GRADIENT-FLOW STRUCTURES FOR SINGULAR JUMP PROCESSES

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Abstract.

1. Introduction

We focus on Markov jump processes

$$aaa$$

featuring a *singular* jump kernel, namely...

We aim to impart to these processes a (generalized) gradient-flow structure in terms of a suibable Energy-Dissipation balance. In particular, we are going to obtain solutions to this variational formulation as limits of a solutions to a suitably regularized jump process.

Plan of the paper.

List of symbols. Throughout the paper we will use the following notation.

$\mathscr{L}$	Lebesgue measure on $\mathbb{R}$	
$\overline{\nabla}$, $\overline{\mathrm{div}}$	graph gradient and divergence	(3.18)
$\mathrm{B_b}(Y;\mathbb{R}^m)$	bounded Borel $\mathbb{R}^m$-valued maps	
$\mathrm{B_c}(Y;\mathbb{R}^m)$	bounded and compactly supported Borel $\mathbb{R}^m$-valued maps	

add symbols

2. Preliminaries of measure theory

Finite measures. Let (Y,d) be separable metric space, and let $\mathfrak{B}(Y)$ be its associated Borel σ-algebra. We denote by $\mathcal{M}(Y;\mathbb{R}^m)$ the space of σ-additive measures on $\mu:\mathfrak{B}(Y)\to\mathbb{R}^m$ of *finite* total variation $\|\mu\|_{TV}:=|\mu|(Y)<+\infty$, where for every $B\in\mathfrak{B}(Y)$

$$|\mu|(B):=\sup\left\{\sum_{i=0}^{+\infty}|\mu(B_i)|\ :\ B_i\in\mathfrak{B}_Y,\ \ B_i \text{ pairwise disjoint},\ B=\bigcup_{i=0}^{+\infty}B_i\right\}.$$

The set function $|\mu|:\mathfrak{B}(Y)\to[0,+\infty)$ is a positive finite measure on $\mathfrak{B}(Y)$ [AFP05, Thm. 1.6] and $(\mathcal{M}(Y;\mathbb{R}^m),\|\cdot\|_{TV})$ is a Banach space. In the case $m=1$, we will simply write $\mathcal{M}(Y)$, and we shall denote the space of *positive* finite measures on $\mathfrak{B}(Y)$ by $\mathcal{M}^+(Y)$. For $m>1$, we will identify any element $\mu\in\mathcal{M}(Y;\mathbb{R}^m)$ with a vector $(\mu^1,\ldots,\mu^m)$, with $\mu^i\in\mathcal{M}(Y)$ for all $i=1,\ldots,m$. If $\varphi=(\varphi^1,\ldots,\varphi^m)\in\mathrm{B_b}(Y;\mathbb{R}^m)$ (the set of bounded $\mathbb{R}^m$-valued $\mathfrak{B}$-measurable maps), the duality between $\mu\in\mathcal{M}(Y;\mathbb{R}^m)$ and φ can be expressed by

$$\langle\mu,\varphi\rangle:=\int_Y\varphi\cdot\mu(\mathrm{d}x)=\sum_{i=1}^m\int_Y\varphi^i(x)\mu^i(\mathrm{d}x). \tag{2.1}$$ vector-dual

Radon measures. We call *Borel measure* any measure $\mu : \mathfrak{B}(Y) \to [0, +\infty]$. If a Borel measure is finite on the compact subsets of Y, we will call it a *positive Radon measure*. We call *Radon (vector) measure* any set function $\mu : \mathfrak{B}_c(Y) \to \mathbb{R}^m$ (where $\mathfrak{B}_c(Y)$ denotes the family of relatively compact Borel subsets of Y) such that its restriction to $\mathfrak{B}(K)$, for any *compact* set $K \subset Y$, is a finite (vector) measure. We denote by $\mathcal{M}^+_{\mathrm{loc}}(Y)$, $\mathcal{M}_{\mathrm{loc}}(Y)$, and $\mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ the spaces of positive, real-valued, and vector-valued Radon measures. For $\mu \in \mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ and $\varphi \in \mathrm{C_c}(Y;\mathbb{R}^m)$, the set of continuous $\mathbb{R}^m$-valued functions with compact support the integral in (2.1) is still well defined and induces a duality pairing between $\mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ and $\varphi \in \mathrm{C_c}(Y;\mathbb{R}^m)$.

We recall that, in addition, if Y is a *locally compact* separable (l.c.s. for short) metric space, any $\mu \in \mathcal{M}^+_{\mathrm{loc}}(Y)$ is inner regular, namely for every $E \in \mathfrak{B}(Y)$

$$\mu(E) = \sup\{\mu(K) \,:\, K \subset E,\ K \text{ compact}\}\,, \tag{2.2}$$

inner-regul

cf. e.g. [AFP05, Prop. 1.43].

Restriction, Lebesgue decomposition. For every $\mu \in \mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ and $B \in \mathfrak{B}(Y)$ we will denote by $\mu \llcorner B$ the restriction of μ to B, i.e. $\mu \llcorner B(A) := \mu(A \cap B)$ for every $A \in \mathfrak{B}(Y)$. Let $(X, \mathfrak{A})$ be another measurable space and let $\mathsf{p} : X \to Y$ a measurable map. For every $\mu \in \mathcal{M}_{\mathrm{loc}}(X;\mathbb{R}^m)$ we will denote by $\mathsf{p}_\sharp\mu$ the push-forward measure obtained by setting

$$\mathsf{p}_\sharp\mu(B) := \mu(\mathsf{p}^{-1}(B)) \quad \text{for every } B \in \mathfrak{B}(Y). \tag{2.3}$$

eq:82

For every pair $\mu \in \mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ and $\gamma \in \mathcal{M}^+_{\mathrm{loc}}(Y)$ there exist a unique (up to the modification in a γ-negligible set) γ-locally integrable map $\frac{\mathrm{d}\mu}{\mathrm{d}\gamma} : Y \to \mathbb{R}^m$, a γ-negligible set $N \in \mathfrak{B}$ and a unique measure $\mu^\perp \in \mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ yielding the *Lebesgue decomposition*

$$\begin{gathered}\mu = \mu^{\mathrm{a}} + \mu^\perp, \quad \mu^{\mathrm{a}} = \frac{\mathrm{d}\mu}{\mathrm{d}\gamma}\gamma = \mu \llcorner (Y \setminus N), \quad \mu^\perp = \mu \llcorner N, \quad \gamma(N) = 0 \\ |\mu^\perp| \perp \gamma, \quad |\mu|(Y) = \int_Y \left|\frac{\mathrm{d}\mu}{\mathrm{d}\gamma}\right| \mathrm{d}\gamma + |\mu^\perp|(Y).\end{gathered} \tag{2.4}$$

eq:Leb

Convergences of measures. The space $\mathcal{M}(Y;\mathbb{R}^m)$ is clearly endowed with the notion of convergence induced by the total variation norm $\|\cdot\|_{TV}$; we shall refer to it as *strong convergence*. On $\mathcal{M}(Y;\mathbb{R}^m)$ we will also consider *setwise convergence*, defined, for $(\mu_n)_n, \mu \in \mathcal{M}(Y;\mathbb{R}^m)$, by

$$\mu_n \to \mu \textit{ setwise} \quad \text{if} \quad \lim_{n\to\infty} \mu_n(B) = \mu(B) \quad \text{for all } B \in \mathfrak{B}(Y)\,. \tag{2.5}$$

setwise-con

In fact, the associated topology is the coarsest one on $\mathcal{M}(Y;\mathbb{R}^m)$ making all the functions $\mathfrak{B}(Y) \ni B \mapsto \mu(B)$ continuous. Among the various characterizations of setwise convergence, we recall [Bog07, §4.7(v)] that (2.5) is equivalent to

(1) *Convergence in duality with* $\mathrm{B_b}(Y;\mathbb{R}^m)$:
$$\lim_{n\to+\infty} \langle \mu_n, \varphi\rangle = \langle \mu, \varphi\rangle \qquad \text{for every } \varphi \in \mathrm{B_b}(Y;\mathbb{R}^m). \tag{2.6}$$
eq:70

(2) *Weak topology convergence in* $\mathcal{M}(Y;\mathbb{R}^m)$: the sequence μ_n converges to μ w.r.t. the weak topology of the Banach space $(\mathcal{M}(Y;\mathbb{R}^m); \|\cdot\|_{TV})$.

Furthermore, we recall that, by [Bog07, Thm. 4.7.25)], given a sequence $(\mu_n)_n \subset \mathcal{M}(Y;\mathbb{R}^m)$, the following properties are equivalent:

(i) $(\mu_n)_n$ is relatively compact w.r.t. setwise convergence;

(ii) there exists $\gamma \in \mathcal{M}^+(Y)$ such that
$$\forall\,\varepsilon > 0\ \exists\,\delta > 0 : \quad B \in \mathfrak{B}(Y),\ \gamma(B) \le \delta \quad \Rightarrow \quad \sup_n \mu_n(B) \le \varepsilon; \tag{2.7}$$
eq:73

(iii) there exists $\gamma \in \mathcal{M}^+(Y)$ such that $\mu_n \ll \gamma$ for all $n \in \mathbb{N}$ and the sequence $\left(\frac{\mathrm{d}\mu_n}{\mathrm{d}\gamma}\right)_n$ admits a subsequence weakly converging in the topology of $\mathrm{L}^1(Y,\gamma;\mathbb{R}^m)$.

Finally, we recall that a sequence $(\mu_n)_n \subset \mathcal{M}(Y;\mathbb{R}^m)$ converges *narrowly* to some $\mu \in \mathcal{M}(Y;\mathbb{R}^m)$ if (2.6) holds for every $\varphi \in \mathrm{C_b}(Y;\mathbb{R}^m)$, the space of continuous *bounded* $\mathbb{R}^m$-valued functions.

When, in addition, Y is also locally compact, on $\mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ we will consider

- *vague* convergence, namely convergence in duality against all functions $\varphi \in \mathrm{C_c}(Y;\mathbb{R}^m)$.

Convex functionals of measures. We will work with functionals depending on pairs of Radon measures defined in this way: let $\Upsilon : \mathbb{R}^m \to [0,+\infty]$ be proper, convex and lower semicontinuous and let us denote by $\Upsilon^\infty : \mathbb{R}^m \to [0,+\infty]$ its recession function

$$\Upsilon^\infty(z) := \lim_{t\to+\infty} \frac{\Upsilon(tz)}{t} = \sup_{t>0} \frac{\Upsilon(tz)-\Upsilon(0)}{t}\,. \tag{2.8}$$

recession-u

We note that Υ^∞ is convex, lower semicontinuous, and positively 1-homogeneous, with $\Upsilon^\infty(0)=0$. Hence, we define

$$\begin{aligned} &\mathscr{F}_\Upsilon : \mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m) \times \mathcal{M}^+_{\mathrm{loc}}(Y) \mapsto [0,+\infty],\\ &\mathscr{F}_\Upsilon(\mu|\nu) := \int_Y \Upsilon\Big(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu + \int_Y \Upsilon^\infty\Big(\frac{\mathrm{d}\mu^\perp}{\mathrm{d}|\mu^\perp|}\Big)\,\mathrm{d}|\mu^\perp|, \qquad \text{for } \mu = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\nu + \mu^\perp. \end{aligned} \tag{2.9}$$

def:F-F

In [PRST22, Lemma 2.3] we collected some properties of functionals of this class, albeit defined on $\mathcal{M}(Y;\mathbb{R}^m)\times\mathcal{M}^+(Y)$; most of them extend to the present case. We highlight the properties that will be used in what follows: in particular, lower semicontinuity extends to *vague* convergence. The proof of the following result will be carried out in Appendix A ahead.

l:crucial-F **Lemma 2.1.** *The functional $\mathscr{F}_\Upsilon : \mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)\times\mathcal{M}^+_{\mathrm{loc}}(Y) \mapsto [0,+\infty]$ enjoys the following properties:*

(1) if $\Upsilon(0)=0$, then for every $\mu \in \mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ and $\nu,\ \nu' \in \mathcal{M}^+_{\mathrm{loc}}(Y)$ there holds

$$\nu \le \nu' \implies \mathscr{F}_\Upsilon(\mu|\nu') \le \mathscr{F}_\Upsilon(\mu|\nu); \tag{2.10}$$

AC-monotoni

(2) if Υ is superlinear, then

$$\mathscr{F}_\Upsilon(\mu|\nu) < \infty \quad \Longrightarrow \quad \mu \ll \nu, \quad \mathscr{F}_\Upsilon(\mu|\nu) = \int_Y \Upsilon\Big(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu; \tag{2.11}$$

superlinear

(3) if Υ positively 1-homogeneous, then $\Upsilon \equiv \Upsilon^\infty$, $\mathscr{F}_\Upsilon(\cdot|\nu) \doteq \mathscr{F}_\Upsilon$ is independent of ν and satisfies

$$\mathscr{F}_\Upsilon(\mu) = \int_Y \Upsilon\left(\frac{\mathrm{d}\mu}{\mathrm{d}\gamma}\right)\mathrm{d}\gamma \quad \text{for every } \gamma \in \mathcal{M}^+_{\mathrm{loc}}(Y) \text{ such that } \mu \ll \gamma; \tag{2.12}$$

eq:78

(4) $\mathscr{F}_\Upsilon(\cdot|\cdot)$ is sequentially lower semicontinuous in $\mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)\times\mathcal{M}^+_{\mathrm{loc}}(Y)$ with respect to vague convergence.

Concave transformations of vector measures. We will also need to extend a construction set forth in [PRST22, Sec. 2.3] for vector and positive finite measures, to the case of vector and positive *Radon* measures. Namely, let $\mathbb{R}_+ := [0,+\infty[$, $\mathbb{R}^m_+ := (\mathbb{R}_+)^m$, and let $\alpha : \mathbb{R}^m_+ \to \mathbb{R}_+$ be a continuous and concave function. It is obvious that α is non-decreasing with respect to each variable. As in (2.8), the recession function α^∞ is defined by

$$\alpha^\infty(z) := \lim_{t\to+\infty} \frac{\alpha(tz)}{t} = \inf_{t>0} \frac{\alpha(tz)-\alpha(0)}{t}, \quad z \in \mathbb{R}^m_+. \tag{2.13}$$

eq:1

We define

$$\alpha : \mathcal{M}_{\rm loc}(Y;\mathbb{R}^m_+) \times \mathcal{M}^+_{\rm loc}(Y) \to \mathcal{M}^+_{\rm loc}(Y),$$
$$\alpha[\mu|\gamma] := \alpha\Big(\frac{\mathrm{d}\mu}{\mathrm{d}\gamma}\Big)\gamma + \alpha^\infty\Big(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu^\perp|}\Big)|\mu^\perp| \quad \mu \in \mathcal{M}_{\rm loc}(Y;\mathbb{R}^m_+),\ \gamma \in \mathcal{M}^+_{\rm loc}(Y), \tag{2.14}$$

alpha-mu-ga

where $\mu = \frac{\mathrm{d}\mu}{\mathrm{d}\gamma}\gamma + \mu^\perp$ is the Lebesgue decomposition of μ with respect to γ.

s:3

3. Setup and gradient system structure

We start by collecting our conditions on the vertex space V, on the reference measure π, which will be invariant under the evolution generated by our generalized gradient system, and on the family of singular kernels $(\kappa(x,\cdot))_{x\in V}$ (whose 'singularity' is encoded in the fact that for each $x \in V$ $\kappa(x,\cdot)$ is a positive Radon measure on $V\setminus\{x\}$). Recall that $E = V\times V$ is the space of edges; a special role will be played by the subset obtained removing from E its diagonal

$$E' := E \setminus \{(x,x)\ :\ x\in V\}$$

and by the measure $\boldsymbol{\vartheta}_\kappa$ on E' defined by

$$\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) := \kappa(x,\mathrm{d}y)\pi(\mathrm{d}x), \qquad \boldsymbol{\vartheta}_\kappa(A{\times}B) = \int_A \kappa(x,B)\pi(\mathrm{d}x) \tag{3.1}$$

teta-coupli

for all *disjoint* (Borel) subsets $A, B \subset V$. We postpone to Lemma 3.3 a thorough discussion of the properties of $\boldsymbol{\vartheta}_\kappa$ deriving from Assumption 3.1 below. For its statement we need to introduce the symmetry map $s : E\to E$, $s(x,y) := (y,x)$.

Ass:V

Assumption 3.1 (Vertex space and kernels)**.** *We suppose that*

(1) the vertex space

$$(V,\mathsf{d}) \text{ is a locally compact separable metric space, with Borel } \sigma\text{-algebra } \mathfrak{B}(V); \tag{3.2}$$

ass:V

(2) the reference measure $\pi \in \mathcal{M}^+(V)$ is a finite positive measure

-conditions

(3) the kernels $(\kappa(x,\cdot))_{x\in V}$ form a Borel family of measures in $\mathcal{M}^+_{\rm loc}(V\setminus\{x\})$, i.e.

$$\text{for all } f \in \mathrm{B_c}(E') \text{ the map } V \ni x \mapsto \int_{V\setminus\{x\}} f(x,y)\kappa(x,\mathrm{d}y) \text{ is Borel,} \tag{3.3a}$$

true-measur

such that for all $x\in V$

$$\forall\,\varepsilon>0 \int_{V\setminus B_\varepsilon(x)} \kappa(x,\mathrm{d}y) < +\infty \qquad \text{with} \qquad \lim_{\varepsilon\downarrow 0}\int_{V\setminus B_\varepsilon(x)} \kappa(x,\mathrm{d}y) = +\infty. \tag{3.3b}$$

bounds

Furthermore,

$$\sup_{x\in V}\int_{V\setminus\{x\}} (1{\wedge}d^2(x,y))\,\kappa(x,\mathrm{d}y) =: c_\kappa < +\infty. \tag{3.3c}$$

mitigation

(4) Finally, the detailed balance condition *holds, i.e. the coupling $\boldsymbol{\vartheta}_\kappa \in \mathcal{M}^+_{\rm loc}(E')$ defined by* (3.1) *satisfies*

$$s_\#\boldsymbol{\vartheta}_\kappa = \boldsymbol{\vartheta}_\kappa \tag{3.4}$$

DBC

As an immediate consequence of (3.3c) we have that, while for every $x\in V$ $\kappa(x,\cdot)$ is positive Radon measure on $V\setminus\{x\}$, the measure $(1{\wedge}d^2(x,y))\kappa(x,\cdot)$ extends to the whole V. Likewise, the formula

$$\boldsymbol{\vartheta}^d_\kappa(A) := \iint_A (1{\wedge}d^2(x,y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \text{ for all } A\in\mathfrak{B}(E), \text{ defines a (\textit{finite}) measure in } \mathcal{M}^+(E), \tag{3.5}$$

tetapi-lamb

where the extension to the whole if E is carried out by setting

$$\boldsymbol{\vartheta}^d_\kappa(E\setminus E') := \lim_{r\downarrow 0}\int_V \int_{B_r(x)\setminus\{x\}} (1\wedge d^2(x,y))\,\kappa(x,\mathrm{d}y)\,\mathrm{d}x\,.$$

rmk:lambda **Remark 3.2.** Let us dig deeper on our conditions on the kernels $(\kappa(x,\cdot))_{x\in V}$:

(1) In fact, by the local compactness of V, the measurability requirement (3.3a) is equivalent to having that the map

$$V \ni x \mapsto \int_{V\setminus\{x\}} (1\wedge d^2(x,y)) f(x,y)\kappa(x,\mathrm{d}y) \text{ is Borel } \textit{for all } f\in \mathrm{B_b}(E').$$

(2) Condition (3.3c) serves the purpose of *taming* the singularity of the kernels at the diagonal of E. In fact, since λ dominates d, (3.3c) implies that

$$\sup_{x\in V}\int_{V\setminus\{x\}} (1\wedge d^2(x,y))\,\kappa(x,\mathrm{d}y) \leq \frac{c_\kappa}{c_\lambda^2} < +\infty\,,$$

-mitigation so that, in particular, the second of (3.3b) is mitigated by the estimate

$$\lim_{\varepsilon\downarrow 0}\sup_{x\in V}\int_{B_1\setminus B_\varepsilon(x)} d^2(x,y)\kappa(x,\mathrm{d}y) < +\infty\,. \tag{3.6a}$$ conse-1

In turn, since for $R\gg 1$ we have $(1\wedge d^2(x,y)) = 1$ for all $y\in V\setminus B_R(x)$, (3.3c) also guarantees that

$$\lim_{R\to\infty}\sup_{x\in V}\int_{V\setminus B_R(x)} \kappa(x,\mathrm{d}y) < +\infty\,. \tag{3.6b}$$ conse-2

We remark that properties (3.6) are weaker than the uniform integrability condition required in [Erb14, Assumption 1.1] in the Euclidean setting of $V=\mathbb{R}^d$, $d(x,y)=|x-y|$, namely that

$$\lim_{R\to\infty}\sup_{x\in V}\left(\int_{V\setminus B_{1/R}(x)} |x-y|^2\,\kappa(x,\mathrm{d}y) + \int_{V\setminus B_R(x)} \kappa(x,\mathrm{d}y)\right) = 0\,.$$

Before moving on, let us pin down an easy, albeit crucial, property of $\boldsymbol{\vartheta}_\kappa$.

If we decide to kill Lemma 7.1, then we can also kill Lemma 3.3...

s-tetakappa **Lemma 3.3.** *Under Assumption 3.2, the coupling measure $\boldsymbol{\vartheta}_\kappa \in \mathcal{M}^+_{\mathrm{loc}}(E')$ enjoys the following property*

$$\textit{for all } N,B\in\mathfrak{B}(V) \textit{ with } N\cap B=\emptyset: \quad \pi(N)=0 \;\Rightarrow\; \boldsymbol{\vartheta}_\kappa(N\times B) = \boldsymbol{\vartheta}_\kappa(B\times N) = 0\,. \tag{3.7}$$ null-proper

In particular, suppose that a given property $\mathfrak{P}$ holds π-almost everywhere in V. Then, the property $\mathfrak{P}^\wedge$ defined for all $(x,y)\in E$ by

$$\mathfrak{P}^\wedge(x,y) := \mathfrak{P}(x)\wedge\mathfrak{P}(y) \quad \textit{holds } \boldsymbol{\vartheta}_\kappa\textit{-almost everywhere in } E'\,. \tag{3.8}$$ Pwedge

Proof. Let $(\varepsilon_n)_n$ be a null sequence. By Fatou's Lemma

$$\boldsymbol{\vartheta}_\kappa(N\times B) = \int_N\int_B \kappa(x,\mathrm{d}y)\pi(\mathrm{d}x) \leq \liminf_{n\to\infty}\int_N\int_{B\setminus B_{\varepsilon_n}(x)} \kappa(x,\mathrm{d}y)\pi(\mathrm{d}x)\,.$$

On the other hand, for $\varepsilon_n<1$ we have, by (3.3c),

$$\begin{aligned}\int_N\int_{B\setminus B_{\varepsilon_n}(x)} \kappa(x,\mathrm{d}y)\pi(\mathrm{d}x) &\leq \int_N\int_{V\setminus B_{\varepsilon_n}(x)} \frac{1}{1\wedge d^2(x,y)}(1\wedge d^2(x,y))\,\kappa(x,\mathrm{d}y)\,\pi(\mathrm{d}x)\\ &\leq c_\kappa\left(\frac{1}{\varepsilon_n^2}+1\right)\pi(N) = 0\,,\end{aligned}$$

and (3.7) follows.

In order to check (3.8), suppose that $\mathfrak{P}$ holds for all $x \in B$, with $B \in \mathfrak{B}(V)$ such that $\pi(V\backslash B) = 0$. Then, $\mathfrak{P}^\wedge$ holds for all $(x, y) \in B \times B$, and by (3.7) we have

$$\boldsymbol{\vartheta}_\kappa(E'\backslash(B\times B)) \leq \boldsymbol{\vartheta}_\kappa(B\times(V\backslash B)) + \boldsymbol{\vartheta}_\kappa((V\backslash B)\times B) = 0\,.$$

$\square$

Our variational structure will also be based on a driving energy functional, on a dissipation potential, and on a flux density map whose properties are collected below.

Ass:E **Assumption 3.4** (Energy). *The energy functional $\mathscr{E} : \mathcal{M}^+(V) \to [0, +\infty]$ is of the form*

$$\mathscr{E}(\rho) = \begin{cases} \displaystyle\int_V \phi\left(\frac{\mathrm{d}\rho}{\mathrm{d}\pi}\right)\mathrm{d}\pi & \text{if } \rho \ll \pi, \\ +\infty & \text{otherwise.} \end{cases}$$

We require the energy density to fulfill

$$\phi \in \mathrm{C}([0, +\infty)) \cap \mathrm{C}^1(]0, +\infty]),\ \ \phi \text{ convex with } \min\phi = 0, \qquad \lim_{r\to+\infty} \frac{\phi(r)}{r} = +\infty. \tag{3.9}$$ ass-phi

Ass:D **Assumption 3.5** (Dissipation). *The dual dissipation density $\psi^* : \mathbb{R} \to [0, +\infty)$ is convex, differentiable, even, with $\psi^*(0) = 0$ and*

$$\lim_{|\xi|\to+\infty} \frac{\psi^*(\xi)}{|\xi|} = +\infty, \tag{3.10}$$ growth-psis

$$\lim_{\xi\to 0} \frac{\psi^*(\xi)}{|\xi|^2} = c_0 \in]0, +\infty[\,. \tag{3.11}$$ quadratic-a

lux-density **Assumption 3.6** (Flux density). The mapping $\alpha : [0, +\infty)\times[0, +\infty) \to [0, +\infty)$ is non-degenerate (not identically null), continuous, concave, symmetric, with

$$\alpha(u_1, u_2) = \alpha(u_2, u_1) \qquad \forall\,(u_1, u_2) \in [0, +\infty) \times [0, +\infty)\,. \tag{3.12}$$ props-alpha

Before moving on, let us pin down two key consequences of Assumption 3.5 that will be extensively used later on. Other outcomes will be highlighted in Section 6.1.

Lemma 3.7. *The function ψ^* is non-decreasing on $[0, \infty)$ and there exists a convex, even, and superlinear function $\mathfrak{f} : \mathbb{R} \to [0, +\infty)$ such that* ded-control

$$\psi^*(\eta\xi) \leq \eta^2\mathfrak{f}(\xi) \qquad \text{for all } \eta \in [-1, 1],\ \xi \in \mathbb{R} \tag{3.13a}$$ needed-cont

or, equivalently,

$$\forall\,\xi \in \mathbb{R}\ \forall\,M > 0\ \forall\,\beta \in [-M, M] : \qquad \psi^*(\beta\xi) \leq \frac{1}{M^2}\eta^2\mathfrak{f}(M\xi)\,. \tag{3.13b}$$ needed-cont

Proof. Since $0 = \psi^*(0) = \min_{\mathbb{R}}\psi^*$, we have $(\psi^*)'(0) = 0$ and thus $(\psi^*)'(\xi) \geq 0$ for all $\xi \geq 0$. Hence, the monotonicity statement for ψ^*.

Now, from (3.11) we infer that

$$\exists\, r > 0\ \ \forall\,\eta \in [-r, r] : \ \frac{1}{2}c_0|\eta|^2\psi^*(\eta) \leq \frac{3}{2}c_0|\eta|^2\,. \tag{3.14}$$ easy-conseq

To show (3.13a) we distinguish two cases:

(1) $|\eta\xi| \leq r$: then

$$\psi^*(\eta\xi) \leq \frac{3}{2}c_0\eta^2\xi^2\,. \tag{3.15}$$ est4needed-

(2) $|\eta\xi| > r$: then,

$$\psi^*(\eta\xi) \overset{(1)}{=} \psi^*(|\eta\xi|) \overset{(2)}{\leq} \psi^*(\xi) \overset{(3)}{\leq} \frac{|\eta\xi|}{r}\psi^*(\xi) \overset{(4)}{\leq} \frac{|\xi|^2}{r}|\eta|^2\psi^*(\xi)\,, \tag{3.16}$$ est4needed-

where (1) follows from the evenness of ψ^*, (2) from its monotonicity on $[0,+\infty)$, and (3), (4) from the fact that $\frac{|\eta\xi|}{r} > 1$.

All in all, we conclude that

$$\psi^*(\eta\xi) \leq \eta^2\left(\frac{3}{2}c_0\xi^2 + \frac{|\xi|^2}{r}\psi^*(\xi)\right) \doteq \eta^2\mathfrak{f}(\xi)\,.$$

It is immediate to check that the thus defined function $\mathfrak{f} : \mathbb{R} \to [0,+\infty)$ is convex, even and with superlinear growth at infinity. This finishes the proof. □

The primal and dual dissipation potentials: formal definition. After specifying our conditions the triple (κ, ψ, α), we are in a position to (formally) introduce the dissipation potentials $\mathscr{R}$ and $\mathscr{R}^*$ via

$$\begin{aligned}
&\mathscr{R}(\rho,\boldsymbol{j}) := \frac{1}{2}\iint_{E'} \psi\left(2\frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}\boldsymbol{\nu}_\rho}\right)\boldsymbol{\nu}_\rho(\mathrm{d}x\mathrm{d}y) \quad \text{with } \boldsymbol{\nu}_\rho(\mathrm{d}x\mathrm{d}y) = \boldsymbol{\alpha}(u(x),u(y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x,\mathrm{d}y) \text{ and } u = \frac{\mathrm{d}\rho}{\mathrm{d}\pi}\\
&\mathscr{R}^*(\rho,\xi) := \frac{1}{2}\iint_{E'} \psi^*(\xi)\,\boldsymbol{\nu}_\rho(\mathrm{d}x\mathrm{d}y)\,.
\end{aligned} \tag{3.17}$$ formal_defi

Their rigorous definition is postponed to Def. 3.13 ahead.

def:ona-div 3.1. **The continuity equation.** Recall the definition of the 'graph gradient' $\overline{\nabla}$

$$(\overline{\nabla}\varphi)(x,y) := \varphi(y) - \varphi(x) \qquad \text{for any Borel function } \varphi : V \to \mathbb{R}\,, \tag{3.18a}$$ eq:def:ona-

and of the 'graph divergence operator' $\overline{\mathrm{div}} : \mathcal{M}_{\mathrm{loc}}(E) \to \mathcal{M}_{\mathrm{loc}}(V)$

$$(\overline{\mathrm{div}}\,\boldsymbol{j})(\mathrm{d}x) := \int_{y\in V}\big[\boldsymbol{j}(\mathrm{d}x,\mathrm{d}y) - \boldsymbol{j}(\mathrm{d}y,\mathrm{d}x)\big] \qquad \text{for any } \boldsymbol{j} \in \mathcal{M}_{\mathrm{loc}}(E)\,. \tag{3.18b}$$ eq:def:div

Clearly, we have that

$$\iint_E \overline{\nabla}\varphi(x,y)\,\boldsymbol{j}(\mathrm{d}x,\mathrm{d}y) = -\int_V \varphi(x)\,\overline{\mathrm{div}}\,\boldsymbol{j}(\mathrm{d}x) \quad \text{for every } \varphi \in \mathrm{C_c}(V). \tag{3.19}$$ eq:nabladiv

Hereafter, for a given function $\mu : I \to \mathcal{M}(V)$, or $\mu : I \to \mathcal{M}(E)$, with $I = [a,b] \subset \mathbb{R}$, we shall often write μ_t in place of $\mu(t)$ for a given $t \in I$ and denote the time-dependent function μ by $(\mu_t)_{t\in I}$.
Lip-bounded Test functions for the continuity equation will be chosen in this space

$$\mathrm{Lip_b}(V) := \mathrm{Lip}(V) \cap \mathrm{B_b}(V), \tag{3.20a}$$ Lip-bounded

which we consider with the norm

$$\|\varphi\|_{\mathrm{Lip_b}(V)} := \sup_{x\in V}|\varphi(x)| + \sup_{x,y\in V,\ x\neq y}\frac{|\overline{\nabla}\varphi\,(x,y)|}{1\wedge d(x,y)}\,. \tag{3.20b}$$ Lip-bounded

Observe that the above norm in indeed equivalent to the usual norm for $\mathrm{Lip_b}(V)$, defined as $\sup_{x\in V}|\varphi(x)| + \sup_{x\neq y\in V}\frac{|\overline{\nabla}\varphi\,(x,y)|}{d(x,y)}$. In fact, we have opted for the definition in (3.20b) only to have more transparent estimates, cf. Remark 3.9 ahead. We are now in a position to introduce the continuity equation we will work with in this paper.

def:CE **Definition 3.8** (Solutions to the continuity equation)**.** Let $[a,b] \subset \mathbb{R}$. We denote by $\mathcal{CE}([a,b])$ the set of pairs $(\rho,\boldsymbol{j})$ such that

- $\rho = (\rho_t)_{t\in[a,b]}$ is a family of time-dependent measures in $\mathcal{M}^+(V)$;
- $\boldsymbol{j} = (\boldsymbol{j}_t)_{t\in[a,b]}$ is a measurable family of measures in $\mathcal{M}_{\mathrm{loc}}(E')$ such that

$$\int_a^b \iint_{E'} (1\wedge d(x,y))\,|\boldsymbol{j}_t|(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}t < +\infty \tag{3.21}$$

crucial-bou

- the continuity equation holds in this sense: For all $\varphi \in \mathrm{Lip_b}(V)$ and all $[s,t] \subset [a,b]$,

$$\int_V \varphi(x)\,\rho_t(\mathrm{d}x) - \int_V \varphi(x)\,\rho_s(\mathrm{d}x) = \int_s^t \iint_{E'} \overline{\nabla}\varphi\,(x,y)\,\boldsymbol{j}_r(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r. \tag{3.22}$$

CE

In what follows, we will use the short-hand notation $\boldsymbol{j}_{\mathscr{L}}$ for the measure on $[a,b]\times E'$ given by $\boldsymbol{j}_{\mathscr{L}}(A) := \iiint_A \boldsymbol{j}_t(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}t$ for all $A \in \mathfrak{B}([a,b]\times E')$.

t-functions **Remark 3.9.** By our choice of test functions, the integral on the right-hand side of (3.22) is well defined. Indeed, for every $[s,t]\subset[a,b]$ we have

$$\left|\int_s^t \iint_{E'} \overline{\nabla}\varphi\,(x,y)\,\boldsymbol{j}_r(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r\right| \le \|\varphi\|_{\mathrm{Lip_b}(V)} \int_a^b \iint_{E'} \cdot(1\wedge d(x,y))\,|\boldsymbol{j}_t|(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}t < +\infty \tag{3.23}$$

obvious-est

thanks to (3.21).

It can easily checked that the concatenation of two solutions to the continuity equation is again a solution to the continuity equation, and that the family of solutions is closed under time rescaling.

n&rescaling **Lemma 3.10** (Concatenation and time rescaling)**.** *(1) Let $(\rho^i, \boldsymbol{j}^i) \in \mathcal{CE}([0,T_i])$, $i=1,2$, with $\rho^1_{T_1} = \rho^2_0$. Define $(\rho_t, \boldsymbol{j}_t)_{t\in[0,T_1+T_2]}$ by*

$$\rho_t := \begin{cases} \rho^1_t & \text{if } t\in[0,T_1],\\ \rho^2_{t-T_1} & \text{if } t \in [T_1, T_1+T_2], \end{cases} \qquad \boldsymbol{j}_t := \begin{cases} \boldsymbol{j}^1_t & \text{if } t\in[0,T_1],\\ \boldsymbol{j}^2_{t-T_1} & \text{if } t \in [T_1, T_1+T_2]. \end{cases}$$

Then, $(\rho,\boldsymbol{j}) \in \mathcal{CE}([0,T_1+T_2])$.

(2) Let $\mathsf{t} : [0,\hat{T}] \to [0,T]$ be strictly increasing and absolutely continuous, with inverse $\mathsf{s} : [0,T] \to [0,\hat{T}]$. Then, $(\rho,\boldsymbol{j}) \in \mathcal{CE}([0,T])$ if and only if $\hat\rho := \rho\circ\mathsf{t}$ and $\hat{\boldsymbol{j}} := \mathsf{t}'(\boldsymbol{j}\circ\mathsf{t})$ fulfill $(\hat\rho,\hat{\boldsymbol{j}}) \in \mathcal{CE}([0,\hat{T}])$.

We consider the norm on $\mathcal{M}(V)$ obtained by taking the duality against bounded Lipschitz functions:

$$\|\rho\|_{\mathrm{BL}} := \sup\left\{ \left|\int_V \varphi(x)\rho(\mathrm{d}x)\right| : \varphi\in\mathrm{Lip_b}(V),\ \|\varphi\|_{\mathrm{Lip_b}(V)} \le 1 \right\}. \tag{3.24}$$

nbl-norm

We have the following result, to be compared with [PRST22, Lemma 4.4].

s-represent **Lemma 3.11.** *Let $(\rho_t)_{t\in I} \subset \mathcal{M}^+(V)$ and $(\boldsymbol{j}_t)_{t\in I}$ be measurable families that are integrable with respect to $\mathscr{L}$, fulfilling*

$$-\int_a^b \eta'(t)\left(\int_V \zeta(x)\rho_t(\mathrm{d}x)\right)\mathrm{d}t = \int_a^b \eta(t)\Big(\iint_E \overline{\nabla}\zeta(x,y)\,\boldsymbol{j}_t(\mathrm{d}x\,\mathrm{d}y)\Big)\,\mathrm{d}t\,, \tag{3.25}$$

eq:90

holds for every $\eta\in \mathrm{C_c^\infty}((a,b))$ and $\zeta \in \mathrm{Lip_b}(V)$.

Then there exists a unique curve $[a,b] \ni t \mapsto \tilde\rho_t \in \mathcal{M}^+(V)$ such that $\tilde\rho_t = \rho_t$ for $\mathscr{L}$-a.e. $t\in[a,b]$. The curve $\tilde\rho$ is continuous in the $\|\cdot\|_{\mathrm{BL}}$-norm, fulfills the estimate

$$\|\tilde\rho_{t_2} - \tilde\rho_{t_1}\|_{\mathrm{BL}} \le \int_{t_1}^{t_2} \iint_{E'} (1\wedge d(x,y))\,|\boldsymbol{j}_t|(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}t \qquad \text{for all } t_1 \le t_2 \in [a,b]\,. \tag{3.26}$$

est:ct-eq-B

The *proof* follows the very same lines as that of [PRST22, Lemma 4.4], and it is thus omitted; we only remark that (3.26) is an immediate consequence of estimate (3.23).

3.2. **The primal and dual dissipation potentials.** Based on Assumptions 3.1, 3.5, and 3.6, we rigorously define the primal dissipation and dual dissipation potentials $\mathscr{R}$ and $\mathscr{R}^*$ formally introduced in (3.17). As in [PRST22], first of all with any $\rho \in \mathcal{M}^+(V)$ we associate the couplings $\boldsymbol{\vartheta}_\rho^\pm$ defined on E' by

$$\boldsymbol{\vartheta}_\rho^-(\mathrm{d}x\,\mathrm{d}y) := \rho(\mathrm{d}x)\kappa(x,\mathrm{d}y), \qquad \boldsymbol{\vartheta}_\rho^+(\mathrm{d}x\,\mathrm{d}y) := \rho(\mathrm{d}y)\kappa(y,\mathrm{d}x) = s_\# \boldsymbol{\vartheta}_\rho^-(\mathrm{d}x\,\mathrm{d}y), \tag{3.27}$$ rig-def:tet

Observe that $\boldsymbol{\vartheta}_\rho^\pm \in \mathcal{M}_{\mathrm{loc}}(E')$. We prove, for later use, the following continuity result, to be compared with [PRST22, Lemma 2.4].

l:3.4 **Lemma 3.12.** *Let $(\rho_n)_n, \rho \in \mathcal{M}^+(V)$ satisfy $\rho_n \to \rho$ setwise. Then, $\boldsymbol{\vartheta}_{\rho_n}^\pm \to \boldsymbol{\vartheta}_\kappa$ vaguely in $\mathcal{M}_{\mathrm{loc}}(E')$.*

Proof. Clearly, it is sufficient to prove the assertion for $(\boldsymbol{\vartheta}_{\rho_n}^-)_n$. Let us fix $f \in \mathrm{C_c}(E')$: then, the map $x \mapsto \int_{V\setminus\{x\}} f(x,y)\kappa(x,\mathrm{d}y)$ is a bounded Borel function on V. Indeed, for every $x \in V$ there exists $\eta > 0$ such that $\mathrm{supp}(f(x,\cdot)) \subset V \setminus B_\eta(x)$, so that

$$\left| \int_{V\setminus\{x\}} f(x,y)\kappa(x,\mathrm{d}y) \right| = \left| \int_{V\setminus B_\eta(x)} f(x,y)\kappa(x,\mathrm{d}y) \right| \le \|f\|_\infty \int_{V\setminus B_\eta(x)} \kappa(x,\mathrm{d}y)$$

with $\|f\|_\infty := \sup_{(x,y)\in E'} |f(x,y)|$. Therefore,

$$\begin{aligned} \iint_{E'} f(x,y)\,\boldsymbol{\vartheta}_{\rho_n}^-(\mathrm{d}x\mathrm{d}y) &= \int_V \left(\int_{V\setminus\{x\}} f(x,y)\,\kappa(x,\mathrm{d}y) \right) \rho_n(\mathrm{d}x) \\ &\longrightarrow \int_V \left(\int_{V\setminus\{x\}} f(x,y)\,\kappa(x,\mathrm{d}y) \right) \rho(\mathrm{d}x) = \iint_{E'} f(x,y)\,\boldsymbol{\vartheta}_\rho^-(\mathrm{d}x\mathrm{d}y)\,. \end{aligned}$$

□

Based on the construction of $\boldsymbol{\vartheta}_\rho^\pm$, we may rigorously define the measure $\boldsymbol{\nu}_\rho \in \mathcal{M}_{\mathrm{loc}}(E')$ from (3.17) as concave transformation of the couplings $\boldsymbol{\vartheta}_\rho^\pm$ (cf. construction set forth in (2.14)), namely

$$\boldsymbol{\nu}_\rho := \upalpha[\boldsymbol{\vartheta}_\rho^-, \boldsymbol{\vartheta}_\rho^+ | \boldsymbol{\vartheta}_\kappa] \tag{3.28}$$ rig-nu-rho

where we have used the simplified notation $\upalpha[\boldsymbol{\vartheta}_\rho^-, \boldsymbol{\vartheta}_\rho^+ | \boldsymbol{\vartheta}_\kappa]$ in place of $\upalpha[(\boldsymbol{\vartheta}_\rho^-, \boldsymbol{\vartheta}_\rho^+) | \boldsymbol{\vartheta}_\kappa]$. In the definition of $\mathscr{R}$ we will also resort to the construction from (2.9).

al-and-dual **Definition 3.13.** We define $\mathscr{R} : \mathcal{M}^+(V) \times \mathcal{M}_{\mathrm{loc}}(E') \to [0,+\infty]$ and $\mathscr{R}^* : \mathcal{M}^+(V) \times \mathrm{C_c}(E') \to [0,+\infty)$ via

$$\mathscr{R}(\rho, \boldsymbol{j}) := \frac{1}{2} \mathscr{F}_\uppsi(2\boldsymbol{j} | \boldsymbol{\nu}_\rho)\,, \tag{3.29}$$ rigR

$$\mathscr{R}^*(\rho, \xi) := \frac{1}{2} \iint_{E'} \uppsi^*(\xi)\,\boldsymbol{\nu}_\rho(\mathrm{d}x\mathrm{d}y)\,, \tag{3.30}$$ Rig-Rstar

The following result collects some key facts about $\mathscr{R}$. Firstly, we provide an equivalent representation, cf. (3.31) below, for $\mathscr{R}$ involving the convex and lower semicontinuous function $\Upsilon : [0,+\infty) \times [0,+\infty) \times \mathbb{R} \to [0,+\infty]$

$$\Upsilon(u,v,w) := \hat{\uppsi}(w, \upalpha(u,v))$$

with $\hat{\uppsi} : \mathbb{R}^2 \to [0,\infty]$ the 1-homogeneous, convex, perspective function associated with $\uppsi$ by

$$\hat{\uppsi}(z,t) := \begin{cases} \uppsi(z/t)t & \text{if } t > 0, \\ \uppsi^\infty(z) & \text{if } t = 0, \\ +\infty & \text{if } t < 0. \end{cases}$$

Based on (3.31) we may prove (3.33), which extends the sequential lower semicontinuity of $\mathscr{R}(\rho,\cdot)$, addressed in [PRST22, Lemma 4.10], from setwise to *vague* convergence.

Lemma 3.14. *The following properties hold:*

(1) For $\mathscr{R}$ we have the equivalent representation

$$\mathscr{R}(\rho,\boldsymbol{j}) = \frac{1}{2}\mathscr{F}_{\Upsilon}((\boldsymbol{\vartheta}_\rho^-,\boldsymbol{\vartheta}_\rho^+,2\boldsymbol{j})|\boldsymbol{\vartheta}_\kappa); \tag{3.31}$$

equivalent-

(2) if $(\rho,\boldsymbol{j})$ fulfill $\mathscr{R}(\rho,\boldsymbol{j})<+\infty$ and $\rho\ll\pi$, then $\boldsymbol{j}\ll\boldsymbol{\vartheta}_\kappa$ and

$$\mathscr{R}(\rho,\boldsymbol{j}) = \begin{cases} \frac{1}{2}\iint_{E_\alpha}\Psi\Big(\frac{w(x,y)}{\alpha(u(x),u(y))}\Big)\alpha(u(x),u(y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x,\mathrm{d}y) & \text{if } |\boldsymbol{j}|(E'\setminus E_\alpha)=0,\\ +\infty & \text{if } |\boldsymbol{j}|(E'\setminus E_\alpha)>0, \end{cases} \tag{3.32}$$

nice-repres

where $w=\frac{\mathrm{d}(2\boldsymbol{j})}{\mathrm{d}\boldsymbol{\vartheta}_\kappa}$ and $E_\alpha:=\{(x,y)\in E' \,:\, \alpha(u(x),u(y))>0\}$;

(3) for all $(\rho_n)\,\rho\in\mathcal{M}^+(V)$ and $(\boldsymbol{j}_n)_n$, $\boldsymbol{j}\in\mathcal{M}_{\mathrm{loc}}(E')$ there holds

$$[\rho_n\to\rho \text{ setwise and } \boldsymbol{j}_n\to\boldsymbol{j} \text{ vaguely }] \Longrightarrow \liminf_{n\to\infty}\mathscr{R}(\rho_n,\boldsymbol{j}_n)\geq\mathscr{R}(\rho,\boldsymbol{j})\,. \tag{3.33}$$

R-lsc

Proof. Formula (3.31) follows by the same arguments as in [PRST22, Sec. 4.2]; likewise, the proof of the first part of Claim **(2)** follows by [PRST22, Lemma 4.10].

Now, when $\rho\ll\pi$ then we also have $\boldsymbol{\vartheta}_\rho^\pm\ll\boldsymbol{\vartheta}_\kappa$ and (recalling that $\boldsymbol{\vartheta}_\rho^+=s_{\#}\boldsymbol{\vartheta}_\rho^-$)

$$\frac{\mathrm{d}\boldsymbol{\vartheta}_\rho^-}{\mathrm{d}\boldsymbol{\vartheta}_\kappa}(x,y)=u(x), \qquad \frac{\mathrm{d}\boldsymbol{\vartheta}_\rho^-}{\mathrm{d}\boldsymbol{\vartheta}_\kappa}(x,y)=u(y),$$

so that

$$\boldsymbol{\nu}_\rho(\mathrm{d}x\,\mathrm{d}y)=\alpha\left(\frac{\mathrm{d}\boldsymbol{\vartheta}_\rho^-}{\mathrm{d}\boldsymbol{\vartheta}_\kappa},\frac{\mathrm{d}\boldsymbol{\vartheta}_\rho^+}{\mathrm{d}\boldsymbol{\vartheta}_\kappa}\right)\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)=\alpha(u(x),u(y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,.$$

Since $\boldsymbol{j}\ll\boldsymbol{\vartheta}_\kappa$, we can write $2\boldsymbol{j}=w\boldsymbol{\vartheta}_\kappa$, so that

$$\Upsilon\left(\frac{\mathrm{d}\boldsymbol{\vartheta}_\rho^-}{\mathrm{d}\boldsymbol{\vartheta}_\kappa},\frac{\mathrm{d}\boldsymbol{\vartheta}_\rho^+}{\mathrm{d}\boldsymbol{\vartheta}_\kappa},\frac{\mathrm{d}2\boldsymbol{j}}{\mathrm{d}\boldsymbol{\vartheta}_\kappa}\right)(x,y)=\hat{\psi}(w(x,y),\alpha(u(x),u(y)))$$

and (3.32) follows by the definition of the perspective function $\hat{\psi}$.

Finally, let $(\rho_n)\,\rho\in\mathcal{M}^+(V)$ and $(\boldsymbol{j}_n)_n$, $\boldsymbol{j}\in\mathcal{M}_{\mathrm{loc}}(E')$ be as in (3.33). By Lemma 3.12 we infer that $(\boldsymbol{\vartheta}_{\rho_n}^-,\boldsymbol{\vartheta}_{\rho_n}^+,2\boldsymbol{j}_n)\to(\boldsymbol{\vartheta}_\rho^-,\boldsymbol{\vartheta}_\rho^+,2\boldsymbol{j})$ vaguely in $\mathcal{M}_{\mathrm{loc}}(E';\mathbb{R}^3)$, and the assertion follows from Lemma 2.1. $\square$

3.3. **The $\mathscr{R}$-$\mathscr{R}^*$ formulation.** Prior to specifying our notion of weak solution, we need to property introduce the Fisher information. Formally, it is given by

$$\mathscr{D}(\rho)=\mathscr{R}^*\Big(\rho,-\overline{\nabla}\phi(u)\Big)=\frac{1}{2}\iint_E\Psi^*\big(-(\phi'(u(y))-\phi'(u(x))\big)\boldsymbol{\nu}_\rho(\mathrm{d}x\,\mathrm{d}y), \qquad \rho=u\pi\,.$$

However, note that ϕ need not be differentiable at 0. Hence, in order to rigorously define $\mathscr{D}$ in the present context, we mimick the ideas of [PRST22]. Firstly, we introduce the function $\Lambda_\phi:\mathbb{R}_+\times\mathbb{R}_+\to[-\infty,+\infty]$

$$\Lambda_\phi(u,v):=\begin{cases}\phi'(v)-\phi'(u) & \text{if } u,v\in\mathbb{R}_+\times\mathbb{R}_+\setminus\{(0,0)\},\\ 0 & \text{if } u=v=0\end{cases}$$

where we have set $\phi'(0) := \lim_{r\downarrow 0} \phi'(r) \in [-\infty, +\infty)$. Hence, we define the function $\mathrm{D}^+_\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty]$ by

$$\mathrm{D}^+_\phi(u,v) := \begin{cases} \psi^*\left(\Lambda_\phi(u,v)\right)\alpha(u,v) & \text{if } \alpha(u,v) > 0, \\ 0 & \text{if } u = v = 0, \\ +\infty & \text{if } \alpha(u,v) = 0 \text{ with } u \neq v\,. \end{cases}$$

Finally, we consider

$$\text{the } \textit{lower semicontinous envelope } \mathrm{D}_\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,+\infty] \text{ of } \mathrm{D}^+_\phi\,.$$

Def:Fisher **Definition 3.15** (Fisher information)**.** The Fisher information $\mathscr{D} : \mathrm{dom}(\mathscr{E}) \to [0, +\infty]$ is defined as

$$\mathscr{D}(\rho) := \begin{cases} \frac{1}{2}\iint_{E'} \mathrm{D}_\phi(u(x), u(y))\, \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) & \text{if } \rho = u\pi \text{ and } \mathrm{D}_\phi(u(x), u(y)) \in L^1(E', \boldsymbol{\vartheta}_\kappa), \\ +\infty \text{ otherwise.} \end{cases}$$

We now highlight the crucial lower semicontinuity property of $\mathscr{D}$.

:Fisher-lsc **Proposition 3.16.** *Suppose that either π is purely atomic, or that the function $\mathrm{D}_\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty]$ is convex. Then, for all $(\rho_n)_n\, \rho \in \mathrm{dom}(\mathscr{E})$ we have*

$$\rho_n \to \rho \textit{ setwise in } \mathcal{M}^+(V) \implies \liminf_{n\to\infty} \mathscr{D}(\rho_n) \geq \mathscr{D}(\rho)\,.$$

The *proof* follows the same lines as the argument for [PRST22, Prop. 5.3], hence it is omitted.

Finally, we are in a position to formalize our concept of solution for the evolution system associated with $(V, \kappa, \phi, \psi, \alpha)$.

tar-balance **Definition 3.17.** We say that a curve $\rho : [0, T] \to \mathcal{M}^+(V)$ is a solution of the $(\mathscr{E}, \mathscr{R}, \mathscr{R}^*)$ evolution system, if it satisfies the following conditions:

(1) $\mathscr{E}(\rho_0) < +\infty$;
(2) There exists a measurable family $(\boldsymbol{j}_t)_{t\in[0,T]} \subset \mathcal{M}_{\mathrm{loc}}(E')$ such that $(\rho, \boldsymbol{j}) \in \mathcal{CE}([0, T])$ and the pair $(\rho, \boldsymbol{j})$ complies with the $(\mathscr{E}, \mathscr{R}, \mathscr{R}^*)$ *Energy-Dissipation balance*:

$$\int_s^t \left(\mathscr{R}(\rho_r, \boldsymbol{j}_r) + \mathscr{D}(\rho_r)\right) \mathrm{d}r + \mathscr{E}(\rho_t) = \mathscr{E}(\rho_s) \qquad \text{for all } 0 \leq s \leq t \leq T. \tag{3.34}$$ R-Rstar-bal

s:4

4. Main result

We will prove our existence result for the evolution system associated with $(V, \kappa, \phi, \psi, \alpha)$ by approximating the singular kernels $(\kappa(x, \cdot))_{x\in V}$ via a family of regularized kernels. More precisely, we define

$$\kappa_\varepsilon(x, \mathrm{d}y) := \frac{1 \wedge d^2(x,y)}{\varepsilon + (1 \wedge d^2(x,y))} \kappa(x, \mathrm{d}y) \qquad \text{for all } x \in V \tag{4.1}$$ kappa-eps-r

In this way, we clearly obtain a *finite* measure on the whole of V, fulfilling

$$\kappa_\varepsilon(x, \cdot) \leq \frac{1}{\varepsilon}(1 \wedge d^2(x, \cdot))\kappa(x, \cdot) \qquad \text{for all } \varepsilon > 0. \tag{4.2}$$ crucial-bou

Thus, by (3.3c) we infer that

$$\|\kappa_\varepsilon\|_\infty := \sup_{x\in V} \kappa_\varepsilon(x, V) \leq \frac{1}{\varepsilon} c_\kappa\,. \tag{4.3}$$ back-2-boun

For later use, we also observe that, since $\kappa_\varepsilon(x, \cdot) \leq \kappa(x, \cdot)$, there holds

$$\sup_{x\in V} \int_V (1 \wedge d^2(x,y))\kappa_\varepsilon(x, \mathrm{d}y) \leq c_\kappa\,. \tag{4.4}$$ mitigated-b

We will denote by $\boldsymbol{\vartheta}_{\kappa_\varepsilon}$ the induced coupling

$$\boldsymbol{\vartheta}_{\kappa_\varepsilon}(\mathrm{d}x\mathrm{d}y) = \kappa_\varepsilon(x, \mathrm{d}y)\pi(\mathrm{d}x); \tag{4.5}$$ tetapie

observe that, now, $\boldsymbol{\vartheta}_{\kappa_\varepsilon} \in \mathcal{M}(E)$ and, clearly, it still satisfies the detailed balance (3.4).

Let $\mathscr{R}_\varepsilon : \mathcal{M}^+(V) \times \mathcal{M}(E) \to [0, +\infty]$ and $\mathscr{R}^*_\varepsilon : \mathcal{M}^+(V) \times \mathrm{C_b}(E) \to [0, +\infty)$ the primal and dual dissipation potentials associated with $\boldsymbol{\vartheta}_{\kappa_\varepsilon}$ via Definition 3.13 (with $\iint_{E'}$ replaced by $\iint_E$), and let $\mathscr{D}_\varepsilon$ the corresponding Fisher information functional

$$\mathscr{D}_\varepsilon(\rho) := \begin{cases} \frac{1}{2}\iint_E \mathrm{D}_\phi(u(x), u(y))\, \boldsymbol{\vartheta}_{\kappa_\varepsilon}(\mathrm{d}x\mathrm{d}y) & \text{if } \rho = u\pi \text{ and } \mathrm{D}_\phi(u(x), u(y)) \in L^1(E, \boldsymbol{\vartheta}_{\kappa_\varepsilon}), \\ +\infty \text{ otherwise.} \end{cases}$$

We may thus consider the $(\mathscr{E}, \mathscr{R}_\varepsilon, \mathscr{R}^*_\varepsilon)$ energy-dissipation balance modelled on (3.34).

Under Assumptions 3.1, 3.4, 3.5, and 3.6, also in view of (4.3) the regularized system $(V, \kappa_\varepsilon, \phi, \psi, \alpha)$ complies with the assumptions of [PRST22, Thm. 5.7], ensuring the existence of a solution to the $(\mathscr{E}, \mathscr{R}_\varepsilon, \mathscr{R}^*_\varepsilon)$ evolution system in the sense of Def. 3.17. We point out that such solution in fact fulfills the continuity equation in an enhanced sense, namely (3.22) holds, along any sub-interval $[s, t] \subset [a, b]$, for all test functions $\varphi \in \mathrm{B_b}(V)$. Besides stating the existence of solutions, the following result also subsumes the maximum principle proved in [PRST22, Thm. 6.5].

Theorem 4.1. [PRST22, Thm. 5.10, Thm. 6.5] *Let $(V, \kappa, \phi, \psi, \alpha)$ comply with Assumptions 3.1, 3.4, 3.5, and 3.6. In addition, suppose that either π is atomic, or the function $\mathrm{D}_\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty]$ is convex. Then,*

(1) For every $\rho^0 \in \mathcal{M}^+(V)$ with $\mathscr{E}(\rho^0) < +\infty$ there exists a curve $\rho : [0, T] \to \mathcal{M}^+(V)$, with $\rho(0) = \rho^0$, solving the $(\mathscr{E}, \mathscr{R}_\varepsilon, \mathscr{R}^*_\varepsilon)$ evolution system, i.e. there exists a measurable family $\boldsymbol{j} = (\boldsymbol{j}_t)_{t\in[a,b]} \subset \mathcal{M}(E)$, with $(\rho, \boldsymbol{j}) \in \mathcal{CE}^{\mathrm{enh}}([0, T])$, such that the pair $(\rho, \boldsymbol{j})$ is an enhanced *solution of the the $(\mathscr{E}, \mathscr{R}_\varepsilon, \mathscr{R}^*_\varepsilon)$ system.*

(2) If $\rho^0 = u^0\pi$ with $u^0 \in L^\infty(V, \pi)$ such that

$$\exists\, 0 \le \underline{U} < \overline{U} \quad \text{for } \pi\text{-a.a. } x \in V: \qquad 0 \le \underline{U} \le u^0(x) \le \overline{U}, \tag{4.6}$$

then $\rho = u\pi$ satisfies

$$0 \le \underline{U} \le u_t(x) \le \overline{U} \qquad \text{for } (\mathscr{L}\times\pi)\text{-a.a. } (t, x) \in (0, T)\times V. \tag{4.7}$$ boundedness

Approximation of solutions to the singular system. Let $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0, +\infty)$ be a null sequence. Correspondingly, let $(\rho^{\varepsilon_n}, \boldsymbol{j}^{\varepsilon_n})_n \subset \mathcal{CE}([0, T])$, satisfying the continuity equation for all test functions $\varphi \in \mathrm{B_b}(V)$, be solutions of the $(\mathscr{E}, \mathscr{R}_{\varepsilon_n}, \mathscr{R}^*_{\varepsilon_n})$ evolution systems starting from initial data $(\rho^0_n)_n \subset \mathcal{M}^+(V)$ satisfying suitable conditions. With the main result of this paper we are going to prove that, up to a subsequence, the pairs $(\rho^{\varepsilon_n}, \boldsymbol{j}^{\varepsilon_n})_n$ converge as $n \to \infty$ to a solution of the $(\mathscr{E}, \mathscr{R}, \mathscr{R}^*)$ system. We recall that, denoting $\mathscr{L}$ the Lebesgue measure $\mathscr{L}|_{(0,T)}$, a measurable family $\nu = (\nu_t)_{t\in[0,T]} \in \mathcal{M}_{\mathrm{loc}}(Y)$, with $Y \in \{V, E, E'\}$, induces a measure in $\mathcal{M}_{\mathrm{loc}}([0, T]\times Y)$ by setting $\nu_{\mathscr{L}}(\mathrm{d}t\mathrm{d}x\mathrm{d}y) := \nu_t(\mathrm{d}x\mathrm{d}y)\mathscr{L}(\mathrm{d}t)$.

Before giving the main result of this paper, we need to introduce one more condition, besides Assumptions 3.1, 3.4, 3.5, and 3.6. Prior to stating it, let us settle some preliminary definitions. First of all, on the measure space $(E', \mathfrak{B}(E'), \boldsymbol{\vartheta}_\kappa)$ we introduce the *Orlicz space* associated with the *Young* function ψ^*, namely

$$\mathrm{L}^{\psi^*}(E'; \boldsymbol{\vartheta}_\kappa) := \left\{ y \in \mathrm{L}^1(E'; \boldsymbol{\vartheta}_\kappa) \,:\, \exists\, \ell > 0 \ \iint_{E'} \psi^*\left(\frac{y(x, y)}{\ell}\right) \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) < +\infty \right\},$$

with the associated Luxemburg norm

$$\|y\|_{\mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)} := \inf\left\{ \ell > 0 \,:\, \iint_{E'} \psi^*\left(\frac{y(x, y)}{\ell}\right) \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \le 1 \right\};$$

we refer to, e.g., [RR91] for a comprehensive presentation of Orlicz spaces. We will also consider the space

$$\mathrm{X}^{\psi^*} := \left\{ \varphi \in \mathrm{B_b}(V) \, : \, \overline{\nabla}\varphi \in \mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa) \right\}. \tag{4.8}$$ X-L-Orli

Furthermore, let us introduce the *small Orlicz* space

$$\mathcal{M}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa) := \left\{ y \in \mathrm{L}^1(E';\boldsymbol{\vartheta}_\kappa) \, : \, \forall\, \ell > 0 \, \iint_{E'} \psi^*\left(\frac{y(x,y)}{\ell}\right) \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) < +\infty \right\},$$

and, accordingly,

$$\mathcal{X}^{\psi^*} := \left\{ \varphi \in \mathrm{B_b}(V) \, : \, \overline{\nabla}\varphi \in \mathcal{M}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa) \right\}. \tag{4.9}$$ X-M-Orli

Clearly, $\mathcal{M}^{\psi^*}(V;\pi)$ is a subspace of $\mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)$, so that $\mathcal{X}^{\psi^*} \subset \mathrm{X}^{\psi^*}$. If, in addition, the Young function satisfies the so-called Δ_2 condition (cf. [RR91, §2.3, Definition 1]), then $\mathcal{M}^{\psi^*}(V;\pi) = \mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)$. We are not going to impose Δ_2-regularity on ψ^* but, still, we will be able to show that, in the present context, the spaces X^{ψ^*} and $\mathcal{X}^{\psi^*}$ coincide (cf. Corollary 6.3). We will also show in Lemma 6.4 that, for every $\varphi \in \mathrm{Lip_b}(V)$, $\overline{\nabla}\varphi \in \mathcal{M}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)$, so that $\mathrm{Lip_b}(V) \subset \mathcal{X}^{\psi^*}$. This makes our additional assumption meaningful.

Ass:F-bis **Assumption 4.2.** The space $\mathrm{Lip_b}(V)$ is dense in $\mathcal{X}^{\psi^*}$ in the following sense: for every $\varphi \in \mathcal{X}^{\psi^*}$ there exists a sequence $(\varphi_n)_n \subset \mathrm{Lip_b}(V)$ such that as $n \to \infty$

(1) $\varphi_n \to \varphi$ in $\mathrm{L}^1(V;\pi)$;
(2) $\overline{\nabla}\varphi_n \to \overline{\nabla}\varphi$ in $\mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)$.

We postpone a discussion on the validity of Assumption 4.2 to Section 7.

I have had to strengthen the previous weak convergence in $\mathrm{L}^1(V)$ to strong convergence in view of the cut-off argument of Lemma 7.1 but, if we decide to kill it, then we can go back to weak convergence...

We are in a position to state the main result of this paper.

th:main
al-thm:main **Theorem 4.3.** *Under Assumptions 3.1, 3.4, 3.5, 3.6, and 4.2, suppose that the function* $\mathrm{D}_\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,+\infty]$ *is convex. Let* $(\rho_n^0)_n, \rho^0 \in \mathcal{M}^+(V)$ *satisfy*

$$\begin{cases} \rho_n^0 \to \rho \text{ setwise in } \mathcal{M}^+(V) \text{ as } n \to \infty\,, \\ \mathscr{E}(\rho_n^0) \to \mathscr{E}(\rho) \text{ as } n \to \infty\,. \end{cases} \tag{4.10a}$$ initial-con

and

$$\exists\, 0 \le \underline{U} < \overline{U} \quad \forall\, n \in \mathbb{N} \text{ for } \pi\text{-a.a. } x \in V: \qquad \underline{U} \le u_n^0(x) \le \overline{U} \tag{4.10b}$$ strong-init

Then,

(1) There exist $(\rho, \boldsymbol{j}) \in \mathcal{CE}([0,T])$ *and (a not relabeled) subsequence such that the following convergences hold as* $n \to \infty$

$$\rho_t^{\varepsilon_n} \to \rho_t \qquad \text{setwise in } \mathcal{M}^+(V) \quad \text{for all } t \in [0,T]; \tag{4.11}$$ DBL

$$\boldsymbol{j}_{\mathscr{L}}^{\varepsilon_n} \to \boldsymbol{j}_{\mathscr{L}} = \boldsymbol{j}_t \mathscr{L} \qquad \text{vaguely in } \mathcal{M}_{\mathrm{loc}}([0,T] \times E'); \tag{4.12}$$ cvg-j

the measure ρ *satisfies* $\rho_t = u_t \pi$ *for all* $t \in [0,T]$*, for some* $u \in \mathrm{L}^1(0,T;\mathrm{L}^1(V;\pi))$ *such that*

$$\underline{U} \le u(t,x) \le \overline{U} \text{ for } (\mathscr{L} \otimes \pi)\text{-a.a. } (t,x) \in (0,T) \times V, \tag{4.13}$$ inherited-b

and the pair $(\rho, \boldsymbol{j})$ *is a solution of the* $(\mathscr{E}, \mathscr{R}, \mathscr{R}^*)$ *evolution system in the sense of Definition 3.17.*

(2) *If, in addition,* (4.10b) *holds with a constant* $\underline{U} > 0$*, then* $(\rho, \boldsymbol{j})$ *is an* enhanced *solution of the* $(\mathscr{E}, \mathscr{R}, \mathscr{R}^*)$ *evolution system, namely the Energy-Dissipation balance* (3.34) *holds in pointwise form*

$$\mathscr{R}(\rho_t, \boldsymbol{j}_t) + \mathscr{D}(\rho_t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho_t) = -\iint_{E'} \overline{\nabla}(\phi'(u_t))\, \boldsymbol{j}_t(\mathrm{d}x\mathrm{d}y) \qquad \text{for a.e. } t \in (0,T). \tag{4.14}$$

enh-EDB

In fact, as it will be clear from the proof, in case **(2)** we will also be able to show that $(\rho, \boldsymbol{j})$ solves the continuity equation in an enhanced sense, namely for a bigger set of test functions than $\mathrm{Lip_b}(V)$, cf. Proposition 6.1.

At this point we might add a remark on the difference between (3.34) and (4.14)

Outline of the proof. In Sec. 5 we are going to prove convergences (4.11)&(4.12) to a pair $(\rho, \boldsymbol{j}) : [0,T] \to \mathcal{M}^+(V) \times \mathcal{M}_{\mathrm{loc}}(E')$ satisfying the

upper energy-dissipation estimate:

$$\int_0^t (\mathscr{R}(\rho_r, \boldsymbol{j}_r) + \mathscr{D}(\rho_r))\, \mathrm{d}r + \mathscr{E}(\rho_t) \le \mathscr{E}(\rho_0) \qquad \text{Efor all } t \in [0,T]\,. \tag{4.15}$$

UEDE

We will conclude the proof in Sec. 6 by showing that the curve $(\rho, \boldsymbol{j})$ satisfies the continuity equation in the sense of Def. 3.8 and fulfills the

lower energy-dissipation estimate:

$$\int_0^t (\mathscr{R}(\rho_r, \boldsymbol{j}_r) + \mathscr{D}(\rho_r))\, \mathrm{d}r + \mathscr{E}(\rho_t) \ge \mathscr{E}(\rho_0) \qquad \text{for all } t \in [0,T]\,. \tag{4.16}$$

LEDE

s:5

5. Compactness and Upper energy-dissipation inequality

Preliminarily, for the reader's convenience we recall a refined version of the Ascoli-Arzelà theorem in metric spafes that will be used later on.

thm:Ascoli

Theorem 5.1. [AGS08, Prop. 3.3.1] *Let* $(\mathscr{S}, \mathsf{d})$ *be a complete metric space, also endowed with a topology* σ *compatible with* d *in the sense that for all* $(x_n)_n$, $(y_n)_n \subset \mathscr{S}$ *there holds*

$$(x_n, y_n) \xrightarrow{\sigma} (x,y) \implies \liminf_{n\to\infty} \mathsf{d}(x_n, y_n) \ge \mathsf{d}(x,y)\,. \tag{5.1}$$

compatibili

-for-Ascoli

Let K *be a* σ*-sequentially compact subset of* $\mathscr{S}$*, and let* $(v_n)_n$ *be a sequence of curves* $v_n : [0,T] \to \mathscr{S}$ *such that*

$$v_n(t) \in \mathsf{K} \qquad \text{for all } t \in [0,T],\ n \in \mathbb{N}, \tag{5.2a}$$

compactness

$$\limsup_{n\to\infty} \mathsf{d}(v_n(t), v_n(s)) \le \omega(s,t) \qquad \text{for all } s,\ t \in [0,T] \tag{5.2b}$$

equicontinu

where $\omega : [0,T]\times[0,T] \to [0,\infty)$ *is a symmetric function for which there exists an (at most) countable subset* $\mathscr{C}$ *of* $[0,T]$ *such that*

$$\lim_{(s,t)\to(r,r)} \omega(s,t) = 0 \quad \text{for all } r \in [0,T]\backslash\mathscr{C}.$$

Then, there exist an increasing subsequence $(n_k)_k$ *and a limit curve* $v : [0,T] \to \mathscr{S}$ *such that*

$$v_{n_k}(t) \xrightarrow{\sigma} v(t) \text{ for all } t \in [0,T] \quad \text{and} \quad v : [0,T] \to \mathscr{S} \text{ is } \mathsf{d}\text{-continuous on } [0,T]\backslash\mathscr{C}\,.$$

Furthermore, we record an observation that will be used several times in what follows.

ious-useful **Lemma 5.2.** *Let* $(\boldsymbol{\eta}_n)_n, \boldsymbol{\eta} \subset \mathcal{M}([0,T]\times E)$ *fulfill*

$$\boldsymbol{\eta}_n \to \boldsymbol{\eta} \qquad \textit{setwise in } \mathcal{M}([0,T]\times E)\,,$$

and set $\boldsymbol{\zeta}_n := \frac{1}{1\wedge d}\boldsymbol{\eta}_n$. *Then,*

$$\boldsymbol{\zeta}_n \to \boldsymbol{\zeta} := \frac{1}{1\wedge d}\boldsymbol{\eta} \qquad \textit{vaguely in } \mathcal{M}([0,T]\times E')\,.$$

Proof. For the measures $\boldsymbol{\zeta}_n$ let us consider cylindrical test functions φ, i.e., $\varphi(t,x,y) = \phi(t)\gamma(x,y)$ with $\eta \in \mathrm{C_c}([0,T])$ and $\gamma \in \mathrm{C_c}(E')$. We have

$$\begin{aligned}
\iiint_{[0,T]\times E'} \varphi\, \boldsymbol{\zeta}_n(\mathrm{d}t\mathrm{d}x\mathrm{d}t) &= \iiint_{[0,T]\times E'} \phi(t)\frac{\gamma(x,y)}{1\wedge d(x,y)}(1\wedge d(x,y))\, \boldsymbol{\zeta}_n(\mathrm{d}t\mathrm{d}x\mathrm{d}t)\\
&= \iiint_{[0,T]\times E'} \phi(t)\frac{\gamma(x,y)}{1\wedge d(x,y)}\, \boldsymbol{\eta}_n(\mathrm{d}t\mathrm{d}x\mathrm{d}t)\\
&\xrightarrow{(1)} \iiint_{[0,T]\times E'} \phi(t)\frac{\gamma(x,y)}{1\wedge d(x,y)}\boldsymbol{\eta}(\mathrm{d}t\mathrm{d}x\mathrm{d}t)\\
&= \iiint_{[0,T]\times E'} \varphi\, \boldsymbol{\zeta}(\mathrm{d}t\mathrm{d}x\mathrm{d}t)
\end{aligned}$$

where convergence (1) follows from the fact that γ has compact support in E' and thus $\gamma/(1\wedge d)$ extends to a function in $\mathrm{B_b}(E)$. □

compactness

5.1. **Compactness.** We now address the compactness properties of the sequence $(\rho^{\varepsilon_n}, \boldsymbol{j}^{\varepsilon_n})_n$. With the short-hand notation $\boldsymbol{\vartheta}_{\varepsilon_n} := \boldsymbol{\vartheta}_{\kappa_{\varepsilon_n}}$, we write

$$\rho^{\varepsilon_n} = u^{\varepsilon_n}\pi \text{ and } 2\boldsymbol{j}^{\varepsilon_n} = w^{\varepsilon_n}\boldsymbol{\vartheta}_{\varepsilon_n} \tag{5.3}$$ absolute-co

(since $\rho_t^{\varepsilon_n} \ll \pi$ and $\boldsymbol{j}_t^{\varepsilon_n} \ll \boldsymbol{\vartheta}_{\varepsilon_n}$ for $\mathscr{L}$-a.a. $t \in (0,T)$, by (3.32)) for some $(u^{\varepsilon_n})_n \subset \mathrm{L}^1(0,T;\mathrm{L}^1(V,\pi))$ and $(w^{\varepsilon_n})_n \subset \mathrm{L}^1(0,T;\mathrm{L}^1(E,\boldsymbol{\vartheta}_{\varepsilon_n}))$. From the $(\mathscr{E}, \mathscr{R}_{\varepsilon_n}, \mathscr{R}^*_{\varepsilon_n})$ energy-dissipation balance written on the interval $[0,t]$ for any $t \in [0,T]$

$$\int_0^t (\mathscr{R}^{\varepsilon_n}(\rho_r^{\varepsilon_n}, \boldsymbol{j}_r^{\varepsilon_n}) + \mathscr{D}^{\varepsilon_n}(\rho_r^{\varepsilon_n}))\,\mathrm{d}r + \mathscr{E}(\rho_t^{\varepsilon_n}) \le \mathscr{E}(\rho_n^0) \tag{5.4}$$ EDB-n

it immediately follows that

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]} \mathscr{E}(\rho_t^{\varepsilon_n}) \le \sup_{n\in\mathbb{N}} \mathscr{E}(\rho_n^0) = \mathsf{E}_0, \tag{5.5}$$ energy-boun

$$\sup_{n\in\mathbb{N}} \int_0^T \mathscr{R}_{\varepsilon_n}(\rho^{\varepsilon_n}, \boldsymbol{j}^{\varepsilon_n})\mathrm{d}t \le \mathsf{E}_0\,, \tag{5.6}$$ action-boun

$$\sup_{n\in\mathbb{N}} \int_0^T \mathscr{D}_{\varepsilon_n}(\rho^{\varepsilon_n})\mathrm{d}t \le \mathsf{E}_0\,, \tag{5.7}$$ Fisher-boun

For later convenience we remark that the action integrals $\int_0^T \mathscr{R}_{\varepsilon_n}(\rho^{\varepsilon_n}, \boldsymbol{j}^{\varepsilon_n})\mathrm{d}t$ rewrite taking into account that, by (5.6) we have for $\mathscr{L}$-a.a. $t \in (0,T)$

$$|\boldsymbol{j}_t^{\varepsilon_n}|(E \setminus E_t^{\varepsilon_n}) = 0 \qquad \text{with } E_t^{\varepsilon_n} = \{(x,y) \in E \,:\, \upalpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y)) > 0\} \tag{5.8}$$ def-set-Eep

and hence

$$\mathscr{R}(\rho_t^{\varepsilon_n}, \boldsymbol{j}_t^{\varepsilon_n}) = \frac{1}{2}\iint_{E_t^{\varepsilon_n}} \uppsi\Big(\frac{w_t^{\varepsilon_n}(x,y)}{\upalpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))}\Big)\upalpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\,\boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x,\mathrm{d}y) \quad \text{for } \mathscr{L}\text{-a.a. } t \in (0,T).$$

We emphasize that the compactness result below holds under the sole conditions (4.10). In particular, we may allow for the constant $\underline{U}$ in (4.10b) to be zero.

compactness **Proposition 5.3** (Compactness)**.** *There exist a curve $\rho : [0,T] \to \mathcal{M}^+(V)$, continuous in the $\|\cdot\|_{\mathrm{BL}}$-norm, a curve $u \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(V;\pi))$ satisfying* (4.13)*, and $\boldsymbol{j}_{\mathscr{L}} \in \mathcal{M}_{\mathrm{loc}}([0,T]\times E')$, such that*

$$\begin{cases} \rho_t(\mathrm{d}x) = u_t\,\pi(\mathrm{d}x) & \text{for } \mathscr{L}\text{-a.a. } t\in(0,T), \\ \boldsymbol{j}_{\mathscr{L}}(\mathrm{d}t\mathrm{d}x\mathrm{d}y) = \boldsymbol{j}_t(\mathrm{d}x\mathrm{d}y)\,\mathscr{L}(\mathrm{d}t) & \text{for a measurable family } \boldsymbol{j} = (\boldsymbol{j}_t)_{t\in[0,T]} \subset \mathcal{M}_{\mathrm{loc}}(E') \end{cases} \tag{5.9}$$ disintegrat

such that, along a (not relabeled) subsequence, convergences (4.11) *and* (4.12) *hold. Moreover, the curves $(u^{\varepsilon_n})_n$ from* (5.3) *fulfill*

$$u^{\varepsilon_n} \overset{*}{\rightharpoonup} u \quad \text{in } \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(V;\pi)). \tag{5.10}$$ added-cvg-u

Proof. Clearly, by (4.10a) we have that $\sup_n \rho_n^0(V) \le M_1$ for some $M_1 > 0$. Now, from the continuity equation we deduce the mass conservation property

$$\rho_t^{\varepsilon_n}(V) = \rho_0^{\varepsilon_n}(V) = \rho_n^0(V) \qquad \text{for all } t\in[0,T] \text{ and } j\in\mathbb{N}.$$

In fact, by (4.7) we also have

$$0 \le \underline{U} \le u_t^{\varepsilon_n}(x) \le \overline{U} \qquad \text{for } (\mathscr{L}\times\pi)\text{-a.a. } (t,x)\in(0,T)\times V\,, \tag{5.11}$$ boundedness

so that there exists $u \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(V;\pi))$ such that convergence (5.10) holds (along a not relabeled subsequence). Clearly, u inherits the bounds (5.11). We will now address the proof of some claims.

Claim 1: *Consider the families of measures $\boldsymbol{g}_{\mathscr{L}}^{\varepsilon_n}(\mathrm{d}x\mathrm{d}y\mathrm{d}t) = \boldsymbol{g}_t^{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\mathscr{L}(\mathrm{d}t) \in \mathcal{M}([0,T]\times E)$, with $\boldsymbol{g}_t^{\varepsilon_n} := (1\wedge d)\,\boldsymbol{j}_t^{\varepsilon_n}$. Then, there exists $\boldsymbol{G} \in \mathcal{M}((0,T)\times E)$ such that, up to a not relabeled subsequence, there holds as $n\to\infty$*

$$\boldsymbol{g}_{\mathscr{L}}^{\varepsilon_n} \to \boldsymbol{G} \qquad \text{setwise in } \mathcal{M}([0,T]\times E)\,. \tag{5.12}$$ setwise-g

The measure $\boldsymbol{G}$ can be disintegrated as $\boldsymbol{G}(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y) = \boldsymbol{g}_{\mathscr{L}}(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y) = \boldsymbol{g}_t(\mathrm{d}x\,\mathrm{d}y)\,\mathscr{L}(\mathrm{d}t)$ for a measurable family $\boldsymbol{g} = (\boldsymbol{g}_t)_{t\in[0,T]} \subset \mathcal{M}(E)$.

We start by observing that, for every $\beta > 0$ and for $\mathscr{L}$-a.a. $t\in(0,T)$

$$\begin{aligned}
|\boldsymbol{g}_t^{\varepsilon}|(E) &= \frac{1}{2}\iint_E \frac{1}{\beta}(1\wedge d(x,y))|w_t^{\varepsilon_n}(x,y)|\boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y) \\
&\overset{(1)}{=} \frac{1}{2}\iint_{E_t^{\varepsilon_n}} \frac{1}{\beta}(1\wedge d(x,y))\frac{|w_t^{\varepsilon_n}(x,y)|}{\upalpha(u_t^{\varepsilon_n}(x),u_t^{\varepsilon_n}(y))}\upalpha(u_t^{\varepsilon_n}(x),u_t^{\varepsilon_n}(y))\,\boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y) \\
&\overset{(2)}{\le} \frac{1}{\beta}\Big[\frac{1}{2}\iint_{E_t^{\varepsilon_n}} \uppsi\left(\frac{w_t^{\varepsilon_n}(x,y)}{\upalpha(u_t^{\varepsilon_n}(x),u_t^{\varepsilon_n}(y))}\right)\upalpha(u_t^{\varepsilon_n}(x),u_t^{\varepsilon_n}(y))\,\boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y) \\
&\qquad + \frac{1}{2}\iint_{E_t^{\varepsilon_n}} \uppsi^*\left(\beta(1\wedge d(x,y))\right)\upalpha(u_t^{\varepsilon_n}(x),u_t^{\varepsilon_n}(y))\,\boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\Big] \\
&\overset{(3)}{\le} \frac{1}{\beta}\Big[\mathscr{R}_{\varepsilon_n}(\rho_t^{\varepsilon_n},\boldsymbol{j}_t^{\varepsilon_n}) + \frac{1}{2}\mathfrak{f}(\beta)\iint_{E_t^{\varepsilon_n}}\left(1\wedge d^2(x,y)\right)\upalpha(u_t^{\varepsilon_n}(x),u_t^{\varepsilon_n}(y))\,\boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\Big]\,.
\end{aligned} \tag{5.13}$$ calc1

Indeed, (1) is due to the fact that $|\boldsymbol{j}_t^{\varepsilon_n}|(E\backslash E_t^{\varepsilon_n}) = 0$ (cf. (5.8)), while (2) follows from the fact that $\uppsi$ is even, and eventually the estimate (3) for $\uppsi^*$ in terms of $\mathfrak{f}$ follows from (3.13). Now, in order to estimate the second integral in the above formula, we use that, by the concavity (and, a fortiori, continuity) of $\upalpha$, by (5.11) we have for π-a.a. $x,y\in V$

$$\upalpha(u_t^{\varepsilon_n}(x),u_t^{\varepsilon_n}(y)) \le C_{\overline{U}} \tag{5.14}$$ for-later-u

for some $C_{\overline{U}} > 0$. Also taking into account that

$$\iint_E (1\wedge d^2(x,y))\,\boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y) \le \iint_E (1\wedge d^2(x,y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \le c_k\pi(V)$$

by (3.3c), we thus obtain

$$\iint_{E_t^{\varepsilon_n}} \big(1\wedge d^2(x,y)\big)\, \boldsymbol{\alpha}(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\, \boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y) \le C_{\overline{U}} c_\kappa \pi(V) \doteq \overline{C}\,. \tag{5.15}$$

calc2

Mimicking the above calculations we conclude that for every Borel subset $A \subset [0,T]\times E$ and every $\beta > 0$

$$\begin{aligned} |\boldsymbol{g}_{\mathscr{L}}^{\varepsilon_n}|(A) &= \frac{1}{2}\iiint_A \frac{1}{\beta}(1\wedge d(x,y)) w_t^{\varepsilon_n}(x,y)\, \boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\mathrm{d}t \\ &\le \frac{1}{\beta}\Big[\int_0^T \mathscr{R}_{\varepsilon_n}(\rho_t^{\varepsilon_n}, \boldsymbol{j}_t^{\varepsilon_n})\mathrm{d}t + \frac{1}{2}\mathfrak{f}(\beta)\iiint_A \big(1\wedge d^2(x,y)\big)\, \boldsymbol{\alpha}(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\, \boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\mathrm{d}t\Big] \\ &\le \frac{1}{\beta}\mathsf{E}_0 + \frac{\mathfrak{f}(\beta)}{\beta}\frac{C_{\overline{U}}}{2}\iiint_A \big(1\wedge d^2(x,y)\big)\, \boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\mathrm{d}t\,, \end{aligned}$$

where we have relied on estimate (5.6). In this way, exploiting the arbitrariness of β we may prove that

$$\forall\,\eta>0\;\exists\,\delta>0:\; A\in\mathfrak{B}([0,T]\times E),\; \boldsymbol{\vartheta}_{\mathscr{L}}^{d}(A)<\delta \implies \sup_n |\boldsymbol{g}_{\mathscr{L}}^{\varepsilon_n}|(A) < \eta \tag{5.16}$$

for-setwise

where we have used the place-holder $\boldsymbol{\vartheta}_{\mathscr{L}}^{d}$ for the positive finite measure $(1\wedge d^2)(\boldsymbol{\vartheta}_\kappa)_{\mathscr{L}}$. Therefore, we have shown that the sequence $(\boldsymbol{g}_{\mathscr{L}}^{\varepsilon_n})_n$ is relatively compact w.r.t. setwise convergence in $\mathcal{M}((0,T)\times E)$.

It remains to show that $\boldsymbol{G} = \boldsymbol{g}_t \mathscr{L}$ for a measurable family $(\boldsymbol{g}_t)_{t\in[0,T]} \subset \mathcal{M}(E)$. For this, it is sufficient to apply the disintegration result from [Bog07, Cor. 10.4.15]. Hence, we need to show that the first marginal of $\boldsymbol{G}$, i.e. the push-forward measure $(\pi_0)_\#\boldsymbol{G}$ via the projection $\pi_0 : [0,T]\times E \to [0,T]$, is absolutely continuous w.r.t. the Lebesgue measure $\mathscr{L}$. It is immediate to deduce this property from estimates (5.13)–(5.15), which yield for all $I \subset [0,T]$ and $\beta > 0$

$$((\pi_0)_\#\boldsymbol{G})(I) \le \liminf_{n\to\infty}\int_I |\boldsymbol{g}_t^{\varepsilon_n}|(E')\mathrm{d}t \le \frac{1}{\beta}\sup_n \int_I \mathscr{R}_{\varepsilon_n}(\rho_t^{\varepsilon_n}, \boldsymbol{j}_t^{\varepsilon_n})\mathrm{d}t + \overline{C}\frac{\mathfrak{f}(\beta)}{2\beta}\mathscr{L}(I)\,.$$

All in all, we have shown **Claim** 1.

Claim 2: *Convergence* (4.11) *holds along a further, not relabeled, subsequence, and* $\rho_t(\mathrm{d}x) = u_t\,\pi(\mathrm{d}x)$ *with* u *from* (5.10).
Indeed, on the one hand, it follows from (5.11) that

$$\forall\,\eta>0\;\exists\,\delta>0:\; A\in\mathfrak{B}(V), \pi(A)<\delta \implies \sup_n \sup_{t\in[0,T]} \rho_t^{\varepsilon_n}(A) < \eta\,. \tag{5.17}$$

setwise-com

This shows that the family $(\rho_t^{\varepsilon_n})_{t\in[0,T],\,n\in\mathbb{N}}$ fulfills the analogue of property (2.7) with respect to the measure $\gamma = \pi$, which characterizes (sequential) compactness w.r.t. setwise convergence, and thus the setwise topology, in $\mathcal{M}(V)$. Therefore, the 'compactness' condition (5.2a) in Theorem 5.1 holds with K a setwise compact subset of $\mathcal{M}(V)$.

On the other hand, estimate (3.26) (which holds, all the more, along the solutions $(\rho^{\varepsilon_n}, \boldsymbol{j}^{\varepsilon_n}) \in \mathcal{CE}^{\mathrm{enh}}([0,T])$ of the continuity equation), guarantees that

$$\|\rho_{t_2}^{\varepsilon_n} - \rho_{t_1}^{\varepsilon_n}\|_{\mathrm{BL}} \le \int_{t_1}^{t_2} |\boldsymbol{g}_t^{\varepsilon_n}|(E)\,\mathrm{d}t \qquad \text{for } \mathscr{L}\text{-a.a. } t_1 \le t_2 \in [0,T]\,,$$

cf. (3.24) for the definition of $\|\cdot\|_{\mathrm{BL}}$. Combining it with estimates (5.13)–(5.15) we conclude that for all $\beta>0$

$$\begin{aligned}\|\rho_{t_2}^{\varepsilon_n}-\rho_{t_1}^{\varepsilon_n}\|_{\mathrm{BL}} &\leq \frac{1}{\beta}\Big[\int_{t_1}^{t_2}\mathscr{R}_{\varepsilon_n}(\rho_t^{\varepsilon_n},\boldsymbol{j}_t^{\varepsilon_n})\mathrm{d}t+\mathfrak{f}(\beta)\int_{t_1}^{t_2}\frac{C_{\overline{U}}}{2}c_\kappa\pi(V)\mathrm{d}t\Big]\\ &\leq \frac{1}{\beta}\big(\mathsf{E}_0+C_{\overline{U}}c_\kappa\pi(V)\mathfrak{f}(\beta)|t_2-t_1|\big)\,.\end{aligned} \tag{5.18}$$

Ascoli

By (5.18), the equicontinuity condition (5.2b) holds with the function $\omega:[0,T]\times[0,T]\to[0,+\infty)$ given by

$$\omega(r,s):=\inf_{\beta>0}\frac{1}{\beta}\big(\mathsf{E}_0+C_{\overline{U}}c_\kappa\pi(V)\mathfrak{f}(\beta)|r-s|\big)\,,$$

which satisfies $\lim_{(r,s)\to(t,t}\omega(r,s)=0$ for all $t\in[0,T]$.

All in all, we are in a position to apply Theorem 5.1 to the sequence $(\rho_n)_n$, in the space $\mathscr{S}:=\mathcal{M}(V)$ endowed with the setwise topology on $\mathcal{M}(V)$ and with the metric d induced by $\|\cdot\|_{\mathrm{BL}}$, for which the compatibility condition (5.1) clearly holds. Hence, by Theorem 5.1 we conclude convergence (4.11) along a suitable subsequence. A straightforward argument yields $\rho_t(\mathrm{d}x)=u_t\,\pi(\mathrm{d}x)$ for $\mathscr{L}$-a.a. $t\in(0,T)$. We thus conclude **Claim** 2.

Claim 3: *define $\boldsymbol{j}_{\mathscr{L}}\in\mathcal{M}_{\mathrm{loc}}([0,T]\times E')$ by*

$$\boldsymbol{j}_{\mathscr{L}}(\mathrm{d}t\mathrm{d}x\mathrm{d}y)=\frac{1}{1\wedge d(x,y)}\boldsymbol{g}_{\mathscr{L}}(\mathrm{d}t\mathrm{d}x\mathrm{d}y)$$

Then, convergence (4.12) *and the disintegration property* (5.9) *(with $\boldsymbol{j}_t(\mathrm{d}x\mathrm{d}y)=\frac{1}{1\wedge d(x,y)}\boldsymbol{g}_t(\mathrm{d}x\mathrm{d}y)\in\mathcal{M}_{\mathrm{loc}}(E')$) hold.*

It suffices to apply Lemma 5.2 to with the choices $\boldsymbol{\eta}_n:=\boldsymbol{g}_{\mathscr{L}}^{\varepsilon_n}$ and $\boldsymbol{\zeta}_n:=\boldsymbol{j}_{\mathscr{L}}^{\varepsilon_n}$. This finishes the proof of Prop. 5.3. □

As an immediate consequence of convergences (4.11) and (4.12) we have the following

Corollary 5.4. *The pair $(\rho,\boldsymbol{j})$ satisfies the continuity equation on the interval $[0,T]$ in the sense of Def. 3.8.*

For later use in the proof of Lemma 5.6, we also record the following convergences.

Corollary 5.5. *Consider the measures $(\boldsymbol{\sigma}_t^{\varepsilon_n})_{t\in[0,T]}\subset\mathcal{M}^+(E)$ defined by*

$$\boldsymbol{\sigma}_t^{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)=\frac{1\wedge d^2(x,y)}{\varepsilon_n+(1\wedge d^2(x,y))}\alpha(u_t^{\varepsilon_n}(x),u_t^{\varepsilon_n}(y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,,$$

and let $\boldsymbol{\sigma}_{\mathscr{L}}^{\varepsilon_n}\subset\mathcal{M}([0,T]\times E)$ be given by $\boldsymbol{\sigma}_{\mathscr{L}}^{\varepsilon_n}(\mathrm{d}t\mathrm{d}x\mathrm{d}y)=\boldsymbol{\sigma}_t^{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\,\mathscr{L}(\mathrm{d}t)$. Then, there exists $\boldsymbol{\sigma}_{\mathscr{L}}\in\mathcal{M}_{\mathrm{loc}}^+([0,T]\times E')$, with $\boldsymbol{\sigma}_{\mathscr{L}}(\mathrm{d}t\mathrm{d}x\mathrm{d}y)=\boldsymbol{\sigma}_t(\mathrm{d}x\mathrm{d}y)\mathscr{L}(\mathrm{d}t)$ for some $\boldsymbol{\sigma}=(\boldsymbol{\sigma}_t)_{t\in[0,T]}\subset\mathcal{M}_{\mathrm{loc}}^+(E')$, such that

$$\boldsymbol{\sigma}_t(A)\leq\boldsymbol{\nu}_{\rho_t}(A)=\alpha[\boldsymbol{\vartheta}_{\rho_t}^-,\boldsymbol{\vartheta}_{\rho_t}^+|\boldsymbol{\vartheta}_\kappa](A)\qquad\text{for all } A\in\mathfrak{B}_{\mathrm{c}}(E'),\ \text{for } \mathscr{L}\text{-a.a. } t\in(0,T), \tag{5.19}$$

key-absolut

and

$$\boldsymbol{\sigma}_{\mathscr{L}}^{\varepsilon_n}\to\boldsymbol{\sigma}_{\mathscr{L}}\qquad\qquad \text{vaguely in } \mathcal{M}_{\mathrm{loc}}([0,T]\times E')\,. \tag{5.20}$$

vague-signa

Proof. Let $(\boldsymbol{\nu}_t^{\varepsilon_n})_{t\in[0,T]}\in\mathcal{M}^+([0,T]\times E)$ be given by

$$\boldsymbol{\nu}_t^{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)=(1\wedge d^2(x,y))\boldsymbol{\sigma}_t^{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\quad\text{and set}\qquad \boldsymbol{\nu}_{\mathscr{L}}^{\varepsilon_n}=\boldsymbol{\nu}_t^{\varepsilon_n}\,\mathscr{L}\,.$$

By repeating the very same calculations as in (5.15), we find for any $A \in \mathfrak{B}(E)$,

$$
\begin{aligned}
\boldsymbol{\nu}_t^{\varepsilon_n}(A) &= \iint_A (1 \wedge d^2(x,y))\, \boldsymbol{\sigma}_t^{\varepsilon_n}(\mathrm{d}x\mathrm{d}y) \\
&\leq \iint_A (1 \wedge d^2(x,y)) \alpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\, \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \leq C_{\overline{U}} \iint_A (1 \wedge d^2(x,y)) \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,.
\end{aligned}
\tag{5.21}
$$

crucial-for

Then, a further integration of (5.21) w.r.t. the Lebesgue measure $\mathscr{L}$ reveals that the sequence $(\boldsymbol{\nu}^{\varepsilon_n})_n$ enjoys the analogue of estimate (5.16). Thus there exists $\boldsymbol{\nu} \in \mathcal{M}^+([0,T] \times E)$ such that $\boldsymbol{\nu}_{\mathscr{L}}^{\varepsilon_n} \to \boldsymbol{\nu}$ setwise in $\mathcal{M}([0,T] \times E)$ along a suitable subsequence. From (5.21) we also infer for every $I \subset [0,T]$

$$
((\pi_0)_\# \boldsymbol{\nu})(I) \leq \liminf_{n \to \infty} \int_I \boldsymbol{\nu}_t^{\varepsilon_n}(E')\mathrm{d}t \leq M_2\, \mathscr{L}(I) \iint_{E'} (1 \wedge d^2(x,y)) \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,,
$$

which shows that the first marginal of $\boldsymbol{\nu}$ is absolutely continuous w.r.t. $\mathscr{L}$. Then, again by [Bog07, Cor. 10.4.15] $\boldsymbol{\nu}$ can be disintegrated w.r.t. $\mathscr{L}$ in terms of a family $(\boldsymbol{\nu}_t)_{t \in [0,T]} \subset \mathcal{M}^+(E)$.

Clearly, applying Lemma 5.2 with $\boldsymbol{\eta}_n = \boldsymbol{\nu}_{\mathscr{L}}^{\varepsilon_n}$ and $\boldsymbol{\zeta}_n = \boldsymbol{\sigma}_{\mathscr{L}}^{\varepsilon_n}$ we infer convergence (5.20). It remains to prove (5.19). For this, we will indeed first show that

$$
\int_a^b \boldsymbol{\nu}_t(A)\, \mathrm{d}t \leq \int_a^b \iint_A (1 \wedge d^2(x,y))\, \mathbf{v}_{\rho_t}(\mathrm{d}x\mathrm{d}y)\mathrm{d}t \qquad \text{for all } A \in \mathfrak{B}(E) \text{ and } (a,b) \subset (0,T)\,. \tag{5.22}
$$

intermediat

Indeed, we have

$$
\begin{aligned}
\int_a^b \boldsymbol{\nu}_t(A)\, \mathrm{d}t \overset{(1)}{=} \lim_{n \to \infty} \int_a^b \boldsymbol{\nu}_t^{\varepsilon_n}(A)\, \mathrm{d}t &\overset{(2)}{\leq} \limsup_{n \to \infty} \iiint_{(a,b) \times A} \alpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\, \boldsymbol{\vartheta}_{\mathscr{L}}^{d}(\mathrm{d}t\mathrm{d}x\mathrm{d}y) \\
&\overset{(3)}{\leq} \iiint_{(a,b) \times A} \alpha(u_t(x), u_t(y))\, \boldsymbol{\vartheta}_{\mathscr{L}}^{d}(\mathrm{d}t\mathrm{d}x\mathrm{d}y)
\end{aligned}
$$

where (1) follows from the setwise convergence $\boldsymbol{\nu}_{\mathscr{L}}^{\varepsilon_n} \to \boldsymbol{\nu}$, (2) is due to (5.21) (recall that $\boldsymbol{\vartheta}_{\mathscr{L}}^{d}$ is a place-holder for $(1 \wedge d^2)(\boldsymbol{\vartheta}_\kappa)_{\mathscr{L}}$). Finally, (3) ensues from combining the fact that, in fact,

$$
u^{\varepsilon_n} \overset{*}{\rightharpoonup} u \text{ in } \mathrm{L}^\infty((0,T) \times V; \boldsymbol{\vartheta}_{\mathscr{L}}^{d})
$$

(cf. (5.10)) due to (4.7), with a variant of the Ioffe theorem, cf. [Val90, Thm. 21, p. 171]. Hence, (5.22) follows, implying that

$$
\begin{aligned}
\int_a^b \iint_E \phi(x,y)\, \boldsymbol{\nu}_t(\mathrm{d}x\mathrm{d}y)\mathrm{d}t &\leq \int_a^b \iint_E (1 \wedge d^2(x,y)) \phi(x,y)\, \mathbf{v}_{\rho_t}(\mathrm{d}x\mathrm{d}y)\mathrm{d}t \\
&\text{for all } \phi \in \mathrm{B_b}(E) \quad \text{with } \phi \geq 0 \text{ and all } (a,b) \subset (0,T).
\end{aligned}
\tag{5.23}
$$

localized-t

Let us now fix an arbitrary test function $\varphi \in \mathrm{C_c}(E')$ with $\varphi \geq 0$: we have that

$$
\begin{aligned}
\int_a^b \iint_E \varphi(x,y)\, \boldsymbol{\sigma}_t(\mathrm{d}x\mathrm{d}y)\, \mathrm{d}t &= \int_a^b \iint_E \frac{\varphi(x,y)}{1 \wedge d^2(x,y)}\, \boldsymbol{\nu}_t(\mathrm{d}x\mathrm{d}y)\, \mathrm{d}t \\
&\overset{(1)}{\leq} \int_a^b \iint_E \frac{\varphi(x,y)}{1 \wedge d^2(x,y)} (1 \wedge d^2(x,y))\, \mathbf{v}_{\rho_t}(\mathrm{d}x\mathrm{d}y)\mathrm{d}t \\
&= \int_a^b \iint_E \varphi(x,y)\, \mathbf{v}_{\rho_t}(\mathrm{d}x\mathrm{d}y)\mathrm{d}t\,,
\end{aligned}
$$

where (1) follows from (5.23), since $\frac{\varphi(x,y)}{1 \wedge d^2(x,y)}$ extends to a function in $\mathrm{B_b}(E)$. Therefore, (5.19) follows.

□

ss:5.2

5.2. **Upper energy-dissipation inequality.** Ultimately, we are in a position to prove the following

l:UEDE

Lemma 5.6. *The pair $(\rho, \boldsymbol{j})$ satisfies the upper energy-dissipation estimate* (4.15).

Again, we emphasize that so far we have relied on the sole conditions (4.10) where, in particular, (4.10b) may hold with $\underline{U} = 0$.

Proof. We take the limit as $n \to \infty$ in (5.4). By (4.11) we have

$$\mathscr{E}(\rho_t) \le \liminf_{n\to\infty} \mathscr{E}(\rho_t^{\varepsilon_n}) \qquad \text{for all } t \in [0,T]. \tag{5.24}$$

lsc-energie

Next, we rewrite $\int_0^t \mathscr{R}^{\varepsilon_n}(\rho^{\varepsilon_n}, \boldsymbol{j}^{\varepsilon_n})\mathrm{d}r$ in terms of the measure $(\boldsymbol{\sigma}_t^{\varepsilon_n})_t$. In what follows, we use the place-holder $m^{\varepsilon_n} = \frac{1\wedge d^2}{\varepsilon_n + (1\wedge d^2)}$: we find for $\mathscr{L}$-a.a. $t \in (0,T)$ and for all $(x,y) \in E_t^{\varepsilon_n}$

$$\frac{w_t^{\varepsilon_n}(x,y)}{\alpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))} = \frac{w_t^{\varepsilon_n}(x,y)\, m^{\varepsilon_n}(x,y)}{\alpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\, m^{\varepsilon_n}(x,y)} = \frac{\mathrm{d}(2\boldsymbol{j}_t^{\varepsilon_n})}{\mathrm{d}\boldsymbol{\sigma}_t^{\varepsilon_n}}(x,y)\,.$$

Therefore

$$\begin{aligned}
\mathscr{R}(\rho_t^{\varepsilon_n}, \boldsymbol{j}_t^{\varepsilon_n}) &= \frac{1}{2}\iint_{E_t^{\varepsilon_n}} \psi\Big(\frac{w_t^{\varepsilon_n}(x,y)}{\alpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))}\Big)\alpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\, \boldsymbol{\vartheta}_\kappa \mathrm{d}x, \mathrm{d}y)\\
&= \frac{1}{2}\iint_{E_t^{\varepsilon_n}} \psi\Big(2\frac{\mathrm{d}\boldsymbol{j}_t^{\varepsilon_n}}{\mathrm{d}\boldsymbol{\sigma}_t^{\varepsilon_n}}(x,y)\Big)\frac{1}{m^{\varepsilon_n}(x,y)}\alpha(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\, m^{\varepsilon_n}(x,y)\, \boldsymbol{\vartheta}_\kappa(\mathrm{d}x, \mathrm{d}y)\\
&\ge \frac{1}{2}\iint_{E_t^{\varepsilon_n}} \psi\Big(2\frac{\mathrm{d}\boldsymbol{j}_t^{\varepsilon_n}}{\mathrm{d}\boldsymbol{\sigma}_t^{\varepsilon_n}}(x,y)\Big)\, \boldsymbol{\sigma}_t^{\varepsilon_n}(\mathrm{d}x, \mathrm{d}y)\,.
\end{aligned}$$

where we have used that $1/m^{\varepsilon_n} \ge 1$. All in all, we have

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^t \mathscr{R}(\rho_r^{\varepsilon_n}, \boldsymbol{j}_r^{\varepsilon_n})\mathrm{d}r &= \frac{1}{2}\liminf_{n\to\infty} \iiint_{[0,t]\times E'} \psi\left(2\frac{\mathrm{d}\boldsymbol{j}_{\mathscr{L}}^{\varepsilon_n}}{\mathrm{d}\boldsymbol{\sigma}_{\mathscr{L}}^{\varepsilon_n}}\right)\boldsymbol{\sigma}_{\mathscr{L}}^{\varepsilon_n}(\mathrm{d}r\mathrm{d}x\mathrm{d}y)\\
&= \liminf_{n\to\infty} \mathscr{F}_\psi\left(\boldsymbol{j}_{\mathscr{L}}^{\varepsilon_n} | \boldsymbol{\sigma}_{\mathscr{L}}^{\varepsilon_n}\right)\\
&\ge \frac{1}{2}\mathscr{F}_\psi\left(2\boldsymbol{j}_{\mathscr{L}} | \boldsymbol{\sigma}_{\mathscr{L}}\right) = \frac{1}{2}\iiint_{[0,t]\times E'} \psi\left(2\frac{\mathrm{d}\boldsymbol{j}_{\mathscr{L}}}{\mathrm{d}\boldsymbol{\sigma}_{\mathscr{L}}}\right)\boldsymbol{\sigma}_{\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y)\\
&= \frac{1}{2}\int_0^t \iint_{E'} \psi\left(2\frac{\mathrm{d}\boldsymbol{j}_r}{\mathrm{d}\boldsymbol{\sigma}_r}(x,y)\right)\boldsymbol{\sigma}_r(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r
\end{aligned} \tag{5.25}$$

lsc-R

since $\boldsymbol{j}_{\mathscr{L}}^{\varepsilon_n} \to \boldsymbol{j}_{\mathscr{L}}$, $\boldsymbol{\sigma}_{\mathscr{L}}^{\varepsilon_n} \to \boldsymbol{\sigma}_{\mathscr{L}}$ vaguely in $\mathcal{M}_{\mathrm{loc}}([0,T]\times E')$, and $\mathcal{F}_\psi$ is lower semicontinuous w.r.t. vague convergence by Lemma 2.1. Now, since $\boldsymbol{j}_t \le \boldsymbol{\nu}_{\rho_t}$ for $\mathscr{L}$-a.a. $t \in (0,T)$ by (5.19), recalling the monotonicity property (2.10) from Lemma 2.1 we have that

$$\begin{aligned}
\frac{1}{2}\int_0^t \iint_{E'} \psi\left(2\frac{\mathrm{d}\boldsymbol{j}_r}{\mathrm{d}\boldsymbol{\sigma}_r}(x,y)\right)\boldsymbol{\sigma}_r(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r &\ge \frac{1}{2}\int_0^t \iint_{E'} \psi\left(2\frac{\mathrm{d}\boldsymbol{j}_r}{\mathrm{d}\boldsymbol{\nu}_{\rho_r}}(x,y)\right)\boldsymbol{\nu}_{\rho_r}(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r\\
&= \int_0^t \mathscr{R}(\rho_r, \boldsymbol{j}_r)\mathrm{d}r\,.
\end{aligned}$$

Finally, in order to take the limit as $n \to \infty$ in the Fisher information term, we use that, by (5.7),

$$\mathscr{D}(\rho_t^{\varepsilon_n}) = \frac{1}{2}\iint_{E'} \mathrm{D}_\phi(u_t^{\varepsilon_n}(x), u_t^{\varepsilon_n}(y))\, \boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)$$

Therefore, introducing the measures $\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho^{\varepsilon_n},\pm}(\mathrm{d}t\mathrm{d}x\mathrm{d}y) = \boldsymbol{\vartheta}^{\pm}_{\rho^{\varepsilon_n},t}(\mathrm{d}x\mathrm{d}y)\,\mathscr{L}(\mathrm{d}t)$ and $\boldsymbol{\vartheta}_{\varepsilon_n,\mathscr{L}}(\mathrm{d}t\mathrm{d}x\mathrm{d}y) = \boldsymbol{\vartheta}_{\varepsilon_n}(\mathrm{d}x\mathrm{d}y)\,\mathscr{L}(\mathrm{d}t)$, we rewrite

$$\int_0^t \mathscr{D}(\rho_r^{\varepsilon_n})\mathrm{d}r = \frac{1}{2}\iiint_{[0,t]\times E'} \mathrm{D}_\phi\left(\frac{\mathrm{d}\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho^{\varepsilon_n},-}}{\mathrm{d}\boldsymbol{\vartheta}_{\varepsilon_n,\mathscr{L}}}, \frac{\mathrm{d}\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho^{\varepsilon_n},+}}{\mathrm{d}\boldsymbol{\vartheta}_{\varepsilon_n,\mathscr{L}}}\right)\boldsymbol{\vartheta}_{\varepsilon_n,\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y) \doteq \mathscr{F}_\Xi(\boldsymbol{\beta}_{\mathscr{L}}^{\varepsilon_n}|\boldsymbol{\vartheta}_{\varepsilon_n,\mathscr{L}})$$

where the convex functional $\Xi : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,+\infty]$ is defined by $\Xi(w,z) = \frac{1}{2}\mathrm{D}_\phi(w,z)$ and we have used the place-holder $\boldsymbol{\beta}_{\mathscr{L}}^{\varepsilon_n} = (\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho^{\varepsilon_n},-}, \boldsymbol{\vartheta}_{\mathscr{L}}^{\rho^{\varepsilon_n},+})$. Now, it can be immediately checked that the setwise convergence of $\rho_t^{\varepsilon_n}$ to ρ_t for all $t\in[0,T]$ gives that

$$(1\wedge d^2)\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho^{\varepsilon_n},\pm} \to (1\wedge d^2)\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho,\pm} \qquad \text{setwise in } \mathcal{M}([0,t]\times E')$$

with

$$\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho,\pm}(\mathrm{d}t\mathrm{d}x\mathrm{d}y) = \boldsymbol{\vartheta}^{\pm}_{\rho_t}(\mathrm{d}x\mathrm{d}y)\,\mathscr{L}(\mathrm{d}t)\,.$$

Thus, by Lemma 5.2 we have

$$\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho^{\varepsilon_n},\pm} \to \boldsymbol{\vartheta}_{\mathscr{L}}^{\rho,\pm} \qquad \text{vaguely in } \mathcal{M}_{\mathrm{loc}}([0,t]\times E')\,.$$

Since we also have $\boldsymbol{\vartheta}_{\varepsilon_n,\mathscr{L}} \to \boldsymbol{\vartheta}_{\mathscr{L}}$ vaguely in $\mathcal{M}_{\mathrm{loc}}([0,t]\times E')$, again by Lemma 2.1 we conclude that (with the place-holder $\boldsymbol{\beta}_{\mathscr{L}} = (\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho,-}, \boldsymbol{\vartheta}_{\mathscr{L}}^{\rho,+})$)

$$\begin{aligned}\liminf_{n\to\infty}\int_0^t \mathscr{D}(\rho_r^{\varepsilon_n})\mathrm{d}r &\geq \mathscr{F}_\Xi(\boldsymbol{\beta}_{\mathscr{L}}|\boldsymbol{\vartheta}_{\mathscr{L}}) \\ &= \frac{1}{2}\iiint_{[0,t]\times E'} \mathrm{D}_\phi\left(\frac{\mathrm{d}\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho,-}}{\mathrm{d}\boldsymbol{\vartheta}_{\mathscr{L}}}, \frac{\mathrm{d}\boldsymbol{\vartheta}_{\mathscr{L}}^{\rho,+}}{\mathrm{d}\boldsymbol{\vartheta}_{\mathscr{L}}}\right)\boldsymbol{\vartheta}_{\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y) = \int_0^t \mathscr{D}(\rho_r)\mathrm{d}r\,.\end{aligned}$$

Taking into account that $\mathscr{E}(\rho_n^0)\to\mathscr{E}(\rho_0)$, the upper energy-dissipation inequality (4.15) follows. □

s:6

6. Chain rule and lower energy-dissipation inequality

In this section

(1) in Proposition 6.1 below we will show that the pair (ρ, j) is in fact a solution of the continuity equation in an *enhanced* sense, in comparison to that of Def. 3.8.

Precisely in the proof of Prop. 6.1, we will resort to the density property assumed in Assumption 4.2. We highlight that we will succeed in proving Proposition 6.1 only under the additional condition that $u = \frac{\mathrm{d}\rho}{\mathrm{d}\pi}$ is bounded from above and away from zero, cf. (6.2) below. Thus, Prop. 6.1 will apply to the pair $(\rho, \boldsymbol{j}) \in \mathcal{CE}([0,T])$ obtained by the approximation procedure set forth in Section 4, only under the condition that the constant $\underline{U}$ from (4.10b) is *strictly positive*.

(2) Exploiting this improved continuity equation, in Proposition 6.5 ahead we will show that, along solutions of the continuity equation satisfying (6.2), the chain rule identity holds.
(3) In this way, we will conclude the proof of part **(2)** of Theorem 4.3.
(4) Eventually, we will prove part **(1)** via an approximation argument.

s:6.1

6.1. **An enhanced continuity equation.** In order to extend the class of test functions for the continuity equation, we will crucially rely on the information that the limit pair $(\rho,\boldsymbol{j})\in\mathcal{CE}([0,T])$ constructed in Sec. 5 have finite entropy and finite action:

$$\sup_{t\in[0,T]}\mathscr{E}(\rho_t) < \infty, \qquad \int_0^T \mathscr{R}(\rho_t,\boldsymbol{j}_t)\,\mathrm{d}t < \infty. \tag{6.1}$$ finite-entr

In view of [PRST22, Rmk. 4.12], we may assume without loss of generality that for $\mathscr{L}$-a.a. $t\in(0,T)$ the measure $\boldsymbol{j}_t$ is skew-symmetric, i.e. $s_\#\boldsymbol{j}_t = -\boldsymbol{j}_t$ with $s: E\to E$ the symmetry map $s(x,y) = (y,x)$.

We no longer need the gray part above, right?

prop:ENH-CE **Proposition 6.1.** *Let $(\rho, \boldsymbol{j}) \in \mathcal{CE}([0,T])$ fulfill (6.1) and suppose that $\rho_t = u_t \pi$ with*

$$\exists\, 0 < \underline{U} \le \overline{U} \ : \quad \underline{U} \le u_t(x) \le \overline{U} \ \textit{for } \pi\textit{-a.a. } x \in V \ \textit{ and for all } t \in [0,T]. \tag{6.2}$$ strong-boun

Then

$$\int_V \varphi(x)\, \rho_t(\mathrm{d}x) - \int_V \varphi(x)\, \rho_s(\mathrm{d}x) = \int_s^t \iint_{E'} \overline{\nabla}\varphi(x,y)\, \boldsymbol{j}_r(\mathrm{d}x\mathrm{d}y)\, \mathrm{d}r \qquad \textit{for any } [s,t] \subset [0,T]. \tag{6.3}$$ desired-CE

for all test functions $\varphi \in \mathrm{B_b}(V)$ fulfilling

$$\int_0^T \iint_{E'} \psi^*(\overline{\nabla}\varphi(x,y))\, \alpha(u_t(x), u_t(y))\, \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\, \mathrm{d}t < \infty. \tag{6.4}$$ eq:test_ext

Prior to carrying out the proof of Prop. 6.1 we need to derive a key estimate from the asymptotic condition $\psi^*(\xi) \simeq |\xi|^2$ as $|\xi| \to 0$.

Lemma 6.2. *Let ψ^* comply with Assumption 3.5. Then,*

$$\forall\, \beta > 0\ \forall\, M > 0\ \exists\, C_{\beta,M} > 0 : \quad \psi^*(\beta\xi) \le C_{\beta,M}\psi^*(\xi) \qquad \forall\, \xi \in [-M, M]. \tag{6.5}$$ Olli-est

Proof. Clearly, if $|\beta| \le 1$ then $\psi^*(\beta\xi) \le \psi^*(\xi)$ since ψ^* is non-decreasing. Now let $|\beta| \ge 1$ and let $r > 0$ be as in (3.14). We estimate $\psi^*(\beta\xi)$ distinguishing two cases:

(1) $|\beta\xi| \le r$: then $|\xi| \le \frac{r}{|\beta|} < r$, since $|\beta| > 1$. By (3.14) we have

$$\psi^*(\beta\xi) \le \frac{3}{2} c_0 |\beta\xi|^2 \le \frac{3}{2} c_0 |\beta|^2 \frac{2}{c_0} \psi^*(\xi) = 3|\beta|^2 \psi^*(\xi)\,. \tag{6.6}$$ easy-case1

(2) $|\beta\xi| > r$: then, $|\xi| > \frac{r}{\beta}$, so that, by monotonicity of ψ^*, we find $\psi^*(\frac{r}{\beta}) \le \psi^*(\xi)$, and thus

$$\psi^*(\beta\xi) \le \frac{\psi^*(\xi)}{\psi^*(\frac{r}{\beta})} \psi^*(\beta\xi) \le \frac{\psi^*(\xi)}{\psi^*(\frac{r}{\beta})} \psi^*(\beta M)\,, \tag{6.7}$$ easy-case2

where for the last estimate we have used that $|\xi| \le M$.

Combining (6.6) and (6.7) we conclude estimate (6.5) with $C_{\beta,M} = \max\left\{ 3|\beta|^2, \frac{\psi^*(\beta M)}{\psi^*(\frac{r}{\beta})} \right\}$. □

As an immediate consequence of this result, we have that the spaces introduced in (4.8) and (4.9) do coincide.

ical-spaces **Corollary 6.3.** *Let ψ^* comply with Assumption 3.5. Then, $\mathcal{X}^{\psi^*} = \mathrm{X}^{\psi^*}$.*

Proof. Clearly, it suffices to prove that $\mathrm{X}^{\psi^*} \subset \mathcal{X}^{\psi^*}$. For this, let us fix $\varphi \in \mathrm{X}^{\psi^*}$. Since $\varphi \in \mathrm{B_b}(V)$, we have that $\overline{\nabla}\varphi \in \mathrm{B_b}(E)$ and thus we can apply estimate (6.5) with $M = \sup_{(x,y)\in E} |\overline{\nabla}\varphi(x,y)|$, which then ensures that

$$\forall\, \beta > 0 : \quad \iint_{E'} \psi^*(\beta \overline{\nabla}\varphi(x,y)) \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \le C_{\beta,M} \iint_{E'} \psi^*(\overline{\nabla}\varphi(x,y)) \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) < \infty\,.$$

Hence, $\overline{\nabla}\varphi \in \mathcal{M}^{\psi^*}(E'; \boldsymbol{\vartheta}_\kappa)$, and thus $\varphi \in \mathcal{X}^{\psi^*}$. □

We now verify that bounded Lipschitz functions belong to the space $\mathcal{X}^{\psi^*}$.

Lipb-r-nice **Lemma 6.4.** *Let ψ^* comply with Assumption 3.5. Then, $\mathrm{Lip_b}(V) \subset \mathcal{X}^{\psi^*}$.*

Proof. Let us fix an arbitrary $\varphi \in \mathrm{Lip_b}(V)$, with norm $\lambda := \|\varphi\|_{\mathrm{Lip_b}(V)}$. Recall (3.14). Then, by the monotonicity of ψ^* we have that

$$\begin{aligned}
&\iint_{E'} \psi^*(\overline{\nabla}\varphi(x,y))\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \\
&\le \iint_{E'} \psi^*(\lambda(1\wedge d(x,y)))\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \\
&= \iint_{A_r} \psi^*(\lambda(1\wedge d(x,y)))\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) + \iint_{E'\setminus A_r} \psi^*(\lambda(1\wedge d(x,y)))\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \doteq I_1 + I_2\,,
\end{aligned}$$

with $A_r = \{(x,y) \in E' \,:\, \lambda(1\wedge d(x,y)) \le r\}$. Now, by (3.14)

$$I_1 \le \frac{3}{2}c_0 \iint_{A_r} \lambda^2 (1\wedge d(x,y))^2\, \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) < +\infty\,,$$

while, again by the monotonicity of ψ^* and condition (3.3c), we have

$$I_2 \le \psi^*(\lambda)\, \boldsymbol{\vartheta}_\kappa(E'\setminus A_r) < +\infty\,.$$

All in all, we have shown that $\iint_{E'} \psi^*(\overline{\nabla}\varphi(x,y))\, \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) < +\infty$, so that $\varphi \in \mathrm{X}^{\psi^*}$. In view of Corollary 6.3, we conclude that $\varphi \in \mathfrak{X}^{\psi^*}$.

Shorten prove above!

□

Finally, prior to carrying out the proof of Prop. 6.1, we pin down a key fact: for any $(\xi_n)_n \subset \mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)$ we have that

$$\|\zeta_n\|_{\mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)} \to 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \iint_{E'} \psi^*(\beta\,\zeta_n(x,y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) = 0 \text{ for all } \beta > 0\,. \tag{6.8}$$ characteriz

Which reference for (6.8)? I have found it only in the (unpublished) lecture notes by C. Léonard...

Proof of Proposition 6.1. Preliminarily, we observe that, since u enjoys the bounds (6.2), showing (6.3) for all test functions $\varphi \in \mathrm{B_b}(V)$ fulfilling (6.4) is equivalent to showing it for all test functions $\varphi \in \mathrm{X}^{\psi^*}$. Thus, let us fix $\varphi \in \mathrm{X}^{\psi^*} = \mathfrak{X}^{\psi^*}$. By Assumption 4.2, there exists a sequence $(\varphi_n)_n \subset \mathrm{Lip_b}(V)$ suitably approximating φ. We now send $n \to \infty$ in the continuity equation tested by φ_n, i.e. in

$$\int_V \varphi_n(x)\,\rho_t(\mathrm{d}x) - \int_V \varphi_n(x)\,\rho_s(\mathrm{d}x) = \iiint_{[s,t]\times E'} \overline{\nabla}\varphi_n(x,y)\,\boldsymbol{j}_{\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y) \text{ for all } 0 \le s \le t \le T\,.$$

Since $\varphi_n \rightharpoonup \varphi$ in $\mathrm{L}^1(V;\pi)$, we pass to the limit on the left-hand side. As for the right-hand side, we will show that

$$\iiint_{[s,t]\times E'} \overline{\nabla}\varphi_n(x,y)\,\boldsymbol{j}_{\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y) \longrightarrow \iiint_{[s,t]\times E'} \overline{\nabla}\varphi(x,y)\,\boldsymbol{j}_{\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y)\,. \tag{6.9}$$ RHSto0

For this, we use that

$$
\begin{aligned}
&\iiint_{[s,t]\times E'} \left|\overline{\nabla}\varphi_n(x,y) - \overline{\nabla}\varphi(x,y)\right| \boldsymbol{j}_{\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y)\\
&= \frac{1}{\beta}\int_s^t \iint_{E^r_{\upalpha}} \beta\left|\overline{\nabla}\varphi_n(x,y) - \overline{\nabla}\varphi(x,y)\right| \frac{w_r(x,y)}{\upalpha(u_r(x),u_r(y))}\upalpha(u_r(x),u_r(y))\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r\\
&\overset{(1)}{\leq} \frac{1}{\beta}\int_s^t \iint_{E^r_{\upalpha}} \uppsi^*(\beta(\overline{\nabla}\varphi_n(x,y) - \overline{\nabla}\varphi(x,y)))\upalpha(u_r(x),u_r(y))\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r\\
&\quad + \frac{1}{\beta}\int_s^t \iint_{E^r_{\upalpha}} \uppsi\left(\frac{w_r(x,y)}{\upalpha(u_r(x),u_r(y))}\right)\upalpha(u_r(x),u_r(y))\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r
\end{aligned}
$$

with the set $E^r_{\upalpha} = \{(x,y)\in E' : \upalpha(u_r(x),u_r(y))>0\}$ fulfilling $|\boldsymbol{j}_r|(E'\backslash E^r_{\upalpha}) = 0$ for $\mathscr{L}$-a.a. $r\in(0,T)$, cf. (3.32), and (1) due to Young's inequality. Now, for every fixed $\beta>0$ we have

$$
\begin{aligned}
&\int_s^t \iint_{E^r_{\upalpha}} \uppsi^*(\beta(\overline{\nabla}\varphi_n(x,y) - \overline{\nabla}\varphi(x,y)))\upalpha(u_r(x),u_r(y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r\\
&\leq C_{\overline{U}}(t-s)\iint_{E'} \uppsi^*(\beta(\overline{\nabla}\varphi_n(x,y) - \overline{\nabla}\varphi(x,y)))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,\mathrm{d}r \doteq C_{\overline{U}}(t-s)I_n
\end{aligned}
$$

with $C_{\overline{U}}$ from (5.14). All in all, we conclude that for every $\beta>0$

$$
\begin{aligned}
&\limsup_{n\to\infty}\iiint_{[s,t]\times E'} \left|\overline{\nabla}\varphi_n(x,y) - \overline{\nabla}\varphi(x,y)\right| \boldsymbol{j}_{\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y)\\
&\leq C_{\overline{U}}(t-s)\lim_{n\to\infty} I_n + \frac{1}{\beta}\int_s^t \mathscr{R}(\rho_r,\boldsymbol{j}_r)\,\mathrm{d}r = \frac{1}{\beta}\int_s^t \mathscr{R}(\rho_r,\boldsymbol{j}_r)\,\mathrm{d}r\,,
\end{aligned}
$$

since $I_n\to 0$ by Assumption 4.2(2) and the characterization of Orlicz convergence provided by (6.8). By the arbitrariness of β, we conclude (6.9). We have thus shown that for all $0\leq s\leq t\leq T$

$$
\int_V \varphi(x)\,\rho_t(\mathrm{d}x) - \int_V \varphi(x)\,\rho_s(\mathrm{d}x) = \iiint_{[s,t]\times E'} \overline{\nabla}\varphi(x,y)\,\boldsymbol{j}_{\mathscr{L}}(\mathrm{d}r\mathrm{d}x\mathrm{d}y) \qquad \forall\,\varphi\in \mathrm{X}^{\uppsi^*},
$$

which finishes the proof. □

6.2. **A chain rule identity.** Exploiting Proposition 6.1, we show that along a solution of the continuity equation satisfying bounds (6.1) *and* (6.2), the chain rule holds.

prop:CR **Proposition 6.5** (Chain-rule)**.** *Let* $(\rho,\boldsymbol{j})\in\mathcal{CE}([0,T])$ *fulfill* (6.1) *and* (6.2)*. Then,*

$$
-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho_t) = \iint_{E'} (-\overline{\nabla}\phi'\circ u_t)(x,y)\,\boldsymbol{j}_t(dxdy) \leq \mathscr{R}(\rho_t,\boldsymbol{j}_t) + \mathscr{D}(\rho_t) \quad \textit{for a.e. } t\in(0,T)\,. \tag{6.10}
$$
chain-rule

In particular, the pair $(\rho,\boldsymbol{j})$ *satisfies the lower energy-dissipation inequality* (4.16)*.*

Proof. We begin by showing the absolute continuity of $t\mapsto\mathscr{E}(\rho_t)$. Preliminarily, we recall that, in view of (6.2),

$$
\upalpha(u_t(x),u_t(y)) \geq \underline{\upalpha} := \min_{(u,v)\in[\underline{U},\overline{U}]\times[\underline{U},\overline{U}]} \upalpha(u,v) > 0 \quad \text{for } \pi\text{-a.a. } x,y\in V \quad \text{for all } t\in[0,T]\,. \tag{6.11}
$$
needed-4-ri

Now, by convexity, we obtain for all $0 \le s \le t \le T$

$$
\begin{aligned}
&\mathscr{E}(\rho_t) - \mathscr{E}(\rho_s) \\
&\le \int_V \phi'(u_t(x))[u_t(x) - u_s(x)]\,\pi(\mathrm{d}x) \\
&\overset{(1)}{=} \int_s^t \iint_{E'} (\overline{\nabla}\phi' \circ u_t)(x,y)\,\boldsymbol{j}_r(\mathrm{d}x\mathrm{d}y)\mathrm{d}r \\
&\overset{(2)}{=} \int_s^t \iint_{E'} \frac{1}{\upalpha(u_r(x),u_r(y))}(\overline{\nabla}\phi' \circ u_t)(x,y)\upalpha(u_r(x),u_r(y))\,\boldsymbol{j}_r(\mathrm{d}x\mathrm{d}y)\mathrm{d}r \\
&\overset{(3)}{\le} \int_s^t \mathscr{R}(\rho_r,\boldsymbol{j}_r)\,dr + \int_s^t \iint_{E'} \uppsi^*(-\overline{\nabla}(\phi' \circ u_t)(x,y))\,\upalpha(u_r(x),u_r(y))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\mathrm{d}r \\
&\le \int_s^t \mathscr{R}(\rho_r,\boldsymbol{j}_r)\,dr + \int_s^t \iint_{E'} \uppsi^*((\overline{\nabla}\phi' \circ u_t)(x,y))\,\upalpha(u_t(x),u_t(y))\frac{\upalpha(u_r(x),u_r(y))}{\upalpha(u_t(x),u_t(y))}\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\mathrm{d}r \\
&\le \int_s^t \mathscr{R}(\rho_r,j_r)\,dr + \frac{C_{\overline{U}}}{\underline{\upalpha}}\mathscr{D}(\rho_t)|t-s|\,.
\end{aligned}
$$

Here, (1) follows from choosing the test function $\varphi = \phi' \circ u_t$ in the continuity equation, thanks to Proposition 6.1; (2) is justified by (6.11), while (3) ensues from Young's inequality and from the fact that $\uppsi^*$ is an even function. Analogously, we have

$$
\begin{aligned}
\mathscr{E}(\rho_t) - \mathscr{E}(\rho_s) &\ge \int_V \phi'(u_s(x))[u_t(x) - u_s(x)]\,\pi(\mathrm{d}x) \\
&= -\int_s^t \iint_{E'} -\overline{\nabla}(\phi' \circ u_s)(x,y)\,\boldsymbol{j}_r(\mathrm{d}x\mathrm{d}y)\mathrm{d}r \\
&\ge -\int_s^t \mathscr{R}(\rho_r,j_r)\,dr - \frac{C_{\overline{U}}}{\underline{\upalpha}}\mathscr{D}(\rho_s)|t-s|\,.
\end{aligned}
$$

All in all, we obtain

$$
|\mathscr{E}(\rho_t) - \mathscr{E}(\rho_s)| \le \int_s^t \mathscr{R}(\rho_r,j_r)\,dr + \frac{C_{\overline{U}}}{\underline{\upalpha}}\Big(\mathscr{D}(\rho_t) + \mathscr{D}(\rho_s)\Big)|t-s| \qquad \text{for all } 0 \le s \le t \le T\,.
$$

Applying [AGS08, Lemma 1.2.6], we infer that the mapping $[0,T] \ni t \mapsto \mathscr{E}(\rho_t)$ is in $W^{1,1}(0,T)$.

Let us now fix a point t, out of a negligible set, where $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho_t)$ exists. With the same calculations as above we obtain

$$
\begin{aligned}
\frac{1}{h}\big[\mathscr{E}(\rho_t) - \mathscr{E}(\rho_{t+h})\big] &\le \frac{1}{h}\int_V \phi'(u_t(x))[u_t(x) - u_{t+h}(x)]\,\pi(\mathrm{d}x) \\
&= -\int_t^{t+h} \iint_{E'} -\overline{\nabla}(\phi' \circ u_t)(x,y)\,\boldsymbol{j}_r(\mathrm{d}x\mathrm{d}y)\mathrm{d}r
\end{aligned}
$$

and, analogously,

$$
\frac{1}{h}\big[\mathscr{E}(\rho_t) - \mathscr{E}(\rho_{t+h})\big] \ge \int_t^{t+h} \iint_{E'} -\overline{\nabla}(\phi' \circ u_{t+h}))(x,y)\,\boldsymbol{j}_r(\mathrm{d}x\mathrm{d}y)\mathrm{d}r
$$

Dividing both inequalities by $h > 0$ and letting $h \downarrow 0$ we conclude that

$$
-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho_t) = \iint_{E'} -\overline{\nabla}(\phi' \circ u_t)(x,y)\,\boldsymbol{j}_t(dxdy)\,,
$$

whence (6.10). $\square$

6.3. **The lower energy-dissipation inequality and conclusion of the proof.** We now extend the validity of (4.16).

prop:LEDE **Proposition 6.6.** *Let $(\rho, \boldsymbol{j}) \in \mathcal{CE}([0,T])$ fulfill (6.1) and suppose that for $u_t = \frac{\mathrm{d}\rho_t}{\mathrm{d}\pi}$ the upper bound*

$$\exists\, \overline{U} > 0 : \quad 0 \le u_t(x) \le \overline{U} \quad \text{for } \pi\text{-a.a. } x \in V \quad \text{for all } t \in [,T]$$

holds. Then, the pair $(\rho, \boldsymbol{j})$ satisfies the lower energy-dissipation inequality

$$\int_0^t (\mathscr{R}(\rho_r, \boldsymbol{j}_r) + \mathscr{D}(\rho_r))\,\mathrm{d}r + \mathscr{E}(\rho_t) \ge \mathscr{E}(\rho_0) \qquad \text{for all } t \in [0,T]\,. \tag{6.12}$$ LEDE-t

Proof. Let us use $\mathcal{L}_{[0,t]}(\rho, \boldsymbol{j})$ as a place-holder for the left-hand side of (6.12) and consider the curve of measures $\rho^\theta : [0,T] \to \mathcal{M}^+(V)$

$$\rho_t^\theta := (1{-}\theta)\rho_t + \theta\pi \quad t \in [0,T],\ \theta \in [0,1]\,.$$

Clearly, $(\rho^\theta, (1{-}\theta)\boldsymbol{j}) \in \mathcal{CE}([0,T])$; furthermore, by convexity of $\mathscr{E}$ and $\mathscr{R}$,

$$\begin{cases} \sup_{t\in[0,T]} \mathscr{E}(\rho_t^\theta) \le (1{-}\theta) \sup_{t\in[0,T]} \mathscr{E}(\rho_t) < +\infty, \\ \displaystyle\int_0^T \mathscr{R}(\rho_t^\theta, (1{-}\theta)\boldsymbol{j}_t)\,\mathrm{d}t \le (1{-}\theta) \int_0^T \mathscr{R}(\rho_t, \boldsymbol{j}_t)\,\mathrm{d}t < +\infty. \end{cases}$$

What is more, $u_t^\theta = \frac{\mathrm{d}\rho_t^\theta}{\mathrm{d}\pi}$ satisfies (6.2). Therefore, Proposition 6.5 applies, yielding $\mathscr{E}(\rho_0^\theta) \le \mathcal{L}_{[0,t]}(\rho^\theta, (1{-}\theta)\boldsymbol{j})$. Thus, by convexity of $\mathcal{L}_{[0,t]}$ we infer

$$\mathscr{E}(\rho_0^\theta) \le \mathcal{L}_{[0,t]}(\rho^\theta, (1{-}\theta)\boldsymbol{j}) \le (1{-}\theta)\mathcal{L}_{[0,t]}(\rho, \boldsymbol{j}) + \theta\mathcal{L}_{[0,t]}(\pi, \mathbf{0}) = (1{-}\theta)\mathcal{L}_{[0,t]}(\rho, \boldsymbol{j}) + \theta\mathscr{E}(\pi)$$

since we can immediately check that $\mathcal{L}_{[0,t]}(\pi, \mathbf{0}) = \mathscr{E}(\pi)$. Then, it suffices to send $\theta \downarrow 0$ in the above inequality, and (6.12) ensues. □

We are now in a position to conclude the **proof of Theorem 4.3.**

- As for part (1), we apply Proposition 6.6 to the pair $(\rho, \boldsymbol{j}) \in \mathcal{CE}([0,T])$ obtained in Section 5. Since $(\rho, \boldsymbol{j})$ also satisfies the upper energy-dissipation estimate thanks to Lemma 5.6, we conclude that $(\rho, \boldsymbol{j})$ is a solution of the $(\mathscr{E}, \mathscr{R}, \mathscr{R}^*)$ system in the sense of Definition 3.17.
- As for part (2), it suffices to apply Proposition 6.5.

■

7. More on Assumption 4.2

s:lip

For the time being I have moved Lemma 7.1 here....

First of all, we improve the approximation property required in Assumption 4.2. Namely, Lemma 7.1 below we show that, if Assumption 4.2 holds, we can indeed construct a sequence $(\widehat{\varphi}_n)_n$ such that, in addition to the convergences in Ass. 4.2, we have that $\|\overline{\nabla}\widehat{\varphi}_n\|_\infty = \sup_{x,y\in V} |\overline{\nabla}\widehat{\varphi}_n(x,y)|$ is uniformly bounded.

l:cutoff **Lemma 7.1.** *Under Assumptions 3.5 and 4.2, for every $\varphi \in \mathcal{X}^{\psi^*}$ there exists a sequence $(\widehat{\varphi}_n)_n \subset \mathrm{Lip}_\mathrm{b}(V)$ such that as $n \to \infty$*

(1) $\sup_{n\in\mathbb{N}} \|\overline{\nabla}\widehat{\varphi}_n\|_\infty \le 4\|\varphi\|_\infty$;
(2) $\widehat{\varphi}_n \to \varphi$ *in* $\mathrm{L}^1(V;\pi)$;
(3) $\overline{\nabla}\widehat{\varphi}_n \to \overline{\nabla}\varphi$ *in* $\mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)$.

Proof. Obviously, we may suppose that $\varphi \not\equiv 0$. Let us introduce the truncation operator

$$\tau_\ell : \mathbb{R} \to \mathbb{R}, \quad \tau_\ell(x) := \begin{cases} x & \text{if } |x| \leq \ell, \\ \ell & \text{if } x > \ell, \\ -\ell & \text{if } x < -\ell, \end{cases} \qquad \text{with the place-holder } \ell := 2\|\varphi\|_\infty,$$

and let us define

$$\widehat{\varphi}_n(x) := \tau_\ell(\varphi_n(x)) \,. \tag{7.1}$$

truncated-p

Clearly, $\widehat{\varphi}_n \subset \mathrm{Lip_b}(V)$, with $\sup_{n\in\mathbb{N}} \|\overline{\nabla}\widehat{\varphi}_n\|_\infty \leq 2\ell$. We now check the remaining items of the statement in separate claims.

***Claim* 1:** *we have* $\widehat{\varphi}_n \to \varphi$ *in* $\mathrm{L}^1(V;\pi)$.
Since $\ell = 2\|\varphi\|_\infty$, we have that

$$A_n := \{x \in V \,:\, |\varphi_n(x)| > \ell\} \subset \left\{x \in V \,:\, |\varphi_n(x) - \varphi(x)| > \frac{\ell}{2}\right\} .$$

Therefore, since $\varphi_n \to \varphi$ in $\mathrm{L}^1(V;\pi)$, we infer that

$$\lim_{n\to\infty} \pi(A_n) \leq \lim_{n\to\infty} \pi\left(\left\{x \in V \,:\, |\varphi_n(x) - \varphi(x)| > \frac{\ell}{2}\right\}\right) = 0,$$

so that

$$\int_{A_n} |\widehat{\varphi_n} - \varphi(x)| \, \pi(\mathrm{d}x) \longrightarrow 0 \text{ as } n \to \infty \,.$$

In this way, we have that

$$\lim_{n\to\infty} \|\widehat{\varphi}_n - \varphi\|_{\mathrm{L}^1(V;\pi)} = \lim_{n\to\infty} \int_{V\setminus A_n} |\varphi_n(x) - \varphi(x)| \, \pi(\mathrm{d}x) = 0 \,.$$

***Claim* 2:** *we have*

$$\lim_{n\to\infty} \iint_{E'} \psi^*\left[\beta\left(\overline{\nabla}\widehat{\varphi}_n(x,y) - \overline{\nabla}\varphi(x,y)\right)\right] \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) = 0 \qquad \textit{for all } \beta > 0 \,. \tag{7.2}$$

cvg-small-0

First of all, we observe that, since $\widehat{\varphi}_n \to \varphi$ π-almost everywhere in V, by Lemma 3.3 we have

$$\overline{\nabla}\widehat{\varphi}_n \to \overline{\nabla}\varphi \qquad \boldsymbol{\vartheta}_\kappa\text{-a.e. in } E' \,. \tag{7.3}$$

pointwise-c

A direct calculation shows that $|\overline{\nabla}\widehat{\varphi}_n(x,y)| \leq |\overline{\nabla}\varphi_n(x,y)|$ for every $(x,y) \in E'$, therefore we have

$$f_n(x,y) := |\overline{\nabla}(\widehat{\varphi}_n - \varphi)(x,y)| \leq |\overline{\nabla}\widehat{\varphi}_n(x,y)| + |\overline{\nabla}\varphi(x,y)| \leq |\overline{\nabla}\varphi_n(x,y)| + |\overline{\nabla}\varphi(x,y)| \doteq g_n(x,y) \qquad \text{for all } (x,y) \in E' \,,$$

and, a fortiori, since ψ^* is non-decreasing on $[0,+\infty)$ we have

$$\psi^*(\beta f_n(x,y)) \leq \psi^*(\beta g_n(x,y)) \qquad \text{for all } (x,y) \in E' \,, \text{ for all } \beta > 0 \,. \tag{7.4}$$

domination

Now, observe that, while $\beta f_n \to 0$ $\boldsymbol{\vartheta}_\kappa$-a.e. in E' by (7.3), we have that

$$\beta g_n \to \beta g \quad \boldsymbol{\vartheta}_\kappa\text{-a.e. in } E' \qquad \text{with } g := 2|\overline{\nabla}\varphi| \,.$$

We will now prove that, indeed,

$$\lim_{n\to\infty} \iint_{E'} |\psi^*(\beta g_n(x,y)) - \psi^*(\beta g(x,y))| \, \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) = 0 \qquad \text{for all } \beta > 0 \,.. \tag{7.5}$$

next-aim-gn

Combining (7.4) and a modified version of the dominated convergence theorem (cf., e.g., [Bog07, §2, Thm. 2.8.8]), we will then conclude that $\lim_{n\to\infty} \iint_{E'} \psi^*(\beta f_n(x,y)) \, \boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) = 0$, namely the desired (7.2). Let us then check (7.5). First of all, we observe that

$$|g_n - g| = ||\overline{\nabla}\varphi_n| - |\overline{\nabla}\varphi|| \leq |\overline{\nabla}\varphi_n - \overline{\nabla}\varphi| \qquad \boldsymbol{\vartheta}_\kappa\text{-a.e. in } E' \,,$$

hence $\psi^*(|g_n-g|) \le \psi^*(|\overline{\nabla}\varphi_n - \overline{\nabla}\varphi|)$ $\boldsymbol{\vartheta}_\kappa$-a.e. in E'. Hence, from $\overline{\nabla}\varphi_n \to \overline{\nabla}\varphi$ in $\mathrm{L}^{\psi^*}(E';\boldsymbol{\vartheta}_\kappa)$, taking into account the characterization of Orlicz convergence provided by (6.8), we deduce that

$$\lim_{n\to\infty} \iint_{E'} \psi^*(\beta|g_n(x,y)-g(x,y)|)\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) = 0 \qquad \text{for all } \beta>0\,. \tag{7.6}$$

only-this

We now use the convexity inequality

$$\psi^*(a) \le \alpha\psi^*(\alpha^{-1}b) + (1-\alpha)\psi^*((1-\alpha)^{-1}(a-b)) \qquad \text{for all } a,b\in[0,+\infty),\ \alpha\in(0,1)\,. \tag{7.7}$$

useful-conv

Plugging in $a=\beta g_n$ and $b=\beta g$ we obtain

$$\begin{aligned}&\iint_{E'} \psi^*(\beta g_n)\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\\ &\le \alpha \iint_{E'} \psi^*(\alpha^{-1}\beta g)\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) + (1-\alpha)\iint_{E'} \psi^*((1-\alpha)^{-1}\beta(g_n-g))\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y)\,.\end{aligned}$$

so that, in view of (7.6), we infer

$$\limsup_{n\to\infty} \iint_{E'} \psi^*(\beta g_n)\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \le \alpha \iint_{E'} \psi^*(\alpha^{-1}\beta g)\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \quad \text{for all } \alpha\in(0,1)\,.$$

Plugging then $a=\beta g$ and $b=\beta g_n$ in (7.7) we get

$$\iint_{E'} \psi^*(\beta g)\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \le \alpha \liminf_{n\to\infty} \iint_{E'} \psi^*(\alpha^{-1}\beta g_n) \quad \text{for all } \alpha\in(0,1)\,,$$

so that, letting $\alpha\uparrow 1$ we conclude

$$\lim_{n\to\infty} \iint_{E'} \psi^*(\beta g_n)\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) = \iint_{E'} \psi^*(\beta g)\,\boldsymbol{\vartheta}_\kappa(\mathrm{d}x\mathrm{d}y) \quad \text{for all } \alpha\in(0,1)\,.$$

Similarly, (7.5) follows from the enhanced convexity inequality

$$|\psi^*(a)-\psi^*(b)| \le \alpha\psi^*(\alpha^{-1}b) - \psi^*(b) + \alpha\psi^*(\alpha^{-1}a) - \psi^*(a) + (1-\alpha)\psi^*((1-\alpha)^{-1}|a-b|) \tag{7.8}$$

enhanced-co

for all $a,b\in[0,+\infty)$, $\alpha\in(0,1)$: it suffices to choose $a=\beta g_n$ and $b=\beta g$ in (7.8), and then to send first $n\to\infty$ and the $\alpha\uparrow 1$.

□

8. Applications

maybe this section could be moved right BEFORE the sections with the proof...

s:a.1

Appendix A. Proof of Lemma 2.1

For statements **(1)** & **(2)** we refer to the proof of the analogous items in [PRST22, Lemma 2.3]. We only address here the proof of **(3)**, by adapting the argument carried out for $V=\mathbb{R}^d$ in the proof of [AFP05, Thm. 2.34]. Exploiting [AFP05, Prop. 2.31, Lemma 2.31] we sequences $(a_j)_j\subset\mathbb{R}^m$ and $(b_j)_j\subset\mathbb{R}$ such that, setting $L_j(x) := \langle a_j,x\rangle + b_j$ for all $x\in\mathbb{R}^m$, we have the representation formulae

$$\psi(x) = \sup_{j\in\mathbb{N}} L_j(x), \qquad \psi^\infty(x) = \sup_{j\in\mathbb{N}} \langle a_j,x\rangle \qquad \text{for all } x\in\mathbb{R}^m. \tag{A.1}$$

representat

Let us now consider sequences $(\mu_n)_n\,\mu\in\mathcal{M}_{\mathrm{loc}}(Y;\mathbb{R}^m)$ and $(\nu_n)_n$, $\nu\in\mathcal{M}^+_{\mathrm{loc}}(Y)$ such that $\mu_n\to\mu$ and $\nu_n\to\nu$ vaguely and $\liminf_{n\to\infty}\mathscr{F}_\psi(\mu_n|\nu_n)<\infty$; let $g_n := \frac{\mathrm{d}\mu_n}{\mathrm{d}\nu_n}$, $g=\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$, $h_n = \frac{\mathrm{d}\mu_n^\perp}{\mathrm{d}|\mu_n^\perp|}$ and $h := \frac{\mathrm{d}\mu^\perp}{\mathrm{d}|\mu^\perp|}$. Furthermore, let $(A_j)_{j=0}^k$, $k\in\mathbb{N}$, be pairwise disjoint open subsets of V with compact closure. For any family of functions $(\phi_j)_{j=0}^k$ such that $\phi_j \in \mathrm{C}^1_{\mathrm{c}}(A_j;\mathbb{R})$ and $0 \le \phi_j \le 1$ for all $j = \{0,\dots,k\}$ there holds

$$\int_{A_j} b_j\phi_j \,\mathrm{d}\nu_n + \left\langle a_j, \int_{A_j} \phi_j \,\mathrm{d}\nu_n \right\rangle \le \int_{A_j} \psi(g_n)\,\mathrm{d}\nu_n + \int_{A_j} \psi^\infty(h_n)\,\mathrm{d}|\mu_n^\perp|$$

so that, since $\mu_n \to \mu$ and $\nu_n \to \nu$ vaguely, we have

$$\sum_{j=0}^k \int_{A_j} b_j\phi_j \,\mathrm{d}\nu + \left\langle a_j, \int_{A_j} \phi_j \,\mathrm{d}\nu \right\rangle \le \liminf_{n\to\infty} \sum_{j=0}^k \int_{A_j} b_j\phi_j \,\mathrm{d}\nu_n + \left\langle a_j, \int_{A_j} \phi_j \,\mathrm{d}\nu_n \right\rangle \le \liminf_{n\to\infty} \mathscr{F}_\psi(\mu_n|\nu_n)\,.$$

We now introduce the functions $f_j : \mathbb{R} \to \mathbb{R}$ defined by

$$f_j(v) := \begin{cases} L_j(g(v)) & \text{if } v \in V \setminus N, \\ \langle a_j, h(v)\rangle & \text{if } v \in N, \end{cases} \quad \text{with } N \text{ a } \nu\text{-negligible set where } |\mu^\perp| \text{ is concentrated,}$$

and rewrite the above estimate as

$$\sum_{j=0}^k \int_{A_j} f_j\phi_j \,\mathrm{d}\eta \le \liminf_{n\to\infty} \mathscr{F}_\psi(\mu_n|\nu_n) \qquad \text{with } \eta := \nu + |\mu^\perp\,. \tag{A.2}$$ estimate-AF

Now, let us set $D_j := \{v \in A_j \ : \ f_j(v) > 0\}$ and let 1_{D_j} be its characteristic function. It follows from (A.2) that $\eta(D_j) < \infty$ for all $j = 0,\dots,k$. Now, by the inner regularity property (2.2), for every $j \in \{0,\dots,k\}$ we find compact sets $K_j \subset D_j \subset A_j$ such that $\eta(D_j) \setminus \eta(K_j) \le \frac{1}{j}$. Let now O_j be open sets such that $K_j \subset O_j \subset A_j$, and let us pick functions $\overline{\phi}_j \in \mathrm{C}^1_{\mathrm{c}}(A_j;\mathbb{R})$ such that $0 \le \overline{\phi}_j \le 1$ and

$$\overline{\phi}_j(v) \begin{cases} \equiv 1 & \text{if } v \in K_j, \\ \equiv 0 & \text{if } v \in A_j \setminus O_j\,. \end{cases}$$

From (A.2) we deduce

$$\sum_{j=0}^k \int_{A_j} (f_j)^+ \,\mathrm{d}\eta = \sum_{j=0}^k \int_{A_j} f_j \sup_{j\in\mathbb{N}} \overline{\phi}_j \,\mathrm{d}\eta \le \liminf_{n\to\infty} \mathscr{F}_\psi(\mu_n|\nu_n)\,. \tag{A.3}$$ almost-fina

Let now

$$f(v) := \begin{cases} \psi(g(v)) & \text{if } v \in V \setminus N, \\ \psi_\infty(h(v)) & \text{if } v \in N, \end{cases}$$

and observe that $f = \sup_{j\in\mathbb{N}} f_j$ by A.1 and hence $f = \sup_{j\in\mathbb{N}} f_j^+$. Since the family $(A_j)_{j=0}^k$ is arbitrary in the family $\mathcal{A}$ of pairwise disjoint open subsets of V with compact closure of all from (A.3) we deduce that

$$\mathscr{F}_\psi(\mu|\nu) = \int_V f\,\mathrm{d}\eta \overset{(1)}{=} \sup_{(A_j)_{j=0}^k \in \mathcal{A}} \sum_{j=0}^k \int_{A_j} (f_j)^+ \,\mathrm{d}\eta \overset{(2)}{\le} \liminf_{n\to\infty} \mathscr{F}_\psi(\mu_n|\nu_n)$$

with (1) from [AFP05, Lemma 2.35] and (2) due to (A.3). This concludes the proof. ∎

References

mFuPa05FBVF [AFP05] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, 2005.

AGS08 [AGS08] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

Bogachev07 [Bog07] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007.

Erbar14 [Erb14] Matthias Erbar. Gradient flows of the entropy for jump processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(3):920–945, 2014.

PRST22 [PRST22] Mark A. Peletier, Riccarda Rossi, Giuseppe Savaré, and Oliver Tse. Jump processes as generalized gradient flows. *Calc. Var. Partial Differential Equations*, 61(1):Paper No. 33, 85, 2022.

RaoRen [RR91] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1991.

Valadier90 [Val90] M. Valadier. Young measures. In A. Cellina, editor, *Methods of nonconvex analysis (Varenna, 1989)*, pages 152–188. Springer, Berlin, 1990.

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