

TU/e, 2MBA70

Solutions to problems for Measure and Probability Theory



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Version 0.5 September 24, 2025

Chapter 2: Measurable spaces (sigma-algebras and measures)

Problem 2.6 (23 points) Let \mathcal{O} denote the open sets in \mathbb{R} .

- (a) (2 points) Note that the interval (a, b) is open for any $a < b \in \mathbb{R}$. Hence $\mathcal{A}_1 \subset \mathcal{A}'_1 \subset \mathcal{O}$ and thus by Lemma 2.1.5 we have that $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}'_1) \subset \sigma(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$.
- (b) (2 points) The inclusion \supset is trivial. So assume that $x \in O$. Then by definition there exist an $r > 0$ such that the ball $B_x(r) \subset O$. But $B_x(r) = (x - r, x + r) \in \mathcal{A}_1$ so $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$.
- (c) (3 points) Take $O \in \mathcal{O}$. If we can show that $O \in \sigma(\mathcal{A})$ then $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. The result then follows from 1.
- From 2 it follows that O is a union over a subset collection of interval (a, b) where $a, b \in \mathbb{Q}$. Since \mathbb{Q} is countable, the collection $\{(a, b) : a < b \in \mathbb{Q}\}$ is also countable and hence $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$, from which it follows that $\mathcal{B}_{\mathbb{R}} \subset \sigma(\mathcal{A})$.
- (d) (1 point) This follows immediately from 1 and 3 since these imply that $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}'_1) \subset \mathcal{B}_{\mathbb{R}}$.
- (e) (3 points) The inclusion \subset is trivial, since $(a, b] \subset (a + b + 1/j)$ for any $j \in \mathbb{N}$. For the other inclusion we argue by contradiction. Suppose that $x \in \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$ but $x \notin (a, b]$. Then $x > b$ and hence there exists a $j \in \mathbb{N}$ such that $(b - x) > 1/j$. But this implies that $x \notin (a, b + 1/j)$ which is a contradiction. So we conclude that $(a, b] \supset \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$.
- (f) (3 points) This time the inclusion \supset is trivial since $(a, b - 1/j] \subset (a, b)$ for every $j \in \mathbb{N}$. For the other inclusion suppose that $x \in (a, b)$. Then there exists a $r > 0$ such that the interval $(x - r, x + r) \subset (a, b)$. In particular, this implies that $b - (x + r) > 0$. Now take any $j \in \mathbb{N}$ such that $j > 1/(b - (x + r))$. Then $b - x > r + 1/j$ which implies that $(x - r, x + r) \subset (x - r, b - 1/j]$ and hence $x \in \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$.
- (g) (4 points) It is clear that $\mathcal{A}_2 \subset \mathcal{A}'_2$. By 5 it follows that any interval $(a, b]$ can be obtained as a countable intersection of intervals of the form $(a, b + 1/j)$. By 4 $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}'_1)$ which by Lemma 2.1.2 contains $\bigcap_{j \in \mathbb{N}} (a, b + 1/j) = (a, b]$. So we conclude that any interval $(a, b] \in \mathcal{B}_{\mathbb{R}}$ from which it now follows that

$$\sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}'_2) \subset \sigma(\mathcal{A}'_1) = \mathcal{B}_{\mathbb{R}}.$$

For the other inclusion we consider a set (a, b) with $a, b \in \mathbb{Q}$. Then by 6 we have that $(a, b) = \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$ where the later is a countable union of sets $(c, d]$ with $c, d \in \mathbb{Q}$ which must be in $\sigma(\mathcal{A}_2)$ by definition of a σ -algebra. Hence, any interval $(a, b) \in \sigma(\mathcal{A}_2)$ and we thus conclude, using 3, that

$$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}'_2) \subset \sigma(\mathcal{A}'_1) = \mathcal{B}_{\mathbb{R}},$$

which implies the result.

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- (h) (2 points) Step 1 is to show that any interval $[a, b]$ can be obtained as a countable intersection of intervals $(a - 1/j, b)$. From this we can conclude that any set $[a, b]$ must be in $\mathcal{B}_{\mathbb{R}}$ proving inclusions \subset .

For the other inclusions we have to show that any interval (a, b) can be obtained as a countable union of intervals $[a + 1/j, b)$, which implies that (a, b) must be in the σ -algebra generated by $[a, b]$.

- (i) (3 points) The main tool is to show that each of the intervals $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$ can be obtained by taking any allowed set operation for σ -algebras, i.e. countable unions/intersections and finite complements. This will help use prove the \subset inclusions.

Then we show that any set of the form (a, b) , $[a, b)$ or $(a, b]$ can also be obtained through countable unions/intersections and finite complements of intervals of the forms $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$. These will then yield the \supset inclusions and finish the proof.

Problem 2.9

First note that if $\mu(A \cap B) = \infty$ then by property 2 we have that also $\mu(A)$, $\mu(B)$ and $\mu(A \cup B) = \infty$ and hence the result holds trivially. So assume now that $\mu(A \cap B) < \infty$. Since

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

it follows from property 1 that

$$\mu(A \cup B) = \mu(A \setminus (A \cap B)) + \mu(A \cap B) + \mu(B \setminus (A \cap B)).$$

Adding $\mu(A \cap B) < \infty$ to both side we get

$$\begin{aligned} \mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus (A \cap B)) + \mu(A \cap B) + \mu(B \setminus (A \cap B)) + \mu(A \cap B) \\ &= \mu(A) + \mu(B), \end{aligned}$$

where the last line follows from applying property 3 twice.

Problem 2.11

The idea is to construct a family of disjoint sets $(E_i)_{i \in \mathbb{N}}$ with the following properties:

1. $E_i \subset A_i$, and
2. $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} A_i$.

If such a sequence exists then we have

$$\begin{aligned} \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) &= \mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) && \text{by 2} \\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{because } E_i \text{ are disjoint and } \mu \text{ is } \sigma\text{-additive} \\ &\leq \sum_{i=1}^{\infty} \mu(A_i) && \text{by 1 and monotone property of } \mu. \end{aligned}$$

So we are left to construct the required family of sets $(E_i)_{i \in \mathbb{N}}$. The following set will do:

$$E_1 = A_1 \quad E_i = A_i \setminus \bigcup_{k < i}^i A_k \text{ for all } i > 1.$$

Note that by definition the set E_i are pair-wise disjoint and property 1 holds. Finally, property 2 holds since $\bigcup_{i=1}^k E_i = \bigcup_{i=1}^k A_i$ holds for all $k \geq 1$.

Problem 2.12

(a) (4 pt)

1 pt We first make the following observations about \mathcal{N} :

- because $\mu(\emptyset) = 0$ it holds that $\emptyset \in \mathcal{N}$,
- if $N, M \in \mathcal{N}$ then $N \setminus M \in \mathcal{N}$ since $N \setminus M \subset N$, and
- if $(N_i)_{i \geq 1}$ is a family of sets in \mathcal{N} then so is $\bigcup_{i \geq 1} N_i$.

1 pt From the first point it follows that $\emptyset = \emptyset \cup \emptyset \in \overline{\mathcal{F}}$ and $\Omega = \Omega \cup \emptyset \in \overline{\mathcal{F}}$.

1 pt Furthermore, if $A, B \in \mathcal{F}$ and $N, M \in \mathcal{N}$, then by the second point and because $A \setminus B \in \mathcal{F}$,

$$(A \cup N) \setminus (B \cup M) = (A \setminus B) \cup (N \setminus M) \in \overline{\mathcal{F}}.$$

1 pt Finally, let $(A_i \cup N_i)_{i \geq 1}$ be a collection of sets in \mathcal{N} . Then using the third point we get

$$\bigcup_{i \geq 1} A_i \cup N_i = \bigcup_{i \geq 1} A_i \cup \bigcup_{i \geq 1} N_i \in \overline{\mathcal{F}}.$$

(b) (1 pt) From the definition we immediately get that $\mu(\emptyset) = 0$. Now, let $(A_i \cup N_i)_{i \geq 1}$ be a collection of disjoint sets in \mathcal{N} . Then

$$\bar{\mu}\left(\bigcup_{i \geq 1} A_i \cup N_i\right) = \bar{\mu}\left(\bigcup_{i \geq 1} A_i \cup \bigcup_{i \geq 1} N_i\right) = \mu\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} \mu(A_i) = \sum_{i \geq 1} \bar{\mu}(A_i \cup N_i).$$

(c) (1 pt) This follows from the fact that $\bar{\mu}|_{\mathcal{F}}(A) = \bar{\mu}(A \cup \emptyset) = \mu(A)$.

(d) (2 pt) Suppose that $N \subset \Omega$ is a null set for $\overline{\mathcal{F}}$. Then there exists an $A \cup M \in \overline{\mathcal{F}}$ such that $N \subset A \cup M$ and $\bar{\mu}(A \cup M) = \mu(A) = 0$. However, since $M \in \mathcal{N}$, there must also exist a $B \in \mathcal{F}$ with $M \subset B$ and $\mu(B) = 0$. But this implies that $N \subset A \cup B \in \mathcal{F}$ which implies that $N \in \mathcal{N}$. Therefore, since $N = \emptyset \cup N$ it follows that $N \in \overline{\mathcal{F}}$ and hence every null set is part of the σ -algebra.

Chapter 3: Measurable functions

Problem 3.2

- (a) First we note that $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}$ and $f^{-1}(E) = \Omega \in \mathcal{F}$. So $\emptyset, E \in \mathcal{H}$.

Next, let $B \in \mathcal{H}$. Then

$$f^{-1}(E \setminus B) = \Omega \setminus f^{-1}(B) \in \mathcal{F},$$

since by definition $f^{-1}(B) \in \mathcal{F}$. So $E \setminus B \in \mathcal{H}$.

Finally, if $(B_i)_{i \in \mathbb{N}}$ is a sequence of sets in \mathcal{H} , then

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) \in \mathcal{F},$$

which shows that $\bigcup_{i=1}^{\infty} B_i \in \mathcal{H}$, completing the proof that \mathcal{H} is a σ -algebra.

- (b) By construction $\mathcal{A} \subseteq \mathcal{H}$. It therefore follows from Lemma 2.5 that $\mathcal{G} = \sigma(\mathcal{A}) \subseteq \mathcal{H}$. But this then implies that $f^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{G}$ which means that f is $(\mathcal{F}, \mathcal{G})$ -measurable.

Problem 3.3 “ \subset ” By definition, the product σ -algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$ is defined as the σ -algebra generated by the collection

$$\mathcal{A} := \left\{ A \times B \subset \Omega_1 \times \Omega_2 : A \in \mathcal{F}_1, B \in \mathcal{F}_2 \right\}.$$

Since $A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B)$, we have that

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \sigma(\pi_1, \pi_2).$$

“ \supset ” Let $C \in \{\pi_i^{-1}(A) : i = 1, 2, A \in \mathcal{F}_1\}$. Then there exist sets $A \in \mathcal{F}_1$ or $B \in \mathcal{F}_2$ such that $C = \pi_1^{-1}(A) = A \times \Omega_2$ or $C = \pi_2^{-1}(B) = \Omega_1 \times B$. Either way, since $\Omega_1 \in \mathcal{F}_1$ and $\Omega_2 \in \mathcal{F}_2$, we have that $C \in \mathcal{A}$.

Problem 3.4 It is clear that $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Suppose a sequence of mutually disjoint sets $B_i \in \mathcal{G}$, $i \in \mathbb{N}$, is given. Then,

$$f_{\#}\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) = \sum_{i=1}^{\infty} f_{\#}\mu(B_i).$$

Problem 3.6

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- (a) By Proposition 2.8, we know that $\mathcal{B}_{\mathbb{R}}$ is generated by intervals of the form $(-\infty, a]$ with $a \in \mathbb{Q}$. As a consequence, $\mathcal{B}_{\mathbb{R}}$ is also generated by intervals of the form $(a, +\infty)$ with $a \in \mathbb{Q}$. Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\}$$

is measurable for every $a \in \mathbb{Q}$. For brevity, we write $\{f + g > a\}$. The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \left(\{f > t\} \cap \{g > a - t\} \right).$$

We first show the inclusion ' \subset '. If $\omega \in \Omega$ is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some $t \in \mathbb{Q}$ such that

$$f(\omega) > t > a - g(\omega),$$

and thus $f(\omega) > t$ and $g(\omega) > a - t$. So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \left(\{f > t\} \cap \{g > a - t\} \right).$$

Now we will show the inclusion ' \supset '. Let $\omega \in \Omega$ be such that $f(\omega) > t$ and $g(\omega) > a - t$. Then, by adding the inequalities, we know that $f(\omega) + g(\omega) > a$.

- (b) The constant function $f(\omega) = a$ is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \quad \forall B \in \mathcal{B}_{\mathbb{R}}.$$

- (c) Similar to the proof of Point (2) of Proposition 3.2.12.

- (d) Let $g(\omega) \neq 0$ for all $\omega \in \Omega$. Then, since g is measurable, we have that

$$\begin{aligned} \{1/g > a\} &= \{g < 1/a, g > 0\} \cup \{g > 1/a, g < 0\} \\ &= \left(\{g < 1/a\} \cap \{g > 0\} \right) \cup \left(\{g > 1/a\} \cap \{g < 0\} \right) \in \mathcal{F}, \end{aligned}$$

thus implying that $1/g$ is measurable.

- (e) The previous part of this exercise together with point (4) of Proposition 3.12 yields Point (5) of Proposition 3.12.

Problem 3.7 From (3.6), we have for any $a \in \mathbb{R}$,

$$\left\{ \sup_{n \geq 1} f_n > a \right\} = \bigcup_{n \geq 1} \{f_n > a\} \in \mathcal{F},$$

Since \mathcal{F} is a σ -algebra and f_n is measurable for all $n \geq 1$, i.e., $\{f_n > a\} \in \mathcal{F}$ for all $n \geq 1$.

Problem 3.8

(a) (3 pts) **1 pt** Note that

$$f_M = M\mathbf{1}_{\{f \geq M\}} + f\mathbf{1}_{\{|f| < M\}} - M\mathbf{1}_{\{f \leq -M\}}.$$

2 pts Since the sets

$$\{f \geq M\}, \quad \{f \leq -M\}, \quad \{|f| < M\} \quad \text{are } \mathcal{F}\text{-measurable,}$$

their corresponding indicator functions are $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable. Since f_M is the sum of products of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable functions, we conclude that f_M is also $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable.

(b) (3 pts) It is easy to see that f_M converges pointwise to f as $M \rightarrow \infty$, i.e.,

$$\lim_{M \rightarrow \infty} f_M(\omega) = f(\omega) \quad \forall \omega \in \Omega.$$

1 pt Indeed, if $\omega \in \Omega$ is such that $f(\omega) = +\infty$, then

$$\lim_{M \rightarrow \infty} f_M(\omega) = \lim_{M \rightarrow \infty} M = +\infty = f(\omega),$$

and similarly for $\omega \in \Omega$ for which $f(\omega) = -\infty$.

2 pts On the other hand, for any $\omega \in \Omega$ with $f(\omega) \in \mathbb{R}$, there is some $N_0(\omega) \in \mathbb{N}$ such that $f_N(\omega) = f(\omega)$ for all $N \geq N_0(\omega)$, and hence,

$$\lim_{M \rightarrow \infty} f_M(\omega) = f(\omega).$$

Since f is the limit of a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable functions, we conclude from Lemma 3.2.13 that f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable.

Chapter 4: The Lebesgue Integral

Problem 4.2

Problem 4.3

(a) The fact that the sets are disjoint is immediate from the definition. Measurability follows from Lemma 3.11

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- (b) Let us fix a $\omega \in \Omega$. Then if $f(\omega) = +\infty$ we get that $f_n(\omega) = 2^n$ holds for all $n \geq 1$ and hence $\lim_{n \rightarrow \infty} f_n(\omega) = +\infty = f(\omega)$. So assume that $f(\omega) < +\infty$. Then there exists an $M \in \mathbb{N}$ such that $f(\omega) < M$. Hence, for all $n \geq M$ we have that

$$\|f_n(\omega) - f(\omega)\| = f(\omega) - f_n(\omega) \leq 2^{-n},$$

which implies that $\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$.

- (c) Fix $n \geq 1$ and $\omega \in \Omega$. Clearly, if $f(\omega) = +\infty$ then $f_n(\omega) = 2^n < +\infty = f(\omega)$.
 (d) Fix $\omega \in \Omega$ such that $f(\omega) < +\infty$ and $\omega \in A_k^n$ for some $0 \leq k < N_n = n2^n$.

Note that $k2^{-n} \leq f(\omega) < (k+1)2^{-n}$ holds and this interval can be split into two intervals as follows:

$$[k2^{-n}, (k+1)2^{-n}) = [(2k)2^{-(n+1)}, (2k+1)2^{-(n+1)}) \cup [(2k+1)2^{-(n+1)}, (2k+2)2^{-(n+1)}).$$

Hence, we conclude that either $\omega \in A_{2k}^{n+1}$ or $\omega \in A_{2k+1}^{n+1}$. In both case we get that

$$f_n(\omega) = k2^{-n} = 2kn^{-(n+1)} \leq f_{n+1}(\omega).$$

- (e) Now let us consider the case where $\omega \in A_k^n$ with $k = n2^n$, so that $n \leq f(\omega) < +\infty$. Then, if $f(\omega) \geq n+1$ it follows that $f_n(\omega) = n < n+1 = f_{n+1}(\omega)$. If, on the other hand, $n \leq f(\omega) < n+1$ there exists an $2n2^n \leq \ell \leq (2n+2)2^n$ such that $\omega \in A_\ell^{n+1}$, which then implies that

$$f_n(\omega) = n = (2n2^n)2^{-(n+1)} \leq f_{n+1}(\omega).$$

Problem 4.5

- (a) First suppose $f = \sum_{i=1}^N a_i \mathbb{1}_{A_i}$ is a simple function. Then $f \mathbb{1}_B = \sum_{i=1}^N a_i \mathbb{1}_{A_i \cap B}$ is also a simple function and thus

$$\int_B f \, d\mu = \int_\Omega f \mathbb{1}_B \, d\mu = \sum_{i=1}^N a_i \mu(A_i \cap B) \leq \mu(B) \sum_{i=1}^N a_i \mu(A_i) = 0.$$

Now let f be a non-negative function and $g \leq f$ be a simple function. Then $g \mathbb{1}_B \leq f \mathbb{1}_B$ and thus by Definition 4.7

$$\int_B f \, d\mu = \int_\Omega f \mathbb{1}_B \, d\mu \geq \int_\Omega g \mathbb{1}_B \, d\mu = 0,$$

which implies the result.

- (b) Suppose $f \leq g$ are non-negative functions and observe that if h is a simple function such that $h \leq f$ then also $h \leq g$. Therefore we get

$$\int_\Omega f \, d\mu = \sup_{h \leq f} \left\{ \int_\Omega h \, d\mu \right\} \leq \sup_{h \leq g} \left\{ \int_\Omega h \, d\mu \right\} = \int_\Omega g \, d\mu.$$

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- (c) Suppose that h is a simple function. Then αh is also simple and it immediately follows that $\int_{\Omega} (\alpha h) d\mu = \alpha \int_{\Omega} h d\mu$. Now let f be non-negative. Then $h \leq f \iff \alpha h \leq \alpha f$ and $h \leq \alpha f \iff \alpha^{-1}h \leq f$. Thus by Definition 4.7 we have

$$\begin{aligned}
\alpha \int_{\Omega} f d\mu &= \alpha \sup_{h \leq f} \left\{ \int_{\Omega} h d\mu \right\} \\
&= \sup_{h \leq f} \alpha \left\{ \int_{\Omega} h d\mu \right\} \\
&= \sup_{h \leq f} \left\{ \int_{\Omega} (\alpha h) d\mu \right\} \\
&= \sup_{\alpha^{-1}h \leq f} \left\{ \int_{\Omega} h d\mu \right\} \\
&= \sup_{h \leq \alpha f} \left\{ \int_{\Omega} (\alpha h) d\mu \right\} = \int_{\Omega} (\alpha f) d\mu.
\end{aligned}$$

Problem 4.7

- (a) (2pts) **1 pt** Note that $(f \mathbb{1}_B)^+ = f^+ \mathbb{1}_B$ and $(f \mathbb{1}_B)^- = f^- \mathbb{1}_B$. **1 pt** Then we get, using Lemma 4.8,

$$\int_B f d\mu = \int_{\Omega} (f \mathbb{1}_B)^+ d\mu - \int_{\Omega} (f \mathbb{1}_B)^- d\mu = \int_{\Omega} f^+ \mathbb{1}_B d\mu - \int_{\Omega} f^- \mathbb{1}_B d\mu = 0 + 0 = 0.$$

- (b) (2pts)

1 pt Here we note that if $f \leq g$ then $f^+ \leq g^+$, while $f^- \geq g^-$.

1 pt Hence, using Lemma 4.8 again,

$$\int_{\Omega} f d\mu = \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu \leq \int_{\Omega} g^+ d\mu - \int_{\Omega} g^- d\mu = \int_{\Omega} g d\mu.$$

- (c) (3 pt)

1 pt Assume first that $\alpha \geq 0$. Then it follows from Lemma 4.8 that

$$\alpha \int_{\Omega} f^{\pm} d\mu = \int_{\Omega} (\alpha f)^{\pm} d\mu,$$

which implies the result.

1 pt Now suppose that $\alpha < 0$ so that $\beta := -\alpha > 0$. Note that for any function f we have

$$(-f)^+ := \max\{-f, 0\} = \min\{f, 0\} = f^-$$

and similarly $(-f)^- = f^+$.

1 pt We then get that

$$-\int_{\Omega} f d\mu = -\int_{\Omega} f^+ d\mu + \int_{\Omega} (f)^- d\mu = -\int_{\Omega} (-f)^- d\mu + \int_{\Omega} (-f)^+ d\mu = \int_{\Omega} (-f) d\mu.$$

The result then follows since

$$\alpha \int_{\Omega} f d\mu = -\beta \int_{\Omega} f d\mu = -\int_{\Omega} (\beta f) d\mu = \int_{\Omega} (-\beta f) d\mu = \int_{\Omega} (\alpha f) d\mu.$$

- (d) (2 pt) This result follows immediately from Lemma 4.8 and the observation that $(f+g)^{\pm} = f^{\pm} + g^{\pm}$.

Problem 4.8

- (a) By definition, we have that $\nu_f(\Omega) = \int_{\Omega} f d\mu = 1$. Now let $(A_n)_{n \in \mathbb{N}}$ be a family of mutually disjoint measurable sets. Then we have that the sequence

$$g_n := \sum_{i=1}^n f \mathbf{1}_{A_i} = f \mathbf{1}_{\bigcup_{i=1}^n A_i} \longrightarrow g := f \mathbf{1}_{\bigcup_{i \in \mathbb{N}} A_i} \quad \text{pointwise monotonically.}$$

By MCT, we then have that

$$\nu_f \left(\bigcup_{i \in \mathbb{N}} A_i \right) = \int_{\bigcup_{i \in \mathbb{N}} A_i} f d\mu = \lim_{n \rightarrow \infty} \int_{\bigcup_{i=1}^n A_i} f d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{A_i} f d\mu = \sum_{i \in \mathbb{N}} \nu_f(A_i),$$

thus showing that ν_f is a probability measure on (Ω, \mathcal{F}) .

- (b) Following the hint, we start by considering nonnegative simple functions g . Suppose $g = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ for $a_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$ mutually disjoint. Then,

$$\int_{\Omega} g d\nu_f = \sum_{i=1}^n a_i = \nu_f(A_i) = \sum_{i=1}^n a_i \int_{A_i} f d\mu = \int_{\Omega} g f d\mu.$$

Now let g be a nonnegative measurable function and $[g]_n$ be a sequence of nonnegative simple functions that converge pointwise monotonically to g . Then MCT yields

$$\int_{\Omega} g d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} [g]_n d\nu_f = \lim_{n \rightarrow \infty} \int_{\Omega} [g]_n f d\mu = \int_{\Omega} g f d\mu,$$

where we used the fact that $[g]_n f$ converges pointwise monotonically to $g f$.

- (c) Let g be measurable. Then $g = g^+ - g^-$, where g^{\pm} are nonnegative measurable functions. Since f is nonnegative, we have that $(fg)^{\pm} = fg^{\pm}$. Due to (b), we deduce

$$\int_{\Omega} g^{\pm} d\nu_f = \int_{\Omega} g^{\pm} f d\mu = \int_{\Omega} (gf)^{\pm} d\mu.$$

Hence, g^{\pm} is ν_f -integrable if and only if $(gf)^{\pm}$ is μ -integrable. Consequently, g is ν_f -integrable if and only if gf is μ -integrable, since

$$\int_{\Omega} |g| d\nu_f = \int_{\Omega} g^+ d\nu_f + \int_{\Omega} g^- d\nu_f = \int_{\Omega} g^+ f d\mu + \int_{\Omega} g^- f d\mu = \int_{\Omega} |gf| d\mu.$$

Problem 4.9

(\Rightarrow) (4pts) **2 pts** Let f be μ -integrable. Then both $|f|\mathbf{1}_{\{|f|<n\}}$ and $|f|\mathbf{1}_{\{|f|\geq n\}}$ are integrable, due to the monotonicity of the integral. By linearity of the integral,

$$\int_{\Omega} |f|\mathbf{1}_{\{|f|\geq n\}} d\mu = \int_{\Omega} |f| d\mu - \int_{\Omega} |f|\mathbf{1}_{\{|f|<n\}} d\mu.$$

2 pts Since the sequence $g_n := |f|\mathbf{1}_{\{|f|<n\}} \geq 0$ converges pointwise monotonically to $|f|$, we can apply MCT to obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f|\mathbf{1}_{\{|f|<n\}} d\mu = \int_{\Omega} |f| d\mu.$$

Hence,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f|\mathbf{1}_{\{|f|\geq n\}} d\mu = \int_{\Omega} |f| d\mu - \lim_{n \rightarrow \infty} \int_{\Omega} |f|\mathbf{1}_{\{|f|<n\}} d\mu = 0.$$

(\Leftarrow) (3 pts)

1 pt By assumption, there is some $N \geq 1$ such that

$$\int_{\Omega} |f|\mathbf{1}_{\{|f|\geq N\}} d\mu \leq 1.$$

2 pts By linearity of the integral,

$$\int_{\Omega} |f| d\mu = \int_{\Omega} |f|\mathbf{1}_{\{|f|<N\}} d\mu + \int_{\Omega} |f|\mathbf{1}_{\{|f|\geq N\}} d\mu \leq N\mu(\{|f|<N\}) + 1.$$

Since μ is a finite measure, the right-hand side is finite, implying that f is μ -integrable.

Problem 4.10

Observe that $\Omega = \bigcup_{n \in \mathbb{N}} \{|f| > n\}$.

We then get that

$$\sum_{n=1}^{\infty} \int_{\{|f|>n\}} |f| d\mu = \int_{\Omega} |f| d\mu < \infty.$$

This implies that for some N and all $n \geq N$: $\int_{\{|f|>n\}} |f| d\mu < 1/n$ or else the sum cannot be finite.

Now let $\varepsilon > 0$, take $M > \max\{N, 2/\varepsilon\}$ and $\delta = \varepsilon/(2M)$. Then

$$\begin{aligned} \int_A |f| d\mu &= \int_A |f|\mathbf{1}_{|f|\leq M} d\mu + \int_A |f|\mathbf{1}_{|f|>M} d\mu \\ &\leq M\mu(A) + \frac{1}{M} \leq M\delta + \frac{1}{M} < \varepsilon. \end{aligned}$$

Chapter 5: Product spaces and Lebesgue integration

Problem 5.2

(a) Note that $\mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{F}_1 \times \mathcal{F}_2$, and hence

$$\sigma(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{F}_1 \otimes \mathcal{F}_2.$$

(b) Let $B \in \mathcal{A}_2$. Then we have that

$$\Omega_1 \times B = \bigcup_{n \geq 1} A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$$

since $A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ for all $n \geq 1$. So $\Omega_1 \in \Sigma$

For the second property, let $C \in \Sigma$ and note that $C^c \times B = (\Omega_1 \times B) \setminus (C \times B)$. Since both these sets are in $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ it follows that $C^c \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ and hence $C^c \in \Sigma$.

Finally consider a countable sequence $(C_n)_{n \geq 1}$ of sets in Σ . Then for any $B \in \mathcal{A}_2$

$$\left(\bigcup_{n \geq 1} C_n \right) \times B = \bigcup_{n \geq 1} (C_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2),$$

since each $C_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(c) Note that $\mathcal{A}_1 \subset \Sigma_1 \subset \mathcal{F}_1$. From which it follows that $\Sigma_1 = \mathcal{F}_1$. But then, from the definition of Σ_1 we have that $\mathcal{F}_1 \times \mathcal{A}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(d) We can show in a similar fashion that

$$\Sigma_2 := \{C \in \mathcal{F}_2 : B \times C \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2) \forall B \in \mathcal{A}_1\}.$$

is a σ -algebra on Ω_2 , from which we conclude that $\mathcal{A}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(e) take any $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Then

$$A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B) = \bigcup_{n, m \geq 1} (A \times B_m) \cap (A_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2).$$

From this we conclude that $\mathcal{F}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$, which finishes the proof.

Chapter 6: Probability I, The basics

Problem 6.2

- (a) The implication from right to left is by definition of \overleftarrow{F} and the fact that F is non-decreasing. The implication from left to right is because F is right continuous.
- (b) Consider the preimage of $(-\infty, t]$ under X . Then, using the above observation, we have

$$\begin{aligned} X^{-1}((-\infty, t]) &= \{\omega \in \Omega : \overleftarrow{F}(U(\omega)) \in (-\infty, t]\} \\ &= \{\omega \in \Omega : U(\omega) \in (-\infty, F(t)]\} = U^{-1}((-\infty, F(t)]) \in \mathcal{B}_{[0,1]}. \end{aligned}$$

Hence, X is measurable. Finally, the above computation, together with Lemma 6.5, also implies that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \mathbb{P}(U^{-1}((-\infty, F(t)])) = F(t).$$

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and U a standard normal random variable. We will show that $X = \overleftarrow{F} \circ U$ is a random variable with the right probability measure. Since we can construct a standard uniform random variable on the probability $([0, 1], \mathcal{B}_{[0,1]}, \lambda|_{[0,1]})$ this also implies the last part.

which finished the proof.

Problem 6.3

- (a) For the probability space, take $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0, 1]$.

Observe that the function $H_\gamma(z)$ is continuous and hence has an inverse $g_\gamma(y) = \gamma \tan(\pi(y - 1/2))$ on $[0, 1]$.

So the function $Y[0, 1] \rightarrow \mathbb{R}$ defined by $Y(x) = g_\gamma(x)$ has the correct distribution as

$$\mathbb{P}(Y^{-1}((-\infty, t])) = \mathbb{P}(g_\gamma^{-1}((-\infty, t])) = \lambda(H_\gamma((-\infty, t])) = H_\gamma(t).$$

- (b) Note that g_γ is continuous on $[0, 1]$ and hence measurable.
- (c) For any $t \geq 0$, the cdf of the Poisson random variable is given by

$$F_\lambda(t) = \sum_{n=0}^{\lceil t \rceil} f_\lambda(n),$$

where $\lceil t \rceil$ is the ceiling of t , i.e. the smallest integer $k \geq t$.

- (d) For the probability space, we again take $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0, 1]$.

Now for any $y \in [0, 1]$ let $k := k(y)$ be such that

$$\sum_{n=1}^k f_\lambda(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_\lambda(n) < y,$$

where the last sum is interpreted as -1 if $k = 0$.

Now define $X(y) = k(y) : [0, 1] \rightarrow \mathbb{R}$. Then $k(y) \leq t$ if and only if $y \leq F_\lambda(t)$ and hence

$$X^{-1}((-\infty, t]) = \{y \in [0, 1] : k(y) \in (0, t]\} = \{y \in [0, 1] : y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

- (e) It follows from the above computation that $X^{-1}((-\infty, t]) = \{y \in [0, 1] : y \in (0, F_\lambda(t)]\}$. Since the latter is a measurable set we conclude that $X^{-1}((-\infty, t])$ is measurable for all t and since these generate the Borel σ -algebra X is measurable.
- (f) for any $\ell \in \mathbb{N}$ define the sets $A_\ell = (n - 1 - 1/\ell, n - 1 + 1/\ell]$. Then A_ℓ is a decreasing set with $\lim_{\ell \rightarrow \infty} A_\ell = \{n\}$. Moreover, $A_\ell = (-\infty, n - 1 + 1/\ell] \setminus (-\infty, n - 1 - 1/\ell]$ and $\mathbb{P}(A_1) < \infty$. It now follows from continuity from above and (d) that

$$\begin{aligned} X_\# \mathbb{P}(\{n\}) &= \lim_{\ell \rightarrow \infty} X_\# \mathbb{P}(A_\ell) \\ &= \lim_{\ell \rightarrow \infty} X_\# \mathbb{P}((-\infty, n - 1 + 1/\ell]) - X_\# \mathbb{P}((-\infty, n - 1 - 1/\ell]) \\ &= F_\lambda(n - 1 + 1/\ell) - F_\lambda(n - 1 - 1/\ell) \\ &= \sum_{k=0}^n f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n). \end{aligned}$$

Problem 6.5

Define for any $j \in \mathbb{Z}$, $p_j := \mathbb{P}(X^{-1}(\{j\}))$. Then, since $(X^{-1}(j))_{j \in \mathbb{Z}}$ is a family of disjoint sets and \mathbb{P} is a probability measure we get that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}\left(\bigcup_{j \in \mathbb{Z}} X^{-1}(j)\right) = \sum_{j \in \mathbb{Z}} p_j.$$

Now let $A \subset \mathbb{R}$ be a measurable set and note that

$$X^{-1}(A) = \bigcup_{j \in \mathbb{Z} \cap A} X^{-1}(j).$$

Then it follows that

$$\mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}\left(\bigcup_{j \in \mathbb{Z} \cap A} X^{-1}(j)\right) = \sum_{j \in \mathbb{Z} \cap A} p_j = \sum_{j \in \mathbb{Z}} \delta_j(A) p_j.$$

Problem 6.7

(a) (3 pts)

1 pt We first observe that

$$\nu((-\infty, t]) = X_{\#}\mathbb{P}((-\infty, t])$$

holds by definition of the pdf.

1 pt Next we note that the collection $\mathcal{A} := \{(-\infty, t] : t \in \mathbb{R}\}$ generates \mathcal{B} and satisfies the conditions of Theorem 2.15:

a) $(-\infty, t] \cap (-\infty, s] = (-\infty, \min\{t, s\}] \in \mathcal{A}$, and

b) the sets $A_n(-\infty, n] \in \mathcal{A}$ for $n \in \mathbb{N}$ satisfies $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{R}$.

1 pt Thus, since $\nu = X_{\#}\mathbb{P}$ on \mathcal{A} , by Theorem 2.15 we conclude that $\nu = X_{\#}\mathbb{P}$ on $\sigma(\mathcal{A}) = \mathcal{B}$.

(b) (2 pts) Let $g = \sum_{i=1}^N a_i \mathbb{1}_{A_i}$ be a simple function. Then by definition of ν and linearity of the integral we get

$$\begin{aligned} \int_{\mathbb{R}} g \, d\nu &= \sum_{i=1}^N a_i \int_{A_i} d\nu \\ &= \sum_{i=1}^N a_i \nu(A_i) \\ &= \sum_{i=1}^N a_i \int_{A_i} \rho \, d\lambda \\ &= \int_{\Omega} \sum_{i=1}^N a_i \mathbb{1}_{A_i} \rho \, d\lambda = \int_{\mathbb{R}} g \rho \, d\lambda. \end{aligned}$$

(c) (2 pts) By using (b) and monotone convergence twice we get

$$\begin{aligned} \int_{\mathbb{R}} h \, d\nu &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} [h]_n \, d\nu \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n \, d\nu \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} [h]_n \rho \, d\lambda \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} [h]_n \rho \, d\lambda = \int_{\mathbb{R}} h \rho \, d\lambda. \end{aligned}$$

(d) (1 pt) Using the change of variables result (Proposition 4.14) (a) and (c) we get

$$\mathbb{E}[h(X)] = \int_{\Omega} h \circ X \, d\mathbb{P} = \int_{\mathbb{R}} h \, dX_{\#}\mathbb{P} = \int_{\mathbb{R}} h \, d\nu = \int_{\mathbb{R}} h \rho \, d\lambda.$$

Problem 6.8

(a) This follows from the following computation

$$\int_{\Omega} |f|^p d\mu \geq \int_{\Omega} |f|^p \mathbb{1}_{|f| \geq t} d\mu \geq t^p \int_{\Omega} \mathbb{1}_{|f| \geq t} d\mu = t^p \mu(\{\omega \in \Omega : |f| \geq t\}).$$

(b) Using the result for $p = 1$ we get

$$\mathbb{P}(|X| \geq t) = \mu(\omega \in \Omega : |X(\omega)| \geq t) \leq \frac{1}{t} \int_{\Omega} X d\mathbb{P} = \frac{1}{t} \mathbb{E}[X].$$

(c) Take $f(\omega) = X(\omega) - \mathbb{E}[X]$, which is measurable. Then using the first result with $p = 2$ gives

$$\begin{aligned} \mathbb{P}(|X - \mathbb{E}[X]| \geq t) &= \mathbb{P}(|X - \mathbb{E}[X]|^2 \geq t^2) \\ &\leq \frac{1}{t^2} \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \frac{1}{t^2} (\mathbb{E}[X^2] - \mathbb{E}[X]^2) = \frac{\text{Var}(X)}{t^2}. \end{aligned}$$

Chapter 7: Convergence of integrals and measures

Problem 7.1

Similar to the proof of Fatou's lemma, we define $g_n = \sup_{k \geq n} f_k$ which are measurable due to Proposition 3.13. Moreover, we have that $\limsup_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} g_n$.

Next we note that $g_n \geq f_\ell$ for all $\ell \geq n$. Thus, by monotonicity of the integral, we have that

$$\int_{\Omega} g_n d\mu \geq \int_{\Omega} f_\ell d\mu,$$

holds for all $\ell \geq n$, which implies that

$$\int_{\Omega} g_n d\mu \geq \sup_{k \geq n} \int_{\Omega} f_k d\mu.$$

In addition, since $g_n < f$ with f being non-negative and integrable we can apply Dominated Convergence to conclude that

$$\int_{\Omega} \lim_{n \rightarrow \infty} g_n d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu.$$

Putting all this together we get

$$\int_{\Omega} \limsup_{n \rightarrow \infty} f_n d\mu = \int_{\Omega} \lim_{n \rightarrow \infty} g_n d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} g_n d\mu \geq \lim_{n \rightarrow \infty} \sup_{k \geq n} \int_{\Omega} f_k d\mu.$$

Problem 7.2

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- (a) Let $t_0 \in (a, b)$ be fixed. It suffices to check the continuity result for arbitrary sequences $(t_n)_{n \geq 1} \subset (a, b)$ such that $t_n \rightarrow t_0$ as $n \rightarrow \infty$. Fix such a sequence and define $g_n(\omega) := f(\omega, t_n)$ for all $\omega \in \Omega$ and $n \geq 1$. Since $\lim_{t \rightarrow t_0} f(\omega, t) = f(\omega, t_0)$ for all $\omega \in \Omega$, we deduce that $\lim_{n \rightarrow \infty} g_n(\omega) = f(\omega, t_0)$ for every $\omega \in \Omega$. Moreover, by assumption $|g_n| \leq g$ for all $n \geq 1$ and g is integrable. By the Dominated Convergence Theorem

$$\lim_{n \rightarrow \infty} \int_{\Omega} g_n(\omega) \mu(d\omega) = \int_{\Omega} f(\omega, t_0) \mu(d\omega).$$

As the chosen sequence was arbitrary, we deduce that $\lim_{t \rightarrow t_0} F(t) = F(t_0)$.

- (b) If $t \mapsto f(\omega, t)$ is continuous on (a, b) for all $\omega \in \Omega$ then $\lim_{t \rightarrow t_0} f(\omega, t) = f(\omega, t_0)$ at every $t_0 \in (a, b)$ for all $\omega \in \Omega$. In particular, (a) applies, showing that $\lim_{t \rightarrow t_0} F(t) = F(t_0)$ for every $t_0 \in (a, b)$, i.e., F is continuous on (a, b) .

Problem 7.3

- (a) (4 pts) We start by showing that $(\partial f / \partial t)(\cdot, t)$ is measurable.

2 pts Let $(t_n)_{n \geq 1} \subset (a, b)$ be an arbitrary sequence with $t_n \neq t$ and $t_n \rightarrow t$ for $n \rightarrow \infty$. We set

$$h_n(\omega) = \frac{f(\omega, t_n) - f(\omega, t)}{t_n - t}.$$

Clearly, h_n is measurable for every $n \geq 1$.

1 pt Moreover, $\lim_{n \rightarrow \infty} h_n(\omega) = (\partial f / \partial t)(\omega, t)$ by the definition of the derivative. Since $(\partial f / \partial t)(\cdot, t)$ is the pointwise limit of a sequence of measurable functions, it is also measurable.

1 pt Finally, $(\partial f / \partial t)(\cdot, t)$ is integrable since

$$\int_{\Omega} |(\partial f / \partial t)(\omega, t)| \mu(d\omega) \leq \int_{\Omega} g d\mu < +\infty.$$

- (b) (3 pts)

2 pts Let $t_0 \in (a, b)$ and suppose w.l.o.g. $t_0 < t$. Since $t \mapsto f(\omega, t)$ is differentiable on (a, b) for all $\omega \in \Omega$, the Mean Value Theorem gives

$$\frac{f(\omega, t) - f(\omega, t_0)}{t - t_0} = (\partial f / \partial t)(\omega, \tau) \quad \text{for some } \tau \in (t_0, t).$$

1 pt Taking the modulus on both sides, we obtain

$$\left| \frac{f(\omega, t) - f(\omega, t_0)}{t - t_0} \right| \leq |(\partial f / \partial t)(\omega, \tau)| \leq g(\omega) \quad \text{for all } \omega \in \Omega.$$

(c) (5 pts)

2 pts Take $t_0 \in (a, b)$ and let $(t_n)_{n \geq 1}$ be a sequence in (a, b) such that $t_n \rightarrow t_0$ and define

$$h(\omega, t) := \frac{f(\omega, t) - f(\omega, t_0)}{t - t_0}.$$

Then by (b) and the conditions in this exercise, h satisfies all the conditions listed in Problem 4.2.

1 pt Next, we note that by linearity of the integral we have that

$$\frac{F(t_n) - F(t_0)}{t_n - t_0} = \int_{\Omega} \frac{f(\omega, t_n) - f(\omega, t_0)}{t_n - t_0} \mu(d\omega) = \int_{\Omega} h(\omega, t_n) \mu(d\omega).$$

1 pt Now by Problem 4.2 it holds that

$$\lim_{t_n \rightarrow t_0} \int_{\Omega} h(\omega, t_n) \mu(d\omega) = \int_{\Omega} \lim_{t_n \rightarrow t_0} h(\omega, t_n) \mu(d\omega) = \int_{\Omega} \frac{\partial f}{\partial t}(\omega, t_0) \mu(d\omega).$$

1 pt Since $t_0 \in (a, b)$ and the sequence $(t_n)_{n \geq 1}$ were arbitrary, we conclude that F is indeed differentiable on (a, b) with

$$\frac{\partial F}{\partial t}(t_0) := \lim_{t_n \rightarrow t_0} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \int_{\Omega} \frac{\partial f}{\partial t}(\omega, t_0) \mu(d\omega).$$

Problem 7.4

(a) Note that the integrand $f_n(x) = \frac{1+nx^2}{(1+x^2)^n}$ is continuous on $[0, 1]$ and non-negative. Hence, the Riemann integral and Lebesgue integral coincide, i.e.,

$$\int_0^1 f_n(x) dx = \int_{[0,1]} f_n d\lambda.$$

Observe that we have the following pointwise limit

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in (0, 1], \end{cases}$$

i.e., $\lim_{n \rightarrow \infty} f_n = 0$ λ -almost everywhere. Moreover, $f_n(x) \leq 1$ for every $x \in [0, 1]$ and $n \geq 1$. Since the constant function $g \equiv 1$ is λ -integrable on $[0, 1]$, it is a valid dominator. Hence, the DCT gives

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \lim_{n \rightarrow \infty} \int_{[0,1]} f_n d\lambda = \int_{[0,1]} \lim_{n \rightarrow \infty} f_n d\lambda = 0$$

- (b) For the purpose of convergence, we consider $n \geq 3$. Note that the integrand $f_n(x) = \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right)$ is continuous on $(0, +\infty)$ with pointwise limit

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{if } x \in (0, 1), \\ 1/2 & \text{if } x = 1, \\ 1/x^2 & \text{if } x > 1, \end{cases}$$

Setting the function

$$g(x) = \begin{cases} 1 & \text{for } x \in (0, 1), \\ \frac{1}{x^2} & \text{for } x \geq 1, \end{cases}$$

we see that $f_n \leq g$ λ -almost everywhere in $(0, +\infty)$ and for all $n \geq 3$. Indeed, for $x \geq 1$, we obtain

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq \frac{x^{n-2}}{x^n} = \frac{1}{x^2},$$

while for $x \in (0, 1)$, we have

$$|f_n(x)| \leq \left| \frac{x^{n-2}}{1+x^n} \cos\left(\frac{\pi x}{n}\right) \right| \leq \frac{x^{n-2}}{1+x^n} \leq 1.$$

Notice that g is non-negative and λ -integrable on $(0, +\infty)$. Indeed, using the MCT,

$$\begin{aligned} \int_{(0, +\infty)} g \, d\lambda &= \int_{(0, 1)} g \, d\lambda + \int_{(1, +\infty)} g \, d\lambda = 1 + \lim_{n \rightarrow \infty} \int_{(1, n)} g \, d\lambda \\ &= 1 + \lim_{n \rightarrow \infty} \int_1^n \frac{1}{x^2} \, dx = 1 + \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 2 < +\infty. \end{aligned}$$

To conclude, we apply DCT to deduce that the limit is 1.

Problem 7.5

Problem 7.6

(a)

(b) By definition

$$\lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} f \, d\mu \right| \leq \varepsilon,$$

implies that for any $\delta > 0$

$$\left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} f \, d\mu \right| < \varepsilon + \delta,$$

holds for large enough n . Note that this holds for any $\varepsilon, \delta > 0$.

Now pick $\eta > 0$ and set $\varepsilon = \eta/2 = \delta$, then the above inequality implies that

$$\lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} f \, d\mu \right| = 0.$$

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- (c) Consider the sequence of sets $A_n = \mathbb{R} \setminus [-n, n]$. Then $A_n \supset A_{n+1}$ and $A_n \downarrow \emptyset$. Hence, it follows from Proposition 2.12 2) that $\lim_{n \rightarrow \infty} \mu(A_n) = 0$. Thus, there exists a N such that $\mu(A_n) < \varepsilon/(2M)$ holds for all $n \geq N$. We can then take any $\alpha > N$.

- (d) The function

$$g(x) = \mathbb{1}_{[-\alpha, \alpha]}(x) + \mathbb{1}_{(-(\alpha+1), -\alpha)}(x)(x + (\alpha + 1)) + \mathbb{1}_{(\alpha, \alpha+1)}(x)(-x + \alpha + 1)$$

does the trick. This is simply a linear increase from zero to one from $-(\alpha + 1)$ to $-\alpha$ and from $\alpha + 1$ to α .

- (e) Observe that g is a non-negative continuous bounded function that is zero outside the interval $[-(\alpha + 1), \alpha + 1]$, and thus we can apply (3). Using linearity of the integral, the fact that $|f| \leq M$ and the definition of g , we get

$$\begin{aligned} \left| \int_{\mathbb{R}} f \, d\mu - \int_{\mathbb{R}} fg \, d\mu \right| &= \left| \int_{\mathbb{R}} f(1 - g) \, d\mu \right| \leq M \int_{\mathbb{R}} (1 - g) \, d\mu \\ &\leq M \int_{\mathbb{R}} (1 - g) \, d\mu \\ &= M \left(1 - \int_{\mathbb{R}} g \, d\mu \right) \\ &\leq M\mu(\mathbb{R} \setminus [-\alpha, \alpha]) < \frac{\varepsilon}{2}. \end{aligned}$$

- (f) Again, using linearity of the integral and the fact that $|f| \leq M$ we get

$$\begin{aligned} \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} fg \, d\mu_n \right| &= \left| \int_{\mathbb{R}} f(1 - g) \, d\mu_n \right| \leq M \int_{\mathbb{R}} (1 - g) \, d\mu_n \\ &\leq M \int_{\mathbb{R}} (1 - g) \, d\mu_n = M \left(1 - \int_{\mathbb{R}} g \, d\mu_n \right) \end{aligned}$$

Now observe that the integral in the last term converges to $\int_{\mathbb{R}} g \, d\mu$ by (3). Thus, we obtain

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} fg \, d\mu_n \right| \leq M \int_{\mathbb{R}} (1 - g) \, d\mu \leq M\mu(\mathbb{R} \setminus [-\alpha, \alpha]) < \frac{\varepsilon}{2}.$$

- (g) Recall that

$$\begin{aligned} \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} f \, d\mu \right| &\leq \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} fg \, d\mu_n \right| + \left| \int_{\mathbb{R}} f \, d\mu - \int_{\mathbb{R}} fg \, d\mu \right| \\ &\quad + \left| \int_{\mathbb{R}} fg \, d\mu_n - \int_{\mathbb{R}} fg \, d\mu \right|. \end{aligned}$$

For the first two terms, the (e) and (f) imply that the $\limsup_{n \rightarrow \infty}$ is bounded by $\varepsilon/2$. For the third term we note that fg is a continuous bounded function and hence this term converges to zero by our assumption that (3) holds.

Together we then have that

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f \, d\mu_n - \int_{\mathbb{R}} f \, d\mu \right| < \varepsilon,$$

which implies the result.

Chapter 8: Convergence of random variables

Problem 8.2

(a) (2 pts)

For this let $h_t(x) = \mathbf{1}_{(-\infty, t]}$ and note that

$$F_n(t) = (X_n)_\# \mathbb{P}_n((-\infty, x]) = \int_{\mathbb{R}} \mathbf{1}_{(-\infty, t]} d(X_n)_\# \mathbb{P}_n = \int_{\mathbb{R}} h_t d\mu_n.$$

and similarly $F(t) = \int_{\mathbb{R}} h_t d\mu$

(b) (3 pts)

1 pt The function h is discontinuous only at t , i.e. $\mathcal{C}_h = \mathbb{R} \setminus \{t\}$.

2 pts Moreover, for any $\varepsilon > 0$

$$\mu((t - \varepsilon, t + \varepsilon)) = \mu((t - \varepsilon, t]) + \mu((t, t + \varepsilon)) = F(t) - F(t - \varepsilon) + F(t + \varepsilon) - F(t).$$

Since F is continuous at t , the right hand side goes to zero as $\varepsilon \rightarrow 0$. Therefore

$$\mu(\{t\}) = \lim_{\varepsilon \rightarrow 0} \mu((t - \varepsilon, t + \varepsilon)) = 0,$$

which implies that $\mu(\mathcal{C}_h) = 1$.

(c) (1 pt) The result follows by applying condition (2) in Theorem 7.6.

(d) (2 pts)

1 pt Without loss of generality assume $x \in I_\ell = (a_\ell, b_\ell]$ for some $1 \leq \ell \leq L$. Then it holds that $\hat{g}(x) = g(b_\ell)$.

1 pt Moreover, since $|x - b_\ell| < \delta$ we have that

$$|g(x) - \hat{g}(x)| = |g(x) - g(b_\ell)| < \varepsilon.$$

(e) (1 pt) Let $M = L$, $\beta_\ell = \sum_{t=1}^\ell h(b_t)$ and $t_\ell = b_\ell$. Then

$$\hat{g} := \sum_{\ell=1}^L \beta_\ell \mathbf{1}_{(-\infty, b_\ell]}.$$

(f) (2 pts) Using the representation in (e) we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[\hat{g}(X_n)] &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell \int_{\mathbb{R}} \mathbf{1}_{X_n^{-1}((-\infty, b_\ell])} d\mathbb{P} \\ &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell (X_n)_\# \mathbb{P}((-\infty, b_\ell]) \\ &= \lim_{n \rightarrow \infty} \sum_{\ell=1}^L \beta_\ell F_n(b_\ell) \\ &= \sum_{\ell=1}^L F(b_\ell) = \mathbb{E}[\hat{g}(X)]. \end{aligned}$$

(g) (4 pts) **1 pt** First we write

$$\begin{aligned} \|\mathbb{E}[g(X_n)] - \mathbb{E}[g(X)]\| &\leq \|\mathbb{E}[g(X_n)] - \mathbb{E}[\hat{g}(X_n)]\| + \|\mathbb{E}[g(X)] - \mathbb{E}[\hat{g}(X)]\| \\ &\quad + \|\mathbb{E}[\hat{g}(X_n)] - \mathbb{E}[\hat{g}(X)]\|. \end{aligned}$$

1 pt We have shown in (f) that the last term goes to zero as $n \rightarrow \infty$.

1 pt Next, using (d) it follows that the other two terms are bounded by ε .

1 pt Since ε was arbitrary we conclude that

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(X_n)] = \mathbb{E}[g(X)].$$

(h) (1 pt) This now follows from Theorem 7.6 (3).

Problem 8.3

The main idea is to use the equivalent version of convergence in distribution.

Suppose that $X_n \xrightarrow{\mathbb{P}} X$ and define $Y_n = |X_n - X|$. We need to show that $\mathbb{P}(Y_n > \varepsilon) \rightarrow 0$ holds for any $\varepsilon > 0$. First recall that $X_n \xrightarrow{\mathbb{P}} X$ is defined as weak convergence of Y_n to the constant zero random variable. By Lemma 8.2 this is equivalent to

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \leq t) = \mathbb{P}(0 \leq t),$$

for all continuity points of the function $\omega \mapsto 0$. We now note that any $\varepsilon > 0$ is a continuity point of this function. Hence, we get

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n > \varepsilon) = 1 - \lim_{n \rightarrow \infty} \mathbb{P}(Y_n \leq \varepsilon) = 1 - \mathbb{P}(0 \leq \varepsilon) = 0$$

Now we prove the other implication. So suppose that $\mathbb{P}(Y_n > \varepsilon) \rightarrow 0$ holds for any $\varepsilon > 0$. We then have to prove that $(Y_n)_\# \mathbb{P} \Rightarrow 0_\# \mathbb{P}$. Due to Lemma 8.2 it is enough to show that

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \leq t) = \mathbb{P}(0 \leq t) = \mathbb{1}_{t \geq 0},$$

holds for all continuity points t of the function $\omega \mapsto 0$. Notice that the only non-continuity point is 0. Moreover, for all $t < 0$ we have that $\mathbb{P}(Y_n \leq t) = 0$ since $Y_n \geq 0$ almost every-where. Finally, for all $t > 0$ we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \leq t) = 1 - \lim_{n \rightarrow \infty} \mathbb{P}(Y_n > t) = 1 = \mathbb{P}(0 \leq t).$$

Problem 8.4

Problem 8.5 Suppose that $X_n \xrightarrow{\text{a.s.}} X$. Then by Lemma 5.2.16 this is equivalent to $\mathbb{P}(\|X_n - X\| > \varepsilon \text{ i.o.}) = 0$ for all $\varepsilon > 0$.

For now fix an $\varepsilon > 0$ and write $A_n := \{\|X_n - X\| > \varepsilon\}$. Recall that

$$\{A_n \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n$$

and note two things:

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-
- (a) The sets $B_k := \bigcup_{n \geq k} A_n$ are non-increasing, i.e. $B_k \supset B_{k+1}$, and
(b) $\mathbb{P}(A_k) \leq \mathbb{P}(\bigcup_{n \geq k} A_n) = \mathbb{P}(B_k)$.

We then have that:

$$\begin{aligned}
0 &= \mathbb{P}(\{A_n \text{ i.o.}\}) && \text{by assumption} \\
&= \mathbb{P}\left(\bigcap_{k=1}^{\infty} B_k\right) && \text{by Lemma 5.2.16} \\
&= \lim_{k \rightarrow \infty} \mathbb{P}(B_k) && \text{by continuity from above (Proposition 2.2.15)} \\
&\geq \lim_{k \rightarrow \infty} \mathbb{P}(A_k) && \text{by (b).}
\end{aligned}$$

Problem 8.6

- (a) Define the sets

$$B_j := \bigcup_{i \geq j} A_i, \quad j \in \mathbb{N}.$$

Clearly the sequence $(B_j)_{j \in \mathbb{N}}$ is decreasing and $\{A_n \text{ i.o.}\} \subset B_j$ for every $j \in \mathbb{N}$.

By assumption, and the σ -subadditivity of \mathbb{P} ,

$$\mathbb{P}(B_1) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i) < +\infty.$$

Moreover, the summability also gives

$$\lim_{j \rightarrow \infty} \mathbb{P}(B_j) \leq \limsup_{j \rightarrow \infty} \sum_{i=j}^{\infty} \mathbb{P}(A_i) = 0.$$

Hence, by the continuity from above of μ , we obtain

$$\mathbb{P}(\{A_n \text{ i.o.}\}) \leq \mathbb{P}\left(\bigcap_{j=1}^{\infty} B_j\right) = \lim_{j \rightarrow \infty} \mathbb{P}(B_j) = 0,$$

i.e., $\{A_n \text{ i.o.}\}$ is a null set. In other words, \mathbb{P} -almost every ω is in only finitely many A_n .

- (b) We will prove that

$$\mathbb{P}(\Omega \setminus \{A_n \text{ i.o.}\}) = 0,$$

from which the result follows since $\mathbb{P}(\Omega) = 1$.

First note that

$$\Omega \setminus \{A_n \text{ i.o.}\} = \bigcup_{k \geq 1} \left(\bigcup_{n \geq k} A_n \right)^c = \bigcup_{k \geq 1} \bigcap_{n \geq k} A_n^c.$$

Next, since A_n are mutually exclusive, so are A_n^c . Thus, for any $k \geq 1$ we have that

$$\begin{aligned}\mathbb{P}\left(\bigcap_{n \geq k} A_n^c\right) &= \prod_{n \geq k} \mathbb{P}(A_n^c) = \prod_{n \geq k} (1 - \mathbb{P}(A_n)) \\ &\leq \prod_{n \geq k} e^{-\mathbb{P}(A_n)} = e^{-\sum_{n \geq k} \mathbb{P}(A_n)} = 0.\end{aligned}$$

Here we used that for any $0 \leq x \leq 1$ it holds that $1 - x \leq e^{-x}$.

Finally, using σ -subadditivity we conclude that

$$\mathbb{P}(\Omega \setminus \{A_n \text{ i.o.}\}) = \mathbb{P}\left(\bigcup_{k \geq 1} \bigcap_{n \geq k} A_n^c\right) \leq \sum_{k \geq 1} \mathbb{P}\left(\bigcap_{n \geq k} A_n^c\right) = 0.$$

Problem 8.7

Fix $\varepsilon > 0$ and define $A_n(\varepsilon) : \{|X_n - X| > \varepsilon\}$. Then the assumption translates to

$$\sum_{n \geq 1} \mathbb{P}(A_n(\varepsilon)) < \infty.$$

By Lemma 8.11 1) this then implies that $\mathbb{P}(A_n(\varepsilon) \text{ i.o.}) = 0$. Since $\varepsilon > 0$ was arbitrary, Lemma 8.9 now implies that $X_n \xrightarrow{\text{a.s.}} X$.