Problem 6.5

Define for any $j \in \mathbb{Z}$, $p_j := \mathbb{P}(X^{-1}(\{j\}))$. Then, since $(X^{-1}(j))_{j\in\mathbb{Z}}$ is a family of disjoint sets and $\mathbb{P}$ is a probability measure we get that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\bigcup_{j\in\mathbb{Z}} X^{-1}(j)) = \sum_{j\in\mathbb{Z}} p_j.$$

Now let $A \subset \mathbb{R}$ be a measurable set and note that

$$X^{-1}(A) = \bigcup_{j\in\mathbb{Z}\cap A} X^{-1}(j).$$

Then it follows that

$$\mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}\left(\bigcup_{j\in\mathbb{Z}\cap A} X^{-1}(j)\right) = \sum_{j\in\mathbb{Z}\cap A} p_j = \sum_{j\in\mathbb{Z}} \delta_j(A) p_j.$$

Problem 6.8

(a) This follows from the following computation

$$\int_\Omega |f|^p \,\mathrm{d}\mu \geq \int_\Omega |f|^p \mathbb{1}_{|f|\geq t} \,\mathrm{d}\mu \geq t^p \int_\Omega \mathbb{1}_{|f|\geq t} \,\mathrm{d}\mu = t^p \mu(\{\omega \in \Omega \,:\, |f| \geq t\}).$$

(b) Using the result for $p = 1$ we get

$$\mathbb{P}(|X| \geq t) = \mu(\omega \in \Omega \,:\, |X(\omega) \geq t\}) \leq \frac{1}{t} \int_\Omega X \,\mathrm{d}\mathbb{P} = \frac{1}{t}\mathbb{E}[X].$$

(c) Take $f(\omega) = X(\omega) - \mathbb{E}[X]$, which is measurable. Then using the first result with $p = 2$ gives

$$\begin{aligned} \mathbb{P}(|X - \mathbb{E}[X]| \geq t) &= \mathbb{P}(|X - \mathbb{E}[X]|^2 \geq t^2) \\ &\leq \frac{1}{t^2}\mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \frac{1}{t^2}(\mathbb{E}[X^2] - \mathbb{E}[X]^2) = \frac{\mathrm{Var}(X)}{t^2}. \end{aligned}$$