Problem 2.6 Let $\mathcal{O}$ denote the open sets in $\mathbb{R}$.

(a) Note that the interval (a, b) is open for any $a < b \in \mathbb{R}$. Hence $\mathcal{A}_1 \subset \mathcal{A}_1' \subset \mathcal{O}$ and thus by Lemma 2.1.5 we have that $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \sigma(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$.

(b) The inclusion $\supset$ is trivial. So assume that $x \in O$. Then by definition there exist an $r > 0$ such that the ball $B_x(r) \subset O$. But $B_x(r) = (x - r, x + r) \in \mathcal{A}_1$ so $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$.

(c) Take $O \in \mathcal{O}$. If we can show that $O \in \sigma(\mathcal{A})$ then $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. The result then follows from 1.

From 2 it follows that O is a union over a subset collection of interval (a, b) where $a, b \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, the collection $\{(a, b) \,:\, a < b \in \mathbb{Q}\}$ is also countable and hence $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$, from which it follows that $\mathcal{B}_{\mathbb{R}} \subset \sigma(\mathcal{A})$.

(d) This follows immediately from 1 and 3 since these imply that $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \mathcal{B}_{\mathbb{R}}$.

Problem 2.12

(a) We first make the following observations about $\mathcal{N}$:

- because $\mu(\emptyset) = 0$ it holds that $\emptyset \in \mathcal{N}$,
- if $N, M \in \mathcal{N}$ then $N \setminus M \in \mathcal{N}$ since $N \setminus M \subset N$, and
- if $(N_i)_{i \geq 1}$ is a family of sets in $\mathcal{N}$ then so is $\bigcup_{i \geq 1} N_i$.

From the first point it follows that $\emptyset = \emptyset \cup \emptyset \in \overline{\mathcal{F}}$ and $\Omega = \Omega \cup \emptyset \in \overline{\mathcal{F}}$.

Furthermore, if $A, B \in \mathcal{F}$ and $N, M \in \mathcal{N}$, then by the second point and because $A \setminus B \in \mathcal{F}$,

$$(A \cup N) \setminus (B \cup M) = (A \setminus B) \cup (N \setminus M) \in \overline{\mathcal{F}}.$$

Finally, let $(A_i \cup N_i)_{i \geq 1}$ be a collection of sets in $\mathcal{N}$. Then using the third point we get

$$\bigcup_{i \geq 1} A_i \cup N_i = \bigcup_{i \geq 1} A_i \cup \bigcup_{i \geq 1} N_i \in \overline{\mathcal{F}}.$$

(b) From the definition we immediately get that $\mu(\emptyset) = 0$. Now, let $(A_i \cup N_i)_{i \geq 1}$ be a collection of disjoint sets in $\mathcal{N}$. Then

$$\bar{\mu}(\bigcup_{i \geq 1} A_i \cup N_i) = \bar{\mu}(\bigcup_{i \geq 1} A_i \cup \bigcup_{i \geq 1} N_i) = \mu(\bigcup_{i \geq 1} A_i) = \sum_{i \geq 1} \mu(A_i) = \sum_{i \geq 1} \bar{\mu}(A_i \cup N_i).$$

(c) This follows from the fact that $\bar{\mu}|_{\mathcal{F}}(A) = \bar{\mu}(A \cup \emptyset) = \mu(A)$.

(d) Suppose that $N \subset \Omega$ is a null set for $\overline{\mathcal{F}}$. Then there exists an $A \cup M \in \overline{\mathcal{F}}$ such that $N \subset A \cup M$ and $\bar{\mu}(A \cup M) = \mu(A) = 0$. However, since $M \in \mathcal{N}$, there must also exist a $B \in \mathcal{F}$ with $M \subset B$ and $\mu(B) = 0$. But this implies that $N \subset A \cup B \in \mathcal{F}$ which implies that $N \in \mathcal{N}$. Therefore, since $N = \emptyset \cup N$ it follows that $N \in \overline{\mathcal{F}}$ and hence every null set is part of the σ-algebra.