

TU/e, 2MBA70

Solutions to problems for Measure and Probability Theory



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Chapter 2: Measurable spaces (sigma-algebras and measures)

Problem 2.6

First note that if $\mu(A \cap B) = \infty$ then by property 2 we have that also $\mu(A)$, $\mu(B)$ and $\mu(A \cup B) = \infty$ and hence the result holds trivially. So assume now that $\mu(A \cap B) < \infty$. Since

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

it follows from property 1 that

$$\mu(A \cup B) = \mu(A \setminus (A \cap B)) + \mu(A \cap B) + \mu(B \setminus (A \cap B)).$$

Adding $\mu(A \cap B) < \infty$ to both side we get

$$\begin{aligned} \mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus (A \cap B)) + \mu(A \cap B) + \mu(B \setminus (A \cap B)) + \mu(A \cap B) \\ &= \mu(A) + \mu(B), \end{aligned}$$

where the last line follows from applying property 3 twice.

Problem 2.7

The idea is to construct a family of disjoint sets $(E_i)_{i \in \mathbb{N}}$ with the following properties:

1. $E_i \subset A_i$, and
2. $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} A_i$.

If such a sequence exists then we have

$$\begin{aligned} \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) &= \mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) && \text{by 2} \\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{because } E_i \text{ are disjoint and } \mu \text{ is } \sigma\text{-additive} \\ &\leq \sum_{i=1}^{\infty} \mu(A_i) && \text{by 1 and monotone property of } \mu. \end{aligned}$$

So we are left to construct the required family of sets $(E_i)_{i \in \mathbb{N}}$. The following set will do:

$$E_1 = A_1 \quad E_i = A_i \setminus \bigcup_{k < i} A_k \text{ for all } i > 1.$$

Note that by definition the set E_i are pair-wise disjoint and property 1 holds. Finally, property 2 holds since $\bigcup_{i=1}^k E_i = \bigcup_{i=1}^k A_i$ holds for all $k \geq 1$.

Problem 2.9 (23 points) Let \mathcal{O} denote the open sets in \mathbb{R} .

1. (2 points) Note that the interval (a, b) is open for any $a < b \in \mathbb{R}$. Hence $\mathcal{A}_1 \subset \mathcal{A}'_1 \subset \mathcal{O}$ and thus by Lemma 2.1.5 we have that $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}'_1) \subset \sigma(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$.

2. (2 points) The inclusion \supset is trivial. So assume that $x \in O$. Then by definition there exist an $r > 0$ such that the ball $B_x(r) \subset O$. But $B_x(r) = (x - r, x + r) \in \mathcal{A}_1$ so $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$.

3. (3 points) Take $O \in \mathcal{O}$. If we can show that $O \in \sigma(\mathcal{A})$ then $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. The result then follows from 1.

From 2 it follows that O is a union over a subset collection of interval (a, b) where $a, b \in \mathbb{Q}$. Since \mathbb{Q} is countable, the collection $\{(a, b) : a < b \in \mathbb{Q}\}$ is also countable and hence $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$, from which it follows that $\mathcal{B}_{\mathbb{R}} \subset \sigma(\mathcal{A})$.

4. (1 point) This follows immediately from 1 and 3 since these imply that $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}'_1) \subset \mathcal{B}_{\mathbb{R}}$.

5. (3 points) The inclusion \subset is trivial, since $(a, b] \subset (a, b + 1/j)$ for any $j \in \mathbb{N}$. For the other inclusion we argue by contradiction. Suppose that $x \in \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$ but $x \notin (a, b]$. Then $x > b$ and hence there exists a $j \in \mathbb{N}$ such that $(b - x) > 1/j$. But this implies that $x \notin (a, b + 1/j)$ which is a contradiction. So we conclude that $(a, b] \supset \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$.

6. (3 points) This time the inclusion \supset is trivial since $(a, b - 1/j] \subset (a, b)$ for every $j \in \mathbb{N}$. For the other inclusion suppose that $x \in (a, b)$. Then there exists a $r > 0$ such that the interval $(x - r, x + r) \subset (a, b)$. In particular, this implies that $b - (x + r) > 0$. Now take any $j \in \mathbb{N}$ such that $j > 1/(b - (x + r))$. Then $b - x > r + 1/j$ which implies that $(x - r, x + r) \subset (x - r, b - 1/j]$ and hence $x \in \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$.

7. (4 points) It is clear that $\mathcal{A}_2 \subset \mathcal{A}'_2$. By 5 it follows that any interval $(a, b]$ can be obtained as a countable intersection of intervals of the form $(a, b + 1/j)$. By 4 $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}'_1)$ which by Lemma 2.1.2 contains $\bigcap_{j \in \mathbb{N}} (a, b + 1/j) = (a, b]$. So we conclude that any interval $(a, b] \in \mathcal{B}_{\mathbb{R}}$ from which it now follows that

$$\sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}'_2) \subset \sigma(\mathcal{A}'_1) = \mathcal{B}_{\mathbb{R}}.$$

For the other inclusion we consider a set (a, b) with $a, b \in \mathbb{Q}$. Then by 6 we have that $(a, b) = \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$ where the later is a countable union of sets $(c, d]$ with $c, d \in \mathbb{Q}$ which must be in $\sigma(\mathcal{A}_2)$ by definition of a σ -algebra. Hence, any interval $(a, b) \in \sigma(\mathcal{A}_2)$ and we thus conclude, using 3, that

$$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}'_2) \subset \sigma(\mathcal{A}'_1) = \mathcal{B}_{\mathbb{R}},$$

which implies the result.

8. (2 points) Step 1 is to show that any interval $[a, b)$ can be obtained as a countable intersection of intervals $(a - 1/j, b)$. From this we can conclude that any set $[a, b)$ must be in $\mathcal{B}_{\mathbb{R}}$ proving inclusions \subset .

For the other inclusions we have to show that any interval (a, b) can be obtained as a countable union of intervals $[a + 1/j, b)$, which implies that (a, b) must be in the σ -algebra generated by $[a, b)$.

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9. (3 points) The main tool is to show that each of the intervals $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$ can be obtained by taking any allowed set operation for σ -algebras, i.e. countable unions/intersections and finite complements. This will help use prove the \subset inclusions.

Then we show that any set of the form (a, b) , $[a, b)$ or $(a, b]$ can also be obtained through countable unions/intersections and finite complements of intervals of the forms $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$. These will then yield the \supset inclusions and finish the proof.

Chapter 3: Measurable functions and stochastic objects

Problem 3.2 “ \subset ” By definition, the product σ -algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$ is defined as the σ -algebra generated by the collection

$$\mathcal{A} := \left\{ A \times B \subset \Omega_1 \times \Omega_2 : A \in \mathcal{F}_1, B \in \mathcal{F}_2 \right\}.$$

Since $A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B)$, we have that

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \sigma(\pi_1, \pi_2).$$

“ \supset ” Let $C \in \{\pi_i^{-1}(A) : i = 1, 2, A \in \mathcal{F}_1\}$. Then there exist sets $A \in \mathcal{F}_1$ or $B \in \mathcal{F}_2$ such that $C = \pi_1^{-1}(A) = A \times \Omega_2$ or $C = \pi_2^{-1}(B) = \Omega_1 \times B$. Either way, since $\Omega_1 \in \mathcal{F}_1$ and $\Omega_2 \in \mathcal{F}_2$, we have that $C \in \mathcal{A}$.

Problem 3.3 It is clear that $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Suppose a sequence of mutually disjoint sets $B_i \in \mathcal{G}$, $i \in \mathbb{N}$, is given. Then,

$$f_{\#}\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) = \sum_{i=1}^{\infty} f_{\#}\mu(B_i).$$

Problem 3.5

- (a) Some meaningful explanation would suffice.
- (b) By Proposition 2.1.8 and Problem 2.9, we know that $\mathcal{B}_{\mathbb{R}}$ is generated by intervals of the form $(-\infty, a]$ with $a \in \mathbb{Q}$. As a consequence, $\mathcal{B}_{\mathbb{R}}$ is also generated by intervals of the form $(a, +\infty)$ with $a \in \mathbb{Q}$. Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\}$$

is measurable for every $a \in \mathbb{Q}$. For brevity, we write $\{f + g > a\}$. The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \left(\{f > t\} \cap \{g > a - t\} \right).$$

We first show the inclusion ‘ \subset ’. If $\omega \in \Omega$ is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some $t \in \mathbb{Q}$ such that

$$f(\omega) > t > a - g(\omega),$$

and thus $f(\omega) > t$ and $g(\omega) > a - t$. So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \left(\{f > t\} \cap \{g > a - t\} \right).$$

Now we will show the inclusion ‘ \supset ’. Let $\omega \in \Omega$ be such that $f(\omega) > t$ and $g(\omega) > a - t$. Then, by adding the inequalities, we know that $f(\omega) + g(\omega) > a$.

(c) The constant function $f(\omega) = a$ is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \quad \forall B \in \mathcal{B}_{\mathbb{R}}.$$

(d) Similar to the proof of Point (2) of Proposition 3.2.12.

(e) Let $g(\omega) \neq 0$ for all $\omega \in \Omega$. Then, since g is measurable, we have that

$$\begin{aligned} \{1/g > a\} &= \{g < 1/a, g > 0\} \cup \{g > 1/a, g < 0\} \\ &= \left(\{g < 1/a\} \cap \{g > 0\} \right) \cup \left(\{g > 1/a\} \cap \{g < 0\} \right) \in \mathcal{F}, \end{aligned}$$

thus implying that $1/g$ is measurable.

(f) Point (e) and Point (4) of Proposition 3.2.12 yields Point (5) of Proposition 3.2.12.

Problem 3.6 From (3.6), we have for any $a \in \mathbb{R}$,

$$\left\{ \sup_{n \geq 1} f_n > a \right\} = \bigcup_{n \geq 1} \{f_n > a\} \in \mathcal{F},$$

Since \mathcal{F} is a σ -algebra and f_n is measurable for all $n \geq 1$, i.e., $\{f_n > a\} \in \mathcal{F}$ for all $n \geq 1$.

Problem 3.7

(a) Note that

$$f_M = M \mathbf{1}_{\{f \geq M\}} + f \mathbf{1}_{\{|f| < M\}} - M \mathbf{1}_{\{f \leq -M\}}.$$

Since the sets

$$\{f \geq M\}, \quad \{f \leq -M\}, \quad \{|f| < M\} \quad \text{are } \mathcal{F}\text{-measurable,}$$

their corresponding indicator functions are $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable. Since f_M is the sum of products of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable functions, we conclude that f_M is also $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable.

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- (b) It is easy to see that f_M converges pointwise to f as $M \rightarrow \infty$, i.e.,

$$\lim_{M \rightarrow \infty} f_M(\omega) = f(\omega) \quad \forall \omega \in \Omega.$$

Indeed, if $\omega \in \Omega$ is such that $f(\omega) = +\infty$, then

$$\lim_{M \rightarrow \infty} f_M(\omega) = \lim_{M \rightarrow \infty} M = +\infty = f(\omega),$$

and similarly for $\omega \in \Omega$ for which $f(\omega) = -\infty$. On the other hand, for any $\omega \in \Omega$ with $f(\omega) \in \mathbb{R}$, there is some $N_0(\omega) \in \mathbb{N}$ such that $f_N(\omega) = f(\omega)$ for all $N \geq N_0(\omega)$, and hence,

$$\lim_{M \rightarrow \infty} f_M(\omega) = f(\omega).$$

Since f is the limit of a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable functions, we conclude from Lemma 3.2.13 that f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable.

Problem 3.9

- (a) For the probability space, take $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0, 1]$.

Observe that the function $H_{\gamma}(z)$ is continuous and hence has an inverse $g_{\gamma}(y) = \gamma \tan(\pi(y - 1/2))$ on $[0, 1]$.

So the function $Y[0, 1] \rightarrow \mathbb{R}$ defined by $Y(x) = g_{\gamma}(x)$ has the correct distribution as

$$\mathbb{P}(Y^{-1}((-\infty, t])) = \mathbb{P}(g_{\gamma}^{-1}((-\infty, t])) = \lambda(H_{\gamma}((-\infty, t])) = H_{\gamma}(t).$$

- (b) Note that g_{γ} is continuous on $[0, 1]$ and hence measurable.
(c) For any $t \geq 0$, the cdf of the Poisson random variable is given by

$$F_{\lambda}(t) = \sum_{n=0}^{\lceil t \rceil} f_{\lambda}(n),$$

where $\lceil t \rceil$ is the ceiling of t , i.e. the smallest integer $k \geq t$.

- (d) For the probability space, we again take $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0, 1]$.

Now for any $y \in [0, 1]$ let $k := k(y)$ be such that

$$\sum_{n=1}^k f_{\lambda}(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_{\lambda}(n) < y,$$

where the last sum is interpreted as -1 if $k = 0$.

Now define $X(y) = k(y) : [0, 1] \rightarrow \mathbb{R}$. Then $k(y) \leq t$ if and only if $y \leq F_\lambda(t)$ and hence

$$X^{-1}((-\infty, t]) = \{y \in [0, 1] : k(y) \in (-\infty, t]\} = \{y \in [0, 1] : y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

- (e) It follows from the above computation that $X^{-1}((-\infty, t]) = \{y \in [0, 1] : y \in (0, F_\lambda(t)]\}$. Since the latter is a measurable set we conclude that $X^{-1}((-\infty, t])$ is measurable for all t and since these generate the Borel σ -algebra X is measurable.
- (f) for any $\ell \in \mathbb{N}$ define the sets $A_\ell = (n - 1 - 1/\ell, n - 1 + 1/\ell]$. Then A_ℓ is a decreasing set with $\lim_{\ell \rightarrow \infty} A_\ell = \{n\}$. Moreover, $A_\ell = (-\infty, n - 1 + 1/\ell] \setminus (-\infty, n - 1 - 1/\ell]$ and $\mathbb{P}(A_1) < \infty$. It now follows from continuity from above and (d) that

$$\begin{aligned} X_\# \mathbb{P}(\{n\}) &= \lim_{\ell \rightarrow \infty} X_\# \mathbb{P}(A_\ell) \\ &= \lim_{\ell \rightarrow \infty} X_\# \mathbb{P}((-\infty, n - 1 + 1/\ell]) - X_\# \mathbb{P}((-\infty, n - 1 - 1/\ell]) \\ &= F_\lambda(n - 1 + 1/\ell) - F_\lambda(n - 1 - 1/\ell) \\ &= \sum_{k=0}^n f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n). \end{aligned}$$

Chapter 4: The Lebesgue Integral

Problem 4.2

The idea is to apply the monotone convergence theorem (Theorem 4.3.4). To this end we first note that

$$\|f_n(\omega) - f(\omega)\| \leq 2^{-n} \quad \text{for all } n \in \mathbb{N}, \omega \in \Omega.$$

From this it follows that $f_n(\omega) \leq 2^{-n} + f(\omega)$ and hence

$$\begin{aligned} \|[f_n](\omega) - f(\omega)\| &= \|2^n - f(\omega)\| \mathbf{1}_{2^n \leq f_n} + \|f_n(\omega) - f(\omega)\| \mathbf{1}_{f_n < 2^n} \\ &\leq 2^{-n} + 2^{-n} \end{aligned}$$

from which we conclude that $[f_n] \rightarrow f$.

The final part is to show that $[f_n] \leq [f_{n+1}]$ which follows if we can show that $f_n \leq f_{n+1}$. For this we first note that for all $k \geq 1$ $(k+1)2^{-(n+1)} \leq k2^{-n}$. We also note that $2^n \leq 2n+1$. Now suppose that there exist an $n \geq 1$ and ω such that $f_n(\omega) > f_{n+1}(\omega)$. Then it must hold that $f_n(\omega) > 0$ and hence $f_n(\omega) = k2^{-n}$ for some $k \geq 1$. This then implies that $f_{n+1}(\omega) = \ell 2^{-n}$ for some $\ell \geq k+1$. But this cannot be the case as $[\ell 2^{-n}, (\ell+1)2^{-n}) \cap [k2^{-n}, (k+1)2^{-n}) = \emptyset$ while $f(\omega)$ should be in both sets.

Problem 4.3

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- (a) By definition, we have that $\nu_f(\Omega) = \int_{\Omega} f \, d\mu = 1$. Now let $(A_n)_{n \in \mathbb{N}}$ be a family of mutually disjoint measurable sets. Then we have that the sequence

$$g_n := \sum_{i=1}^n f \mathbf{1}_{A_i} = f \mathbf{1}_{\bigcup_{i=1}^n A_i} \longrightarrow g := f \mathbf{1}_{\bigcup_{i \in \mathbb{N}} A_i} \quad \text{pointwise monotonically.}$$

By MCT, we then have that

$$\nu_f \left(\bigcup_{i \in \mathbb{N}} A_i \right) = \int_{\bigcup_{i \in \mathbb{N}} A_i} f \, d\mu = \lim_{n \rightarrow \infty} \int_{\bigcup_{i=1}^n A_i} f \, d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{A_i} f \, d\mu = \sum_{i \in \mathbb{N}} \nu_f(A_i),$$

thus showing that ν_f is a probability measure on (Ω, \mathcal{F}) .

- (b) Following the hint, we start by considering nonnegative simple functions g . Suppose $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ for $a_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$ mutually disjoint. Then,

$$\int_{\Omega} g \, d\nu_f = \sum_{i=1}^n a_i = \nu_f(A_i) = \sum_{i=1}^n a_i \int_{A_i} f \, d\mu = \int_{\Omega} g f \, d\mu.$$

Now let g be a nonnegative measurable function and $[g]_n$ be a sequence of nonnegative simple functions that converge pointwise monotonically to g . Then MCT yields

$$\int_{\Omega} g \, d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} [g]_n \, d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} [g]_n f \, d\mu = \int_{\Omega} g f \, d\mu,$$

where we used the fact that $[g]_n f$ converges pointwise monotonically to $g f$.

- (c) Let g be measurable. Then $g = g^+ - g^-$, where g^{\pm} are nonnegative measurable functions. Since f is nonnegative, we have that $(fg)^{\pm} = fg^{\pm}$. Due to (b), we deduce

$$\int_{\Omega} g^{\pm} \, d\nu_f = \int_{\Omega} g^{\pm} f \, d\mu = \int_{\Omega} (gf)^{\pm} \, d\mu.$$

Hence, g^{\pm} is ν_f -integrable if and only if $(gf)^{\pm}$ is μ -integrable. Consequently, g is ν_f -integrable if and only if gf is μ -integrable, since

$$\int_{\Omega} |g| \, d\nu_f = \int_{\Omega} g^+ \, d\nu_f + \int_{\Omega} g^- \, d\nu_f = \int_{\Omega} g^+ f \, d\mu + \int_{\Omega} g^- f \, d\mu = \int_{\Omega} |gf| \, d\mu.$$

Problem 4.4

(\Rightarrow) Let f be μ -integrable. Then both $|f| \mathbf{1}_{\{|f| < n\}}$ and $|f| \mathbf{1}_{\{|f| \geq n\}}$ are integrable, due to the monotonicity of the integral. By linearity of the integral,

$$\int_{\Omega} |f| \mathbf{1}_{\{|f| \geq n\}} \, d\mu = \int_{\Omega} |f| \, d\mu - \int_{\Omega} |f| \mathbf{1}_{\{|f| < n\}} \, d\mu.$$

Since the sequence $g_n := |f| \mathbf{1}_{\{|f| < n\}} \geq 0$ converges pointwise monotonically to $|f|$, we can apply MCT to obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f| \mathbf{1}_{\{|f| < n\}} d\mu = \int_{\Omega} |f| d\mu.$$

Hence,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f| \mathbf{1}_{\{|f| \geq n\}} d\mu = \int_{\Omega} |f| d\mu - \lim_{n \rightarrow \infty} \int_{\Omega} |f| \mathbf{1}_{\{|f| < n\}} d\mu = 0.$$

(\Leftarrow) By assumption, there is some $N \geq 1$ such that

$$\int_{\Omega} |f| \mathbf{1}_{\{|f| \geq N\}} d\mu \leq 1.$$

By linearity of the integral,

$$\int_{\Omega} |f| d\mu = \int_{\Omega} |f| \mathbf{1}_{\{|f| < N\}} d\mu + \int_{\Omega} |f| \mathbf{1}_{\{|f| \geq N\}} d\mu \leq N\mu(\{|f| < N\}) + 1.$$

Since μ is a finite measure, the right-hand side is finite, implying that f is μ -integrable.

Problem 4.5

Observe that $\Omega = \bigcup_{n \in \mathbb{N}} \{|f| > n\}$.

We then get that

$$\sum_{n=1}^{\infty} \int_{\{|f| > n\}} |f| d\mu = \int_{\Omega} |f| d\mu < \infty.$$

This implies that for some N and all $n \geq N$: $\int_{\{|f| > n\}} |f| d\mu < 1/n$ or else the sum cannot be finite.

Now let $\varepsilon > 0$, take $M > \max\{N, 2/\varepsilon\}$ and $\delta = \varepsilon/(2M)$. Then

$$\begin{aligned} \int_A |f| d\mu &= \int_A |f| \mathbf{1}_{|f| \leq M} d\mu + \int_A |f| \mathbf{1}_{|f| > M} d\mu \\ &\leq M\mu(A) + \frac{1}{M} \leq M\delta + \frac{1}{M} < \varepsilon. \end{aligned}$$

Problem 4.6

- (a) Let $t \in \mathbb{R}$ and consider the set $A_t = (-\infty, t]$. Then by definition of the probability density function

$$\nu(A_t) = \int_{-\infty}^t \rho d\lambda = (X_{\#}\mathbb{P})((-\infty, t]).$$

We thus conclude that ν and $X_{\#}\mathbb{P}$ coincide on the family of set A_t and since these generate \mathcal{B} Theorem 2.2.17 implies that $\nu = X_{\#}\mathbb{P}$.

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- (b) Since g is a simple function, there exist an $N \in \mathbb{N}$, constants $(a_n)_{1 \leq n \leq N}$ and measurable sets $(A_n)_{1 \leq n \leq N}$ such that

$$g = \sum_{n=1}^N a_n \mathbf{1}_{A_n}.$$

Now, by first applying Proposition 4.8.11 and then part (a), we get that

$$\begin{aligned} \mathbb{E}[g(X)] &= \int_{\Omega} g(X) \, d\mathbb{P} = \int_{\Omega} g \, dX_{\#}\mathbb{P} = \int_{\Omega} g \, d\nu \\ &= \int_{\Omega} \sum_{n=1}^N a_n \mathbf{1}_{A_n} \, d\nu = \sum_{n=1}^N a_n \nu(A_n) = \sum_{n=1}^N a_n \int_{A_n} \rho \, d\lambda \\ &= \int_{\mathbb{R}} \sum_{n=1}^N a_n \mathbf{1}_{A_n} \rho \, d\lambda = \int_{\mathbb{R}} g \rho \, d\lambda \end{aligned}$$

- (c) First note that by part (b) we have that

$$\int_{\Omega} [h]_n(X) \, d\mathbb{P} = \int_{\mathbb{R}} [h_n] \rho \, d\lambda.$$

Now we split the function $[h_n]\rho$ into its positive and negative part and note that

$$([h_n]\rho)^+ = [h]_n^+ \rho^+ + [h]_n^- \rho^- \quad \text{and} \quad ([h_n]\rho)^- = [h]_n^+ \rho^- + [h]_n^- \rho^+,$$

where $[h]_n^{\pm}$ and ρ^{\pm} denote the positive and negative parts of $[h]_n$ and ρ .

We will show that

$$\int_{\Omega} h^+(X) \, d\mathbb{P} = \int_{\mathbb{R}} h^+ \rho \, d\lambda.$$

The proof for the negative part is similar.

$$\begin{aligned} \int_{\mathbb{R}} h^+ \, d\nu &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n^+ \, d\nu && \text{by Theorem 4.3.4} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n^+ \rho \, d\lambda && \text{by part (b)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n^+ \rho^+ \, d\lambda - \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n^+ \rho^- \, d\lambda && \text{by linearity of integration} \\ &= \int_{\mathbb{R}} h^+ \rho^+ \, d\lambda - \int_{\mathbb{R}} h^+ \rho^- \, d\lambda && \text{by Theorem 4.3.4} \\ &= \int_{\mathbb{R}} h^+ \rho \, d\lambda && \text{by linearity of integration} \end{aligned}$$

(d)

$$\begin{aligned}\mathbb{E}[h(X)] &= \int_{\Omega} h(X) \, d\mathbb{P} \\ &= \int_{\mathbb{R}} h \, dX_{\#}\mathbb{P} && \text{by Proposition 4.8.11} \\ &= \int_{\mathbb{R}} h \, d\nu && \text{by part (a)} \\ &= \int_{\mathbb{R}} h\rho \, d\lambda && \text{by part (c).}\end{aligned}$$

Problem 4.7 This follows from the following inequalities:

$$\int_{\mathbb{R}} |f|^p \, d\mu \geq \int_{\{|f| \geq t\}} |f|^p \, d\mu \geq t^p \mu(\{|f| \geq t\}).$$

Chapter 5: Convergence of integrals and functions

Problem 5.2

- (a) Let $t_0 \in (a, b)$ be fixed. It suffices to check the continuity result for arbitrary sequences $(t_n)_{n \geq 1} \subset (a, b)$ such that $t_n \rightarrow t_0$ as $n \rightarrow \infty$. Fix such a sequence and define $g_n(\omega) := f(\omega, t_n)$ for all $\omega \in \Omega$ and $n \geq 1$. Since $\lim_{t \rightarrow t_0} f(\omega, t) = f(\omega, t_0)$ for all $\omega \in \Omega$, we deduce that $\lim_{n \rightarrow \infty} g_n(\omega) = f(\omega, t_0)$ for every $\omega \in \Omega$. Moreover, by assumption $|g_n| \leq g$ for all $n \geq 1$ and g is integrable. By the Dominated Convergence Theorem

$$\lim_{n \rightarrow \infty} \int_{\Omega} g_n(\omega) \, \mu(d\omega) = \int_{\Omega} f(\omega, t_0) \, \mu(d\omega).$$

As the chosen sequence was arbitrary, we deduce that $\lim_{t \rightarrow t_0} F(t) = F(t_0)$.

- (b) If $t \mapsto f(\omega, t)$ is continuous on (a, b) for all $\omega \in \Omega$ then $\lim_{t \rightarrow t_0} f(\omega, t) = f(\omega, t_0)$ at every $t_0 \in (a, b)$ for all $\omega \in \Omega$. In particular, (a) applies, showing that $\lim_{t \rightarrow t_0} F(t) = F(t_0)$ for every $t_0 \in (a, b)$, i.e., F is continuous on (a, b) .

Problem 5.3

- (1) We start by showing that $(\partial f / \partial t)(\cdot, t)$ is measurable. Let $(t_n)_{n \geq 1} \subset (a, b)$ be an arbitrary sequence with $t_n \neq t$ and $t_n \rightarrow t$ for $n \rightarrow \infty$. We set

$$g_n(\omega) = \frac{f(\omega, t_n) - f(\omega, t)}{t_n - t}.$$

Clearly, g_n is measurable for every $n \geq 1$. Moreover, $\lim_{n \rightarrow \infty} g_n(\omega) = (\partial f / \partial t)(\omega, t)$ by the definition of the derivative. Since $(\partial f / \partial t)(\cdot, t)$ is the pointwise limit of a sequence of measurable functions, it is also measurable. Clearly, $(\partial f / \partial t)(\cdot, t)$ is integrable since

$$\int_{\Omega} |(\partial f / \partial t)(\omega, t)| \, \mu(d\omega) \leq \int_{\Omega} g \, d\mu < +\infty.$$

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- (2) Let $t_0 \in (a, b)$ and suppose w.l.o.g. $t_0 < t$. Since $t \mapsto f(\omega, t)$ is differentiable on (a, b) for all $\omega \in \Omega$, the Mean Value Theorem gives

$$\frac{f(\omega, t) - f(\omega, t_0)}{t - t_0} = (\partial f / \partial t)(\omega, \tau) \quad \text{for some } \tau \in (t_0, t).$$

Taking the modulus on both sides, we obtain

$$\left| \frac{f(\omega, t) - f(\omega, t_0)}{t - t_0} \right| \leq |(\partial f / \partial t)(\omega, \tau)| \leq g(\omega) \quad \text{for all } \omega \in \Omega.$$

- (3) We now have all the ingredients needed to apply the DCT, which yields

$$\lim_{n \rightarrow \infty} \frac{F(t_n) - F(t)}{t_n - t} = \lim_{n \rightarrow \infty} \int_{\Omega} g_n \, d\mu = \int_{\Omega} (\partial f / \partial t)(\omega, t) \, \mu(d\omega).$$

Since $t \in (a, b)$ and the sequence $(t_n)_{n \geq 1}$ was arbitrary, we conclude that F is differentiable on (a, b) with

$$F'(t) = \int_{\Omega} (\partial f / \partial t)(\omega, t) \, \mu(d\omega).$$

Problem 5.3

- (a) Note that the integrand $f_n(x) = \frac{1+nx^2}{(1+x^2)^n}$ is continuous on $[0, 1]$ and non-negative. Hence, the Riemann integral and Lebesgue integral coincide, i.e.,

$$\int_0^1 f_n(x) \, dx = \int_{[0,1]} f_n \, d\lambda.$$

Observe that we have the following pointwise limit

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0, 1], \end{cases}$$

i.e., $\lim_{n \rightarrow \infty} f_n = 0$ λ -almost everywhere. Moreover, $f_n(x) \leq 1$ for every $x \in [0, 1]$ and $n \geq 1$. Since the constant function $g \equiv 1$ is λ -integrable on $[0, 1]$, it is a valid dominator. Hence, the DCT gives

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \, dx = \lim_{n \rightarrow \infty} \int_{[0,1]} f_n \, d\lambda = \int_{[0,1]} \lim_{n \rightarrow \infty} f_n \, d\lambda = 0$$

- (b) For the purpose of convergence, we consider $n \geq 3$. Note that the integrand $f_n(x) = \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right)$ is continuous on $(0, +\infty)$ with pointwise limit

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{if } x \in (0, 1), \\ 1/2 & \text{if } x = 1, \\ 1/x^2 & \text{if } x > 1, \end{cases}$$

Setting the function

$$g(x) = \begin{cases} 1 & \text{for } x \in (0, 1), \\ \frac{1}{x^2} & \text{for } x \geq 1, \end{cases}$$

we see that $f_n \leq g$ λ -almost everywhere in $(0, +\infty)$ and for all $n \geq 3$. Indeed, for $x \geq 1$, we obtain

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq \frac{x^{n-2}}{x^n} = \frac{1}{x^2},$$

while for $x \in (0, 1)$, we have

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq 1.$$

Notice that g is non-negative and λ -integrable on $(0, +\infty)$. Indeed, using the MCT,

$$\begin{aligned} \int_{(0, +\infty)} g \, d\lambda &= \int_{(0,1)} g \, d\lambda + \int_{(1, +\infty)} g \, d\lambda = 1 + \lim_{n \rightarrow \infty} \int_{(1,n)} g \, d\lambda \\ &= 1 + \lim_{n \rightarrow \infty} \int_1^n \frac{1}{x^2} \, dx = 1 + \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 2 < +\infty. \end{aligned}$$

To conclude, we apply DCT to deduce that the limit is 1.

Problem 5.4

The proof follows verbatim to the proof of the Dominated Convergence Theorem.

Problem 5.7

Let F_n denote the cdf of $Y_n = \|X_n - X\|$ and F_0 denote the cdf of 0. By Definition 5.2.9 and Lemma 5.2.8 we have that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $F_n(t) \rightarrow F_0(t)$ for all continuity points t of F_0 . This is equivalent to showing that $1 - F_n(t) \rightarrow 1 - F_0(t)$, where

$$1 - F_0(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ 1 & \text{else.} \end{cases}$$

Now note that the only discontinuity point of F_0 is 0. Moreover, $1 - F_n(t) = 0 = F_0(t)$ for all $t < 0$. Hence it follows that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $1 - F_n(t) \rightarrow 0$ for all $t > 0$, which is what we needed to show.

Problem 5.8

(a) For this let $h_t(x) = \mathbf{1}_{(-\infty, t]}$ and note that

$$F_n(t) = (X_n)_\# \mathbb{P}_n((-\infty, x]) = \int_{\mathbb{R}} \mathbb{1}_{(-\infty, t]} \, d(X_n)_\# \mathbb{P}_n = \int_{\mathbb{R}} h_t \, d\mu_n.$$

and similarly $F(t) = \int_{\mathbb{R}} h_t \, d\mu$

-
- (b) The function h is discontinuous only at t , i.e. $\mathcal{C}_h = \mathbb{R} \setminus \{t\}$. Moreover, for any $\varepsilon > 0$
- $$\mu((t - \varepsilon, t + \varepsilon)) = \mu((t - \varepsilon, t]) + \mu((t, t + \varepsilon)) = F(t) - F(t - \varepsilon) + F(t + \varepsilon) - F(t).$$

Since F is continuous at t , the right hand side goes to zero as $\varepsilon \rightarrow 0$. Therefore

$$\mu(\{t\}) = \lim_{\varepsilon \rightarrow 0} \mu((t - \varepsilon, t + \varepsilon)) = 0,$$

which implies that $\mu(\mathcal{C}_h) = 1$.

- (c) The result follows by applying condition (2) in Theorem 5.2.7.
- (d) Let $\varepsilon > 0$, pick such a δ and partition the interval $[-K, K]$ into $L_\delta := \lceil \frac{4K}{\delta} \rceil$ intervals $I_\ell = (a_\ell, b_\ell]$ of equal length, which is $\leq \delta/2 < \delta$. Now we define the simple function

$$\hat{g} := \sum_{\ell=1}^L h(b_\ell) \mathbb{1}_{I_\ell},$$

- (e) Let $M = L$, $\beta_\ell = \sum_{t=1}^\ell h(b_t)$ and $t_\ell = b_\ell$. Then

$$\hat{g} := \sum_{\ell=1}^L \beta_\ell \mathbb{1}_{(-\infty, b_\ell]}.$$

- (f) Using the representation in (e) we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[\hat{g}(X_n)] &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell \int_{\mathbb{R}} \mathbb{1}_{X_n^{-1}((-\infty, b_\ell])} d\mathbb{P} \\ &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell (X_n)_\# \mathbb{P}((-\infty, b_\ell]) \\ &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell F_n(b_\ell) \\ &= \sum_{\ell=1}^L F(b_\ell) = \mathbb{E}[\hat{g}(X)]. \end{aligned}$$

- (g) Using the representation of \hat{g} in (d) we note that $\|x - y\| < \varepsilon$ for all $x, y \in I_\ell$. This then implies that $\|g(x) - \hat{g}(y)\| \leq \varepsilon$ from which it follows that

$$\begin{aligned} \|\mathbb{E}[g(X_n)] - \mathbb{E}[g(X)]\| &\leq \|\mathbb{E}[g(X_n)] - \mathbb{E}[\hat{g}(X_n)]\| + \|\mathbb{E}[g(X)] - \mathbb{E}[\hat{g}(X)]\| \\ &\quad + \|\mathbb{E}[\hat{g}(X_n)] - \mathbb{E}[\hat{g}(X)]\| \\ &\leq 2\varepsilon + \|\mathbb{E}[\hat{g}(X_n)] - \mathbb{E}[\hat{g}(X)]\|. \end{aligned}$$

We have shown in (f) that the last term goes to zero as $n \rightarrow \infty$. Since ε was arbitrary we conclude that (??) holds.

(h) This now follows from Theorem 5.2.7 (3).

Problem 5.9 Suppose that $X_n \xrightarrow{\text{a.s.}} X$. Then by Lemma 5.2.16 this is equivalent to $\mathbb{P}(\|X_n - X\| > \varepsilon \text{ i.o.}) = 0$ for all $\varepsilon > 0$.

For now fix an $\varepsilon > 0$ and write $A_n := \{\|X_n - X\| > \varepsilon\}$. Recall that

$$\{A_n \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n$$

and note two things:

(a) The sets $B_k := \bigcup_{n \geq k} A_n$ are non-increasing, i.e. $B_k \supset B_{k+1}$, and

(b) $\mathbb{P}(A_k) \leq \mathbb{P}(\bigcup_{n \geq k} A_n) = \mathbb{P}(B_k)$.

We then have that:

$$\begin{aligned} 0 &= \mathbb{P}(\{A_n \text{ i.o.}\}) && \text{by assumption} \\ &= \mathbb{P}\left(\bigcap_{k=1}^{\infty} B_k\right) && \text{by Lemma 5.2.16} \\ &= \lim_{k \rightarrow \infty} \mathbb{P}(B_k) && \text{by continuity from above (Proposition 2.2.15)} \\ &\geq \lim_{k \rightarrow \infty} \mathbb{P}(A_k) && \text{by (b).} \end{aligned}$$

Chapter 6: L^p -spaces

Problem 6.2

Problem 6.4

Let $E_n := \{\omega \in \mathbb{R}^d : |f(\omega)| \geq n\}$. Since $\mathbf{1}_{E_n} f \rightarrow 0$ as $n \rightarrow \infty$, and $\mathbf{1}_{E_n} |f| \leq |f|$ for every $n \geq 1$, we can apply DCT to conclude that

$$\int_{E_n} |f| \, d\mu = \int_{\mathbb{R}^d} \mathbf{1}_{E_n} |f| \, d\mu \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now pick some $n \geq 1$ such that $\int_{E_n} |f| \, d\mu < \varepsilon/3$ and define

$$f_n(\omega) := \max\{-n, \min\{f(\omega), n\}\}, \quad \omega \in \mathbb{R}^d,$$

i.e., f_n is a truncation of f . From Lusin's theorem, we find a continuous function g such that $f_n \equiv g$ on a compact set $K \subset \mathbb{R}^d$ with $\mu(\mathbb{R}^d \setminus K) < (2\varepsilon)/(3n)$. We assume w.l.o.g. that $|g| \leq n$, since otherwise, we can consider a truncation of g . Altogether, this yields

$$\begin{aligned} \int_{\mathbb{R}^d} |f - g| \, d\mu &= \int_{\mathbb{R}^d} |f - f_n| \, d\mu + \int_{\mathbb{R}^d} |f_n - g| \, d\mu \\ &= \int_{E_n} |f| \, d\mu + \int_{\mathbb{R}^d \setminus K} |f_n - g| \, d\mu \\ &\leq \frac{\varepsilon}{3} + 2n \mu(\mathbb{R}^d \setminus K) \leq \varepsilon. \end{aligned}$$

Finally, $g \in L^1(\mathbb{R}^d, \mu)$ holds simply due to the triangle inequality.

Chapter 7: Fubini-Tonelli

Problem 7.1

1. Note that $\mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{F}_1 \times \mathcal{F}_2$, and hence

$$\sigma(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{F}_1 \otimes \mathcal{F}_2.$$

2. Let $B \in \mathcal{A}_2$. Then we have that

$$\Omega_1 \times B = \bigcup_{n \geq 1} A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$$

since $A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ for all $n \geq 1$. So $\Omega_1 \in \Sigma$

For the second property, let $C \in \Sigma$ and note that $C^c \times B = (\Omega_1 \times B) \setminus (C \times B)$. Since both these sets are in $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ it follows that $C^c \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ and hence $C^c \in \Sigma$.

Finally consider a countable sequence $(C_n)_{n \geq 1}$ of sets in Σ . Then for any $B \in \mathcal{A}_2$

$$\left(\bigcup_{n \geq 1} C_n \right) \times B = \bigcup_{n \geq 1} (C_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2),$$

since each $C_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

3. Note that $\mathcal{A}_1 \subset \Sigma_1 \subset \mathcal{F}_1$. From which it follows that $\Sigma_1 = \mathcal{F}_1$. But then, from the definition of Σ_1 we have that $\mathcal{F}_1 \times \mathcal{A}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.
4. We can show in a similar fashion that

$$\Sigma_2 := \{C \in \mathcal{F}_2 : B \times C \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2) \forall B \in \mathcal{A}_1\}.$$

is a σ -algebra on Ω_2 , from which we conclude that $\mathcal{A}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

5. take any $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Then

$$A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B) = \bigcup_{n, m \geq 1} (A \times B_m) \cap (A_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2).$$

From this we conclude that $\mathcal{F}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$, which finishes the proof.