TU/e, 2MBA70

Solutions to problems for Measure and Probability Theory

Pim van der Hoorn and Oliver Tse

Version 0.2 September 16 2024

Chapter 2: Measurable spaces (sigma-algebras and measures)

Problem 2.6
First note that if $\mu(A \cap B) = \infty$ then by property 2 we have that also $\mu(A)$, $\mu(B)$ and $\mu(A \cup B) = \infty$ and hence the result holds trivially. So assume now that $\mu(A \cap B) < \infty$. Since

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

it follows from property 1 that

$$\mu(A \cup B) = \mu(A \setminus (A \cap B))) + \mu(A \cap B) + \mu(B \setminus (A \cap B)).$$

Adding $\mu(A \cap B) < \infty$ to both side we get

$$\begin{aligned}\mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus (A \cap B))) + \mu(A \cap B) + \mu(B \setminus (A \cap B)) + \mu(A \cap B) \\ &= \mu(A) + \mu(B),\end{aligned}$$

where the last line follows from applying property 3 twice.

Problem 2.7
The idea is to construct a family of disjoint sets $(E_i)_{i \in \mathbb{N}}$ with the following properties:

1. $E_i \subset A_i$, and
2. $\bigcup_{i \in \mathbb{N}} E_i = \bigcup_{i \in \mathbb{N}} A_i$.

If such a sequence exists then we have

$$\begin{aligned}\mu(\bigcup_{i \in \mathbb{N}} A_i) &= \mu(\bigcup_{i \in \mathbb{N}} E_i) && \text{by 2} \\ &= \sum_{i=1}^{\infty} \mu(A_i) && \text{because } E_i \text{ are disjoint and } \mu \text{ is } \sigma\text{-additive} \\ &\leq \sum_{i=1}^{\infty} \mu(A_i) && \text{by 1 and monotone property of } \mu.\end{aligned}$$

So we are left to construct the required family of sets $(E_i)_{i \in \mathbb{N}}$. The following set will do:

$$E_1 = A_1 \quad E_i = A_i \setminus \bigcup_{k<i}^{i} A_k \text{ for all } i > 1.$$

Note that by definition the set E_i are pair-wise disjoint and property1 holds. Finally, property 2 holds since $\bigcup_{i=1}^{k} E_i = \bigcup_{i=1}^{k} A_i$ holds for all $k \geq 1$.

Problem 2.9 (23 points) Let $\mathcal{O}$ denote the open sets in $\mathbb{R}$.

1. (2 points) Note that the interval (a, b) is open for any $a < b \in \mathbb{R}$. Hence $\mathcal{A}_1 \subset \mathcal{A}_1' \subset \mathcal{O}$ and thus by Lemma 2.1.5 we have that $\sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \sigma(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$.

2. (2 points) The inclusion $\supset$ is trivial. So assume that $x \in O$. Then by definition there exist an $r > 0$ such that the ball $B_x(r) \subset O$. But $B_x(r) = (x-r, x+r) \in \mathcal{A}_1$ so $x \in \bigcup_{I \in \mathcal{A}, I \subset O} I$.

3. (3 points) Take $O \in \mathcal{O}$. If we can show that $O \in \sigma(\mathcal{A})$ then $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. The result then follows from 1.

   From 2 it follows that O is a union over a subset collection of interval (a, b) where $a, b \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, the collection $\{(a, b) \,:\, a < b \in \mathbb{Q}\}$ is also countable and hence $O = \bigcup_{I \in \mathcal{A}, I \subset O} I \in \sigma(\mathcal{A})$, from which it follows that $\mathcal{B}_{\mathbb{R}} \subset \sigma(\mathcal{A})$.

4. (1 point) This follows immediately from 1 and 3 since these imply that $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_1') \subset \mathcal{B}_{\mathbb{R}}$.

5. (3 points) The inclusion $\subset$ is trivial, since $(a, b] \subset (a + b + 1/j)$ for any $j \in \mathbb{N}$. For the other inclusion we argue by contradiction. Suppose that $x \in \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$ but $x \notin (a, b]$. Then $x > b$ and hence there exists a $j \in \mathbb{N}$ such that $(b - x) > 1/j$. But this implies that $x \notin (a, b + 1/j)$ which is a contradiction. So we conclude that $(a, b] \supset \bigcap_{j \in \mathbb{N}} (a, b + 1/j)$.

6. (3 points) This time the inclusion $\supset$ is trivial since $(a, b - 1/j] \subset (a, b)$ for every $j \in \mathbb{N}$. For the other inclusion suppose that $x \in (a, b)$. Then there exists a $r > 0$ such that the interval $(x - r, x + r) \subset (a, b)$. In particular, this implies that $b - (x + r) > 0$. Now take any $j \in \mathbb{N}$ such that $j > 1/(b - (x + r))$. Then $b - x > r + 1/j$ which implies that $(x - r, x + r) \subset (x - r, b - 1/j]$ and hence $x \in \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$.

7. (4 points) It is clear that $\mathcal{A}_2 \subset \mathcal{A}_2'$. By 5 it follows that any interval $(a, b]$ can be obtained as a countable intersection of intervals of the form $(a, b + 1/j)$. By 4 $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1')$ which by Lemma 2.1.2 contains $\bigcap_{j \in \mathbb{N}} (a, b + 1/j) = (a, b]$. So we conclude that any interval $(a, b] \in \mathcal{B}_{\mathbb{R}}$ from which it now follows that

   $$\sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}_2') \subset \sigma(\mathcal{A}_1') = \mathcal{B}_{\mathbb{R}}.$$

   For the other inclusion we consider a set (a, b) with $a, b \in \mathbb{Q}$. Then by 6 we have that $(a, b) = \bigcup_{j \in \mathbb{N}} (a, b - 1/j]$ where the later is a countable union of sets $(c, d]$ with $c, d \in \mathbb{Q}$ which must be in $\sigma(\mathcal{A}_2)$ by definition of a σ-algebra. Hence, any interval $(a, b) \in \sigma(\mathcal{A}_2)$ and we thus conclude, using 3, that

   $$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) \subset \sigma(\mathcal{A}_2') \subset \sigma(\mathcal{A}_1') = \mathcal{B}_{\mathbb{R}},$$

   which implies the result.

8. (2 points) Step 1 is to show that any interval $[a, b)$ can be obtained as a countable intersection of intervals $(a - 1/j, b)$. From this we can conclude that any set $[a, b)$ must be in $\mathcal{B}_{\mathbb{R}}$ proving inclusions $\subset$.

   For the other inclusions we have to show that any interval (a, b) can be obtained as a countable union of intervals $[a + 1/j, b)$, which implies that (a, b) must be in the σ-algebra generated by $[a, b)$.

9. (3 points) The main tool is to show that each of the intervals $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$ can be obtained by taking any allowed set operation for σ-algebras, i.e. countable unions/intersections and finite complements. This will help use prove the $\subset$ inclusions.

   Then we show that any set of the form (a, b), $[a, b)$ or $(a, b]$ can also be obtained through countable unions/intersections and finite complements of intervals of the forms $(-\infty, a]$, $(-\infty, a)$, (a, ∞) and $[a, \infty)$. These will then yield the $\supset$ inclusions and finish the proof.

Chapter 3: Measurable functions and stochastic objects

Problem 3.2 "$\subset$" By definition, the product σ-algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$ is defined as the σ-algebra generated by the collection

$$\mathcal{A} := \Big\{ A \times B \subset \Omega_1 \times \Omega_2 : A \in \mathcal{F}_1,\, B \in \mathcal{F}_2 \Big\}.$$

Since $A \times B = (A \times \Omega_2) \cap (\Omega_1 \times B)$, we have that

$$A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \sigma(\pi_1, \pi_2).$$

"$\supset$" Let $C \in \{\pi_i^{-1}(A) : i = 1, 2,\ A \in \mathcal{F}_1\}$. Then there exist sets $A \in \mathcal{F}_1$ or $B \in \mathcal{F}_2$ such that $C = \pi_1^{-1}(A) = A \times \Omega_2$ or $C = \pi_2^{-1}(B) = \Omega_1 \times B$. Either way, since $\Omega_1 \in \mathcal{F}_1$ and $\Omega_2 \in \mathcal{F}_2$, we have that $C \in \mathcal{A}$.

Problem 3.3 It is clear that $f_{\#}\mu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Suppose a sequence of mutually disjoint sets $B_i \in \mathcal{G}$, $i \in \mathbb{N}$, is given. Then,

$$f_{\#}\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) = \sum_{i=1}^{\infty} f_{\#}\mu(B_i).$$

Problem 3.5

(a) Some meaningful explanation would suffice.

(b) By Proposition 2.1.8 and Problem 2.9, we know that $\mathcal{B}_{\mathbb{R}}$ is generated by intervals of the form $(-\infty, a]$ with $a \in \mathbb{Q}$. As a consequence, $\mathcal{B}_{\mathbb{R}}$ is also generated by intervals of the form $(a, +\infty)$ with $a \in \mathbb{Q}$. Therefore, by Lemma 3.1.4, it suffices to show that the set

$$\big\{\omega \in \Omega : f(\omega) + g(\omega) \in (a, +\infty)\big\}$$

is measurable for every $a \in \mathbb{Q}$. For brevity, we write $\{f + g > a\}$. The trick is to express this set as a countable union of sets of which we already know are measurable.

In fact, we will show that

$$\{f + g > a\} = \bigcup_{t \in \mathbb{Q}} \Big(\{f > t\} \cap \{g > a - t\}\Big).$$

We first show the inclusion '$\subset$'. If $\omega \in \Omega$ is such that

$$f(\omega) + g(\omega) > a,$$

then

$$f(\omega) > a - g(\omega),$$

so there exists some $t \in \mathbb{Q}$ such that

$$f(\omega) > t > a - g(\omega),$$

and thus $f(\omega) > t$ and $g(\omega) > a - t$. So in that case

$$\omega \in \bigcup_{t \in \mathbb{Q}} \Big(\{f > t\} \cap \{g > a - t\} \Big).$$

Now we will show the inclusion '$\supset$'. Let $\omega \in \Omega$ be such that $f(\omega) > t$ and $g(\omega) > a - t$. Then, by adding the inequalities, we know that $f(\omega) + g(\omega) > a$.

(c) The constant function $f(\omega) = a$ is measurable since

$$f^{-1}(B) = f^{-1}(B \cap \{a\}) \cup f^{-1}(B \setminus \{a\}) = \Omega \cup \emptyset = \Omega \in \mathcal{F} \qquad \forall\, B \in \mathcal{B}_{\mathbb{R}}.$$

(d) Similar to the proof of Point (2) of Proposition 3.2.12.

(e) Let $g(\omega) \neq 0$ for all $\omega \in \Omega$. Then, since g is measurable, we have that

$$\begin{aligned} \{1/g > a\} &= \{g < 1/a,\ g > 0\} \cup \{g > 1/a,\ g < 0\} \\ &= \Big(\{g < 1/a\} \cap \{g > 0\}\Big) \cup \Big(\{g > 1/a\} \cap \{g < 0\}\Big) \in \mathcal{F}, \end{aligned}$$

thus implying that $1/g$ is measurable.

(f) Point (e) and Point (4) of Proposition 3.2.12 yields Point (5) of Proposition 3.2.12.

Problem 3.6 From (3.6), we have for any $a \in \mathbb{R}$,

$$\Big\{ \sup_{n \geq 1} f_n > a \Big\} = \bigcup_{n \geq 1} \{f_n > a\} \in \mathcal{F},$$

Since $\mathcal{F}$ is a σ-algebra and f_n is measurable for all $n \geq 1$, i.e., $\{f_n > a\} \in \mathcal{F}$ for all $n \geq 1$.

Problem 3.7

(a) Note that

$$f_M = M\mathbf{1}_{\{f \geq M\}} + f\mathbf{1}_{\{|f| < M\}} - M\mathbf{1}_{\{f \leq -M\}}.$$

Since the sets

$$\{f \geq M\}, \quad \{f \leq -M\}, \quad \{|f| < M\} \qquad \text{are } \mathcal{F}\text{-measurable},$$

their corresponding indicator functions are $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable. Since f_M is the sum of products of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable functions, we conclude that f_M is also $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

(b) It is easy to see that f_M converges pointwise to f as $M \to \infty$, i.e.,

$$\lim_{M\to\infty} f_M(\omega) = f(\omega) \qquad \forall\, \omega \in \Omega.$$

Indeed, if $\omega\Omega$ is such that $f(\omega) = +\infty$, then

$$\lim_{M\to\infty} f_M(\omega) = \lim_{M\to\infty} M = +\infty = f(\omega),$$

and similarly for $\omega \in \Omega$ for which $f(\omega) = -\infty$. On the other hand, for any $\omega \in \Omega$ with $f(\omega) \in \mathbb{R}$, there is some $N_0(\omega) \in \mathbb{N}$ such that $f_N(\omega) = f(\omega)$ for all $N \geq N_0(\omega)$, and hence,

$$\lim_{M\to\infty} f_M(\omega) = f(\omega).$$

Since f is the limit of a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable functions, we conclude from Lemma 3.2.13 that f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

Problem 3.9

(a) For the probability space, take $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0, 1]$.

Observe that the function $H_\gamma(z)$ is continuous and hence has an inverse $g_\gamma(y) = \gamma \tan(\pi(y - 1/2))$ on $[0, 1]$.

So the function $Y[0, 1] \to \mathbb{R}$ defined by $Y(x) = g_\gamma(x)$ has the correct distribution as

$$\mathbb{P}(Y^{-1}((-\infty, t])) = \mathbb{P}(g_\gamma^{-1}((-\infty, t])) = \lambda(H_\gamma((-\infty, t])) = H_\gamma(t).$$

(b) Note that g_γ is continuous on $[0, 1]$ and hence measurable.

(c) For any $t \geq 0$, the cdf of the Poisson random variable is given by

$$F_\lambda(t) = \sum_{n=0}^{\lceil t \rceil} f_\lambda(n),$$

where $\lceil t \rceil$ is the ceiling of t, i.e. the smallest integer $k \geq t$.

(d) For the probability space, we again take $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}_{[0,1]}$ and $\mathbb{P} = \lambda$ the Lebesgue measure restricted to $[0, 1]$.

Now for any $y \in [0, 1]$ let $k := k(y)$ be such that

$$\sum_{n=1}^{k} f_\lambda(n) \geq y \quad \text{and} \quad \sum_{n=1}^{k-1} f_\lambda(n) < y,$$

where the last sum is interpreted as -1 if $k = 0$.

Now define $X(y) = k(y) : [0,1] \to \mathbb{R}$. Then $k(y) \le t$ if and only if $y \le F_\lambda(t)$ and hence

$$X^{-1}((-\infty, t]) = \{y \in [0,1] : k(y) \in (0,t]\} = \{y \in [0,1] : y \in (0, F_\lambda(t)]\},$$

from which it follows that

$$\mathbb{P}(X^{-1}((-\infty, t])) = \lambda((0, F_\lambda(t)]) = F_\lambda(t).$$

(e) It follows from the above computation that $X^{-1}((-\infty, t]) = \{y \in [0,1] : y \in (0, F_\lambda(t)]\}$. Since the latter is a measurable set we conclude that $X^{-1}((-\infty, t])$ is measurable for all t and since these generate the Borel σ-algebra X is measurable.

(f) for any $\ell \in \mathbb{N}$ define the sets $A_\ell = (n-1-1/\ell), n-1+1/\ell]$. Then A_ℓ is a decreasing set with $\lim_{\ell\to\infty} A_\ell = \{n\}$. Moreover, $A_\ell = (-\infty, n-1+1/\ell] \setminus (-\infty, n-1-1/\ell]$ and $\mathbb{P}(A_1) < \infty$. It now follows from continuity from above and (d) that

$$\begin{aligned}
X_\#\mathbb{P}(\{n\}) &= \lim_{\ell\to\infty} X_\#\mathbb{P}(A_\ell) \\
&= \lim_{\ell\to\infty} X_\#\mathbb{P}((-\infty, n-1+1/\ell]) - X_\#\mathbb{P}((-\infty, n-1-1/\ell]) \\
&= F_\lambda(n-1+1/\ell) - F_\lambda(n-1-1/\ell) \\
&= \sum_{k=0}^{n} f_\lambda(k) - \sum_{k=0}^{n-1} f_\lambda(k) = f_\lambda(n).
\end{aligned}$$

Chapter 4: The Lebesgue Integral

Problem 4.2

The idea is to apply the monotone convergence theorem (Theorem 4.3.4). To this end we first note that

$$\|f_n(\omega) - f(\omega)\| \le 2^{-n} \quad \text{for all } n \in \mathbb{N},\ \omega \in \Omega.$$

From this it follows that $f_n(\omega) \le 2^{-n} + f(\omega)$ and hence

$$\begin{aligned}
\|[f_n](\omega) - f(\omega)\| &= \|2^n - f(\omega)\|\mathbf{1}_{2^n \le f_n} + \|f_n(\omega) - f(\omega)\|\mathbf{1}_{f_n < 2^n} \\
&\le 2^{-n} + 2^{-n}
\end{aligned}$$

form which we conclude that $[f_n] \to f$.

The final part is to show that $[f_n] \le [f_{n+1}]$ which follows if we can show that $f_n \le f_{n+1}$. For this we first note that for all $k \ge 1$ $(k+1)2^{-(n+1)} \le k2^{-n}$. We also note that $2^n \le 2n+1$. Now suppose that there exist an $n \ge 1$ and ω such that $f_n(\omega) > f_{n+1}(\omega)$. Then it must hold that $f_n(\omega) > 0$ and hence $f_n(\omega) = k2^{-n}$ for some $k \ge 1$. This then implies that $f_{n+1}(\omega) = \ell 2^{-n}$ for some $\ell \ge k+1$. But this cannot be the case as $[\ell 2^{-n}, (\ell+1)2^{-n}) \cap [k2^{-n}, (k+1)2^{-n}) = \emptyset$ while $f(\omega)$ should be in both sets.

Problem 4.3

Problem 4.4

Problem 4.6

(a) Let $t \in \mathbb{R}$ and consider the set $A_t = (-\infty, t]$. Then by definition of the probability density function

$$\nu(A_t) = \int_{-\infty}^{t} \rho \, \mathrm{d}\lambda = (X_{\#}\mathbb{P})((-\infty, t]).$$

We thus conclude that ν and $X_{\#}\mathbb{P}$ coincide on the family of set A_t and since these generate $\mathcal{B}$ Theorem 2.2.17 implies that $\nu = X_{\#}\mathbb{P}$.

(b) Since g is a simple function, there exist an $N \in \mathbb{N}$, constants $(a_n)_{1\le n\le N}$ and measurable sets $(A_n)_{1\le n\le N}$ such that

$$g = \sum_{n=1}^{N} a_n \mathbf{1}_{A_n}.$$

Now, by first applying Proposition 4.8.11 and then part (a), we get that

$$\begin{aligned}
\mathbb{E}[g(X)] &= \int_\Omega g(X) \, \mathrm{d}\mathbb{P} = \int_\Omega g \, \mathrm{d}X_{\#}\mathbb{P} = \int_\Omega g \, \mathrm{d}\nu \\
&= \int_\Omega \sum_{n=1}^{N} a_n \mathbf{1}_{A_n} \, \mathrm{d}\nu = \sum_{n=1}^{N} a_n \nu(A_n) = \sum_{n=1}^{N} a_n \int_{A_n} \rho \, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{n=1}^{N} a_n \mathbf{1}_{A_n} \rho \, \mathrm{d}\lambda = \int_{\mathbb{R}} g\rho \, \mathrm{d}\lambda
\end{aligned}$$

(c) First note that by part (b) we have that

$$\int_\Omega [h]_n(X) \, \mathrm{d}\mathbb{P} = \int_{\mathbb{R}} [h_n]\rho \, \mathrm{d}\lambda.$$

Now we split the function $[h_n]\rho$ into its positive and negative part and note that

$$([h_n]\rho)^+ = [h]_n^+ \rho^+ + [h]_n^- \rho^- \quad \text{and} \quad ([h_n]\rho)^- = [h]_n^+ \rho^- + [h]_n^- \rho^+,$$

where $[h]_n^\pm$ and $\rho^\pm$ denote the positive and negative parts of $[h]_n$ and ρ.

We will show that

$$\int_\Omega h^+(X) \, \mathrm{d}\mathbb{P} = \int_{\mathbb{R}} h^+ \rho \, \mathrm{d}\lambda.$$

The proof for the negative part is similar.

$$
\begin{aligned}
\int_{\mathbb{R}} h^+ \,\mathrm{d}\nu &= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n^+ \,\mathrm{d}\nu && \text{by Theorem 4.3.4} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n^+ \rho \,\mathrm{d}\lambda && \text{by part (b)} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n^+ \rho^+ \,\mathrm{d}\lambda - \lim_{n\to\infty} \int_{\mathbb{R}} [h]_n^+ \rho^- \,\mathrm{d}\lambda && \text{by linearity of integration} \\
&= \int_{\mathbb{R}} h + \rho^+ \,\mathrm{d}\lambda - \int_{\mathbb{R}} h + \rho^- \,\mathrm{d}\lambda && \text{by Theorem 4.3.4} \\
&= \int_{\mathbb{R}} h^+ \rho \,\mathrm{d}\lambda && \text{by linearity of integration}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathbb{E}[h(X)] &= \int_{\Omega} h(X) \,\mathrm{d}\mathbb{P} \\
&= \int_{\mathbb{R}} h \,\mathrm{d}X_{\#}\mathbb{P} && \text{by Proposition 4.8.11} \\
&= \int_{\mathbb{R}} h \,\mathrm{d}\nu && \text{by part (a)} \\
&= \int_{\mathbb{R}} h\rho \,\mathrm{d}\lambda && \text{by part (c).}
\end{aligned}
$$

Chapter 5: Convergence of integrals and functions