Gradient Structures from Classical to Quantum

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Abstract.

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As we have seen, a quantum circuit is essentially a noisy, controlled Schrödinger evolution. From a physical perspective, this noise arises from the unavoidable interaction between the system of interest and its environment. As a result, realistic quantum devices cannot be modeled as closed systems evolving unitarily on a Hilbert space, but rather as *open quantum systems*, whose effective dynamics are irreversible.

Throughout these notes, we denote by $\mathcal{H}$ a complex Hilbert space and $\mathscr{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. We further consider the following subspaces of $\mathscr{B}(\mathcal{H})$ that play a distinguishing role:

$$\begin{aligned}
\mathscr{U}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger \mathsf{a} = \mathsf{I}_{\mathcal{H}}\} && \textit{(unitary operators)}\\
\mathscr{O}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger = \mathsf{a}\} && \textit{(observables)}\\
\mathscr{D}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{O}(\mathcal{H}) : \mathsf{a} \succeq 0,\ \mathrm{tr}[\mathsf{a}] = 1\} && \textit{(density operators)}\\
\mathscr{P}(\mathcal{H}) &:= \{\mathsf{a} \in \mathscr{O}(\mathcal{H}) : \mathsf{a} = |\psi\rangle\langle\psi|,\ \|\psi\|_{\mathcal{H}} = 1\} && \textit{(pure states)}
\end{aligned}$$

where $\mathsf{a}^\dagger := \bar{\mathsf{a}}^\top$ is the hermitian conjugate of an operator $\mathsf{a} \in \mathscr{B}(\mathcal{H})$ and we will adopt the *bra-ket* notation to distinguish between primal vectors $|\psi\rangle$ (*ket*) and dual vectors $\langle\psi|$ (*bra*) for any vector $\psi \in \mathcal{H}$. Notice that $\mathscr{P}(\mathcal{H})$ is simply the subset of rank-1 projection operators.

1.1 Lindblad equation

A wide range of mathematical and physical models have been developed to describe open quantum systems, spanning microscopic Hamiltonian descriptions of system-environment interactions to phenomenological effective equations. Among these, one of the simplest and most widely used frameworks is provided by the *Lindblad (or Gorini-Kossakowski-Sudarshan-Lindblad, GKSL) equation.* Its importance stems from the fact that it gives the most general form of a Markovian, time-homogeneous quantum evolution that is compatible with the basic principles of quantum mechanics.

Rather than describing the system by a wave function, the Lindblad equation governs the evolution of the density operator $t \mapsto \rho_t \in \mathscr{D}(\mathcal{H})$, which encodes both classical and quantum uncertainty. The evolution with initial datum $\rho_0 \in \mathscr{D}(\mathcal{H})$ is given by

eq:lindblad

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -i[H, \rho_t] + \mathscr{L}(\rho_t), \qquad \mathscr{L}(\rho) := -\tfrac{1}{2}\sum\nolimits_j [L_j^\dagger, [L_j, \rho]], \tag{GKSL}$$

where $H \in \mathscr{O}(\mathcal{H})$ is a given system Hamiltonian, $\mathscr{L} : \mathscr{D}(\mathcal{H}) \to \mathscr{D}(\mathcal{H})$ is the *Lindblad (super)-operator*, and $L_j \in \mathscr{B}(\mathcal{H})$, called *jump operators*, describe the coupling to the environment. The first term represents coherent unitary evolution, while the second term captures irreversible processes such as decoherence, relaxation, and dissipation.

A thorough derivation of the Lindblad equation is lengthy and complex, so we will skip it here. However, from an operational point of view, the density operator ρ is an ensemble of pure states $\mathsf{a} = |\psi\rangle\langle\psi| \in \mathscr{P}(\mathcal{H})$. In other words, ρ can be interpreted as the expectation of the canonical random variable under a probability measure P on pure states $\mathscr{P}(\mathcal{H})$, i.e.,

$$\rho = \int_{\mathscr{P}(\mathcal{H})} \mathsf{a}\,\mathsf{P}(\mathrm{d}\mathsf{a}) \;\in \mathscr{D}(\mathcal{H}).$$

In the following, we will be interested in *stochastic dilations* or *stochastic unravellings* of the Lindblad equation, i.e., we will look at $\mathscr{P}(\mathcal{H})$-valued processes (a_t) for which their expectation solves the Lindblad equation. The following sections provide examples of such processes, where information about the environment is embedded into the model.

1.2 Stochastic Schrödinger equation

In the context of Rydberg atoms, the optical control system is a primary source of noise. In the semiclassical limit of light-matter interactions, such noise sources can be considered classical. Without going into the details of the physics, we introduce the stochastic Schrödinger equation, which describes the evolution of a quantum system driven by classical noise sources.

Given an observable $L \in \mathscr{O}(\mathcal{H})$ and a real-valued smooth process α_t, the Schrödinger equation driven by the time-dependent observable $\dot{\alpha}_t L \in \mathscr{O}(\mathcal{H})$ may be expressed as an evolution in the space of unitaries $\mathscr{U}(\mathcal{H})$:

$$\mathrm{d}U_t = iLU_t\,\mathrm{d}\alpha_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}.$$

In this simple case, the solution may be explicitly expressed as

$$U_t = \exp\bigl(i(\alpha_t - \alpha_0)L\bigr)\mathsf{I}_{\mathcal{H}}.$$

Remark 1.1 When $\mathcal{H} = \mathbb{C}^n$, $\mathscr{U}(\mathcal{H})$ is a compact Lie group with Lie algebra

$$\mathscr{A}(\mathcal{H}) := \bigl\{\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a}^\dagger = -\mathsf{a}\bigr\} = \bigl\{i\mathsf{a} \in \mathscr{B}(\mathcal{H}) : \mathsf{a} \in \mathscr{O}(\mathcal{H})\bigr\},$$

i.e., the space of skew-hermitian matrices. The Lie algebra $\mathscr{A}(\mathcal{H})$ can be equipped with a real inner product given by the Hilbert-Schmidt scalar product

$$\langle \mathsf{a}, \mathsf{b}\rangle_{\mathscr{A}(\mathcal{H})} = -\operatorname{tr}(\mathsf{a}^\dagger \mathsf{b}) = \Re \operatorname{tr}(\mathsf{a}\mathsf{b}^\dagger),$$

which is positive-definite on $\mathscr{A}(\mathcal{H})$. The associated norm is then

$$|\mathsf{a}|^2_{\mathscr{A}(n)} = \langle \mathsf{a}, \mathsf{a}\rangle_{\mathscr{A}(n)} = \operatorname{tr}(\mathsf{a}\mathsf{a}^\dagger).$$

Similarly, the space $\mathscr{O}(\mathcal{H})$ of observables may be equipped with the Hilbert-Schmidt scalar product $\langle \mathsf{a}, \mathsf{b}\rangle_{\mathscr{O}(\mathcal{H})} = \operatorname{tr}(\mathsf{a}^\dagger \mathsf{b}) \in \mathbb{R}$.

From a geometrical perspective, the unitary evolution is simply the exponential map applied to the time-dependent right-invariant vector field $\mathfrak{a}_t{:}\mathscr{U}(\mathcal{H}) \to \mathfrak{A}(\mathcal{H})$, $\mathfrak{a}_t(U) = i\dot{\alpha}_t HU = \mathfrak{a}_t(\mathsf{I}_{\mathcal{H}})U$. Throughout, we will consider right-multiplication, with U on the left-hand side. $\diamondsuit$

In the presence of noise, however, α_t may no longer be smooth. Nevertheless, if $\alpha_t = \omega_t$ is the Brownian motion, then the Itô formula applies and we get for $U_t := \exp(i\omega_t L)\mathsf{I}_{\mathcal{H}}$,

$$\mathrm{d}U_t = \bigl(iL\,\mathrm{d}\omega_t - \tfrac{1}{2}L^2\,\mathrm{d}t\bigr)U_t = iLU_t \circ \mathrm{d}\omega_t,$$

where $\circ\,\mathrm{d}\omega_t$ denotes the Stratonovich integral. This is precisely a stochastic Schrödinger equation with one noise operator L and one driving noise ω_t.

1.3 Unitary-valued processes

More generally, we consider a set $\mathscr{X} = \{L_1, \dots, L_d\} \subset \mathscr{O}(\mathcal{H})$ of orthonormal observables on the n-dimensional complex Hilbert space $\mathcal{H}$ under the Hilbert-Schmidt scalar product on $\mathscr{O}(\mathcal{H})$, where $1 \le d \le n^2$ is the number of noise channels. These observables will often be called noise operators. Further, let $\omega_t^1, \dots, \omega_t^d$ be independent standard real-valued Brownian motions and $\alpha_t^1, \dots, \alpha_t^d$ be Itô processes of the form

$$\alpha_t^j = \alpha_0^j + \mathsf{b}_t^j + \sqrt{\gamma_j}\, \omega_t^j, \qquad j = 1, \dots, d,$$

where b_t^j is an absolutely continuous process and $\gamma_j > 0$.

Define the $\mathscr{O}(\mathcal{H})$–valued (possibly degenerate) Brownian driver

$$\chi_t = \sum\nolimits_j iL_j\, \alpha_t^j.$$

The intrinsic $\mathscr{U}(\mathcal{H})$-valued diffusion process solves the Stratonovich SDE

eq:unitary-sse

$$\mathrm{d}U_t = \big(-iH\,\mathrm{d}t + \circ\, \mathrm{d}\chi_t\big)U_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}, \tag{SSE}$$

or in components

$$\mathrm{d}U_t = -iHU_t\,\mathrm{d}t + \sum\nolimits_j iL_jU_t\,\mathrm{d}\mathsf{b}_t^j + \sum\nolimits_j \sqrt{\gamma_j}\, iL_jU_t \circ \mathrm{d}\omega_t^j,$$

where H is a system Hamiltonian. The matrix-valued quadratic variation of χ is

$$\mathrm{d}\langle\chi\rangle_t = \sum\nolimits_j \gamma_j\, iL_j \otimes iL_j\,\mathrm{d}t.$$

If $d = n^2$, the driver is elliptic (non-degenerate). If $d < n^2$, the covariance has rank d and the process is *degenerate* (hypoelliptic), exploring only the connected subgroup

$$\exp(\mathscr{A}_{\mathscr{X}}) \subset \mathscr{U}(\mathcal{H}), \quad \mathscr{A}_{\mathscr{X}} = \mathrm{Lie}\{iL_1, \dots, iL_r\} \subset \mathscr{A}(\mathcal{H}).$$

Clearly, there are situations where $\mathscr{A}_{\mathscr{X}} = \mathscr{A}(\mathcal{H})$ for $d < n^2$, in which case, the full group of unitaries is explored, i.e., $\exp(\mathscr{A}_{\mathscr{X}}) = \mathscr{U}(\mathcal{H})$.

In Itô form, the equivalent SDE reads

$$\mathrm{d}U_t = \big(\mathrm{d}\chi_t + (-iH + \mathfrak{I})\,\mathrm{d}t\big)U_t, \qquad \mathfrak{I} := -\tfrac{1}{2}\sum\nolimits_j \gamma_j L_j^2,$$

where $\mathfrak{I}$ is the Laplace-Beltrami operator associated with the right-invariant connection.

rem:explicit-sse

Remark 1.2 In the case $H = 0$, $d = 1$, we find, as in the deterministic case, the explicit solution

$$U_t = \exp\big(i(\mathsf{b}_t + \sqrt{\gamma}\omega_t)L\big)\mathsf{I}_{\mathcal{H}}.$$

In particular, U_t commutes with L for all $t \ge 0$. $\diamondsuit$

Towards the Lindblad equation To obtain the Lindblad equation (GKSL) from the stochastic Schrödinger equation (SSE), we consider noise profiles of the form

$$\alpha_t^j = \int_0^t u^j(r)\,\mathrm{d}r + \omega_t^j, \qquad j = 1, \dots, d.$$

Now let $\rho_0 \in \mathscr{D}(\mathcal{H})$ be an initial datum for the Lindblad equation and U_t be the solution to (SSE). Then, the $\mathscr{D}(\mathcal{H})$-valued process $\mathsf{a}_t := U_t\,\rho_0\,U_t^\dagger$ satisfies

$$\begin{aligned}
\mathrm{d}\mathsf{a}_t &= \mathrm{d}U_t\,\rho_0\,U_t^\dagger + U_t\,\rho_0\,\mathrm{d}U_t^\dagger + \mathrm{d}U_t\,\rho_0\,\mathrm{d}U_t^\dagger \\
&= (\mathrm{d}\chi_t + (-iH + \mathfrak{I})\,\mathrm{d}t)\,\mathsf{a}_t + \mathsf{a}_t(\mathrm{d}\chi_t^\dagger + (-iH + \mathfrak{I})^\dagger \mathrm{d}t) + \mathrm{d}\chi_t\,\mathsf{a}_t\,\mathrm{d}\chi_t^\dagger \\
&= -i[H_t^u, \mathsf{a}_t]\,\mathrm{d}t + i\sum\nolimits_j [L_j, \mathsf{a}_t]\sqrt{\gamma_j}\,\mathrm{d}\omega_t^j + \tfrac{1}{2}\sum\nolimits_j \gamma_j\big(2L_j\,\mathsf{a}_t\,L_j - L_j^2\mathsf{a}_t - \mathsf{a}_t L_j^2\big)\,\mathrm{d}t \\
&= -i[H_t^u, \mathsf{a}_t]\,\mathrm{d}t - \tfrac{1}{2}\sum\nolimits_j \gamma_j[L_j, [L_j, \mathsf{a}_t]]\,\mathrm{d}t + i\sum\nolimits_j \sqrt{\gamma_j}[L_j, \mathsf{a}_t]\,\mathrm{d}\omega_t^j,
\end{aligned}$$

where we set the Hamiltonian $H_t^u := H - \sum_j u_j(t)L_j$. Hence, taking the expectation, we obtain for the density operator $\rho_t := \mathbb{E}[\mathsf{a}_t] \in \mathscr{D}(\mathcal{H})$ the deterministic evolution

$$\mathrm{d}\rho_t = -i[H_t^u, \rho_t]\,\mathrm{d}t - \tfrac{1}{2}\sum\nolimits_j \gamma_j[L_j, [L_j, \rho_t]]\,\mathrm{d}t,$$

which is precisely the Lindblad equation (GKSL) after absorbing $\sqrt{\gamma_j}$ into L_j.

Note, however, that we recover the Lindblad equation with hermitian jump operators in this way, i.e., $L_j^\dagger = L_j$. To consider general jump operators, we have to leave the realm of classical noise and talk about quantum noise, which will be the main topic of the remaining sections in this lecture series.

Remark 1.3 We note that if $\rho_0 \in \mathscr{P}(\mathcal{H})$ is a pure state, then a_t is a $\in \mathscr{P}(\mathcal{H})$-valued process, i.e., a_t is almost surely a pure state for all times. $\diamondsuit$

1.4 Example: 1-qubit fidelity of the Hadamard gate

In quantum computing, one is often interested in the *fidelity* of a quantum gate, i.e., a unitary operation, where the fidelity of two density operators $\rho, \sigma \in \mathscr{D}(\mathcal{H})$ is defined by

$$\mathsf{F}(\rho, \sigma) := \mathrm{tr}\Big[(\sqrt{\rho}\sigma\sqrt{\rho})^{\frac{1}{2}}\Big].$$

When both $\rho = |\psi\rangle\langle\psi|, \sigma = |\varphi\rangle\langle\varphi| \in \mathscr{P}(\mathcal{H})$ are pure states, the fidelity between the two reduces to $\mathsf{F}(\rho, \sigma) = |\langle\psi, \varphi\rangle|^2$, which is much simpler than the case for general density operators. Let's see how the SSE can be used to compute the fidelity of a quantum gate.

Say, we would like to implement a Hadamard gate on a single qubit on the Hilbert space $\mathcal{H} = \mathbb{C}^2 = \mathrm{span}\{\mathbf{e}_0, \mathbf{e}_1\}$. The Hadamard gate is given by

$$U_{\mathsf{h}} = \frac{|\mathbf{e}_0\rangle + |\mathbf{e}_1\rangle}{\sqrt{2}}\langle\mathbf{e}_0| + \frac{|\mathbf{e}_0\rangle - |\mathbf{e}_1\rangle}{\sqrt{2}}\langle\mathbf{e}_1| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

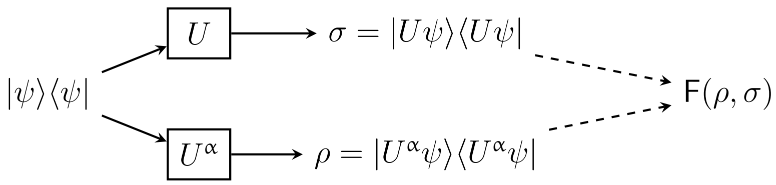

Figure 1: Computing the fidelity of a quantum gate

with the corresponding eigenpairs $(1, \mathbf{e}_+)$ and $(-1, \mathbf{e}_-)$, where

$$\mathbf{e}_+ = \frac{\mathbf{e}_0 + \mathbf{e}_1}{\sqrt{2}}, \qquad \mathbf{e}_- = \frac{\mathbf{e}_0 - \mathbf{e}_1}{\sqrt{2}}.$$

By spectral calculus, we obtain

$$\begin{aligned} U_{\mathsf{h}} &= (1)|\mathbf{e}_+\rangle\langle\mathbf{e}_+| + (-1)|\mathbf{e}_-\rangle\langle\mathbf{e}_-| = e^0|\mathbf{e}_+\rangle\langle\mathbf{e}_+| + e^{-i\pi}|\mathbf{e}_-\rangle\langle\mathbf{e}_-| \\ &= \exp(-i\pi|\mathbf{e}_-\rangle\langle\mathbf{e}_-|) = e^{-i\pi H_{\mathsf{h}}}, \qquad H_{\mathsf{h}} \coloneqq |\mathbf{e}_-\rangle\langle\mathbf{e}_-| = \tfrac{1}{2}(\mathsf{I}_{\mathcal{H}} - \sigma_{\mathsf{x}}). \end{aligned}$$

In other words, $iH_{\mathsf{h}} \in \mathscr{A}(\mathcal{H})$ generates the Hadamard gate after evolving for time $t = \pi$, i.e., $U_t = \exp(itH_{\mathsf{h}})$ solves the Schrödinger equation

$$\mathrm{d}U_t = iH_{\mathsf{h}}U_t\,\mathrm{d}t, \qquad U_0 = \mathsf{I}_{\mathcal{H}}.$$

Notice that $H_{\mathsf{h}} \in \mathscr{P}(\mathcal{H})$ happens to be a rank-1 projection on the unit vector $\mathbf{e}_- \in \mathcal{H}$.

In the presence of noise, however, we instead have

$$\mathrm{d}U_t^{\boldsymbol{\alpha}} = iH_{\mathsf{h}}U_t^{\boldsymbol{\alpha}} \circ \mathrm{d}\boldsymbol{\alpha}_t, \qquad U_0 = \mathsf{I}_{\mathcal{H}},$$

From Remark 1.2, we obtain the explicit form $U_t^{\boldsymbol{\alpha}} = \exp(i\boldsymbol{\alpha}_t H_{\mathsf{h}})$.

Suppose we were to simply apply the noisy pulse according to $\boldsymbol{\alpha}_t = 1 + \sqrt{\gamma}\boldsymbol{\omega}_t$. Then the fidelity between the desired pure state $|U_t\psi\rangle\langle U_t\psi|$ and the noisy pure state $|U_t^{\boldsymbol{\alpha}}\psi\rangle\langle U_t^{\boldsymbol{\alpha}}\psi|$ for any unit vector $\psi \in \mathcal{H}$ is given by

$$\mathsf{F}_t^{\boldsymbol{\alpha}}(\psi) \coloneqq |\langle U_t^{\boldsymbol{\alpha}}\psi, U_t\psi\rangle|^2 = |\langle U_t^\dagger U_t^{\boldsymbol{\alpha}}\psi, \psi\rangle|^2 \quad \in [0,1].$$

Using the forms obtained above, we easily deduce that

$$U_t^\dagger U_t^{\boldsymbol{\alpha}} = \exp\big(i\sqrt{\gamma}\boldsymbol{\omega}_t H_{\mathsf{h}}\big) = |\mathbf{e}_+\rangle\langle\mathbf{e}_+| + e^{i\sqrt{\gamma}\boldsymbol{\omega}_t}|\mathbf{e}_-\rangle\langle\mathbf{e}_-|.$$

Since $\{\mathbf{e}_+, \mathbf{e}_-\}$ forms an orthonormal basis for $\mathcal{H}$, $\psi = \psi_+\mathbf{e}_+ + \psi_-\mathbf{e}_-$, for which we obtain

$$\begin{aligned} \mathsf{F}_t^{\boldsymbol{\alpha}}(\psi) &= |\langle \psi_+\mathbf{e}_+ + \psi_- e^{i\sqrt{\gamma}\boldsymbol{\omega}_t}\mathbf{e}_-, \psi_+\mathbf{e}_+ + \psi_-\mathbf{e}_-\rangle|^2 \\ &= ||\psi_+|^2 + |\psi_-|^2 e^{-i\sqrt{\gamma}\boldsymbol{\omega}_t}|^2 = 1 - 2\big(1 - \cos(\sqrt{\gamma}\boldsymbol{\omega}_t)\big)|\psi_+|^2|\psi_-|^2. \end{aligned}$$

From this, we can deduce all statistical properties of the fidelity, e.g., its expectation

$$\mathbb{E}[\mathsf{F}_t^{\boldsymbol{\alpha}}(\psi)] = 1 - 2\big(1 - \mathbb{E}[\cos(\sqrt{\gamma}\boldsymbol{\omega}_t)]\big)|\psi_+|^2|\psi_-|^2 = 1 - 2\big(1 - e^{-\gamma t/2}\big)|\psi_+|^2|\psi_-|^2,$$

and variance

$$\begin{aligned} \mathbb{V}[\mathsf{F}_t^{\boldsymbol{\alpha}}(\psi)] &= \mathbb{E}\Big[\big(\mathsf{F}_t^{\boldsymbol{\alpha}}(\psi) - \mathbb{E}[\mathsf{F}_t^{\boldsymbol{\alpha}}(\psi)]\big)^2\Big] = 4\mathbb{E}\Big[\big(\cos(\sqrt{\gamma}\boldsymbol{\omega}_t) - e^{-\gamma t/2}\big)^2\Big]|\psi_+|^4|\psi_-|^4 \\ &= 4\big(1 - 2\mathbb{E}[\cos(\sqrt{\gamma}\boldsymbol{\omega}_t)]e^{-\gamma t/2} + e^{-\gamma t}\big)^2|\psi_+|^4|\psi_-|^4 = 4\big(1 - e^{-\gamma t}\big)^2|\psi_+|^4|\psi_-|^4, \end{aligned}$$

where we used the fact that $\mathbb{E}[\cos(\sqrt{\gamma}\boldsymbol{\omega}_t)] = \exp(-\gamma t/2)$.

Exercise 1.1 Compute the fidelity of the Hadamard gate using the Lindblad equation.

Question: Can we improve the fidelity by implementing a different pulse?

In theory, one could try to solve the maximization problem

$$\max_{\mathsf{b}} \Big\{ \mathbb{E}[\mathsf{F}_\pi^\alpha(\psi)] : \alpha = \mathsf{b} + \sqrt{\gamma}\omega,\ \mathsf{b}_0 = 0 \Big\},$$

with

$$\mathbb{E}[\mathsf{F}_t^\alpha(\psi)] = 1 - 2\big(1 - \mathbb{E}[\cos(\mathsf{b}_t - 1 + \sqrt{\gamma}\omega_t)]\big)|\psi_+|^2|\psi_-|^2.$$

Therefore, the optimal solution is essentially $\mathsf{b}_t = 1 - \sqrt{\gamma}\omega_t$. Unfortunately, we do not know the noise ω_t. However, if we can measure the noise and use it with a slight delay, i.e., we use $\mathsf{b}_t = 1 - \sqrt{\gamma}\omega_{t-\Delta t}$ instead, we may be able to slightly correct for it. Indeed, with this, we obtain the expected fidelity

$$\begin{aligned}\mathbb{E}[\mathsf{F}_t^\alpha(\psi)] &= 1 - 2\big(1 - \mathbb{E}[\cos(\sqrt{\gamma}(\omega_t - \omega_{t-\Delta t}))]\big)|\psi_+|^2|\psi_-|^2 \\ &= 1 - 2\big(1 - e^{-\gamma\Delta t/2}\big)|\psi_+|^2|\psi_-|^2,\end{aligned}$$

where we used the fact that $\sqrt{\gamma}(\omega_t - \omega_{t-\Delta t}) \sim \mathsf{N}(0, \gamma\Delta t)$. In particular, the fidelity is uniform in $t \geq 0$, which provides an improvement if $\Delta t \ll \pi$. [1]

[1] R.J.P.T. de Keijzer, L.Y. Visser, O. Tse, S.J.J.M.F. Kokkelmans, Qubit fidelity distribution under stochastic Schrödinger equations driven by classical noise, *Physical Review Research* 7, 023063 (2025).

2 Quantum Noise and Fock Space

sec:quantum-noise

As shown in the previous section, stochastic Schrödinger equations driven by classical noise provide a useful class of stochastic unravellings of Lindblad dynamics. However, this approach is fundamentally limited: it produces only Lindblad generators with *self-adjoint* jump operators. To describe general irreversible quantum dynamics—including spontaneous emission, particle loss, and counting processes—we must move beyond classical noise and introduce *quantum noise.*

Recall that in Section 1, the stochastic dilation $\mathsf{a}_t = U_t \rho_0 U_t^\dagger$, led, after taking expectations, to a Lindblad equation with Hermitian noise operators $L_j^\dagger = L_j$. While such Lindblad operators model dephasing and diffusion-type noise, they cannot describe dissipative processes such as amplitude damping, where the jump operator is non-Hermitian. An important example is the jump operator $L = \sigma_- = |0\rangle\langle 1|$ on a single qubit describing *spontaneous emission* due to black body radiation.

From a physical perspective, this reflects the fact that classical noise models only randomize *phases* or *energies.* Truly quantum processes involve the exchange of quanta with an environment, and therefore require a non-commutative noise model.

Unitary dilations and infinite environments

A guiding principle in the theory of open quantum systems is that irreversibility arises from neglecting environmental degrees of freedom in the following sense. One considers the composite Hilbert space $\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_{\mathrm{sys}} \otimes \mathcal{H}_{\mathrm{env}}$, describing the *universe* and consists of the system $\mathcal{H}_{\mathrm{sys}}$ and an environment $\mathcal{H}_{\mathrm{env}}$. In this setting, one obtains a unitary evolution

$$U_t : \mathcal{H}_{\mathrm{tot}} \to \mathcal{H}_{\mathrm{tot}}$$

The reduced evolution of the system state is then obtained by tracing out the environment,

$$\rho_t = \mathrm{tr}_{\mathcal{K}}\big[U_t(\rho_0 \otimes \rho_{\mathrm{env}})U_t^\dagger\big] \ \in \mathscr{D}(\mathcal{H}_{\mathrm{sys}}),$$

where $\rho_0 \in \mathscr{D}(\mathcal{H}_{\mathrm{sys}})$ is the initial state on the system and $\rho_{\mathrm{env}} \in \mathscr{D}(\mathcal{H}_{\mathrm{env}})$ is a given state on the environment. Requiring *Markovianity* and time-homogeneity of the evolution inevitably forces the environment to possess infinitely many degrees of freedom. In continuous time, this naturally leads to a description in terms of bosonic quantum fields.

2.1 Bosonic Fock space

The *bosonic (or symmetric) Fock space* of a complex Hilbert space $\mathcal{K}$ is defined as

$$\mathfrak{F}(\mathcal{K}) \coloneqq \bigoplus_{n \in \mathbb{N}_0} \mathcal{K}^{\odot n},$$

where $\mathcal{K}^{\odot n}$ denotes the n-fold symmetric tensor product, i.e.,

$$f \in \mathcal{K}^{\odot n} \quad \Leftrightarrow \quad f(\sigma(u_1, \dots, u_n)) = f(u_1, \dots, u_n) \quad \text{for any permutation } \sigma,$$

and by convention $\mathcal{K}^{\odot 0} := \mathbb{C}$. The bosonic Fock space $\mathfrak{F}(\mathcal{K})$ inherits the scalar product from $\mathcal{K}$ defined by

$$\langle \oplus u^{(n)}, \oplus v^{(n)} \rangle_{\mathfrak{F}(\mathcal{K})} := \sum_{n \in \mathbb{N}_0} \langle u^{(n)}, v^{(n)} \rangle_{\mathcal{K}^{\otimes n}}.$$

We define the *exponential vectors*

$$\mathsf{e}(u) = \bigoplus_{n \in \mathbb{N}_0} \frac{1}{\sqrt{n!}} u^{\otimes n}, \qquad u \in \mathcal{K},$$

and the distinguished vector $\Omega := \mathsf{e}(0) = 1 \oplus 0 \oplus \cdots \in \mathfrak{F}(\mathcal{K})$, called the *vacuum vector.*

It turns out that the set of exponential vectors $\mathfrak{E}(\mathcal{K})$ is *total* in $\mathfrak{F}(\mathcal{K})$, i.e., the linear span of $\mathfrak{E}(\mathcal{K})$ is dense in $\mathfrak{F}(\mathcal{K})$. This fact will be helpful for us in the future. Since,

$$\langle \mathsf{e}(u), \mathsf{e}(v) \rangle_{\mathfrak{F}(\mathcal{K})} = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} \langle u, v \rangle_{\mathcal{K}}^n = e^{\langle u, v \rangle_{\mathcal{K}}}, \qquad u, v \in \mathcal{K},$$

the exponential vectors are normalizable. These normalized exponential vectors

$$\psi(u) = e^{-\frac{1}{2}\|u\|_{\mathcal{K}}^2} \mathsf{e}(u), \qquad u \in \mathcal{K},$$

are called *coherent vectors.*

For time-continuous noise, the *canonical* choice is

$$\mathcal{K} = L^2(\mathbb{R}_+, \lambda; \mathbb{C}^d) \cong L^2(\mathbb{R}_+, \lambda) \otimes \mathbb{C}^d,$$

where $d \in \mathbb{N}$ represents the number of *noise channels.* From now on, we will only consider this canonical choice and call $\mathfrak{F}(\mathcal{K})$ our noise environment.

Definition 2.1 Let $\mathcal{D}$ be a total subset of a complex Hilbert space $\mathcal{H}$.

(i) A *random variable* X is an element of $\mathscr{L}(\mathcal{D}; \mathcal{H})$, where

$$\mathscr{L}(\mathcal{D}; \mathcal{H}) := \Big\{ \mathsf{Z} : \mathcal{D} \to \mathcal{H} \text{ linear} : \mathcal{D} \subset \operatorname{dom}(\mathsf{Z}) \cap \operatorname{dom}(\mathsf{Z}^\dagger) \Big\}.$$

(ii) A *stochastic process* in $\mathcal{H}$ is a family $(\mathsf{X}(t))_{t \in \mathbb{R}_+}$ of random variables such that

$$\mathbb{R}_+ \ni t \mapsto \mathsf{X}(t)\eta \quad \text{is Borel measurable for every } \eta \in \mathcal{D}.$$

Notice that nothing about the definition above is stochastic in the usual sense.

Field operators: Creation, annihilation, and gauge processes

On the bosonic Fock space $\mathfrak{F}(\mathcal{K})$, one defines operator-valued processes on $\mathfrak{E}(\mathcal{K})$:

the *annihilation* processes $\mathsf{A}_j(t)$, the *creation* processes $\mathsf{A}_j^\dagger(t)$,
the *gauge (or counting)* processes $\Lambda_{ij}(t)$ for $i, j = 1, \ldots, d$.

Heuristically, $\mathsf{A}_j(t)$ annihilates a quantum in channel j arriving before time $t \geq 0$, $\mathsf{A}_j^\dagger(t)$ creates such a quantum, and $\Lambda_{ij}(t)$ counts quanta between channels. Together, these processes encode absorption, emission, and counting statistics in a unified operator-theoretic framework and form the building blocks of quantum stochastic calculus, serving as driving noises in the Hudson-Parthasarathy theory of quantum processes.

Annihilation and creation processes. For each channel $j = 1, \ldots, d$, the annihilation and creation operators are defined on exponential vectors $\mathsf{e}(u)$ by

$$\mathsf{A}_j(t)\mathsf{e}(u) \coloneqq \langle \mathbf{1}_{[0,t]}, u_j\rangle \mathsf{e}(u), \qquad \mathsf{A}_j^\dagger(t)\mathsf{e}(u) \coloneqq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathsf{e}(u + \varepsilon \mathbf{1}_{[0,t]} e_j),$$

where $u = (u_1, \ldots, u_d) \in \mathcal{K}$ and e_j, $j = 1, \ldots, d$, denotes the canonical basis of $\mathbb{C}^d$.

These operators are densely defined, mutually adjoint, and they satisfy the *canonical commutation relations*

$$[\mathsf{A}_i(t), \mathsf{A}_j^\dagger(s)] = \delta_{ij} \min(t,s)\, \mathsf{I}_{\mathfrak{F}(\mathcal{K})}, \qquad [\mathsf{A}_i(t), \mathsf{A}_j(s)] = [\mathsf{A}_i^\dagger(t), \mathsf{A}_j^\dagger(s)] = 0. \tag{CCR}$$

These relations give rise to the quantum Itô table that we will see in the following section.

With respect to the coherent vector $\psi(u) \in \mathfrak{F}(\mathcal{K})$, $u \in \mathcal{K}$, one has

$$\begin{aligned} \langle \psi(u), \mathsf{A}_j(t)\psi(u)\rangle &= \langle \mathbf{1}_{[0,t]}, u_j\rangle = \overline{\langle \psi(u), \mathsf{A}_j^\dagger(t)\psi(u)\rangle}, \\ \langle \psi(u), [\mathsf{A}_i(t), \mathsf{A}_j^\dagger(s)]\psi(u)\rangle &= \delta_{ij} \min(t,s). \end{aligned}$$

In particular, in the vacuum vector $\Omega = \psi(0)$, we find

$$\langle \Omega, \mathsf{A}_j(t)\Omega\rangle = \overline{\langle \Omega, \mathsf{A}_j^\dagger(t)\Omega\rangle} = 0, \qquad \langle \Omega, \mathsf{A}_i(t)\mathsf{A}_j^\dagger(s)\Omega\rangle = \delta_{ij}\min(t,s),$$

which shows that the self-adjoint field operators $\mathsf{B}_j = \mathsf{A}_j + \mathsf{A}_j^\dagger$ reproduce the covariance structure of classical Brownian motion, i.e.,

$$\langle \Omega, \mathsf{B}_i(t)\mathsf{B}_j^\dagger(s)\Omega\rangle = \delta_{ij}\min(t,s).$$

Thus, classical noise is recovered as a commutative subtheory of quantum noise.

Exercise 2.1 Use the *Zassenhaus formula* and the fact that

$$[\mathsf{A}_j(t), [\mathsf{A}_j(t), \mathsf{A}_j^\dagger(t)]] = [\mathsf{A}_j(t), [\mathsf{A}_j(t), \mathsf{A}_j^\dagger(t)]] = 0 \qquad \text{for all } t \geq 0,$$

to show that for every $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$,

$$\mathbb{E}_\Omega\big[e^{i\sum_j r_j \mathsf{B}_j(t)}\big] \coloneqq \langle \Omega, e^{i\sum_j r_j \mathsf{B}_j(t)}\Omega\rangle = \exp\Big(-\tfrac{1}{2}\sum\nolimits_j r_j^2\, t\Big).$$

Conclude from this that the vector of field operators $(\mathsf{B}_1(t), \ldots, \mathsf{B}_d(t))$ is a d-dimensional Gaussian *random variable* with mean 0 and covariance $\Sigma = t\mathsf{I}_{\mathbb{R}^d}$.

What would change if we replace the vacuum Ω with a coherent vector $\psi(u)$?

Gauge processes. The gauge processes $\Lambda_{ij}(t)$ describe the flow of quanta between channels i and j. Their expectation on coherent vectors $\psi(u)$ is given by

$$\langle \psi(u), \Lambda_{ij}(t)\psi(u)\rangle = \int_{[0,t]} \overline{u}_j u_i \,\mathrm{d}\lambda.$$

In particular, the diagonal processes $\Lambda_{jj}(t)$ *counts* the number of quanta in channel j up to time $t \geq 0$ with

$$\mathbb{E}_{\psi(u)}\big[e^{ir\Lambda_{jj}(t)}\big] := \langle \psi(u), e^{ir\Lambda_{jj}(t)}\psi(u)\rangle = \exp\Big((e^{ir}-1)\|u_j\|^2_{L^2([0,t],\lambda)}\Big),$$

i.e., in the coherent state $\psi(u)$, $\Lambda_{jj}(t)$ has a Poisson statistics with intensity $|u_j(t)|^2$.

In these lectures, we only have time to focus on the creation and annihilation processes. A proper treatise of the gauge process requires more preparation, but its inclusion in the following theory may be done without too much trouble.

2.2 Fock-Wiener isometry

subsec:fock-wiener

The Wiener-Itô-Segal isomorphism provides a precise mathematical link between classical and quantum noise, which identifies the canonical bosonic Fock space with an L^2-space over classical Wiener space.

More precisely, let $(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d), \mathcal{F}, \mathbb{W}_d)$ be a classical Wiener probability space with canonical process $(W_t^1, \ldots, W_t^d)$. Then there exists a unitary isomorphism

$$\mathcal{U} : \mathfrak{F}(\mathcal{K}) \to L^2(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d), \mathbb{W}_d),$$

called the *Fock-Wiener (or Wiener-Itô-Segal) isometry*, with the following properties:

(i) The vacuum vector Ω is mapped to the constant function 1.

(ii) Exponential vectors correspond to stochastic exponentials of Brownian motion, i.e.,

$$\mathcal{U}\big(\psi(u\mathbf{1}_{[0,t]})\big)(\omega) = \exp\left(\sum\nolimits_j \int_0^t u_j(s)\,\mathrm{d}W_s^j(\omega) - \frac{1}{2}\sum\nolimits_j \|u_j\mathbf{1}_{[0,t]}\|^2_{\mathcal{K}}\right) \quad \text{for } t \geq 0.$$

Recall that the right-hand side is an exponential martingale and that the family of such functions is total in $L^2(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d), \mathbb{W}_d)$.

(iii) Multiple Wiener integrals of order n correspond to the n-particle sector $\mathcal{K}^{\odot n}$.

Under this isometry, the self-adjoint field operator $\mathsf{B}_j(t)$ acts as multiplication by the classical Brownian motion W_t^j. In particular,

$$\mathcal{U}\mathsf{B}_j(t)\mathcal{U}^{-1} = W_t^j,$$

viewed as a multiplication operator on $L^2(\mathcal{C}(\mathbb{R}_+;\mathbb{R}^d), \mathbb{W}_d)$. This identification shows that classical stochastic calculus is faithfully embedded into quantum stochastic calculus as the restriction to a commuting subalgebra of field operators.

Remark 2.2 (1) Notice that the coherent states $\psi(u) \in \mathfrak{F}(\mathcal{K})$ play the role of changing the reference measure $\mathbb{W}_d$ by the drift field $u \in \mathcal{K}$.

(2) A similar construction holds for Poisson processes on the Skorokhod space. $\diamondsuit$

3 Hudson-Parthasarathy Theory

sec:HP

Having introduced quantum noise, we now describe the dynamics of systems driven by such noise. This is accomplished by the Hudson-Parthasarathy (HP) theory of quantum stochastic differential equations (QSDE), where the bosonic Fock space is used as a model for the environment Hilbert space, i.e., $\mathcal{H}_{\text{env}} = \mathfrak{F}(\mathcal{K})$. For simplicity, we consider a single noise channel, i.e., $d = 1$, and focus on the creation and annihilation processes. Henceforth, we consider an n-dimensional complex Hilbert space $\mathcal{H}_{\text{sys}}$ as our system and $\mathcal{H}_{\text{env}} = \mathfrak{F}(\mathcal{K})$ with $\mathcal{K} = L^2(\mathbb{R}_+, \lambda)$ is our environment, which gives $\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}$.

Recall from Section 1 that the basic unitary evolution was derived by simply applying Itô's formula to $U_t = \exp(i\omega_t L)\mathsf{I}_{\mathcal{H}}$, where $L \in \mathscr{O}(\mathcal{H})$ is a Hermitian noise operator such that $i\omega_t L \in \mathscr{A}(\mathcal{H})$ is in the Lie algebra. Here, we can do the same by considering

$$U_t = \exp(L \otimes \mathsf{A}^\dagger(t) - L^\dagger \otimes \mathsf{A}(t))\mathsf{I}_{\mathcal{H}_{\text{tot}}},$$

where $L \in \mathscr{B}(\mathcal{H}_{\text{sys}})$ is a system operator and $\mathsf{A}(t)$, $\mathsf{A}^\dagger(t)$ are the field operators on $\mathcal{H}_{\text{env}}$. Notice that

$$\left(L \otimes \mathsf{A}^\dagger - L^\dagger \otimes \mathsf{A}\right)^\dagger = L^\dagger \otimes \mathsf{A} - L \otimes \mathsf{A}^\dagger = -\left(L \otimes \mathsf{A}^\dagger - L^\dagger \otimes \mathsf{A}\right),$$

i.e., $L \otimes \mathsf{A}^\dagger - L^\dagger \otimes \mathsf{A} \in \mathscr{A}(\mathcal{H}_{\text{tot}})$ is skew-Hermitian. Thus, U_t is a unitary evolution.

If we knew of an Itô formula for the creation and annihilation operators, then we could derive a QSDE for the unitary process U_t. In general, one would also like to consider time-dependent system operators, i.e., $L = L(t)$, which will require us to introduce stochastic integrals of the form

$$\int_0^t L(s) \otimes \mathrm{d}\mathsf{A}^\dagger(s), \quad \int_0^t L^\dagger(s) \otimes \mathrm{d}\mathsf{A}(s).$$

To do so, we will need to understand the type of processes $L(t)$ we can integrate against. As in the classical Itô integral, we will require the notion of an *adapted process*.

3.1 Quantum Itô integrals and Itô calculus

Before we talk about quantum stochastic integrals, we need to introduce the quantum analog of a *filtration*. We begin by noting that for any $0 \leq t_1 < \cdots < t_n < +\infty$, we have the direct sum factorization

$$\mathcal{K} = L^2([0, t_1]) \oplus L^2([t_1, t_2]) \oplus \cdots \oplus L^2([t_n, +\infty)) =: \mathcal{K}_{t_1]} \oplus \mathcal{K}_{[t_1, t_2]} \oplus \cdots \oplus \mathcal{K}_{[t_n},$$

which gives the factorization $\mathfrak{F}(\mathcal{K}) = \mathfrak{F}(\mathcal{K}_{t_1]}) \otimes \mathfrak{F}(\mathcal{K}_{[t_1, t_2]}) \otimes \cdots \otimes \mathfrak{F}(\mathcal{K}_{[t_n})$. Denoting

$$\mathcal{F}_0 := \mathcal{H}_{\text{sys}}, \quad \mathcal{F}_{t]} := \mathcal{H}_{\text{sys}} \otimes \mathfrak{F}(\mathcal{K}_{t]}), \quad \mathcal{F}_{[s,t]} := \mathfrak{F}(\mathcal{K}_{[s,t]}), \quad \mathcal{F}_{[t} := \mathfrak{F}(\mathcal{K}_{[t}),$$

we then have the tensor product factorization of the time index

$$\mathcal{H}_{\text{tot}} = \mathcal{F}_{t_1]} \otimes \mathcal{F}_{[t_1, t_2]} \otimes \cdots \otimes \mathcal{F}_{[t_n},$$

which is associated with the factorization on the level of the vectors

$$\psi \otimes \mathsf{e}(u) = \psi \otimes \mathsf{e}(u\mathbf{1}_{[0,t_1]}) \otimes \mathsf{e}(u\mathbf{1}_{[t_1,t_2]}) \otimes \cdots \otimes \mathsf{e}(u\mathbf{1}_{[t_n,+\infty)})$$

The increasing family $(\mathcal{F}_{t]})_{t\geq 0}$ may then be used to define an adapted process. Roughly speaking, a process $(\mathsf{X}(t))_{t\in\mathbb{R}_+}$ is adapted if for each $t \geq 0$, it leaves the space $\mathcal{F}_{[t}$ *ontouched*, i.e., it takes the form $\mathsf{X}(t)\otimes\mathsf{I}_{[t}$ for every $t \geq 0$.

Definition 3.1 A stochastic process $(\mathsf{X}(t))_{t\in\mathbb{R}_+}$ in $\mathcal{H}_{\text{tot}}$ is said to be $\mathcal{F}$-*adapted* to if the map $t \mapsto \mathsf{X}(t)\psi\otimes\mathsf{e}(u)$ is measurable and there exists an operator $\mathsf{Z}(t)$ in $\mathcal{F}_{t]}$ such that

$$\mathsf{X}(t)\psi\otimes\mathsf{e}(u) = \big(\mathsf{Z}(t)\psi\otimes\mathsf{e}(u\mathbf{1}_{[0,t]})\big) \otimes \mathsf{e}(u\mathbf{1}_{[t,+\infty)}),$$

for every $t \geq 0$, $\psi \in \mathcal{H}_{\text{sys}}$ and $u \in \mathcal{K}$.

Such a process is said to be *regular* if the map $t \mapsto \mathsf{X}(t)\psi\otimes\mathsf{e}(u)$ is continuous.

Remark 3.2 (i) The processes $L\otimes\mathsf{A}^\dagger(t)$, $L^\dagger\otimes\mathsf{A}(t)$ are $\mathcal{F}$-adapted. Indeed,

$$\begin{aligned} L^\dagger\otimes\mathsf{A}(t)\psi \otimes \mathsf{e}(u) &= (L^\dagger\psi) \otimes (\mathsf{A}(t)\mathsf{e}(u)) \\ &= (L^\dagger\psi) \otimes (\langle 1, u\mathbf{1}_{[0,t]}\rangle \mathsf{e}(u\mathbf{1}_{[0,t]})) \otimes \mathsf{e}(u\mathbf{1}_{[t,+\infty)}). \end{aligned}$$

A similar result holds for $L\otimes\mathsf{A}^\dagger(t)$.

(ii) For any $0 \leq s < t < +\infty$, we have that

$$\begin{aligned} \big(\mathsf{A}(t) - \mathsf{A}(s)\big)\mathsf{e}(u) &= \big(\langle \mathbf{1}_{[0,t]}, u\rangle - \langle \mathbf{1}_{[0,s]}, u\rangle\big)\mathsf{e}(u) = \langle \mathbf{1}_{[s,t]}, u\rangle \mathsf{e}(u) \\ &= \mathsf{e}(u\mathbf{1}_{[0,s]}) \otimes \big(\mathsf{A}(t) - \mathsf{A}(s)\big)\mathsf{e}(u\mathbf{1}_{[s,t]}) \otimes \mathsf{e}(u\mathbf{1}_{[t,+\infty]}), \end{aligned}$$

i.e., $\mathsf{A}(t) - \mathsf{A}(s)$ acts only on the part $\mathcal{F}_{[s,t]}$, while leaving the rest untouched. This holds similarly for $\mathsf{A}^\dagger(t) - \mathsf{A}^\dagger(s)$. ♢

In the following, let M be either A or $\mathsf{A}^\dagger$ and for $0 \leq t_1 < \cdots < t_n < +\infty$, we consider the *simple process*

$$L := \sum_{j=0}^{n-1} L_j \mathbf{1}_{[t_j,t_{j+1})} + L_n \mathbf{1}_{[t_j,+\infty)}, \qquad L_j \in \mathscr{B}(\mathcal{F}_{t_j]}).$$

By construction, $(L(t))_{t\in\mathbb{R}_+}$ is an $\mathcal{F}$-adapted process in $\mathcal{H}_{\text{tot}}$. For these simple processes, we define the Itô integral w.r.t. M as

$$\int_0^t L(s)\otimes \mathrm{d}\mathsf{M}(s) := \sum_{j=0}^{n} L_j\otimes\big(\mathsf{M}(t\wedge t_{j+1}) - \mathsf{M}(t\wedge t_j)\big),$$

with the convention $t_{n+1} = +\infty$. Setting the left-hand side as $\mathsf{X}(t)$, we then have that

$$\langle \psi\otimes\Omega, \mathsf{X}(t)\psi\otimes\Omega\rangle = 0 \qquad \text{for all } t \geq 0 \text{ and } \psi \in \mathcal{H}_{\text{sys}},$$

since each component of the sum gives

$$\langle \psi\otimes\Omega, L_j\psi\otimes\Omega\rangle \langle \Omega, \big(\mathsf{M}(t\wedge t_{j+1}) - \mathsf{M}(t\wedge t_j)\big)\Omega\rangle = 0.$$

If Y is another simple process with

$$\mathsf{Y}(t) := \int_0^t H(s) \otimes \mathrm{d}\mathsf{N}(s),$$

then

$$\langle \psi \otimes \Omega, \mathsf{X}(t)\mathsf{Y}(t)\psi \otimes \Omega \rangle = \begin{cases} 0 & \text{if } \mathsf{M} = \mathsf{A}^\dagger \text{ or } \mathsf{N} = \mathsf{A}, \\ \Big\langle \psi \otimes \Omega, \Big(\int_{[0,t]} LH \,\mathrm{d}\lambda \Big) \psi \otimes \Omega \Big\rangle & \text{if } \mathsf{M} = \mathsf{A} \text{ and } \mathsf{N} = \mathsf{A}^\dagger, \end{cases}$$

and the Itô product rule becomes

$$\mathrm{d}\big(\mathsf{X}(t)\mathsf{Y}(t)\big) = (\mathrm{d}\mathsf{X}(t))\mathsf{Y}(t) + \mathsf{X}(t)\mathrm{d}\mathsf{Y}(t) + L(t)H(t)\,\mathrm{d}\langle \mathsf{MN}\rangle(t),$$

with the following Itô table, which allows us to further obtain an Itô formula.

$\mathrm{d}\mathsf{M}/\mathrm{d}\mathsf{N}$	$\mathrm{d}\mathsf{A}(t)$	$\mathrm{d}\mathsf{A}^\dagger(t)$	$\mathrm{d}\Lambda(t)$
$\mathrm{d}\mathsf{A}(t)$	0	$\mathrm{d}t$	$\mathrm{d}\mathsf{A}(t)$
$\mathrm{d}\mathsf{A}^\dagger(t)$	0	0	0
$\mathrm{d}\Lambda(t)$	0	$\mathrm{d}\mathsf{A}^\dagger(t)$	0

3.2 Hudson-Parthasarathy equation

Let $H \in \mathcal{O}(\mathcal{H}_{\mathrm{sys}})$ and $L \in \mathcal{B}(\mathcal{H}_{\mathrm{sys}})$. Setting

$$\mathsf{X}(t) = -itH \otimes \mathsf{I}_{\mathcal{H}_{\mathrm{env}}} + \int_0^t L(s) \otimes \mathrm{d}\mathsf{A}^\dagger(s) - \int_0^t L^\dagger(s) \otimes \mathrm{d}\mathsf{A}(s) \quad \in \mathscr{A}(\mathcal{H}_{\mathrm{tot}}),$$

the unitary process $U_t := \exp(\mathsf{X}(t))$ satisfies the Hudson-Parthasarathy QSDE

$$\mathrm{d}U_t = \Big(L \otimes \mathrm{d}\mathsf{A}^\dagger(t) - L^\dagger \otimes \mathrm{d}\mathsf{A}(t) - \Big(iH + \tfrac{1}{2} L^\dagger L \Big) dt \Big) U_t, \qquad U_0 = I. \tag{3.1}$$

representing the joint evolution of system and environment.

3.3 Stratonovich versus Itô formulations

subsec:ito-strat-hp

As in the classical theory, quantum stochastic differential equations admit both Itô and Stratonovich formulations. The Itô form is algebraically convenient and is intrinsic to the Hudson-Parthasarathy framework, while the Stratonovich form is often conceptually closer to Hamiltonian dynamics.

Let us write the HP equation in Itô form as

$$dU_t = G_{\text{Itô}}(t), U_t,$$

where

$$G_{\text{Itô}}(t) := \sum_j L_j, dA_j^\dagger(t) * \sum_j L_j^\dagger, dA_j(t) * \Big(iH + \tfrac{1}{2}\sum_j L_j^\dagger L_j\Big)dt.$$

The corresponding Stratonovich form is defined by the requirement that the usual Leibniz and chain rules hold. One may formally write

$$dU_t = G_{\text{Strat}}(t) \circ U_t,$$

where the Stratonovich generator is

$$G_{\text{Strat}}(t) := \sum_j L_j, \circ dA_j^\dagger(t) * \sum_j L_j^\dagger, \circ dA_j(t) * iH; dt.$$

The relation between the two forms mirrors the classical correction term: the Itô drift contains the additional dissipative contribution $-\frac{1}{2}\sum_j L_j^\dagger L_j$, which arises from the non-vanishing quadratic products $dA_j(t)dA_j^\dagger(t) = dt$ in the quantum Itô table.

In particular, the Stratonovich form makes explicit that the unitary dynamics are generated by a *stochastic Hamiltonian*

$$H_{\text{eff}}(t) = H + i\sum_j \big(L_j, \dot{A}_j^\dagger(t) - L_j^\dagger, \dot{A}_j(t)\big),$$

while the Itô form encodes the same dynamics together with the irreversible drift required for complete positivity.

This distinction is especially useful when comparing HP equations with the classical stochastic Schrödinger equations of Section 1, which were naturally written in Stratonovich form.

Remark 3.3 (Classical versus quantum noise) In the classical setting of Section 1, a unitary-valued stochastic evolution driven by Brownian motion takes the Stratonovich form

$$\mathrm{d}U_t = iL \circ dW_t, U_t,$$

which corresponds to a random Hamiltonian perturbation and preserves unitarity pathwise. Passing to Itô form introduces the familiar correction $-\frac{1}{2}H^2dt$, reflecting the quadratic variation of Brownian motion.

In the quantum setting, the Stratonovich HP equation

$$dU_t = \sum_j L_j \otimes \circ\, \mathsf{dA}_j^\dagger(t) - \sum_j L_j^\dagger \otimes \circ\ \mathsf{dA}_j(t), U_t - iH\, \mathrm{d}t U_t$$

plays an analogous role: it represents a stochastic Hamiltonian coupling between system and field. However, the non-commutativity of the quantum noise leads, in Itô form, to the additional drift term $-\frac{1}{2}\sum_j L_j^\dagger L_j dt$, which has no classical analogue and is ultimately responsible for irreversible Lindblad dynamics after tracing out the environment.

Thus, classical noise produces diffusion on the unitary group, while quantum noise produces both diffusion and dissipation—a distinction that lies at the heart of open quantum system theory.

3.4 From HP equations to Lindblad dynamics

Let $\rho_0 \in \mathcal{D}(\mathcal{H})$ and consider the vacuum state on the Fock space. The reduced system state is given by

$$\rho_t = \mathrm{Tr}_{\mathfrak{F}}\big(U_t(\rho_0 \otimes |\Omega\rangle\langle\Omega|)U_t^\dagger\big).$$

A direct computation using the quantum Itô calculus yields the Lindblad equation

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] + \mathscr{L}(\rho_t), \qquad \mathscr{L}(\rho) := -\tfrac{1}{2}\sum\nolimits_j [L_j^\dagger, [L_j, \rho]]$$

Thus, every GKSL generator admits a unitary dilation driven by quantum noise.

3.5 Worked example: amplitude damping

Consider a single qubit with $H = 0$ and $L = \sqrt{\gamma}\sigma_- \in \mathscr{B}(\mathcal{H}_{\mathrm{sys}})$. The QSDE becomes

$$dU_t = \big(\sqrt{\gamma}\sigma_- \otimes \mathsf{dA}^\dagger(t) - \sqrt{\gamma}\sigma_+ \otimes \mathsf{dA}(t) - \tfrac{\gamma}{2}\sigma_+\sigma_-\, \mathrm{d}t\big)U_t.$$

Tracing out the Fock space yields the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\tfrac{\gamma}{2}[\sigma_+, [\sigma_-, \rho_t]],$$

which describes spontaneous emission. This dynamics cannot be obtained from any classical-noise-driven stochastic Schrödinger equation.

3.6 Outlook: control and measurement

Hudson-Parthasarathy equations provide the natural starting point for quantum filtering, continuous measurement, and feedback control. In this framework, control inputs appear as adapted processes acting on the system operators, while measurement corresponds to observing commuting output fields derived from the quantum noise.

3.7 Boson Fock spaces

Consider a (complex) Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The symmetric Fock space associated with $\mathcal{H}$ is

$$\mathfrak{F} = \mathfrak{F}_{\mathsf{sym}}(\mathcal{H}) \coloneqq \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}^{\odot n}, \qquad \mathcal{H}^{\odot 0} = \mathbb{C},$$

where $\odot$ denotes the symmetric tensor product such that

$$\mathcal{H}^{\odot n} = \Big\{ f \in \mathcal{H}^{\otimes n} : f(x_{\sigma_1}, \dots, x_{\sigma_n}) = f(x_1, \dots, x_n) \text{ for every permutation } \sigma \Big\}.$$

The Fock space $\mathfrak{F}$ inherits the scalar product from $\mathcal{H}$ defined by

$$\langle \oplus f^{(n)}, \oplus g^{(n)} \rangle_{\mathfrak{F}} = \sum_{n \in \mathbb{N}_0} \langle f^{(n)}, g^{(n)} \rangle_{\mathcal{H}^{\otimes n}}.$$

We define the *vacuum vector* $\Omega = 1 \oplus 0 \oplus 0^{\otimes 2} \oplus \cdots \in \mathfrak{F}$, and the *exponential vectors*

$$\mathsf{e}(f) = \bigoplus_{n \in \mathbb{N}_0} \frac{1}{\sqrt{n!}} f^{\otimes n}, \qquad f \in \mathcal{H}.$$

It turns out that the family of exponential vectors $\mathfrak{E}$ is *total* in $\mathfrak{F}$, i.e., the linear span of $\mathfrak{E}$ is dense in $\mathfrak{F}$. This fact will be helpful for us in the future. Since,

$$\langle \mathsf{e}(f), \mathsf{e}(g) \rangle_{\mathfrak{F}} = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} \langle f, g \rangle_{\mathcal{H}}^n = e^{\langle f, g \rangle_{\mathcal{H}}}, \qquad f, g \in \mathcal{H},$$

the exponential vectors are normalizable. These normalized exponential vectors

$$\psi(f) = e^{-\frac{1}{2}\|f\|_{\mathcal{H}}^2} \mathsf{e}(f), \qquad f \in \mathcal{H},$$

are commonly known as *coherent vectors*.

3.7.1 Weyl and field operators

For any $f \in \mathcal{H}$, we define the *Weyl operator* on exponential vectors by

$$\mathsf{W}(f)\mathsf{e}(g) \coloneqq \exp\left(-\langle f, g \rangle_{\mathcal{H}} - \frac{1}{2}\|f\|_{\mathcal{H}}^2 \right) \mathsf{e}(f + g), \qquad g \in \mathcal{H}.$$

Weyl operators play an essential role in the setup of Fock spaces. For one, they generate coherent states by acting on the vacuum state, i.e.,

$$\mathsf{W}(f)\Omega = e^{-\frac{1}{2}\|f\|_{\mathcal{H}}^2} \mathsf{e}(f) = \psi(f), \qquad f \in \mathcal{H}.$$

Moreover, they give the means to map any element $f \in \mathcal{H}$ to unitary operators on $\mathfrak{F}$ that satisfy the *canonical commutation relation* (CCR):

Proposition 3.4. *The Weyl operator* $\mathsf{W}(f)$ *is a unitary operator and satisfies*

(i) $\mathsf{W}^\dagger(f) = \mathsf{W}(-f)$.

(ii) $\mathsf{W}(f)\mathsf{W}(g) = e^{-i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})}\mathsf{W}(f+g) = e^{-2i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})}\mathsf{W}(g)\mathsf{W}(f)$.

Property (ii) is the Weyl form of the canonical commutation relation (CCR).

Proof. For any $g \in \mathcal{H}$,

$$\begin{aligned}\langle \mathsf{W}(f)\mathsf{e}(g), \mathsf{W}(f)\mathsf{e}(g)\rangle_{\mathfrak{F}} &= \exp\left(-2\langle f,g\rangle_{\mathcal{H}} - \|f\|^2_{\mathcal{H}}\right)\langle \mathsf{e}(f+g), \mathsf{e}(f+g)\rangle_{\mathfrak{F}} \\ &= \exp\left(-2\langle f,g\rangle_{\mathcal{H}} - \|f\|^2_{\mathcal{H}} + \|f+g\|^2_{\mathcal{H}}\right) \\ &= e^{\|g\|^2_{\mathcal{H}}} = \langle \mathsf{e}(g), \mathsf{e}(g)\rangle_{\mathfrak{F}}.\end{aligned}$$

Hence, $\mathsf{W}(f)$ preserves inner products on $\mathfrak{E}$. Since $\mathfrak{E}$ is dense in $\mathfrak{F}$, $\mathsf{W}(f)$ extends uniquely to an isometry on $\mathfrak{F}$.

In a similar fashion, we compute

$$\mathsf{W}(-f)\mathsf{W}(f)\mathsf{e}(g) = \mathsf{e}(g) = \mathsf{W}(f)\mathsf{W}(-f)\mathsf{e}(g) \qquad \forall g \in \mathcal{H},$$

i.e., $\mathsf{W}(-f)\mathsf{W}(f)$ is the identity on the dense set $\mathfrak{E}$. In particular, $\mathsf{W}(f)$ is surjective and an isometry, i.e., $\mathsf{W}(f)$ is unitary with $\mathsf{W}^\dagger(f) = \mathsf{W}(-f)$.

As for the last property, we observe that

$$\begin{aligned}\mathsf{W}(f)\mathsf{W}(g)W(-(f+g))\mathsf{e}(h) &= e^{\langle f+g,h\rangle_{\mathcal{H}} - \frac{1}{2}\|f+g\|^2_{\mathcal{H}}}\mathsf{W}(f)\mathsf{W}(g)\mathsf{e}(-(f+g)+h) \\ &= e^{\langle f+g,h\rangle_{\mathcal{H}} - \frac{1}{2}\|f\|^2_{\mathcal{H}} - \langle g,-f+h\rangle_{\mathcal{H}} - \mathrm{Re}(\langle g,f\rangle_{\mathcal{H}})}\mathsf{W}(f)\mathsf{e}(-f+h) \\ &= e^{-i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})}\mathsf{e}(h),\end{aligned}$$

and hence, $\mathsf{W}(f)\mathsf{W}(g)\mathsf{W}(-(f+g)) = e^{-i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})}I_{\mathfrak{F}}$. We then conclude by using property *(i)* of Weyl operators. $\square$

Since $\mathsf{W}(f)$ is unitary for every $f \in \mathcal{H}$, the family $\{\mathsf{W}(tf)\}_{t\in\mathbb{R}}$ forms a one-parameter (strongly continuous) group of unitaries. In particular, due to Stone's theorem, it has a corresponding Hermitian operator $\mathsf{P}(f)$ such that

$$\mathsf{W}(tf) = \exp(it\mathsf{P}(f)).$$

We further define the following operators

$$\mathsf{Q}(f) \coloneqq -\mathsf{P}(if), \qquad \mathsf{A}^-(f) \coloneqq \frac{\mathsf{Q}(f) + i\mathsf{P}(f)}{2}, \qquad \mathsf{A}^+(f) \coloneqq \frac{\mathsf{Q}(f) - i\mathsf{P}(f)}{2}.$$

The operators $\mathsf{A}^\pm$ are called the *field operators* and will play an essential role as they form the creation/annihilation operators on Fock spaces.

Proposition 3.5. *The following are true: For any* $f, g \in \mathcal{H}$,

(i) $\mathfrak{E}$ *is a core for* $\mathsf{P}(f)$ *and* $[\mathsf{P}(f), \mathsf{P}(g)] = 2i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})\mathsf{I}_{\mathfrak{F}}$.

(ii) $\mathsf{A}^-(f)\mathsf{e}(g) = \langle f, g\rangle_{\mathcal{H}}\,\mathsf{e}(g), \quad \mathsf{A}^+(f)\mathsf{e}(g) = \frac{d}{dt}\mathsf{e}(g + tf)|_{t=0}$.

(iii) $\mathsf{W}^\dagger(f)\mathsf{A}^-(g)\mathsf{W}(f) = \mathsf{A}^-(g) + \langle g, f\rangle_{\mathcal{H}}\mathsf{I}_{\mathfrak{F}}, \quad \mathsf{W}^\dagger(f)\mathsf{A}^+(g)\mathsf{W}(f) = \mathsf{A}^+(g) + \langle f, g\rangle_{\mathcal{H}}\mathsf{I}_{\mathfrak{F}}$.

(iv) $[\mathsf{A}^-(f), \mathsf{A}^-(g)] = [\mathsf{A}^+(f), \mathsf{A}^+(g)] = 0, \quad [\mathsf{A}^-(f), \mathsf{A}^+(g)] = \langle f, g\rangle_{\mathcal{H}}\mathsf{I}_{\mathfrak{F}}$,

i.e., the field operators $\mathsf{A}^\pm$ *satisfy the canonical commutation relation.*

Remark 3.6 On the finite particle vectors, the field operators act as

$$\mathsf{A}^-(f)g^{\otimes n} = \sqrt{n}\langle f, g\rangle_{\mathcal{H}} g^{\otimes(n-1)}, \qquad \mathsf{A}^+(f)g^{\otimes(n-1)} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} g^{\otimes k}\otimes f\otimes g^{\otimes(n-1-k)}.$$

Example 3.7 Let $\mathcal{H} = L^2_{\mathbb{C}}([-\pi, \pi])$ be the space of square-integrable functions. Then the countable set $\{\psi_\ell(x) = e^{i\ell x} : \ell \in \mathbb{Z}\}$ forms an orthonormal basis for $\mathcal{H}$.

$$\psi(x) = \sum\nolimits_\ell e^{i\ell x}\mathsf{A}^-_\ell$$

3.7.2 Second quantization

The term *second quantization* is associated with the action of lifting operators on an *k-particle* space $\mathcal{H}^{\otimes k}$ to an associated operator on the Fock space.

We begin our discussion with the 1-*particle* case. For any bounded operator $A \in \mathcal{B}(\mathcal{H})$ one defines the map

$$\Gamma(A) \coloneqq I \oplus \bigoplus_{n\in\mathbb{N}} A^{\otimes n},$$

which acts on $\mathcal{H}^{\otimes n}$ by

$$\Gamma(A)g_1 \otimes \cdots \otimes g_n = Ag_1 \otimes \cdots\cdots \otimes Ag_n.$$

Clearly, if A is unitary, then so is $\Gamma(A)$. Indeed, in this case, one has

$$\langle \Gamma(A)e(g), \Gamma(A)e(g)\rangle_{\mathfrak{F}(\mathcal{H})} = \sum_{n\in\mathbb{N}_0}\frac{1}{n!}\langle Ag, Ag\rangle^n_{\mathcal{H}} = \sum_{n\in\mathbb{N}_0}\frac{1}{n!}\|g\|^n_{\mathcal{H}} = \langle e(g), e(g)\rangle_{\mathfrak{F}(\mathcal{H})}.$$

Now let H be a self-adjoint operator on $\mathcal{H}$ and $U_t \coloneqq \exp(itH)$ be its unitary evolution. Then, $\Gamma(U_t)$ is a one-parameter group of unitary operators on $\mathfrak{F}(\mathcal{H})$. Stone's theorem then provides the existence of a densely defined Hermitian operator $\mathsf{d}\Gamma(H)$ such that

$$\Gamma(U_t) = \exp(it\,\mathsf{d}\Gamma(H)).$$

The generator $\mathsf{d}\Gamma(H)$ is called the *second quantization of* H, and takes the explicit form

$$\mathsf{d}\Gamma(H)g^{(n)} = \sum_{j=1}^{n} g_1 \otimes \cdots \otimes g_{j-1} \otimes Hg_j \otimes g_{j+1} \otimes \cdots \otimes g_n = \sum_{j=1}^{n} H_j g^{(n)},$$

for any $g^{(n)} = g_1 \otimes \cdots \otimes g_n \in \mathcal{H}^{\otimes n}$ with $g_j \in D(H)$ and

$$H_j = I_{\mathcal{H}} \otimes \cdots \otimes I_{\mathcal{H}} \otimes H \otimes I_{\mathcal{H}} \otimes \cdots \otimes I_{\mathcal{H}},$$

where H acts on the j-th tensor product. The special case $H = I_{\mathcal{H}}$ yields

$$\mathsf{N} g^{(n)} := \mathsf{d}\Gamma(I_{\mathcal{H}}) g^{(n)} = n g^{(n)}, \qquad n \in \mathbb{N},$$

and is called the *number operator* due to its diagonal nature, with eigenvalues representing the number of particles in each configuration. Its domain is given by

$$D(\mathsf{N}) = \Big\{ \{f^{(n)}\}_{n \in \mathbb{N}_0} : \sum_{n \in \mathbb{N}_0} n^2 \|f^{(n)}\|^2_{\mathcal{H}^{\otimes n}} < +\infty \Big\}.$$

$$\mathsf{A}_j^- \mathsf{A}_j^+ \psi_j^{\otimes n} = \sqrt{n}\, \mathsf{A}_j^- \psi_j^{\otimes (n-1)} = \sum_{k=0}^{n-1} \psi_j^{\otimes k} \otimes \psi_j \otimes \psi_j^{\otimes (n-1-k)} = n \psi_j^{\otimes n}$$

$$\mathsf{A}_j^+ \mathsf{A}_j^- \mathsf{e}(\psi_k) = \delta_{jk} \mathsf{N}_j\, \mathsf{e}(\psi_j)$$

This construction can be performed similarly for the general k-particle case.

$$\mathsf{d}\Gamma(H^{(k)}) f^{(n)} = \sum_{j_1 \neq \cdots \neq j_k} H_{j_1 \cdots j_k} f^{(n)},$$

where $H_{j_1,\dots,j_k}$ denotes the operator where $H^{(k)}$ acts on the $(j_1, \dots, j_k)$-th tensor product.

Example 3.8 Consider the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{T})$ and the Hamiltonians,

$$H^{(1)} = -\Delta \in \mathcal{O}(\mathcal{H}), \qquad H^{(2)} = \mathsf{M}_W \in \mathcal{O}(\mathcal{H}^{\otimes 2}),$$

where $W : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is an interaction potential, and M_f denotes the multiplication operator corresponding to f.

Then, their second quantization is given by

$$\mathsf{d}\Gamma(H^{(1)}) = \sum_{j=1}^{n} H_j^{(1)}, \qquad \mathsf{d}\Gamma(H^{(2)}) = \sum_{j \neq \ell} H_{j\ell}^{(2)} \qquad \text{on } \mathcal{H}^{\otimes n}.$$

$$\mathsf{d}\Gamma(H^{(1)}) \psi_j^{\otimes n} = n j^2 \psi_j^{\otimes n} = j^2 \mathsf{A}_j^+ \mathsf{A}_j^- \psi_j^{\otimes n} = j^2 \mathsf{A}_j^+ \mathsf{A}_j^- \psi_j^{\otimes n}$$

$$\psi^{(n)} = \psi_{k_1} \otimes \cdots \otimes \psi_{k_n}$$

$$\mathsf{d}\Gamma(H^{(1)}) \psi^{(n)} = \sum_{j=1}^{n} k_j^2\, \psi^{(n)}$$

3.7.3 Free field operators

For each $f \in \mathcal{H}$, we consider the pair $\{a^-(f), a^+(f)\}$ of operators on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ defined by

$$a^-(f)\Psi =$$

satisfying the commutation relations: For any $f, g \in \mathcal{H}$,

$$[a^\pm(f), a^\pm(g)] = 0, \qquad [a^-(f), a^+(g)] = \langle f, g\rangle,$$

where $f \mapsto a^-(f)$ is conjugate linear and $f \mapsto a^+(f)$ is linear. Moreover, if $\Omega \in \mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is the *vacuum vector*, then $a^-(f)\Omega = 0$ for every $f \in \mathcal{H}$. The *field operators* a^- and a^+ are called *creation* and *annihilation* operators, respectively. On appropriate domains, the field operators are adjoints of one another, i.e., $(a^-(f))^\dagger = a^+(f)$ for every $f \in \mathcal{H}$.

It is common in the physics literature to consider operator-valued distributions $\{a_x^-, a_x^+\}$ instead, where if $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^d)$, then $x \in \mathbb{R}^d$ and

$$a^-(f) = \int_{\mathbb{R}^d} \overline{f(x)}\, a_x \,\mathrm{d}x, \qquad a^+(f) = \int_{\mathbb{R}^d} f(x)\, a_x \,\mathrm{d}x.$$

The commutation relations then simply read

$$[a_x^\pm, a_y^\pm] = 0, \qquad [a_x^-, a_y^+] = \delta(x-y).$$

The *number operator* is formally defined by $N_x = a_x^+ a_x$, $x \in \mathbb{R}^d$.

On the rigorous not, if $\mathcal{H}$ is separable with orthonormal basis $\{\psi_i\}$, then one obtains a family of field operators $\{a_i^-, a_i^+\}$ with $a_i^\pm : a^\pm(\psi_i)$.

3.7.4 Gaussian states

Definition 3.9 (Gaussian states) A state Ψ on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is said to be a mean-zero *Gaussian* (or *quasi-free*) state if it can be uniquely determined from the field operators $\{a, a^\dagger\}$ by its covariance

$$\Sigma_\Psi(f,g) := \begin{pmatrix} \Psi(a^+(f)\, a^-(g)) & \Psi(a^-(f)\, a^-(g)) \\ \Psi(a^+(f)\, a^+(g)) & \Psi(a^-(f)\, a^+(g)) \end{pmatrix}, \qquad f, g \in \mathcal{H}.$$

If the off-diagonal elements of the covariance are zero, the state Ψ is called *gauge-invariant* since it is invariant under the so-called gauge transformations of the first kind, i.e.,

$$a^\pm(f) = e^{\pm i\alpha} a^\pm(f) \qquad \text{for any } \alpha \in \mathbb{R}.$$

If the off-diagonal elements of the covariance are nonzero, the state Ψ is called *squeezed.*

3.7.5 States invariant under free evolutions

Consider a Hamiltonian H on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ with $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^d)$ and its associated 1-parameter automorphism group

$$\mathfrak{u}_t(a) = e^{itH} a e^{-itH}, \qquad t \in \mathbb{R}.$$

Definition 3.10 The Hamiltonian H on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is called *free* if there exists a real-valued function $\omega\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathfrak{u}_t(a_x^-) = e^{-it\omega(x)} a_x^-, \qquad x \in \mathbb{R}^d.$$

In this case, the function ω is called the *free 1-particle Hamiltonian*, and H is said to be the *second quantization* of ω. Accordingly, $\mathfrak{u}_t$ is called a *free evolution.*

Example 3.11 Consider a Hamiltonian on $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R})$ and its eigensystem $\{(\lambda_i, \psi_i)\}_i$ such that $\{\psi_i\}_i$ forms an orthonormal basis for $\mathcal{H}$. Setting $a_i^\pm = a^\pm(\psi_i)$, we see that

$$[a_i^\pm, a_j^\pm] = 0, \qquad [a_i^-, a_j^+] = \delta_{ij},$$

i.e., $a_i^\pm$ define field operators on the symmetric Fock space $\mathfrak{F}_{\mathsf{sym}}(\mathcal{K})$ with $\mathcal{K} = \ell^2_{\mathbb{C}}$

Defining the field operators

$$a^- = \frac{1}{\sqrt{2}}\big(Q + iP\big), \qquad a^+ = \frac{1}{\sqrt{2}}\big(Q - iP\big), \qquad N^+a^-$$

such that $[a^-, a^+] = I$, we find that

$$H = a^+a^- + \frac{1}{2} = N + \frac{1}{2}.$$

Clearly, the eigenvectors of N and H coincide

Definition 3.12 A Gaussian state Ψ on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ with $\mathcal{H} = L^2(\mathbb{R}^d)$ is said to be invariant under a free evolution $\mathfrak{u}_t$ if

$$\Psi(\mathfrak{u}_t(a_x^-)\, \mathfrak{u}_s(a_y^+)) = \Psi(\mathfrak{u}_{t-s}(a_x^-)\, a_y^+) = \Psi(a_x^-\, \mathfrak{u}_{s-t}(a_y^+)) \qquad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^d.$$

A *Gaussian free state* is a Gaussian state that is invariant under *all* free evolutions.

Theorem 3.13. *A Gaussian state Ψ on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is a Gaussian free state if and only if it is gauge-invariant and its diagonal correlations are supported on the diagonal, i.e.,*

$$\Psi(a_x^-\, a_y^+) = m(x)\delta(x-y), \qquad \Psi(a_x^+\, a_y^-) = n(x)\delta(x-y).$$

Theorem 3.14. *The field operators $\{a_x^+, a_x^-\}$ satisfying the commutation relations*

$$[a_x^\pm, a_y^\pm] = 0, \qquad [a_x^-, a_y^+] = m(x)\delta(x-y).$$

are mean-zero Gaussian random variables w.r.t. the Fock vacuum state $\Psi_\Omega = \langle \Omega, \cdot\, \Omega\rangle$, where $\Omega \in \mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is the Fock vacuum vector, with covariance

$$\Sigma_\Psi(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & m(x) \end{pmatrix} \delta(x-y), \qquad x, y \in \mathbb{R}^d.$$

Conversely, if $\{a_x^+, a_x^-\}$ are random variables with these properties, then they satisfy the commutation relations above.

3.7.6 Boson Fock white noise

def:boson-white-noise

Definition 3.15 A boson Fock white noise on $\mathbb{R}^d$ is a boson Fock field $\{b^+_{t,x}, b^-_{t,x}\}$ on $\mathbb{R}^{d+1}$ with vacuum vector Ω satisfying the commutation relations

$$[b^{\pm}_{t,x}, b^{\pm}_{s,y}] = 0, \qquad [b^-_{t,x}, b^+_{s,y}] = \delta(t-s)m(x)\delta(x-y), \qquad b^-_{t,x}\Omega = 0.$$

thm:stochastic-limit

Theorem 3.16. *Let $\{a^+_x, a^-_x\}$ be Gaussian free fields w.r.t. the Fock vacuum state Ψ_Ω with*

$$\mathfrak{u}_t(a^-_x) = e^{-it\omega(x)}a^-_x.$$

Then the rescaled field operators

$$b^{\lambda,\pm}_{t,x} \coloneqq \frac{1}{\lambda}\mathfrak{u}_{t/\lambda^2}(a^{\pm}_x)$$

converges in the sense of correlator distributions to a boson Fock white noise, i.e.,

$$\lim_{\lambda\to 0}\Psi(b^{\lambda,\varepsilon_1}_{t,x} b^{\lambda,\varepsilon_2}_{t,y}) = \Psi(b^{\varepsilon_1}_{t,x} b^{\varepsilon_2}_{t,y}) \qquad \varepsilon_1, \varepsilon_2 \in \{+,-\},$$

where $b^{\pm}_{t,x}$ as defined in Definition 3.15 with $m = 2\pi\delta(\omega)$.

Proof. Using the invariance of Ψ_Ω under free evolutions, we find

$$\begin{aligned}\Psi_\Omega(b^{\lambda,-}_{t,x}\, b^{\lambda,+}_{s,y}) &= \frac{1}{\lambda^2}\Psi_\Omega(\mathfrak{u}_{(t-s)/\lambda^2}(a^-_x)\, a^+_y) \\ &= \frac{1}{\lambda^2}e^{-i\omega(x)(t-s)/\lambda^2}\,\Psi_\Omega(a^-_x\, a^+_y) = \frac{1}{\lambda^2}e^{-i\omega(x)(t-s)/\lambda^2}\delta(x-y).\end{aligned}$$

Passing to the limit $\lambda \to 0$ recovers the desired limit. All the other terms vanish. □

Remark 3.17 Let $\mathcal{K} \subset L^2(\mathbb{R}^d)$ be a set of functions for which

$$\int_{\mathbb{R}} |\langle f, e^{it\omega} g\rangle|\, \mathrm{d}t < +\infty \qquad \forall f, g \in \mathcal{K}.$$

Since $t \mapsto \langle f, e^{it\omega} f\rangle$ is positive definite for each $f \in \mathcal{K}$, Bochner's theorem implies that the sesquilinear form

$$\langle f, 2\pi\delta(\omega)g\rangle \coloneqq \int_{\mathbb{R}} \langle f, e^{it\omega} g\rangle\, \mathrm{d}t,$$

is a pre-scalar product. With $(\cdot|\cdot)$, the set $\mathcal{K}$ becomes a pre-Hilbert space, which can be completed to obtain a Hilbert space, still denoted by $\mathcal{K}$. The function $m = 2\pi\delta(\omega)$ has to be understood in this sense, and only makes sense for functions in $\mathcal{K}$.

The operators

$$B^-_t(f) \coloneqq \int_0^t\int_{\mathbb{R}^d} \overline{f(x)}\, b^-_{s,x}\, \mathrm{d}x\, \mathrm{d}s, \qquad B^+_t(f) \coloneqq \int_0^t\int_{\mathbb{R}^d} f(x)\, b^+_{s,x}\, \mathrm{d}x\, \mathrm{d}s,$$

define *quantum Brownian motions.* The self-adjoint *(momentum)* operators

$$P_t(f) := \frac{1}{i}\big[B_t^-(f) - B_t^+(f)\big],$$

form a commuting family of classical random variables whose statistics in the Fock vacuum state $\Psi_\Omega = \langle \Omega, \cdot\, \Omega\rangle$ is completely determined by the relation

$$\Psi(e^{iP_t(f)}) = \exp\left(-\frac{t}{2}\|f\|^2_{L^2(\mathbb{R}^d)}\right). \qquad \text{check!}$$

3.7.7 Gaussian equilibrium states: The KMS condition

For any states $a, b \in \mathcal{A}$, the map $t \mapsto \Psi(a\, \mathfrak{u}_t(b))$ can be analytically continued and satisfies the so-called *KMS condition* at inverse temperature $\beta > 0$:

$$\Psi(a\, \mathfrak{u}_{t+i\beta}(b)) = \Psi(\mathfrak{u}_t(a)\, b) \qquad \forall\, a, b \in \mathcal{A}. \qquad \text{check!}$$

3.8 Composite systems

Definition 3.18 A composite system of two given quantum dynamical systems $S = \{\mathcal{H}_S, H_S\}$, $R = \{\mathcal{H}_R, H_R\}$ is a quantum dynamical system of the form

$$\{\mathcal{H}_S \otimes \mathcal{H}_R, H_{SR}\}, \qquad H_{SR} = H_S \otimes 1_R + 1_S \otimes H_R + H_I,$$

where H_I is called the *interaction Hamiltonian* and contains all the new physics of the composite system, while $H_0 := H_S \otimes 1_R + 1_S \otimes H_R$ is called the free Hamiltonian.

We will consider *scaled* total Hamiltonians

$$H^\lambda := H_0 + \lambda H_I,$$

and the following unitary evolutions:

$$\textit{free evolution} \quad V_t^0 = e^{-itH_0}, \qquad \textit{total evolution} \quad V_t^\lambda = e^{-itH^\lambda},$$
$$\textit{interacting representation evolution} \quad U_t^\lambda = (V_t^0)^\dagger V_t^\lambda,$$

where U_t^λ satisfies the Schrödinger equation in the interaction picture:

$$\partial_t U_t^\lambda = -i\lambda H_I(t) U_t^\lambda, \qquad U_0^\lambda = I,$$

with the time dependent Hamiltonian $H_I(t) = (V_t^0)^\dagger H_I V_t^0$.

For simplicity, we will make the following assumptions on system $S = \{\mathcal{H}_S, H_S\}$, the reservoir $R = \{\mathcal{H}_R, H_R\}$, and the interaction Hamiltonian H_I.

3.8.1 The reservoir

The reservoir $R = \{\mathcal{H}_R, H_R\}$ is given by the Hilbert space $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ with $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^d)$, a free Hamiltonian H_R with continuous spectrum $\mathbb{R}$, and a mean-zero Gaussian free state Ψ_R such that

$$\int_{\mathbb{R}} |\Psi_R(a_x^{\varepsilon_1}\, \mathfrak{u}_t(a_y^{\varepsilon_2}))|\, \mathrm{d}t < +\infty, \qquad \varepsilon_1, \varepsilon_2 \in \{+, -\}.$$

In particular, Ψ_R is characterized by the covariances

$$\Psi_R(a_x^- a_y^+) = m(x)\delta(x-y), \qquad \Psi_R(a_x^+ a_y^-) = n(x)\delta(x-y).$$

Since H_R is free, there exists a function ω, for which

$$\mathfrak{u}_t(a_x^-) = e^{itH_R} a_x^- e^{-itH_R} = e^{-it\omega(x)} a_x^-,$$

where ω describes the 1-particle evolution.

An example of a free reservoir Hamiltonian is given by

$$H_R = \int_{\mathbb{R}^d} \omega(x)\, a_x^+ a_x^-\, \mathrm{d}x,$$

where ω is a smooth cutoff function.

3.8.2 The system Hamiltonian

For simplicity, we shall assume that the system Hamiltonian H_S has a discrete spectrum such that

$$H_S = \sum_j \lambda_j P_j,$$

where λ_j are the eigenvalues and P_j are their corresponding spectral projections.

3.8.3 The interaction Hamiltonian

We consider dipole-type interaction Hamiltonians of the form

$$H_I = \int_{\mathbb{R}^d} \left[D(x) \otimes a_x^+ + D^\dagger(x) \otimes a_x^- \right] \mathrm{d}x,$$

where $\{D(x) : x \in \mathbb{R}^d\}$ is a family of system operators called the *response terms*.

With the spectral projections of H_S, we may express H_I as

$$H_I = \sum_{j,\,k} \int_{\mathbb{R}^s} \left[P_j D(x) P_k \otimes a_x^+ + P_k D^\dagger(x) P_j \otimes a_x^- \right] \mathrm{d}x,$$

and hence, the time-dependent Hamiltonian reads

$$
\begin{aligned}
H_I(t) &= \sum_{j,k} \int_{\mathbb{R}^d} \left[P_j D(x) P_k \otimes e^{it(\omega(x)+\lambda_j-\lambda_k)} a_x^+ + P_k D^\dagger(x) P_j \otimes e^{-it(\omega(x)+\lambda_j-\lambda_k)} a_x^- \right] \mathrm{d}x \\
&= \sum_q \sum_{\lambda_k-\lambda_j=\eta_q} \int_{\mathbb{R}^d} \left[P_j D(x) P_k \otimes e^{it(\omega(x)-\eta_q)} a_x^+ + P_k D^\dagger(x) P_j \otimes e^{-it(\omega(x)-\eta_q)} a_x^- \right] \mathrm{d}x \\
&= \sum_q \int_{\mathbb{R}^d} \left[D_q(x) \otimes e^{it(\omega(x)-\eta_q)} a_x^+ + D_q^\dagger(x) \otimes e^{-it(\omega(x)-\eta_q)} a_x^- \right] \mathrm{d}x,
\end{aligned}
$$

where the system operators

$$
D_q(x) := \sum_{\lambda_k-\lambda_j=\eta_q} P_j D(x) P_k \qquad \text{satisfy} \quad e^{itH_S} D_q(x) e^{-itH_S} = e^{-it\eta_q} D_q(x).
$$

To simplify things drastically, we assume that $q = 1$ and $D_1(x) = \chi(x) D$ for some smooth cutoff function χ and a fixed system operator D. In this case, we obtain

$$
H_I(t) = \int_{\mathbb{R}^d} \left[D \otimes \chi(x) e^{it(\omega(x)-\eta)} a_x^+ + D^\dagger \otimes \overline{\chi(x)} e^{-it(\omega(x)-\eta)} a_x^- \right] \mathrm{d}x.
$$

3.9 The weak interaction stochastic limit

Altogether, we arrive at the rescaled Schrödinger equation in the interaction picture

$$
U^\lambda_{t/\lambda^2} = I - i \int_0^t H_I^\lambda(s)\, U^\lambda_{s/\lambda^2}\, \mathrm{d}s, \qquad \texttt{eq:rescaled-schrodinger-interaction} \tag{3.2}
$$

with $H_I^\lambda(t) = D \otimes b_t^{\lambda,+} + D^\dagger \otimes b_t^{\lambda,-}$, where, due to Theorem 3.16,

$$
b_t^{\lambda,\pm} = \frac{1}{\lambda} a^\pm\big(\chi e^{i(t/\lambda^2)(\omega-\eta)}\big) \longrightarrow b_t^\pm \qquad \text{in the sense of correlators,}
$$

and therefore,

$$
H_I^\lambda(t) \longrightarrow H_t = D \otimes b_t^+ + D^\dagger \otimes b_t^-, \qquad U^\lambda_{t/\lambda^2} \longrightarrow U_t,
$$

where U_t satisfies the formal stochastic differential equation

$$
\mathrm{d}U_t = -i(D \otimes \mathrm{d}B_t^+ + D^\dagger \otimes \mathrm{d}B_t^-) U_t, \qquad B_t^\pm = \int_0^t b_s^\pm \,\mathrm{d}s. \qquad \texttt{eq:sde} \tag{3.3}
$$

Under certain assumptions on ω, this SDE has a unique solution.

The SDE (3.3) in natural-time order (or Itô form) is the SDE given by

$$
\mathrm{d}U_t = -i(D\,\mathrm{d}B_t^+ U_t + D^\dagger U_t\,\mathrm{d}B_t^-) - \gamma_- D^\dagger D U_t\,\mathrm{d}t, \qquad \texttt{eq:sde-normal} \tag{3.4}
$$

obtained by commuting $\mathrm{d}B_t^- U_t$ and using the fact that the solution U_t to (3.3) satisfies

$$\begin{aligned} [b_t^-, U_t] &= -i\gamma_- D U_t, \\ [U_t^\dagger, b_t^+] &= i\overline{\gamma}_- U_t^\dagger D^\dagger, \\ [b_t^-, U_t^\dagger] &= i\gamma_- U_t^\dagger D, \end{aligned} \qquad \text{where} \quad \gamma_- := \int_{-\infty}^0 \langle \chi, e^{-it(\omega-\eta)}\chi\rangle\,\mathrm{d}t.$$

The additional term is known as the Itô correction term in quantum stochastic calculus. Morally, the natural-time order is the order induced by the filtration for which one expects U_t to be adapted to. Expand on the concept of normal order!

With the unitary evolution U_t one can derive an evolution for any system observable $X_t = U_t^\dagger (X \otimes 1_{\mathcal{H}_R}) U_t$ given by

$$\mathrm{d}X_t = -i\,\mathrm{d}B_t^+ [D, X_t] + i[X_t, D^\dagger]\,\mathrm{d}B_t^- + LX_t\,\mathrm{d}t$$

where

$$LX = 2\Re(\gamma_-) D^\dagger X D - \gamma_- D^\dagger D X - \overline{\gamma}_- X D^\dagger D,$$

is the corresponding Lindblad operator.

Remark 3.19 The rescaled equation (3.2) may be formally expressed as

$$U_{t/\lambda^2}^\lambda = I + \sum_{n=1}^\infty (-i)^n \lambda^n \int_0^{t/\lambda^2} \cdots \int_0^{t_{n-1}} H_I(t_1)\cdots H_I(t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

In the case when $H_I(t)$ commutes for every $t \geq 0$, the integrand would be a symmetric function of $t_1, \ldots, t_n$, and gives

$$U_{t/\lambda^2}^\lambda = I + \sum_{n=1}^\infty (-i)^n \frac{\lambda^n}{n!} \left(\int_0^{t/\lambda^2} H_I(s)\,\mathrm{d}s \right)^n = \exp\left(-i\lambda \int_0^{t/\lambda^2} H_I(s)\,\mathrm{d}s \right),$$

i.e., the expectation of U_{t/λ^2}^λ is the characteristic function of the process

$$W_t^\lambda := \lambda \int_0^{t/\lambda^2} H_I(s)\,\mathrm{d}s.$$

3.10 Damped harmonic oscillator

We consider a simple setup in which a single atom interacts with an electromagnetic field. The total system is given by the composite of the atom and the reservoir systems

$$S = \{\mathfrak{F}_{\mathsf{sym}}(\mathbb{C}^2), H_S\}, \qquad R = \{\mathfrak{F}_{\mathsf{sym}}(L^2(\mathbb{R}^d)), H_R\},$$

with the free Hamiltonians

$$H_S = \omega_0\, c^+ c^-, \qquad H_R = \int_{\mathbb{R}^d} \omega(x)\, a_x^+ a_x^-\,\mathrm{d}x,$$

where $\{c^+, c^-\}$ is the field operator for $\mathcal{H}_S = \mathfrak{F}_{\mathsf{sym}}(\mathbb{C}^2)$, $\{a_x^+, a_x^-\}$ are the field operators for $\mathcal{H}_R = \mathfrak{F}_{\mathsf{sym}}(L^2(\mathbb{R}^d))$, and ω is a suitable cutoff function.

For the interaction Hamiltonian H_I, we consider a dipole approximation of the form

$$H_I = \int_{\mathbb{R}^d} \chi(x) \big[c^- \otimes a_x^+ + c^+ \otimes a_x\big] \,\mathrm{d}x = c^- \otimes A^+ + c^+ \otimes A^-,$$

where χ are suitable cutoff function and

$$A^\pm = \int_{\mathbb{R}^d} \chi(x)\, a_x^\pm \,\mathrm{d}x.$$

such that the rescaled total Hamiltonian is given by

$$H^\lambda = H_S \otimes 1_{\mathcal{H}_R} + 1_{\mathcal{H}_S} \otimes H_R + \lambda H_I.$$

Define the evolution

$$\mathfrak{u}_t^\lambda(a) = e^{itH^\lambda} a e^{-itH^\lambda}.$$

Then, the Heisenberg equation for $c(t) = \mathfrak{u}_t^\lambda(c \otimes 1_{\mathcal{H}_R})$ and $a_x^-(t) = \mathfrak{u}_t^\lambda(1_{\mathcal{H}_S} \otimes a_x)$ reads

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} c^-(t) &= -i\omega_0\, c^-(t) - i\lambda \int_{\mathbb{R}^d} \chi(x)\, a_x^-(t) \,\mathrm{d}x \\
\frac{\mathrm{d}}{\mathrm{d}t} a_x^-(t) &= -i\omega(x)\, a_x^-(t) - \lambda\, \chi(x)\, c(t).
\end{aligned}$$

Solving for $a_x^-(t)$, one obtains

$$a_x^-(t) = a_x^- e^{-it\omega(x)} - i\lambda\chi(x) \int_0^t e^{-i(t-s)\omega(x)} c^-(s) \,\mathrm{d}s.$$

Inserting $a_x^-(t)$ into the equation for $c(t)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} c^-(t) = -i\omega_0 c^-(t) - \int_0^t \gamma(t-s)\, c^-(s) \,\mathrm{d}s - i\,\xi_t^-.$$

where we defined the quantities

$$\gamma(r) := \lambda^2 \int_{\mathbb{R}^d} e^{-ir\omega(x)} \chi^2(x) \,\mathrm{d}x, \qquad \xi_t^- := \lambda \int_{\mathbb{R}^d} e^{-it\omega(x)} \chi(x)\, a_x^- \,\mathrm{d}x.$$

Observe that ξ_t^- depends only on the reservoir field operator a_x^-, and therefore, acts as an *external force* to the atomic system.

Now let us consider the statistics of ξ_t^- for the Fock vacuum state free state $\Psi_\Omega = \langle \Omega, \cdot\, \Omega \rangle$ on $\mathcal{H}_R$. Since Ψ_Ω is a Gaussian free state, the statistics of ξ_t^- are uniquely determined by the 2-points correlation functions. Clearly, Ψ_Ω has zero mean, and therefore,

$$\Psi_\Omega(\xi_t^-) = \lambda \int_{\mathbb{R}^d} e^{-it\omega(x)} \chi(x)\, \Psi_\Omega(a_x^-) \,\mathrm{d}x = 0, \qquad \Psi_\Omega((\xi_t^-)^\dagger) = 0.$$

Moreover, we find

$$\Psi_\Omega((\xi_t^-)^\dagger\,\xi_s^-) = \Psi_\Omega(\xi_t^-\,\xi_s^-) = \Psi_\Omega((\xi_t^-)^\dagger\,(\xi_s^-)^\dagger) = 0,$$

and

$$\begin{aligned}\Psi_\Omega(\xi_t^-\,(\xi_s^-)^\dagger) &= \lambda^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-it(\omega(x)-\omega(y))}\chi(x)\chi(y)\,\Psi_\Omega(a_x^-a_y^+)\,\mathrm{d}x\,\mathrm{d}y\\ &= \lambda^2\int_{\mathbb{R}^d} e^{-i(t-s)\omega(x)}\chi^2(x)\,\mathrm{d}x = \gamma(t-s).\end{aligned}$$

Hence, ξ_t^- is a mean-zero γ-correlated *random process* under Ψ.

To obtain the stochastic limit, we start by rescaling time $t \mapsto t/\lambda^2$ and consider the rescaled quantity $c^{\lambda,\pm}(t) \coloneqq c^\pm(t/\lambda^2)$, which yields

$$\dot{c}^{\lambda,-}(t) = -i\omega_0 c^{\lambda,-}(t) - \int_0^t \gamma^\lambda(t-s)\,c^{\lambda,-}(s)\,\mathrm{d}s - i\,\xi_t^{\lambda,-}, \qquad \text{eq:quantum-pre-langevin} \quad (3.5)$$

with

$$\gamma^\lambda(r) \coloneqq \frac{1}{\lambda^2}\langle\chi, e^{-i(r/\lambda^2)\omega}\chi\rangle, \qquad \xi_t^{\lambda,-} \coloneqq \lambda^{-2}\xi_{t/\lambda^2}^-.$$

From Theorem 3.16, we then establish that

$$\gamma^\lambda(r) \longrightarrow \gamma\,\delta(r), \qquad \xi_t^{\lambda,-} \longrightarrow b_t^- \qquad \text{in the sense of correlators,}$$

with $\gamma = \langle\chi, \delta(\omega)\chi\rangle$. Consequently, we obtain

$$\mathrm{d}c_t^- = -(i\omega_0 + \gamma)\,c_t^-\,\mathrm{d}t - i\,\mathrm{d}B_t^-, \qquad B_t^- = \int_0^t b_s^-\,\mathrm{d}s.$$

$$\mathrm{d}U_t = -i(c^-\,\mathrm{d}B_t^+U_t + c^-U_t\,\mathrm{d}B_t^-) - \gamma_- c^+c^-U_t\,\mathrm{d}t$$

$$\begin{aligned}\mathrm{d}(U_t^\dagger c^- U_t) &= i(c^-\,\mathrm{d}B_t^+U_t + c^+U_t\,\mathrm{d}B_t^-)^\dagger c^-U_t - \overline{\gamma}_- U_t^\dagger c^-c^+c^-U_t\,\mathrm{d}t\\ &\quad - iU_t^\dagger c^-(c^-\,\mathrm{d}B_t^+U_t + c^+U_t\,\mathrm{d}B_t^-) - \gamma_- U_t^\dagger c^-c^+c^-U_t\,\mathrm{d}t\\ &= iU_t^\dagger\,\mathrm{d}B_t^-c^+c^-U_t + i\,\mathrm{d}B_t^+U_t^\dagger c^-c^-U_t - \overline{\gamma}_- U_t^\dagger c^-c^+c^-U_t\,\mathrm{d}t\end{aligned}$$