

Gradient Structures from Classical to Quantum

Oliver Tse

Eindhoven University of Technology

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Abstract.

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1 Stochastic Schrödinger Equation

As we have seen, a quantum circuit is essentially a noisy controlled Schrödinger equation. In this section, we consider classical noise sources arising from the control mechanism in terms of light-matter interactions. Without going into the details of the physics, we introduce the stochastic Schrödinger equation, which captures the evolution of a quantum system driven by classical noise sources.

Let \mathcal{H} be a Hilbert space and $\mathcal{B}(\mathcal{H})$

$$dU_t = HU_t dt + \sum_j V_j U_t \circ d\alpha_t,$$

where H, V_j are

1.1 Quantum measurements

Let $\varrho \in \mathcal{S}(\mathcal{H})$ be a state on a Hilbert space \mathcal{H} and $t \mapsto U_t \in \mathcal{U}(\mathcal{H})$ be a group of unitary actions on \mathcal{H} . Suppose \mathbf{Q} is an \mathbb{R} -valued observable with spectral measure $\pi^{\mathbf{Q}} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{P}(\mathcal{H})$, i.e., $\pi^{\mathbf{Q}}$ is a projection-valued measure (PVM). Then the probability of measuring an event $A \in \mathcal{B}(\mathbb{R})$ at time $t^- > 0$ is given by Born's rule:

$$\mathbb{P}_{\varrho}(\lambda_{t^-}^{\mathbf{Q}} \in A) = \text{tr}[\pi^{\mathbf{Q}}(A)\varrho_{t^-}] = \text{tr}[U_t^{\dagger}\pi^{\mathbf{Q}}(A)U_t\varrho].$$

As measurements generally change the state of the system, Lüders' rule postulates that, *conditioned* on the event $A \in \mathcal{B}(\mathbb{R})$, the state “collapses” to

$$\varrho_{t^+} = \mathcal{S}(\pi^{\mathbf{Q}}(A), \varrho_{t^-}), \quad \mathcal{S}(\pi, \varrho) := \frac{\pi\varrho\pi^{\dagger}}{\text{tr}[\pi\varrho\pi^{\dagger}]} \in \mathcal{S}(\mathcal{H}).$$

Notice that by definition, $\mathcal{S}(\pi, U\varrho U^{\dagger}) = \mathcal{S}(\pi U, \varrho)$ and $U\mathcal{S}(\pi, \varrho)U^{\dagger} = \mathcal{S}(U\pi, \varrho)$.

The probability of measuring the event $B \in \mathcal{B}(\mathbb{R})$ at time $t^+ > 0$ given that the event $A \in \mathcal{B}(\mathbb{R})$ was measured at time $t > 0$ is

$$\mathbb{P}_{\varrho}(\lambda_{t^+}^{\mathbf{Q}} \in B \mid \lambda_{t^-}^{\mathbf{Q}} \in A) = \text{tr}[\pi^{\mathbf{Q}}(B)\mathcal{S}(\pi^{\mathbf{Q}}(A), \varrho_{t^-})] = \frac{\text{tr}[\pi^{\mathbf{Q}}(A \cap B)\varrho_{t^-}]}{\text{tr}[\pi^{\mathbf{Q}}(A)\varrho_{t^-}]}.$$

In particular, the conditional probability of measuring event $A \in \mathcal{B}(\mathbb{R})$ at time $t^+ > 0$ given it was measured at time $t^- > 0$ is 1. If \mathbf{R} is another \mathbb{R} -valued observable, then

$$\mathbb{P}_{\varrho}(\lambda_{t^+}^{\mathbf{R}} \in B \mid \lambda_{t^-}^{\mathbf{Q}} \in A) = \text{tr}[\pi^{\mathbf{R}}(B)\varrho_{t^+}] = \frac{\text{tr}[\pi^{\mathbf{R}}(B)\pi^{\mathbf{Q}}(A)\varrho_{t^-}\pi^{\mathbf{Q}}(A)\pi^{\mathbf{R}}(B)]}{\text{tr}[\pi^{\mathbf{Q}}(A)\varrho_{t^-}]}.$$

When \mathbf{R} and \mathbf{Q} commute, then $\pi^{\mathbf{R}}(B)\pi^{\mathbf{Q}}(A) = \pi^{\mathbf{Q}}(A)\pi^{\mathbf{R}}(B)$, and hence,

$$\mathbb{P}_{\varrho}(\lambda_{t^+}^{\mathbf{R}} \in B \mid \lambda_{t^-}^{\mathbf{Q}} \in A)\mathbb{P}_{\varrho}(\lambda_{t^-}^{\mathbf{Q}} \in A) = \mathbb{P}_{\varrho}(\lambda_{t^+}^{\mathbf{Q}} \in A \mid \lambda_{t^-}^{\mathbf{R}} \in B)\mathbb{P}_{\varrho}(\lambda_{t^-}^{\mathbf{R}} \in B),$$

i.e., Bayes' theorem holds and the usual notion of joint probability is well defined. Clearly, this does not hold when \mathbf{R} and \mathbf{Q} do not commute. It is this failure of the Bayes theorem that distinguishes quantum probability from classical probability.

In a similar fashion, one determines the probability of measuring the event A_2 at time $t_2^- > 0$ for R given that the event A_1 was measured at time $t_1 > 0$ for Q from

$$\begin{aligned}\mathbb{P}_\varrho(\lambda_{t_2^-}^R \in A_2 \mid \lambda_{t_1^-}^Q \in A_1) &= \text{tr}[\pi^R(A_2)U_{t_2-t_1}\mathcal{S}(\pi^Q(A_1), \varrho_{t_1^-})U_{t_2-t_1}^\dagger] \\ &= \text{tr}[\pi^R(A_2)U_{t_2-t_1}\mathcal{S}(\pi^Q(A_1)U_{t_1}, \varrho)U_{t_2-t_1}^\dagger] \\ &= \text{tr}[\pi^R(A_2)\mathcal{S}(U_{t_2-t_1}\pi^Q(A_1)U_{t_1}, \varrho)] = \text{tr}[\pi^R(A_2)\varrho_{t_2^-}].\end{aligned}$$

As before, upon measuring the event A_2 at time $t_2 > 0$, Lüders' rule implies

$$\varrho_{t_2^+} = \mathcal{S}(\pi^R(A_2), \varrho_{t_2^-}) = \mathcal{S}(\pi^R(A_2)U_{t_2-t_1}\pi^Q(A_1)U_{t_1}, \varrho).$$

Restricting ourselves to a single observable Q ,

2 Noncommutative probability

2.1 From commutative to noncommutative

Consider the Hilbert space $\mathcal{H} = L_{\mathbb{C}}^2(\Omega, \mu)$ over the complete probability space $(\Omega, \mathcal{F}, \mu)$. Then any function $f \in L_{\mathbb{C}}^\infty(\Omega, \mu)$ gives rise to a multiplication operator $M_f \in \mathcal{B}(\mathcal{H})$:

$$M_f g = fg \in \mathcal{H} \quad \forall g \in \mathcal{H},$$

with $\|M_f\|_\infty = \|f\|_{L^\infty(\mu)}$. The collection of all such multiplication operators

$$\mathcal{A} := \{M_f : f \in L_{\mathbb{C}}^\infty(\Omega, \mu)\} \subset \mathcal{B}(\mathcal{H})$$

forms a *commutative* subalgebra of $\mathcal{B}(\mathcal{H})$.

In fact, this subalgebra is a possibly noncommutative *von Neumann algebra*:

Definition 2.1 (von Neumann algebra) A (unital) *von Neumann algebra* is a $*$ -subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ that contains $I_{\mathcal{H}}$ and is closed in the *weak operator topology* (WOT), i.e.,

$$\text{WOT-lim } \mathbf{a}_n = \mathbf{a} \iff \langle f, \mathbf{a}_n g \rangle_{\mathcal{H}} \rightarrow \langle f, \mathbf{a} g \rangle_{\mathcal{H}} \quad \forall f, g \in \mathcal{H}.$$

Equivalently, is it a C^* -algebra with a predual \mathcal{A}_* that is a Banach space.

There are sub-families of \mathcal{A} that play a distinguished role, namely,

$$\begin{aligned}\mathcal{A}_+ &:= \{\mathbf{a} \in \mathcal{A} : \mathbf{a} \succeq 0\} && \text{(nonnegative operators)} \\ \mathcal{O} &:= \{\mathbf{a} \in \mathcal{A} : \mathbf{a}^\dagger = \mathbf{a}\} && \text{(Hermitian operators)} \\ \mathcal{P} &:= \{\mathbf{a} \in \mathcal{O} : \mathbf{a}^2 = \mathbf{a}\} && \text{(projection operators)}\end{aligned}$$

Here, $\mathbf{a} \succeq 0$ if and only if $\langle f, \mathbf{a} f \rangle_{\mathcal{H}} \geq 0$ for all $f \in \mathcal{H}$. Additionally, for von Neumann algebras, we also have the following useful density result:

Proposition 2.2. *Let \mathcal{A} be a von Neumann algebra. Then*

$$\mathcal{A} = \overline{\text{span}(\mathcal{P})}^{\text{WOT}},$$

i.e., \mathcal{A} is the WOT-closure of the linear span of its projections.

Let us return to our previous example with $\mathcal{A} = \{\mathbf{M}_f : f \in L_{\mathbb{C}}^{\infty}(\Omega, \mu)\}$. It is not difficult to see that \mathcal{A} is a W^* -algebra. Notice that since \mathbf{M}_f is self-adjoint for $f \in L_{\mathbb{R}}^{\infty}(\Omega, \mu)$, the family of self-adjoint operators is given by $\mathcal{O} = \{\mathbf{M}_f : f \in L_{\mathbb{R}}^{\infty}(\Omega, \mu)\}$ and the family of projection operators is given by $\mathcal{P} = \{\mathbf{M}_f \in \mathcal{O} : f = \mathbf{1}_A, A \in \mathcal{F}\}$.

On a von Neumann algebra \mathcal{A} , we can talk about special types of continuous linear functionals on \mathcal{A} , called states.

Definition 2.3 A *state* on a von Neumann algebra \mathcal{A} is a linear functional $\psi : \mathcal{A} \rightarrow \mathbb{C}$ that is positive and normalized, i.e., $\psi(\mathbf{a}^{\dagger}\mathbf{a}) \geq 0$ for all $\mathbf{a} \in \mathcal{A}$ and $\psi(I_{\mathcal{B}(\mathcal{H})}) = 1$.

A state ψ is said to be

faithful if $\psi(\mathbf{a}^{\dagger}\mathbf{a}) = 0 \Leftrightarrow \mathbf{a} = 0$,

tracial if $\psi(\mathbf{a}\mathbf{b}) = \psi(\mathbf{b}\mathbf{a})$ for all $\mathbf{a}, \mathbf{b} \in \mathcal{A}$, and

normal if $\psi \in \mathcal{A}_*$, i.e., if it is an element of the predual \mathcal{A}_* .

For any probability measure $\nu \ll \mu$, we set

$$\psi_{\nu}(\mathbf{M}_f) := \int_{\Omega} f \, d\nu, \quad f \in L_{\mathbb{C}}^{\infty}(\Omega, \mu),$$

we find that ψ is a linear functional that is positive and normalized, i.e., ψ is a state. Moreover, it is tracial. It is normal if $\omega := d\nu/d\mu \in L^1(\Omega, \mu)$ and faithful if $\omega > 0$.

Normal states play an essential role, serving as a counterpart to classical measures, as made explicit by the following proposition.

prop:normal-state

Proposition 2.4. *Let ψ be a state on a von Neumann algebra \mathcal{A} . The following are equivalent:*

(i) ψ is a normal state.

(ii) (σ -additivity) If $(\mathbf{a}_n) \subset \mathcal{P}$ are mutually orthogonal projections, i.e., $\mathbf{a}_n(\mathcal{H}) \perp \mathbf{a}_m(\mathcal{H})$ for all $n \neq m$, and $\mathbf{a} = \vee_n \mathbf{a}_n$ being the projection on the smallest closed subspace containing $\cup_n \mathbf{a}_n(\mathcal{H})$, then

$$\psi(\mathbf{a}) = \sum_n \psi(\mathbf{a}_n).$$

(iii) (Continuity from below) For any increasing net $0 \preceq \mathbf{a}_n \uparrow \mathbf{a}$ in \mathcal{A}_+ , one has the increasing limit $\psi(\mathbf{a}_n) \uparrow \psi(\mathbf{a})$.

(iv) There exists a family $\{\xi_n\} \subset \mathcal{H}$ with $\sum_n \|\xi_n\|_{\mathcal{H}}^2 = 1$ such that¹

$$\psi = \sum_n \langle \xi_n, \cdot \xi_n \rangle \quad \text{in the sense of norm convergence.}$$

¹[Theorem 7.1.8, Fundamentals of the Theory of Operator Algebras, V.II, Kadison-Ringrose]

Consider an arbitrary normal state ψ and set

$$\mu(A) := \psi(M_{\mathbf{1}_A}) \quad \text{for every } A \in \mathcal{F}.$$

Then, clearly, $\mu(\emptyset) = 0$, $\mu(\Omega) = \psi(I_{\mathcal{B}(\mathcal{H})}) = 1$ and μ is σ -additive due to the equivalent characterization of a normal state given by Proposition 2.4(ii). In particular, one obtains a classical measure on (Ω, \mathcal{F}) . In this sense, a state on a noncommutative von Neumann algebra generalizes that of a classical measure.

2.2 Observables

In classical probability, we are often interested in computing expressions like $\mathbb{P}(X \in A)$, where X is a random variable and $A \subset \mathbb{R}$ is a Borel set.

In the quantum world, a random variable is modelled by a Hermitian operator $\mathbf{a} \in \mathcal{O}$ and is called an *observable*. Observables have their spectrum in \mathbb{R} and can therefore be *measured*.

Given a ψ on a von Neumann algebra \mathcal{A} , the expectation of an observable \mathbf{a} w.r.t. the state ψ is given by $\psi(\mathbf{a})$. Since every observable $\mathbf{a} \in \mathcal{O}$ has a spectral decomposition

$$\mathbf{a} = \int_{\mathbb{R}} \lambda E_{\mathbf{a}}(d\lambda), \quad E_{\mathbf{a}} \hat{=} \mathcal{P}\text{-valued measure},$$

we can associate a classical probability measure with the observable \mathbf{a} and state ψ :

$$\mathbb{P}_{\psi}(\mathbf{a} \in A) := \psi(E_{\mathbf{a}}(A)) \quad \text{for all Borel set } A \subset \mathbb{R}.$$

For two observables $\mathbf{a}, \mathbf{b} \in \mathcal{O}$ that do not commute, one would like to be able to write $\mathbb{P}_{\psi}(\mathbf{a} \in A, \mathbf{b} \in B)$ for two Borel sets $A, B \subset \mathbb{R}$. The simple argument for this is that if $[\mathbf{a}, \mathbf{b}] \neq 0$, then $E_{\mathbf{a}}(A)$ and $E_{\mathbf{b}}(B)$ may not commute for all Borel sets A, B . In particular,

$$\psi(E_{\mathbf{a}}(A)E_{\mathbf{b}}(B)) \neq \psi(E_{\mathbf{b}}(B)E_{\mathbf{a}}(A)),$$

so there is no consistent way of writing $\mathbb{P}_{\psi}(\mathbf{a} \in A, \mathbf{b} \in B)$. Yet, when they do commute, then $\psi(E_{\mathbf{a}}(A)E_{\mathbf{b}}(B)) = \psi(E_{\mathbf{b}}(B)E_{\mathbf{a}}(A))$ and a joint distribution exists and we are back to the classical scenario.

However, this turns out not to be possible, as the following example portrays.

Example 2.5 (Stern-Gerlach measurements) Take a beam of atoms (each with spin- $\frac{1}{2}$) and perform the following steps:

SG_z Measure spin in z -direction: Send the beam through a Stern-Gerlach magnet oriented along z . The beam splits into spin-up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$ paths. Keep only the spin-up branch, which gives a pure state $|\uparrow_z\rangle$.

SG_x Measure spin in x -direction: Now send the filtered beam through a second magnet, oriented along x . The beam splits again into $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ outcomes with 50%–50% probability. So far, this can *still* be explained with classical probability.

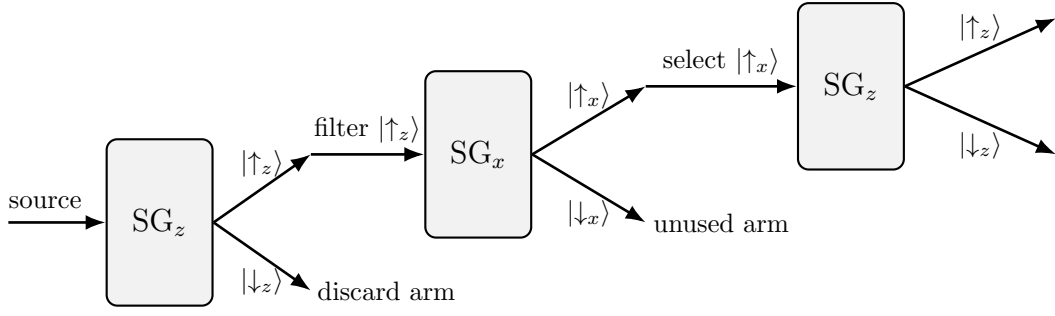


Figure 1: Stern-Gerlach experiment

SG_z Measure spin in z -direction again: Send either of the filtered beams $|\uparrow_x\rangle$ or $|\downarrow_x\rangle$ through a z -magnet again. You do *not* get back the original result, i.e., instead of remaining spin-up, you see another 50%–50% split between $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$

Morally, if spin- x and spin- z were classical random variables, measuring x would *erase* knowledge of z . In classical probability, observing one property never randomizes another unless there is *hidden causal disturbance*.

Quantum mechanically, the two observables (or random variables)

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{spin in } z\text{-direction}, \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{spin in } x\text{-direction}.$$

do not commute, i.e., $[\sigma_x, \sigma_y] \neq 0$, and measuring σ_x *changes* information about σ_z . In this sense, they cannot be jointly sampled, i.e., there is no classical joint probability $p(\sigma_x, \sigma_z)$ for these observables, which illustrates the need for noncommutative probability.

3 Stochastic Dilation

3.1 Noncommutative random variables

def:random-variable

Definition 3.1 A *noncommutative \mathfrak{X} -valued random variable* on a quantum probability space (\mathfrak{A}, μ) is an identity preserving $*$ -homomorphism

$$\mathfrak{z}: \mathfrak{X} \rightarrow \mathfrak{A}.$$

Correspondingly, a *noncommutative \mathfrak{X} -valued stochastic process* on a quantum probability space (\mathfrak{A}, μ) is a family of random variables

$$\mathfrak{z}_t: \mathfrak{X} \rightarrow \mathfrak{A}, \quad t \in \mathbb{T},$$

with \mathbb{T} being a (possibly uncountable) index set.

Example 3.2 (Classical random variable) Let X be a classical E -valued random variable on (Ω, \mathbb{P}) . Setting

$$\mathfrak{X} := B_b(E), \quad \mathfrak{A} := L^\infty(\Omega, \mathbb{P}), \quad \mu(g) := \int_{\Omega} g \, d\mathbb{P}, \quad g \in \mathfrak{A},$$

we have that

$$\mathfrak{X} \ni f \mapsto \mathfrak{z}(f) := f \circ X \in L^\infty(\Omega, \mathbb{P})$$

is a noncommutative \mathfrak{X} -valued random variable. We see here that the noncommutative notion of a random variable is ‘dual’ to the classical notion of a random variable.

ex:stochastic-classical

Example 3.3 (Classical stochastic process) Consider the space of continuous paths (or trajectories) $\Omega := \mathcal{C}_0(\mathbb{R}_+; E)$ starting at 0 and the Wiener measure \mathbb{R} . Let $X = (X_t)_{t \in \mathbb{T}}$ be the canonical stochastic process $X_t(\omega) = \omega(t)$, $t \in \mathbb{R}_+$. As before, we set

$$\mathfrak{X} := B_b(E), \quad \mathfrak{A} := L^\infty(\Omega, \mathbb{R}), \quad \mu(G) := \int_{\Omega} G \, d\mathbb{R}, \quad G \in \mathfrak{A}.$$

Then,

$$\mathfrak{X} \ni f \mapsto \mathfrak{z}_t(f) := f \circ X_t \in L^\infty(\Omega, \mathbb{R})$$

defines a noncommutative \mathfrak{X} -valued stochastic process on (\mathfrak{A}, μ) .

Definition 3.4 A stochastic process $\mathfrak{z}_t: \mathfrak{X} \rightarrow (\mathfrak{A}, \mu)$, $t \in \mathbb{T}$ admits a *time translation* if there are $*$ -homomorphisms $\alpha_t: \mathfrak{A} \rightarrow \mathfrak{A}$, $t \in \mathbb{T}$ such that

- (1) $\alpha_{t+s} = \alpha_t \circ \alpha_s$ for all $s, t \in \mathbb{T}$, and
- (2) $\mathfrak{z}_t = \alpha_t \circ \mathfrak{z}_0$ for all $t \in \mathbb{T}$.

Example 3.5 Consider the Wiener space (Ω, \mathbb{R}) in Example 3.3. Then, \mathbb{R} is stationary under the time-shift operator $\mathfrak{s}_t \omega = \omega(\cdot - t)$, i.e., $(\mathfrak{s}_t)_\# \mathbb{R} = \mathbb{R}$ for every $t \in \mathbb{T}$. Hence, the $*$ -homomorphism

$$\alpha_t(G) = G \circ \mathfrak{s}_t, \quad G \in \mathfrak{A}, \quad t \in \mathbb{T}$$

is a time translation for the corresponding process $(\mathfrak{z}_t)_{t \in \mathbb{T}}$ and leaves the state μ invariant. Indeed, property (1) holds trivially, while property (2) follows from

$$\alpha_t \circ \mathfrak{z}_0(f) = f \circ X_0 \circ \mathfrak{s}_t = f \circ X_t = \mathfrak{z}_t(f), \quad f \in \mathfrak{X}.$$

Finally, we conclude with

$$\mu(\alpha_t(G)) = \int_{\Omega} \alpha_t(G) dR = \int_{\Omega} G \circ \mathfrak{s}_t dR = \int_{\Omega} G d(\mathfrak{s}_t)_{\#} R = \int_{\Omega} G dR = \mu(G),$$

thus proving the invariance of μ .

For $h \in L^2(\mathbb{T})$ we consider the exponential martingale

$$Z_t = \exp \left(- \int_0^t h_r dB_r - \frac{1}{2} \int_0^t |h_r|^2 dr \right), \quad t \in \mathbb{T}.$$

It is easy to see that $(Z_t)_{t \in \mathbb{T}}$ satisfies

$$dZ_t = -h_t Z_t dB_t, \quad Z_0 = 1.$$

Setting $Q := Z_T P$, we claim that

$$W_t = B_t + \int_0^t h_r dr \quad \text{is a } Q\text{-martingale.}$$

To see this, we note that W is a Q -martingale if and only if $Z_T W$ is a P -martingale. Since

$$\begin{aligned} d(Z_t W_t) &= dZ_t W_t + Z_t dW_t + d[Z, W]_t \\ &= -h_t Z_t dB_t + Z_t dB_t = -(h_t - 1) Z_t dB_t, \end{aligned}$$

we conclude that $Z_T W$ is indeed a P -martingale.

$$dU_t = -(a^* a dt + i\sqrt{2} a dB_t) U_t$$

In the classical case, the notion of adaptedness plays an important role for stochastic processes. In this regard, we choose to simply consider the natural filtration induced by a stochastic process.

Definition 3.6 Let $\mathfrak{z}_t: \mathfrak{X} \rightarrow (\mathfrak{A}, \mu)$, $t \in \mathbb{T}$ be a stochastic process. For $\mathbb{I} \subset \mathbb{T}$, we denote by $\mathfrak{A}_{\mathbb{I}}$ the subalgebra of \mathfrak{A} generated by $\{\mathfrak{z}_t(x) : x \in \mathfrak{X}, t \in \mathbb{I}\}$. The filtration induced by $(\mathfrak{z}_t)_{t \in \mathbb{T}}$ is then given by $\mathfrak{F} = (\mathfrak{A}_{[0, t]})_{t \in \mathbb{T}}$.

3.2 From commutative to noncommutative

Consider the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\Omega, \mu)$ over the complete probability space $(\Omega, \mathcal{F}, \mu)$. Then any function $f \in L^{\infty}_{\mathbb{C}}(\Omega, \mu)$ gives rise to a multiplication operator $M_f \in \mathcal{B}(\mathcal{H})$:

$$M_f g = fg \in \mathcal{H} \quad \forall g \in \mathcal{H},$$

with $\|M_f\|_{\infty} = \|f\|_{L^{\infty}(\mu)}$. The collection of all such multiplication operators

$$\mathcal{A} := \{M_f : f \in L^{\infty}_{\mathbb{C}}(\Omega, \mu)\} \subset \mathcal{B}(\mathcal{H})$$

forms a commutative subalgebra of $\mathcal{B}(\mathcal{H})$.

In fact, this subalgebra is a *von Neumann algebra*:

Definition 3.7 (von Neumann algebra) A (unital) *von Neumann algebra* (or W^* -algebra) is a $*$ -subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ that contains $I_{\mathcal{H}}$ and is closed in the *weak operator topology* (WOT), i.e.,

$$\text{WOT-lim } a_n = a \iff \langle f, a_n g \rangle_{\mathcal{H}} \rightarrow \langle f, a g \rangle_{\mathcal{H}} \quad \forall f, g \in \mathcal{H}.$$

Equivalently, is it a C^* -algebra with a predual \mathcal{A}_* that is a Banach space.

There are sub-families of \mathcal{A} that play a distinguished role, namely,

$$\begin{aligned} \mathcal{A}_+ &:= \{a \in \mathcal{A} : a \succeq 0\} && \text{(nonnegative operators)} \\ \mathcal{O} &:= \{a \in \mathcal{A} : a^* = a\} && \text{(self-adjoint operators)} \\ \mathcal{P} &:= \{a \in \mathcal{O} : a^2 = a\} && \text{(projection operators)} \end{aligned}$$

Here, $a \succeq 0$ if and only if $\langle f, a f \rangle_{\mathcal{H}} \geq 0$ for all $f \in \mathcal{H}$.

On a von Neumann algebra \mathcal{A} , we can talk about special types of continuous linear functionals on \mathcal{A} , called states.

Definition 3.8 A *state* on a von Neumann algebra \mathcal{A} is a linear functional $\psi : \mathcal{A} \rightarrow \mathbb{C}$ that is positive and normalized, i.e., $\psi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and $\psi(I_{\mathcal{B}(\mathcal{H})}) = 1$.

A state ψ is said to be

faithful if $\psi(a^*a) = 0 \iff a = 0$,

tracial if $\psi(ab) = \psi(ba)$ for all $a, b \in \mathcal{A}$, and

normal if $\psi \in \mathcal{A}_*$, i.e., if it is an element of the predual \mathcal{A}_* .

Let us return to our previous example with $\mathcal{A} = \{M_f : f \in L^{\infty}_{\mathbb{C}}(\Omega, \mu)\}$. It is not difficult to see that \mathcal{A} is a W^* -algebra. Notice that since M_f is self-adjoint for $f \in L^{\infty}_{\mathbb{R}}(\Omega, \mu)$, the family of self-adjoint operators is given by $\mathcal{O} = \{M_f : f \in L^{\infty}_{\mathbb{R}}(\Omega, \mu)\}$ and the family of projection operators is given by $\mathcal{P} = \{M_f : f = \mathbf{1}_A, A \in \mathcal{F}\}$. Additionally,

For any probability measure $\nu \ll \mu$, we set

$$\psi_{\nu}(M_f) := \int_{\Omega} f \, d\nu, \quad f \in L^{\infty}_{\mathbb{C}}(\Omega, \mu),$$

we find that ψ is a linear functional that is positive and normalized, i.e., ψ is a state. Moreover, it is tracial. It is normal if $\omega := d\nu/d\mu \in L^1(\Omega, \mu)$ and faithful if $\omega > 0$.

Normal states play an essential role, serving as a counterpart to classical measures, as made explicit by the following proposition.

Proposition 3.9. *Let $\psi : \mathcal{A} \rightarrow \mathbb{C}$ be a state on a von Neumann algebra \mathcal{A} . The following are equivalent:*

- (i) ψ is a normal state.
- (ii) (σ -additivity) If $(a_n) \subset \mathcal{P}$ are mutually orthogonal projections, i.e., $a_n(\mathcal{H}) \perp a_m(\mathcal{H})$ for all $n \neq m$, and $a = \vee_n a_n$ being the projection on the smallest closed subspace containing $\cup_n a_n(\mathcal{H})$, then

$$\psi(a) = \sum_n \psi(a_n).$$

- (iii) (Continuity from above) For any increasing net $0 \preceq a_n \uparrow a$ in $\mathcal{A}_+ \Rightarrow \psi(a_n) \uparrow \psi(a)$.

Consider an arbitrary normal state ψ and set

$$\mu(A) := \psi(\mathbf{M}_{1_A}) \quad \text{for every } A \in \mathcal{F}.$$

In particular, we recover the probability \mathbb{P} by

$$\mathbb{P}(A) = \psi(\mathbf{M}_{1_A})$$

$$\psi(\mathbf{M}_{1_A} \mathbf{M}_{1_B}) = \psi(\mathbf{M}_{1_{A \cap B}}) = \mathbb{P}(A \cap B)$$

$$\mathbb{P}(\cup_i A_i) = \psi(\mathbf{M}_{1_{\cup_i A_i}}) = \psi(\sum_i \mathbf{M}_{1_{A_i}}) = \sum_i \psi(\mathbf{M}_{1_{A_i}}) = \sum_i \mathbb{P}(A_i)$$

Let $(X_t)_{t \in \mathbb{T}}$ be a stochastic process on the Wiener space (Ω, \mathbb{R})

Consider a path measure \mathbf{P} on $\Omega := \mathcal{D}(\mathbb{R}_+; E)$ and the canonical process $(X_t)_{t \geq 0}$ given by $X_t(\omega) = \omega(t)$, $\omega \in \Omega$, $t \in \mathbb{R}_+$. Suppose that for every $f \in B_b(E)$,

$$f(X_t) - f(X_0) - \int_0^t Lf(X_{r-}) dr \quad \text{is an } (\mathfrak{F}, \mathbf{P})\text{-martingale,} \quad \text{eq: classical-martingale} \quad (3.1)$$

where $\mathfrak{F} = (\mathcal{F}_t)_{t \geq 0}$ is the canonical filtration $\mathcal{F}_t = \sigma(X_s : s \leq t)$, and $L : B_b(E) \rightarrow B_b(E)$ is a bounded Markov generator such that $L^*\pi = 0$ for some stationary measure $\pi \in \mathcal{P}(E)$.

We now write (3.1) in the form above. Set $\mathfrak{X} := L^\infty(E, \pi)$ and $\mathfrak{A} := L^\infty(\Omega, \mathbf{P}_\pi)$, which are both commutative von Neumann algebras and where

$$\mathbf{P}_\pi(A) = \int_E \mathbf{P}(x + A) \pi(dx).$$

Now, define the stochastic process

$$\mathfrak{z}_t : \mathfrak{X} \rightarrow (\mathfrak{A}, \mu), \quad \mathfrak{z}_t(f) = f \circ X_t,$$

with the state

$$\mu(G) := \int_\Omega G(\omega) \mathbf{P}_\pi(d\omega), \quad G \in \mathfrak{A},$$

which makes (\mathfrak{A}, μ) a quantum probability space.

$$\mathbf{E}_t[\mathfrak{z}]$$

and the $*$ -homomorphism

$$\begin{aligned} \mathfrak{z}_t(f) - \mathfrak{z}_0(f) - \int_0^t \mathfrak{z}_{r-}(Lf) dr \\ \alpha_t(F) = F \circ \mathfrak{s}_t \\ \alpha_t \circ \mathfrak{z}_0(f) = \alpha_t(f \circ X_0 \otimes \mathbf{1}_\Omega) = f \circ X_0 \circ \mathfrak{s}_t(\omega) \\ \alpha_t : \mathcal{A} \times \Omega \rightarrow \mathcal{A}; \quad \alpha_t(M_f, \omega) = M_{f \circ X_t(\omega)}. \end{aligned}$$

The martingale identity (3.1) above can then be expressed as

$$\mathfrak{m}_t[M_f] := \alpha_t(M_f) - \alpha_0(M_f) - \int_0^t \alpha_{r-}(M_{Lf}) dr.$$

Setting $\mathcal{L}M_f := M_{Lf}$, the previous identity allows one to write (3.1) as

$$\mathfrak{m}_t[A] = \alpha_t(A) - \alpha_0(A) - \int_0^t \alpha_{r-}(\mathcal{L}A) dr, \quad A \in \mathcal{A},$$

thereby generalizing the martingale problem to a non-commutative setting.

$$\mathfrak{m}_t[A] - \mathfrak{m}_s[A] = \alpha_t(A) - \alpha_s(A) - \int_s^t \alpha_{r-}(\mathcal{L}A) dr$$

$$\mathcal{L}_a = a^*[\bullet, a] + [a^*, \bullet]a$$

Example 3.10 (Diffusion) Consider the formal example of the (possibly degenerate) diffusion process with the generator

$$Lf = \operatorname{div}(A\nabla f) - A\nabla V \cdot \nabla f, \quad f \in \mathcal{C}_c^\infty(\mathbb{R}^d),$$

i.e., it is the generator of the Itô diffusion

$$dX_t = -\sigma(X_t)\sigma^\top(X_t)\nabla V(X_t)dt + \sigma(X_t) \circ dB_t,$$

with $A = \sigma\sigma^\top \in \mathbb{R}^{d \times d}$, $\sigma \in \mathbb{R}^{d \times m}$

$$a_i = \sum_{j=1}^d \sigma_{ji} \partial_j, \quad a_i^* = - \sum_{j=1}^d \partial_j(\sigma_{ji} \bullet), \quad i = 1, \dots, m.$$

It is not difficult to see that $[a_i^*, M_f] = -[a_i, M_f]$. Moreover, for any $g \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} a_i^*[M_f, a_i]g &= \sum_{j=1}^d \partial_j(\sigma_{ji}g) \sum_{k=1}^d \sigma_{ki} \partial_k f \\ &= \sum_{j=1}^d g \partial_j(\sigma_{ji} \sum_{k=1}^d \sigma_{ki} \partial_k f) + \sum_{j,k=1}^d \sigma_{ji} \sigma_{ki} \partial_j g \partial_k f \\ &= \sum_{j=1}^d g \partial_j(\sigma_{ji} \sum_{k=1}^d \sigma_{ki} \partial_k f) - [M_f, a_i]a_i g. \end{aligned}$$

Consequently, we find that

$$\sum_{i=1}^m \mathcal{L}_{a_i}(M_f) = \sum_{i=1}^m (a_i^*[M_f, a_i] + [a_i^*, M_f]a_i) = M_{\operatorname{div}(A\nabla f)}$$

Moreover, setting

$$H_i = \sum_{j,k=1}^d \sigma_{ji} \sigma_{ki} \partial_j V \partial_k, \quad i = 1, \dots, m,$$

we have that

$$\begin{aligned} \sum_{i=1}^m [H_i, M_f] &= M_{\sigma\sigma^\top \nabla V \cdot \nabla f} \\ &= [a_i, M_V][a_i, \bullet] \end{aligned}$$

Therefore, we obtain

$$\mathcal{L}M_f = \sum_{i=1}^m ([H_i, M_f] + \mathcal{L}_{a_i}M_f) = M_{Lf} \quad \forall f \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

However, we know that the diffusion process constitutes a gradient flow with driving energy $\mathcal{F} = \operatorname{Ent}(\bullet|\pi)$, where $\pi = e^{-V} \operatorname{Leb}$ is the invariant measure. Formally, one defines a state on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ by

$$\tau(M_f) := \int_{\mathbb{R}^d} f d\pi, \quad f \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

$$\begin{aligned}\tau_\pi(M_f M_g) &= \tau(M_f M_g M_\pi) = \tau(M_{fg\pi}) = \tau_\pi(M_g M_f) \\ \Delta_\pi A &= M_\pi A M_\pi^{-1}\end{aligned}$$

$$\begin{aligned}(\Delta_\pi V_j)(g) &= M_\pi V_j M_\pi^{-1} g = \pi \sum_{j=1}^d \sigma_{ji} \partial_j (\pi^{-1} g) - \sigma_{ji} \partial_j V (\pi^{-1} g) \\ &= \sum_{j=1}^d \sigma_{ji} [g \partial_j V + \partial_j g] - \sigma_{ji} \partial_j V g =\end{aligned}$$

$$V_j = \sum_{j=1}^d \sigma_{ji} \partial_j - M_{\sigma_{ji} \partial_j V}$$

$$\tau_\sigma(M_g \mathcal{L} M_f) = \hat{\sigma}(M_{gLf}) = \int_{\mathbb{R}^d} g Lf \, \mathrm{d}\sigma =$$

$$\begin{aligned}\mathcal{L} M_{e^{-V}} g &= (\operatorname{div}(A \nabla e^{-V}) + A \nabla V \cdot \nabla e^{-V}) g \\ &= (-\operatorname{div}(A \nabla V e^{-V}) - A \nabla V \cdot \nabla V e^{-V}) g\end{aligned}$$

$$\begin{aligned}
[a_i, M_f]g &= \sum_{j=1}^d \sigma_{ji} \partial_j (fg) - \sum_{j=1}^d f \sigma_{ji} \partial_j g = \sum_{j=1}^d \sigma_{ji} g \partial_j f = M_{a_i(f)} g \\
[a_i^*, M_f]g &= - \sum_{j=1}^d \partial_j (\sigma_{ji} fg) + \sum_{j=1}^d f \partial_j (\sigma_{ji} g) = - \sum_{j=1}^d \sigma_{ji} \partial_j fg = -M_{a_i(f)} g = -[a_i^*, M_f]g
\end{aligned}$$

Consequently, we find that

$$\begin{aligned}
\mathcal{L}_{a_i} M_f &= \\
[a_i, M_f] a_i g &= \sum_{j,k=1}^d \sigma_{ji} \sigma_{ki} \partial_j f \partial_k g \\
[a_i, M_V] &= \sum_{j=1}^d \sigma_{ji} \partial_j V
\end{aligned}$$

$$[a_i, [a_i, M_f]] = [a_i, M_{a_i(f)}] = M_{a_i(a_i(f))},$$

where we used the fact that $[a_i, M_f] = M_{a_i(f)}$ for every $i = 1, \dots, d$.

$$\begin{aligned}
\sum_{i=1}^m a_i \circ a_i(f) &= \sum_{i=1}^m \sum_{j=1}^d \sigma_{ji} \partial_j \left(\sum_{k=1}^d \sigma_{ki} \partial_k f \right) = \sum_{i=1}^m \sum_{j,k=1}^d \sigma_{ji} \partial_j (\sigma_{ki} \partial_k f) \\
&= \sum_{i=1}^m \sum_{j,k=1}^d \sigma_{ji} \partial_j (\sigma_{ki} \partial_k f)
\end{aligned}$$

$$\begin{aligned}
\operatorname{div}(\sigma \sigma^\top \nabla f) &= \sum_{j=1}^d \sum_{i=1}^m \sum_{k=1}^d \partial_j (\sigma_{ji} \sigma_{ki} \partial_k f) \\
&= \sum_{j=1}^d \sum_{i=1}^m \sum_{k=1}^d \partial_j (\sigma_{ji} \sigma_{ki} \partial_k f)
\end{aligned}$$

$$\begin{aligned}
\langle \psi | M_{L_f} | \psi \rangle &= \int (Lf)(x) |\psi(x)|^2 \pi(dx) \\
&= - \int \sigma^2 \partial_x f \partial_x (\psi^2 \pi) dx - \int \sigma^2 \partial_x f \partial_x V \psi^2 d\pi \\
&= -2 \int \sigma^2 \partial_x f \psi \partial_x \psi d\pi \\
&= -2 \int \\
&\quad a\psi = \sigma \partial_x \psi
\end{aligned}$$

$$\begin{aligned}
\langle \varphi | a \psi \rangle &= \int \varphi \sigma \partial_x \psi \, d\pi \\
&= - \int \partial_x (\sigma \varphi \pi) \psi(x) \, dx \\
&= - \int \partial_x (\sigma \varphi) \psi \, d\pi + \int \varphi \psi \sigma \partial_x V \, d\pi \\
&= \int [-\partial_x (\sigma \varphi) + \varphi \sigma \partial_x V] \psi \, d\pi = \langle a^* \varphi | \psi \rangle
\end{aligned}$$

$$\begin{aligned}
(a^* M_f a \psi)(x) &= -\partial_x (f \sigma^2 \partial_x \psi) + f \sigma^2 \partial_x \psi \partial_x V \\
(a^* a M_f \psi)(x) &= -\partial_x (\sigma^2 \partial_x (f \psi)) + \sigma^2 \partial_x (f \psi) \partial_x V \\
(M_f a^* a \psi)(x) &= -f \partial (\sigma^2 \partial_x \psi) + \sigma^2 f \partial_x \psi \partial_x V
\end{aligned}$$

$$\int \psi(x) (a^* M_f a \psi)(x) \, d\pi = \int f |\sigma \partial_x \psi|^2 \, d\pi$$

3.3 Stochastic processes on matrix algebras

In this section, we consider $\mathfrak{X} = \text{Mat}(\mathbb{C}, n)$. Let $a \in \mathcal{O}(\mathfrak{X})$ be an observable on \mathfrak{X} and set

$$\mathcal{L}x = a^*[x, a] + [a^*, x]a, \quad x \in \mathfrak{X}.$$

We then consider the solution of the

$$d\mathcal{U}_t = (-a^* a \, dt + i\sqrt{2}a \, dB_t) \mathcal{U}_t,$$

which, due to $a = a^*$, can be explicitly expressed as

$$\mathcal{U}_t(x, e(f)) := \exp(i\sqrt{2}a B_t(\omega))x, \quad t \in \mathbb{T},$$

which is a $*$ -automorphism on \mathfrak{X} , which extends to an isometry

$$\mathcal{U}_t: \mathfrak{X} \otimes \mathcal{A}$$

$$B \in \mathcal{A}$$

$$\mathfrak{z}_t(x) - \mathfrak{z}_0(x) - \int_0^t \mathfrak{z}_r(\mathcal{L}x) \, dr$$

3.4 Stochastic processes on the unitary group

3.4.1 The unitary group and its Lie structure

The unitary group is defined as

$$U(n) = \{U \in \mathbb{C}^{n \times n} : U^* U = I\}.$$

It is a compact Lie group with Lie algebra (tangent space at identity)

$$\mathcal{U}(n) = \mathsf{T}_I U(n) = \{\mathfrak{a} \in \mathbb{C}^{n \times n} : \mathfrak{a}^* = -\mathfrak{a}\},$$

i.e., the space of skew-Hermitian matrices. A vector field $\mathfrak{a} : U(n) \rightarrow \mathcal{U}(n)$ is *left-invariant* if $\mathfrak{a}(U) = U\mathfrak{a}(I) \in \mathcal{U}(n)$ for all $U \in U(n)$.

The Lie algebra $\mathcal{U}(n)$ can be equipped with a real inner product given by the Hilbert-Schmidt inner product

$$\langle \mathfrak{a}, \mathfrak{b} \rangle_{\mathcal{U}(n)} = -\operatorname{tr}(\mathfrak{a}^* \mathfrak{b}) = \Re \operatorname{tr}(\mathfrak{a} \mathfrak{b}^*),$$

which is positive-definite on $\mathcal{U}(n)$. The associated norm is then

$$|\mathfrak{a}|_{\mathcal{U}(n)}^2 = \langle \mathfrak{a}, \mathfrak{a} \rangle_{\mathcal{U}(n)} = \operatorname{Tr}(\mathfrak{a} \mathfrak{a}^*).$$

3.4.2 Unitary-valued Brownian motion (rank- r noise)

Let $\mathfrak{a}_1, \dots, \mathfrak{a}_r \in \mathcal{U}(n)$ be orthonormal under the inner product $\langle \cdot, \cdot \rangle_{\mathcal{U}(n)}$, where $1 \leq r \leq n^2$. Further, let $\beta_t^1, \dots, \beta_t^r$ be independent standard real Brownian motions.

Define the $\mathcal{U}(n)$ -valued (possibly degenerate) Brownian driver

$$\mathfrak{w}_t = \sum_j \mathfrak{a}_j \beta_t^j.$$

If $r = n^2$, the driver is elliptic (non-degenerate). If $r < n^2$, the covariance has rank r and the process is *degenerate* (hypoelliptic), exploring only the connected subgroup

$$H = \exp(\mathcal{H}) \subset U(n), \quad \mathcal{H} = \operatorname{Lie}\{\mathfrak{a}_1, \dots, \mathfrak{a}_r\} \subset \mathcal{U}(n).$$

The intrinsic $U(n)$ -valued diffusion process solves the Stratonovich SDE

$$dU_t = U_t \circ d\mathfrak{w}_t, \quad U_0 = I,$$

or in components

$$dU_t = \sum_j U_t \mathfrak{a}_j \circ d\beta_t^j.$$

The matrix-valued quadratic variation of \mathfrak{w} is given by

$$d[\mathfrak{w}] = \sum_j \mathfrak{a}_j \otimes \mathfrak{a}_j dt.$$

In Itô form, the equivalent SDE reads

$$dU_t = U_t (d\mathfrak{w}_t + \mathfrak{l} dt), \quad \mathfrak{l} := \frac{1}{2} \sum_j \mathfrak{a}_j^2,$$

where \mathfrak{l} is the Laplace-Beltrami generator associated with the left-invariant connection.

In the following, we denote its law on path space $C([0, T]; U(n))$ by \mathbf{P} .

3.4.3 Girsanov transform on $U(n)$ (Karandikar type)

Let $\mathbf{u} \in L^2((0, T); \mathcal{U}(n))$ be adapted and square-integrable.

Define the drifted group motion by

$$dU_t^{\mathbf{u}} = U_t^{\mathbf{u}} \circ (d\mathbf{w}_t + \mathbf{u}_t dt), \quad U_0^{\mathbf{u}} = I.$$

Assuming a Novikov condition in the active noise directions,

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T |\mathbf{u}_s|^2 ds \right) \right] < +\infty,$$

the process defined below is a true martingale.

Decompose the drift against the active algebra frame:

$$\mathbf{u}_t = \sum_j \mathbf{a}_j v_t^j, \quad v = (v^1, \dots, v^r) \in L^2((0, T); \mathbb{R}^r).$$

Define the exponential Cameron-Martin density process

$$\begin{aligned} M_T &= \exp \left(\int_0^T \langle \mathbf{u}_t, d\mathbf{w}_t \rangle - \frac{1}{2} \sum_j \int_0^T \langle \mathbf{u}_t, \mathbf{a}_j \otimes \mathbf{a}_j \mathbf{u}_t \rangle dt \right) \\ &= \exp \left(\sum_j \int_0^T v_t^j d\beta_t^j - \frac{1}{2} \sum_j \int_0^T |v_t^j|^2 dt \right). \end{aligned}$$

This expression depends only on the directions with nonzero quadratic variation. It is the standard Cameron-Martin shift density on \mathbb{R}^r lifted to the group via left multiplication.

We then define a new path measure on the same probability space by

$$\mathbb{Q} = M_T \mathbb{P}.$$

Under \mathbb{Q} :

(i) The shifted drivers

$$\beta_t^{j, \mathbb{Q}} := \beta_t^j - \int_0^t v_s^j ds$$

are Brownian motions again on \mathbb{R} .

(ii) The algebra noise

$$d\mathbf{w}_t^{\mathbb{Q}} := d\mathbf{w}_t - \mathbf{u}_t dt = \sum_j \mathbf{a}_j d\beta_t^{j, \mathbb{Q}}$$

is a $\mathcal{U}(n)$ -valued Brownian motion again with same covariance

$$d[\mathbf{w}^{\mathbb{Q}}] = d[\mathbf{w}] = \sum_j \mathbf{a}_j \otimes \mathbf{a}_j dt.$$

(iii) The group paths satisfy the same *driftless* intrinsic SDE

$$dU_t^{\mathbf{u}} = U_t^{\mathbf{u}} \circ dW_t^{\mathbb{Q}} = \sum_j U_t^{\mathbf{u}} \mathbf{a}_j \circ d\beta_t^{j, \mathbb{Q}},$$

so $U_t^{\mathbf{u}}$ is an intrinsic Brownian motion on the same symmetry group H .

Hence, M_T is the Radon-Nikodym derivative between the laws of the drifted and driftless group diffusions on path space.

3.4.4 Induced Girsanov on pure states

Rank-1 projections evolve by the conjugation map

$$P_t = U_t P_0 U_t^*, \quad P_0 = \psi_0 \psi_0^*.$$

This gives a valid reference probability on paths of $C([0, T]; \mathcal{M})$, where

$$\mathcal{M} = \{V P_0 V^* : V \in H\} \subset \mathcal{P}_1(n) \simeq \mathbb{CP}^{n-1}$$

is the homogeneous orbit that the degenerate noise can reach.

Linearising conjugation and applying the Stratonovich chain rule yields the SDE

$$dP_t^u = [\circ dW_t^Q, P_t^u] = \sum_j [\mathbf{a}_j, P_t^u] \circ d\beta_t^{j,Q},$$

so the pure-state path law remains Brownian on the same orbit manifold \mathcal{M} .

If a drift \mathbf{u} has components outside $\text{span}\{\mathbf{a}_j\}$, only the projection of the drift onto that span can be removed. Directions with zero quadratic variation cannot absorb drifts.

$$\int F(U_t P_0 U_t^*) \mathbf{P}(dU) =$$

3.5 Dilation of semigroups (Sz.Nagy)

3.6 Stochastic dilations: How to make a heat bath

3.7 Interpretation of LDPs on unitary evolutions

The task of this subsection is to reinterpret the large deviations of empirical measures in a functional analytic framework.

Consider a family of iid E -valued random variables $(X_i)_{i \in \mathbb{N}}$ with $\sigma = \text{Law} X_1$. Setting $\Omega = E^{\mathbb{N}}$ and $\mathbf{R} = \sigma^{\otimes \mathbb{N}}$, we see that $\text{Law}(X_i)_{i \in \mathbb{N}} = \mathbf{R}$. Moreover, we can disregard the initial probability space for X_i and consider instead the canonical random variable on Ω .

For every $f \in \mathfrak{X} := \mathcal{C}_b(E)$ and $n \in \mathbb{N}$, we define the noncommutative random variable

$$\mathfrak{X} \ni f \mapsto \mathfrak{z}^n(f) = \frac{1}{n} \sum_{i=1}^n f(X_i) \in \mathfrak{A} := L^\infty(\Omega, \mathbf{R}),$$

where \mathfrak{A} is equipped with the state $\mu(G) = \mathbf{E}_{\mathbf{R}}[G]$, $G \in \mathfrak{A}$.

$$\mu(\mathfrak{z}^n(f)) = \frac{1}{n} \sum_{i=1}^n \mathbf{E}_{\mathbf{R}}[f(X_i)] = \frac{1}{n} \sum_{i=1}^n \int_E f \, d\sigma = \int_E f \, d\sigma = \mu(\mathfrak{z}_1(f)).$$

$$\begin{aligned} \mathbf{R}(\sup_f \{\mu^n(f) - \nu(f)\} > \varepsilon) &= \mathbf{R}(n\mu^n(f) > n(\nu(f) + \varepsilon)) \\ &= \mathbf{R}(e^{n\mu^n(f)} > e^{n(\nu(f) + \varepsilon)}) \leq e^{-n(\nu(f) + \varepsilon)} \int_{\Omega} e^{n\mu^n(f)} \, d\mathbf{R} \end{aligned}$$

$$\mathfrak{z}: \mathfrak{X} \rightarrow (\mathfrak{X} \otimes \mathfrak{C}, \text{tr} \otimes \mu)$$

$$\mathfrak{A} = \mathfrak{X} \otimes \mathfrak{C}^{\otimes \mathbb{Z}}, \quad \varphi = \text{tr} \otimes \mu^{\otimes \mathbb{Z}}$$

Let X_i

$$\mathfrak{z}_i(x) = x \otimes c_i, \quad c_i = \cdots 1 \otimes c$$

$$\mathfrak{z}^n = \frac{1}{n} \sum_{i=1}^n \mathfrak{z}_i, \quad \mathfrak{z}_i = \cdots \otimes \mathbb{1} \otimes \mathfrak{z} \otimes \mathbb{1} \otimes \cdots$$

$$\varphi(\mathfrak{z}^n(x)) = \frac{1}{n} \sum_{i=1}^n \varphi(\mathfrak{z}_i(x)) = \varphi(\mathfrak{z}_1(x)) = \mu(\mathfrak{z}(x))$$

$$\mathfrak{z}_i(x) = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \mathfrak{z}(x) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

$$\mathfrak{z}^n(x) \in \mathfrak{A} \cong \mathfrak{X} \otimes \mathfrak{Y}$$

$$\mu = \text{tr} \otimes \mu_{\mathfrak{C}}^{\otimes \mathbb{N}}$$

$$\mu(\mathbf{1}_A(\mathfrak{z}^n(x) - \mathfrak{y}_1(x))) = \mu_{\mathfrak{C}}^{\otimes \mathbb{N}}(\text{tr}[\mathbf{1}_A(\mathfrak{z}^n(x) - \mathfrak{y}(x) \otimes \mathbb{1}_{\mathfrak{C}^{\otimes \mathbb{N}}})])$$

$$\sup_f \{\mu^n(f) - \nu(f)\} > \varepsilon \quad \Rightarrow \quad \mu^n(f^\delta) - \nu(f^\delta) > \varepsilon - \delta$$

$$\begin{aligned} \mathbf{R}(\mu^n(f) - \nu(f) > \varepsilon) &= \mathbf{R}(e^{n\mu^n(f^\delta)} > e^{n(\nu(f^\delta) + \varepsilon - \delta)}) \\ &\leq e^{-n(\nu(f^\delta) + \varepsilon - \delta)} \mathbf{E}_{\mathbf{R}}[e^{n\mu^n(f^\delta)}] \end{aligned}$$

$$\begin{aligned} -\frac{1}{n} \log \mathbf{R}(\mu^n(f) - \mu(f) > \varepsilon) &\geq \mu(f) + \varepsilon - \frac{1}{n} \log \int_{\Omega} e^{n\mu^n(f)} d\mathbf{R} \\ &= \mu(f) + \varepsilon - \log \int_E e^f d\mu \\ -\frac{1}{n} \log \mathbf{R}(\mu^n(f) - \mu(f) > \varepsilon) &\geq \end{aligned}$$

Taking the sup

$$\begin{aligned} \int_{\Omega} e^{n\mu^n(f)} d\mathbf{R} &= \int_{\Omega} \prod_{i=1}^n e^{f(X_i)} d\mathbf{R} \\ &= \int_{E^N} \prod_{i=1}^n e^{f(x_i)} \mu(dx_1) \cdots \mu(dx_n) = \left(\int_E e^{f(x)} \mu(dx) \right)^n \end{aligned}$$

3.8 LDP via Gärtner-Ellis revisited

Let $\Omega = \mathcal{C}(I; \mathfrak{h})$ be the space of \mathfrak{h} -valued continuous path and consider the canonical process $(X_t)_{t \in I}$ defined by $X_t(\omega) := \omega(t)$ for all $t \in I$. Further, let $\mathcal{F} = \cup_{t \geq 0} \mathcal{F}_t$, where the filtration $\mathfrak{F} := \{\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)\}_{t \in I}$ is generated by $(X_t)_{t \in I}$. Throughout, we consider a reference path measure $\mathbf{R} \in \mathcal{P}(\Omega)$ such that X admits the \mathbf{R} -semimartingale decomposition

$$X = X_0 + B^{\mathbf{R}} + M^{\mathbf{R}} \quad \mathbf{R}\text{-almost surely,}$$

where $B^{\mathbf{R}}$ is an adapted process with absolutely continuous sample paths \mathbf{R} -almost surely, and $M^{\mathbf{R}}$ is an \mathbf{R} -martingale. For simplicity of presentation, we assume that the quadratic variation of $M^{\mathbf{R}}$ is an $\mathcal{S}_+(\mathfrak{X})$ -valued random measure on I with Lebesgue density, i.e.,

$$\frac{d[M^{\mathbf{R}}, M^{\mathbf{R}}]}{d\lambda}(t) = \mathbf{a}_t(X_t) \in \mathcal{S}_+(\mathfrak{h}) \quad \mathbf{R}\text{-almost surely,}$$

and also that $t \mapsto \mathbf{a}_t(X_t)$ is an \mathfrak{F} -adapted process.

and has the quadratic variation $\langle M^{\mathbf{R}}[\varphi] \rangle$ for every smooth function φ .

Let X^1, X^2, \dots be independent copies of (X^1, \dots, X^n)

$$\begin{aligned} \frac{dQ^n}{dP^{\otimes n}} &= \exp \left(\sum_{i=1}^n \int_I \langle \beta_t^i(X_t), dM_t^{\mathbf{R}} \rangle - \frac{1}{2} \sum_{i=1}^n \int_I \langle \beta_t^i(X_t), \mathbf{a}_t(X_t) \beta_t^i(X_t) \rangle dt \right) =: Z^{(n)} \\ Y_t^{(n)} &:= \sum_{i=1}^n \int_0^t \langle \beta_s^i(X_s), dM_s^{\mathbf{R}} \rangle - \frac{1}{2} \sum_{i=1}^n \int_0^t \langle \beta_s^i(X_s), \mathbf{a}_s(X_s) \beta_s^i(X_s) \rangle ds \end{aligned}$$

$$dZ_t^{(n)} = dY_t^{(n)} = Z_t^{(n)} \left(dY_t^{(n)} + \frac{1}{2} d[Y^{(n)}, Y^{(n)}]_t \right) = \sum_{i=1}^n Z_t^{(n)} \langle \beta_t^i(X_t), dM_t^{\mathbf{R}} \rangle$$

Define $\rho^n := \frac{1}{n} \sum_{i=1}^n X^i$. Then,

$$d\rho^n = \frac{1}{n} \sum_{i=1}^n dX_t^i = \frac{1}{n} \sum_{i=1}^n b_t^{\mathbf{R}}(X_t^i) dt + \frac{1}{n} \sum_{i=1}^n dM_t^{\mathbf{R},i}$$

$$\begin{aligned} \mathbf{P}^{\otimes n} \left(\frac{1}{n} \sum_{i=1}^n X^i \in A \right) &= \int_A e^{-\int_I \langle \beta_t^i(X_t), dM_t^{\mathbf{R}} \rangle + \frac{1}{2} \int_I \langle \beta_t^i(X_t), \mathbf{a}_t(X_t) \beta_t^i(X_t) \rangle dt} d\mathbf{Q}^{(n)} \\ &= \int_A e^{-\int_I \langle \beta_t^i(X_t), dM_t^{\mathbf{R}} \rangle + \frac{1}{2} \int_I \langle \beta_t^i(X_t), \mathbf{a}_t(X_t) \beta_t^i(X_t) \rangle dt} d\mathbf{Q}^{(n)} \end{aligned}$$

$$A := \{\rho : \partial_t \rho + \overline{\operatorname{div}} j = L\rho\}$$

$$D_{\beta^i} \exp \left(\int_I \langle \beta_t^i(X_t), dM_t^{\mathbf{R}} \rangle + \frac{1}{2} \int_I \langle \beta_t^i(X_t), \mathbf{a}_t(X_t) \beta_t^i(X_t) \rangle dt \right) [h^i]$$

$$\int_I \langle \beta_t, j_t \rangle dt - \log \int_A Z^{(n)} d\mathbf{P}^{\otimes n}$$

3.8.1 Example

We consider the Hilbert space $\mathfrak{h} = \mathbb{M}$ equipped with the scalar product $\langle A, B \rangle = \operatorname{tr}[A^* B]$. Consider the stochastic process

$$P_t = P_0 + \int_0^t LP_s ds + \int_0^t \overline{\nabla} P_s dW_s,$$

where $\overline{\nabla} P = [i\sigma_X, P]$ and $LP = (1/2)[i\sigma_X, [i\sigma_X, P]]$. In other words,

$$B_t^{\mathbf{R}} = \int_0^t LP_s ds, \quad M_t^{\mathbf{R}} = \int_0^t \overline{\nabla} P_s dW_s,$$

where $M^{\mathbf{R}}$ has the quadratic variation

$$\mathbf{a}(P) = \langle (\overline{\nabla} P)^*, \bullet \rangle \overline{\nabla} P \quad \text{for every } P \in \mathfrak{h}.$$

Let P^1, P^2, \dots be independent copies and define $\rho^n := \frac{1}{n} \sum_{i=1}^n P^i$. Then,

$$d\rho_t^n = L\rho_t^n dt + \frac{1}{n} \sum_{i=1}^n \overline{\nabla} P_t^i dW_t^i = L\rho_t^n dt + \frac{1}{n} dM_t^{\mathbf{R},n}.$$

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n \bar{\nabla} P_t^i dW_t^i &= \frac{1}{n} \sum_{i=1}^n \left(\bar{\nabla} P_t^i dW_t^i - \mathbf{a}(P_t^i) H_t dt \right) + \frac{1}{n} \sum_{i=1}^n \mathbf{a}(P_t^i) H_t dt \\ d\rho^n &= L\rho^n dt + \frac{1}{n} \sum_{i=1}^n \mathbf{a}(P_t^i) H_t dt + \frac{1}{n} dM_t^{\mathbf{Q},n}\end{aligned}$$

For each $n \in \mathbb{N}$, we set

$$\frac{d\mathbf{Q}^n}{d\mathbf{R}^{\otimes n}} = \exp \left(\sum_{i=1}^n \int_I \langle \bar{\nabla} P_t^i, H_t \rangle dW_t^i - \frac{1}{2} \sum_{i=1}^n \int_I \langle H_t, \mathbf{a}(P_t^i) H_t \rangle dt \right) =: Z^{(n)}.$$

Then, for curves η satisfying $\partial_t \eta + \bar{\text{div}} J^\eta = L\eta$ for some J^η , we find that

$$J^\eta = \sum \langle \sigma_j, J^\eta \rangle \sigma_j$$

Suppose

$$-\bar{\text{div}} = \sum_{j \in J} \alpha_j [i\sigma_j, \bullet],$$

then

$$-\bar{\text{div}} J^\eta = \sum_{j,k} \alpha_j \langle \sigma_k, J^\eta \rangle [i\sigma_j, \sigma_k] = \sum_{j,k \neq j} \alpha_j \langle \sigma_k, J^\eta \rangle [i\sigma_j, \sigma_k] \in \text{span}\{[i\sigma_j, \sigma_k] : j \in J, k \neq j\}$$

$$\begin{aligned}\mathbf{R}^{\otimes n}(\rho^n \in A) &= \int_A \exp \left(- \sum_{i=1}^n \int_I \langle \bar{\nabla} P_t^i, H_t \rangle dW_t^i + \frac{1}{2} \sum_{i=1}^n \int_I \langle H_t, \mathbf{a}(P_t^i) H_t \rangle dt \right) d\mathbf{Q}^{(n)} \\ &= \int_A \exp \left(- \int_I \langle dM_t^{\mathbf{R},n}, H_t \rangle + \frac{1}{2} \sum_{i=1}^n \int_I \langle H_t, \mathbf{a}(P_t^i) H_t \rangle dt \right) d\mathbf{Q}^{(n)} \\ &= \int_A \exp \left(- \int_I \langle dM_t^{\mathbf{Q},n}, H_t \rangle - \frac{1}{2} \sum_{i=1}^n \int_I \langle H_t, \mathbf{a}(P_t^i) H_t \rangle dt \right) d\mathbf{Q}^{(n)}\end{aligned}$$

$$H^\eta \in \underset{H}{\text{argmin}} \left\{ \mathbb{E}_{\mathbf{Q}} \left[\frac{1}{2} \int_I \langle H_t, \mathbf{a}(P_t) H_t \rangle dt \right] - \int_I \langle H_t, \bar{\text{div}} J_t^\eta \rangle dt \right\}$$

$$\mathbb{E}_{\mathbf{Q}} \left[\int_I (\text{tr}[P_t \bar{\nabla} H_t])^2 dt \right] = \int_I \mathbb{E}_{\mathbf{Q}} \left[(\text{tr}[P_t \bar{\nabla} H_t])^2 \right] dt$$

$$\text{tr}[\rho_t^{\mathbf{Q}} \bar{\nabla} H_t] = \sum \lambda_i(t) \langle v_i(t), \bar{\nabla} H_t v_i(t) \rangle \geq \gamma \sum \lambda_i(t)$$

$$\mathbb{E}_{\mathbf{R}} \left[\int_I \langle G_t, \mathbf{a}(P_t) H_t \rangle dt \right] = \int_I \langle G_t, \bar{\text{div}} J_t \rangle dt$$

$$\int_I \langle H_t, \bar{\text{div}} J_t \rangle dt - \log \mathbb{E}_{\mathbf{R}} \left[\exp \left(\int_I \langle H_t, dM_t^{\mathbf{R},n} \rangle - \frac{1}{2} \sum_{i=1}^n \int_I \langle H_t, \mathbf{a}(P_t^i) H_t \rangle dt \right) \right]$$

$$\int_I \langle G_t, \overline{\text{div}} J_t \rangle dt = -\mathbb{E}_{\mathbb{R}^{\otimes n}} \left[\left(\int_I \langle G_t, dM_t^{\mathbb{R},n} \rangle - \sum_{i=1}^n \int_I \langle G_t, \mathbf{a}(P_t^i) H_t \rangle dt \right) \exp \left(\int_I \langle H_t, dM_t^{\mathbb{R},n} \rangle - \frac{1}{2} \sum_{i=1}^n \int_I \langle H_t, \mathbf{a}(P_t^i) H_t \rangle dt \right) \right]$$

Let $Y_t^{(n)} = \sum_{i=1}^n \int_0^t \langle \overline{\nabla} H_s, \overline{\nabla} P_s^i \rangle dW_s^i$ and $Z_t^{(n)} := \exp(Y_t^{(n)})$. Then,

$$dZ_t^{(n)} = Z_t^{(n)} \left(\sum_{i=1}^n \langle \overline{\nabla} H_t, \overline{\nabla} P_t^i \rangle dW_t^i + \frac{1}{2} \sum_{i=1}^n \langle \overline{\nabla} H_t, \mathbf{a}(P_t^i) \overline{\nabla} H_t \rangle dt \right)$$

$$Z_t^{(n)} = 1 + \sum_{i=1}^n \int_0^t Z_s^{(n)} \langle \overline{\nabla} H_s, \overline{\nabla} P_s^i \rangle dW_s^i + \frac{1}{2} \sum_{i=1}^n \int_0^t Z_s^{(n)} \langle \overline{\nabla} H_s, \mathbf{a}(P_s^i) \overline{\nabla} H_s \rangle ds$$

Let $\varrho \in \mathcal{S}(\mathcal{H})$ be state and $(\varrho^n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mathcal{H})$ be a sequence of states. We say that $\varrho^n \rightarrow \varrho$ setwise if

$$\varrho^n(E) \rightarrow \varrho(E) \quad \text{for all projections } E \in \mathcal{P}(\mathcal{H}).$$

Further, let $(\mathfrak{z}_t)_{t \in \mathbb{T}}$ be a stochastic process in the sense of Definition 3.1 with

$$\mathfrak{z}_t: \mathfrak{X} \rightarrow (\mathfrak{X} \otimes \mathfrak{C}, \varphi), \quad \varphi = \varrho \otimes \mu.$$

$$\mathfrak{z}_t(x) = \alpha_t^* x \alpha_t$$

and $\mathfrak{z}_0(x) = x \otimes 1$, $x \in \mathfrak{X}$. By construction, we have that

$$\varphi(\mathfrak{z}_0(x)) = \varphi(x \otimes 1) = \varrho(x)\mu(1) = \varrho(x) \quad \forall x \in \mathfrak{X}.$$

We suppose that

$$\begin{aligned} \mathcal{S}(\mathfrak{X} \otimes \mathfrak{C}^{\otimes n}) &\rightarrow \mathcal{S}(\mathfrak{X} \otimes \mathfrak{C}) \\ \mu_\varphi &= \frac{1}{n} \sum_{i=1}^n \varphi_i = \frac{1}{n} \sum_{i=1}^n \rho \otimes \varphi \\ \mathbf{x}_i: \mathfrak{C}^{\otimes n} &\rightarrow \mathfrak{C}; \quad \mathbf{x}_i(\omega_1, \dots, \omega_n) = \omega_i \\ \delta_{\mathbf{x}_i}(E) & \end{aligned}$$

3.9 Groups to algebras

Let \mathbb{T} be the 1-dimensional torus. Further, let \mathcal{H} be a Hilbert space and $\mathfrak{h} \in \mathcal{O}(\mathcal{H})$ be an observable on \mathcal{H} . Consider the unitary $\mathbf{u}: \mathbb{T} \rightarrow \mathcal{U}(\mathcal{H})$ such that $\mathbf{u}(\exp(tX)) = e^{it\mathfrak{h}}$.

$$\left. \frac{d}{dt} f(\exp(tX)) \right|_{t=0} = X(f)$$

$$dX_t =$$

$$\mathcal{B}(\mathcal{H}) \rightarrow L^\infty(\mathbb{T}; \mathcal{B}(\mathcal{H})); \quad a \mapsto \mathbf{u}^*(z) a \mathbf{u}(z)$$

Let $\rho_0 \in \mathcal{S}(\mathcal{H})$ be a given fixed state

$$\begin{aligned} \varrho_\mu(x) &= \int_{\mathbb{T}} \text{tr}[\mathbf{u}^*(z) x \mathbf{u}(z) \varrho_0] \mu(dz) = \int_{\mathbb{T}} f_{\rho_0, x}(z) \mu(dz) \\ &= \text{tr} \left[x \int_{\mathbb{T}} \mathbf{u}(z) \varrho_0 \mathbf{u}^*(z) \mu(dz) \right] = \text{tr}[x \varrho_\mu] \end{aligned}$$

Consider iid random variables $(Z^i)_{i \in \mathbb{N}}$, $Z^i \sim \mu \in \mathcal{P}(\mathbb{T})$ and the corresponding sequence of empirical measures

$$\mu^n = \frac{1}{n} \sum_{i=1}^n \delta_{Z^i}, \quad \varrho_{\mu^n}(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{u}(Z^i) \varrho_0 \mathbf{u}^*(Z^i)$$

$$\begin{aligned}\frac{1}{n} \log \mathbb{E}[e^{n\langle f, \mu^n \rangle}] &= \frac{1}{n} \log \mathbb{E} \left[\exp \left(\sum_{i=1}^n f(Z^i) \right) \right] \\ &= \log \left(\int_{\mathbb{T}} e^{f(z)} \mu(dz) \right)\end{aligned}$$

$$\sup_{f=f_{\varrho_0, x}} \left\{ \langle f, \nu \rangle - \log \int_{\mathbb{T}} e^{f(z)} \mu(dz) \right\} = \sup_{x \in \mathcal{O}(\mathcal{H})} \left\{ \text{tr}[x \varrho_\nu] - \log \int_{\mathbb{T}} e^{\text{tr}[\mathbf{u}^*(z) x \mathbf{u}(z) \varrho_0]} \mu(dz) \right\}$$

$$\begin{aligned}& \sum_{j,k} x_{jk} \text{tr}[E_{jk} \varrho_\nu] - \log \int_{\mathbb{T}} e^{\sum_{j,k} x_{jk} \text{tr}[\mathbf{u}^*(z) E_{jk} \mathbf{u}(z) \varrho_0]} \mu(dz) \\ &= \sum_{j,k} x_{jk} \text{tr}[E_{jk} \varrho_\nu] - \log \int_{\mathbb{T}} \prod_{j,k} e^{x_{jk} \text{tr}[\mathbf{u}^*(z) E_{jk} \mathbf{u}(z) \varrho_0]} \mu(dz)\end{aligned}$$

$$\text{tr}[E_{jk} \varrho_\nu] = \frac{1}{c} \int_{\mathbb{T}} \text{tr}[\mathbf{u}^*(z) E_{jk} \mathbf{u}(z) \varrho_0] e^{\sum_{j,k} x_{jk} \text{tr}[\mathbf{u}^*(z) E_{jk} \mathbf{u}(z) \varrho_0]} \mu(dz)$$

$$b = \frac{1}{c} \int_{\mathbb{T}} v(z) e^{\langle \mathbf{x}, v(z) \rangle} \mu(dz)$$

Consider the map

$$\mathbf{x} \mapsto \Lambda(\mathbf{x}) = \int_{\mathbb{T}} v(z) e^{\langle \mathbf{x}, v(z) \rangle} \mu(dz)$$

$$\langle \Lambda(\mathbf{x}) - \Lambda(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle = \int_0^1 \langle D\Lambda((1-\tau)\mathbf{x} + \tau\mathbf{y})[\mathbf{x} - \mathbf{y}], \mathbf{x} - \mathbf{y} \rangle d\tau \geq 0$$

$$\frac{d}{d\lambda} \int_{\mathbb{T}} v(z) e^{\langle \lambda, v(z) \rangle} \mu(dz) = \int_{\mathbb{T}} v(z) \otimes v(z) e^{\langle \lambda, v(z) \rangle} \mu(dz)$$

$$\text{tr}[E_{jk} \mathbf{u}(z) \varrho_0 \mathbf{u}^*(z)] = \langle \psi_k | \mathbf{u}(z) \varrho_0 \mathbf{u}^*(z) | \psi_j \rangle = \sum_i \lambda_i \langle \psi_k | \mathbf{u}(z) | \psi_i \rangle \langle \psi_i | \mathbf{u}^*(z) | \psi_j \rangle$$

If $\varrho_0 = \mathbb{1}$, then

$$\text{tr}[E_{jk} \mathbf{u}(z) \varrho_0 \mathbf{u}^*(z)] = \delta_{kj}.$$

Hence,

$$\text{tr}[E_{jk} \varrho_\nu] = \frac{1}{c} \delta_{kj} e^{\sum_j x_j}$$

4 Boson Fock spaces and quantum noise

4.1 Boson Fock spaces

Consider a (complex) Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The symmetric Fock space associated with \mathcal{H} is

$$\mathfrak{F} = \mathfrak{F}_{\text{sym}}(\mathcal{H}) := \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}^{\odot n}, \quad \mathcal{H}^{\odot 0} = \mathbb{C},$$

where \odot denotes the symmetric tensor product such that

$$\mathcal{H}^{\odot n} = \left\{ f \in \mathcal{H}^{\otimes n} : f(x_{\sigma_1}, \dots, x_{\sigma_n}) = f(x_1, \dots, x_n) \text{ for every permutation } \sigma \right\}.$$

The Fock space \mathfrak{F} inherits the scalar product from \mathcal{H} defined by

$$\langle \oplus f^{(n)}, \oplus g^{(n)} \rangle_{\mathfrak{F}} = \sum_{n \in \mathbb{N}_0} \langle f^{(n)}, g^{(n)} \rangle_{\mathcal{H}^{\otimes n}}.$$

We define the *vacuum vector* $\Omega = 1 \oplus 0 \oplus 0^{\otimes 2} \oplus \dots \in \mathfrak{F}$, and the *exponential vectors*

$$\mathbf{e}(f) = \bigoplus_{n \in \mathbb{N}_0} \frac{1}{\sqrt{n!}} f^{\otimes n}, \quad f \in \mathcal{H}.$$

It turns out that the family of exponential vectors \mathfrak{E} is *total* in \mathfrak{F} , i.e., the linear span of \mathfrak{E} is dense in \mathfrak{F} . This fact will be helpful for us in the future. Since,

$$\langle \mathbf{e}(f), \mathbf{e}(g) \rangle_{\mathfrak{F}} = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} \langle f, g \rangle_{\mathcal{H}}^n = e^{\langle f, g \rangle_{\mathcal{H}}}, \quad f, g \in \mathcal{H},$$

the exponential vectors are normalizable. These normalized exponential vectors

$$\psi(f) = e^{-\frac{1}{2}\|f\|_{\mathcal{H}}^2} \mathbf{e}(f), \quad f \in \mathcal{H},$$

are commonly known as *coherent vectors*.

4.1.1 Weyl and field operators

For any $f \in \mathcal{H}$, we define the *Weyl operator* on exponential vectors by

$$\mathbf{W}(f)\mathbf{e}(g) := \exp\left(-\langle f, g \rangle_{\mathcal{H}} - \frac{1}{2}\|f\|_{\mathcal{H}}^2\right) \mathbf{e}(f+g), \quad g \in \mathcal{H}.$$

Weyl operators play an essential role in the setup of Fock spaces. For one, they generate coherent states by acting on the vacuum state, i.e.,

$$\mathbf{W}(f)\Omega = e^{-\frac{1}{2}\|f\|_{\mathcal{H}}^2} \mathbf{e}(f) = \psi(f), \quad f \in \mathcal{H}.$$

Moreover, they give the means to map any element $f \in \mathcal{H}$ to unitary operators on \mathfrak{F} that satisfy the *canonical commutation relation* (CCR):

Proposition 4.1. *The Weyl operator $W(f)$ is a unitary operator and satisfies*

$$(i) \quad W^\dagger(f) = W(-f).$$

$$(ii) \quad W(f)W(g) = e^{-i\text{Im}(\langle f, g \rangle_{\mathcal{H}})} W(f+g) = e^{-2i\text{Im}(\langle f, g \rangle_{\mathcal{H}})} W(g)W(f).$$

Property (ii) is the Weyl form of the canonical commutation relation (CCR).

Proof. For any $g \in \mathcal{H}$,

$$\begin{aligned} \langle W(f)e(g), W(f)e(g) \rangle_{\mathfrak{F}} &= \exp(-2\langle f, g \rangle_{\mathcal{H}} - \|f\|_{\mathcal{H}}^2) \langle e(f+g), e(f+g) \rangle_{\mathfrak{F}} \\ &= \exp(-2\langle f, g \rangle_{\mathcal{H}} - \|f\|_{\mathcal{H}}^2 + \|f+g\|_{\mathcal{H}}^2) \\ &= e^{\|g\|_{\mathcal{H}}^2} = \langle e(g), e(g) \rangle_{\mathfrak{F}}. \end{aligned}$$

Hence, $W(f)$ preserves inner products on \mathfrak{E} . Since \mathfrak{E} is dense in \mathfrak{F} , $W(f)$ extends uniquely to an isometry on \mathfrak{F} .

In a similar fashion, we compute

$$W(-f)W(f)e(g) = e(g) = W(f)W(-f)e(g) \quad \forall g \in \mathcal{H},$$

i.e., $W(-f)W(f)$ is the identity on the dense set \mathfrak{E} . In particular, $W(f)$ is surjective and an isometry, i.e., $W(f)$ is unitary with $W^\dagger(f) = W(-f)$.

As for the last property, we observe that

$$\begin{aligned} W(f)W(g)W(-(f+g))e(h) &= e^{\langle f+g, h \rangle_{\mathcal{H}} - \frac{1}{2}\|f+g\|_{\mathcal{H}}^2} W(f)W(g)e(-(f+g)+h) \\ &= e^{\langle f+g, h \rangle_{\mathcal{H}} - \frac{1}{2}\|f\|_{\mathcal{H}}^2 - \langle g, -f+h \rangle_{\mathcal{H}} - \text{Re}(\langle g, f \rangle_{\mathcal{H}})} W(f)e(-f+h) \\ &= e^{-i\text{Im}(\langle f, g \rangle_{\mathcal{H}})} e(h), \end{aligned}$$

and hence, $W(f)W(g)W(-(f+g)) = e^{-i\text{Im}(\langle f, g \rangle_{\mathcal{H}})} I_{\mathfrak{F}}$. We then conclude by using property (i) of Weyl operators. \square

Since $W(f)$ is unitary for every $f \in \mathcal{H}$, the family $\{W(tf)\}_{t \in \mathbb{R}}$ forms a one-parameter (strongly continuous) group of unitaries. In particular, due to Stone's theorem, it has a corresponding Hermitian operator $P(f)$ such that

$$W(tf) = \exp(itP(f)).$$

We further define the following operators

$$Q(f) := -P(if), \quad A^-(f) := \frac{Q(f) + iP(f)}{2}, \quad A^+(f) := \frac{Q(f) - iP(f)}{2}.$$

The operators A^\pm are called the *field operators* and will play an essential role as they form the creation/annihilation operators on Fock spaces.

Proposition 4.2. *The following are true: For any $f, g \in \mathcal{H}$,*

$$(i) \quad \mathfrak{E} \text{ is a core for } P(f) \text{ and } [P(f), P(g)] = 2i\text{Im}(\langle f, g \rangle_{\mathcal{H}})I_{\mathfrak{F}}.$$

$$(ii) \quad A^-(f)e(g) = \langle f, g \rangle_{\mathcal{H}} e(g), \quad A^+(f)e(g) = \frac{d}{dt} e(g + tf)|_{t=0}.$$

$$(iii) \quad W^\dagger(f)A^-(g)W(f) = A^-(g) + \langle g, f \rangle_{\mathcal{H}} \mathbf{l}_{\mathfrak{F}}, \quad W^\dagger(f)A^+(g)W(f) = A^+(g) + \langle f, g \rangle_{\mathcal{H}} \mathbf{l}_{\mathfrak{F}}.$$

$$(iv) \quad [A^-(f), A^-(g)] = [A^+(f), A^+(g)] = 0, \quad [A^-(f), A^+(g)] = \langle f, g \rangle_{\mathcal{H}} \mathbf{l}_{\mathfrak{F}},$$

i.e., the field operators A^\pm satisfy the canonical commutation relation.

Remark 4.3 On the finite particle vectors, the field operators act as

$$A^-(f)g^{\otimes n} = \sqrt{n} \langle f, g \rangle_{\mathcal{H}} g^{\otimes(n-1)}, \quad A^+(f)g^{\otimes(n-1)} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g^{\otimes k} \otimes f \otimes g^{\otimes(n-1-k)}.$$

Example 4.4 Let $\mathcal{H} = L_{\mathbb{C}}^2([-\pi, \pi])$ be the space of square-integrable functions. Then the countable set $\{\psi_\ell(x) = e^{i\ell x} : \ell \in \mathbb{Z}\}$ forms an orthonormal basis for \mathcal{H} .

$$\psi(x) = \sum_{\ell} e^{i\ell x} A_{\ell}^-$$

4.1.2 Second quantization

The term *second quantization* is associated with the action of lifting operators on an k -particle space $\mathcal{H}^{\otimes k}$ to an associated operator on the Fock space.

We begin our discussion with the 1-particle case. For any bounded operator $A \in \mathcal{B}(\mathcal{H})$ one defines the map

$$\Gamma(A) := I \oplus \bigoplus_{n \in \mathbb{N}} A^{\otimes n},$$

which acts on $\mathcal{H}^{\otimes n}$ by

$$\Gamma(A)g_1 \otimes \cdots \otimes g_n = Ag_1 \otimes \cdots \otimes Ag_n.$$

Clearly, if A is unitary, then so is $\Gamma(A)$. Indeed, in this case, one has

$$\langle \Gamma(A)e(g), \Gamma(A)e(g) \rangle_{\mathfrak{F}(\mathcal{H})} = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} \langle Ag, Ag \rangle_{\mathcal{H}}^n = \sum_{n \in \mathbb{N}_0} \frac{1}{n!} \|g\|_{\mathcal{H}}^n = \langle e(g), e(g) \rangle_{\mathfrak{F}(\mathcal{H})}.$$

Now let H be a self-adjoint operator on \mathcal{H} and $U_t := \exp(itH)$ be its unitary evolution. Then, $\Gamma(U_t)$ is a one-parameter group of unitary operators on $\mathfrak{F}(\mathcal{H})$. Stone's theorem then provides the existence of a densely defined Hermitian operator $d\Gamma(H)$ such that

$$\Gamma(U_t) = \exp(it d\Gamma(H)).$$

The generator $d\Gamma(H)$ is called the *second quantization of H* , and takes the explicit form

$$d\Gamma(H)g^{(n)} = \sum_{j=1}^n g_1 \otimes \cdots \otimes g_{j-1} \otimes Hg_j \otimes g_{j+1} \otimes \cdots \otimes g_n = \sum_{j=1}^n H_j g^{(n)},$$

for any $g^{(n)} = g_1 \otimes \cdots \otimes g_n \in \mathcal{H}^{\otimes n}$ with $g_j \in D(H)$ and

$$H_j = I_{\mathcal{H}} \otimes \cdots \otimes I_{\mathcal{H}} \otimes H \otimes I_{\mathcal{H}} \otimes \cdots \otimes I_{\mathcal{H}},$$

where H acts on the j -th tensor product. The special case $H = I_{\mathcal{H}}$ yields

$$\mathbf{N}g^{(n)} := d\Gamma(I_{\mathcal{H}})g^{(n)} = ng^{(n)}, \quad n \in \mathbb{N},$$

and is called the *number operator* due to its diagonal nature, with eigenvalues representing the number of particles in each configuration. Its domain is given by

$$D(\mathbf{N}) = \left\{ \{f^{(n)}\}_{n \in \mathbb{N}_0} : \sum_{n \in \mathbb{N}_0} n^2 \|f^{(n)}\|_{\mathcal{H}^{\otimes n}}^2 < +\infty \right\}.$$

$$\mathbf{A}_j^- \mathbf{A}_j^+ \psi_j^{\otimes n} = \sqrt{n} \mathbf{A}_j^- \psi_j^{\otimes (n-1)} = \sum_{k=0}^{n-1} \psi_j^{\otimes k} \otimes \psi_j \otimes \psi_j^{\otimes (n-1-k)} = n \psi_j^{\otimes n}$$

$$\mathbf{A}_j^+ \mathbf{A}_j^- \mathbf{e}(\psi_j) = \delta_{jk} \mathbf{N}_j \mathbf{e}(\psi_j)$$

This construction can be performed similarly for the general k -particle case.

$$d\Gamma(H^{(k)})f^{(n)} = \sum_{j_1 \neq \dots \neq j_k} H_{j_1 \dots j_k} f^{(n)},$$

where H_{j_1, \dots, j_k} denotes the operator where $H^{(k)}$ acts on the (j_1, \dots, j_k) -th tensor product.

Example 4.5 Consider the Hilbert space $\mathcal{H} = L_{\mathbb{C}}^2(\mathbb{T})$ and the Hamiltonians,

$$H^{(1)} = -\Delta \in \mathcal{O}(\mathcal{H}), \quad H^{(2)} = \mathbf{M}_W \in \mathcal{O}(\mathcal{H}^{\otimes 2}),$$

where $W : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ is an interaction potential, and \mathbf{M}_f denotes the multiplication operator corresponding to f .

Then, their second quantization is given by

$$d\Gamma(H^{(1)}) = \sum_{j=1}^n H_j^{(1)}, \quad d\Gamma(H^{(2)}) = \sum_{j \neq \ell} H_{j\ell}^{(2)} \quad \text{on } \mathcal{H}^{\otimes n}.$$

$$d\Gamma(H^{(1)})\psi_j^{\otimes n} = nj^2\psi_j^{\otimes n} = j^2\mathbf{A}_j^+\mathbf{A}_j^-\psi_j^{\otimes n} = j^2\mathbf{A}_j^+\mathbf{A}_j^-\psi_j^{\otimes n}$$

$$\psi^{(n)} = \psi_{k_1} \otimes \dots \otimes \psi_{k_n}$$

$$d\Gamma(H^{(1)})\psi^{(n)} = \sum_{j=1}^n k_j^2 \psi^{(n)}$$

4.1.3 Free field operators

For each $f \in \mathcal{H}$, we consider the pair $\{a^-(f), a^+(f)\}$ of operators on $\mathfrak{F}_{\text{sym}}(\mathcal{H})$ defined by

$$a^-(f)\Psi =$$

satisfying the commutation relations: For any $f, g \in \mathcal{H}$,

$$[a^\pm(f), a^\pm(g)] = 0, \quad [a^-(f), a^+(g)] = \langle f, g \rangle,$$

where $f \mapsto a^-(f)$ is conjugate linear and $f \mapsto a^+(f)$ is linear. Moreover, if $\Omega \in \mathfrak{F}_{\text{sym}}(\mathcal{H})$ is the *vacuum vector*, then $a^-(f)\Omega = 0$ for every $f \in \mathcal{H}$. The *field operators* a^- and a^+ are called *creation* and *annihilation* operators, respectively. On appropriate domains, the field operators are adjoints of one another, i.e., $(a^-(f))^\dagger = a^+(f)$ for every $f \in \mathcal{H}$.

It is common in the physics literature to consider operator-valued distributions $\{a_x^-, a_x^+\}$ instead, where if $\mathcal{H} = L_{\mathbb{C}}^2(\mathbb{R}^d)$, then $x \in \mathbb{R}^d$ and

$$a^-(f) = \int_{\mathbb{R}^d} \overline{f(x)} a_x dx, \quad a^+(f) = \int_{\mathbb{R}^d} f(x) a_x dx.$$

The commutation relations then simply read

$$[a_x^\pm, a_y^\pm] = 0, \quad [a_x^-, a_y^+] = \delta(x - y).$$

The *number operator* is formally defined by $N_x = a_x^+ a_x$, $x \in \mathbb{R}^d$.

On the rigorous not, if \mathcal{H} is separable with orthonormal basis $\{\psi_i\}$, then one obtains a family of field operators $\{a_i^-, a_i^+\}$ with $a_i^\pm : a^\pm(\psi_i)$.

4.1.4 Gaussian states

Definition 4.6 (Gaussian states) A state Ψ on $\mathfrak{F}_{\text{sym}}(\mathcal{H})$ is said to be a mean-zero *Gaussian* (or *quasi-free*) state if it can be uniquely determined from the field operators $\{a, a^\dagger\}$ by its covariance

$$\Sigma_\Psi(f, g) := \begin{pmatrix} \Psi(a^+(f) a^-(g)) & \Psi(a^-(f) a^-(g)) \\ \Psi(a^+(f) a^+(g)) & \Psi(a^-(f) a^+(g)) \end{pmatrix}, \quad f, g \in \mathcal{H}.$$

If the off-diagonal elements of the covariance are zero, the state Ψ is called *gauge-invariant* since it is invariant under the so-called gauge transformations of the first kind, i.e.,

$$a^\pm(f) = e^{\pm i\alpha} a^\pm(f) \quad \text{for any } \alpha \in \mathbb{R}.$$

If the off-diagonal elements of the covariance are nonzero, the state Ψ is called *squeezed*.

4.1.5 States invariant under free evolutions

Consider a Hamiltonian H on $\mathfrak{F}_{\text{sym}}(\mathcal{H})$ with $\mathcal{H} = L_{\mathbb{C}}^2(\mathbb{R}^d)$ and its associated 1-parameter automorphism group

$$\mathbf{u}_t(a) = e^{itH} a e^{-itH}, \quad t \in \mathbb{R}.$$

Definition 4.7 The Hamiltonian H on $\mathfrak{F}_{\text{sym}}(\mathcal{H})$ is called *free* if there exists a real-valued function $\omega: \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\mathbf{u}_t(a_x^-) = e^{-it\omega(x)} a_x^-, \quad x \in \mathbb{R}^d.$$

In this case, the function ω is called the *free 1-particle Hamiltonian*, and H is said to be the *second quantization* of ω . Accordingly, \mathbf{u}_t is called a *free evolution*.

Example 4.8 Consider a Hamiltonian on $\mathcal{H} = L_{\mathbb{C}}^2(\mathbb{R})$ and its eigensystem $\{(\lambda_i, \psi_i)\}_i$ such that $\{\psi_i\}_i$ forms an orthonormal basis for \mathcal{H} . Setting $a_i^{\pm} = a^{\pm}(\psi_i)$, we see that

$$[a_i^{\pm}, a_j^{\pm}] = 0, \quad [a_i^-, a_j^+] = \delta_{ij},$$

i.e., a_i^{\pm} define field operators on the symmetric Fock space $\mathfrak{F}_{\text{sym}}(\mathcal{K})$ with $\mathcal{K} = \ell_{\mathbb{C}}^2$. Defining the field operators

$$a^- = \frac{1}{\sqrt{2}}(Q + iP), \quad a^+ = \frac{1}{\sqrt{2}}(Q - iP), \quad N := a^+ a^-$$

such that $[a^-, a^+] = I$, we find that

$$H = a^+ a^- + \frac{1}{2} = N + \frac{1}{2}.$$

Clearly, the eigenvectors of N and H coincide

Definition 4.9 A Gaussian state Ψ on $\mathfrak{F}_{\text{sym}}(\mathcal{H})$ with $\mathcal{H} = L^2(\mathbb{R}^d)$ is said to be invariant under a free evolution \mathbf{u}_t if

$$\Psi(\mathbf{u}_t(a_x^-) \mathbf{u}_s(a_y^+)) = \Psi(\mathbf{u}_{t-s}(a_x^-) a_y^+) = \Psi(a_x^- \mathbf{u}_{s-t}(a_y^+)) \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^d.$$

A *Gaussian free state* is a Gaussian state that is invariant under *all* free evolutions.

Theorem 4.10. A Gaussian state Ψ on $\mathfrak{F}_{\text{sym}}(\mathcal{H})$ is a Gaussian free state if and only if it is gauge-invariant and its diagonal correlations are supported on the diagonal, i.e.,

$$\Psi(a_x^- a_y^+) = m(x) \delta(x - y), \quad \Psi(a_x^+ a_y^-) = n(x) \delta(x - y).$$

Theorem 4.11. The field operators $\{a_x^{\pm}, a_x^{\mp}\}$ satisfying the commutation relations

$$[a_x^{\pm}, a_y^{\pm}] = 0, \quad [a_x^-, a_y^+] = m(x) \delta(x - y).$$

are mean-zero Gaussian random variables w.r.t. the Fock vacuum state $\Psi_{\Omega} = \langle \Omega, \cdot \Omega \rangle$, where $\Omega \in \mathfrak{F}_{\text{sym}}(\mathcal{H})$ is the Fock vacuum vector, with covariance

$$\Sigma_{\Psi}(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & m(x) \end{pmatrix} \delta(x - y), \quad x, y \in \mathbb{R}^d.$$

Conversely, if $\{a_x^{\pm}, a_x^{\mp}\}$ are random variables with these properties, then they satisfy the commutation relations above.

4.1.6 Boson Fock white noise

def:boson-white-noise

Definition 4.12 A boson Fock white noise on \mathbb{R}^d is a boson Fock field $\{b_{t,x}^+, b_{t,x}^-\}$ on \mathbb{R}^{d+1} with vacuum vector Ω satisfying the commutation relations

$$[b_{t,x}^\pm, b_{s,y}^\pm] = 0, \quad [b_{t,x}^-, b_{s,y}^+] = \delta(t-s)m(x)\delta(x-y), \quad b_{t,x}^-\Omega = 0.$$

thm:stochastic-limit

Theorem 4.13. Let $\{a_x^+, a_x^-\}$ be Gaussian free fields w.r.t. the Fock vacuum state Ψ_Ω with

$$\mathbf{u}_t(a_x^-) = e^{-it\omega(x)}a_x^-.$$

Then the rescaled field operators

$$b_{t,x}^{\lambda,\pm} := \frac{1}{\lambda} \mathbf{u}_{t/\lambda^2}(a_x^\pm)$$

converges in the sense of correlator distributions to a boson Fock white noise, i.e.,

$$\lim_{\lambda \rightarrow 0} \Psi(b_{t,x}^{\lambda,\varepsilon_1} b_{t,y}^{\lambda,\varepsilon_2}) = \Psi(b_{t,x}^{\varepsilon_1} b_{t,y}^{\varepsilon_2}) \quad \varepsilon_1, \varepsilon_2 \in \{+, -\},$$

where $b_{t,x}^\pm$ as defined in Definition 4.12 with $m = 2\pi\delta(\omega)$.

Proof. Using the invariance of Ψ_Ω under free evolutions, we find

$$\begin{aligned} \Psi_\Omega(b_{t,x}^{\lambda,-} b_{s,y}^{\lambda,+}) &= \frac{1}{\lambda^2} \Psi_\Omega(\mathbf{u}_{(t-s)/\lambda^2}(a_x^-) a_y^+) \\ &= \frac{1}{\lambda^2} e^{-i\omega(x)(t-s)/\lambda^2} \Psi_\Omega(a_x^- a_y^+) = \frac{1}{\lambda^2} e^{-i\omega(x)(t-s)/\lambda^2} \delta(x-y). \end{aligned}$$

Passing to the limit $\lambda \rightarrow 0$ recovers the desired limit. All the other terms vanish. \square

Remark 4.14 Let $\mathcal{K} \subset L^2(\mathbb{R}^d)$ be a set of functions for which

$$\int_{\mathbb{R}} |\langle f, e^{it\omega} g \rangle| dt < +\infty \quad \forall f, g \in \mathcal{K}.$$

Since $t \mapsto \langle f, e^{it\omega} f \rangle$ is positive definite for each $f \in \mathcal{K}$, Bochner's theorem implies that the sesquilinear form

$$\langle f, 2\pi\delta(\omega)g \rangle := \int_{\mathbb{R}} \langle f, e^{it\omega} g \rangle dt,$$

is a pre-scalar product. With $(\cdot|\cdot)$, the set \mathcal{K} becomes a pre-Hilbert space, which can be completed to obtain a Hilbert space, still denoted by \mathcal{K} . The function $m = 2\pi\delta(\omega)$ has to be understood in this sense, and only makes sense for functions in \mathcal{K} .

The operators

$$B_t^-(f) := \int_0^t \int_{\mathbb{R}^d} \overline{f(x)} b_{s,x}^- dx ds, \quad B_t^+(f) := \int_0^t \int_{\mathbb{R}^d} f(x) b_{s,x}^+ dx ds,$$

define *quantum Brownian motions*. The self-adjoint (*momentum*) operators

$$P_t(f) := \frac{1}{i} [B_t^-(f) - B_t^+(f)],$$

form a commuting family of classical random variables whose statistics in the Fock vacuum state $\Psi_\Omega = \langle \Omega, \cdot \Omega \rangle$ is completely determined by the relation

$$\Psi(e^{iP_t(f)}) = \exp\left(-\frac{t}{2} \|f\|_{L^2(\mathbb{R}^d)}^2\right). \quad \text{check!}$$

4.1.7 Gaussian equilibrium states: The KMS condition

For any states $a, b \in \mathcal{A}$, the map $t \mapsto \Psi(a \mathbf{u}_t(b))$ can be analytically continued and satisfies the so-called *KMS condition* at inverse temperature $\beta > 0$:

$$\Psi(a \mathbf{u}_{t+i\beta}(b)) = \Psi(\mathbf{u}_t(a) b) \quad \forall a, b \in \mathcal{A}. \quad \text{check!}$$

4.2 Composite systems

Definition 4.15 A composite system of two given quantum dynamical systems $S = \{\mathcal{H}_S, H_S\}$, $R = \{\mathcal{H}_R, H_R\}$ is a quantum dynamical system of the form

$$\{\mathcal{H}_S \otimes \mathcal{H}_R, H_{SR}\}, \quad H_{SR} = H_S \otimes 1_R + 1_S \otimes H_R + H_I,$$

where H_I is called the *interaction Hamiltonian* and contains all the new physics of the composite system, while $H_0 := H_S \otimes 1_R + 1_S \otimes H_R$ is called the free Hamiltonian.

We will consider *scaled* total Hamiltonians

$$H^\lambda := H_0 + \lambda H_I,$$

and the following unitary evolutions:

$$\begin{aligned} \text{free evolution} \quad V_t^0 &= e^{-itH_0}, & \text{total evolution} \quad V_t^\lambda &= e^{-itH^\lambda}, \\ \text{interacting representation evolution} \quad U_t^\lambda &= (V_t^0)^\dagger V_t^\lambda, \end{aligned}$$

where U_t^λ satisfies the Schrödinger equation in the interaction picture:

$$\partial_t U_t^\lambda = -i\lambda H_I(t) U_t^\lambda, \quad U_0^\lambda = I,$$

with the time dependent Hamiltonian $H_I(t) = (V_t^0)^\dagger H_I V_t^0$.

For simplicity, we will make the following assumptions on system $S = \{\mathcal{H}_S, H_S\}$, the reservoir $R = \{\mathcal{H}_R, H_R\}$, and the interaction Hamiltonian H_I .

4.2.1 The reservoir

The reservoir $R = \{\mathcal{H}_R, H_R\}$ is given by the Hilbert space $\mathfrak{H}_{\text{sym}}(\mathcal{H})$ with $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^d)$, a free Hamiltonian H_R with continuous spectrum \mathbb{R} , and a mean-zero Gaussian free state Ψ_R such that

$$\int_{\mathbb{R}} |\Psi_R(a_x^{\varepsilon_1} \mathbf{u}_t(a_y^{\varepsilon_2}))| dt < +\infty, \quad \varepsilon_1, \varepsilon_2 \in \{+, -\}.$$

In particular, Ψ_R is characterized by the covariances

$$\Psi_R(a_x^- a_y^+) = m(x) \delta(x - y), \quad \Psi_R(a_x^+ a_y^-) = n(x) \delta(x - y).$$

Since H_R is free, there exists a function ω , for which

$$\mathbf{u}_t(a_x^-) = e^{itH_R} a_x^- e^{-itH_R} = e^{-it\omega(x)} a_x^-,$$

where ω describes the 1-particle evolution.

An example of a free reservoir Hamiltonian is given by

$$H_R = \int_{\mathbb{R}^d} \omega(x) a_x^+ a_x^- dx,$$

where ω is a smooth cutoff function.

4.2.2 The system Hamiltonian

For simplicity, we shall assume that the system Hamiltonian H_S has a discrete spectrum such that

$$H_S = \sum_j \lambda_j P_j,$$

where λ_j are the eigenvalues and P_j are their corresponding spectral projections.

4.2.3 The interaction Hamiltonian

We consider dipole-type interaction Hamiltonians of the form

$$H_I = \int_{\mathbb{R}^d} [D(x) \otimes a_x^+ + D^\dagger(x) \otimes a_x^-] dx,$$

where $\{D(x) : x \in \mathbb{R}^d\}$ is a family of system operators called the *response terms*.

With the spectral projections of H_S , we may express H_I as

$$H_I = \sum_{j,k} \int_{\mathbb{R}^s} [P_j D(x) P_k \otimes a_x^+ + P_k D^\dagger(x) P_j \otimes a_x^-] dx,$$

and hence, the time-dependent Hamiltonian reads

$$\begin{aligned} H_I(t) &= \sum_{j,k} \int_{\mathbb{R}^d} [P_j D(x) P_k \otimes e^{it(\omega(x)+\lambda_j-\lambda_k)} a_x^+ + P_k D^\dagger(x) P_j \otimes e^{-it(\omega(x)+\lambda_j-\lambda_k)} a_x^-] dx \\ &= \sum_q \sum_{\lambda_k-\lambda_j=\eta_q} \int_{\mathbb{R}^d} [P_j D(x) P_k \otimes e^{it(\omega(x)-\eta_q)} a_x^+ + P_k D^\dagger(x) P_j \otimes e^{-it(\omega(x)-\eta_q)} a_x^-] dx \\ &= \sum_q \int_{\mathbb{R}^d} [D_q(x) \otimes e^{it(\omega(x)-\eta_q)} a_x^+ + D_q^\dagger(x) \otimes e^{-it(\omega(x)-\eta_q)} a_x^-] dx, \end{aligned}$$

where the system operators

$$D_q(x) := \sum_{\lambda_k-\lambda_j=\eta_q} P_j D(x) P_k \quad \text{satisfy} \quad e^{itH_S} D_q(x) e^{-itH_S} = e^{-it\eta_q} D_q(x).$$

To simplify things drastically, we assume that $q = 1$ and $D_1(x) = \chi(x)D$ for some smooth cutoff function χ and a fixed system operator D . In this case, we obtain

$$H_I(t) = \int_{\mathbb{R}^d} [D \otimes \chi(x) e^{it(\omega(x)-\eta)} a_x^+ + D^\dagger \otimes \overline{\chi(x)} e^{-it(\omega(x)-\eta)} a_x^-] dx.$$

4.3 The weak interaction stochastic limit

Altogether, we arrive at the rescaled Schrödinger equation in the interaction picture

$$U_{t/\lambda^2}^\lambda = I - i \int_0^t H_I^\lambda(s) U_{s/\lambda^2}^\lambda ds, \quad \text{eq:rescaled-schrodinger-interaction} \quad (4.1)$$

with $H_I^\lambda(t) = D \otimes b_t^{\lambda,+} + D^\dagger \otimes b_t^{\lambda,-}$, where, due to Theorem 4.13,

$$b_t^{\lambda,\pm} = \frac{1}{\lambda} a^\pm(\chi e^{i(t/\lambda^2)(\omega-\eta)}) \longrightarrow b_t^\pm \quad \text{in the sense of correlators,}$$

and therefore,

$$H_I^\lambda(t) \longrightarrow H_t = D \otimes b_t^+ + D^\dagger \otimes b_t^-, \quad U_{t/\lambda^2}^\lambda \longrightarrow U_t,$$

where U_t satisfies the formal stochastic differential equation

$$dU_t = -i(D \otimes dB_t^+ + D^\dagger \otimes dB_t^-)U_t, \quad B_t^\pm = \int_0^t b_s^\pm ds. \quad (\text{eq:sde}) \quad (4.2)$$

Under certain assumptions on ω , this SDE has a unique solution.

The SDE (4.2) in natural-time order (or Itô form) is the SDE given by

$$dU_t = -i(D dB_t^+ U_t + D^\dagger U_t dB_t^-) - \gamma_- D^\dagger D U_t dt, \quad (\text{eq:sde}^{\text{normal}}) \quad (4.3)$$

obtained by commuting $dB_t^- U_t$ and using the fact that the solution U_t to (4.2) satisfies

$$\begin{aligned} [b_t^-, U_t] &= -i\gamma_- D U_t, \\ [U_t^\dagger, b_t^+] &= i\bar{\gamma}_- U_t^\dagger D^\dagger, \quad \text{where } \gamma_- := \int_{-\infty}^0 \langle \chi, e^{-it(\omega-\eta)} \chi \rangle dt, \\ [b_t^-, U_t^\dagger] &= i\gamma_- U_t^\dagger D, \end{aligned}$$

The additional term is known as the Itô correction term in quantum stochastic calculus. Morally, the natural-time order is the order induced by the filtration for which one expects U_t to be adapted to. **Expand on the concept of normal order!**

With the unitary evolution U_t one can derive an evolution for any system observable $X_t = U_t^\dagger (X \otimes 1_{\mathcal{H}_R}) U_t$ given by

$$dX_t = -i dB_t^+ [D, X_t] + i[X_t, D^\dagger] dB_t^- + LX_t dt$$

where

$$LX = 2\Re(\gamma_-) D^\dagger X D - \gamma_- D^\dagger D X - \bar{\gamma}_- X D^\dagger D,$$

is the corresponding Lindblad operator.

Remark 4.16 The rescaled equation (4.1) may be formally expressed as

$$U_{t/\lambda^2}^\lambda = I + \sum_{n=1}^{\infty} (-i)^n \lambda^n \int_0^{t/\lambda^2} \cdots \int_0^{t_{n-1}} H_I(t_1) \cdots H_I(t_n) dt_1 \cdots dt_n.$$

In the case when $H_I(t)$ commutes for every $t \geq 0$, the integrand would be a symmetric function of t_1, \dots, t_n , and gives

$$U_{t/\lambda^2}^\lambda = I + \sum_{n=1}^{\infty} (-i)^n \frac{\lambda^n}{n!} \left(\int_0^{t/\lambda^2} H_I(s) ds \right)^n = \exp \left(-i\lambda \int_0^{t/\lambda^2} H_I(s) ds \right),$$

i.e., the expectation of U_{t/λ^2}^λ is the characteristic function of the process

$$W_t^\lambda := \lambda \int_0^{t/\lambda^2} H_I(s) ds.$$

4.4 Damped harmonic oscillator

We consider a simple setup in which a single atom interacts with an electromagnetic field. The total system is given by the composite of the atom and the reservoir systems

$$S = \{\mathfrak{F}_{\text{sym}}(\mathbb{C}^2), H_S\}, \quad R = \{\mathfrak{F}_{\text{sym}}(L^2(\mathbb{R}^d)), H_R\},$$

with the free Hamiltonians

$$H_S = \omega_0 c^+ c^-, \quad H_R = \int_{\mathbb{R}^d} \omega(x) a_x^+ a_x^- dx,$$

where $\{c^+, c^-\}$ is the field operator for $\mathcal{H}_S = \mathfrak{F}_{\text{sym}}(\mathbb{C}^2)$, $\{a_x^+, a_x^-\}$ are the field operators for $\mathcal{H}_R = \mathfrak{F}_{\text{sym}}(L^2(\mathbb{R}^d))$, and ω is a suitable cutoff function.

For the interaction Hamiltonian H_I , we consider a dipole approximation of the form

$$H_I = \int_{\mathbb{R}^d} \chi(x) [c^- \otimes a_x^+ + c^+ \otimes a_x^-] dx = c^- \otimes A^+ + c^+ \otimes A^-,$$

where χ are suitable cutoff function and

$$A^\pm = \int_{\mathbb{R}^d} \chi(x) a_x^\pm dx.$$

such that the rescaled total Hamiltonian is given by

$$H^\lambda = H_S \otimes 1_{\mathcal{H}_R} + 1_{\mathcal{H}_S} \otimes H_R + \lambda H_I.$$

Define the evolution

$$\mathbf{u}_t^\lambda(a) = e^{itH^\lambda} a e^{-itH^\lambda}.$$

Then, the Heisenberg equation for $c(t) = \mathbf{u}_t^\lambda(c \otimes 1_{\mathcal{H}_R})$ and $a_x^-(t) = \mathbf{u}_t^\lambda(1_{\mathcal{H}_S} \otimes a_x^-)$ reads

$$\begin{aligned} \frac{d}{dt} c^-(t) &= -i\omega_0 c^-(t) - i\lambda \int_{\mathbb{R}^d} \chi(x) a_x^-(t) dx \\ \frac{d}{dt} a_x^-(t) &= -i\omega(x) a_x^-(t) - \lambda \chi(x) c(t). \end{aligned}$$

Solving for $a_x^-(t)$, one obtains

$$a_x^-(t) = a_x^- e^{-it\omega(x)} - i\lambda \chi(x) \int_0^t e^{-i(t-s)\omega(x)} c^-(s) ds.$$

Inserting $a_x^-(t)$ into the equation for $c(t)$, we find

$$\frac{d}{dt} c^-(t) = -i\omega_0 c^-(t) - \int_0^t \gamma(t-s) c^-(s) ds - i\xi_t^-.$$

where we defined the quantities

$$\gamma(r) := \lambda^2 \int_{\mathbb{R}^d} e^{-ir\omega(x)} \chi^2(x) dx, \quad \xi_t^- := \lambda \int_{\mathbb{R}^d} e^{-it\omega(x)} \chi(x) a_x^- dx.$$

Observe that ξ_t^- depends only on the reservoir field operator a_x^- , and therefore, acts as an *external force* to the atomic system.

Now let us consider the statistics of ξ_t^- for the Fock vacuum state free state $\Psi_\Omega = \langle \Omega, \cdot \Omega \rangle$ on \mathcal{H}_R . Since Ψ_Ω is a Gaussian free state, the statistics of ξ_t^- are uniquely determined by the 2-points correlation functions. Clearly, Ψ_Ω has zero mean, and therefore,

$$\Psi_\Omega(\xi_t^-) = \lambda \int_{\mathbb{R}^d} e^{-it\omega(x)} \chi(x) \Psi_\Omega(a_x^-) dx = 0, \quad \Psi_\Omega((\xi_t^-)^\dagger) = 0.$$

Moreover, we find

$$\Psi_\Omega((\xi_t^-)^\dagger \xi_s^-) = \Psi_\Omega(\xi_t^- \xi_s^-) = \Psi_\Omega((\xi_t^-)^\dagger (\xi_s^-)^\dagger) = 0,$$

and

$$\begin{aligned} \Psi_\Omega(\xi_t^- (\xi_s^-)^\dagger) &= \lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-it(\omega(x)-\omega(y))} \chi(x) \chi(y) \Psi_\Omega(a_x^- a_y^+) dx dy \\ &= \lambda^2 \int_{\mathbb{R}^d} e^{-i(t-s)\omega(x)} \chi^2(x) dx = \gamma(t-s). \end{aligned}$$

Hence, ξ_t^- is a mean-zero γ -correlated *random process* under Ψ .

To obtain the stochastic limit, we start by rescaling time $t \mapsto t/\lambda^2$ and consider the rescaled quantity $c^{\lambda,\pm}(t) := c^\pm(t/\lambda^2)$, which yields

$$\dot{c}^{\lambda,-}(t) = -i\omega_0 c^{\lambda,-}(t) - \int_0^t \gamma^\lambda(t-s) c^{\lambda,-}(s) ds - i \xi_t^{\lambda,-}, \quad \text{eq:quantum-pre-langevin} \quad (4.4)$$

with

$$\gamma^\lambda(r) := \frac{1}{\lambda^2} \langle \chi, e^{-i(r/\lambda^2)\omega} \chi \rangle, \quad \xi_t^{\lambda,-} := \lambda^{-2} \xi_{t/\lambda^2}^-.$$

From Theorem 4.13, we then establish that

$$\gamma^\lambda(r) \longrightarrow \gamma \delta(r), \quad \xi_t^{\lambda,-} \longrightarrow b_t^- \quad \text{in the sense of correlators,}$$

with $\gamma = \langle \chi, \delta(\omega) \chi \rangle$. Consequently, we obtain

$$dc_t^- = -(i\omega_0 + \gamma) c_t^- dt - i dB_t^-, \quad B_t^- = \int_0^t b_s^- ds.$$

$$dU_t = -i(c^- dB_t^+ U_t + c^- U_t dB_t^-) - \gamma_- c^+ c^- U_t dt$$

$$\begin{aligned} d(U_t^\dagger c^- U_t) &= i(c^- dB_t^+ U_t + c^+ U_t dB_t^-)^\dagger c^- U_t - \bar{\gamma}_- U_t^\dagger c^- c^+ c^- U_t dt \\ &\quad - iU_t^\dagger c^- (c^- dB_t^+ U_t + c^+ U_t dB_t^-) - \gamma_- U_t^\dagger c^- c^+ c^- U_t dt \\ &= iU_t^\dagger dB_t^- c^+ c^- U_t + i dB_t^+ U_t^\dagger c^- c^- U_t - \bar{\gamma}_- U_t^\dagger c^- c^+ c^- U_t dt \end{aligned}$$