Problem 8.6

(a) Define the sets

$$B_j := \bigcup_{i \geq j} A_i, \qquad j \in \mathbb{N}.$$

Clearly the sequence $(B_j)_{j\in\mathbb{N}}$ is decreasing and $\{A_n \text{ i.o.}\} \subset B_j$ for every $j \in \mathbb{N}$.

By assumption, and the σ-subadditivity of $\mathbb{P}$,

$$\mathbb{P}(B_1) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i) < +\infty.$$

Moreover, the summability also gives

$$\lim_{j\to\infty} \mathbb{P}(B_j) \leq \limsup_{j\to\infty} \sum_{i=j}^{\infty} \mathbb{P}(A_i) = 0.$$

Hence, by the continuity from above of μ, we obtain

$$\mathbb{P}(\{A_n \text{ i.o.}\}) \leq \mathbb{P}\left(\bigcup_{j=1}^{\infty} B_j\right) = \lim_{j\to\infty} \mathbb{P}(B_j) = 0,$$

i.e., $\{A_n \text{ i.o.}\}$ is a null set. In other words, $\mathbb{P}$-almost every ω is in only finitely many A_n.

(b) We will prove that

$$\mathbb{P}(\Omega \setminus \{A_n \text{ i.o.}\}) = 0,$$

from which the result follows since $\mathbb{P}(\Omega) = 1$.

First note that

$$\Omega \setminus \{A_n \text{ i.o.}\} = \bigcup_{k\geq 1}\left(\bigcup_{n\geq k} A_n\right)^c = \bigcup_{k\geq 1}\bigcap_{n\geq k} A_n^c.$$

Next, since A_n are independent, so are A_n^c. Thus, for any $k \geq 1$ we have that

$$\begin{aligned}\mathbb{P}\left(\bigcap_{n\geq k} A_n^c\right) &= \prod_{n\geq k} \mathbb{P}(A_n^c) = \prod_{n\geq k}(1 - \mathbb{P}(A_m)) \\ &\leq \prod_{n\geq k} e^{-\mathbb{P}(A_n)} = e^{-\sum_{n\geq k}\mathbb{P}(A_n)} = 0.\end{aligned}$$

Here we used that for any $0 \leq x \leq 1$ it holds that $1 - x \leq e^{-x}$.

Finally, using σ-subadditivity we conclude that

$$\mathbb{P}(\Omega \setminus \{A_n \text{ i.o.}\}) = \mathbb{P}\left(\bigcup_{k\geq 1}\bigcap_{n\geq k} A_n^c\right) \leq \sum_{k\geq 1}\mathbb{P}\left(\bigcap_{n\geq k} A_n^c\right) = 0.$$

Problem 8.8

One direction is easy. Assume that X_1 and X_2 are independent according to Definition 7.1.4. Now take any $a, b \in \mathbb{R}$ and note that $A_1 := X_1^{-1}((-\infty, a]) \in \sigma(X_1)$ and $A_2 := X_2^{-1}((-\infty, b]) \in \sigma(X_2)$. Then by the definition of independence we have that

$$\mathbb{P}(X_1 \leq a, X_2 \leq b) = \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \leq b).$$

So let us focus now on the other direction. Assume that for all $a, b \in \mathbb{R}$

$$\mathbb{P}(X_1 \leq a, X_2 \leq b) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \leq b).$$

We now have to show that X_1 and X_2 are independent according to Definition 7.1.4.
First note that since the family $(-\infty, a] \times (-\infty, b]$ generate the 2-dimensional Borel σ-algebra we have, using Theorem 2.2.17, that

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2) = \mathbb{P}(X_1 \in B_1)\mathbb{P}(X_2 \in B_2)$$

for all $B_1, B_2 \in \mathcal{B}_{\mathbb{R}}$.
Now fix a set $B_2 \in \mathcal{B}_{\mathbb{R}}$, set $A_2 := X_2^{-1}(B_2) \in \sigma(X_2)$, and define the following two measures on the space $(\Omega, \sigma(X_1))$

$$\mu_1(A) = \mathbb{P}(A \cap A_2) \quad \text{and} \quad \mu_2(A) = \mathbb{P}(A)\mathbb{P}(A_2).$$

Let $a \in \mathbb{R}$ and consider the set $A_1 := X_1^{-1}((-\infty, a]) \in \sigma(X_1)$. Then, by our assumption we have that

$$\mu_1(A_1) = \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(X_1 \leq a, X_2 \in B_2) = \mathbb{P}(X_1 \leq a)\mathbb{P}(X_2 \in B_2) = \mathbb{P}(A_1)\mathbb{P}(A_2) = \mu_2(A_1).$$

In other words, the measures μ_1, μ_2 coincide on the set $\{X_1^{-1}((-\infty, a]) \,:\, a \in \mathbb{R}\}$.
Since the set $(-\infty, a]$ generate $\mathcal{B}_{\mathbb{R}}$ it follows that

$$\sigma(X_1) = \sigma(\{X_1^{-1}((-\infty, a]) \,:\, a \in \mathbb{R}\}).$$

In addition, this set satisfies the conditions of Theorem 2.2.17 and hence we conclude that $\mu_1(A) = \mu_2(A)$ for all $A \in \sigma(X_1)$.
We can repeat this argument for the two measures on $(\Omega, \sigma(X_2))$

$$\nu_1(A) = \mathbb{P}(A_1 \cap A) \quad \text{and} \quad \nu_2(A) = \mathbb{P}(A_1)\mathbb{P}(A),$$

where $A_1 \in \{X_1^{-1}((-\infty, a]) \,:\, a \in \mathbb{R}\}$ is fixed.
From this we conclude that for any $A_1 \in \sigma(X_1)$ and $A_2 \in \sigma(X_2)$

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$$

and hence X_1 and X_2 are independent.