Problem 4.3

(a) The fact that the sets are disjoint is immediate from the definition. Measurability follows from Lemma 3.11

(b) Let us fix a $\omega \in \Omega$. Then if $f(\omega) = +\infty$ we get that $f_n(\omega) = 2^n$ holds for all $n \geq 1$ and hence $\lim_{n\to\infty} f_n(\omega) = +\infty = f(\omega)$. So assume that $f(\omega) < +\infty$. Then there exists an $M \in \mathbb{N}$ such that $f(\omega) < M$. Hence, for all $n \geq M$ we have that

$$\|f_n(\omega) - f(\omega)\| = f(\omega) - f_n(\omega) \leq 2^{-n},$$

which implies that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

(c) Fix $n \geq 1$ and $\omega \in \Omega$. Clearly, if $f(\omega) = +\infty$ then $f_n(\omega) = 2^n < +\infty = f(\omega)$.

(d) Fix $\omega \in \Omega$ such that $f(\omega) < +\infty$ and $\omega \in A_k^n$ for some $0 \leq k < N_n = n2^n$.

Note that $k2^{-n} \leq f(\omega) < (k+1)2^{-n}$ holds and this interval can be split into two intervals as follows:

$$[k2^{-n}, (k+1)2^{-n}) = [(2k)2^{-(n+1)}, (2k+1)2^{-(n+1)}) \cup [(2k+1)2^{-(n+1)}, (2k+2)2^{-(n+1)}).$$

Hence, we conclude that either $\omega \in A_{2k}^{n+1}$ or $\omega \in A_{2k+1}^{n+1}$. In both case we get that

$$f_n(\omega) = k2^{-n} = 2kn^{-(n+1)} \leq f_{n+1}(\omega).$$

(e) Now let us consider the case where $\omega \in A_k^n$ with $k = n2^n$, so that $n \leq f(\omega) < +\infty$. Then, if $f(\omega) \geq n+1$ it follows that $f_n(\omega) = n < n+1 = f_{n+1}(\omega)$. If, on the other hand, $n \leq f(\omega) < n+1$ there exists an $2n\,2^n \leq \ell \leq (2n+2)\,2^n$ such that $\omega \in A_\ell^{n+1}$, which then implies that

$$f_n(\omega) = n = (2n2^n)\,2^{-(n+1)} \leq f_{n+1}(\omega).$$

Problem 4.5

(a) First suppose $f = \sum_{i=1}^N a_i \mathbb{1}_{A_i}$ is a simple function. Then $f\mathbb{1}_B = \sum_{i=1}^N a_i \mathbb{1}_{A_i \cap B}$ is also a simple function and thus

$$\int_B f \,\mathrm{d}\mu = \int_\Omega f\mathbb{1}_B \,\mathrm{d}\mu = \sum_{i=1}^N a_i \mu(A_i \cap B) \leq \mu(B) \sum_{i=1}^N a_i \mu(A_i) = 0.$$

Now let f be a non-negative function and $g \leq f$ be a simple function. Then $g\mathbb{1}_B \leq f\mathbb{1}_B$ and thus by Definition 4.7

$$\int_B f \,\mathrm{d}\mu = \int_\Omega f\mathbb{1}_B \,\mathrm{d}\mu \geq \int_\Omega g\mathbb{1}_B \,\mathrm{d}\mu = 0,$$

which implies the result.

(b) Suppose $f \leq g$ are non-negative functions and observe that if h is a simple function such that $h \leq f$ then also $h \leq g$. Therefore we get

$$\int_\Omega f \, \mathrm{d}\mu = \sup_{h \leq f} \left\{ \int_\Omega h \, \mathrm{d}\mu \right\} \leq \sup_{h \leq g} \left\{ \int_\Omega h \, \mathrm{d}\mu \right\} = \int_\Omega g \, \mathrm{d}\mu.$$

(c) Suppose that h is a simple function. Then αh is also simple and it immediately follows that $\int_\Omega (\alpha h) \, \mathrm{d}\mu = \alpha \int_\Omega h \, \mathrm{d}\mu$. Now let f be non-negative. Then $h \leq f \iff \alpha h \leq \alpha f$ and $h \leq \alpha f \iff \alpha^{-1} h \leq f$. Thus by Definition 4.7 we have

$$\begin{aligned} \alpha \int_\Omega f \, \mathrm{d}\mu &= \alpha \sup_{h \leq f} \left\{ \int_\Omega h \, \mathrm{d}\mu \right\} \\ &= \sup_{h \leq f} \alpha \left\{ \int_\Omega h \, \mathrm{d}\mu \right\} \\ &= \sup_{h \leq f} \left\{ \int_\Omega (\alpha h) \, \mathrm{d}\mu \right\} \\ &= \sup_{\alpha^{-1} h \leq f} \left\{ \int_\Omega h) \, \mathrm{d}\mu \right\} \\ &= \sup_{h \leq \alpha f} \left\{ \int_\Omega (\alpha h) \, \mathrm{d}\mu \right\} = \int_\Omega (\alpha f) \, \mathrm{d}\mu. \end{aligned}$$