Problem 7.1

Similar to the proof of Fatou's lemma, we define $g_n = \sup_{k \geq n} f_n$ which are measurable due to Proposition 3.13. Moreover, we have that $\limsup_{n\to\infty} f_n = \lim_{n\to\infty} g_n$.

Next we note that $g_n \geq f_\ell$ for all $\ell \geq n$. Thus, by monotonicity of the integral, we have that

$$\int_\Omega g_n \,\mathrm{d}\mu \geq \int_\Omega f_\ell \,\mathrm{d}\mu,$$

holds for all $\ell \geq n$, which implies that

$$\int_\Omega g_n \,\mathrm{d}\mu \geq \sup_{k\geq n} \int_\Omega f_n \,\mathrm{d}\mu.$$

In addition, since $g_n < f$ with f being non-negative and integrable we can apply Dominated Convergence to conclude that

$$\int_\Omega \lim_{n\to\infty} g_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega g_n \,\mathrm{d}\mu.$$

Putting all this together we get

$$\int_\Omega \limsup_{n\to\infty} f_n \,\mathrm{d}\mu = \int_\Omega \lim_{n\to\infty} g_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega g_n \,\mathrm{d}\mu \geq \lim_{n\to\infty} \sup_{k\geq n} \int_\Omega f_n \,\mathrm{d}\mu.$$

Problem 7.2

(a) Let $t_0 \in (a,b)$ be fixed. It suffices to check the continuity result for arbitrary sequences $(t_n)_{n\geq 1} \subset (a,b)$ such that $t_n \to t_0$ as $n \to \infty$. Fix such a sequence and define $g_n(\omega) := f(\omega, t_n)$ for all $\omega \in \Omega$ and $n \geq 1$. Since $\lim_{t\to t_0} f(\omega,t) = f(\omega,t_0)$ for all $\omega \in \Omega$, we deduce that $\lim_{n\to\infty} g_n(\omega) = f(\omega, t_0)$ for every $\omega \in \Omega$. Moreover, by assumption $|g_n| \leq g$ for all $n \geq 1$ and g is integrable. By the Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_\Omega g_n(\omega)\,\mu(\mathrm{d}\omega) = \int_\Omega f(\omega, t_0)\,\mu(\mathrm{d}\omega).$$

As the chosen sequence was arbitrary, we deduce that $\lim_{t\to t_0} F(t) = F(t_0)$.

(b) If $t \mapsto f(\omega,t)$ is continuous on (a,b) for all $\omega \in \Omega$ then $\lim_{t\to t_0} f(\omega,t) = f(\omega,t_0)$ at every $t_0 \in (a,b)$ for all $\omega \in \Omega$. In particular, (a) applies, showing that $\lim_{t\to t_0} F(t) = F(t_0)$ for every $t_0 \in (a,b)$, i.e., F is continuous on (a,b).