Problem 3.2

(a) First we note that $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}$ and $f^{-1}(E) = \Omega \in \mathcal{F}$. So $\emptyset, E \in \mathcal{H}$.

Next, let $B \in \mathcal{H}$. Then

$$f^{-1}(E \setminus B) = \Omega \setminus f^{-1}(B) \in \mathcal{F},$$

since by definition $f^{-1}(B) \in \mathcal{F}$. So $E \setminus B \in \mathcal{H}$.

Finally, if $(B_i)_{i \in \mathbb{N}}$ is a sequence of sets in $\mathcal{H}$, then

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) \in \mathcal{F},$$

which shows that $\bigcup_{i=1}^{\infty} B_i \in \mathcal{H}$, completing the proof that $\mathcal{H}$ is a σ-algebra.

(b) By construction $\mathcal{A} \subseteq \mathcal{H}$. It therefore follows from Lemma 2.5 that $\mathcal{G} = \sigma(\mathcal{A}) \subseteq \mathcal{H}$. But this then implies that $f^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{G}$ which means that f is $(\mathcal{F}, \mathcal{G})$-measurable.

Problem 3.6

(a) By Proposition 2.8, we know that $\mathcal{B}_{\mathbb{R}}$ is generated by intervals of the form $(-\infty, a]$ with $a \in \mathbb{Q}$. As a consequence, $\mathcal{B}_{\mathbb{R}}$ is also generated by intervals of the form $(a, +\infty)$ with $a \in \mathbb{Q}$. Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\big\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\big\}$$

is measurable for every $a \in \mathbb{Q}$. For brevity, we write $\{f + g > a\}$. The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \Big(\{f > t\} \cap \{g > a - t\}\Big).$$

We first show the inclusion '$\subset$'. If $\omega \in \Omega$ is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some $t \in \mathbb{Q}$ such that

$$f(\omega) > t > a - g(\omega),$$

and thus $f(\omega) > t$ and $g(\omega) > a - t$. So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \Big(\{f > t\} \cap \{g > a - t\}\Big).$$

Now we will show the inclusion '$\supset$'. Let $\omega \in \Omega$ be such that $f(\omega) > t$ and $g(\omega) > a - t$. Then, by adding the inequalities, we know that $f(\omega) + g(\omega) > a$.

(b) The constant function $f(\omega) = a$ is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \qquad \forall\, B \in \mathcal{B}_{\mathbb{R}}.$$

(c) Similar to the proof of Point (2) of Proposition 3.2.12.

(d) Let $g(\omega) \neq 0$ for all $\omega \in \Omega$. Then, since g is measurable, we have that

$$\begin{aligned} \{1/g > a\} &= \{g < 1/a,\ g > 0\} \cup \{g > 1/a,\ g < 0\} \\ &= \Big(\{g < 1/a\} \cap \{g > 0\}\Big) \cup \Big(\{g > 1/a\} \cap \{g < 0\}\Big) \in \mathcal{F}, \end{aligned}$$

thus implying that $1/g$ is measurable.

(e) The previous part of this exercise together with point (4) of Proposition 3.12 yields Point (5) of Proposition 3.12.