

TU/e, 2MBA70

# Solutions to problems for Measure and Probability Theory



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Version 0.5 October 20, 2024

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## Chapter 2: Measurable spaces (sigma-algebras and measures)

### Problem 2.6

First note that if  $\mu(A \cap B) = \infty$  then by property 2 we have that also  $\mu(A)$ ,  $\mu(B)$  and  $\mu(A \cup B) = \infty$  and hence the result holds trivially. So assume now that  $\mu(A \cap B) < \infty$ . Since

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

it follows from property 1 that

$$\mu(A \cup B) = \mu(A \setminus (A \cap B)) + \mu(A \cap B) + \mu(B \setminus (A \cap B)).$$

Adding  $\mu(A \cap B) < \infty$  to both side we get

$$\begin{aligned} \mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus (A \cap B)) + \mu(A \cap B) + \mu(B \setminus (A \cap B)) + \mu(A \cap B) \\ &= \mu(A) + \mu(B), \end{aligned}$$

where the last line follows from applying property 3 twice.

### Problem 2.7

The idea is to construct a family of disjoint sets  $(E_i)_{i \in \mathbb{N}}$  with the following properties:

1.  $E_i \subset A_i$ , and
2.  $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} A_i$ .

If such a sequence exists then we have

$$\begin{aligned} \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) &= \mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) && \text{by 2} \\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{because } E_i \text{ are disjoint and } \mu \text{ is } \sigma\text{-additive} \\ &\leq \sum_{i=1}^{\infty} \mu(A_i) && \text{by 1 and monotone property of } \mu. \end{aligned}$$

So we are left to construct the required family of sets  $(E_i)_{i \in \mathbb{N}}$ . The following set will do:

$$E_1 = A_1 \quad E_i = A_i \setminus \bigcup_{k < i} A_k \text{ for all } i > 1.$$

Note that by definition the set  $E_i$  are pair-wise disjoint and property 1 holds. Finally, property 2 holds since  $\bigcup_{i=1}^k E_i = \bigcup_{i=1}^k A_i$  holds for all  $k \geq 1$ .

**Problem 2.9** (23 points) Let  $\mathcal{O}$  denote the open sets in  $\mathbb{R}$ .

1. (2 points) Note that the interval  $(a, b)$  is open for any  $a < b \in \mathbb{R}$ . Hence  $\mathcal{A}_1 \subset \mathcal{A}'_1 \subset \mathcal{O}$  and thus by Lemma 2.1.5 we have that  $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}'_1) \subset \sigma(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$ .

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2. (2 points) The inclusion  $\supset$  is trivial. So assume that  $x \in O$ . Then by definition there exist an  $r > 0$  such that the ball  $B_x(r) \subset O$ . But  $B_x(r) = (x - r, x + r) \in \mathcal{A}_1$  so  $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$ .

3. (3 points) Take  $O \in \mathcal{O}$ . If we can show that  $O \in \sigma(\mathcal{A})$  then  $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$ . The result then follows from 1.

From 2 it follows that  $O$  is a union over a subset collection of interval  $(a, b)$  where  $a, b \in \mathbb{Q}$ . Since  $\mathbb{Q}$  is countable, the collection  $\{(a, b) : a < b \in \mathbb{Q}\}$  is also countable and hence  $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$ , from which it follows that  $\mathcal{B}_{\mathbb{R}} \subset \sigma(\mathcal{A})$ .

4. (1 point) This follows immediately from 1 and 3 since these imply that  $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}'_1) \subset \mathcal{B}_{\mathbb{R}}$ .

5. (3 points) The inclusion  $\subset$  is trivial, since  $(a, b] \subset (a, b + 1/j)$  for any  $j \in \mathbb{N}$ . For the other inclusion we argue by contradiction. Suppose that  $x \in \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$  but  $x \notin (a, b]$ . Then  $x > b$  and hence there exists a  $j \in \mathbb{N}$  such that  $(b - x) > 1/j$ . But this implies that  $x \notin (a, b + 1/j)$  which is a contradiction. So we conclude that  $(a, b] \supset \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$ .

6. (3 points) This time the inclusion  $\supset$  is trivial since  $(a, b - 1/j] \subset (a, b)$  for every  $j \in \mathbb{N}$ . For the other inclusion suppose that  $x \in (a, b)$ . Then there exists a  $r > 0$  such that the interval  $(x - r, x + r) \subset (a, b)$ . In particular, this implies that  $b - (x + r) > 0$ . Now take any  $j \in \mathbb{N}$  such that  $j > 1/(b - (x + r))$ . Then  $b - x > r + 1/j$  which implies that  $(x - r, x + r) \subset (x - r, b - 1/j]$  and hence  $x \in \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$ .

7. (4 points) It is clear that  $\mathcal{A}_2 \subset \mathcal{A}'_2$ . By 5 it follows that any interval  $(a, b]$  can be obtained as a countable intersection of intervals of the form  $(a, b + 1/j)$ . By 4  $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}'_1)$  which by Lemma 2.1.2 contains  $\bigcap_{j \in \mathbb{N}} (a, b + 1/j) = (a, b]$ . So we conclude that any interval  $(a, b] \in \mathcal{B}_{\mathbb{R}}$  from which it now follows that

$$\sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}'_2) \subset \sigma(\mathcal{A}'_1) = \mathcal{B}_{\mathbb{R}}.$$

For the other inclusion we consider a set  $(a, b)$  with  $a, b \in \mathbb{Q}$ . Then by 6 we have that  $(a, b) = \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$  where the later is a countable union of sets  $(c, d]$  with  $c, d \in \mathbb{Q}$  which must be in  $\sigma(\mathcal{A}_2)$  by definition of a  $\sigma$ -algebra. Hence, any interval  $(a, b) \in \sigma(\mathcal{A}_2)$  and we thus conclude, using 3, that

$$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}'_2) \subset \sigma(\mathcal{A}'_1) = \mathcal{B}_{\mathbb{R}},$$

which implies the result.

8. (2 points) Step 1 is to show that any interval  $[a, b)$  can be obtained as a countable intersection of intervals  $(a - 1/j, b)$ . From this we can conclude that any set  $[a, b)$  must be in  $\mathcal{B}_{\mathbb{R}}$  proving inclusions  $\subset$ .

For the other inclusions we have to show that any interval  $(a, b)$  can be obtained as a countable union of intervals  $[a + 1/j, b)$ , which implies that  $(a, b)$  must be in the  $\sigma$ -algebra generated by  $[a, b)$ .

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9. (3 points) The main tool is to show that each of the intervals  $(-\infty, a]$ ,  $(-\infty, a)$ ,  $(a, \infty)$  and  $[a, \infty)$  can be obtained by taking any allowed set operation for  $\sigma$ -algebras, i.e. countable unions/intersections and finite complements. This will help use prove the  $\subset$  inclusions.

Then we show that any set of the form  $(a, b)$ ,  $[a, b)$  or  $(a, b]$  can also be obtained through countable unions/intersections and finite complements of intervals of the forms  $(-\infty, a]$ ,  $(-\infty, a)$ ,  $(a, \infty)$  and  $[a, \infty)$ . These will then yield the  $\supset$  inclusions and finish the proof.

## Chapter 3: Measurable functions and stochastic objects

**Problem 3.2** “ $\subset$ ” By definition, the product  $\sigma$ -algebra  $\mathcal{F}_1 \otimes \mathcal{F}_2$  is defined as the  $\sigma$ -algebra generated by the collection

$$\mathcal{A} := \left\{ A \times B \subset \Omega_1 \times \Omega_2 : A \in \mathcal{F}_1, B \in \mathcal{F}_2 \right\}.$$

Since  $A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B)$ , we have that

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \sigma(\pi_1, \pi_2).$$

“ $\supset$ ” Let  $C \in \{\pi_i^{-1}(A) : i = 1, 2, A \in \mathcal{F}_1\}$ . Then there exist sets  $A \in \mathcal{F}_1$  or  $B \in \mathcal{F}_2$  such that  $C = \pi_1^{-1}(A) = A \times \Omega_2$  or  $C = \pi_2^{-1}(B) = \Omega_1 \times B$ . Either way, since  $\Omega_1 \in \mathcal{F}_1$  and  $\Omega_2 \in \mathcal{F}_2$ , we have that  $C \in \mathcal{A}$ .

**Problem 3.3** It is clear that  $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$ . Suppose a sequence of mutually disjoint sets  $B_i \in \mathcal{G}$ ,  $i \in \mathbb{N}$ , is given. Then,

$$f_{\#}\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) = \sum_{i=1}^{\infty} f_{\#}\mu(B_i).$$

### Problem 3.5

- (a) Some meaningful explanation would suffice.
- (b) By Proposition 2.1.8 and Problem 2.9, we know that  $\mathcal{B}_{\mathbb{R}}$  is generated by intervals of the form  $(-\infty, a]$  with  $a \in \mathbb{Q}$ . As a consequence,  $\mathcal{B}_{\mathbb{R}}$  is also generated by intervals of the form  $(a, +\infty)$  with  $a \in \mathbb{Q}$ . Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\}$$

is measurable for every  $a \in \mathbb{Q}$ . For brevity, we write  $\{f + g > a\}$ . The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \left( \{f > t\} \cap \{g > a - t\} \right).$$

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We first show the inclusion ‘ $\subset$ ’. If  $\omega \in \Omega$  is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some  $t \in \mathbb{Q}$  such that

$$f(\omega) > t > a - g(\omega),$$

and thus  $f(\omega) > t$  and  $g(\omega) > a - t$ . So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \left( \{f > t\} \cap \{g > a - t\} \right).$$

Now we will show the inclusion ‘ $\supset$ ’. Let  $\omega \in \Omega$  be such that  $f(\omega) > t$  and  $g(\omega) > a - t$ . Then, by adding the inequalities, we know that  $f(\omega) + g(\omega) > a$ .

(c) The constant function  $f(\omega) = a$  is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \quad \forall B \in \mathcal{B}_{\mathbb{R}}.$$

(d) Similar to the proof of Point (2) of Proposition 3.2.12.

(e) Let  $g(\omega) \neq 0$  for all  $\omega \in \Omega$ . Then, since  $g$  is measurable, we have that

$$\begin{aligned} \{1/g > a\} &= \{g < 1/a, g > 0\} \cup \{g > 1/a, g < 0\} \\ &= \left( \{g < 1/a\} \cap \{g > 0\} \right) \cup \left( \{g > 1/a\} \cap \{g < 0\} \right) \in \mathcal{F}, \end{aligned}$$

thus implying that  $1/g$  is measurable.

(f) Point (e) and Point (4) of Proposition 3.2.12 yields Point (5) of Proposition 3.2.12.

**Problem 3.6** From (3.6), we have for any  $a \in \mathbb{R}$ ,

$$\left\{ \sup_{n \geq 1} f_n > a \right\} = \bigcup_{n \geq 1} \{f_n > a\} \in \mathcal{F},$$

Since  $\mathcal{F}$  is a  $\sigma$ -algebra and  $f_n$  is measurable for all  $n \geq 1$ , i.e.,  $\{f_n > a\} \in \mathcal{F}$  for all  $n \geq 1$ .

**Problem 3.7**

(a) Note that

$$f_M = M \mathbf{1}_{\{f \geq M\}} + f \mathbf{1}_{\{|f| < M\}} - M \mathbf{1}_{\{f \leq -M\}}.$$

Since the sets

$$\{f \geq M\}, \quad \{f \leq -M\}, \quad \{|f| < M\} \quad \text{are } \mathcal{F}\text{-measurable,}$$

their corresponding indicator functions are  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable. Since  $f_M$  is the sum of products of  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable functions, we conclude that  $f_M$  is also  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable.

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- (b) It is easy to see that  $f_M$  converges pointwise to  $f$  as  $M \rightarrow \infty$ , i.e.,

$$\lim_{M \rightarrow \infty} f_M(\omega) = f(\omega) \quad \forall \omega \in \Omega.$$

Indeed, if  $\omega \in \Omega$  is such that  $f(\omega) = +\infty$ , then

$$\lim_{M \rightarrow \infty} f_M(\omega) = \lim_{M \rightarrow \infty} M = +\infty = f(\omega),$$

and similarly for  $\omega \in \Omega$  for which  $f(\omega) = -\infty$ . On the other hand, for any  $\omega \in \Omega$  with  $f(\omega) \in \mathbb{R}$ , there is some  $N_0(\omega) \in \mathbb{N}$  such that  $f_N(\omega) = f(\omega)$  for all  $N \geq N_0(\omega)$ , and hence,

$$\lim_{M \rightarrow \infty} f_M(\omega) = f(\omega).$$

Since  $f$  is the limit of a sequence of  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable functions, we conclude from Lemma 3.2.13 that  $f$  is  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable.

### Problem 3.9

- (a) For the probability space, take  $\Omega = [0, 1]$ ,  $\mathcal{F} = \mathcal{B}_{[0,1]}$  and  $\mathbb{P} = \lambda$  the Lebesgue measure restricted to  $[0, 1]$ .

Observe that the function  $H_{\gamma}(z)$  is continuous and hence has an inverse  $g_{\gamma}(y) = \gamma \tan(\pi(y - 1/2))$  on  $[0, 1]$ .

So the function  $Y[0, 1] \rightarrow \mathbb{R}$  defined by  $Y(x) = g_{\gamma}(x)$  has the correct distribution as

$$\mathbb{P}(Y^{-1}((-\infty, t])) = \mathbb{P}(g_{\gamma}^{-1}((-\infty, t])) = \lambda(H_{\gamma}((-\infty, t])) = H_{\gamma}(t).$$

- (b) Note that  $g_{\gamma}$  is continuous on  $[0, 1]$  and hence measurable.  
(c) For any  $t \geq 0$ , the cdf of the Poisson random variable is given by

$$F_{\lambda}(t) = \sum_{n=0}^{\lceil t \rceil} f_{\lambda}(n),$$

where  $\lceil t \rceil$  is the ceiling of  $t$ , i.e. the smallest integer  $k \geq t$ .

- (d) For the probability space, we again take  $\Omega = [0, 1]$ ,  $\mathcal{F} = \mathcal{B}_{[0,1]}$  and  $\mathbb{P} = \lambda$  the Lebesgue measure restricted to  $[0, 1]$ .

Now for any  $y \in [0, 1]$  let  $k := k(y)$  be such that

$$\sum_{n=1}^k f_{\lambda}(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_{\lambda}(n) < y,$$

where the last sum is interpreted as  $-1$  if  $k = 0$ .

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Now define  $X(y) = k(y) : [0, 1] \rightarrow \mathbb{R}$ . Then  $k(y) \leq t$  if and only if  $y \leq F_\lambda(t)$  and hence

$$X^{-1}((-\infty, t]) = \{y \in [0, 1] : k(y) \in (0, t]\} = \{y \in [0, 1] : y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

- (e) It follows from the above computation that  $X^{-1}((-\infty, t]) = \{y \in [0, 1] : y \in (0, F_\lambda(t)]\}$ . Since the latter is a measurable set we conclude that  $X^{-1}((-\infty, t])$  is measurable for all  $t$  and since these generate the Borel  $\sigma$ -algebra  $X$  is measurable.
- (f) for any  $\ell \in \mathbb{N}$  define the sets  $A_\ell = (n - 1 - 1/\ell, n - 1 + 1/\ell]$ . Then  $A_\ell$  is a decreasing set with  $\lim_{\ell \rightarrow \infty} A_\ell = \{n\}$ . Moreover,  $A_\ell = (-\infty, n - 1 + 1/\ell] \setminus (-\infty, n - 1 - 1/\ell]$  and  $\mathbb{P}(A_1) < \infty$ . It now follows from continuity from above and (d) that

$$\begin{aligned} X_\# \mathbb{P}(\{n\}) &= \lim_{\ell \rightarrow \infty} X_\# \mathbb{P}(A_\ell) \\ &= \lim_{\ell \rightarrow \infty} X_\# \mathbb{P}((-\infty, n - 1 + 1/\ell]) - X_\# \mathbb{P}((-\infty, n - 1 - 1/\ell]) \\ &= F_\lambda(n - 1 + 1/\ell) - F_\lambda(n - 1 - 1/\ell) \\ &= \sum_{k=0}^n f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n). \end{aligned}$$

## Chapter 4: The Lebesgue Integral

### Problem 4.2

The idea is to apply the monotone convergence theorem (Theorem 4.3.4). To this end we first note that

$$\|f_n(\omega) - f(\omega)\| \leq 2^{-n} \quad \text{for all } n \in \mathbb{N}, \omega \in \Omega.$$

From this it follows that  $f_n(\omega) \leq 2^{-n} + f(\omega)$  and hence

$$\begin{aligned} \|[f_n](\omega) - f(\omega)\| &= \|2^n - f(\omega)\| \mathbf{1}_{2^n \leq f_n} + \|f_n(\omega) - f(\omega)\| \mathbf{1}_{f_n < 2^n} \\ &\leq 2^{-n} + 2^{-n} \end{aligned}$$

from which we conclude that  $[f_n] \rightarrow f$ .

The final part is to show that  $[f_n] \leq [f_{n+1}]$  which follows if we can show that  $f_n \leq f_{n+1}$ . For this we first note that for all  $k \geq 1$   $(k+1)2^{-(n+1)} \leq k2^{-n}$ . We also note that  $2^n \leq 2n+1$ . Now suppose that there exist an  $n \geq 1$  and  $\omega$  such that  $f_n(\omega) > f_{n+1}(\omega)$ . Then it must hold that  $f_n(\omega) > 0$  and hence  $f_n(\omega) = k2^{-n}$  for some  $k \geq 1$ . This then implies that  $f_{n+1}(\omega) = \ell 2^{-n}$  for some  $\ell \geq k+1$ . But this cannot be the case as  $[\ell 2^{-n}, (\ell+1)2^{-n}) \cap [k2^{-n}, (k+1)2^{-n}) = \emptyset$  while  $f(\omega)$  should be in both sets.

### Problem 4.3

- (a) By definition, we have that  $\nu_f(\Omega) = \int_{\Omega} f \, d\mu = 1$ . Now let  $(A_n)_{n \in \mathbb{N}}$  be a family of mutually disjoint measurable sets. Then we have that the sequence

$$g_n := \sum_{i=1}^n f \mathbf{1}_{A_i} = f \mathbf{1}_{\bigcup_{i=1}^n A_i} \longrightarrow g := f \mathbf{1}_{\bigcup_{i \in \mathbb{N}} A_i} \quad \text{pointwise monotonically.}$$

By MCT, we then have that

$$\nu_f \left( \bigcup_{i \in \mathbb{N}} A_i \right) = \int_{\bigcup_{i \in \mathbb{N}} A_i} f \, d\mu = \lim_{n \rightarrow \infty} \int_{\bigcup_{i=1}^n A_i} f \, d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{A_i} f \, d\mu = \sum_{i \in \mathbb{N}} \nu_f(A_i),$$

thus showing that  $\nu_f$  is a probability measure on  $(\Omega, \mathcal{F})$ .

- (b) Following the hint, we start by considering nonnegative simple functions  $g$ . Suppose  $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$  for  $a_i \in \mathbb{R}$  and  $A_i \in \mathcal{F}$  mutually disjoint. Then,

$$\int_{\Omega} g \, d\nu_f = \sum_{i=1}^n a_i = \nu_f(A_i) = \sum_{i=1}^n a_i \int_{A_i} f \, d\mu = \int_{\Omega} g f \, d\mu.$$

Now let  $g$  be a nonnegative measurable function and  $[g]_n$  be a sequence of nonnegative simple functions that converge pointwise monotonically to  $g$ . Then MCT yields

$$\int_{\Omega} g \, d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} [g]_n \, d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} [g]_n f \, d\mu = \int_{\Omega} g f \, d\mu,$$

where we used the fact that  $[g]_n f$  converges pointwise monotonically to  $g f$ .

- (c) Let  $g$  be measurable. Then  $g = g^+ - g^-$ , where  $g^{\pm}$  are nonnegative measurable functions. Since  $f$  is nonnegative, we have that  $(fg)^{\pm} = fg^{\pm}$ . Due to (b), we deduce

$$\int_{\Omega} g^{\pm} \, d\nu_f = \int_{\Omega} g^{\pm} f \, d\mu = \int_{\Omega} (gf)^{\pm} \, d\mu.$$

Hence,  $g^{\pm}$  is  $\nu_f$ -integrable if and only if  $(gf)^{\pm}$  is  $\mu$ -integrable. Consequently,  $g$  is  $\nu_f$ -integrable if and only if  $gf$  is  $\mu$ -integrable, since

$$\int_{\Omega} |g| \, d\nu_f = \int_{\Omega} g^+ \, d\nu_f + \int_{\Omega} g^- \, d\nu_f = \int_{\Omega} g^+ f \, d\mu + \int_{\Omega} g^- f \, d\mu = \int_{\Omega} |gf| \, d\mu.$$

#### Problem 4.4

( $\Rightarrow$ ) Let  $f$  be  $\mu$ -integrable. Then both  $|f| \mathbf{1}_{\{|f| < n\}}$  and  $|f| \mathbf{1}_{\{|f| \geq n\}}$  are integrable, due to the monotonicity of the integral. By linearity of the integral,

$$\int_{\Omega} |f| \mathbf{1}_{\{|f| \geq n\}} \, d\mu = \int_{\Omega} |f| \, d\mu - \int_{\Omega} |f| \mathbf{1}_{\{|f| < n\}} \, d\mu.$$



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Since the sequence  $g_n := |f| \mathbf{1}_{\{|f| < n\}} \geq 0$  converges pointwise monotonically to  $|f|$ , we can apply MCT to obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f| \mathbf{1}_{\{|f| < n\}} d\mu = \int_{\Omega} |f| d\mu.$$

Hence,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f| \mathbf{1}_{\{|f| \geq n\}} d\mu = \int_{\Omega} |f| d\mu - \lim_{n \rightarrow \infty} \int_{\Omega} |f| \mathbf{1}_{\{|f| < n\}} d\mu = 0.$$

( $\Leftarrow$ ) By assumption, there is some  $N \geq 1$  such that

$$\int_{\Omega} |f| \mathbf{1}_{\{|f| \geq N\}} d\mu \leq 1.$$

By linearity of the integral,

$$\int_{\Omega} |f| d\mu = \int_{\Omega} |f| \mathbf{1}_{\{|f| < N\}} d\mu + \int_{\Omega} |f| \mathbf{1}_{\{|f| \geq N\}} d\mu \leq N\mu(\{|f| < N\}) + 1.$$

Since  $\mu$  is a finite measure, the right-hand side is finite, implying that  $f$  is  $\mu$ -integrable.

#### Problem 4.5

Observe that  $\Omega = \bigcup_{n \in \mathbb{N}} \{|f| > n\}$ .

We then get that

$$\sum_{n=1}^{\infty} \int_{\{|f| > n\}} |f| d\mu = \int_{\Omega} |f| d\mu < \infty.$$

This implies that for some  $N$  and all  $n \geq N$ :  $\int_{\{|f| > n\}} |f| d\mu < 1/n$  or else the sum cannot be finite.

Now let  $\varepsilon > 0$ , take  $M > \max\{N, 2/\varepsilon\}$  and  $\delta = \varepsilon/(2M)$ . Then

$$\begin{aligned} \int_A |f| d\mu &= \int_A |f| \mathbf{1}_{|f| \leq M} d\mu + \int_A |f| \mathbf{1}_{|f| > M} d\mu \\ &\leq M\mu(A) + \frac{1}{M} \leq M\delta + \frac{1}{M} < \varepsilon. \end{aligned}$$

#### Problem 4.6

- (a) Let  $t \in \mathbb{R}$  and consider the set  $A_t = (-\infty, t]$ . Then by definition of the probability density function

$$\nu(A_t) = \int_{-\infty}^t \rho d\lambda = (X_{\#}\mathbb{P})((-\infty, t]).$$

We thus conclude that  $\nu$  and  $X_{\#}\mathbb{P}$  coincide on the family of set  $A_t$  and since these generate  $\mathcal{B}$  Theorem 2.2.17 implies that  $\nu = X_{\#}\mathbb{P}$ .

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- (b) Since  $g$  is a simple function, there exist an  $N \in \mathbb{N}$ , constants  $(a_n)_{1 \leq n \leq N}$  and measurable sets  $(A_n)_{1 \leq n \leq N}$  such that

$$g = \sum_{n=1}^N a_n \mathbf{1}_{A_n}.$$

Now, by first applying Proposition 4.8.11 and then part (a), we get that

$$\begin{aligned} \mathbb{E}[g(X)] &= \int_{\Omega} g(X) \, d\mathbb{P} = \int_{\Omega} g \, dX_{\#}\mathbb{P} = \int_{\Omega} g \, d\nu \\ &= \int_{\Omega} \sum_{n=1}^N a_n \mathbf{1}_{A_n} \, d\nu = \sum_{n=1}^N a_n \nu(A_n) = \sum_{n=1}^N a_n \int_{A_n} \rho \, d\lambda \\ &= \int_{\mathbb{R}} \sum_{n=1}^N a_n \mathbf{1}_{A_n} \rho \, d\lambda = \int_{\mathbb{R}} g \rho \, d\lambda \end{aligned}$$

- (c) First note that by part (b) we have that

$$\int_{\Omega} [h]_n(X) \, d\mathbb{P} = \int_{\mathbb{R}} [h_n] \rho \, d\lambda.$$

Now we split the function  $[h_n]\rho$  into its positive and negative part and note that

$$([h_n]\rho)^+ = [h]_n^+ \rho^+ + [h]_n^- \rho^- \quad \text{and} \quad ([h_n]\rho)^- = [h]_n^+ \rho^- + [h]_n^- \rho^+,$$

where  $[h]_n^{\pm}$  and  $\rho^{\pm}$  denote the positive and negative parts of  $[h]_n$  and  $\rho$ .

We will show that

$$\int_{\Omega} h^+(X) \, d\mathbb{P} = \int_{\mathbb{R}} h^+ \rho \, d\lambda.$$

The proof for the negative part is similar.

$$\begin{aligned} \int_{\mathbb{R}} h^+ \, d\nu &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n^+ \, d\nu && \text{by Theorem 4.3.4} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n^+ \rho \, d\lambda && \text{by part (b)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n^+ \rho^+ \, d\lambda - \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n^+ \rho^- \, d\lambda && \text{by linearity of integration} \\ &= \int_{\mathbb{R}} h^+ \rho^+ \, d\lambda - \int_{\mathbb{R}} h^+ \rho^- \, d\lambda && \text{by Theorem 4.3.4} \\ &= \int_{\mathbb{R}} h^+ \rho \, d\lambda && \text{by linearity of integration} \end{aligned}$$

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(d)

$$\begin{aligned}\mathbb{E}[h(X)] &= \int_{\Omega} h(X) \, d\mathbb{P} \\ &= \int_{\mathbb{R}} h \, dX_{\#}\mathbb{P} && \text{by Proposition 4.8.11} \\ &= \int_{\mathbb{R}} h \, d\nu && \text{by part (a)} \\ &= \int_{\mathbb{R}} h\rho \, d\lambda && \text{by part (c).}\end{aligned}$$

**Problem 4.7** This follows from the following inequalities:

$$\int_{\mathbb{R}} |f|^p \, d\mu \geq \int_{\{|f| \geq t\}} |f|^p \, d\mu \geq t^p \mu(\{|f| \geq t\}).$$

## Chapter 5: Convergence of integrals and functions

### Problem 5.2

- (a) Let  $t_0 \in (a, b)$  be fixed. It suffices to check the continuity result for arbitrary sequences  $(t_n)_{n \geq 1} \subset (a, b)$  such that  $t_n \rightarrow t_0$  as  $n \rightarrow \infty$ . Fix such a sequence and define  $g_n(\omega) := f(\omega, t_n)$  for all  $\omega \in \Omega$  and  $n \geq 1$ . Since  $\lim_{t \rightarrow t_0} f(\omega, t) = f(\omega, t_0)$  for all  $\omega \in \Omega$ , we deduce that  $\lim_{n \rightarrow \infty} g_n(\omega) = f(\omega, t_0)$  for every  $\omega \in \Omega$ . Moreover, by assumption  $|g_n| \leq g$  for all  $n \geq 1$  and  $g$  is integrable. By the Dominated Convergence Theorem

$$\lim_{n \rightarrow \infty} \int_{\Omega} g_n(\omega) \, \mu(d\omega) = \int_{\Omega} f(\omega, t_0) \, \mu(d\omega).$$

As the chosen sequence was arbitrary, we deduce that  $\lim_{t \rightarrow t_0} F(t) = F(t_0)$ .

- (b) If  $t \mapsto f(\omega, t)$  is continuous on  $(a, b)$  for all  $\omega \in \Omega$  then  $\lim_{t \rightarrow t_0} f(\omega, t) = f(\omega, t_0)$  at every  $t_0 \in (a, b)$  for all  $\omega \in \Omega$ . In particular, (a) applies, showing that  $\lim_{t \rightarrow t_0} F(t) = F(t_0)$  for every  $t_0 \in (a, b)$ , i.e.,  $F$  is continuous on  $(a, b)$ .

### Problem 5.3

- (1) We start by showing that  $(\partial f / \partial t)(\cdot, t)$  is measurable. Let  $(t_n)_{n \geq 1} \subset (a, b)$  be an arbitrary sequence with  $t_n \neq t$  and  $t_n \rightarrow t$  for  $n \rightarrow \infty$ . We set

$$g_n(\omega) = \frac{f(\omega, t_n) - f(\omega, t)}{t_n - t}.$$

Clearly,  $g_n$  is measurable for every  $n \geq 1$ . Moreover,  $\lim_{n \rightarrow \infty} g_n(\omega) = (\partial f / \partial t)(\omega, t)$  by the definition of the derivative. Since  $(\partial f / \partial t)(\cdot, t)$  is the pointwise limit of a sequence of measurable functions, it is also measurable. Clearly,  $(\partial f / \partial t)(\cdot, t)$  is integrable since

$$\int_{\Omega} |(\partial f / \partial t)(\omega, t)| \, \mu(d\omega) \leq \int_{\Omega} g \, d\mu < +\infty.$$

- 
- 
- (2) Let  $t_0 \in (a, b)$  and suppose w.l.o.g.  $t_0 < t$ . Since  $t \mapsto f(\omega, t)$  is differentiable on  $(a, b)$  for all  $\omega \in \Omega$ , the Mean Value Theorem gives

$$\frac{f(\omega, t) - f(\omega, t_0)}{t - t_0} = (\partial f / \partial t)(\omega, \tau) \quad \text{for some } \tau \in (t_0, t).$$

Taking the modulus on both sides, we obtain

$$\left| \frac{f(\omega, t) - f(\omega, t_0)}{t - t_0} \right| \leq |(\partial f / \partial t)(\omega, \tau)| \leq g(\omega) \quad \text{for all } \omega \in \Omega.$$

- (3) We now have all the ingredients needed to apply the DCT, which yields

$$\lim_{n \rightarrow \infty} \frac{F(t_n) - F(t)}{t_n - t} = \lim_{n \rightarrow \infty} \int_{\Omega} g_n \, d\mu = \int_{\Omega} (\partial f / \partial t)(\omega, t) \, \mu(d\omega).$$

Since  $t \in (a, b)$  and the sequence  $(t_n)_{n \geq 1}$  was arbitrary, we conclude that  $F$  is differentiable on  $(a, b)$  with

$$F'(t) = \int_{\Omega} (\partial f / \partial t)(\omega, t) \, \mu(d\omega).$$

### Problem 5.3

- (a) Note that the integrand  $f_n(x) = \frac{1+nx^2}{(1+x^2)^n}$  is continuous on  $[0, 1]$  and non-negative. Hence, the Riemann integral and Lebesgue integral coincide, i.e.,

$$\int_0^1 f_n(x) \, dx = \int_{[0,1]} f_n \, d\lambda.$$

Observe that we have the following pointwise limit

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0, 1], \end{cases}$$

i.e.,  $\lim_{n \rightarrow \infty} f_n = 0$   $\lambda$ -almost everywhere. Moreover,  $f_n(x) \leq 1$  for every  $x \in [0, 1]$  and  $n \geq 1$ . Since the constant function  $g \equiv 1$  is  $\lambda$ -integrable on  $[0, 1]$ , it is a valid dominator. Hence, the DCT gives

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \, dx = \lim_{n \rightarrow \infty} \int_{[0,1]} f_n \, d\lambda = \int_{[0,1]} \lim_{n \rightarrow \infty} f_n \, d\lambda = 0$$

- (b) For the purpose of convergence, we consider  $n \geq 3$ . Note that the integrand  $f_n(x) = \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right)$  is continuous on  $(0, +\infty)$  with pointwise limit

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{if } x \in (0, 1), \\ 1/2 & \text{if } x = 1, \\ 1/x^2 & \text{if } x > 1, \end{cases}$$

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Setting the function

$$g(x) = \begin{cases} 1 & \text{for } x \in (0, 1), \\ \frac{1}{x^2} & \text{for } x \geq 1, \end{cases}$$

we see that  $f_n \leq g$   $\lambda$ -almost everywhere in  $(0, +\infty)$  and for all  $n \geq 3$ . Indeed, for  $x \geq 1$ , we obtain

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq \frac{x^{n-2}}{x^n} = \frac{1}{x^2},$$

while for  $x \in (0, 1)$ , we have

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq 1.$$

Notice that  $g$  is non-negative and  $\lambda$ -integrable on  $(0, +\infty)$ . Indeed, using the MCT,

$$\begin{aligned} \int_{(0,+\infty)} g \, d\lambda &= \int_{(0,1)} g \, d\lambda + \int_{(1,+\infty)} g \, d\lambda = 1 + \lim_{n \rightarrow \infty} \int_{(1,n)} g \, d\lambda \\ &= 1 + \lim_{n \rightarrow \infty} \int_1^n \frac{1}{x^2} \, dx = 1 + \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 2 < +\infty. \end{aligned}$$

To conclude, we apply DCT to deduce that the limit is 1.

#### Problem 5.4

The proof follows verbatim to the proof of the Dominated Convergence Theorem.

#### Problem 5.7

Let  $F_n$  denote the cdf of  $Y_n = \|X_n - X\|$  and  $F_0$  denote the cdf of 0. By Definition 5.2.9 and Lemma 5.2.8 we have that  $X_n \xrightarrow{\mathbb{P}} X$  if and only if  $F_n(t) \rightarrow F_0(t)$  for all continuity points  $t$  of  $F_0$ . This is equivalent to showing that  $1 - F_n(t) \rightarrow 1 - F_0(t)$ , where

$$1 - F_0(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ 1 & \text{else.} \end{cases}$$

Now note that the only discontinuity point of  $F_0$  is 0. Moreover,  $1 - F_n(t) = 0 = F_0(t)$  for all  $t < 0$ . Hence it follows that  $X_n \xrightarrow{\mathbb{P}} X$  if and only if  $1 - F_n(t) \rightarrow 0$  for all  $t > 0$ , which is what we needed to show.

#### Problem 5.8

(a) For this let  $h_t(x) = \mathbf{1}_{(-\infty, t]}$  and note that

$$F_n(t) = (X_n)_\# \mathbb{P}_n((-\infty, x]) = \int_{\mathbb{R}} \mathbb{1}_{(-\infty, t]} \, d(X_n)_\# \mathbb{P}_n = \int_{\mathbb{R}} h_t \, d\mu_n.$$

and similarly  $F(t) = \int_{\mathbb{R}} h_t \, d\mu$

- 
- (b) The function  $h$  is discontinuous only at  $t$ , i.e.  $\mathcal{C}_h = \mathbb{R} \setminus \{t\}$ . Moreover, for any  $\varepsilon > 0$
- $$\mu((t - \varepsilon, t + \varepsilon)) = \mu((t - \varepsilon, t]) + \mu((t, t + \varepsilon)) = F(t) - F(t - \varepsilon) + F(t + \varepsilon) - F(t).$$

Since  $F$  is continuous at  $t$ , the right hand side goes to zero as  $\varepsilon \rightarrow 0$ . Therefore

$$\mu(\{t\}) = \lim_{\varepsilon \rightarrow 0} \mu((t - \varepsilon, t + \varepsilon)) = 0,$$

which implies that  $\mu(\mathcal{C}_h) = 1$ .

- (c) The result follows by applying condition (2) in Theorem 5.2.7.
- (d) Let  $\varepsilon > 0$ , pick such a  $\delta$  and partition the interval  $[-K, K]$  into  $L_\delta := \lceil \frac{4K}{\delta} \rceil$  intervals  $I_\ell = (a_\ell, b_\ell]$  of equal length, which is  $\leq \delta/2 < \delta$ . Now we define the simple function

$$\hat{g} := \sum_{\ell=1}^L h(b_\ell) \mathbb{1}_{I_\ell},$$

- (e) Let  $M = L$ ,  $\beta_\ell = \sum_{t=1}^\ell h(b_t)$  and  $t_\ell = b_\ell$ . Then

$$\hat{g} := \sum_{\ell=1}^L \beta_\ell \mathbb{1}_{(-\infty, b_\ell]}.$$

- (f) Using the representation in (e) we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[\hat{g}(X_n)] &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell \int_{\mathbb{R}} \mathbb{1}_{X_n^{-1}((-\infty, b_\ell])} d\mathbb{P} \\ &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell (X_n)_\# \mathbb{P}((-\infty, b_\ell]) \\ &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell F_n(b_\ell) \\ &= \sum_{\ell=1}^L F(b_\ell) = \mathbb{E}[\hat{g}(X)]. \end{aligned}$$

- (g) Using the representation of  $\hat{g}$  in (d) we note that  $\|x - y\| < \varepsilon$  for all  $x, y \in I_\ell$ . This then implies that  $\|g(x) - \hat{g}(y)\| \leq \varepsilon$  from which it follows that

$$\begin{aligned} \|\mathbb{E}[g(X_n)] - \mathbb{E}[g(X)]\| &\leq \|\mathbb{E}[g(X_n)] - \mathbb{E}[\hat{g}(X_n)]\| + \|\mathbb{E}[g(X)] - \mathbb{E}[\hat{g}(X)]\| \\ &\quad + \|\mathbb{E}[\hat{g}(X_n)] - \mathbb{E}[\hat{g}(X)]\| \\ &\leq 2\varepsilon + \|\mathbb{E}[\hat{g}(X_n)] - \mathbb{E}[\hat{g}(X)]\|. \end{aligned}$$

We have shown in (f) that the last term goes to zero as  $n \rightarrow \infty$ . Since  $\varepsilon$  was arbitrary we conclude that (??) holds.

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(h) This now follows from Theorem 5.2.7 (3).

**Problem 5.9** Suppose that  $X_n \xrightarrow{\text{a.s.}} X$ . Then by Lemma 5.2.16 this is equivalent to  $\mathbb{P}(\|X_n - X\| > \varepsilon \text{ i.o.}) = 0$  for all  $\varepsilon > 0$ .

For now fix an  $\varepsilon > 0$  and write  $A_n := \{\|X_n - X\| > \varepsilon\}$ . Recall that

$$\{A_n \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n$$

and note two things:

(a) The sets  $B_k := \bigcup_{n \geq k} A_n$  are non-increasing, i.e.  $B_k \supset B_{k+1}$ , and

(b)  $\mathbb{P}(A_k) \leq \mathbb{P}(\bigcup_{n \geq k} A_n) = \mathbb{P}(B_k)$ .

We then have that:

$$\begin{aligned} 0 &= \mathbb{P}(\{A_n \text{ i.o.}\}) && \text{by assumption} \\ &= \mathbb{P}\left(\bigcap_{k=1}^{\infty} B_k\right) && \text{by Lemma 5.2.16} \\ &= \lim_{k \rightarrow \infty} \mathbb{P}(B_k) && \text{by continuity from above (Proposition 2.2.15)} \\ &\geq \lim_{k \rightarrow \infty} \mathbb{P}(A_k) && \text{by (b).} \end{aligned}$$

## Chapter 6: $L^p$ -spaces

### Problem 6.2

#### Problem 6.4

Let  $E_n := \{\omega \in \mathbb{R}^d : |f(\omega)| \geq n\}$ . Since  $\mathbf{1}_{E_n} f \rightarrow 0$  as  $n \rightarrow \infty$ , and  $\mathbf{1}_{E_n} |f| \leq |f|$  for every  $n \geq 1$ , we can apply DCT to conclude that

$$\int_{E_n} |f| \, d\mu = \int_{\mathbb{R}^d} \mathbf{1}_{E_n} |f| \, d\mu \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now pick some  $n \geq 1$  such that  $\int_{E_n} |f| \, d\mu < \varepsilon/3$  and define

$$f_n(\omega) := \max\{-n, \min\{f(\omega), n\}\}, \quad \omega \in \mathbb{R}^d,$$

i.e.,  $f_n$  is a truncation of  $f$ . From Lusin's theorem, we find a continuous function  $g$  such that  $f_n \equiv g$  on a compact set  $K \subset \mathbb{R}^d$  with  $\mu(\mathbb{R}^d \setminus K) < (2\varepsilon)/(3n)$ . We assume w.l.o.g. that  $|g| \leq n$ , since otherwise, we can consider a truncation of  $g$ . Altogether, this yields

$$\begin{aligned} \int_{\mathbb{R}^d} |f - g| \, d\mu &= \int_{\mathbb{R}^d} |f - f_n| \, d\mu + \int_{\mathbb{R}^d} |f_n - g| \, d\mu \\ &= \int_{E_n} |f| \, d\mu + \int_{\mathbb{R}^d \setminus K} |f_n - g| \, d\mu \\ &\leq \frac{\varepsilon}{3} + 2n \mu(\mathbb{R}^d \setminus K) \leq \varepsilon. \end{aligned}$$

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Finally,  $g \in L^1(\mathbb{R}^d, \mu)$  holds simply due to the triangle inequality.

## Chapter 7: Fubini-Tonelli

### Problem 7.1

- (a) Note that  $\mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{F}_1 \times \mathcal{F}_2$ , and hence

$$\sigma(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{F}_1 \otimes \mathcal{F}_2.$$

- (b) Let  $B \in \mathcal{A}_2$ . Then we have that

$$\Omega_1 \times B = \bigcup_{n \geq 1} A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$$

since  $A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$  for all  $n \geq 1$ . So  $\Omega_1 \in \Sigma$

For the second property, let  $C \in \Sigma$  and note that  $C^c \times B = (\Omega_1 \times B) \setminus (C \times B)$ . Since both these sets are in  $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$  it follows that  $C^c \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$  and hence  $C^c \in \Sigma$ .

Finally consider a countable sequence  $(C_n)_{n \geq 1}$  of sets in  $\Sigma$ . Then for any  $B \in \mathcal{A}_2$

$$\left( \bigcup_{n \geq 1} C_n \right) \times B = \bigcup_{n \geq 1} (C_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2),$$

since each  $C_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ .

- (c) Note that  $\mathcal{A}_1 \subset \Sigma_1 \subset \mathcal{F}_1$ . From which it follows that  $\Sigma_1 = \mathcal{F}_1$ . But then, from the definition of  $\Sigma_1$  we have that  $\mathcal{F}_1 \times \mathcal{A}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ .
- (d) We can show in a similar fashion that

$$\Sigma_2 := \{C \in \mathcal{F}_2 : B \times C \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2) \forall B \in \mathcal{A}_1\}.$$

is a  $\sigma$ -algebra on  $\Omega_2$ , from which we conclude that  $\mathcal{A}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ .

- (e) take any  $A \in \mathcal{F}_1$  and  $B \in \mathcal{F}_2$ . Then

$$A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B) = \bigcup_{n, m \geq 1} (A \times B_m) \cap (A_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2).$$

From this we conclude that  $\mathcal{F}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ , which finishes the proof.

### Problem 7.4



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One direction is easy. Assume that  $X_1$  and  $X_2$  are independent according to Definition 7.1.4. Now take any  $a, b \in \mathbb{R}$  and note that  $A_1 := X_1^{-1}((-\infty, a]) \in \sigma(X_1)$  and  $A_2 := X_2^{-1}((-\infty, b]) \in \sigma(X_2)$ . Then by the definition of independence we have that

$$\mathbb{P}(X_1 \leq a, X_2 \leq b) = \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \leq b).$$

So let us focus now on the other direction. Assume that for all  $a, b \in \mathbb{R}$

$$\mathbb{P}(X_1 \leq a, X_2 \leq b) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \leq b).$$

We now have to show that  $X_1$  and  $X_2$  are independent according to Definition 7.1.4. First note that since the family  $(-\infty, a] \times (-\infty, b]$  generate the 2-dimensional Borel  $\sigma$ -algebra we have, using Theorem 2.2.17, that

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2) = \mathbb{P}(X_1 \in B_1)\mathbb{P}(X_2 \in B_2)$$

for all  $B_1, B_2 \in \mathcal{B}_{\mathbb{R}}$ .

Now fix a set  $B_2 \in \mathcal{B}_{\mathbb{R}}$ , set  $A_2 := X_2^{-1}(B_2) \in \sigma(X_2)$ , and define the following two measures on the space  $(\Omega, \sigma(X_1))$

$$\mu_1(A) = \mathbb{P}(A \cap A_2) \quad \text{and} \quad \mu_2(A) = \mathbb{P}(A)\mathbb{P}(A_2).$$

Let  $a \in \mathbb{R}$  and consider the set  $A_1 := X_1^{-1}((-\infty, a]) \in \sigma(X_1)$ . Then, by our assumption we have that

$$\mu_1(A_1) = \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(X_1 \leq a, X_2 \in B_2) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \in B_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = \mu_2(A_1).$$

In other words, the measures  $\mu_1, \mu_2$  coincide on the set  $\{X_1^{-1}((-\infty, a]) : a \in \mathbb{R}\}$ . Since the set  $(-\infty, a]$  generate  $\mathcal{B}_{\mathbb{R}}$  it follows that

$$\sigma(X_1) = \sigma(\{X_1^{-1}((-\infty, a]) : a \in \mathbb{R}\}).$$

In addition, this set satisfies the conditions of Theorem 2.2.17 and hence we conclude that  $\mu_1(A) = \mu_2(A)$  for all  $A \in \sigma(X_1)$ .

We can repeat this argument for the two measures on  $(\Omega, \sigma(X_2))$

$$\nu_1(A) = \mathbb{P}(A_1 \cap A) \quad \text{and} \quad \nu_2(A) = \mathbb{P}(A_1)\mathbb{P}(A),$$

where  $A_1 \in \{X_1^{-1}((-\infty, a]) : a \in \mathbb{R}\}$  is fixed.

From this we conclude that for any  $A_1 \in \sigma(X_1)$  and  $A_2 \in \sigma(X_2)$

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$$

and hence  $X_1$  and  $X_2$  are independent.

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## Chapter 8: Radon-Nikodym

**Problem 8.2** First note that since  $f - g$  is  $\mathcal{H}$ -measurable, we have that  $\{f \geq g\}, \{f < g\} \in \mathcal{H}$ . We then write

$$\int_{\Omega} \|f - g\| \, d\mathbb{P} = \int_{f \geq g} (f - g) \, d\mathbb{P} - \int_{f < g} (f - g) \, d\mathbb{P}.$$

Since  $\int_B f \, d\mathbb{P} = \int_B g \, d\mathbb{P}$  holds for all  $B \in \mathcal{H}$  both integrals on the right hand side are zero.

### Problem 8.4

(a) By definition we have that

$$\int_B \mathbb{E}[X|\mathcal{H}] \, d\mathbb{P} = \int_B X \, d\mathbb{P},$$

holds for all  $B \in \mathcal{H}$ . Since by assumption both  $\mathbb{E}[X|\mathcal{H}]$  and  $X$  are  $\mathcal{H}$ -measurable, the result follows from problem 8.2.

(b) Note that  $a\mathbb{E}[X|\mathcal{H}]$  is  $\mathcal{H}$ -measurable. Moreover,

$$\int_B a\mathbb{E}[X|\mathcal{H}] \, d\mathbb{P} = a \int_B \mathbb{E}[X|\mathcal{H}] \, d\mathbb{P} = a \int_B X \, d\mathbb{P} = \int_B aX \, d\mathbb{P}.$$

This proves the claim.

(c) Similarly to the previous point, we first note that since  $\mathbb{E}[X|\mathcal{H}]$  and  $\mathbb{E}[Y|\mathcal{H}]$  are  $\mathcal{H}$ -measurable so is  $\mathbb{E}[X|\mathcal{H}] + \mathbb{E}[Y|\mathcal{H}]$ . The result then follows because

$$\begin{aligned} \int_B \mathbb{E}[X|\mathcal{H}] + \mathbb{E}[Y|\mathcal{H}] \, d\mathbb{P} &= \int_B \mathbb{E}[X|\mathcal{H}] \, d\mathbb{P} + \int_B \mathbb{E}[Y|\mathcal{H}] \, d\mathbb{P} \\ &= \int_B X \, d\mathbb{P} + \int_B Y \, d\mathbb{P} = \int_B X + Y \, d\mathbb{P}. \end{aligned}$$

(d) First we observe that for any  $B \in \mathcal{H}$

$$\int_B \mathbb{E}[X|\mathcal{H}] \, d\mathbb{P} = \int_B X \, d\mathbb{P} \leq \int_B Y \, d\mathbb{P} = \int_B \mathbb{E}[Y|\mathcal{H}] \, d\mathbb{P}.$$

Now consider the event  $A := \{\mathbb{E}[X|\mathcal{H}] > \mathbb{E}[Y|\mathcal{H}]\} \in \mathcal{H}$ . If this event has non-zero measure then it would follow that

$$\int_A \mathbb{E}[X|\mathcal{H}] \, d\mathbb{P} > \int_A \mathbb{E}[Y|\mathcal{H}] \, d\mathbb{P},$$

which is a contradiction. Hence we conclude that  $\mathbb{E}[X|\mathcal{H}] \leq \mathbb{E}[Y|\mathcal{H}]$  holds  $\mathbb{P}$ -almost everywhere.