Gradient Structures
from Classical to Quantum

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Abstract.

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1 Noncommutative probability

1.1 From commutative to noncommutative

Consider the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\Omega, \mu)$ over the complete probability space $(\Omega, \mathcal{F}, \mu)$. Then any function $f \in L^\infty_{\mathbb{C}}(\Omega, \mu)$ gives rise to a multiplication operator $\mathsf{M}_f \in \mathcal{B}(\mathcal{H})$:

$$\mathsf{M}_f\, g = fg \in \mathcal{H} \qquad \forall\, g \in \mathcal{H},$$

with $\|\mathsf{M}_f\|_\infty = \|f\|_{L^\infty(\mu)}$. The collection of all such multiplication operators

$$\mathscr{A} := \left\{\mathsf{M}_f : f \in L^\infty_{\mathbb{C}}(\Omega, \mu)\right\} \subset \mathcal{B}(\mathcal{H})$$

forms a *commutative* subalgebra of $\mathcal{B}(\mathcal{H})$.

In fact, this subalgebra is a possibly noncommutative *von Neumann* algebra:

Definition 1.1 (von Neumann algebra) A (unital) *von Neumann algebra* is a $*$-subalgebra $\mathscr{A} \subset \mathcal{B}(\mathcal{H})$ that contains $I_{\mathcal{H}}$ and is closed in the *weak operator topology* (WOT), i.e.,

$$\text{WOT-}\lim \mathsf{a}_n = \mathsf{a} \quad \Longleftrightarrow \quad \langle f, \mathsf{a}_n g\rangle_{\mathcal{H}} \to \langle f, \mathsf{a} g\rangle_{\mathcal{H}} \quad \forall f, g \in \mathcal{H}.$$

Equivalently, is it a C^*-algebra with a predual $\mathscr{A}_*$ that is a Banach space.

There are sub-families of $\mathscr{A}$ that play a distinguished role, namely,

$$\begin{aligned} \mathscr{A}_+ &:= \left\{\mathsf{a} \in \mathscr{A} : \mathsf{a} \succeq 0\right\} && \text{(nonnegative operators)} \\ \mathscr{O} &:= \left\{\mathsf{a} \in \mathscr{A} : \mathsf{a}^\dagger = \mathsf{a}\right\} && \text{(Hermitian operators)} \\ \mathscr{P} &:= \left\{\mathsf{a} \in \mathcal{O} : \mathsf{a}^2 = \mathsf{a}\right\} && \text{(projection operators)} \end{aligned}$$

Here, $\mathsf{a} \succeq 0$ if and only if $\langle f, \mathsf{a} f\rangle_{\mathcal{H}} \geq 0$ for all $f \in \mathcal{H}$. Additionally, for von Neumann algebras, we also have the following useful density result:

Proposition 1.2. *Let $\mathscr{A}$ be a von Neumann algebra. Then*

$$\mathscr{A} = \overline{span\,(\mathscr{P})}^{\text{WOT}},$$

i.e., $\mathscr{A}$ is the WOT-*closure of the linear span of its projections.*

Let us return to our previous example with $\mathscr{A} = \{\mathsf{M}_f : f \in L^\infty_{\mathbb{C}}(\Omega, \mu)\}$. It is not difficult to see that $\mathscr{A}$ is a W^*-algebra. Notice that since M_f is self-adjoint for $f \in L^\infty_{\mathbb{R}}(\Omega, \mu)$, the family of self-adjoint operators is given by $\mathscr{O} = \{\mathsf{M}_f : f \in L^\infty_{\mathbb{R}}(\Omega, \mu)\}$ and the family of projection operators is given by $\mathscr{P} = \{\mathsf{M}_f \in \mathscr{O} : f = \mathbf{1}_A,\ A \in \mathcal{F}\}$.

On a von Neumann algebra $\mathscr{A}$, we can talk about special types of continuous linear functionals on $\mathscr{A}$, called states.

Definition 1.3 A *state* on a von Neumann algebra $\mathscr{A}$ is a linear functional $\psi : \mathscr{A} \to \mathbb{C}$ that is positive and normalized, i.e., $\psi(\mathsf{a}^\dagger \mathsf{a}) \geq 0$ for all $\mathsf{a} \in \mathscr{A}$ and $\psi(I_{\mathcal{B}(\mathcal{H})}) = 1$.
A state ψ is said to be

faithful if $\psi(\mathsf{a}^\dagger \mathsf{a}) = 0 \ \Leftrightarrow\ \mathsf{a} = 0$,

tracial if $\psi(\mathsf{ab}) = \psi(\mathsf{ba})$ for all $\mathsf{a}, \mathsf{b} \in \mathscr{A}$, and

normal if $\psi \in \mathscr{A}_*$, i.e., if it is an element of the predual $\mathscr{A}_*$.

For any probability measure $\nu \ll \mu$, we set

$$\psi_\nu(\mathsf{M}_f) := \int_\Omega f \, \mathrm{d}\nu, \qquad f \in L^\infty_{\mathbb{C}}(\Omega, \mu),$$

we find that ψ is a linear functional that is positive and normalized, i.e., ψ is a state. Moreover, it is tracial. It is normal if $\omega := d\nu/d\mu \in L^1(\Omega, \mu)$ and faithful if $\omega > 0$.

Normal states play an essential role, serving as a counterpart to classical measures, as made explicit by the following proposition.

prop:normal-state

Proposition 1.4. *Let ψ be a state on a von Neumann algebra $\mathscr{A}$. The following are equivalent:*

(i) *ψ is a normal state.*

(ii) *(σ-additivity) If $(\mathsf{a}_n) \subset \mathcal{P}$ are mutually orthogonal projections, i.e., $\mathsf{a}_n(\mathcal{H}) \perp \mathsf{a}_m(\mathcal{H})$ for all $n \neq m$, and $\mathsf{a} = \vee_n \mathsf{a}_n$ being the projection on the smallest closed subspace containing $\cup_n \mathsf{a}_n(\mathcal{H})$, then*

$$\psi(\mathsf{a}) = \sum\nolimits_n \psi(\mathsf{a}_n).$$

(iii) *(Continuity from below) For any increasing net $0 \preceq \mathsf{a}_n \uparrow \mathsf{a}$ in $\mathscr{A}_+$, one has the increasing limit $\psi(\mathsf{a}_n) \uparrow \psi(\mathsf{a})$.*

(iv) *There exists a family $\{\xi_n\} \subset \mathcal{H}$ with $\sum_n \|\xi_n\|^2_{\mathcal{H}} = 1$ such that*[1]

$$\psi = \sum\nolimits_n \langle \xi_n, \cdot\, \xi_n \rangle \qquad \textit{in the sense of norm convergence.}$$

Consider an arbitrary normal state ψ and set

$$\mu(A) := \psi(\mathsf{M}_{\mathbf{1}_A}) \qquad \text{for every } A \in \mathcal{F}.$$

Then, clearly, $\mu(\emptyset) = 0$, $\mu(\Omega) = \psi(I_{\mathcal{B}(\mathcal{H})}) = 1$ and μ is σ-additive due to the equivalent characterization of a normal state given by Proposition 1.4(ii). In particular, one obtains a classical measure on $(\Omega, \mathcal{F})$. In this sense, a state on a noncommutative von Neumann algebra generalizes that of a classical measure.

[1] [Theorem 7.1.8, Fundamentals of the Theory of Operator Algebras, V.II, Kadison-Ringrose]

1.2 Observables

In classical probability, we are often interested in computing expressions like $\mathbb{P}(X \in A)$, where X is a random variable and $A \subset \mathbb{R}$ is a Borel set.

In the quantum world, a random variable is modelled by a Hermitian operator $\mathsf{a} \in \mathscr{O}$ and is called an *observable*. Observables have their spectrum in $\mathbb{R}$ and can therefore be *measured*.

Given a ψ on a von Neumann algebra $\mathscr{A}$, the expectation of an observable a w.r.t. the state ψ is given by $\psi(\mathsf{a})$. Since every observable $\mathsf{a} \in \mathscr{O}$ has a spectral decomposition

$$\mathsf{a} = \int_{\mathbb{R}} \lambda\, E_{\mathsf{a}}(d\lambda), \qquad E_{\mathsf{a}} \mathrel{\widehat{=}} \mathscr{P}\text{-valued measure},$$

we can associate a classical probability measure with the observable a and state ψ:

$$\mathbb{P}_{\psi}(\mathsf{a} \in A) := \psi(E_{\mathsf{a}}(A)) \qquad \text{for all Borel set } A \subset \mathbb{R}.$$

For two observables $\mathsf{a}, \mathsf{b} \in \mathscr{O}$ that do not commute, one would like to be able to write $\mathbb{P}_{\psi}(\mathsf{a} \in A, \mathsf{b} \in B)$ for two Borel sets $A, B \subset \mathbb{R}$. The simple argument for this is that if $[\mathsf{a}, \mathsf{b}] \neq 0$, then $E_{\mathsf{a}}(A)$ and $E_{\mathsf{b}}(B)$ may not commute for all Borel sets A, B. In particular,

$$\psi(E_{\mathsf{a}}(A)E_{\mathsf{b}}(B)) \neq \psi(E_{\mathsf{b}}(B)E_{\mathsf{a}}(A)),$$

so there is no consistent way of writing $\mathbb{P}_{\psi}(\mathsf{a} \in A, \mathsf{b} \in B)$. Yet, when they do commute, then $\psi(E_{\mathsf{a}}(A)E_{\mathsf{b}}(B)) = \psi(E_{\mathsf{b}}(B)E_{\mathsf{a}}(A))$ and a joint distribution exists and we are back to the classical scenario.

However, this turns out not to be possible, as the following example portrays.

Example 1.5 (Stern-Gerlach measurements) Take a beam of atoms (each with spin-$\frac{1}{2}$) and perform the following steps:

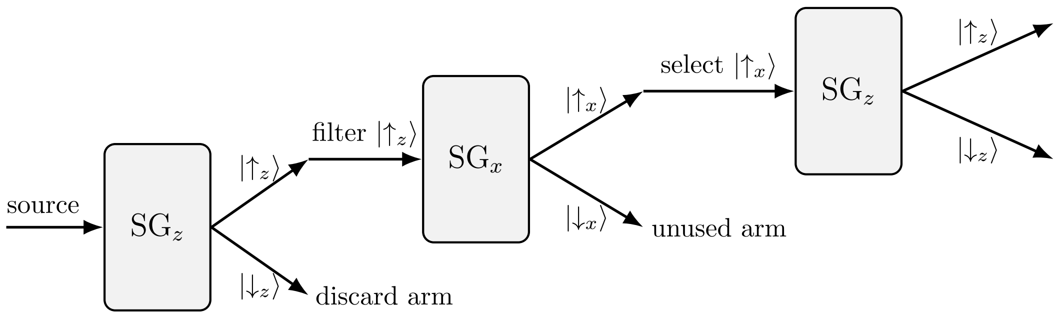

Figure 1: Stern-Gerlach experiment

SG$_z$ Measure spin in z-direction: Send the beam through a Stern-Gerlach magnet oriented along z. The beam splits into spin-up $|\uparrow\rangle$ and spin-down $|\uparrow\rangle$ paths. Keep only the spin-up branch, which gives a pure state $|\uparrow_z\rangle$.

SG$_x$ Measure spin in x-direction: Now send the filtered beam through a second magnet, oriented along x. The beam splits again into $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ outcomes with 50%–50% probability. So far, this can *still* be explained with classical probability.

SG$_z$ Measure spin in z-direction again: Send either of the filtered beams $|\uparrow_x\rangle$ or $|\downarrow_x\rangle$ through a z-magnet again. You do *not* get back the original result, i.e., instead of remaining spin-up, you see another 50%–50% split between $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$

Morally, if spin-x and spin-z were classical random variables, measuring x would *erase* knowledge of z. In classical probability, observing one property never randomizes another unless there is *hidden causal disturbance*.

Quantum mechanically, the two observables (or random variables)

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{spin in } z\text{-direction}, \qquad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{spin in } x\text{-direction}.$$

do not commute, i.e., $[\sigma_x, \sigma_y] \neq 0$, and measuring σ_x *changes* information about σ_z. In this sense, they cannot be jointly sampled, i.e., there is no classical joint probability $p(\sigma_x, \sigma_z)$ for these observables, which illustrates the need for noncommutative probability.

2 Stochastic Dilation

2.1 Noncommutative random variables

def:random-variable

Definition 2.1 A *noncommutative $\mathfrak{X}$-valued random variable* on a quantum probability space $(\mathfrak{A}, \mu)$ is an identity preserving $*$-homomorphism

$$\mathfrak{z}\colon \mathfrak{X} \to \mathfrak{A}.$$

Correspondingly, a *noncommutative $\mathfrak{X}$-valued stochastic process* on a quantum probability space $(\mathfrak{A}, \mu)$ is a family of random variables

$$\mathfrak{z}_t\colon \mathfrak{X} \to \mathfrak{A}, \quad t \in \mathbb{T},$$

with $\mathbb{T}$ being a (possibly uncountable) index set.

Example 2.2 (Classical random variable) Let X be a classical E-valued random variable on $(\Omega, \mathbb{P})$. Setting

$$\mathfrak{X} := B_b(E), \qquad \mathfrak{A} := L^\infty(\Omega, \mathbb{P}), \qquad \mu(g) := \int_\Omega g \,\mathrm{d}\mathbb{P}, \quad g \in \mathfrak{A},$$

we have that

$$\mathfrak{X} \ni f \mapsto \mathfrak{z}(f) := f \circ X \in L^\infty(\Omega, \mathbb{P})$$

is a noncommutative $\mathfrak{X}$-valued random variable. We see here that the noncommutative notion of a random variable is 'dual' to the classical notion of a random variable.

ex:stochastic-classical

Example 2.3 (Classical stochastic process) Consider the space of continuous paths (or trajectories) $\Omega := \mathcal{C}_0(\mathbb{R}_+; E)$ starting at 0 and the Wiener measure R. Let $X = (X_t)_{t\in\mathbb{T}}$ be the canonical stochastic process $X_t(\omega) = \omega(t)$, $t \in \mathbb{R}_+$. As before, we set

$$\mathfrak{X} := B_b(E), \qquad \mathfrak{A} := L^\infty(\Omega, \mathsf{R}), \qquad \mu(G) := \int_\Omega G \,\mathrm{d}\mathsf{R}, \quad G \in \mathfrak{A}.$$

Then,

$$\mathfrak{X} \ni f \mapsto \mathfrak{z}_t(f) := f \circ X_t \in L^\infty(\Omega, \mathsf{R})$$

defines a noncommutative $\mathfrak{X}$-valued stochastic process on $(\mathfrak{A}, \mu)$.

Definition 2.4 A stochastic process $\mathfrak{z}_t\colon \mathfrak{X} \to (\mathfrak{A}, \mu)$, $t \in \mathbb{T}$ admits a *time translation* if there are $*$-homomorphisms $\alpha_t\colon \mathfrak{A} \to \mathfrak{A}$, $t \in \mathbb{T}$ such that

(1) $\alpha_{t+s} = \alpha_t \circ \alpha_s$ for all $s, t \in \mathbb{T}$, and

(2) $\mathfrak{z}_t = \alpha_t \circ \mathfrak{z}_0$ for all $t \in \mathbb{T}$.

Example 2.5 Consider the Wiener space (Ω, R) in Example 2.3. Then, R is stationary under the time-shift operator $\mathfrak{s}_t\omega = \omega(\cdot - t)$, i.e., $(\mathfrak{s}_t)_\#\mathsf{R} = \mathsf{R}$ for every $t \in \mathbb{T}$. Hence, the $*$-homomorphism

$$\alpha_t(G) = G \circ \mathfrak{s}_t, \quad G \in \mathfrak{A}, \ t \in \mathbb{T}$$

is a time translation for the corresponding process $(\mathfrak{z}_t)_{t\in\mathbb{T}}$ and leaves the state μ invariant. Indeed, property (1) holds trivially, while property (2) follows from

$$\alpha_t \circ \mathfrak{z}_0(f) = f \circ X_0 \circ \mathfrak{s}_t = f \circ X_t = \mathfrak{z}_t(f), \qquad f \in \mathfrak{X}.$$

Finally, we conclude with

$$\mu(\alpha_t(G)) = \int_\Omega \alpha_t(G)\,\mathrm{dR} = \int_\Omega G \circ \mathfrak{s}_t\,\mathrm{dR} = \int_\Omega G\,\mathrm{d}(\mathfrak{s}_t)_\#\mathsf{R} = \int_\Omega G\,\mathrm{dR} = \mu(G),$$

thus proving the invariance of μ.

For $h \in L^2(\mathbb{T})$ we consider the exponential martingale

$$\mathsf{Z}_t = \exp\left(-\int_0^t h_r\,\mathrm{d}B_r - \frac{1}{2}\int_0^t |h_r|^2\,\mathrm{d}r\right), \quad t \in \mathbb{T}.$$

It is easy to see that $(\mathsf{Z}_t)_{t\in\mathbb{T}}$ satisfies

$$\mathrm{d}\mathsf{Z}_t = -h_t \mathsf{Z}_t\,\mathrm{d}B_t, \qquad \mathsf{Z}_0 = 1.$$

Setting $\mathsf{Q} \coloneqq \mathsf{Z}_T\mathsf{P}$, we claim that

$$W_t = B_t + \int_0^t h_r\,\mathrm{d}r \quad \text{is a } \mathsf{Q}\text{-martingale.}$$

To see this, we note that W is a Q-martingale if and only if $\mathsf{Z}_T W$ is a P-martingale. Since

$$\begin{aligned}\mathrm{d}(Z_t W_t) &= \mathrm{d}Z_t W_t + Z_t \mathrm{d}W_t + \mathrm{d}[Z, W]_t \\ &= -h_t Z_t \mathrm{d}B_t + Z_t \mathrm{d}B_t = -(h_t - 1) Z_t \mathrm{d}B_t,\end{aligned}$$

we conclude that $\mathsf{Z}_T W$ is indeed a P-martingale.

$$\mathrm{d}U_t = -\big(a^* a\,\mathrm{d}t + i\sqrt{2}\,a\,\mathrm{d}B_t\big)U_t$$

In the classical case, the notion of adaptedness plays an important role for stochastic processes. In this regard, we choose to simply consider the natural filtration induced by a stochastic process.

Definition 2.6 Let $\mathfrak{z}_t \colon \mathfrak{X} \to (\mathfrak{A}, \mu)$, $t \in \mathbb{T}$ be a stochastic process. For $\mathbb{I} \subset \mathbb{T}$, we denote by $\mathfrak{A}_\mathbb{I}$ the subalgebra of $\mathfrak{A}$ generated by $\{\mathfrak{z}_t(x) : x \in \mathfrak{X},\ t \in \mathbb{I}\}$. The filtration induced by $(\mathfrak{z}_t)_{t\in\mathbb{T}}$ is then given by $\mathfrak{F} = (\mathfrak{A}_{[0,t]})_{t\in\mathbb{T}}$.

2.2 From commutative to noncommutative

Consider the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\Omega, \mu)$ over the complete probability space $(\Omega, \mathcal{F}, \mu)$. Then any function $f \in L^\infty_{\mathbb{C}}(\Omega, \mu)$ gives rise to a multiplication operator $\mathsf{M}_f \in \mathcal{B}(\mathcal{H})$:

$$\mathsf{M}_f\, g = fg \in \mathcal{H} \qquad \forall\, g \in \mathcal{H},$$

with $\|\mathsf{M}_f\|_\infty = \|f\|_{L^\infty(\mu)}$. The collection of all such multiplication operators

$$\mathcal{A} := \left\{\mathsf{M}_f : f \in L^\infty_{\mathbb{C}}(\Omega, \mu)\right\} \subset \mathcal{B}(\mathcal{H})$$

forms a commutative subalgebra of $\mathcal{B}(\mathcal{H})$.

In fact, this subalgebra is a *von Neumann* algebra:

Definition 2.7 (von Neumann algebra) A (unital) *von Neumann algebra* (or W^*-algebra) is a $*$-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ that contains $I_{\mathcal{H}}$ and is closed in the *weak operator topology* (WOT), i.e.,

$$\text{WOT-}\lim a_n = a \quad \iff \quad \langle f, a_n g\rangle_{\mathcal{H}} \to \langle f, ag\rangle_{\mathcal{H}} \quad \forall f, g \in \mathcal{H}.$$

Equivalently, is it a C^*-algebra with a predual $\mathcal{A}_*$ that is a Banach space.

There are sub-families of $\mathcal{A}$ that play a distinguished role, namely,

$$\begin{aligned} \mathscr{A}_+ &:= \left\{a \in \mathcal{A} : a \succeq 0\right\} && \text{(nonnegative operators)} \\ \mathcal{O} &:= \left\{a \in \mathcal{A} : a^* = a\right\} && \text{(self-adjoint operators)} \\ \mathcal{P} &:= \left\{a \in \mathcal{O} : a^2 = a\right\} && \text{(projection operators)} \end{aligned}$$

Here, $a \succeq 0$ if and only if $\langle f, af\rangle_{\mathcal{H}} \geq 0$ for all $f \in \mathcal{H}$.

On a von Neumann algebra $\mathcal{A}$, we can talk about special types of continuous linear functionals on $\mathcal{A}$, called states.

Definition 2.8 A *state* on a von Neumann algebra $\mathcal{A}$ is a linear functional $\psi : \mathcal{A} \to \mathbb{C}$ that is positive and normalized, i.e., $\psi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and $\psi(I_{\mathcal{B}(\mathcal{H})}) = 1$.
A state ψ is said to be

faithful if $\psi(a^*a) = 0 \;\Leftrightarrow\; a = 0$,

tracial if $\psi(ab) = \psi(ba)$ for all $a, b \in \mathcal{A}$, and

normal if $\psi \in \mathcal{A}_*$, i.e., if it is an element of the predual $\mathcal{A}_*$.

Let us return to our previous example with $\mathcal{A} = \{\mathsf{M}_f : f \in L^\infty_{\mathbb{C}}(\Omega, \mu)\}$. It is not difficult to see that $\mathcal{A}$ is a W^*-algebra. Notice that since M_f is self-adjoint for $f \in L^\infty_{\mathbb{R}}(\Omega, \mu)$, the family of self-adjoint operators is given by $\mathcal{O} = \{\mathsf{M}_f : f \in L^\infty_{\mathbb{R}}(\Omega, \mu)\}$ and the family of projection operators is given by $\mathcal{P} = \{\mathsf{M}_f : f = \mathbf{1}_A,\ A \in \mathcal{F}\}$. Additionally,
For any probability measure $\nu \ll \mu$, we set

$$\psi_\nu(\mathsf{M}_f) := \int_\Omega f \, \mathrm{d}\nu, \qquad f \in L^\infty_{\mathbb{C}}(\Omega, \mu),$$

we find that ψ is a linear functional that is positive and normalized, i.e., ψ is a state. Moreover, it is tracial. It is normal if $\omega := d\nu/d\mu \in L^1(\Omega, \mu)$ and faithful if $\omega > 0$.

Normal states play an essential role, serving as a counterpart to classical measures, as made explicit by the following proposition.

Proposition 2.9. *Let $\psi : \mathcal{A} \to \mathbb{C}$ be a state on a von Neumann algebra $\mathcal{A}$. The following are equivalent:*

(i) *ψ is a normal state.*

(ii) *(σ-additivity) If $(a_n) \subset \mathcal{P}$ are mutually orthogonal projections, i.e., $a_n(\mathcal{H}) \perp a_m(\mathcal{H})$ for all $n \neq m$, and $a = \vee_n a_n$ being the projection on the smallest closed subspace containing $\cup_n a_n(\mathcal{H})$, then*

$$\psi(a) = \sum\nolimits_n \psi(a_n).$$

(iii) *(Continuity from above) For any increasing net $0 \preceq a_n \uparrow a$ in $\mathcal{A}_+ \Rightarrow \psi(a_n) \uparrow \psi(a)$.*

Consider an arbitrary normal state ψ and set

$$\mu(A) := \psi(\mathsf{M}_{\mathbf{1}_A}) \qquad \text{for every } A \in \mathcal{F}.$$

In particular, we recover the probability $\mathbb{P}$ by

$$\mathbb{P}(A) = \psi(\mathsf{M}_{\mathbf{1}_A})$$

$$\psi(\mathsf{M}_{\mathbf{1}_A}\mathsf{M}_{\mathbf{1}_B}) = \psi(\mathsf{M}_{\mathbf{1}_{A\cap B}}) = \mathbb{P}(A \cap B)$$

$$\mathbb{P}\big(\cup_i A_i\big) = \psi\big(\mathsf{M}_{\mathbf{1}_{\cup_i A_i}}\big) = \psi\big(\sum\nolimits_i \mathsf{M}_{\mathbf{1}_{A_i}}\big) = \sum_i \psi(\mathsf{M}_{\mathbf{1}_{A_i}}) = \sum_i \mathbb{P}(A_i)$$

Let $(X_t)_{t\in\mathbb{T}}$ be a stochastic process on the Wiener space (Ω, R)

Consider a path measure P on $\Omega := \mathcal{D}(\mathbb{R}_+; E)$ and the canonical process $(X_t)_{t\geq 0}$ given by $X_t(\omega) = \omega(t)$, $\omega \in \Omega$, $t \in \mathbb{R}_+$. Suppose that for every $f \in B_b(E)$,

$$f(X_t) - f(X_0) - \int_0^t Lf(X_{r-})\,\mathrm{d}r \qquad \text{is an } (\mathfrak{F}, \mathsf{P})\text{-martingale,} \tag{2.1}$$

eq:classical-martingale

where $\mathfrak{F} = (\mathcal{F}_t)_{t\geq 0}$ is the canonical filtration $\mathcal{F}_t = \sigma(X_s : s \leq t)$, and $L\colon B_b(E) \to B_b(E)$ is a bounded Markov generator such that $L^*\pi = 0$ for some stationary measure $\pi \in \mathcal{P}(E)$.

We now write (2.1) in the form above. Set $\mathfrak{X} := L^\infty(E, \pi)$ and $\mathfrak{A} := L^\infty(\Omega, \mathsf{P}_\pi)$, which are both commutative von Neumann algebras and where

$$\mathsf{P}_\pi(A) = \int_E \mathsf{P}(x + A)\,\pi(\mathrm{d}x).$$

Now, define the stochastic process

$$\mathfrak{z}_t\colon \mathfrak{X} \to (\mathfrak{A}, \mu), \qquad \mathfrak{z}_t(f) = f \circ X_t,$$

with the state

$$\mu(G) := \int_\Omega G(\omega)\,\mathsf{P}_\pi(\mathrm{d}\omega), \qquad G \in \mathfrak{A},$$

which makes $(\mathfrak{A}, \mu)$ a quantum probability space.

$$\mathsf{E}_t[\mathfrak{z}]$$

and the $*$-homomorphism

$$\mathfrak{z}_t(f) - \mathfrak{z}_0(f) - \int_0^t \mathfrak{z}_{r-}(Lf)\,\mathrm{d}r$$

$$\alpha_t(F) = F \circ \mathfrak{s}_t$$

$$\alpha_t \circ \mathfrak{z}_0(f) = \alpha_t(f \circ X_0 \otimes \mathbf{1}_{\mathfrak{Y}}) = f \circ X_0 \circ \mathfrak{s}_t(\omega)$$

$$\alpha_t\colon \mathcal{A}\times\Omega \to \mathcal{A}; \quad \alpha_t(M_f, \omega) = M_{f\circ X_t(\omega)}.$$

The martingale identity (2.1) above can then be expressed as

$$\mathfrak{m}_t[M_f] := \alpha_t(M_f) - \alpha_0(M_f) - \int_0^t \alpha_{r-}(M_{Lf})\,\mathrm{d}r.$$

Setting $\mathscr{L}M_f := M_{Lf}$, the previous identity allows one to write (2.1) as

$$\mathfrak{m}_t[A] = \alpha_t(A) - \alpha_0(A) - \int_0^t \alpha_{r-}(\mathscr{L}A)\,\mathrm{d}r, \qquad A \in \mathcal{A},$$

thereby generalizing the martingale problem to a non-commutative setting.

$$\mathfrak{m}_t[A] - \mathfrak{m}_s[A] = \alpha_t(A) - \alpha_s(A) - \int_s^t \alpha_{r-}(\mathscr{L}A)\,\mathrm{d}r$$

$$\mathscr{L}_a = a^*[\bullet, a] + [a^*, \bullet]a$$

Example 2.10 (Diffusion) Consider the formal example of the (possibly degenerate) diffusion process with the generator

$$Lf = \mathrm{div}(A\nabla f) - A\nabla V \cdot \nabla f, \qquad f \in \mathcal{C}_c^\infty(\mathbb{R}^d),$$

i.e., it is the generator of the Itô diffusion

$$\mathrm{d}X_t = -\sigma(X_t)\sigma^\top(X_t)\nabla V(X_t)\mathrm{d}t + \sigma(X_t) \circ \mathrm{d}B_t,$$

with $A = \sigma\sigma^\top \in \mathbb{R}^{d\times d}$, $\sigma \in \mathbb{R}^{d\times m}$

$$a_i = \sum_{j=1}^d \sigma_{ji}\partial_j, \qquad a_i^* = -\sum_{j=1}^d \partial_j(\sigma_{ji}\,\bullet), \qquad i = 1,\dots,m.$$

It is not difficult to see that $[a_i^*, M_f] = -[a_i, M_f]$. Moreover, for any $g \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} a_i^*[M_f, a_i]g &= \sum_{j=1}^d \partial_j(\sigma_{ji} g \sum_{k=1}^d \sigma_{ki}\partial_k f) \\ &= \sum_{j=1}^d g\partial_j(\sigma_{ji} \sum_{k=1}^d \sigma_{ki}\partial_k f) + \sum_{j,k=1}^d \sigma_{ji}\sigma_{ki}\partial_j g \partial_k f \\ &= \sum_{j=1}^d g\partial_j(\sigma_{ji} \sum_{k=1}^d \sigma_{ki}\partial_k f) - [M_f, a_i]a_i g. \end{aligned}$$

Consequently, we find that

$$\sum_{i=1}^m \mathscr{L}_{a_i}(M_f) = \sum_{i=1}^m \big(a_i^*[M_f, a_i] + [a_i^*, M_f]a_i\big) = M_{\mathrm{div}(A\nabla f)}$$

Moreover, setting

$$H_i = \sum_{j,k=1}^d \sigma_{ji}\sigma_{ki}\partial_j V \partial_k, \qquad i = 1,\dots,m,$$

we have that

$$\sum_{i=1}^m [H_i, M_f] = M_{\sigma\sigma^\top\nabla V\cdot\nabla f}$$

$$[a_i, M_V][a_i, \bullet]$$

Therefore, we obtain

$$\mathscr{L}M_f = \sum_{i=1}^m \big([H_i, M_f] + \mathscr{L}_{a_i} M_f\big) = M_{Lf} \qquad \forall f \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

However, we know that the diffusion process constitutes a gradient flow with driving energy $\mathcal{F} = \mathsf{Ent}(\bullet|\pi)$, where $\pi = e^{-V}\mathsf{Leb}$ is the invariant measure. Formally, one defines a state on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ by

$$\tau(M_f) := \int_{\mathbb{R}^d} f\,\mathrm{d}\pi, \qquad f \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

$$\tau_\pi(M_f M_g) = \tau(M_f M_g M_\pi) = \tau(M_{fg\pi}) = \tau_\pi(M_g M_f)$$
$$\Delta_\pi A = M_\pi A M_\pi^{-1}$$

$$(\Delta_\pi V_j)(g) = M_\pi V_j M_\pi^{-1} g = \pi \sum_{j=1}^{d} \sigma_{ji}\partial_j(\pi^{-1} g) - \sigma_{ji}\partial_j V(\pi^{-1} g)$$
$$= \sum_{j=1}^{d} \sigma_{ji}[g\partial_j V + \partial_j g] - \sigma_{ji}\partial_j V g =$$

$$V_j = \sum_{j=1}^{d} \sigma_{ji}\partial_j - M_{\sigma_{ji}\partial_j V}$$
$$\tau_\sigma(M_g \mathscr{L} M_f) = \hat{\sigma}(M_{gLf}) = \int_{\mathbb{R}^d} gLf \, \mathrm{d}\sigma =$$

$$\begin{aligned} \mathscr{L} M_{e^{-V}} g &= \big(\mathrm{div}(A\nabla e^{-V}) + A\nabla V \cdot \nabla e^{-V}\big) g \\ &= \big(-\mathrm{div}(A\nabla V e^{-V}) - A\nabla V \cdot \nabla V e^{-V}\big) g \end{aligned}$$

$$[a_i, M_f]g = \sum_{j=1}^d \sigma_{ji}\partial_j(fg) - \sum_{j=1}^d f\sigma_{ji}\partial_j g = \sum_{j=1}^d \sigma_{ji} g\partial_j f = M_{a_i(f)}g$$

$$[a_i^*, M_f]g = -\sum_{j=1}^d \partial_j(\sigma_{ji}\, fg) + \sum_{j=1}^d f\partial_j(\sigma_{ji}\, g) = -\sum_{j=1}^d \sigma_{ji}\partial_j f g = -M_{a_i(f)}g = -[a_i^*, M_f]g$$

Consequently, we find that

$$\mathscr{L}_{a_i} M_f =$$

$$[a_i, M_f]a_i g = \sum_{j,k=1}^d \sigma_{ji}\sigma_{ki}\partial_j f\partial_j g$$

$$[a_i, M_V] = \sum_{j=1}^d \sigma_{ji}\partial_j V$$

$$[a_i, [a_i, M_f]] = [a_i, M_{a_i(f)}] = M_{a_i(a_i(f))},$$

where we used the fact that $[a_i, M_f] = M_{a_i(f)}$ for every $i = 1, \ldots, d$.

$$\begin{aligned}\sum_{i=1}^m a_i \circ a_i(f) &= \sum_{i=1}^m \sum_{j=1}^d \sigma_{ji}\partial_j \left(\sum_{k=1}^d \sigma_{ki}\partial_k f\right) = \sum_{i=1}^m \sum_{j,k=1}^d \sigma_{ji}\partial_j(\sigma_{ki}\partial_k f) \\ &= \sum_{i=1}^m \sum_{j,k=1}^d \sigma_{ji}\partial_j(\sigma_{ki}\partial_k f)\end{aligned}$$

$$\begin{aligned}\operatorname{div}(\sigma\sigma^\top \nabla f) &= \sum_{j=1}^d \sum_{i=1}^m \sum_{k=1}^d \partial_j(\sigma_{ji}\sigma_{ki}\partial_k f) \\ &= \sum_{j=1}^d \sum_{i=1}^m \sum_{k=1}^d \partial_j(\sigma_{ji}\sigma_{ki}\partial_k f)\end{aligned}$$

$$\begin{aligned}\langle\psi|M_{Lf}|\psi\rangle &= \int (Lf)(x)|\psi(x)|^2\,\pi(\mathrm{d}x) \\ &= -\int \sigma^2\partial_x f\,\partial_x(\psi^2\pi)\,\mathrm{d}x - \int \sigma^2\partial_x f\,\partial_x V\,\psi^2\,\mathrm{d}\pi \\ &= -2\int \sigma^2\partial_x f\psi\partial_x\psi\,\mathrm{d}\pi \\ &= -2\int\end{aligned}$$

$$a\psi = \sigma\partial_x\psi$$

$$
\begin{aligned}
\langle \varphi | a\psi \rangle &= \int \varphi\, \sigma\, \partial_x \psi \, \mathrm{d}\pi \\
&= -\int \partial_x(\sigma\varphi\pi)\psi(x)\,\mathrm{d}x \\
&= -\int \partial_x(\sigma\varphi)\psi\,\mathrm{d}\pi + \int \varphi\psi\,\sigma\partial_x V\,\mathrm{d}\pi \\
&= \int \bigl[-\partial_x(\sigma\varphi) + \varphi\sigma\partial_x V\bigr]\psi\,\mathrm{d}\pi = \langle a^*\varphi|\psi\rangle
\end{aligned}
$$

$$
\begin{aligned}
(a^* M_f a\psi)(x) &= -\partial_x(f\sigma^2\partial_x\psi) + f\sigma^2\partial_x\psi\partial_x V \\
(a^* a M_f \psi)(x) &= -\partial_x(\sigma^2\partial_x(f\psi)) + \sigma^2\partial_x(f\psi)\partial_x V \\
(M_f a^* a\psi)(x) &= -f\partial(\sigma^2\partial_x\psi) + \sigma^2 f\partial_x\psi\partial_x V
\end{aligned}
$$

$$
\int \psi(x)(a^* M_f a\psi)(x)\,\mathrm{d}\pi = \int f|\sigma\partial_x\psi|^2\,\mathrm{d}\pi
$$

2.3 Stochastic processes on matrix algebras

In this section, we consider $\mathfrak{X} = \mathsf{Mat}(\mathbb{C}, n)$. Let $a \in \mathcal{O}(\mathfrak{X})$ be an observable on $\mathfrak{X}$ and set

$$
\mathscr{L}x = a^*[x, a] + [a^*, x]a, \qquad x \in \mathfrak{X}.
$$

We then consider the solution of the

$$
\mathrm{d}\mathscr{U}_t = \bigl(-a^* a\,\mathrm{d}t + i\sqrt{2}a\,\mathrm{d}B_t\bigr)\mathscr{U}_t,
$$

which, due to $a = a^*$, can be explicitly expressed as

$$
\mathscr{U}_t(x, e(f)) \coloneqq \exp(i\sqrt{2}aB_t(\omega))x, \qquad t \in \mathbb{T},
$$

which is a $*$-automorphism on $\mathfrak{X}$, which extends to an isometry

$$
\mathscr{U}_t \colon \mathfrak{X} \otimes \mathcal{A}
$$

$$
B \in \mathcal{A}
$$

$$
\mathfrak{z}_t(x) - \mathfrak{z}_0(x) - \int_0^t \mathfrak{z}_r(\mathscr{L}x)\,\mathrm{d}r
$$

2.4 Stochastic processes on the unitary group

2.4.1 The unitary group and its Lie structure

The unitary group is defined as

$$
U(n) = \bigl\{U \in \mathbb{C}^{n\times n} : U^* U = I\bigr\}.
$$

It is a compact Lie group with Lie algebra (tangent space at identity)

$$\mathfrak{U}(n) = \mathsf{T}_I U(n) = \{\mathfrak{a} \in \mathbb{C}^{n\times n} : \mathfrak{a}^* = -\mathfrak{a}\},$$

i.e., the space of skew-Hermitian matrices. A vector field $\mathfrak{a}\colon U(n) \to \mathfrak{U}(n)$ is *left–invariant* if $\mathfrak{a}(U) = U\mathfrak{a}(I) \in \mathfrak{U}(n)$ for all $U \in U(n)$.

The Lie algebra $\mathfrak{U}(n)$ can be equipped with a real inner product given by the Hilbert-Schmidt inner product

$$\langle \mathfrak{a}, \mathfrak{b}\rangle_{\mathfrak{U}(n)} = -\operatorname{tr}(\mathfrak{a}^*\mathfrak{b}) = \Re \operatorname{tr}(\mathfrak{a}\mathfrak{b}^*),$$

which is positive-definite on $\mathfrak{U}(n)$. The associated norm is then

$$|\mathfrak{a}|^2_{\mathfrak{U}(n)} = \langle \mathfrak{a}, \mathfrak{a}\rangle_{\mathfrak{U}(n)} = \operatorname{Tr}(\mathfrak{a}\mathfrak{a}^*).$$

2.4.2 Unitary-valued Brownian motion (rank-r noise)

Let $\mathfrak{a}_1, \dots, \mathfrak{a}_r \in \mathfrak{U}(n)$ be orthonormal under the inner product $\langle \cdot, \cdot\rangle_{\mathfrak{U}(n)}$, where $1 \le r \le n^2$. Further, let $\beta^1_t, \dots, \beta^r_t$ be independent standard real Brownian motions.

Define the $\mathfrak{U}(n)$–valued (possibly degenerate) Brownian driver

$$\mathfrak{w}_t = \sum\nolimits_j \mathfrak{a}_j\, \beta^j_t.$$

If $r = n^2$, the driver is elliptic (non-degenerate). If $r < n^2$, the covariance has rank r and the process is *degenerate* (hypoelliptic), exploring only the connected subgroup

$$H = \exp(\mathcal{H}) \subset U(n), \quad \mathcal{H} = \operatorname{Lie}\{\mathfrak{a}_1, \dots, \mathfrak{a}_r\} \subset \mathfrak{U}(n).$$

The intrinsic $U(n)$-valued diffusion process solves the Stratonovich SDE

$$dU_t = U_t \circ d\mathfrak{w}_t, \qquad U_0 = I,$$

or in components

$$dU_t = \sum\nolimits_j U_t\mathfrak{a}_j \circ d\beta^j_t.$$

The matrix-valued quadratic variation of $\mathfrak{w}$ is given by

$$d[\mathfrak{w}] = \sum\nolimits_j \mathfrak{a}_j \otimes \mathfrak{a}_j\, dt.$$

In Itô form, the equivalent SDE reads

$$dU_t = U_t\big(d\mathfrak{w}_t + \mathfrak{l}\, dt\big), \qquad \mathfrak{l} := \frac{1}{2}\sum\nolimits_j \mathfrak{a}_j^2,$$

where $\mathfrak{l}$ is the Laplace-Beltrami generator associated with the left-invariant connection.

In the following, we denote its law on path space $C([0,T]; U(n))$ by P.

2.4.3 Girsanov transform on $U(n)$ (Karandikar type)

Let $\mathfrak{u} \in L^2((0,T); \mathfrak{U}(n))$ be adapted and square-integrable.

Define the drifted group motion by

$$dU_t^{\mathfrak{u}} = U_t^{\mathfrak{u}} \circ (d\mathfrak{w}_t + \mathfrak{u}_t\, dt), \qquad U_0^{\mathfrak{u}} = I.$$

Assuming a Novikov condition in the active noise directions,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\mathfrak{u}_s|^2\, ds\right)\right] < +\infty,$$

the process defined below is a true martingale.

Decompose the drift against the active algebra frame:

$$\mathfrak{u}_t = \sum\nolimits_j \mathfrak{a}_j\, v_t^j, \qquad v = (v^1, \dots, v^r) \in L^2((0,T); \mathbb{R}^r).$$

Define the exponential Cameron-Martin density process

$$\begin{aligned} M_T &= \exp\left(\int_0^T \langle \mathfrak{u}_t, d\mathfrak{w}_t\rangle - \frac{1}{2}\sum\nolimits_j \int_0^T \langle \mathfrak{u}_t, \mathfrak{a}_j \otimes \mathfrak{a}_j\, \mathfrak{u}_t\rangle\, dt\right) \\ &= \exp\left(\sum\nolimits_j \int_0^T v_t^j\, d\beta_t^j - \frac{1}{2}\sum\nolimits_j \int_0^T |v_t^j|^2 dt\right). \end{aligned}$$

This expression depends only on the directions with nonzero quadratic variation. It is the standard Cameron-Martin shift density on $\mathbb{R}^r$ lifted to the group via left multiplication.

We then define a new path measure on the same probability space by

$$\mathsf{Q} = M_T\, \mathsf{P}.$$

Under Q:

(i) The shifted drivers

$$\beta_t^{j,\mathsf{Q}} := \beta_t^j - \int_0^t v_s^j\, ds$$

are Brownian motions again on $\mathbb{R}$.

(ii) The algebra noise

$$d\mathfrak{w}_t^{\mathsf{Q}} := d\mathfrak{w}_t - \mathfrak{u}_s\, dt = \sum\nolimits_j \mathfrak{a}_j\, d\beta_t^{j,\mathsf{Q}}$$

is a $\mathfrak{U}(n)$-valued Brownian motion again with same covariance

$$d[\mathfrak{w}^{\mathsf{Q}}] = d[\mathfrak{w}] = \sum\nolimits_j \mathfrak{a}_j \otimes \mathfrak{a}_j\, dt.$$

(iii) The group paths satisfy the same *driftless* intrinsic SDE

$$dU_t^{\mathfrak{u}} = U_t^{\mathfrak{u}} \circ dW_t^{\mathsf{Q}} = \sum\nolimits_j U_t^{\mathfrak{u}} \mathfrak{a}_j \circ d\beta_t^{j,\mathsf{Q}},$$

so $U_t^{\mathfrak{u}}$ is an intrinsic Brownian motion on the same symmetry group H.

Hence, M_T is the Radon-Nikodym derivative between the laws of the drifted and driftless group diffusions on path space.

2.4.4 Induced Girsanov on pure states

Rank-1 projections evolve by the conjugation map

$$P_t = U_t P_0 U_t^*, \qquad P_0 = \psi_0 \psi_0^*.$$

This gives a valid reference probability on paths of $C([0,T];\mathcal{M})$, where

$$\mathcal{M} = \{V P_0 V^* : V \in H\} \subset \mathcal{P}_1(n) \simeq \mathbb{CP}^{n-1}$$

is the homogeneous orbit that the degenerate noise can reach.

Linearising conjugation and applying the Stratonovich chain rule yields the SDE

$$dP_t^{\mathfrak{u}} = [\circ dW_t^{\mathsf{Q}},\, P_t^{\mathfrak{u}}] = \sum\nolimits_j [\mathfrak{a}_j, P_t^{\mathfrak{u}}] \circ d\beta_t^{j,\mathsf{Q}},$$

so the pure-state path law remains Brownian on the same orbit manifold $\mathcal{M}$.

If a drift $\mathfrak{u}$ has components outside $\mathrm{span}\{\mathfrak{a}_j\}$, only the projection of the drift onto that span can be removed. Directions with zero quadratic variation cannot absorb drifts.

$$\int F(U_t P_0 U_t^*)\, \mathsf{P}(dU) =$$

2.5 Dilation of semigroups (Sz.Nagy)

2.6 Stochastic dilations: How to make a heat bath

2.7 Interpretation of LDPs on unitary evolutions

The task of this subsection is to reinterpret the large deviations of empirical measures in a functional analytic framework.

Consider a family of iid E-valued random variables $(X_i)_{i\in\mathbb{N}}$ with $\sigma = \mathrm{Law} X_1$. Setting $\Omega = E^{\mathbb{N}}$ and $\mathsf{R} = \sigma^{\otimes\mathbb{N}}$, we see that $\mathrm{Law}\,(X_i)_{i\in\mathbb{N}} = \mathsf{R}$. Moreover, we can disregard the initial probability space for X_i and consider instead the canonical random variable on Ω.

For every $f \in \mathfrak{X} \coloneqq \mathcal{C}_b(E)$ and $n \in \mathbb{N}$, we define the noncommutative random variable

$$\mathfrak{X} \ni f \mapsto \mathfrak{z}^n(f) = \frac{1}{n}\sum_{i=1}^{n} f(X_i) \in \mathfrak{A} \coloneqq L^\infty(\Omega, \mathsf{R}),$$

where $\mathfrak{A}$ is equipped with the state $\mu(G) = \mathsf{E}_{\mathsf{R}}[G]$, $G \in \mathfrak{A}$.

$$\mu(\mathfrak{z}^n(f)) = \frac{1}{n}\sum_{i=1}^{n} \mathsf{E}_{\mathsf{R}}[f(X_i)] = \frac{1}{n}\sum_{i=1}^{n}\int_E f\,\mathrm{d}\sigma = \int_E f\,\mathrm{d}\sigma = \mu(\mathfrak{z}_1(f)).$$

$$\begin{aligned}\mathsf{R}(\sup_f\{\mu^n(f) - \nu(f)\} > \varepsilon) &= \mathsf{R}(n\mu^n(f) > n(\nu(f)+\varepsilon)) \\ &= \mathsf{R}(e^{n\mu^n(f)} > e^{n(\nu(f)+\varepsilon)}) \le e^{-n(\nu(f)+\varepsilon)}\int_\Omega e^{n\mu^n(f)}\,\mathrm{d}\mathsf{R}\end{aligned}$$

$$\mathfrak{z}\colon \mathfrak{X} \to (\mathfrak{X}\otimes\mathfrak{C}, \mathrm{tr}\otimes\mu)$$

$$\mathfrak{A} = \mathfrak{X}\otimes\mathfrak{C}^{\otimes\mathbb{Z}}, \quad \varphi = \mathrm{tr}\otimes\mu^{\otimes\mathbb{Z}}$$

Let X_i

$$\mathfrak{z}_i(x) = x\otimes c_i, \qquad c_i = \cdots 1\otimes c$$

$$\mathfrak{z}^n = \frac{1}{n}\sum_{i=1}^{n}\mathfrak{z}_i, \qquad \mathfrak{z}_i = \cdots\otimes\mathbb{1}\otimes\mathfrak{z}\otimes\mathbb{1}\otimes\cdots$$

$$\varphi(\mathfrak{z}^n(x)) = \frac{1}{n}\sum_{i=1}^{n}\varphi(\mathfrak{z}_i(x)) = \varphi(\mathfrak{z}_1(x)) = \mu(\mathfrak{z}(x))$$

$$\mathfrak{z}_i(x) = \mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes\mathfrak{z}(x)\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}$$

$$\mathfrak{z}^n(x) \in \mathfrak{A} \cong \mathfrak{X}\otimes\mathfrak{Y}$$

$$\mu = \mathrm{tr}\otimes\mu_{\mathfrak{C}}^{\otimes\mathbb{N}}$$

$$\mu(\mathbf{1}_A(\mathfrak{z}^n(x) - \mathfrak{y}_1(x))) = \mu_{\mathfrak{C}}^{\otimes\mathbb{N}}\big(\mathrm{tr}[\mathbf{1}_A(\mathfrak{z}^n(x) - \mathfrak{y}(x)\otimes\mathbb{1}_{\mathfrak{C}^{\otimes\mathbb{N}}})]\big)$$

$$\sup_f\{\mu^n(f) - \nu(f)\} > \varepsilon \quad\Rightarrow\quad \mu^n(f^\delta) - \nu(f^\delta) > \varepsilon - \delta$$

$$\begin{aligned}\mathsf{R}(\mu^n(f)-\nu(f)>\varepsilon) &= \mathsf{R}(e^{n\mu^n(f^\delta)} > e^{n(\nu(f^\delta)+\varepsilon-\delta)}) \\ &\le e^{-n(\nu(f^\delta)+\varepsilon-\delta)}\mathsf{E}_{\mathsf{R}}[e^{n\mu^n(f^\delta)}]\end{aligned}$$

$$\begin{aligned}-\frac{1}{n}\log \mathsf{R}(\mu^n(f)-\mu(f)>\varepsilon) &\ge \mu(f)+\varepsilon-\frac{1}{n}\log\int_\Omega e^{n\mu^n(f)}\,\mathrm{d}\mathsf{R} \\ &= \mu(f)+\varepsilon-\log\int_E e^f\mathrm{d}\mu\end{aligned}$$

$$-\frac{1}{n}\log \mathsf{R}(\mu^n(f)-\mu(f)>\varepsilon) \ge$$

Taking the sup

$$\begin{aligned}\int_\Omega e^{n\mu^n(f)}\,\mathrm{d}\mathsf{R} &= \int_\Omega \prod_{i=1}^n e^{f(X_i)}\,\mathrm{d}\mathsf{R} \\ &= \int_{E^N}\prod_{i=1}^n e^{f(x_i)}\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_n) = \left(\int_E e^{f(x)}\mu(\mathrm{d}x)\right)^n\end{aligned}$$

2.8 LDP via Gärtner-Ellis revisited

Let $\Omega = \mathcal{C}(I;\mathfrak{h})$ be the space of $\mathfrak{h}$-valued continuous path and consider the canonical process $(X_t)_{t\in I}$ defined by $X_t(\omega) := \omega(t)$ for all $t\in I$. Further, let $\mathcal{F} = \cup_{t\ge 0}\mathcal{F}_t$, where the filtration $\mathfrak{F} := \{\mathcal{F}_t := \sigma(X_s : 0\le s\le t)\}_{t\in I}$ is generated by $(X_t)_{t\in I}$. Throughout, we consider a reference path measure $\mathsf{R}\in\mathcal{P}(\Omega)$ such that X admits the R-semimartingale decomposition

$$X = X_0 + B^{\mathsf{R}} + M^{\mathsf{R}} \qquad \mathsf{R}\text{-almost surely,}$$

where B^{R} is an adapted process with absolutely continuous sample paths R-almost surely, and M^{R} is an R-martingale. For simplicity of presentation, we assume that the quadratic variation of M^{R} is an $\mathcal{S}_+(\mathfrak{X})$-valued random measure on I with Lebesgue dentisy, i.e.,

$$\frac{\mathrm{d}[M^{\mathsf{R}},M^{\mathsf{R}}]}{\mathrm{d}\lambda}(t) = \mathsf{a}_t(X_t)\in\mathcal{S}_+(\mathfrak{h}) \qquad \mathsf{R}\text{-almost surely,}$$

and also that $t\mapsto \mathsf{a}_t(X_t)$ is an $\mathfrak{F}$-adapted process.

and has the quadratic variation $\langle M^{\mathsf{R}}[\varphi]\rangle$ for every smooth function φ.

Let $X^1, X^2,\dots$ be independent copies of

$(X^1,\dots,X^n)$

$$\frac{\mathrm{d}\mathsf{Q}^n}{\mathrm{d}\mathsf{P}^{\otimes n}} = \exp\left(\sum_{i=1}^n\int_I\langle\beta_t^i(X_t),\,\mathrm{d}M_t^{\mathsf{R}}\rangle - \frac{1}{2}\sum_{i=1}^n\int_I\langle\beta_t^i(X_t),\mathsf{a}_t(X_t)\beta_t^i(X_t)\rangle\,\mathrm{d}t\right) =: Z^{(n)}$$

$$Y_t^{(n)} := \sum_{i=1}^n\int_0^t\langle\beta_s^i(X_s),\,\mathrm{d}M_s^{\mathsf{R}}\rangle - \frac{1}{2}\sum_{i=1}^n\int_0^t\langle\beta_s^i(X_s),\mathsf{a}_s(X_s)\beta_s^i(X_s)\rangle\,\mathrm{d}s$$

$$\mathrm{d}Z_t^{(n)} = \mathrm{d}e^{Y_t^{(n)}} = Z_t^{(n)}\left(\mathrm{d}Y_t^{(n)} + \frac{1}{2}\,\mathrm{d}[Y^{(n)}, Y^{(n)}]_t\right) = \sum_{i=1}^n Z_t^{(n)}\langle \beta_t^i(X_t),\, \mathrm{d}M_t^{\mathsf{R}}\rangle$$

Define $\rho^n := \frac{1}{n}\sum_{i=1}^n X^i$. Then,

$$\mathrm{d}\rho^n = \frac{1}{n}\sum_{i=1}^n \mathrm{d}X_t^i = \frac{1}{n}\sum_{i=1}^n b_t^{\mathsf{R}}(X_t^i)\,\mathrm{d}t + \frac{1}{n}\sum_{i=1}^n \mathrm{d}M_t^{\mathsf{R},i}$$

$$\begin{aligned}
\mathsf{P}^{\otimes n}\left(\frac{1}{n}\sum_{i=1}^n X^i \in A\right) &= \int_A e^{-\int_I \langle \beta_t^i(X_t),\, \mathrm{d}M_t^{\mathsf{R}}\rangle + \frac{1}{2}\int_I \langle \beta_t^i(X_t), \mathsf{a}_t(X_t)\beta_t^i(X_t)\rangle\,\mathrm{d}t}\,\mathrm{d}\mathsf{Q}^{(n)} \\
&= \int_A e^{-\int_I \langle \beta_t^i(X_t),\, \mathrm{d}M_t^{\mathsf{R}}\rangle + \frac{1}{2}\int_I \langle \beta_t^i(X_t), \mathsf{a}_t(X_t)\beta_t^i(X_t)\rangle\,\mathrm{d}t}\,\mathrm{d}\mathsf{Q}^{(n)}
\end{aligned}$$

$$A := \{\rho : \partial_t\rho + \overline{\mathsf{div}}\, j = L\rho\}$$

$$D_{\beta^i}\exp\left(\int_I \langle \beta_t^i(X_t),\, \mathrm{d}M_t^{\mathsf{R}}\rangle + \frac{1}{2}\int_I \langle \beta_t^i(X_t), \mathsf{a}_t(X_t)\beta_t^i(X_t)\rangle\,\mathrm{d}t\right)[h^i]$$

$$\int_I \langle \beta_t, j_t\rangle\,\mathrm{d}t - \log\int_A Z^{(n)}\,\mathrm{d}\mathsf{P}^{\otimes n}$$

2.8.1 Example

We consider the Hilbert space $\mathfrak{h} = \mathbb{M}$ equipped with the scalar product $\langle A, B\rangle = \mathrm{tr}[A^*B]$. Consider the stochastic process

$$P_t = P_0 + \int_0^t LP_s\,\mathrm{d}s + \int_0^t \overline{\nabla}P_s\,\mathrm{d}W_s,$$

where $\overline{\nabla}P = [i\sigma_X, P]$ and $LP = (1/2)[i\sigma_X, [i\sigma_X, P]]$. In other words,

$$B_t^{\mathsf{R}} = \int_0^t LP_s\,\mathrm{d}s, \qquad M_t^{\mathsf{R}} = \int_0^t \overline{\nabla}P_s\,\mathrm{d}W_s,$$

where M^{R} has the quadratic variation

$$\mathsf{a}(P) = \langle (\overline{\nabla}P)^*, \bullet\,\rangle\,\overline{\nabla}P \qquad \text{for every } P \in \mathfrak{h}.$$

Let $P^1, P^2, \ldots$ be independent copies and define $\rho^n := \frac{1}{n}\sum_{i=1}^n P^i$. Then,

$$\mathrm{d}\rho_t^n = L\rho_t^n\,\mathrm{d}t + \frac{1}{n}\sum_{i=1}^n \overline{\nabla}P_t^i\,\mathrm{d}W_t^i = L\rho_t^n\,\mathrm{d}t + \frac{1}{n}\,\mathrm{d}M_t^{\mathsf{R},n}.$$

$$\frac{1}{n}\sum_{i=1}^{n}\overline{\nabla}P_t^i\,\mathrm{d}W_t^i = \frac{1}{n}\sum_{i=1}^{n}\Big(\overline{\nabla}P_t^i\,\mathrm{d}W_t^i - \mathsf{a}(P_t^i)H_t\,\mathrm{d}t\Big) + \frac{1}{n}\sum_{i=1}^{n}\mathsf{a}(P_t^i)H_t\,\mathrm{d}t$$

$$\mathrm{d}\rho^n = L\rho^n\,\mathrm{d}t + \frac{1}{n}\sum_{i=1}^{n}\mathsf{a}(P_t^i)H_t\,\mathrm{d}t + \frac{1}{n}\,\mathrm{d}M_t^{\mathsf{Q},n}$$

For each $n \in \mathbb{N}$, we set

$$\frac{\mathrm{d}\mathsf{Q}^n}{\mathrm{d}\mathsf{R}^{\otimes n}} = \exp\left(\sum_{i=1}^{n}\int_I \langle\overline{\nabla}P_t^i, H_t\rangle\,\mathrm{d}W_t^i - \frac{1}{2}\sum_{i=1}^{n}\int_I\langle H_t, \mathsf{a}(P_t^i)H_t\rangle\,\mathrm{d}t\right) =: Z^{(n)}.$$

Then, for curves η satisfying $\partial_t\eta + \overline{\mathsf{div}}\,J^\eta = L\eta$ for some J^η, we find that

$$J^\eta = \sum\langle\sigma_j, J^\eta\rangle\sigma_j$$

Suppose

$$-\overline{\mathsf{div}} = \sum_{j\in J}\alpha_j[i\sigma_j, \bullet],$$

then

$$-\overline{\mathsf{div}}\,J^\eta = \sum_{j,k}\alpha_j\langle\sigma_k, J^\eta\rangle[i\sigma_j,\sigma_k] = \sum_{j,k\neq j}\alpha_j\langle\sigma_k, J^\eta\rangle[i\sigma_j,\sigma_k] \in \operatorname{span}\big\{[i\sigma_j,\sigma_k] : j\in J, k\neq j\big\}$$

$$\begin{aligned}
\mathsf{R}^{\otimes n}\big(\rho^n\in A\big) &= \int_A \exp\Big(-\sum_{i=1}^{n}\int_I\langle\overline{\nabla}P_t^i, H_t\rangle\,\mathrm{d}W_t^i + \frac{1}{2}\sum_{i=1}^{n}\int_I\langle H_t, \mathsf{a}(P_t^i)H_t\rangle\,\mathrm{d}t\Big)\,\mathrm{d}\mathsf{Q}^{(n)}\\
&= \int_A \exp\Big(-\int_I\langle\,\mathrm{d}M_t^{\mathsf{R},n}, H_t\rangle + \frac{1}{2}\sum_{i=1}^{n}\int_I\langle H_t, \mathsf{a}(P_t^i)H_t\rangle\,\mathrm{d}t\Big)\,\mathrm{d}\mathsf{Q}^{(n)}\\
&= \int_A \exp\Big(-\int_I\langle\,\mathrm{d}M_t^{\mathsf{Q},n}, H_t\rangle - \frac{1}{2}\sum_{i=1}^{n}\int_I\langle H_t, \mathsf{a}(P_t^i)H_t\rangle\,\mathrm{d}t\Big)\,\mathrm{d}\mathsf{Q}^{(n)}
\end{aligned}$$

$$H^\eta \in \operatorname*{argmin}_H\left\{\mathsf{E}_{\mathsf{Q}}\left[\frac{1}{2}\int_I\langle H_t, \mathsf{a}(P_t)H_t\rangle\,\mathrm{d}t\right] - \int_I\langle H_t, \overline{\mathsf{div}}\,J_t^\eta\rangle\,\mathrm{d}t\right\}$$

$$\mathsf{E}_{\mathsf{Q}}\left[\int_I\big(\mathrm{tr}[P_t\overline{\nabla}H_t]\big)^2\,\mathrm{d}t\right] = \int_I\mathsf{E}_{\mathsf{Q}}\Big[\big(\mathrm{tr}[P_t\overline{\nabla}H_t]\big)^2\Big]\,\mathrm{d}t$$

$$\mathrm{tr}[\rho_t^{\mathsf{Q}}\overline{\nabla}H_t] = \sum\lambda_i(t)\langle v_i(t), \overline{\nabla}H_t v_i(t)\rangle \geq \gamma\sum\lambda_i(t)$$

$$\mathsf{E}_{\mathsf{R}}\left[\int_I\langle G_t, \mathsf{a}(P_t)H_t\rangle\,\mathrm{d}t\right] = \int_I\langle G_t, \overline{\mathsf{div}}\,J_t\rangle\,\mathrm{d}t$$

$$\int_I\langle H_t, \overline{\mathsf{div}}\,J_t\rangle\,\mathrm{d}t - \log\mathsf{E}_{\mathsf{R}}\left[\exp\Big(\int_I\langle H_t,\,\mathrm{d}M_t^{\mathsf{R},n}\rangle - \frac{1}{2}\sum_{i=1}^{n}\int_I\langle H_t, \mathsf{a}(P_t^i)H_t\rangle\,\mathrm{d}t\Big)\right]$$

$$\int_I \langle G_t, \overline{\mathsf{div}}\, J_t \rangle \,\mathrm{d}t = -\mathsf{E}_{\mathsf{R}^{\otimes n}}\Bigg[\bigg(\int_I \langle G_t,\, \mathrm{d}M_t^{\mathsf{R},n}\rangle - \sum_{i=1}^{n}\int_I \langle G_t, \mathsf{a}(P_t^i)H_t\rangle \,\mathrm{d}t\bigg) \exp\bigg(\int_I \langle H_t,\, \mathrm{d}M_t^{\mathsf{R},n}\rangle - \frac{1}{2}\sum_{i=1}^{n}\int_I \langle H_t$$

Let $Y_t^{(n)} = \sum_{i=1}^{n}\int_0^t \langle \overline{\nabla} H_s, \overline{\nabla} P_s^i\rangle \,\mathrm{d}W_s^i$ and $Z_t^{(n)} := \exp(Y_t^{(n)})$. Then,

$$\mathrm{d}Z_t^{(n)} = Z_t^{(n)}\left(\sum_{i=1}^{n}\langle \overline{\nabla} H_t, \overline{\nabla} P_t^i\rangle \,\mathrm{d}W_t^i + \frac{1}{2}\sum_{i=1}^{n}\langle \overline{\nabla} H_t, \mathsf{a}(P_t)\overline{\nabla} H_t\rangle \,\mathrm{d}t\right)$$

$$Z_t^{(n)} = 1 + \sum_{i=1}^{n}\int_0^t Z_s^{(n)}\langle \overline{\nabla} H_s, \overline{\nabla} P_s^i\rangle \,\mathrm{d}W_s^i + \frac{1}{2}\sum_{i=1}^{n}\int_0^t Z_s^{(n)}\langle \overline{\nabla} H_s, \mathsf{a}(P_s)\overline{\nabla} H_s\rangle \,\mathrm{d}s$$

Let $\varrho \in \mathcal{S}(\mathcal{H})$ be state and $(\varrho^n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathcal{H})$ be a sequence of states. We say that $\varrho^n \to \varrho$ setwise if

$$\varrho^n(E) \to \varrho(E) \quad \text{for all projections } E \in \mathcal{P}(\mathcal{H}).$$

Further, let $(\mathfrak{z}_t)_{t\in\mathbb{T}}$ be a stochastic process in the sense of Definition 2.1 with

$$\mathfrak{z}_t \colon \mathfrak{X} \to (\mathfrak{X} \otimes \mathfrak{C}, \varphi), \quad \varphi = \varrho \otimes \mu.$$

$$\mathfrak{z}_t(x) = \alpha_t^* x \alpha_t$$

and $\mathfrak{z}_0(x) = x \otimes 1$, $x \in \mathfrak{X}$. By construction, we have that

$$\varphi(\mathfrak{z}_0(x)) = \varphi(x \otimes 1) = \varrho(x)\mu(1) = \varrho(x) \quad \forall\, x \in \mathfrak{X}.$$

We suppose that

$$\mathcal{S}(\mathfrak{X} \otimes \mathfrak{C}^{\otimes n}) \to \mathcal{S}(\mathfrak{X} \otimes \mathfrak{C})$$

$$\mu_\varphi = \frac{1}{n}\sum_{i=1}^n \varphi_i = \frac{1}{n}\sum_{i=1}^n \rho \otimes \varphi$$

$$\mathbf{x}_i : \mathfrak{C}^{\otimes n} \to \mathfrak{C}; \quad \mathbf{x}_i(\omega_1, \dots, \omega_n) = \omega_i$$

$$\delta_{\mathbf{x}_i}(E)$$

2.9 Groups to algebras

Let $\mathbb{T}$ be the 1-dimensional torus. Further, let $\mathcal{H}$ be a Hilbert space and $\mathsf{h} \in \mathcal{O}(\mathcal{H})$ be an observable on $\mathcal{H}$. Consider the unitary $\mathfrak{u} \colon \mathbb{T} \to \mathcal{U}(\mathcal{H})$ such that $\mathfrak{u}(\exp(tX)) = e^{it\mathsf{h}}$.

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tX))\Big|_{t=0} = X(f)$$

$$\mathrm{d}X_t =$$

$$\mathcal{B}(\mathcal{H}) \to L^\infty(\mathbb{T}; \mathcal{B}(\mathcal{H})); \qquad a \mapsto \mathfrak{u}^*(z) a \mathfrak{u}(z)$$

Let $\rho_0 \in \mathcal{S}(\mathcal{H})$ be a given fixed state

$$\begin{aligned}\varrho_\mu(x) &= \int_{\mathbb{T}} \mathrm{tr}[\mathfrak{u}^*(z) x \mathfrak{u}(z) \varrho_0]\, \mu(\mathrm{d}z) = \int_{\mathbb{T}} f_{\rho_0, x}(z)\, \mu(\mathrm{d}z) \\ &= \mathrm{tr}\left[x \int_{\mathbb{T}} \mathfrak{u}(z) \varrho_0 \mathfrak{u}^*(z)\, \mu(\mathrm{d}z)\right] = \mathrm{tr}[x \varrho_\mu]\end{aligned}$$

Consider iid random variables $(Z^i)_{i\in\mathbb{N}}$, $Z^i \sim \mu \in \mathcal{P}(\mathbb{T})$ and the corresponding sequence of empirical measures

$$\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{Z^i}, \qquad \varrho_{\mu^n}(x) = \frac{1}{n}\sum_{i=1}^n \mathfrak{u}(Z^i) \varrho_0 \mathfrak{u}^*(Z^i)$$

$$\begin{aligned}\frac{1}{n}\log\mathbb{E}[e^{n\langle f,\mu^n\rangle}] &= \frac{1}{n}\log\mathbb{E}\left[\exp\left(\sum_{i=1}^n f(Z^i)\right)\right]\\ &= \log\left(\int_{\mathbb{T}} e^{f(z)}\mu(\mathrm{d}z)\right)\end{aligned}$$

$$\sup_{f=f_{\varrho_0,x}}\left\{\langle f,\nu\rangle - \log\int_{\mathbb{T}} e^{f(z)}\mu(\mathrm{d}z)\right\} = \sup_{x\in\mathcal{O}(\mathcal{H})}\left\{\mathrm{tr}[x\varrho_\nu] - \log\int_{\mathbb{T}} e^{\mathrm{tr}[\mathfrak{u}^*(z)x\mathfrak{u}(z)\varrho_0]}\mu(\mathrm{d}z)\right\}$$

$$\begin{aligned}&\sum_{j,k} x_{jk}\,\mathrm{tr}[E_{jk}\varrho_\nu] - \log\int_{\mathbb{T}} e^{\sum_{j,k} x_{jk}\,\mathrm{tr}[\mathfrak{u}^*(z)E_{jk}\mathfrak{u}(z)\varrho_0]}\mu(\mathrm{d}z)\\ &= \sum_{j,k} x_{jk}\,\mathrm{tr}[E_{jk}\varrho_\nu] - \log\int_{\mathbb{T}}\prod_{j,k} e^{x_{jk}\,\mathrm{tr}[\mathfrak{u}^*(z)E_{jk}\mathfrak{u}(z)\varrho_0]}\mu(\mathrm{d}z)\end{aligned}$$

$$\mathrm{tr}[E_{jk}\varrho_\nu] = \frac{1}{c}\int_{\mathbb{T}}\mathrm{tr}[\mathfrak{u}^*(z)E_{jk}\mathfrak{u}(z)\varrho_0]e^{\sum_{j,k} x_{jk}\,\mathrm{tr}[\mathfrak{u}^*(z)E_{jk}\mathfrak{u}(z)\varrho_0]}\mu(\mathrm{d}z)$$

$$b = \frac{1}{c}\int_{\mathbb{T}} v(z)e^{\langle\mathbf{x},v(z)\rangle}\mu(\mathrm{d}z)$$

Consider the map

$$\mathbf{x}\mapsto\Lambda(\mathbf{x}) = \int_{\mathbb{T}} v(z)e^{\langle\mathbf{x},v(z)\rangle}\mu(\mathrm{d}z)$$

$$\langle\Lambda(\mathbf{x})-\Lambda(\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle = \int_0^1\langle D\Lambda((1-\tau)\mathbf{x}+\tau\mathbf{y})[\mathbf{x}-\mathbf{y}],\mathbf{x}-\mathbf{y}\rangle\,\mathrm{d}\tau \geq 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{T}} v(z)e^{\langle\lambda,v(z)\rangle}\mu(\mathrm{d}z) = \int_{\mathbb{T}} v(z)\otimes v(z)e^{\langle\lambda,v(z)\rangle}\mu(\mathrm{d}z)$$

$$\mathrm{tr}[E_{jk}\mathfrak{u}(z)\varrho_0\mathfrak{u}^*(z)] = \langle\psi_k|\mathfrak{u}(z)\varrho_0\mathfrak{u}^*(z)|\psi_j\rangle = \sum_i \lambda_i\langle\psi_k|\mathfrak{u}(z)|\psi_i\rangle\langle\psi_i|\mathfrak{u}^*(z)|\psi_j\rangle$$

If $\varrho_0 = \mathbb{1}$, then

$$\mathrm{tr}[E_{jk}\mathfrak{u}(z)\varrho_0\mathfrak{u}^*(z)] = \delta_{kj}.$$

Hence,

$$\mathrm{tr}[E_{jk}\varrho_\nu] = \frac{1}{c}\delta_{kj}e^{\sum_j x_j}$$

3 Boson Fock spaces and quantum noise

3.1 Boson Fock spaces

Consider a (complex) Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. The symmetric Fock space associated with $\mathcal{H}$ is

$$\mathfrak{F} = \mathfrak{F}_{\mathsf{sym}}(\mathcal{H}) := \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}^{\odot n}, \qquad \mathcal{H}^{\odot 0} = \mathbb{C},$$

where $\odot$ denotes the symmetric tensor product such that

$$\mathcal{H}^{\odot n} = \Big\{ f \in \mathcal{H}^{\otimes n} : f(x_{\sigma_1},\dots,x_{\sigma_n}) = f(x_1,\dots,f_n) \text{ for every permutation } \sigma \Big\}.$$

The Fock space $\mathfrak{F}$ inherits the scalar product from $\mathcal{H}$ defined by

$$\langle \oplus f^{(n)}, \oplus g^{(n)}\rangle_{\mathfrak{F}} = \sum_{n\in\mathbb{N}_0} \langle f^{(n)}, g^{(n)}\rangle_{\mathcal{H}^{\otimes n}}.$$

We define the *vacuum vector* $\Omega = 1 \oplus 0 \oplus 0^{\otimes 2} \oplus \cdots \in \mathfrak{F}$, and the *exponential vectors*

$$\mathsf{e}(f) = \bigoplus_{n\in\mathbb{N}_0} \frac{1}{\sqrt{n!}} f^{\otimes n}, \qquad f \in \mathcal{H}.$$

It turns out that the family of exponential vectors $\mathfrak{E}$ is *total* in $\mathfrak{F}$, i.e., the linear span of $\mathfrak{E}$ is dense in $\mathfrak{F}$. This fact will be helpful for us in the future. Since,

$$\langle \mathsf{e}(f), \mathsf{e}(g)\rangle_{\mathfrak{F}} = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}\langle f, g\rangle_{\mathcal{H}}^n = e^{\langle f,g\rangle_{\mathcal{H}}}, \qquad f, g \in \mathcal{H},$$

the exponential vectors are normalizable. These normalized exponential vectors

$$\psi(f) = e^{-\frac{1}{2}\|f\|_{\mathcal{H}}^2} \mathsf{e}(f), \qquad f \in \mathcal{H},$$

are commonly known as *coherent vectors*.

3.1.1 Weyl and field operators

For any $f \in \mathcal{H}$, we define the *Weyl operator* on exponential vectors by

$$\mathsf{W}(f)\mathsf{e}(g) := \exp\left(-\langle f, g\rangle_{\mathcal{H}} - \frac{1}{2}\|f\|_{\mathcal{H}}^2 \right) \mathsf{e}(f+g), \qquad g \in \mathcal{H}.$$

Weyl operators play an essential role in the setup of Fock spaces. For one, they generate coherent states by acting on the vacuum state, i.e.,

$$\mathsf{W}(f)\Omega = e^{-\frac{1}{2}\|f\|_{\mathcal{H}}^2}\mathsf{e}(f) = \psi(f), \qquad f \in \mathcal{H}.$$

Moreover, they give the means to map any element $f \in \mathcal{H}$ to unitary operators on $\mathfrak{F}$ that satisfy the *canonical commutation relation* (CCR):

Proposition 3.1. *The Weyl operator* $\mathsf{W}(f)$ *is a unitary operator and satisfies*

(i) $\mathsf{W}^\dagger(f) = \mathsf{W}(-f)$.

(ii) $\mathsf{W}(f)\mathsf{W}(g) = e^{-i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})}\mathsf{W}(f+g) = e^{-2i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})}\mathsf{W}(g)\mathsf{W}(f)$.

Property (ii) is the Weyl form of the canonical commutation relation (CCR).

Proof. For any $g \in \mathcal{H}$,

$$\begin{aligned}\langle \mathsf{W}(f)\mathsf{e}(g), \mathsf{W}(f)\mathsf{e}(g)\rangle_{\mathfrak{F}} &= \exp\left(-2\langle f,g\rangle_{\mathcal{H}} - \|f\|^2_{\mathcal{H}}\right)\langle \mathsf{e}(f+g), \mathsf{e}(f+g)\rangle_{\mathfrak{F}} \\ &= \exp\left(-2\langle f,g\rangle_{\mathcal{H}} - \|f\|^2_{\mathcal{H}} + \|f+g\|^2_{\mathcal{H}}\right) \\ &= e^{\|g\|^2_{\mathcal{H}}} = \langle \mathsf{e}(g), \mathsf{e}(g)\rangle_{\mathfrak{F}}.\end{aligned}$$

Hence, $\mathsf{W}(f)$ preserves inner products on $\mathfrak{E}$. Since $\mathfrak{E}$ is dense in $\mathfrak{F}$, $\mathsf{W}(f)$ extends uniquely to an isometry on $\mathfrak{F}$.

In a similar fashion, we compute

$$\mathsf{W}(-f)\mathsf{W}(f)\mathsf{e}(g) = \mathsf{e}(g) = \mathsf{W}(f)\mathsf{W}(-f)\mathsf{e}(g) \qquad \forall g \in \mathcal{H},$$

i.e., $\mathsf{W}(-f)\mathsf{W}(f)$ is the identity on the dense set $\mathfrak{E}$. In particular, $\mathsf{W}(f)$ is surjective and an isometry, i.e., $\mathsf{W}(f)$ is unitary with $\mathsf{W}^\dagger(f) = \mathsf{W}(-f)$.

As for the last property, we observe that

$$\begin{aligned}\mathsf{W}(f)\mathsf{W}(g)W(-(f+g))\mathsf{e}(h) &= e^{\langle f+g,h\rangle_{\mathcal{H}} - \frac{1}{2}\|f+g\|^2_{\mathcal{H}}}\mathsf{W}(f)\mathsf{W}(g)\mathsf{e}(-(f+g)+h) \\ &= e^{\langle f+g,h\rangle_{\mathcal{H}} - \frac{1}{2}\|f\|^2_{\mathcal{H}} - \langle g,-f+h\rangle_{\mathcal{H}} - \mathrm{Re}(\langle g,f\rangle_{\mathcal{H}})}\mathsf{W}(f)\mathsf{e}(-f+h) \\ &= e^{-i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})}\mathsf{e}(h),\end{aligned}$$

and hence, $\mathsf{W}(f)\mathsf{W}(g)\mathsf{W}(-(f+g)) = e^{-i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})}I_{\mathfrak{F}}$. We then conclude by using property *(i)* of Weyl operators. ☐

Since $\mathsf{W}(f)$ is unitary for every $f \in \mathcal{H}$, the family $\{\mathsf{W}(tf)\}_{t\in\mathbb{R}}$ forms a one-parameter (strongly continuous) group of unitaries. In particular, due to Stone's theorem, it has a corresponding Hermitian operator $\mathsf{P}(f)$ such that

$$\mathsf{W}(tf) = \exp(it\mathsf{P}(f)).$$

We further define the following operators

$$\mathsf{Q}(f) := -\mathsf{P}(if), \qquad \mathsf{A}^-(f) := \frac{\mathsf{Q}(f) + i\mathsf{P}(f)}{2}, \qquad \mathsf{A}^+(f) := \frac{\mathsf{Q}(f) - i\mathsf{P}(f)}{2}.$$

The operators $\mathsf{A}^\pm$ are called the *field operators* and will play an essential role as they form the creation/annihilation operators on Fock spaces.

Proposition 3.2. *The following are true: For any* $f, g \in \mathcal{H}$,

(i) $\mathfrak{E}$ *is a core for* $\mathsf{P}(f)$ *and* $[\mathsf{P}(f), \mathsf{P}(g)] = 2i\mathrm{Im}(\langle f,g\rangle_{\mathcal{H}})\mathsf{I}_{\mathfrak{F}}$.

(ii) $\mathsf{A}^-(f)\mathsf{e}(g) = \langle f, g\rangle_{\mathcal{H}}\, \mathsf{e}(g), \quad \mathsf{A}^+(f)\mathsf{e}(g) = \frac{d}{dt}\mathsf{e}(g + tf)|_{t=0}$.

(iii) $\mathsf{W}^\dagger(f)\mathsf{A}^-(g)\mathsf{W}(f) = \mathsf{A}^-(g) + \langle g, f\rangle_{\mathcal{H}}\mathsf{I}_{\mathfrak{F}}, \quad \mathsf{W}^\dagger(f)\mathsf{A}^+(g)\mathsf{W}(f) = \mathsf{A}^+(g) + \langle f, g\rangle_{\mathcal{H}}\mathsf{I}_{\mathfrak{F}}$.

(iv) $[\mathsf{A}^-(f), \mathsf{A}^-(g)] = [\mathsf{A}^+(f), \mathsf{A}^+(g)] = 0, \quad [\mathsf{A}^-(f), \mathsf{A}^+(g)] = \langle f, g\rangle_{\mathcal{H}}\mathsf{I}_{\mathfrak{F}}$,

i.e., the field operators $\mathsf{A}^\pm$ *satisfy the canonical commutation relation.*

Remark 3.3 On the finite particle vectors, the field operators act as

$$\mathsf{A}^-(f)g^{\otimes n} = \sqrt{n}\langle f, g\rangle_{\mathcal{H}} g^{\otimes(n-1)}, \qquad \mathsf{A}^+(f)g^{\otimes(n-1)} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} g^{\otimes k} \otimes f \otimes g^{\otimes(n-1-k)}.$$

Example 3.4 Let $\mathcal{H} = L^2_{\mathbb{C}}([-\pi, \pi])$ be the space of square-integrable functions. Then the countable set $\{\psi_\ell(x) = e^{i\ell x} : \ell \in \mathbb{Z}\}$ forms an orthonormal basis for $\mathcal{H}$.

$$\psi(x) = \sum\nolimits_\ell e^{i\ell x}\mathsf{A}^-_\ell$$

3.1.2 Second quantization

The term *second quantization* is associated with the action of lifting operators on an *k-particle* space $\mathcal{H}^{\otimes k}$ to an associated operator on the Fock space.

We begin our discussion with the 1-*particle* case. For any bounded operator $A \in \mathcal{B}(\mathcal{H})$ one defines the map

$$\Gamma(A) := I \oplus \bigoplus_{n\in\mathbb{N}} A^{\otimes n},$$

which acts on $\mathcal{H}^{\otimes n}$ by

$$\Gamma(A)g_1 \otimes \cdots \otimes g_n = Ag_1 \otimes \cdots\cdots \otimes Ag_n.$$

Clearly, if A is unitary, then so is $\Gamma(A)$. Indeed, in this case, one has

$$\langle \Gamma(A)e(g), \Gamma(A)e(g)\rangle_{\mathfrak{F}(\mathcal{H})} = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}\langle Ag, Ag\rangle^n_{\mathcal{H}} = \sum_{n\in\mathbb{N}_0} \frac{1}{n!}\|g\|^n_{\mathcal{H}} = \langle e(g), e(g)\rangle_{\mathfrak{F}(\mathcal{H})}.$$

Now let H be a self-adjoint operator on $\mathcal{H}$ and $U_t := \exp(itH)$ be its unitary evolution. Then, $\Gamma(U_t)$ is a one-parameter group of unitary operators on $\mathfrak{F}(\mathcal{H})$. Stone's theorem then provides the existence of a densely defined Hermitian operator $\mathsf{d}\Gamma(H)$ such that

$$\Gamma(U_t) = \exp(it\, \mathsf{d}\Gamma(H)).$$

The generator $\mathsf{d}\Gamma(H)$ is called the *second quantization of* H, and takes the explicit form

$$\mathsf{d}\Gamma(H)g^{(n)} = \sum_{j=1}^{n} g_1 \otimes \cdots \otimes g_{j-1} \otimes Hg_j \otimes g_{j+1} \otimes \cdots \otimes g_n = \sum_{j=1}^{n} H_j g^{(n)},$$

for any $g^{(n)} = g_1 \otimes \cdots \otimes g_n \in \mathcal{H}^{\otimes n}$ with $g_j \in D(H)$ and

$$H_j = I_{\mathcal{H}} \otimes \cdots \otimes I_{\mathcal{H}} \otimes H \otimes I_{\mathcal{H}} \otimes \cdots \otimes I_{\mathcal{H}},$$

where H acts on the j-th tensor product. The special case $H = I_{\mathcal{H}}$ yields

$$\mathsf{N}g^{(n)} := \mathsf{d}\Gamma(I_{\mathcal{H}})g^{(n)} = ng^{(n)}, \qquad n \in \mathbb{N},$$

and is called the *number operator* due to its diagonal nature, with eigenvalues representing the number of particles in each configuration. Its domain is given by

$$D(\mathsf{N}) = \Big\{ \{f^{(n)}\}_{n\in\mathbb{N}_0} : \sum_{n\in\mathbb{N}_0} n^2 \|f^{(n)}\|^2_{\mathcal{H}^{\otimes n}} < +\infty \Big\}.$$

$$\mathsf{A}_j^- \mathsf{A}_j^+ \psi_j^{\otimes n} = \sqrt{n}\, \mathsf{A}_j^- \psi_j^{\otimes(n-1)} = \sum_{k=0}^{n-1} \psi_j^{\otimes k} \otimes \psi_j \otimes \psi_j^{\otimes(n-1-k)} = n\psi_j^{\otimes n}$$

$$\mathsf{A}_j^+ \mathsf{A}_j^- \mathsf{e}(\psi_k) = \delta_{jk} \mathsf{N}_j\, \mathsf{e}(\psi_j)$$

This construction can be performed similarly for the general k-particle case.

$$\mathsf{d}\Gamma(H^{(k)}) f^{(n)} = \sum_{j_1 \neq \cdots \neq j_k} H_{j_1 \cdots j_k} f^{(n)},$$

where $H_{j_1,\ldots,j_k}$ denotes the operator where $H^{(k)}$ acts on the $(j_1, \ldots, j_k)$-th tensor product.

Example 3.5 Consider the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{T})$ and the Hamiltonians,

$$H^{(1)} = -\Delta \in \mathcal{O}(\mathcal{H}), \qquad H^{(2)} = \mathsf{M}_W \in \mathcal{O}(\mathcal{H}^{\otimes 2}),$$

where $W : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is an interaction potential, and M_f denotes the multiplication operator corresponding to f.

Then, their second quantization is given by

$$\mathsf{d}\Gamma(H^{(1)}) = \sum_{j=1}^{n} H_j^{(1)}, \qquad \mathsf{d}\Gamma(H^{(2)}) = \sum_{j\neq\ell} H_{j\ell}^{(2)} \qquad \text{on } \mathcal{H}^{\otimes n}.$$

$$\mathsf{d}\Gamma(H^{(1)})\psi_j^{\otimes n} = nj^2 \psi_j^{\otimes n} = j^2 \mathsf{A}_j^+ \mathsf{A}_j^- \psi_j^{\otimes n} = j^2 \mathsf{A}_j^+ \mathsf{A}_j^- \psi_j^{\otimes n}$$

$$\psi^{(n)} = \psi_{k_1} \otimes \cdots \otimes \psi_{k_n}$$

$$\mathsf{d}\Gamma(H^{(1)})\psi^{(n)} = \sum_{j=1}^{n} k_j^2\, \psi^{(n)}$$

3.1.3 Free field operators

For each $f \in \mathcal{H}$, we consider the pair $\{a^-(f), a^+(f)\}$ of operators on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ defined by

$$a^-(f)\Psi =$$

satisfying the commutation relations: For any $f, g \in \mathcal{H}$,

$$[a^\pm(f), a^\pm(g)] = 0, \qquad [a^-(f), a^+(g)] = \langle f, g \rangle,$$

where $f \mapsto a^-(f)$ is conjugate linear and $f \mapsto a^+(f)$ is linear. Moreover, if $\Omega \in \mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is the *vacuum vector*, then $a^-(f)\Omega = 0$ for every $f \in \mathcal{H}$. The *field operators* a^- and a^+ are called *creation* and *annihilation* operators, respectively. On appropriate domains, the field operators are adjoints of one another, i.e., $(a^-(f))^\dagger = a^+(f)$ for every $f \in \mathcal{H}$.

It is common in the physics literature to consider operator-valued distributions $\{a_x^-, a_x^+\}$ instead, where if $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^d)$, then $x \in \mathbb{R}^d$ and

$$a^-(f) = \int_{\mathbb{R}^d} \overline{f(x)}\, a_x \,\mathrm{d}x, \qquad a^+(f) = \int_{\mathbb{R}^d} f(x)\, a_x \,\mathrm{d}x.$$

The commutation relations then simply read

$$[a_x^\pm, a_y^\pm] = 0, \qquad [a_x^-, a_y^+] = \delta(x-y).$$

The *number operator* is formally defined by $N_x = a_x^+ a_x$, $x \in \mathbb{R}^d$.

On the rigorous not, if $\mathcal{H}$ is separable with orthonormal basis $\{\psi_i\}$, then one obtains a family of field operators $\{a_i^-, a_i^+\}$ with $a_i^\pm : a^\pm(\psi_i)$.

3.1.4 Gaussian states

Definition 3.6 (Gaussian states) A state Ψ on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is said to be a mean-zero *Gaussian* (or *quasi-free*) state if it can be uniquely determined from the field operators $\{a, a^\dagger\}$ by its covariance

$$\Sigma_\Psi(f,g) := \begin{pmatrix} \Psi(a^+(f)\, a^-(g)) & \Psi(a^-(f)\, a^-(g)) \\ \Psi(a^+(f)\, a^+(g)) & \Psi(a^-(f)\, a^+(g)) \end{pmatrix}, \qquad f, g \in \mathcal{H}.$$

If the off-diagonal elements of the covariance are zero, the state Ψ is called *gauge-invariant* since it is invariant under the so-called gauge transformations of the first kind, i.e.,

$$a^\pm(f) = e^{\pm i\alpha} a^\pm(f) \qquad \text{for any } \alpha \in \mathbb{R}.$$

If the off-diagonal elements of the covariance are nonzero, the state Ψ is called *squeezed.*

3.1.5 States invariant under free evolutions

Consider a Hamiltonian H on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ with $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^d)$ and its associated 1-parameter automorphism group

$$\mathfrak{u}_t(a) = e^{itH} a e^{-itH}, \qquad t \in \mathbb{R}.$$

Definition 3.7 The Hamiltonian H on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is called *free* if there exists a real-valued function $\omega\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathfrak{u}_t(a_x^-) = e^{-it\omega(x)} a_x^-, \qquad x \in \mathbb{R}^d.$$

In this case, the function ω is called the *free 1-particle Hamiltonian*, and H is said to be the *second quantization* of ω. Accordingly, $\mathfrak{u}_t$ is called a *free evolution.*

Example 3.8 Consider a Hamiltonian on $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R})$ and its eigensystem $\{(\lambda_i, \psi_i)\}_i$ such that $\{\psi_i\}_i$ forms an orthonormal basis for $\mathcal{H}$. Setting $a_i^{\pm} = a^{\pm}(\psi_i)$, we see that

$$[a_i^{\pm}, a_j^{\pm}] = 0, \qquad [a_i^-, a_j^+] = \delta_{ij},$$

i.e., $a_i^{\pm}$ define field operators on the symmetric Fock space $\mathfrak{F}_{\mathsf{sym}}(\mathcal{K})$ with $\mathcal{K} = \ell^2_{\mathbb{C}}$
Defining the field operators

$$a^- = \frac{1}{\sqrt{2}}\big(Q + iP\big), \qquad a^+ = \frac{1}{\sqrt{2}}\big(Q - iP\big), \qquad N := a^+a^-$$

such that $[a^-, a^+] = I$, we find that

$$H = a^+a^- + \frac{1}{2} = N + \frac{1}{2}.$$

Clearly, the eigenvectors of N and H coincide

Definition 3.9 A Gaussian state Ψ on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ with $\mathcal{H} = L^2(\mathbb{R}^d)$ is said to be invariant under a free evolution $\mathfrak{u}_t$ if

$$\Psi(\mathfrak{u}_t(a_x^-)\,\mathfrak{u}_s(a_y^+)) = \Psi(\mathfrak{u}_{t-s}(a_x^-)\,a_y^+) = \Psi(a_x^-\,\mathfrak{u}_{s-t}(a_y^+)) \qquad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^d.$$

A *Gaussian free state* is a Gaussian state that is invariant under *all* free evolutions.

Theorem 3.10. *A Gaussian state Ψ on $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is a Gaussian free state if and only if it is gauge-invariant and its diagonal correlations are supported on the diagonal, i.e.,*

$$\Psi(a_x^-\,a_y^+) = m(x)\delta(x-y), \qquad \Psi(a_x^+\,a_y^-) = n(x)\delta(x-y).$$

Theorem 3.11. *The field operators $\{a_x^+, a_x^-\}$ satisfying the commutation relations*

$$[a_x^{\pm}, a_y^{\pm}] = 0, \qquad [a_x^-, a_y^+] = m(x)\delta(x-y).$$

are mean-zero Gaussian random variables w.r.t. the Fock vacuum state $\Psi_\Omega = \langle \Omega, \cdot\, \Omega\rangle$, where $\Omega \in \mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ is the Fock vacuum vector, with covariance

$$\Sigma_\Psi(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & m(x) \end{pmatrix} \delta(x-y), \qquad x, y \in \mathbb{R}^d.$$

Conversely, if $\{a_x^+, a_x^-\}$ are random variables with these properties, then they satisfy the commutation relations above.

3.1.6 Boson Fock white noise

def:boson-white-noise

Definition 3.12 A boson Fock white noise on $\mathbb{R}^d$ is a boson Fock field $\{b^+_{t,x}, b^-_{t,x}\}$ on $\mathbb{R}^{d+1}$ with vacuum vector Ω satisfying the commutation relations

$$[b^\pm_{t,x}, b^\pm_{s,y}] = 0, \qquad [b^-_{t,x}, b^+_{s,y}] = \delta(t-s)m(x)\delta(x-y), \qquad b^-_{t,x}\Omega = 0.$$

thm:stochastic-limit

Theorem 3.13. *Let $\{a^+_x, a^-_x\}$ be Gaussian free fields w.r.t. the Fock vacuum state Ψ_Ω with*

$$\mathfrak{u}_t(a^-_x) = e^{-it\omega(x)}a^-_x.$$

Then the rescaled field operators

$$b^{\lambda,\pm}_{t,x} := \frac{1}{\lambda}\mathfrak{u}_{t/\lambda^2}(a^\pm_x)$$

converges in the sense of correlator distributions to a boson Fock white noise, i.e.,

$$\lim_{\lambda\to 0}\Psi(b^{\lambda,\varepsilon_1}_{t,x} b^{\lambda,\varepsilon_2}_{t,y}) = \Psi(b^{\varepsilon_1}_{t,x} b^{\varepsilon_2}_{t,y}) \qquad \varepsilon_1, \varepsilon_2 \in \{+,-\},$$

where $b^\pm_{t,x}$ as defined in Definition 3.12 with $m = 2\pi\delta(\omega)$.

Proof. Using the invariance of Ψ_Ω under free evolutions, we find

$$\begin{aligned}\Psi_\Omega(b^{\lambda,-}_{t,x} b^{\lambda,+}_{s,y}) &= \frac{1}{\lambda^2}\Psi_\Omega(\mathfrak{u}_{(t-s)/\lambda^2}(a^-_x)\,a^+_y) \\ &= \frac{1}{\lambda^2}e^{-i\omega(x)(t-s)/\lambda^2}\,\Psi_\Omega(a^-_x\,a^+_y) = \frac{1}{\lambda^2}e^{-i\omega(x)(t-s)/\lambda^2}\delta(x-y).\end{aligned}$$

Passing to the limit $\lambda\to 0$ recovers the desired limit. All the other terms vanish. □

Remark 3.14 Let $\mathcal{K} \subset L^2(\mathbb{R}^d)$ be a set of functions for which

$$\int_{\mathbb{R}} |\langle f, e^{it\omega}g\rangle|\,\mathrm{d}t < +\infty \qquad \forall f, g \in \mathcal{K}.$$

Since $t \mapsto \langle f, e^{it\omega}f\rangle$ is positive definite for each $f \in \mathcal{K}$, Bochner's theorem implies that the sesquilinear form

$$\langle f, 2\pi\delta(\omega)g\rangle := \int_{\mathbb{R}}\langle f, e^{it\omega}g\rangle\,\mathrm{d}t,$$

is a pre-scalar product. With $(\cdot|\cdot)$, the set $\mathcal{K}$ becomes a pre-Hilbert space, which can be completed to obtain a Hilbert space, still denoted by $\mathcal{K}$. The function $m = 2\pi\delta(\omega)$ has to be understood in this sense, and only makes sense for functions in $\mathcal{K}$.

The operators

$$B^-_t(f) := \int_0^t\int_{\mathbb{R}^d}\overline{f(x)}\,b^-_{s,x}\,\mathrm{d}x\,\mathrm{d}s, \qquad B^+_t(f) := \int_0^t\int_{\mathbb{R}^d} f(x)\,b^+_{s,x}\,\mathrm{d}x\,\mathrm{d}s,$$

define *quantum Brownian motions*. The self-adjoint *(momentum)* operators

$$P_t(f) := \frac{1}{i}\big[B^-_t(f) - B^+_t(f)\big],$$

form a commuting family of classical random variables whose statistics in the Fock vacuum state $\Psi_\Omega = \langle\Omega, \cdot\,\Omega\rangle$ is completely determined by the relation

$$\Psi(e^{iP_t(f)}) = \exp\left(-\frac{t}{2}\|f\|^2_{L^2(\mathbb{R}^d)}\right). \qquad \text{check!}$$

3.1.7 Gaussian equilibrium states: The KMS condition

For any states $a, b \in \mathcal{A}$, the map $t \mapsto \Psi(a\, \mathfrak{u}_t(b))$ can be analytically continued and satisfies the so-called *KMS condition* at inverse temperature $\beta > 0$:

$$\Psi(a\, \mathfrak{u}_{t+i\beta}(b)) = \Psi(\mathfrak{u}_t(a)\, b) \qquad \forall\, a, b \in \mathcal{A}. \qquad \text{check!}$$

3.2 Composite systems

Definition 3.15 A composite system of two given quantum dynamical systems $S = \{\mathcal{H}_S, H_S\}$, $R = \{\mathcal{H}_R, H_R\}$ is a quantum dynamical system of the form

$$\{\mathcal{H}_S \otimes \mathcal{H}_R, H_{SR}\}, \qquad H_{SR} = H_S \otimes 1_R + 1_S \otimes H_R + H_I,$$

where H_I is called the *interaction Hamiltonian* and contains all the new physics of the composite system, while $H_0 \coloneqq H_S \otimes 1_R + 1_S \otimes H_R$ is called the free Hamiltonian.

We will consider *scaled* total Hamiltonians

$$H^\lambda \coloneqq H_0 + \lambda H_I,$$

and the following unitary evolutions:

$$\textit{free evolution} \quad V_t^0 = e^{-itH_0}, \qquad \textit{total evolution} \quad V_t^\lambda = e^{-itH^\lambda},$$
$$\textit{interacting representation evolution} \quad U_t^\lambda = (V_t^0)^\dagger V_t^\lambda,$$

where U_t^λ satisfies the Schrödinger equation in the interaction picture:

$$\partial_t U_t^\lambda = -i\lambda H_I(t) U_t^\lambda, \qquad U_0^\lambda = I,$$

with the time dependent Hamiltonian $H_I(t) = (V_t^0)^\dagger H_I V_t^0$.

For simplicity, we will make the following assumptions on system $S = \{\mathcal{H}_S, H_S\}$, the reservoir $R = \{\mathcal{H}_R, H_R\}$, and the interaction Hamiltonian H_I.

3.2.1 The reservoir

The reservoir $R = \{\mathcal{H}_R, H_R\}$ is given by the Hilbert space $\mathfrak{F}_{\mathsf{sym}}(\mathcal{H})$ with $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}^d)$, a free Hamiltonian H_R with continuous spectrum $\mathbb{R}$, and a mean-zero Gaussian free state Ψ_R such that

$$\int_{\mathbb{R}} |\Psi_R(a_x^{\varepsilon_1}\, \mathfrak{u}_t(a_y^{\varepsilon_2}))|\, \mathrm{d}t < +\infty, \qquad \varepsilon_1, \varepsilon_2 \in \{+, -\}.$$

In particular, Ψ_R is characterized by the covariances

$$\Psi_R(a_x^- a_y^+) = m(x)\delta(x-y), \qquad \Psi_R(a_x^+ a_y^-) = n(x)\delta(x-y).$$

Since H_R is free, there exists a function ω, for which

$$\mathfrak{u}_t(a_x^-) = e^{itH_R} a_x^- e^{-itH_R} = e^{-it\omega(x)} a_x^-,$$

where ω describes the 1-particle evolution.

An example of a free reservoir Hamiltonian is given by

$$H_R = \int_{\mathbb{R}^d} \omega(x)\, a_x^+ a_x^-\, \mathrm{d}x,$$

where ω is a smooth cutoff function.

3.2.2 The system Hamiltonian

For simplicity, we shall assume that the system Hamiltonian H_S has a discrete spectrum such that

$$H_S = \sum_j \lambda_j P_j,$$

where λ_j are the eigenvalues and P_j are their corresponding spectral projections.

3.2.3 The interaction Hamiltonian

We consider dipole-type interaction Hamiltonians of the form

$$H_I = \int_{\mathbb{R}^d} \left[D(x) \otimes a_x^+ + D^\dagger(x) \otimes a_x^- \right] \mathrm{d}x,$$

where $\{D(x) : x \in \mathbb{R}^d\}$ is a family of system operators called the *response terms*.

With the spectral projections of H_S, we may express H_I as

$$H_I = \sum_{j,\,k} \int_{\mathbb{R}^s} \left[P_j D(x) P_k \otimes a_x^+ + P_k D^\dagger(x) P_j \otimes a_x^- \right] \mathrm{d}x,$$

and hence, the time-dependent Hamiltonian reads

$$\begin{aligned} H_I(t) &= \sum_{j,\,k} \int_{\mathbb{R}^d} \left[P_j D(x) P_k \otimes e^{it(\omega(x)+\lambda_j-\lambda_k)} a_x^+ + P_k D^\dagger(x) P_j \otimes e^{-it(\omega(x)+\lambda_j-\lambda_k)} a_x^- \right] \mathrm{d}x \\ &= \sum_q \sum_{\lambda_k-\lambda_j=\eta_q} \int_{\mathbb{R}^d} \left[P_j D(x) P_k \otimes e^{it(\omega(x)-\eta_q)} a_x^+ + P_k D^\dagger(x) P_j \otimes e^{-it(\omega(x)-\eta_q)} a_x^- \right] \mathrm{d}x \\ &= \sum_q \int_{\mathbb{R}^d} \left[D_q(x) \otimes e^{it(\omega(x)-\eta_q)} a_x^+ + D_q^\dagger(x) \otimes e^{-it(\omega(x)-\eta_q)} a_x^- \right] \mathrm{d}x, \end{aligned}$$

where the system operators

$$D_q(x) \coloneqq \sum_{\lambda_k-\lambda_j=\eta_q} P_j D(x) P_k \qquad \text{satisfy} \quad e^{itH_S} D_q(x) e^{-itH_S} = e^{-it\eta_q} D_q(x).$$

To simplify things drastically, we assume that $q = 1$ and $D_1(x) = \chi(x) D$ for some smooth cutoff function χ and a fixed system operator D. In this case, we obtain

$$H_I(t) = \int_{\mathbb{R}^d} \left[D \otimes \chi(x) e^{it(\omega(x)-\eta)} a_x^+ + D^\dagger \otimes \overline{\chi(x)} e^{-it(\omega(x)-\eta)} a_x^- \right] \mathrm{d}x.$$

3.3 The weak interaction stochastic limit

Altogether, we arrive at the rescaled Schrödinger equation in the interaction picture

eq:rescaled-schrodinger-interaction

$$U^\lambda_{t/\lambda^2} = I - i \int_0^t H_I^\lambda(s)\, U^\lambda_{s/\lambda^2}\, \mathrm{d}s, \tag{3.1}$$

with $H_I^\lambda(t) = D \otimes b_t^{\lambda,+} + D^\dagger \otimes b_t^{\lambda,-}$, where, due to Theorem 3.13,

$$b_t^{\lambda,\pm} = \frac{1}{\lambda} a^\pm(\chi e^{i(t/\lambda^2)(\omega-\eta)}) \longrightarrow b_t^\pm \qquad \text{in the sense of correlators,}$$

and therefore,

$$H_I^\lambda(t) \longrightarrow H_t = D \otimes b_t^+ + D^\dagger \otimes b_t^-, \qquad U_{t/\lambda^2}^\lambda \longrightarrow U_t,$$

where U_t satisfies the formal stochastic differential equation

$$\mathrm{d}U_t = -i(D \otimes \mathrm{d}B_t^+ + D^\dagger \otimes \mathrm{d}B_t^-)U_t, \qquad B_t^\pm = \int_0^t b_s^\pm \,\mathrm{d}s. \tag{3.2}$$

eq:sde

Under certain assumptions on ω, this SDE has a unique solution.

The SDE (3.2) in natural-time order (or Itô form) is the SDE given by

$$\mathrm{d}U_t = -i(D\,\mathrm{d}B_t^+ U_t + D^\dagger U_t\,\mathrm{d}B_t^-) - \gamma_- D^\dagger D U_t\,\mathrm{d}t, \tag{3.3}$$

eq:sde-normal

obtained by commuting $\mathrm{d}B_t^- U_t$ and using the fact that the solution U_t to (3.2) satisfies

$$\begin{aligned} [b_t^-, U_t] &= -i\gamma_- D U_t, \\ [U_t^\dagger, b_t^+] &= i\overline{\gamma}_- U_t^\dagger D^\dagger, \\ [b_t^-, U_t^\dagger] &= i\gamma_- U_t^\dagger D, \end{aligned} \qquad \text{where} \quad \gamma_- := \int_{-\infty}^0 \langle \chi, e^{-it(\omega-\eta)}\chi\rangle \,\mathrm{d}t.$$

The additional term is known as the Itô correction term in quantum stochastic calculus. Morally, the natural-time order is the order induced by the filtration for which one expects U_t to be adapted to. Expand on the concept of normal order!

With the unitary evolution U_t one can derive an evolution for any system observable $X_t = U_t^\dagger(X \otimes 1_{\mathcal{H}_R})U_t$ given by

$$\mathrm{d}X_t = -i\,\mathrm{d}B_t^+[D, X_t] + i[X_t, D^\dagger]\,\mathrm{d}B_t^- + LX_t\,\mathrm{d}t$$

where

$$LX = 2\Re(\gamma_-)D^\dagger X D - \gamma_- D^\dagger D X - \overline{\gamma}_- X D^\dagger D,$$

is the corresponding Lindblad operator.

Remark 3.16 The rescaled equation (3.1) may be formally expressed as

$$U_{t/\lambda^2}^\lambda = I + \sum_{n=1}^\infty (-i)^n \lambda^n \int_0^{t/\lambda^2} \cdots \int_0^{t_{n-1}} H_I(t_1)\cdots H_I(t_n)\,\mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

In the case when $H_I(t)$ commutes for every $t \geq 0$, the integrand would be a symmetric function of $t_1, \ldots, t_n$, and gives

$$U_{t/\lambda^2}^\lambda = I + \sum_{n=1}^\infty (-i)^n \frac{\lambda^n}{n!} \left(\int_0^{t/\lambda^2} H_I(s)\,\mathrm{d}s \right)^n = \exp\left(-i\lambda \int_0^{t/\lambda^2} H_I(s)\,\mathrm{d}s \right),$$

i.e., the expectation of U_{t/λ^2}^λ is the characteristic function of the process

$$W_t^\lambda := \lambda \int_0^{t/\lambda^2} H_I(s)\,\mathrm{d}s.$$

3.4 Damped harmonic oscillator

We consider a simple setup in which a single atom interacts with an electromagnetic field. The total system is given by the composite of the atom and the reservoir systems

$$S = \{\mathfrak{F}_{\mathsf{sym}}(\mathbb{C}^2), H_S\}, \qquad R = \{\mathfrak{F}_{\mathsf{sym}}(L^2(\mathbb{R}^d)), H_R\},$$

with the free Hamiltonians

$$H_S = \omega_0\, c^+ c^-, \qquad H_R = \int_{\mathbb{R}^d} \omega(x)\, a_x^+ a_x^- \,\mathrm{d}x,$$

where $\{c^+, c^-\}$ is the field operator for $\mathcal{H}_S = \mathfrak{F}_{\mathsf{sym}}(\mathbb{C}^2)$, $\{a_x^+, a_x^-\}$ are the field operators for $\mathcal{H}_R = \mathfrak{F}_{\mathsf{sym}}(L^2(\mathbb{R}^d))$, and ω is a suitable cutoff function.

For the interaction Hamiltonian H_I, we consider a dipole approximation of the form

$$H_I = \int_{\mathbb{R}^d} \chi(x)\big[c^- \otimes a_x^+ + c^+ \otimes a_x\big]\,\mathrm{d}x = c^- \otimes A^+ + c^+ \otimes A^-,$$

where χ are suitable cutoff function and

$$A^{\pm} = \int_{\mathbb{R}^d} \chi(x)\, a_x^{\pm}\,\mathrm{d}x.$$

such that the rescaled total Hamiltonian is given by

$$H^{\lambda} = H_S \otimes 1_{\mathcal{H}_R} + 1_{\mathcal{H}_S} \otimes H_R + \lambda H_I.$$

Define the evolution

$$\mathfrak{u}_t^{\lambda}(a) = e^{itH^{\lambda}} a e^{-itH^{\lambda}}.$$

Then, the Heisenberg equation for $c(t) = \mathfrak{u}_t^{\lambda}(c \otimes 1_{\mathcal{H}_R})$ and $a_x^-(t) = \mathfrak{u}_t^{\lambda}(1_{\mathcal{H}_S} \otimes a_x)$ reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} c^-(t) &= -i\omega_0\, c^-(t) - i\lambda \int_{\mathbb{R}^d} \chi(x)\, a_x^-(t)\,\mathrm{d}x\\ \frac{\mathrm{d}}{\mathrm{d}t} a_x^-(t) &= -i\omega(x)\, a_x^-(t) - \lambda\,\chi(x)\, c(t).\end{aligned}$$

Solving for $a_x^-(t)$, one obtains

$$a_x^-(t) = a_x^- e^{-it\omega(x)} - i\lambda\chi(x)\int_0^t e^{-i(t-s)\omega(x)} c^-(s)\,\mathrm{d}s.$$

Inserting $a_x^-(t)$ into the equation for $c(t)$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} c^-(t) = -i\omega_0 c^-(t) - \int_0^t \gamma(t-s)\, c^-(s)\,\mathrm{d}s - i\,\xi_t^-.$$

where we defined the quantities

$$\gamma(r) := \lambda^2 \int_{\mathbb{R}^d} e^{-ir\omega(x)} \chi^2(x)\,\mathrm{d}x, \qquad \xi_t^- := \lambda \int_{\mathbb{R}^d} e^{-it\omega(x)} \chi(x)\, a_x^-\,\mathrm{d}x.$$

Observe that ξ_t^- depends only on the reservoir field operator a_x^-, and therefore, acts as an *external force* to the atomic system.

Now let us consider the statistics of ξ_t^- for the Fock vacuum state free state $\Psi_\Omega = \langle \Omega, \cdot\, \Omega\rangle$ on $\mathcal{H}_R$. Since Ψ_Ω is a Gaussian free state, the statistics of ξ_t^- are uniquely determined by the 2-points correlation functions. Clearly, Ψ_Ω has zero mean, and therefore,

$$\Psi_\Omega(\xi_t^-) = \lambda \int_{\mathbb{R}^d} e^{-it\omega(x)} \chi(x)\, \Psi_\Omega(a_x^-)\,\mathrm{d}x = 0, \qquad \Psi_\Omega((\xi_t^-)^\dagger) = 0.$$

Moreover, we find

$$\Psi_\Omega((\xi_t^-)^\dagger\, \xi_s^-) = \Psi_\Omega(\xi_t^-\, \xi_s^-) = \Psi_\Omega((\xi_t^-)^\dagger\, (\xi_s^-)^\dagger) = 0,$$

and

$$\begin{aligned}\Psi_\Omega(\xi_t^-\, (\xi_s^-)^\dagger) &= \lambda^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-it(\omega(x)-\omega(y))}\chi(x)\chi(y)\,\Psi_\Omega(a_x^- a_y^+)\,\mathrm{d}x\,\mathrm{d}y \\ &= \lambda^2 \int_{\mathbb{R}^d} e^{-i(t-s)\omega(x)}\chi^2(x)\,\mathrm{d}x = \gamma(t-s).\end{aligned}$$

Hence, ξ_t^- is a mean-zero γ-correlated *random process* under Ψ.

To obtain the stochastic limit, we start by rescaling time $t \mapsto t/\lambda^2$ and consider the rescaled quantity $c^{\lambda,\pm}(t) \coloneqq c^\pm(t/\lambda^2)$, which yields

$$\dot{c}^{\lambda,-}(t) = -i\omega_0 c^{\lambda,-}(t) - \int_0^t \gamma^\lambda(t-s)\, c^{\lambda,-}(s)\,\mathrm{d}s - i\,\xi_t^{\lambda,-}, \qquad \text{eq:quantum-pre-langevin} \quad (3.4)$$

with

$$\gamma^\lambda(r) \coloneqq \frac{1}{\lambda^2}\langle \chi, e^{-i(r/\lambda^2)\omega}\chi\rangle, \qquad \xi_t^{\lambda,-} \coloneqq \lambda^{-2}\xi^-_{t/\lambda^2}.$$

From Theorem 3.13, we then establish that

$$\gamma^\lambda(r) \longrightarrow \gamma\,\delta(r), \qquad \xi_t^{\lambda,-} \longrightarrow b_t^- \qquad \text{in the sense of correlators,}$$

with $\gamma = \langle \chi, \delta(\omega)\chi\rangle$. Consequently, we obtain

$$\mathrm{d}c_t^- = -(i\omega_0 + \gamma)\, c_t^-\,\mathrm{d}t - i\,\mathrm{d}B_t^-, \qquad B_t^- = \int_0^t b_s^-\,\mathrm{d}s.$$

$$\mathrm{d}U_t = -i(c^-\,\mathrm{d}B_t^+ U_t + c^- U_t\,\mathrm{d}B_t^-) - \gamma_- c^+ c^- U_t\,\mathrm{d}t$$

$$\begin{aligned}\mathrm{d}(U_t^\dagger c^- U_t) &= i(c^-\,\mathrm{d}B_t^+ U_t + c^+ U_t\,\mathrm{d}B_t^-)^\dagger c^- U_t - \overline{\gamma}_- U_t^\dagger c^- c^+ c^- U_t\,\mathrm{d}t \\ &\quad - iU_t^\dagger c^-(c^-\,\mathrm{d}B_t^+ U_t + c^+ U_t\,\mathrm{d}B_t^-) - \gamma_- U_t^\dagger c^- c^+ c^- U_t\,\mathrm{d}t \\ &= iU_t^\dagger\,\mathrm{d}B_t^- c^+ c^- U_t + i\,\mathrm{d}B_t^+ U_t^\dagger c^- c^- U_t - \overline{\gamma}_- U_t^\dagger c^- c^+ c^- U_t\,\mathrm{d}t\end{aligned}$$