Problem 4.9

(⇒) Let f be μ-integrable. Then both $|f|\mathbf{1}_{\{|f|<n\}}$ and $|f|\mathbf{1}_{\{|f|\geq n\}}$ are integrable, due to the monotonicity of the integral. By linearity of the integral,

$$\int_\Omega |f|\mathbf{1}_{\{|f|\geq n\}}\,\mathrm{d}\mu = \int_\Omega |f|\,\mathrm{d}\mu - \int_\Omega |f|\mathbf{1}_{\{|f|<n\}}\,\mathrm{d}\mu.$$

Since the sequence $g_n := |f|\mathbf{1}_{\{|f|<n\}} \geq 0$ converges pointwise monotonically to $|f|$, we can apply MCT to obtain

$$\lim_{n\to\infty}\int_\Omega |f|\mathbf{1}_{\{|f|<n\}}\,dd\mu = \int_\Omega |f|\,\mathrm{d}\mu.$$

Hence,

$$\lim_{n\to\infty}\int_\Omega |f|\mathbf{1}_{\{|f|\geq n\}}\,\mathrm{d}\mu = \int_\Omega |f|\,\mathrm{d}\mu - \lim_{n\to\infty}\int_\Omega |f|\mathbf{1}_{\{|f|<n\}}\,\mathrm{d}\mu = 0.$$

(⇐) By assumption, there is some $N \geq 1$ such that

$$\int_\Omega |f|\mathbf{1}_{\{|f|\geq N\}}\,\mathrm{d}\mu \leq 1.$$

By linearity of the integral,

$$\int_\Omega |f|\,\mathrm{d}\mu = \int_\Omega |f|\mathbf{1}_{\{|f|<N\}}\,\mathrm{d}\mu + \int_\Omega |f|\mathbf{1}_{\{|f|\geq N\}}\,\mathrm{d}\mu \leq N\mu\big(\{|f|<N\}\big) + 1.$$

Since μ is a finite measure, the right-hand side is finite, implying that f is μ-integrable.

Problem 5.2

(a) Note that $\mathcal{A}_1 \times \mathcal{A}_2 \subset \mathcal{F}_1 \times \mathcal{F}_2$, and hence

$$\sigma(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{F}_1 \otimes \mathcal{F}_2.$$

(b) Let $B \in \mathcal{A}_2$. Then we have that

$$\Omega_1 \times B = \bigcup_{n\geq 1} A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$$

since $A_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ for all $n \geq 1$. So $\Omega_1 \in \Sigma$

For the second property, let $C \in \Sigma$ and note that $C^c \times B = (\Omega_1 \times B) \setminus (C \times B)$. Since both these sets are in $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ it follows that $C^c \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ and hence $C^c \in \Sigma$.

Finally consider a countable sequence $(C_n)_{n\geq 1}$ of sets in Σ. Then for any $B \in \mathcal{A}_2$

$$\left(\bigcup_{n\geq 1} C_n\right) \times B = \bigcup_{n\geq 1}(C_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2),$$

since each $C_n \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(c) Note that $\mathcal{A}_1 \subset \Sigma_1 \subset \mathcal{F}_1$. From which it follows that $\Sigma_1 = \mathcal{F}_1$. But then, from the definition of Σ_1 we have that $\mathcal{F}_1 \times \mathcal{A}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(d) We can show in a similar fashion that

$$\Sigma_2 := \{C \in \mathcal{F}_2 \,:\, B \times C \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2) \, \forall B \in \mathcal{A}_1\}.$$

is a σ-algebra on Ω_2, from which we conclude that $\mathcal{A}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

(e) take any $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Then

$$A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B) = \bigcup_{n,m \geq 1} (A \times B_m) \cap (A_n \times B) \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2).$$

From this we conclude that $\mathcal{F}_1 \times \mathcal{F}_2 \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$, which finishes the proof.