Answers to the exercises

Chapter 1

Exercise 1 (p. 5)

The function f and the components of y_0 are

$$
\begin{aligned}
f^0 &= 1, & y_0^0 &= 1,\\
f^1 &= y^2, & y_0^1 &= 2,\\
f^2 &= 2y^1 - 3y^2 + y^3 + \cos(y^0), & y_0^2 &= -2,\\
f^3 &= y^4, & y_0^3 &= 1,\\
f^4 &= y^1 - y^2 + (y^3)^2 + y^4 + \sin(y^0), & y_0^4 &= 4.
\end{aligned}
$$

Exercise 2 (p. 6)

Substitute

$$z = A\exp(2t) + B\exp(\mathrm{i}t) + C\exp(-\mathrm{i}t)$$

into

$$\mathrm{d}z/\,\mathrm{d}t - 2z - 2\mathrm{i}\exp(\mathrm{i}z) - \mathrm{i}\exp(-\mathrm{i}z)$$

and obtain

$$(2A - 2A)\exp(2t) + (\mathrm{i}B - 2B - 2)\exp(\mathrm{i}t) + (-\mathrm{i}C - 2C - 1)\exp(-\mathrm{i}t).$$

This is zero for all t iff $B = -\frac{4}{5} = \frac{2}{5}\mathrm{i}$ and $C = -\frac{2}{5} + \frac{1}{5}\mathrm{i}$. Add the condition $z(0) = 1$ to obtain $A + B + C = 1$. Hence, $A = \frac{11}{5} + \frac{1}{5}\mathrm{i}$.

Exercise 3 (p. 6)

The real and imaginary components are $x = \frac{11}{5}\exp(2t) - \frac{6}{5}\cos(t) + \frac{3}{5}\sin(t)$, $y = \frac{1}{5}\exp(2t) - \frac{2}{5}\cos(t) - \frac{1}{5}\sin(t)$.

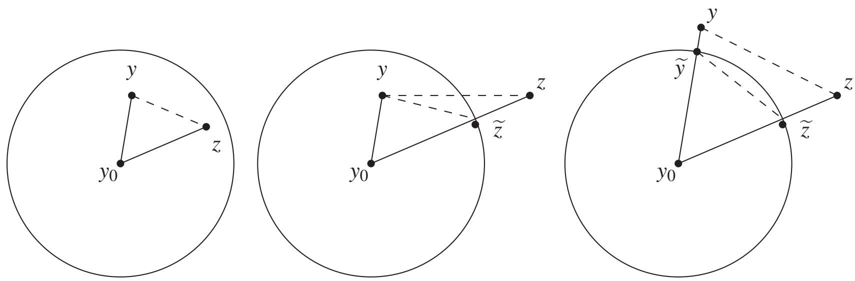
J. C. Butcher, *B-Series*, Springer Series in Computational Mathematics 55,
https://doi.org/10.1007/978-3-030-70956-3

Exercise 4 (p. 7)

Given $y,z \in \mathbb{R}^N$, let

$$\widetilde{y} = y_0 + \frac{R}{\max(\|y-y_0\|,R)}(y-y_0), \qquad \widetilde{z} = y_0 + \frac{R}{\max(\|z-y_0\|,R)}(z-y_0),$$

where $\widetilde{y}$ and $\widetilde{z}$ are shown in three cases, relative to $\{y : \|y-y_0\| \le R\}$,



In each case the Lipschitz condition follows from

$$\|\widetilde{f}(y)-\widetilde{f}(z)\| \le \|f(\widetilde{y})-f(\widetilde{z})\| \le L\|\widetilde{y}-\widetilde{z}\| \le L\|y-z\|.$$

Exercise 5 (p. 11)

$$\widehat{F}(u) = \begin{bmatrix} \frac{u^1-hu^2}{1+h^2} + \frac{0.40001h}{(1+h^2)(1+100h)^2} \\ \frac{u^2+hu^1}{1+h^2} + \frac{0.40001h^2}{(1+h^2)(1+100h)^2} \\ \frac{u^3}{1+100h} \end{bmatrix}.$$

Stability is guaranteed by the power-boundedness of the matrix

$$\begin{bmatrix} \frac{1}{1+h^2} & -\frac{h}{1+h^2} \\ \frac{h}{1+h^2} & \frac{1}{1+h^2} \end{bmatrix},$$

and the boundedness of $(1+100h)^{-n}$ for positive integral n.

Exercise 6 (p. 13)

In this and the following answer, $r := \big((y^1)^2+(y_2)^2)\big)^{1/2}$ so that
$H(x) = \frac{1}{2}\big((y^3)^2+(y^4)^2\big)-r^{-1}, \quad \partial r^{-1}/\partial y^1 = -y^1/r^3 \quad \partial r^{-1}/\partial y^2 = -y^2/r^3.$
We now find
$H' = (\partial H/\partial y^1)(y^1)' + (\partial H/\partial y^2)(y^2)' + (\partial H/\partial y^3)(y^3)' + (\partial H/\partial y^4)(y^3)'$
$\quad = -(y^1/r^3)y^3 - (y^2/r^3)y^4 + y^3y^1/r^3 + y^4y^2/r^3 = 0.$

Exercise 7 (p. 13)

$A' = (\partial A/\partial y^1)(y^1)' + (\partial A/\partial y^2)(y^2)' + (\partial A/\partial y^3)(y^3)' + (\partial A/\partial y^4)(y^3)'$
$\quad = y^4y^3 - y^3y^4 + y^2y^1/r^3 - y^1y^2/r^3 = 0.$

Exercise 8 (p. 14)

Evaluate in turn

$$\begin{aligned}
y' &= y + \sin(x), \\
y'' &= y' + \cos(x) \quad = y + \sin(x) + \cos(x), \\
y^{(3)} &= y'' - \sin(x) \quad = y + \cos(x), \\
y^{(4)} &= y^{(3)} - \cos(x) = y, \\
y^{(5)} &= y^{(4)} + \sin(x) \ = y + \sin(x), \\
y^{(6)} &= y^{(5)} + \cos(x) = y + \sin(x) + \cos(x), \\
y^{(7)} &= y^{(6)} - \sin(x) \ = y + \cos(x).
\end{aligned}$$

Exercise 9 (p. 15)

(a) It is possible that the result `error` vanishes so that the evaluation of `r` fails because of the zero division.
(b) Even if `error` is non-zero but small, the value of `r` might be very large, resulting in an unreasonably large value of `yout`. In practical solvers, the value of the stepsize ratio is not allowed to exceed some heuristic bound such as 2.
(c) Similarly a very small value of `r` needs to be avoided and a heuristic lower bound, such as 0.5 is imposed in practical solvers.

Exercise 10 (p. 18)

For 2 orbits with n steps, $h = 8/n$. The number of steps in successive quadrants are $m+1$, $m+1$, $m+2$, $m+2$, $m+3$, $m+3$, $m+4$, $m+k-16$, giving a final position

$$\frac{2m+4}{n/8}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{2m+4}{n/8}\begin{bmatrix}-1\\-1\end{bmatrix} + \frac{2m+6}{n/8}\begin{bmatrix}1\\-1\end{bmatrix} + \frac{2m+k-14}{n/8}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{n}\begin{bmatrix}8m+9k-128\\8(k-20)\end{bmatrix},$$

which is

$$\frac{8}{n}\begin{bmatrix}k-16\\k-20\end{bmatrix}$$

from the starting point.

Exercise 11 (p. 21)

$$\begin{aligned}
&y(x_0+h) - y(x_0) - hF_2 \\
&= \big(y(x_0+h) - y(x_0) - hy'(x_0+\tfrac{1}{2}h)\big) + \big(y'(x_0+\tfrac{1}{2}h) - F_2\big) \\
&= \big(hy'(x_0) + \tfrac{1}{2}h^2y''(x_0) + \tfrac{1}{6}h^3y^{(3)}(x_0) - hy'(x_0) - \tfrac{1}{2}h^2y''(x_0) \\
&\qquad - \tfrac{1}{8}h^3y^{(3)}(x_0)\big) + h\big(\tfrac{1}{8}h^2 f_y(x_0,y_0)y''(x_0)\big) + \mathscr{O}(h^4) \\
&= \tfrac{1}{24}h^3y^{(3)}(x_0) + \tfrac{1}{8}h^3 f_y(x_0,y_0)y''(x_0) + \mathscr{O}(h^4).
\end{aligned}$$

Exercise 12 (p. 21)

$$
\begin{aligned}
y(x_0+\tfrac{1}{3}h)-Y_2 &= \mathcal{O}(h^2), hy'(x_0+\tfrac{1}{3}h)-hF_2=\mathcal{O}(h^3),\\
y(x_0+\tfrac{2}{3}h)-Y_3 &= \mathcal{O}(h^3), hy'(x_0+\tfrac{2}{3}h)-hF_3=\mathcal{O}(h^4),\\
y(x_0+h)-y_1 &= y(x_0+h)-y_0-\tfrac{1}{4}hy'(x_0)-\tfrac{3}{4}hy'(x_0+\tfrac{2}{3}h)+\mathcal{O}(h^4)\\
&= \mathcal{O}(h^4).
\end{aligned}
$$

Exercise 13 (p. 21)

In this answer $J := f_y(x_0,y_0)$, $\Delta_2 := \frac{1}{32}h^3Jy''(x_0)$, $\Delta_3 := \frac{1}{192}h^4Jy^{(3)}+\frac{1}{64}h^4J^2y''(x_0)$,

$$
\begin{aligned}
y(x_0+\tfrac{1}{4}h)-Y_2 &= y(x_0+\tfrac{1}{4}h)-y_0-\tfrac{1}{4}hy'(x_0) = \tfrac{1}{32}h^2y''(x_0)+\mathcal{O}(h^3),\\
hy'(x_0+\tfrac{1}{4}h)-hF_2 &= \Delta_2+\mathcal{O}(h^4),\\
y(x_0+\tfrac{1}{2}h)-Y_3 &= y(x_0+\tfrac{1}{2}h)-y_0-\tfrac{1}{2}hy'(x_0+\tfrac{1}{4}h)+\tfrac{1}{2}\Delta_2+\mathcal{O}(h^4)\\
&= \tfrac{1}{192}h^3y^{(3)}+\tfrac{1}{64}h^3Jy''(x_0)+\mathcal{O}(h^4),\\
hy'(x_0+\tfrac{1}{2}h)-hF_3 &= \Delta_3+\mathcal{O}(h^5),\\
y(x_0+h)-Y_4 &= y(x_0+h)-hy_0'+2hy'(x_0+\tfrac{1}{4}h)\\
&\quad -2hy'(x_0+\tfrac{1}{2}h)-2\Delta_2+\mathcal{O}(h^4)\\
&= -\tfrac{1}{48}h^3y^{(3)}-\tfrac{1}{16}h^3Jy''(x_0)+\mathcal{O}(h^4)\ \ hy'(x_0+h)-hF_4 \quad = -4\Delta_3+\mathcal{O}(h^4),\\
y(x_0+h)-y_1 &= y(x_0+h)-y_0\\
&\quad -\tfrac{1}{6}hy'(x_0)-\tfrac{2}{3}hy'(x_0+\tfrac{1}{2}h)-\tfrac{1}{6}hy'(x_0+h)+\mathcal{O}(h^5)\\
&= \mathcal{O}(h^5).
\end{aligned}
$$

Exercise 14 (p. 30)

The preconsisitency condition is $\rho(1)=\frac{3}{2}-a_1=0$, implying $a_1=\frac{3}{2}$. The consistency condition then becomes $\rho'(1)-\sigma(1)=(2-\frac{3}{2})-(b_1+1)=0$, implying $b_1=-\frac{1}{2}$. The method $(w^2-\frac{3}{2}w+\frac{1}{2},-\frac{1}{2}w+1)$ is stable because the roots of $\rho(w)=0$ are 1 and $\frac{1}{2}$.

Exercise 15 (p. 30)

Using the relation $w=1+z$ and writing every series in z only to z^2 terms, we have

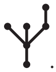
$$
\begin{aligned}
\rho(1+z)/z &= (w^3-w^2)/(w-1)=w^2=1+2z+z^2,\\
\sigma(1+z) &= (1+2z+z^2)(1+\tfrac{1}{2}z-\tfrac{1}{12}z^2)\\
&= 1+\tfrac{5}{12}z+\tfrac{23}{12}=\tfrac{23}{12}w^2-\tfrac{4}{3}w+\tfrac{5}{12}.
\end{aligned}
$$

Exercise 16 (p. 31)

Use the relation $w=1+z$ and write every series up to terms in z^3.

$$
\begin{aligned}
\rho(1+z)/z &= (1+z)^2;\\
\sigma(1+z) &= (1+2z+z^2)(1+\tfrac{1}{2}z-\tfrac{1}{12}z^2+\tfrac{1}{24}z^3)\\
&= 1+\tfrac{5}{2}z+\tfrac{23}{12}z^2+\tfrac{3}{8}z^3\\
&= \tfrac{3}{8}w^3+\tfrac{19}{24}w^2-\tfrac{5}{24}w+\tfrac{1}{24}.
\end{aligned}
$$

Exercise 17 (p. 36)

(a) , (*b*) , (c) .

Exercise 18 (p. 36)

(a) $\mathbf{f}''\mathbf{f}\mathbf{f}''\mathbf{f}^2$, (b) $\mathbf{f}''\mathbf{f}'\mathbf{f}\mathbf{f}'\mathbf{f}'\mathbf{f}$, (c) $\mathbf{f}'''\mathbf{f}(\mathbf{f}'\mathbf{f})^2$.

Chapter 2

Exercise 19 (p. 40)

The result uses induction on $n = \#V$. For $n = 1$ there are no edges and each of the statements is true. For $n > 1$, the result is assumed for $\#V = n-1$. Add an additional vertex and an additional edge is also required to maintain connectivity without creating a loop. However, any additional edge will produce a loop.

Exercise 20 (p. 47)

$$\begin{aligned} \mathrm{t} &= [[\tau^2][_2\tau^2]_2 \\ &= (\tau * ((\tau * \tau) * \tau)) * (\tau * ((\tau * \tau) * \tau)) \\ &= \tau_2\tau_2\tau^2\tau_1\tau_2\tau^2. \end{aligned}$$

Exercise 21 (p. 47)

$[[\tau^3]^2]$, $\qquad \tau * \big(((\tau * \tau) * \tau)\big) * \big((\tau * \tau\big) * \tau)$.

Exercise 22 (p. 49)

The four trees, with the $\sim$ links shown symbolically, are

$\mathbf{t}_{33} = \mathbf{t}_1 * \mathbf{t}_{13} \sim \mathbf{t}_{13} * \mathbf{t}_1 = \mathbf{t}_{22} = \mathbf{t}_6 * \mathbf{t}_2 \sim \mathbf{t}_2 * \mathbf{t}_6 = \mathbf{t}_{24} = \mathbf{t}_{15} * \mathbf{t}_1 \sim \mathbf{t}_1 * \mathbf{t}_{15} = \mathbf{t}_{35}$

Exercise 23 (p. 49)

The five trees, with the $\sim$ links shown symbolically, are

$\mathbf{t}_{32} = \mathbf{t}_1 * \mathbf{t}_6 \sim \mathbf{t}_6 * \mathbf{t}_1 = \mathbf{t}_{21} = \mathbf{t}_3 * \mathbf{t}_4 \sim \mathbf{t}_4 * \mathbf{t}_3 = \mathbf{t}_{27}$

$= \mathbf{t}_7 * \mathbf{t}_2 \sim \mathbf{t}_2 * \mathbf{t}_7 = \mathbf{t}_{25} = \mathbf{t}_{16} * \mathbf{t}_1 \sim \mathbf{t}_1 * \mathbf{t}_{16} = \mathbf{t}_{35}$

Exercise 24 (p. 56)

In the factors on the left of (2.4 a), the factor $(1 - [\tau])^{-1}$ must be removed because no descendants of any vertexcan contain .

Exercise 25 (p. 69)

First calculate p-weight$(1+2^2) = 5!/1!2!2!^2 = 15$. The 15 results are
1+23+45, 1+24+35, 1+25+34, 2+13+45, 2+14+35, 2+15+34, 3+12+45, 3+14+25, 3+15+24, 4+12+35, 4+13+25, 4+15+23, 5+12+34, 5+13+24, 5+14+23.

Exercise 26 (p. 78)

$\mathbf{s}_{01}\mathbf{s}_{21}\mathbf{s}_{01}\mathbf{s}_{10}$ and $\mathbf{s}_{01}\mathbf{s}_{01}\mathbf{s}_{21}\mathbf{s}_{10}$.

Exercise 27 (p. 79)

	1	τ_1	$\tau_2\tau$	$\tau_1\tau_1$	$\tau_3\tau\tau$	$\tau_2\tau\tau_1$	$\tau_1\tau_2\tau$	$\tau_1\tau_1\tau_1$
1	1	τ_1	$\tau_2\tau$	$\tau_1\tau_1$	$\tau_3\tau\tau$	$\tau_2\tau\tau_1$	$\tau_1\tau_2\tau$	$\tau_1\tau_1\tau_1$
τ_1	τ_1	$\tau_1\tau_1$	$\tau_1\tau_2\tau$	$\tau_1\tau_1\tau_1$	$\tau_1\tau_3\tau\tau$	$\tau_1\tau_2\tau\tau_1$	$\tau_1\tau_1\tau_2\tau$	$\tau_1\tau_1\tau_1\tau_1$
$\tau_2\tau$	$\tau_2\tau$	$\tau_2\tau\tau_1$	$\tau_2\tau\tau_2\tau$	$\tau_2\tau\tau_1\tau_1$	$\tau_2\tau\tau_3\tau\tau$	$\tau_2\tau\tau_2\tau\tau_1$	$\tau_2\tau\tau_1\tau_2\tau$	$\tau_2\tau\tau_1\tau_1\tau_1$
$\tau_1\tau_1$	$\tau_1\tau_1$	$\tau_1\tau_1\tau_1$	$\tau_1\tau_1\tau_2\tau$	$\tau_1\tau_1\tau_1\tau_1$	$\tau_1\tau_1\tau_3\tau\tau$	$\tau_1\tau_1\tau_2\tau\tau_1$	$\tau_1\tau_1\tau_1\tau_2\tau$	$\tau_1\tau_1\tau_1\tau_1\tau_1$
$\tau_3\tau\tau$	$\tau_3\tau\tau$	$\tau_3\tau\tau\tau_1$	$\tau_3\tau\tau\tau_2\tau$	$\tau_3\tau\tau\tau_1\tau_1$	$\tau_3\tau\tau\tau_3\tau\tau$	$\tau_3\tau\tau\tau_2\tau\tau_1$	$\tau_3\tau\tau\tau_1\tau_2\tau$	$\tau_3\tau\tau\tau_1\tau_1\tau_1$
$\tau_2\tau\tau_1$	$\tau_2\tau\tau_1$	$\tau_2\tau\tau_1\tau_1$	$\tau_2\tau\tau_1\tau_2\tau$	$\tau_2\tau\tau_1\tau_1\tau_1$	$\tau_2\tau\tau_1\tau_3\tau\tau$	$\tau_2\tau\tau_1\tau_2\tau\tau_1$	$\tau_2\tau\tau_1\tau_1\tau_2\tau$	$\tau_2\tau\tau_1\tau_1\tau_1\tau_1$
$\tau_1\tau_2\tau$	$\tau_1\tau_2\tau$	$\tau_1\tau_2\tau\tau_1$	$\tau_1\tau_2\tau\tau_2\tau$	$\tau_1\tau_2\tau\tau_1\tau_1$	$\tau_1\tau_2\tau\tau_3\tau\tau$	$\tau_1\tau_2\tau\tau_2\tau\tau_1$	$\tau_1\tau_2\tau\tau_1\tau_2\tau$	$\tau_1\tau_2\tau\tau_1\tau_1\tau_1$
$\tau_1\tau_1\tau_1$	$\tau_1\tau_1\tau_1$	$\tau_1\tau_1\tau_1\tau_1$	$\tau_1\tau_1\tau_1\tau_2\tau$	$\tau_1\tau_1\tau_1\tau_1\tau_1$	$\tau_1\tau_1\tau_1\tau_3\tau\tau$	$\tau_1\tau_1\tau_1\tau_2\tau\tau_1$	$\tau_1\tau_1\tau_1\tau_1\tau_2\tau$	$\tau_1\tau_1\tau_1\tau_1\tau_1\tau_1$

Exercise 28 (p. 87)

Use the recursion $\mathrm{t}_n = \mathrm{t}_{n-1} * \tau$ starting with $\Delta(\mathrm{t}_0) = \Delta(\tau) = 1 \otimes \tau + \tau \otimes \varnothing$. By Theorem 2.8D,

$$\begin{aligned}
\Delta(\mathrm{t}_n) &= \Delta(\mathrm{t}_{n-1}) * \Delta(\tau)\\
&= \Big(\textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} \tau^{n-1-i} \otimes \mathrm{t}_i + \mathrm{t}_{n-1} \otimes \varnothing\Big) * \big(1 \otimes \tau + \tau \otimes \varnothing\big)\\
&= \Big(\textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} \tau^{n-1-i} \otimes \mathrm{t}_i\Big) * \big(1 \otimes \tau + \tau \otimes \varnothing\big) + (\mathrm{t}_{n-1} \otimes \varnothing) * \big(1 \otimes \tau + \tau \otimes \varnothing\big)\\
&= \Big(\textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} \tau^{n-1-i} \otimes \mathrm{t}_i\Big) * (1 \otimes \tau) + \Big(\textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} \tau^{n-1-i} \otimes \mathrm{t}_i\Big) * (\tau \otimes \varnothing) + (\mathrm{t}_{n-1} \otimes \varnothing) * (\tau \otimes \varnothing)\\
&= \textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} \tau^{n-1-i} \otimes \mathrm{t}_{i+1} + \textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} \tau^{n-i} \otimes \mathrm{t}_i + (\mathrm{t}_n \otimes \varnothing)\\
&= \textstyle\sum_{i=0}^{n} \binom{n}{i} \tau^{n-i} \otimes \mathrm{t}_i + (\mathrm{t}_n \otimes \varnothing).
\end{aligned}$$

Exercise 29 (p. 87)

Write $\Delta(\mathrm{t}_n) = D_n + \mathrm{t}_n \otimes \varnothing$, with $D_0 = 1 \otimes \tau$. To find D_n,

$$\begin{aligned}
\Delta(\mathrm{t}_n) &= (1 \otimes \tau + \tau \otimes \varnothing) * (D_{n-1} + \mathrm{t}_{n-1} \otimes \varnothing)\\
&= (1 \otimes \tau) * D_{n-1} + \mathrm{t}_{n-1} \otimes \tau + \mathrm{t}_n \otimes \varnothing,
\end{aligned}$$

and it follows that $D_n = (1 \otimes \tau) * D_{n-1} + \mathrm{t}_{n-1} \otimes \tau$. It can be verified by induction that $D_n = \sum_{i=1}^{n-1} \mathrm{t}_{n-i} \otimes \mathrm{t}_i$ so that

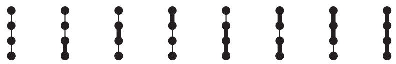
$$\Delta(\mathrm{t}_n) = \sum_{i=1}^{n-1} \mathrm{t}_I \otimes \mathrm{t}_{n-i} + \mathrm{t}_n \otimes \varnothing.$$

Exercise 30 (p. 90)

Denote the vertices of $\mathrm{t} = [\tau^n]$ by $0, 1, 2, \ldots, n$, where 0 is the root. The partitions of t are
(a) $n+1$ singleton vertices,
(b) $n-i$ singleton vertices and an additional tree $[\tau^i]$, $i = 1, 2, \ldots n-1$, and
(c) the one element partition t.
The signed partition contributed by (a) is $(-1)^{n+1}\tau^{n+1}$, the signed partitions contributed by (b), with $1 \le i \le n-1$, are $\binom{n}{i}$ copies of $-(-1)^{n-i}[\tau^i]\tau^{n-i}$, and (c) contributes $-[\tau^n]$.

Exercise 31 (p. 90)

The partitions of $[_3\tau]_3$ are



and the signed partitions, term by term, and then totalled, are

$$\begin{aligned} &\tau^4 - \tau^2[\tau] - \tau^2[\tau] - \tau^2[\tau] + \tau[_2\tau]_2 + [\tau]^2 + \tau[_2\tau]_2 - [_3\tau]_3 \\ &= \tau^4 - 3\tau^2[\tau] + 2\tau[_2\tau]_2 + [\tau]^2 - [_3\tau]_3. \end{aligned}$$

Chapter 3

Exercise 32 (p. 105)

Write the solution in the form

$$y_1 = y_0 + a_1 hF_1 + a_2 h^2 F_2 + \tfrac{1}{2} a_3 h^3 F_3 + a_4 F_4$$

so that $y_1 = y_0 + hf(\frac{1}{2}(y_0 + y_1))$ implies

$$\begin{aligned} &a_1 hF_1 + a_2 h^2 F_2 + \tfrac{1}{2} a_3 h^3 F_3 + a_4 F_4 \\ &= hF_1 + \tfrac{1}{2} a_1 h^2 F_2 + \tfrac{1}{8} a_1^2 h^3 F_3 + \tfrac{1}{2} a_2 F_4 + \mathscr{O}(h^3). \end{aligned}$$

By comparing coefficients, it follows that $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = a_4 = \frac{1}{4}$.

Exercise 33 (p. 105)

$$\begin{aligned} Y_1 &= y_0 && = y_0, \\ hF_1 &= hf(Y_1) && = \quad hE_1, \\ Y_2 &= y_0 + \tfrac{1}{2} hF_1 && = y_0 + \tfrac{1}{2} hE_1, \\ hF_2 &= hf(Y_2) && = \quad hE_1 + \tfrac{1}{4} h^2 E_2 + \tfrac{1}{24} h^3 E_3, \\ y_1 &= y_0 + hF_2 && = y_0 + hE_1 + \tfrac{1}{4} h^2 E_2 + \tfrac{1}{24} h^3 E_3, \end{aligned}$$

giving a result identical with flow_h to within $\mathscr{O}(h^3)$.

Exercise 34 (p. 105)

Write the output from flow_h as y_1 and derive the coefficients a_1, a_2, a_3, a_4 in the following lines

$$\begin{aligned} y_1 &= y_0 + ha_1\mathbf{f} + h^2a_2\mathbf{f}'\mathbf{f} + \tfrac{1}{2}h^3a_3\mathbf{f}''\mathbf{f}\mathbf{f} + h^3a_4\mathbf{f}'\mathbf{f}'\mathbf{f} + \mathscr{O}(h^3), \\ hf(y_1) &= h\mathbf{f} + ha_1\mathbf{f}'\mathbf{f} + \tfrac{1}{2}h^3a_1^2\mathbf{f}''\mathbf{f}\mathbf{f} + h^3a_2\mathbf{f}'\mathbf{f}'\mathbf{f} + \mathscr{O}(h^3), && (1) \\ h(\mathrm{d}/\mathrm{d}h)y_1 &= ha_1\mathbf{f} + ha_2\mathbf{f}'\mathbf{f} + \tfrac{1}{2}h^3a_3\mathbf{f}''\mathbf{f}\mathbf{f} + h^3a_4\mathbf{f}'\mathbf{f}'\mathbf{f} + \mathscr{O}(h^3). && (2) \end{aligned}$$

Compare the coefficients in (1) and (2) to find $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, a_4 = \frac{1}{6}$. Finally substitute into (1) to give

$$hf(y_1) = h\mathbf{f} + h\mathbf{f}'\mathbf{f} + \tfrac{1}{2}h^3\mathbf{f}''\mathbf{f}\mathbf{f} + h^3\tfrac{1}{2}\mathbf{f}'\mathbf{f}'\mathbf{f} + \mathscr{O}(h^3).$$

Exercise 35 (p. 117)

Let $\mathrm{t} = [\mathrm{t}_1\mathrm{t}_2\cdots\mathrm{t}_n]$. Then

$$\begin{aligned} (\mathrm{ED})(\varnothing) &= 0 = |\varnothing|/\varnothing!, \\ (\mathrm{ED})(\tau) &= 1 = |\tau|/\tau!, \\ (\mathrm{ED})(\mathrm{t}) &= \prod_{i=1}^{n}\mathrm{E}(\mathrm{t}_i) = 1\Big/\prod_{i=1}^{n}\mathrm{t}_i! = |\mathrm{t}|\Big/|\mathrm{t}|\prod_{i=1}^{n}\mathrm{t}_i! = |\mathrm{t}|/\mathrm{t}!. \end{aligned}$$

Exercise 36 (p. 118)

Differentiate $\mathbf{y}^{(4)} = \mathbf{f}^{(3)}\mathbf{y}'\mathbf{y}'\mathbf{y}' + 3\mathbf{f}''\mathbf{y}'\mathbf{y}'' + \mathbf{f}'\mathbf{y}^{(3)}$, to obtain

$$\begin{aligned} \mathbf{y}^{(5)} &= \big(\mathbf{f}^{(4)}\mathbf{y}'\mathbf{y}'\mathbf{y}'\mathbf{y}' + 3\mathbf{f}^{(3)}\mathbf{y}'\mathbf{y}'\mathbf{y}''\big) \\ &\qquad + 3\big(\mathbf{f}^{(3)}\mathbf{y}'\mathbf{y}'\mathbf{y}'' + \mathbf{f}''\mathbf{y}''\mathbf{y}'' + \mathbf{f}''\mathbf{y}'\mathbf{y}^{(3)}\big) + \big(\mathbf{f}''\mathbf{y}'\mathbf{y}^{(3)} + \mathbf{f}'\mathbf{y}^{(4)}\big) \\ &= \mathbf{f}^{(4)}\mathbf{y}'\mathbf{y}'\mathbf{y}'\mathbf{y}' + 6\mathbf{f}^{(3)}\mathbf{y}'\mathbf{y}'\mathbf{y}'' + 4\mathbf{f}''\mathbf{y}'\mathbf{y}^{(3)} + 3\mathbf{f}''\mathbf{y}''\mathbf{y}'' + \mathbf{f}'\mathbf{y}^{(4)}. \end{aligned}$$

Exercise 37 (p. 142)

$$\begin{aligned} \lambda(a,\mathbf{t}_6) &= a_1(a_2\mathbf{t}_1 + a_1\mathbf{t}_2 + \mathbf{t}_4) + (a_2\mathbf{t}_1 + a_1\mathbf{t}_2 + \mathbf{t}_4) * (\mathbf{t}_1) \\ &= a_1a_2\mathbf{t}_1 + a_1^2\mathbf{t}_2 + a_1\mathbf{t}_4 + a_2\mathbf{t}_2 + a_1\mathbf{t}_3 + \mathbf{t}_6 \\ &= a_1a_2\mathbf{t}_1 + (a_1^2 + a_2)\mathbf{t}_2 + a_1\mathbf{t}_4 + a_1\mathbf{t}_3 + \mathbf{t}_6. \end{aligned}$$

Exercise 38 (p. 142)

$$\begin{aligned} \lambda(a,\mathbf{t}_6) &= a_2(a_1\mathbf{t}_1 + \mathbf{t}_2) + (a_1\mathbf{t}_1 + \mathbf{t}_2) * (a_1\mathbf{t}_1 + \mathbf{t}_2) \\ &= a_1a_2\mathbf{t}_1 + a_2\mathbf{t}_2 + a_1^2\mathbf{t}_2 + a_1\mathbf{t}_3 + a_1\mathbf{t}_4 + \mathbf{t}_6 \\ &= a_1a_2\mathbf{t}_1 + (a_1^2 + a_2)\mathbf{t}_2 + a_1\mathbf{t}_4 + a_1\mathbf{t}_3 + \mathbf{t}_6. \end{aligned}$$

Chapter 4

Exercise 39 (p. 155)

$$\begin{aligned}
\varphi_\xi(\tau) &= \int_0^\xi \mathrm{d}\xi, &&= \xi, & \varphi_\xi([\tau]) &= \int_0^\xi \xi\,\mathrm{d}\xi, &&= \frac{1}{2}\xi^2,\\
\varphi_\xi([\tau^2]) &= \int_0^\xi \xi^2\,\mathrm{d}\xi, &&= \frac{1}{3}\xi^3, & \varphi_\xi([[\tau]] &= \int_0^\xi \frac{1}{2}\xi^2\,\mathrm{d}\xi, &&= \frac{1}{6}\xi^3,\\
\varphi_\xi([\tau^3]) &= \int_0^\xi \xi^3\,\mathrm{d}\xi, &&= \frac{1}{4}\xi^4, & \varphi_\xi([\tau[\tau]]) &= \int_0^\xi \frac{1}{2}\xi^3\,\mathrm{d}\xi, &&= \frac{1}{8}\xi^4,\\
\varphi_\xi([[\tau^2]]) &= \int_0^\xi \frac{1}{2}\xi^3\,\mathrm{d}\xi, &&= \frac{1}{12}\xi^4, & \varphi_\xi([[[\tau]]]) &= \int_0^\xi \frac{1}{6}\xi^3\,\mathrm{d}\xi, &&= \frac{1}{24}\xi^4.
\end{aligned}$$

To find the $\Phi(\mathrm{t})$, substitute $\xi = 1$. The results are $\Phi(\tau) = 1$, $\Phi([\tau]) = \frac{1}{2}$, $\Phi([\tau^2]) = \frac{1}{3}$, $\Phi([[\tau]]) = \frac{1}{6}$, $\Phi([\tau^3]) = \frac{1}{4}$, $\Phi([\tau[\tau]]) = \frac{1}{8}$, $\Phi([[\tau^2]]) = \frac{1}{12}$, $\Phi([[[\tau]]]) = \frac{1}{24}$.

Exercise 40 (p. 155)

$$\begin{aligned}
\varphi_\xi(\tau) &= \xi\int_0^1 \mathrm{d}\xi, &&= \xi, & \varphi_\xi([\tau]) &= \xi\int_0^1 \xi\,\mathrm{d}\xi, &&= \frac{1}{2}\xi,\\
\varphi_\xi([\tau^2]) &= \xi\int_0^1 \xi^2\,\mathrm{d}\xi, &&= \frac{1}{3}\xi, & \varphi_\xi([[\tau]] &= \xi\int_0^1 \frac{1}{2}\xi\,\mathrm{d}\xi, &&= \frac{1}{4}\xi,\\
\varphi_\xi([\tau^3]) &= \xi\int_0^1 \xi^3\,\mathrm{d}\xi, &&= \frac{1}{4}\xi, & \varphi_\xi([\tau[\tau]]) &= \xi\int_0^1 \frac{1}{2}\xi^2\,\mathrm{d}\xi, &&= \frac{1}{6}\xi,\\
\varphi_\xi([[\tau^2]]) &= \xi\int_0^1 \frac{1}{2}\xi\,\mathrm{d}\xi, &&= \frac{1}{4}\xi, & \varphi_\xi([[[\tau]]]) &= \xi\int_0^1 \frac{1}{4}\xi\,\mathrm{d}\xi, &&= \frac{1}{8}\xi.
\end{aligned}$$

To find the $\Phi(\mathrm{t})$, substitute $\xi = 1$. The results are $\Phi(\tau) = 1$, $\Phi([\tau]) = \frac{1}{2}$, $\Phi([\tau^2]) = \frac{1}{3}$, $\Phi([[\tau]]) = \frac{1}{4}$, $\Phi([\tau^3]) = \frac{1}{4}$, $\Phi([\tau[\tau]]) = \frac{1}{6}$, $\Phi([[\tau^2]]) = \frac{1}{4}$, $\Phi([[[\tau]]]) = \frac{1}{8}$.

Exercise 41 (p. 158)

It is observed that the stages can be reducd using $P_1 = \{1,4\}$. $P_2 = \{2,3\}$, giving the tableau

$$\begin{array}{c|cc}
\frac{1}{2} & \frac{1}{2} & 0\\
\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\
\hline
 & 1 & 0
\end{array}.$$

Only the first reduced stage is essential, and we get the final result

$$\begin{array}{c|c}
\frac{1}{2} & \frac{1}{2}\\
\hline
 & 1
\end{array}.$$

Exercise 42 (p. 169)

The given set is a subgroup because

$$\begin{aligned}
&\left[\begin{array}{cccc}\alpha_1 & \frac{1}{2}\alpha^2 & \alpha_3 & \frac{1}{2}\alpha_3\end{array}\right]\left[\begin{array}{cccc}\beta_1 & \frac{1}{2}\beta^2 & \beta_3 & \frac{1}{2}\beta_3\end{array}\right]\\
&\qquad = \left[\alpha_1+\beta_1 \quad \tfrac{1}{2}(\alpha_1+\beta_1)^2 \quad \alpha_1\beta_1(\alpha_1+\beta_1)+\alpha_3+\beta_3\,\tfrac{1}{2}\big(\alpha_1\beta_1(\alpha_1+\beta_1)+\alpha_3+\beta_3\big)\right].
\end{aligned}$$

Exercise 43 (p. 169)

The H_4 is a subgroup because $(ab)_1 = a_1 + b_1$, $(ab)_2 = a_2 + a_1 b_1 + b_2 = (a_1 + b_1)^2 = (ab)_1^2$. To be a normal subgroup, x must exist such that $xa = ab$. This is solved by writing $x_1 = b_1$, $x_2 = b_2$, with x_i, $i = 3, 4, \dots$, found recursively.

Chapter 5

Exercise 44 (p. 188)

Expand $(I - zA)^{-1}$ as a geometric series noting that $A^s = 0$. This gives
$1 + \sum_{n=1}^{s} b^\mathsf{T} A^{n-1} = 1 + \sum_{n=1}^{s} \Phi([_n 1]_n) z^n$.

Exercise 45 (p. 188)

Since $p = s$, $\Phi([_n 1]_n) = 1/[_n 1]_n!$ and it is only ncessary to verify by induction that $[_n 1]_n! = n!$.

Exercise 46 (p. 189)

Use (5.3 f).

$$R(z) = \frac{\det(I + z(\mathbf{1}b^\mathsf{T} - A))}{\det(I - zA)} = \frac{\det\left(\begin{bmatrix} 1 + \frac{3}{8}z & \frac{3}{8}z \\ 0 & 1 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 1 - \frac{7}{24}z & \frac{1}{24}z \\ -\frac{2}{3}z & 1 - \frac{1}{3}z \end{bmatrix}\right)} = \frac{1 + \frac{3}{8}z}{1 - \frac{5}{8}z + \frac{1}{8}z^2}.$$

Exercise 47 (p. 191)

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array} .$$

Exercise 48 (p. 192)

$$\begin{array}{c|ccc} 0 & & & \\ \frac{2}{3} & \frac{2}{3} & & \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array} .$$

Exercise 49 (p. 194)

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{3} & \frac{1}{3} & & & \\ \frac{3}{4} & -\frac{21}{32} & \frac{45}{32} & & \\ 1 & \frac{7}{3} & -\frac{12}{5} & \frac{16}{15} & \\ \hline & \frac{1}{9} & \frac{9}{20} & \frac{16}{45} & \frac{1}{12} \end{array} .$$

Exercise 50 (p. 203)

$$\begin{array}{c|ccc} \frac{1}{2}-\frac{1}{10}\sqrt{15} & \frac{5}{36} & \frac{2}{9}-\frac{1}{30}\sqrt{15} & \frac{5}{36}-\frac{1}{20}\sqrt{15} \\ \frac{1}{2} & \frac{5}{36}+\frac{1}{24}\sqrt{15} & \frac{2}{9} & \frac{5}{36}-\frac{1}{24}\sqrt{15} \\ \frac{1}{2}+\frac{1}{10}\sqrt{15} & \frac{5}{36}+\frac{1}{20}\sqrt{15} & \frac{2}{9}+\frac{1}{30}\sqrt{15} & \frac{5}{36} \\ \hline & \frac{5}{18} & \frac{4}{9} & \frac{5}{18} \end{array} .$$

Exercise 51 (p. 203)

$\widetilde{P}_n(1) = 1$, for all n. Therefore, $\widetilde{P}_s(1) - \widetilde{P}_{s-1}(1) = 0$, for $s \geq 1$.

Exercise 52 (p. 203)

The zeros of $\widetilde{P}_3 - \widetilde{P}_2 = 20x^3 - 36x^2 + 18x - 2$ are $\frac{2}{5} \mp \frac{1}{10}\sqrt{6}$ and 1. Solve linear equations for A and b^{T}. The final tableau is

$$\begin{array}{c|ccc} \frac{2}{5}-\frac{1}{10}\sqrt{6} & \frac{11}{45}-\frac{7}{360}\sqrt{6} & \frac{37}{225}-\frac{169}{1800}\sqrt{6} & -\frac{2}{225}+\frac{1}{75}\sqrt{6} \\ \frac{2}{5}+\frac{1}{10}\sqrt{6} & \frac{37}{225}+\frac{169}{1800}\sqrt{6} & \frac{11}{45}+\frac{7}{360}\sqrt{6} & -\frac{2}{225}-\frac{1}{75}\sqrt{6} \\ 1 & \frac{4}{9}-\frac{1}{36}\sqrt{6} & \frac{4}{9}+\frac{1}{36}\sqrt{6} & \frac{1}{9} \\ \hline & \frac{4}{9}-\frac{1}{36}\sqrt{6} & \frac{4}{9}+\frac{1}{36}\sqrt{6} & \frac{1}{9} \end{array} .$$

Exercise 53 (p. 208)

From the equations in (5.7 c), it follows that $\sum_{ij} b_i(1-c_i)a_{ij}c_j(c_j - c_3) = \frac{1}{60} - \frac{1}{24}c_3$. Since the left-hand side is zero, $c_3 = \frac{2}{5}$.

Chapter 6

Exercise 54 (p. 216)

In each case, z is in the stability region if the difference equation $(1 - b_0 z)y_k = \sum_{i=1}^{k}(a_i + b_i z)y_{k-i}$ has only bounded solutions.

Exercise 55 (p. 218)

The characteristic polynomial of M is found to be $w(w-1)(w-\frac{240\mu+361}{121})$. The zeros of this polynomial are $0, 1, w'$, where $w' = \frac{240\mu+361}{121}$, which satisfies $|w'| < 1$ for $\mu \in (-\frac{241}{120}, -1)$.

Exercise 56 (p. 229)

$$T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Exercise 57 (p. 235)

$(\theta_5 - (c_2+c_4)\theta_3 + c_2c_4\theta_2)\theta_8 - (\theta_6 - c_4\theta_4)(\theta_7 - c_4\theta_2) = 0$ simplifies to $c_2(1+2c_4) = 0$.

Chapter 7

Exercise 58 (p. 269)

$$\begin{array}{c|ccccc}
0 & & & & & \\
-\frac{1}{3} & -\frac{1}{3} & & & & \\
-\frac{1}{3} & \frac{896237}{950913} & -\frac{1213208}{950913} & & & \\
-\frac{2}{3} & -\frac{15257}{23193} & 0 & -\frac{205}{23193} & & \\
-1 & -\frac{4736}{3591} & 0 & 0 & \frac{1145}{3591} & \\
\hline
0 & -\frac{4537}{12800} & \frac{17759035623}{15529062400} & -\frac{89068851}{3105812480} & -\frac{7731}{12800} & \frac{1197}{12800}
\end{array}.$$

Exercise 59 (p. 269)

The matrix

$$\begin{aligned}
\begin{bmatrix} b^{\mathsf{T}}(c-c_4) \\ b^{\mathsf{T}}A \end{bmatrix} \begin{bmatrix} (c-c_2)c & Ac \end{bmatrix} &= \begin{bmatrix} b^{\mathsf{T}}(c-c_4)(c-c_2)c & b^{\mathsf{T}}(c-c_4)Ac \\ b^{\mathsf{T}}A(c-c_2)c & b^{\mathsf{T}}A^2c \end{bmatrix} \\
&= \begin{bmatrix} b^{\mathsf{T}}(c-c_4)(c-c_2)c & b^{\mathsf{T}}(c-c_4)Ac \\ b^{\mathsf{T}}A(c-c_2)c & b^{\mathsf{T}}A^2c \end{bmatrix} \\
&= \begin{bmatrix} \zeta_5 - (c_2+c_4)\zeta_3 + c_2c_4\zeta 2 & \zeta_6 - c_4\zeta_4 \\ \zeta_7 - c_2\zeta_4 & \zeta_8 \end{bmatrix}
\end{aligned}$$

has rank 1 and its determinant is zero. This simplifies to

$$2574900c_2c_4 + 1453965c_2 + 967440c_4 + 688748 = 0.$$

Substitute $c_2 = \frac{15924}{14305}$, with the result $c_4 = -\frac{10331}{17166}$.

Exercise 60 (p. 282)

Because $\mathbf{t}_{13} = [\mathbf{t}_2^2]$, the result is $H''\mathbf{f}'\mathbf{f}\mathbf{f}'\mathbf{f}$.