)$.

(d) (2 pt) Suppose that $N \subset \Omega$ is a null set for $\overline{\mathcal{F}}$. Then there exists an $A \cup M \in \overline{\mathcal{F}}$ such that $N \subset A \cup M$ and $\bar{\mu}(A \cup M) = \mu(A) = 0$. However, since $M \in \mathcal{N}$, there must also exist a $B \in \mathcal{F}$ with $M \subset B$ and $\mu(B) = 0$. But this implies that $N \subset A \cup B \in \mathcal{F}$ which implies that $N \in \mathcal{N}$. Therefore, since $N = \emptyset \cup N$ it follows that $N \in \overline{\mathcal{F}}$ and hence every null set is part of the σ-algebra.

Chapter 3: Measurable functions

Problem 3.2

(a) First we note that $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}$ and $f^{-1}(E) = \Omega \in \mathcal{F}$. So $\emptyset, E \in \mathcal{H}$.

Next, let $B \in \mathcal{H}$. Then

$$f^{-1}(E \setminus B) = \Omega \setminus f^{-1}(B) \in \mathcal{F},$$

since by definition $f^{-1}(B) \in \mathcal{F}$. So $E \setminus B \in \mathcal{H}$.

Finally, if $(B_i)_{i\in\mathbb{N}}$ is a sequence of sets in $\mathcal{H}$, then

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) \in \mathcal{F},$$

which shows that $\bigcup_{i=1}^{\infty} B_i \in \mathcal{H}$, completing the proof that $\mathcal{H}$ is a σ-algebra.

(b) By construction $\mathcal{A} \subseteq \mathcal{H}$. It therefore follows from Lemma 2.5 that $\mathcal{G} = \sigma(\mathcal{A}) \subseteq \mathcal{H}$. But this then implies that $f^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{G}$ which means that f is $(\mathcal{F}, \mathcal{G})$-measurable.

Problem 3.3 "$\subset$" By definition, the product σ-algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$ is defined as the σ-algebra generated by the collection

$$\mathcal{A} := \Big\{A \times B \subset \Omega_1 \times \Omega_2 : A \in \mathcal{F}_1,\ B \in \mathcal{F}_2\Big\}.$$

Since $A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B)$, we have that

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \sigma(\pi_1, \pi_2).$$

"$\supset$" Let $C \in \{\pi_i^{-1}(A) : i = 1, 2,\ A \in \mathcal{F}_1\}$. Then there exist sets $A \in \mathcal{F}_1$ or $B \in \mathcal{F}_2$ such that $C = \pi_1^{-1}(A) = A \times \Omega_2$ or $C = \pi_2^{-1}(B) = \Omega_1 \times B$. Either way, since $\Omega_1 \in \mathcal{F}_1$ and $\Omega_2 \in \mathcal{F}_2$, we have that $C \in \mathcal{A}$.

Problem 3.4 It is clear that $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Suppose a sequence of mutually disjoint sets $B_i \in \mathcal{G}$, $i \in \mathbb{N}$, is given. Then,

$$f_{\#}\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) = \sum_{i=1}^{\infty} f_{\#}\mu(B_i).$$

Problem 3.6

(a) By Proposition 2.8, we know that $\mathcal{B}_{\mathbb{R}}$ is generated by intervals of the form $(-\infty, a]$ with $a \in \mathbb{Q}$. As a consequence, $\mathcal{B}_{\mathbb{R}}$ is also generated by intervals of the form $(a, +\infty)$ with $a \in \mathbb{Q}$. Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\big\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\big\}$$

is measurable for every $a \in \mathbb{Q}$. For brevity, we write $\{f + g > a\}$. The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \Big(\{f > t\} \cap \{g > a - t\}\Big).$$

We first show the inclusion '$\subset$'. If $\omega \in \Omega$ is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some $t \in \mathbb{Q}$ such that

$$f(\omega) > t > a - g(\omega),$$

and thus $f(\omega) > t$ and $g(\omega) > a - t$. So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \Big(\{f > t\} \cap \{g > a - t\}\Big).$$

Now we will show the inclusion '$\supset$'. Let $\omega \in \Omega$ be such that $f(\omega) > t$ and $g(\omega) > a - t$. Then, by adding the inequalities, we know that $f(\omega) + g(\omega) > a$.

(b) The constant function $f(\omega) = a$ is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \qquad \forall\, B \in \mathcal{B}_{\mathbb{R}}.$$

(c) Similar to the proof of Point (2) of Proposition 3.2.12.

(d) Let $g(\omega) \neq 0$ for all $\omega \in \Omega$. Then, since g is measurable, we have that

$$\begin{aligned}\{1/g > a\} &= \{g < 1/a,\ g > 0\} \cup \{g > 1/a,\ g < 0\} \\ &= \Big(\{g < 1/a\} \cap \{g > 0\}\Big) \cup \Big(\{g > 1/a\} \cap \{g < 0\}\Big) \in \mathcal{F},\end{aligned}$$

thus implying that $1/g$ is measurable.

(e) The previous part of this exercise together with point (4) of Proposition 3.12 yields Point (5) of Proposition 3.12.

Problem 3.7 From (3.6), we have for any $a \in \mathbb{R}$,

$$\left\{\sup_{n\geq 1} f_n > a\right\} = \bigcup_{n\geq 1}\{f_n > a\} \in \mathcal{F},$$

Since $\mathcal{F}$ is a σ-algebra and f_n is measurable for all $n \geq 1$, i.e., $\{f_n > a\} \in \mathcal{F}$ for all $n \geq 1$.

Problem 3.8

(a) (3 pts) **1 pt** Note that

$$f_M = M\mathbf{1}_{\{f\geq M\}} + f\mathbf{1}_{\{|f|<M\}} - M\mathbf{1}_{\{f\leq -M\}}.$$

2 pts Since the sets

$$\{f \geq M\}, \quad \{f \leq -M\}, \quad \{|f| < M\} \qquad \text{are } \mathcal{F}\text{-measurable},$$

their corresponding indicator functions are $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable. Since f_M is the sum of products of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable functions, we conclude that f_M is also $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

(b) (3 pts) It is easy to see that f_M converges pointwise to f as $M \to \infty$, i.e.,

$$\lim_{M\to\infty} f_M(\omega) = f(\omega) \qquad \forall\, \omega \in \Omega.$$

1 pt Indeed, if $\omega \in \Omega$ is such that $f(\omega) = +\infty$, then

$$\lim_{M\to\infty} f_M(\omega) = \lim_{M\to\infty} M = +\infty = f(\omega),$$

and similarly for $\omega \in \Omega$ for which $f(\omega) = -\infty$.

2 pts On the other hand, for any $\omega \in \Omega$ with $f(\omega) \in \mathbb{R}$, there is some $N_0(\omega) \in \mathbb{N}$ such that $f_N(\omega) = f(\omega)$ for all $N \geq N_0(\omega)$, and hence,

$$\lim_{M\to\infty} f_M(\omega) = f(\omega).$$

Since f is the limit of a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable functions, we conclude from Lemma 3.2.13 that f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

Chapter 4: The Lebesgue Integral

Problem 4.2

Problem 4.3

(a) The fact that the sets are disjoint is immediate from the definition. Measurability follows from Lemma 3.11

(b) Let us fix a $\omega \in \Omega$. Then if $f(\omega) = +\infty$ we get that $f_n(\omega) = 2^n$ holds for all $n \geq 1$ and hence $\lim_{n\to\infty} f_n(\omega) = +\infty = f(\omega)$. So assume that $f(\omega) < +\infty$. Then there exists an $M \in \mathbb{N}$ such that $f(\omega) < M$. Hence, for all $n \geq M$ we have that

$$\|f_n(\omega) - f(\omega)\| = f(\omega) - f_n(\omega) \leq 2^{-n},$$

which implies that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

(c) Fix $n \geq 1$ and $\omega \in \Omega$. Clearly, if $f(\omega) = +\infty$ then $f_n(\omega) = 2^n < +\infty = f(\omega)$.

(d) Fix $\omega \in \Omega$ such that $f(\omega) < +\infty$ and $\omega \in A_k^n$ for some $0 \leq k < N_n = n2^n$.

Note that $k2^{-n} \leq f(\omega) < (k+1)2^{-n}$ holds and this interval can be split into two intervals as follows:

$$[k2^{-n}, (k+1)2^{-n}) = [(2k)2^{-(n+1)}, (2k+1)2^{-(n+1)}) \cup [(2k+1)2^{-(n+1)}, (2k+2)2^{-(n+1)}).$$

Hence, we conclude that either $\omega \in A_{2k}^{n+1}$ or $\omega \in A_{2k+1}^{n+1}$. In both case we get that

$$f_n(\omega) = k2^{-n} = 2kn^{-(n+1)} \leq f_{n+1}(\omega).$$

(e) Now let us consider the case where $\omega \in A_k^n$ with $k = n2^n$, so that $n \leq f(\omega) < +\infty$. Then, if $f(\omega) \geq n+1$ it follows that $f_n(\omega) = n < n+1 = f_{n+1}(\omega)$. If, on the other hand, $n \leq f(\omega) < n+1$ there exists an $2n\,2^n \leq \ell \leq (2n+2)\,2^n$ such that $\omega \in A_\ell^{n+1}$, which then implies that

$$f_n(\omega) = n = (2n2^n)\,2^{-(n+1)} \leq f_{n+1}(\omega).$$

Problem 4.5

(a) First suppose $f = \sum_{i=1}^N a_i \mathbb{1}_{A_i}$ is a simple function. Then $f\mathbb{1}_B = \sum_{i=1}^N a_i \mathbb{1}_{A_i \cap B}$ is also a simple function and thus

$$\int_B f \,\mathrm{d}\mu = \int_\Omega f\mathbb{1}_B \,\mathrm{d}\mu = \sum_{i=1}^N a_i \mu(A_i \cap B) \leq \mu(B) \sum_{i=1}^N a_i \mu(A_i) = 0.$$

Now let f be a non-negative function and $g \leq f$ be a simple function. Then $g\mathbb{1}_B \leq f\mathbb{1}_B$ and thus by Definition 4.7

$$\int_B f \,\mathrm{d}\mu = \int_\Omega f\mathbb{1}_B \,\mathrm{d}\mu \geq \int_\Omega g\mathbb{1}_B \,\mathrm{d}\mu = 0,$$

which implies the result.

(b) Suppose $f \leq g$ are non-negative functions and observe that if h is a simple function such that $h \leq f$ then also $h \leq g$. Therefore we get

$$\int_\Omega f \,\mathrm{d}\mu = \sup_{h \leq f} \left\{ \int_\Omega h \,\mathrm{d}\mu \right\} \leq \sup_{h \leq g} \left\{ \int_\Omega h \,\mathrm{d}\mu \right\} = \int_\Omega g \,\mathrm{d}\mu.$$

(c) Suppose that h is a simple function. Then αh is also simple and it immediately follows that $\int_\Omega (\alpha h)\,\mathrm{d}\mu = \alpha \int_\Omega h\,\mathrm{d}\mu$. Now let f be non-negative. Then $h \le f \iff \alpha h \le \alpha f$ and $h \le \alpha f \iff \alpha^{-1} h \le f$. Thus by Definition 4.7 we have

$$\begin{aligned}
\alpha \int_\Omega f\,\mathrm{d}\mu &= \alpha \sup_{h\le f} \left\{ \int_\Omega h\,\mathrm{d}\mu \right\} \\
&= \sup_{h\le f} \alpha \left\{ \int_\Omega h\,\mathrm{d}\mu \right\} \\
&= \sup_{h\le f} \left\{ \int_\Omega (\alpha h)\,\mathrm{d}\mu \right\} \\
&= \sup_{\alpha^{-1}h\le f} \left\{ \int_\Omega h)\,\mathrm{d}\mu \right\} \\
&= \sup_{h\le \alpha f} \left\{ \int_\Omega (\alpha h)\,\mathrm{d}\mu \right\} = \int_\Omega (\alpha f)\,\mathrm{d}\mu.
\end{aligned}$$

Problem 4.7

(a) (2pts) **1 pt** Note that $(f\mathbb{1}_B)^+ = f^+\mathbb{1}_B$ and $(f\mathbb{1}_B)^- = f^-\mathbb{1}_B$. **1 pt** Then we get, using Lemma 4.8,

$$\int_B f\,\mathrm{d}\mu = \int_\Omega (f\mathbb{1}_B)^+\,\mathrm{d}\mu - \int_\Omega (f\mathbb{1}_B)^-\,\mathrm{d}\mu = \int_\Omega f^+\mathbb{1}_B\,\mathrm{d}\mu - \int_\Omega f^-\mathbb{1}_B\,\mathrm{d}\mu = 0+0=0.$$

(b) (2pts)

1 pt Here we note that if $f \le g$ then $f^+ \le g^+$, while $f^- \ge g^-$.

1 pt Hence, using Lemma 4.8 again,

$$\int_\Omega f\,\mathrm{d}\mu = \int_\Omega f^+\,\mathrm{d}\mu - \int_\Omega f^-\,\mathrm{d}\mu \le \int_\Omega g^+\,\mathrm{d}\mu - \int_\Omega g^-\,\mathrm{d}\mu = \int_\Omega g\,\mathrm{d}\mu.$$

(c) (3 pt)

1 pt Assume first that $\alpha \ge 0$. Then it follows from Lemma 4.8 that

$$\alpha \int_\Omega f^\pm\,\mathrm{d}\mu = \int_\Omega (\alpha f)^\pm\,\mathrm{d}\mu,$$

which implies the result.

1 pt Now suppose that $\alpha < 0$ so that $\beta := -\alpha > 0$. Note that for any function f we have

$$(-f)^+ := \max\{-f, 0\} = \min\{f, 0\} = f^-$$

and similarly $(-f)^- = f^+$.

1 pt We then get that

$$-\int_\Omega f \mathrm{d}\mu = -\int_\Omega f^+ \,\mathrm{d}\mu + \int_\Omega (f)^- \,\mathrm{d}\mu = -\int_\Omega (-f)^- \,\mathrm{d}\mu + \int_\Omega (-f)^+ \,\mathrm{d}\mu = \int_\Omega (-f)\,\mathrm{d}\mu.$$

The result then follows since

$$\alpha \int_\Omega f \,\mathrm{d}\mu = -\beta \int_\Omega f \,\mathrm{d}\mu = -\int_\Omega (\beta f)\,\mathrm{d}\mu = \int_\Omega (-\beta f)\,\mathrm{d}\mu = \int_\Omega (\alpha f)\,\mathrm{d}\mu.$$

(d) (2 pt) This result follows immediately from Lemma 4.8 and the observation that $(f+g)^\pm = f^\pm + g^\pm$.

Problem 4.8

(a) By definition, we have that $\nu_f(\Omega) = \int_\Omega f \,\mathrm{d}\mu = 1$. Now let $(A_n)_{n\in\mathbb{N}}$ be a family of mutually disjoint measurable sets. Then we have that the sequence

$$g_n := \sum_{i=1}^n f\,\mathbf{1}_{A_i} = f\,\mathbf{1}_{\bigcup_{i=1}^n A_i} \;\longrightarrow\; g := f\,\mathbf{1}_{\bigcup_{i\in\mathbb{N}} A_i} \qquad \text{pointwise monotonically.}$$

By MCT, we then have that

$$\nu_f\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \int_{\bigcup_{i\in\mathbb{N}} A_i} f\,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\bigcup_{i=1}^n A_i} f\,\mathrm{d}\mu = \lim_{n\to\infty} \sum_{i=1}^n \int_{A_i} f\,\mathrm{d}\mu = \sum_{i\in\mathbb{N}} \nu_f(A_i),$$

thus showing that ν_f is a probability measure on $(\Omega, \mathcal{F})$.

(b) Following the hint, we start by considering nonnegative simple functions g. Suppose $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ for $a_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$ mutually disjoint. Then,

$$\int_\Omega g\, d\nu_f = \sum_{i=1}^n a_i \; \nu_f(A_i) = \sum_{i=1}^n a_i \int_{A_i} f\,\mathrm{d}\mu = \int_\Omega gf\,\mathrm{d}\mu.$$

Now let g be a nonnegative measurable function and $[g]_n$ be a sequence of nonnegative simple functions that converge pointwise monotonically to g. Then MCT yields

$$\int_\Omega g\,\mathrm{d}\nu_f = \lim_{n\to\infty} \int_\Omega [g]_n \,\mathrm{d}\nu_f = \lim_{n\to\infty} \int_\Omega [g]_n f\,\mathrm{d}\mu = \int_\Omega gf\,\mathrm{d}\mu,$$

where we used the fact that $[g]_n f$ converges pointwise monotonically to gf.

(c) Let g be measurable. Then $g = g^+ - g^-$, where $g^\pm$ are nonnegative measurable functions. Since f is nonnegative, we have that $(fg)^\pm = fg^\pm$. Due to (b), we deduce

$$\int_\Omega g^\pm \,\mathrm{d}\nu_f = \int_\Omega g^\pm f\,\mathrm{d}\mu = \int_\Omega (gf)^\pm \,\mathrm{d}\mu.$$

Hence, $g^\pm$ is ν_f-integrable if and only if $(gf)^\pm$ is μ-integrable. Consequently, g is ν_f-integrable if and only if gf is μ-integrable, since

$$\int_\Omega |g|\,\mathrm{d}\nu_f = \int_\Omega g^+ \,\mathrm{d}\nu_f + \int_\Omega g^- \,\mathrm{d}\nu_f = \int_\Omega g^+ f\,\mathrm{d}\mu + \int_\Omega g^- f\,\mathrm{d}\mu = \int_\Omega |gf|\,\mathrm{d}\mu.$$

Problem 4.9

(⇒) (4pts) **2 pts** Let f be μ-integrable. Then both $|f|\mathbf{1}_{\{|f|<n\}}$ and $|f|\mathbf{1}_{\{|f|\geq n\}}$ are integrable, due to the monotonicity of the integral. By linearity of the integral,

$$\int_\Omega |f|\mathbf{1}_{\{|f|\geq n\}}\,\mathrm{d}\mu = \int_\Omega |f|\,\mathrm{d}\mu - \int_\Omega |f|\mathbf{1}_{\{|f|<n\}}\,\mathrm{d}\mu.$$

2 pts Since the sequence $g_n := |f|\mathbf{1}_{\{|f|<n\}} \geq 0$ converges pointwise monotonically to $|f|$, we can apply MCT to obtain

$$\lim_{n\to\infty}\int_\Omega |f|\mathbf{1}_{\{|f|<n\}}\,dd\mu = \int_\Omega |f|\,\mathrm{d}\mu.$$

Hence,

$$\lim_{n\to\infty}\int_\Omega |f|\mathbf{1}_{\{|f|\geq n\}}\,\mathrm{d}\mu = \int_\Omega |f|\,\mathrm{d}\mu - \lim_{n\to\infty}\int_\Omega |f|\mathbf{1}_{\{|f|<n\}}\,\mathrm{d}\mu = 0.$$

(⇐) (3 pts)
1 pt By assumption, there is some $N \geq 1$ such that

$$\int_\Omega |f|\mathbf{1}_{\{|f|\geq N\}}\,\mathrm{d}\mu \leq 1.$$

2 pts By linearity of the integral,

$$\int_\Omega |f|\,\mathrm{d}\mu = \int_\Omega |f|\mathbf{1}_{\{|f|<N\}}\,\mathrm{d}\mu + \int_\Omega |f|\mathbf{1}_{\{|f|\geq N\}}\,\mathrm{d}\mu \leq N\mu\big(\{|f|<N\}\big) + 1.$$

Since μ is a finite measure, the right-hand side is finite, implying that f is μ-integrable.

Problem 4.10
Observe that $\Omega = \bigcup_{n\in\mathbb{N}}\{|f|>n\}$.
We then get that

$$\sum_{n=1}^{\infty}\int_{\{|f|>n\}}|f|\,\mathrm{d}\mu = \int_\Omega |f|\,\mathrm{d}\mu < \infty.$$

This implies that for some N and all $n \geq N$: $\int_{\{|f|>n\}}|f|\,\mathrm{d}\mu < 1/n$ or else the sum cannot be finite.
Now let $\varepsilon > 0$, take $M > \max\{N, 2/\varepsilon\}$ and $\delta = \varepsilon/(2M)$. Then

$$\begin{aligned}\int_A |f|\,\mathrm{d}\mu &= \int_A |f|\mathbf{1}_{|f|\leq M}\,\mathrm{d}\mu + \int_A |f|\mathbf{1}_{|f|>M}\,\mathrm{d}\mu\\ &\leq M\mu(A) + \frac{1}{M} \leq M\delta + \frac{1}{M} < \varepsilon.\end{aligned}$$

Chapter 5: Product spaces and Lebesgue integration

Problem 5.2

(a) Note that $\mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{F}_1 \times \mathcal{F}_2$, and hence

$$\sigma(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{F}_1 \otimes \mathcal{F}_2.$$

(b) Let $B \in \mathcal{A}_2$. Then we have that

$$\Omega_1 \times B = \bigcup_{n \geq 1} A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$$

since $A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ for all $n \geq 1$. So $\Omega_1 \in \Sigma$

For the second property, let $C \in \Sigma$ and note that $C^c \times B = (\Omega_1 \times B) \setminus (C \times B)$. Since both these sets are in $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ it follows that $C^c \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ and hence $C^c \in \Sigma$.

Finally consider a countable sequence $(C_n)_{n \geq 1}$ of sets in Σ. Then for any $B \in \mathcal{A}_2$

$$\left(\bigcup_{n \geq 1} C_n \right) \times B = \bigcup_{n \geq 1} (C_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2),$$

since each $C_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(c) Note that $\mathcal{A}_1 \subset \Sigma_1 \subset \mathcal{F}_1$. From which it follows that $\Sigma_1 = \mathcal{F}_1$. But then, from the definition of Σ_1 we have that $\mathcal{F}_1 \times \mathcal{A}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(d) We can show in a similar fashion that

$$\Sigma_2 := \{C \in \mathcal{F}_2 \,:\, B \times C \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2) \, \forall B \in \mathcal{A}_1\}.$$

is a σ-algebra on Ω_2, from which we conclude that $\mathcal{A}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(e) take any $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Then

$$A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B) = \bigcup_{n,m \geq 1} (A \times B_m) \cap (A_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2).$$

From this we conclude that $\mathcal{F}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$, which finishes the proof.

Chapter 6: Probability I, The basics

Problem 6.2

(a) The implication from right to left is by definition of $\overleftarrow{F}$ and the fact that F is non-decreasing. The implication from left to right is because F is right continuous.

(b) Consider the preimage of $(-\infty, t]$ under X. Then, using the above observation, we have

$$\begin{aligned}X^{-1}((-\infty,t]) &= \{\omega \in \Omega \,:\, \overleftarrow{F}(U(\omega)) \in (-\infty,t]\} \\ &= \{\omega \in \Omega \,:, U(\omega) \in (-\infty, F(t)]\} = U^{-1}((-\infty, F(t)]) \in \mathcal{B}_{[0,1]}.\end{aligned}$$

Hence, X is measurable. Finally, the above computation, together with Lemma 6.5, also implies that

$$\mathbb{P}\left(X^{-1}((-\infty,t])\right) = \mathbb{P}\left(U^{-1}((-\infty,F(t)])\right) = F(t).$$

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and U a standard normal random variable. We will show that $X = \overleftarrow{F} \circ U$ is a random variable with the right probability measure. Since we can construct a standard uniform random variable on the probability $([0,1], \mathcal{B}_{[0,1]}, \lambda|_{[0,1]})$ this also implies the last part.
which finished the proof.

Problem 6.3

(a) For the probability space, take $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0,1]$.

Observe that the function $H_\gamma(z)$ is continuous and hence has an inverse $g_\gamma(y) = \gamma \tan(\pi(y - 1/2))$ on $[0,1]$.

So the function $Y[0,1] \to \mathbb{R}$ defined by $Y(x) = g_\gamma(x)$ has the correct distribution as

$$\mathbb{P}(Y^{-1}((-\infty,t])) = \mathbb{P}(g_\gamma^{-1}((-\infty,t])) = \lambda(H_\gamma((-\infty,t])) = H_\gamma(t).$$

(b) Note that g_γ is continuous on $[0,1]$ and hence measurable.

(c) For any $t \geq 0$, the cdf of the Poisson random variable is given by

$$F_\lambda(t) = \sum_{n=0}^{\lceil t \rceil} f_\lambda(n),$$

where $\lceil t \rceil$ is the ceiling of t, i.e. the smallest integer $k \geq t$.

(d) For the probability space, we again take $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0,1]$.

Now for any $y \in [0,1]$ let $k := k(y)$ be such that

$$\sum_{n=1}^{k} f_\lambda(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_\lambda(n) < y,$$

where the last sum is interpreted as -1 if $k = 0$.

Now define $X(y) = k(y) : [0,1] \to \mathbb{R}$. Then $k(y) \leq t$ if and only if $y \leq F_\lambda(t)$ and hence

$$X^{-1}((-\infty, t]) = \{y \in [0,1] \,:\, k(y) \in (0,t]\} = \{y \in [0,1] \,:\, y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty,t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

(e) It follows from the above computation that $X^{-1}((-\infty,t]) = \{y \in [0,1] \,:\, y \in (0, F_\lambda(t)]\}$. Since the latter is a measurable set we conclude that $X^{-1}((-\infty,t])$ is measurable for all t and since these generate the Borel σ-algebra X is measurable.

(f) for any $\ell \in \mathbb{N}$ define the sets $A_\ell = (n-1-1/\ell), n-1+1/\ell]$. Then A_ℓ is a decreasing set with $\lim_{\ell\to\infty} A_\ell = \{n\}$. Moreover, $A_\ell = (-\infty, n-1+1/\ell] \setminus (-\infty, n-1-1/\ell]$ and $\mathbb{P}(A_1) < \infty$. It now follows from continuity from above and (d) that

$$\begin{aligned}
X_\#\mathbb{P}(\{n\}) &= \lim_{\ell\to\infty} X_\#\mathbb{P}(A_\ell) \\
&= \lim_{\ell\to\infty} X_\#\mathbb{P}((-\infty, n-1+1/\ell]) - X_\#\mathbb{P}((-\infty, n-1-1/\ell]) \\
&= F_\lambda(n-1+1/\ell) - F_\lambda(n-1-1/\ell) \\
&= \sum_{k=0}^{n} f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n).
\end{aligned}$$

Problem 6.5

Define for any $j \in \mathbb{Z}$, $p_j := \mathbb{P}(X^{-1}(\{j\}))$. Then, since $(X^{-1}(j))_{j\in\mathbb{Z}}$ is a family of disjoint sets and $\mathbb{P}$ is a probability measure we get that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\bigcup_{j\in\mathbb{Z}} X^{-1}(j)) = \sum_{j\in\mathbb{Z}} p_j.$$

Now let $A \subset \mathbb{R}$ be a measurable set and note that

$$X^{-1}(A) = \bigcup_{j\in\mathbb{Z}\cap A} X^{-1}(j).$$

Then it follows that

$$\mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}\left(\bigcup_{j\in\mathbb{Z}\cap A} X^{-1}(j)\right) = \sum_{j\in\mathbb{Z}\cap A} p_j = \sum_{j\in\mathbb{Z}} \delta_j(A) p_j.$$

Problem 6.7

(a) (3 pts)

1 pt We first observe that

$$\nu((-\infty, t]) = X_{\#}\mathbb{P}((-\infty, t])$$

holds by definition of the pdf.

1 pt Next we note that the collection $\mathcal{A} := \{(-\infty, t] \,:\, t \in \mathbb{R}\}$ generates $\mathcal{B}$ and satisfies the conditions of Theorem 2.15:

a) $(-\infty, t] \cap (-\infty, s] = (-\infty, \min\{t, s\}] \in \mathcal{A}$, and

b) the sets $A_n(-\infty, n] \in \mathcal{A}$ for $n \in \mathbb{N}$ satisfies $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$.

1 pt Thus, since $\nu = X_{\#}\mathbb{P}$ on $\mathcal{A}$, by Theorem 2.15 we conclude that $\nu = X_{\#}\mathbb{P}$ on $\sigma(\mathcal{A}_2) = \mathcal{B}$.

(b) (2 pts) Let $g = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i}$ be a simple function. Then by definition of ν and linearity of the integral we get

$$\begin{aligned}
\int_{\mathbb{R}} g \,\mathrm{d}\nu &= \sum_{i=1}^{N} a_i \int_{A_i} \mathrm{d}\nu \\
&= \sum_{i=1}^{N} a_i \nu(A_i) \\
&= \sum_{i=1}^{N} a_i \int_{A_i} \rho \,\mathrm{d}\lambda \\
&= \int_{\Omega} \sum_{i=1}^{N} a_i \mathbb{1}_{A_i} \rho \,\mathrm{d}\lambda = \int_{\mathbb{R}} g\rho \,\mathrm{d}\lambda.
\end{aligned}$$

(c) (2 pts) By using (b) and monotone convergence twice we get

$$\begin{aligned}
\int_{\mathbb{R}} h \,\mathrm{d}\nu &= \int_{\mathbb{R}} \lim_{n\to\infty} [h]_n \,\mathrm{d}\nu \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n \,\mathrm{d}\nu \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n \rho \,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} [h]_n \rho \,\mathrm{d}\lambda = \int_{\mathbb{R}} h\rho \,\mathrm{d}\lambda.
\end{aligned}$$

(d) (1 pt) Using the change of variables result (Proposition 4.14) (a) and (c) we get

$$\mathbb{E}[h(X)] = \int_{\Omega} h \circ X \,\mathrm{d}\mathbb{P} = \int_{\mathbb{R}} h \,\mathrm{d}X_{\#}\mathbb{P} = \int_{\mathbb{R}} h \,\mathrm{d}\nu = \int_{\mathbb{R}} h\rho \,\mathrm{d}\lambda.$$

Problem 6.8

(a) This follows from the following computation

$$\int_\Omega |f|^p \,\mathrm{d}\mu \geq \int_\Omega |f|^p \mathbb{1}_{|f|\geq t} \,\mathrm{d}\mu \geq t^p \int_\Omega \mathbb{1}_{|f|\geq t} \,\mathrm{d}\mu = t^p \mu(\{\omega \in \Omega \,:\, |f| \geq t\}).$$

(b) Using the result for $p = 1$ we get

$$\mathbb{P}(|X| \geq t) = \mu(\omega \in \Omega \,:\, |X(\omega) \geq t\}) \leq \frac{1}{t} \int_\Omega X \,\mathrm{d}\mathbb{P} = \frac{1}{t}\mathbb{E}[X].$$

(c) Take $f(\omega) = X(\omega) - \mathbb{E}[X]$, which is measurable. Then using the first result with $p = 2$ gives

$$\begin{aligned} \mathbb{P}(|X - \mathbb{E}[X]| \geq t) &= \mathbb{P}(|X - \mathbb{E}[X]|^2 \geq t^2) \\ &\leq \frac{1}{t^2}\mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \frac{1}{t^2}(\mathbb{E}[X^2] - \mathbb{E}[X]^2) = \frac{\mathrm{Var}(X)}{t^2}. \end{aligned}$$

Chapter 7: Convergence of integrals and measures

Problem 7.1

Similar to the proof of Fatou's lemma, we define $g_n = \sup_{k\geq n} f_n$ which are measurable due to Proposition 3.13. Moreover, we have that $\limsup_{n\to\infty} f_n = \lim_{n\to\infty} g_n$.
Next we note that $g_n \geq f_\ell$ for all $\ell \geq n$. Thus, by monotonicity of the integral, we have that

$$\int_\Omega g_n \,\mathrm{d}\mu \geq \int_\Omega f_\ell \,\mathrm{d}\mu,$$

holds for all $\ell \geq n$, which implies that

$$\int_\Omega g_n \,\mathrm{d}\mu \geq \sup_{k\geq n} \int_\Omega f_n \,\mathrm{d}\mu.$$

In addition, since $g_n < f$ with f being non-negative and integrable we can apply Dominated Convergence to conclude that

$$\int_\Omega \lim_{n\to\infty} g_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega g_n \,\mathrm{d}\mu.$$

Putting all this together we get

$$\int_\Omega \limsup_{n\to\infty} f_n \,\mathrm{d}\mu = \int_\Omega \lim_{n\to\infty} g_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega g_n \,\mathrm{d}\mu \geq \lim_{n\to\infty} \sup_{k\geq n} \int_\Omega f_n \,\mathrm{d}\mu.$$

Problem 7.2

(a) Let $t_0 \in (a,b)$ be fixed. It suffices to check the continuity result for arbitrary sequences $(t_n)_{n\geq 1} \subset (a,b)$ such that $t_n \to t_0$ as $n \to \infty$. Fix such a sequence and define $g_n(\omega) := f(\omega, t_n)$ for all $\omega \in \Omega$ and $n \geq 1$. Since $\lim_{t\to t_0} f(\omega,t) = f(\omega,t_0)$ for all $\omega \in \Omega$, we deduce that $\lim_{n\to\infty} g_n(\omega) = f(\omega,t_0)$ for every $\omega \in \Omega$. Moreover, by assumption $|g_n| \leq g$ for all $n \geq 1$ and g is integrable. By the Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_\Omega g_n(\omega)\,\mu(\mathrm{d}\omega) = \int_\Omega f(\omega,t_0)\,\mu(\mathrm{d}\omega).$$

As the chosen sequence was arbitrary, we deduce that $\lim_{t\to t_0} F(t) = F(t_0)$.

(b) If $t \mapsto f(\omega,t)$ is continuous on (a,b) for all $\omega \in \Omega$ then $\lim_{t\to t_0} f(\omega,t) = f(\omega,t_0)$ at every $t_0 \in (a,b)$ for all $\omega \in \Omega$. In particular, (a) applies, showing that $\lim_{t\to t_0} F(t) = F(t_0)$ for every $t_0 \in (a,b)$, i.e., F is continuous on (a,b).

Problem 7.3

(a) (4 pts) We start by showing that $(\partial f/\partial t)(\cdot,t)$ is measurable.

2 pts Let $(t_n)_{n\geq 1} \subset (a,b)$ be an arbitrary sequence with $t_n \neq t$ and $t_n \to t$ for $n \to \infty$. We set

$$h_n(\omega) = \frac{f(\omega,t_n) - f(\omega,t)}{t_n - t}.$$

Clearly, h_n is measurable for every $n \geq 1$.

1 pt Moreover, $\lim_{n\to\infty} h_n(\omega) = (\partial f/\partial t)(\omega,t)$ by the definition of the derivative. Since $(\partial f/\partial t)(\cdot,t)$ is the pointwise limit of a sequence of measurable functions, it is also measurable.

1 pt Finally, $(\partial f/\partial t)(\cdot,t)$ is integrable since

$$\int_\Omega |(\partial f/\partial t)(\omega,t)|\,\mu(\mathrm{d}\omega) \leq \int_\Omega g\,\mathrm{d}\mu < +\infty.$$

(b) (3 pts)

2 pts Let $t_0 \in (a,b)$ and suppose w.l.o.g. $t_0 < t$. Since $t \mapsto f(\omega,t)$ is differentiable on (a,b) for all $\omega \in \Omega$, the Mean Value Theorem gives

$$\frac{f(\omega,t) - f(\omega,t_0)}{t - t_0} = (\partial f/\partial t)(\omega,\tau) \qquad \text{for some } \tau \in (t_0,t).$$

1 pt Taking the modulus on both sides, we obtain

$$\left|\frac{f(\omega,t) - f(\omega,t_0)}{t - t_0}\right| \leq |(\partial f/\partial t)(\omega,\tau)| \leq g(\omega) \qquad \text{for all } \omega \in \Omega.$$

(c) (5 pts)

2 pts Take $t_0 \in (a,b)$ and let $(t_n)_{n\geq 1}$ be a sequence in (a,b) such that $t_n \to t_0$ and define

$$h(\omega,t) := \frac{f(\omega,t) - f(\omega,t_0)}{t - t_0}.$$

Then by (b) and the conditions in this exercise, h satisfies all the conditions listed in Problem 4.2.

1 pt Next, we note that by linearity of the integral we have that

$$\frac{F(t_n) - F(t_0)}{t_n - t_0} = \int_\Omega \frac{f(\omega,t_n) - f(\omega,t_0)}{t_n - t_0}\,\mu(d\omega) = \int_\Omega h(\omega,t_n)\,\mu(\mathrm{d}\omega).$$

1 pt Now by Problem 4.2 it holds that

$$\lim_{t_n\to t_0} \int_\Omega h(\omega,t_n)\,\mu(\mathrm{d}\omega) = \int_\Omega \lim_{t_n\to t_0} h(\omega,t_n)\,\mu(\mathrm{d}\omega) = \int_\Omega \frac{\partial f}{\partial t}(\omega,t_0)\,\mu(\mathrm{d}\omega).$$

1 pt Since $t_0 \in (a,b)$ and the sequence $(t_n)_{n\geq 1}$ were arbitrary, we conclude that F is indeed differentiable on (a,b) with

$$\frac{\partial F}{\partial t}(t_0) := \lim_{t_n\to t_0} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \int_\Omega \frac{\partial f}{\partial t}(\omega,t_0)\,\mu(\mathrm{d}\omega).$$

Problem 7.4

(a) Note that the integrand $f_n(x) = \frac{1+nx^2}{(1+x^2)^n}$ is continuous on $[0,1]$ and non-negative. Hence, the Riemann integral and Lebesgue integral coincide, i.e.,

$$\int_0^1 f_n(x)\,\mathrm{d}x = \int_{[0,1]} f_n\,\mathrm{d}\lambda.$$

Observe that we have the following pointwise limit

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0,1], \end{cases}$$

i.e., $\lim_{n\to\infty} f_n = 0$ λ-almost everywhere. Moreover, $f_n(x) \leq 1$ for every $x \in [0,1]$ and $n \geq 1$. Since the constant function $g \equiv 1$ is λ-integrable on $[0,1]$, it is a valid dominator. Hence, the DCT gives

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{[0,1]} f_n\,\mathrm{d}\lambda = \int_{[0,1]} \lim_{n\to\infty} f_n\,\mathrm{d}\lambda = 0$$

(b) For the purpose of convergence, we consider $n \geq 3$. Note that the integrand $f_n(x) = \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right)$ is continuous on $(0, +\infty)$ with pointwise limit

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ 1/2 & \text{if } x = 1, \\ 1/x^2 & \text{if } x > 1, \end{cases}$$

Setting the function

$$g(x) = \begin{cases} 1 & \text{for } x \in (0,1), \\ \frac{1}{x^2} & \text{for } x \geq 1, \end{cases}$$

we see that $f_n \leq g$ λ-almost everywhere in $(0, +\infty)$ and for all $n \geq 3$. Indeed, for $x \geq 1$, we obtain

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq \frac{x^{n-2}}{x^n} = \frac{1}{x^2},$$

while for $x \in (0,1)$, we have

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq 1.$$

Notice that g is non-negative and λ-integrable on $(0, +\infty)$. Indeed, using the MCT,

$$\begin{aligned} \int_{(0,+\infty)} g \,\mathrm{d}\lambda &= \int_{(0,1)} g \,\mathrm{d}\lambda + \int_{(1,+\infty)} g \,\mathrm{d}\lambda = 1 + \lim_{n\to\infty} \int_{(1,n)} g \,\mathrm{d}\lambda \\ &= 1 + \lim_{n\to\infty} \int_1^n \frac{1}{x^2} \,\mathrm{d}x = 1 + \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 2 < +\infty. \end{aligned}$$

To conclude, we apply DCT to deduce that the limit is 1.

Problem 7.5

Problem 7.6

(a)

(b) By definition

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} f \,\mathrm{d}\mu_n - \int_{\mathbb{R}} f \,\mathrm{d}\mu \right| \leq \varepsilon,$$

implies that for any $\delta > 0$

$$\left| \int_{\mathbb{R}} f \,\mathrm{d}\mu_n - \int_{\mathbb{R}} f \,\mathrm{d}\mu \right| < \varepsilon + \delta,$$

holds for large enough n. Note that this holds for any $\varepsilon, \delta > 0$.

Now pick $\eta > 0$ and set $\varepsilon = \eta/2 = \delta$, then the above inequality implies that

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} f \,\mathrm{d}\mu_n - \int_{\mathbb{R}} f \,\mathrm{d}\mu \right| = 0.$$

(c) Consider the sequence of sets $A_n = \mathbb{R} \setminus [-n, n]$. Then $A_n \supset A_{n+1}$ and $A_n \downarrow \emptyset$. Hence, it follows from Proposition 2.12 2) that $\lim_{n\to\infty} \mu(A_n) = 0$. Thus, there exists a N such that $\mu(A_n) < \varepsilon/(2M)$ holds for all $n \geq N$. We can then take any $\alpha > N$.

(d) The function

$$g(x) = \mathbb{1}_{[-\alpha,\alpha]}(x) + \mathbb{1}_{(-(\alpha+1),-\alpha)}(x)\,(x + (\alpha + 1)) + \mathbb{1}_{(\alpha,\alpha+1)}(x)\,(-x + \alpha + 1)$$

does the trick. This is simply a linear increase from zero to one from $-(\alpha+1)$ to $-\alpha$ and from $\alpha + 1$ to α.

(e) Observe that g is a non-negative continuous bounded function that is zero outside the interval $[-(\alpha+1), \alpha+1]$, and thus we can apply (3). Using linearity of the integral, the fact that $|f| \leq M$ and the definition of g, we get

$$\begin{aligned}
\left| \int_{\mathbb{R}} f \,\mathrm{d}\mu - \int_{\mathbb{R}} fg \,\mathrm{d}\mu \right| &= \left| \int_{\mathbb{R}} f(1-g) \,\mathrm{d}\mu \right| \leq M \int_{\mathbb{R}} (1-g) \,\mathrm{d}\mu \\
&\leq M \int_{\mathbb{R}} (1-g) \,\mathrm{d}\mu \\
&= M \left(1 - \int_{\mathbb{R}} g \,\mathrm{d}\mu \right) \\
&\leq M\mu(\mathbb{R}\setminus[-\alpha,\alpha]) < \frac{\varepsilon}{2}.
\end{aligned}$$

(f) Again, using linearity of the integral and the fact that $|f| \leq M$ we get

$$\begin{aligned}
\left| \int_{\mathbb{R}} f \,\mathrm{d}\mu_n - \int_{\mathbb{R}} fg \,\mathrm{d}\mu_n \right| &= \left| \int_{\mathbb{R}} f(1-g) \,\mathrm{d}\mu_n \right| \leq M \int_{\mathbb{R}} (1-g) \,\mathrm{d}\mu_n \\
&\leq M \int_{\mathbb{R}} (1-g) \,\mathrm{d}\mu_n = M \left(1 - \int_{\mathbb{R}} g \,\mathrm{d}\mu_n \right)
\end{aligned}$$

Now observe that the integral in the last term converges to $\int_{\mathbb{R}} g \,\mathrm{d}\mu$ by (3). Thus, we obtain

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}} f \,\mathrm{d}\mu_n - \int_{\mathbb{R}} fg \,\mathrm{d}\mu_n \right| \leq M \int_{\mathbb{R}} (1-g) \,\mathrm{d}\mu \leq M\mu(\mathbb{R}\setminus[-\alpha,\alpha]) < \frac{\varepsilon}{2}.$$

(g) Recall that

$$\begin{aligned}
\left| \int_{\mathbb{R}} f \,\mathrm{d}\mu_n - \int_{\mathbb{R}} f \,\mathrm{d}\mu \right| \leq{}& \left| \int_{\mathbb{R}} f \,\mathrm{d}\mu_n - \int_{\mathbb{R}} fg \,\mathrm{d}\mu_n \right| + \left| \int_{\mathbb{R}} f \,\mathrm{d}\mu - \int_{\mathbb{R}} fg \,\mathrm{d}\mu \right| \\
&+ \left| \int_{\mathbb{R}} fg \,\mathrm{d}\mu_n - \int_{\mathbb{R}} fg \,\mathrm{d}\mu \right|.
\end{aligned}$$

For the first two terms, the (e) and (f) imply that the $\limsup_{n\to\infty}$ is bounded by $\varepsilon/2$. For the third term we not that fg is a continuous bounded function and hence this term converges to zero by our assumption that (3) holds.

Together we then have that

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}} f\,\mathrm{d}\mu_n - \int_{\mathbb{R}} f\,\mathrm{d}\mu\right| < \varepsilon,$$

which implies the result.

Chapter 8: Convergence of random variables

Problem 8.2

(a) (2 pts)

For this let $h_t(x) = \mathbf{1}_{(-\infty,t]}$ and note that

$$F_n(t) = (X_n)_\# \mathbb{P}_n((-\infty, x]) = \int_{\mathbb{R}} \mathbb{1}_{(-\infty,t]} \,\mathrm{d}(X_n)_\# \mathbb{P}_n = \int_{\mathbb{R}} h_t \,\mathrm{d}\mu_n.$$

and similarly $F(t) = \int_{\mathbb{R}} h_t \,\mathrm{d}\mu$

(b) (3 pts)

1 pt The function h is discontinuous only at t, i.e. $\mathcal{C}_h = \mathbb{R} \setminus \{t\}$.

2 pts Moreover, for any $\varepsilon > 0$

$\mu((t-\varepsilon, t+\varepsilon)) = \mu((t-\varepsilon, t]) + \mu((t, t+\varepsilon)) = F(t) - F(t-\varepsilon) + F(t+\varepsilon) - F(t).$

Since F is continuous at t, the right hand side goes to zero as $\varepsilon \to 0$. Therefore

$$\mu(\{t\}) = \lim_{\varepsilon \to 0} \mu((t-\varepsilon, t+\varepsilon)) = 0,$$

which implies that $\mu(\mathcal{C}_h) = 1$.

(c) (1 pt) The result follows by applying condition (2) in Theorem 7.6.

(d) (2 pts)

1 pt Without loss of generality assume $x \in I_\ell = (a_\ell, b_\ell]$ for some $1 \leq \ell \leq L$. Then it holds that $\hat{g}(x) = g(b_\ell)$.

1 pt Moreover, since $|x - b_\ell| < \delta$ we have that

$$|g(x) - \hat{g}(x)| = |g(x) - g(b_\ell)| < \varepsilon.$$

(e) (1 pt) Let $M = L$, $\beta_\ell = \sum_{t=1}^{\ell} h(b_t)$ and $t_\ell = b_\ell$. Then

$$\hat{g} := \sum_{\ell=1}^{L} \beta_\ell \mathbf{1}_{(-\infty, b_\ell]}.$$

(f) (2 pts) Using the representation in (e) we get

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[\hat{g}(X_n)] &= \lim_{n\to\infty} \sum_{\ell=1}^{L} \beta_\ell \int_{\mathbb{R}} \mathbf{1}_{X_n^{-1}((-\infty, b_\ell])} \,\mathrm{d}\mathbb{P} \\
&= \lim_{n\to\infty} \sum_{\ell=1}^{L} \beta_\ell (X_n)_\# \mathbb{P}((-\infty, b_\ell]) \\
&= \lim_{n\to\infty} \sum_{\ell=1}^{L} \beta_\ell F_n(b_\ell) \\
&= \sum_{\ell=1}^{L} F(b_\ell) = \mathbb{E}[\hat{g}(X)].
\end{aligned}$$

(g) (4 pts) **1 pt** First we write

$$\|\mathbb{E}[g(X_n)] - \mathbb{E}[g(X)]\| \le \|\mathbb{E}[g(X_n)] - \mathbb{E}[\hat{g}(X_n)]\| + \|\mathbb{E}[g(X)] - \mathbb{E}[\hat{g}(X)]\| \\ + \|\mathbb{E}[\hat{g}(X_n)] - \mathbb{E}[\hat{g}(X)]\|.$$

1 pt We have shown in (f) that the last term goes to zero as $n \to \infty$.

1 pt Next, using (d) it follows that the other two terms are bounded by ε.

1 pt Since ε was arbitrary we conclude that

$$\lim_{n\to\infty} \mathbb{E}[g(X_n)] = \mathbb{E}[g(X)].$$

(h) (1 pt) This now follows from Theorem 7.6 (3).

Problem 8.3

The main idea is to use the equivalent version of convergence in distribution.
Suppose that $X_n \xrightarrow{\mathbb{P}} X$ and define $Y_n = |X_n - X|$. We need to show that $\mathbb{P}(Y_n > \varepsilon) \to 0$ holds for any $\varepsilon > 0$. First recall that $X_n \xrightarrow{\mathbb{P}} X$ is defined as weak convergence of Y_n to the constant zero random variable. By Lemma 8.2 this is equivalent to

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le t) = \mathbb{P}(0 \le t),$$

for all continuity points of the function $\omega \mapsto 0$. We now note that any $\varepsilon > 0$ is a continuity point of this function. Hence, we get

$$\lim_{n\to\infty} \mathbb{P}(Y_n > \varepsilon) = 1 - \lim_{n\to\infty} \mathbb{P}(Y_n \le \varepsilon) = 1 - \mathbb{P}(0 \le \varepsilon) = 0$$

Now we prove the other implication. So suppose that $\mathbb{P}(Y_n > \varepsilon) \to 0$ holds for any $\varepsilon > 0$. We then have to prove that $(Y_n)_{\#}\mathbb{P} \Rightarrow 0_{\#}\mathbb{P}$. Due to Lemma 8.2 it is enough to show that

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le t) = \mathbb{P}(0 \le t) = \mathbb{1}_{t\ge 0},$$

holds for all continuity points t of the function $\omega \mapsto 0$. Notice that the only non-continuity point is 0. Moreover, for all $t < 0$ we have that $\mathbb{P}(Y_n \le t) = 0$ since $Y_n \ge 0$ almost every-where. Finally, for all $t > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le t) = 1 - \lim_{n\to\infty} \mathbb{P}(Y_n > t) = 1 = \mathbb{P}(0 \le t).$$

Problem 8.4

Problem 8.5 Suppose that $X_n \xrightarrow{\text{a.s.}} X$. Then by Lemma 5.2.16 this is equivalent to $\mathbb{P}(\|X_n - X\| > \varepsilon \text{ i.o.}) = 0$ for all $\varepsilon > 0$.
For now fix an $\varepsilon > 0$ and write $A_n := \{\|X_n - X\| > \varepsilon\}$. Recall that

$$\{A_n \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \bigcup_{k\ge n} A_n$$

and note two things:

(a) The sets $B_k := \bigcup_{n \geq k} A_n$ are non-increasing, i.e. $B_k \supset B_{k+1}$, and

(b) $\mathbb{P}(A_k) \leq \mathbb{P}(\bigcup_{n \geq k} A_n) = \mathbb{P}(B_k)$.

We then have that:

$$\begin{aligned}
0 &= \mathbb{P}(\{A_n \text{ i.o.}\}) && \text{by assumption} \\
&= \mathbb{P}(\bigcap_{k=1}^{\infty} B_k) && \text{by Lemma 5.2.16} \\
&= \lim_{k \to \infty} \mathbb{P}(B_k) && \text{by continuity form above (Proposition 2.2.15)} \\
&\geq \lim_{k \to \infty} \mathbb{P}(A_k) && \text{by (b).}
\end{aligned}$$

Problem 8.6

(a) Define the sets

$$B_j := \bigcup_{i \geq j} A_i, \qquad j \in \mathbb{N}.$$

Clearly the sequence $(B_j)_{j \in \mathbb{N}}$ is decreasing and $\{A_n \text{ i.o.}\} \subset B_j$ for every $j \in \mathbb{N}$.

By assumption, and the σ-subadditivity of $\mathbb{P}$,

$$\mathbb{P}(B_1) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i) < +\infty.$$

Moreover, the summability also gives

$$\lim_{j \to \infty} \mathbb{P}(B_j) \leq \limsup_{j \to \infty} \sum_{i=j}^{\infty} \mathbb{P}(A_i) = 0.$$

Hence, by the continuity from above of μ, we obtain

$$\mathbb{P}(\{A_n \text{ i.o.}\}) \leq \mathbb{P}\left(\bigcup_{j=1}^{\infty} B_j\right) = \lim_{j \to \infty} \mathbb{P}(B_j) = 0,$$

i.e., $\{A_n \text{ i.o.}\}$ is a null set. In other words, $\mathbb{P}$-almost every ω is in only finitely many A_n.

(b) We will prove that

$$\mathbb{P}(\Omega \setminus \{A_n \text{ i.o.}\}) = 0,$$

from which the result follows since $\mathbb{P}(\Omega) = 1$.

First note that

$$\Omega \setminus \{A_n \text{ i.o.}\} = \bigcup_{k \geq 1} \left(\bigcup_{n \geq k} A_n\right)^c = \bigcup_{k \geq 1} \bigcap_{n \geq k} A_n^c.$$

Next, since A_n are mutually exclusive, so are A_n^c. Thus, for any $k \geq 1$ we have that

$$\begin{aligned}\mathbb{P}\left(\bigcap_{n\geq k} A_n^c\right) &= \prod_{n\geq k} \mathbb{P}(A_n^c) = \prod_{n\geq k}(1-\mathbb{P}(A_m)\\ &\leq \prod_{n\geq k} e^{-\mathbb{P}(A_n)} = e^{-\sum_{n\geq k}\mathbb{P}(A_n)} = 0.\end{aligned}$$

Here we used that for any $0 \leq x \leq 1$ it holds that $1 - x \leq e^{-x}$.

Finally, using σ-subadditivity we conclude that

$$\mathbb{P}(\Omega \setminus \{A_n \text{ i.o.}\}) = \mathbb{P}\left(\bigcup_{k\geq 1}\bigcap_{n\geq k} A_n^c\right) \leq \sum_{k\geq 1}\mathbb{P}\left(\bigcap_{n\geq k} A_n^c\right) = 0.$$

Problem 8.7
Fix $\varepsilon > 0$ and define $A_n(\varepsilon) : \{|X_n - X| > \varepsilon\}$. Then the assumption translates to

$$\sum_{n\geq 1}\mathbb{P}(A_n(\varepsilon)) < \infty.$$

By Lemma 8.11 1) this then implies that $\mathbb{P}(A_n(\varepsilon) \text{ i.o.}) = 0$. Since $\varepsilon > 0$ was arbitrary, Lemma 8.9 now implies that $X_n \overset{\text{a.s.}}{\to} X$.

Chapter 11: Conditional Expectation and Probability

Problem 11.2

(a) By definition we have that

$$\int_B \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} = \int_B X\,\mathrm{d}\mathbb{P},$$

holds for all $B \in \mathcal{H}$. Since by assumption both $\mathbb{E}[X|\mathcal{H}]$ and X are $\mathcal{H}$-measurable, the result follows from problem 8.2.

(b) Note that $a\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable. Moreover,

$$\int_B a\mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} = a\int_B \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} = a\int_B X\,\mathrm{d}\mathbb{P} = \int_B aX\,\mathrm{d}\mathbb{P}.$$

This proves the claim.

(c) Similarly to the previous point, we first note that since $\mathbb{E}[X|\mathcal{H}]$ and $\mathbb{E}[Y|\mathcal{H}]$ are $\mathcal{H}$-measurable so is $\mathbb{E}[X|\mathcal{H}] + \mathbb{E}[Y|\mathcal{H}]$. The result then follows because

$$\begin{aligned}\int_B \mathbb{E}[X|\mathcal{H}] + \mathbb{E}[Y|\mathcal{H}]\,\mathrm{d}\mathbb{P} &= \int_B \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} + \int_B \mathbb{E}[Y|\mathcal{H}]\,\mathrm{d}\mathbb{P} \\ &= \int_B X\,\mathrm{d}\mathbb{P} + \int_B Y\,\mathrm{d}\mathbb{P} = \int_B X + Y\,\mathrm{d}\mathbb{P}.\end{aligned}$$

(d) First we observe that for any $B \in \mathcal{H}$

$$\int_B \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} = \int_B X\,\mathrm{d}\mathbb{P} \leq \int_B Y\,\mathrm{d}\mathbb{P} = \int_B \mathbb{E}[Y|\mathcal{H}]\,\mathrm{d}\mathbb{P}.$$

Now consider the event $A := \{\mathbb{E}[X|\mathcal{H}] > \mathbb{E}[Y|\mathcal{H}]\} \in \mathcal{H}$. If this event has non-zero measure then it would follow that

$$\int_A \mathbb{E}[X|\mathcal{H}]\,\mathrm{d}\mathbb{P} > \int_A \mathbb{E}[Y|\mathcal{H}]\,\mathrm{d}\mathbb{P},$$

which is a contradiction. Hence we conclude that $\mathbb{E}[X|\mathcal{H}] \leq \mathbb{E}[Y|\mathcal{H}]$ holds $\mathbb{P}$-almost everywhere.

Problem 11.3

(a) This follows by repeating the step for the solution to Problem 4.8 a).

(b) We start by observing that the result will directly follow from Theorem 2.15 if we can show that the σ-algebra $\sigma(X)$ satisfies the two properties.

The first one is immediate, from the fact that $X^{-1}(A) \cap X^{-1}(B) = X^{-1}(A\ capB)$. For the second one consider the intervals $I_n = (-n, n)$ and define $A_n := X^{-1}(I_n) \in \sigma(X)$.

Since $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{R}$ it follows that $\bigcup_{n\in\mathbb{N}} A_n = \Omega$. Moreover since $|X| \leq n$ on the set A_n it holds that

$$\mu(A_n) = \nu_X(A_n) = \int_{A_n} X \,\mathrm{d}\mathbb{P} \leq n \int_{A_n} \mathrm{d}\mathbb{P} = n\mathbb{P}(A_n) \leq n < \infty.$$

Thus the second condition of Theorem 2.15 is also satisfied and the result now follows.

Problem 11.4

The solution to this problem will closely follow the proof of Lemma 11.7. To this end let $g(x,y)$ denote the conditional density of X given $Y = y$ and define the function $\phi(y) := \int_{\mathbb{R}} h(x)g(x,y)\,\lambda(\mathrm{d}x)$. We now need to show that for any $A \in \mathcal{B}_{\mathbb{R}}$ it holds that

$$\int_{Y^{-1}(A)} \phi(Y)\,\mathrm{d}\mathbb{P} = \mathbb{E}[\mathbb{1}_A(Y)h(X)].$$

Using the change of variables formula and the definition of ϕ and g we get

$$\begin{aligned}
\int_{Y^{-1}(A)} \phi(Y)\,\mathrm{d}\mathbb{P} &= \int_{\Omega} \mathbb{1}_{Y^{-1}(A)}\phi(Y)\,\mathrm{d}\mathbb{P} \\
&= \int_{\mathbb{R}} \int_{A} (y)\phi(y) f_Y(y)\,\lambda(\mathrm{d}y) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_A(y)h(x)g(x,y)f_Y(y)\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y) \\
&= \int_{\mathbb{R}^2} \mathbb{1}_A(y)h(x)f(x,y)\,\lambda(\mathrm{d}x)\,\lambda(\mathrm{d}y) \\
&= \mathbb{E}[\mathbb{1}_A(Y)h(X)].
\end{aligned}$$

Problem 11.5

(a) We first observe that

$$\int_B Z\,\mathrm{d}\mathbb{P} = \int_{\Omega} f \circ [(X,Y) \circ \Delta \mathbb{1}_B]\,\mathrm{d}\mathbb{P} = \int_{\mathbb{R}^2} f\,\mathrm{d}[(X,Y) \circ \Delta\mathbb{1}_B]_{\#}\mathbb{P}.$$

On the other hand

$$\int_{\Omega\times B} f\circ(X,Y)\,\mathrm{d}\mathbb{P}\otimes\mathbb{P} = \int_{\Omega\times\Omega} f\circ(X,Y)\mathbb{1}_{\Omega\times B}\,\mathrm{d}\mathbb{P}\otimes\mathbb{P} = \int_{\mathbb{R}^2} f\,\mathrm{d}[(X,Y)\mathbb{1}_{\Omega\times B}]_{\#}\mathbb{P}\otimes\mathbb{P}.$$

Thus, the result follows if we can show that

$$[(X,Y)\mathbb{1}_{\Omega\times B}]_{\#}\mathbb{P}\otimes\mathbb{P} = [(X,Y)\circ\Delta\mathbb{1}_B]_{\#}\mathbb{P}$$

as measures on $\mathcal{B}^2 = \mathcal{B}\otimes\mathcal{B}$. Since both measure are finite it follows from Theorem 2.15 that is suffices to check equality on the generator set $\mathcal{B}\times\mathcal{B}$.

To this end, let $C, D \in \mathcal{B}$. Then it holds that

$$\begin{aligned}[(X,Y)\circ\Delta\mathbb{1}_B]^{-1}(C\times D) &= \{\omega\in\Omega : \omega\in B, X(\omega)\in C, Y(\omega)\in D\}\\ &= X^{-1}(C)\cap Y^{-1}(D)\cap B.\end{aligned}$$

Recall that $B \in \sigma(Y)$ so that also $Y^{-1}(D)\cap B \in \sigma(Y)$. Thus, since X and Y are independent

$$\begin{aligned}[(X,Y)\circ\Delta\mathbb{1}_B]_{\#}\mathbb{P}(C\times D) &= \mathbb{P}(X^{-1}(C)\cap Y^{-1}(D)\cap B)\\ &= \mathbb{P}(X^{-1}(C))\mathbb{P}(Y^{-1}(D)\cap B).\end{aligned}$$

On the other hand, since

$$[(X,Y)\mathbb{1}_{\Omega\times B}]^{-1}(C\times D) = X^{-1}(C)\times Y^{-1}(D)\cap B,$$

we have that

$$\begin{aligned}[(X,Y)\mathbb{1}_{\Omega\times B}]_{\#}\mathbb{P}\otimes\mathbb{P}(C\times D) &= \mathbb{P}\otimes\mathbb{P}(X^{-1}(C)\times Y^{-1}(D)\cap B)\\ &= \mathbb{P}(X^{-1}(C))\mathbb{P}(Y^{-1}(D)\cap B).\end{aligned}$$

We therefore conclude that both measures are equal on the generator set of $\mathcal{B}^2$ and hence are equal on the entire σ-algebra.

(b) As noted in the proof of Lemma 11.8 the measures $(X,Y)_{\#}\mathbb{P}\otimes\mathbb{P}$ and $X_{\#}\mathbb{P}\otimes Y_{\#}\mathbb{P}$ agree on the generator set of $\mathcal{B}^2$. Thus, using Theorem 11.5 the are equal on the entire σ-algebra. Using Fubini-Tonelli twice we then have that

$$\begin{aligned}\int_{R\times C} f\mathrm{d}(X,Y)_{\#}\mathbb{P}\otimes\mathbb{P} &= \int_{R\times C} f\,\mathrm{d}X_{\#}\mathbb{P}\otimes Y_{\#}\mathbb{P}\\ &= \int_{\mathbb{R}}\int_C f(x,y)Y_{\#}\mathbb{P}(\mathrm{d}y)X_{\#}\mathbb{P}(\mathrm{d}x)\\ &= \int_C\int_{\mathbb{R}} f(x,y)X_{\#}\mathbb{P}(\mathrm{d}X)Y_{\#}\mathbb{P}(\mathrm{d}y).\end{aligned}$$