Solution Manual to *Linear Algebra Done Right*, 4th Edition by Sheldon Axler

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CONTENTS

CHAPTER 1

VECTOR SPACES

1 $\mathbb{R}^n$ and $\mathbb{C}^n$

We skip this section.

2 Difinition of Vector Space

Exercise 2.1: (1B-1)

Prove that $-(-v) = v$ for every $v \in V$.

Solution:

For $v \in V$, we have

$$-(-v) = -(-v) + (-v) + v = v. \tag{1.1}$$

Thus we know the additive inverse of the additive inverse of v is itself. ■

Exercise 2.2: (1B-2)

Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution:

If $a = 0$, then we are done.
If $a \neq 0$, then

$$v = (\frac{1}{a} \cdot a)v = \frac{1}{a}(av) = 0. \tag{1.2}$$

■

Exercise 2.3: (1B-3)

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution:

Let $x = \frac{w-v}{3}$, then

$$v + 3x = w. \tag{1.3}$$

This show existence. Now we show the uniqueness. Suppose there is an x' that satisfies $v + 3x' = w$, then

$$3(x - x') = 3x - 3x' = (w - v) - (w - v) = 0. \tag{1.4}$$

By Exercise 2.2, we must have $x - x' = 0$, thus $x = x'$. ■

Exercise 2.4: (1B-4)

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution:

Additive identity. In an empty set $\emptyset$, there does not exist an element 0 that $v + 0 = v$ for all $v \in \emptyset$. ■

Exercise 2.4 shows that the additive identity condition can be replaced with the condition that the set is not empty(because then taking $u \in U$ and multiplying it by 0 would imply that $0 \in U$).

Exercise 2.5: (1B-6)

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & if t < 0, \\ 0 & if t = 0, \\ \infty & if t > 0, \end{cases} t(-\infty) = \begin{cases} \infty & if t < 0, \\ 0 & if t = 0, \\ -\infty & if t > 0, \end{cases}$$

and

$$\begin{gathered} t + \infty = \infty + t = \infty + \infty = \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty, \\ \infty + (-\infty) = (-\infty) + \infty = 0. \end{gathered}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution:

We can notice that

$$\infty = (2 + (-1))\infty = 2\infty + (-1)\infty = \infty + (-\infty) = 0. \tag{1.5}$$

For $\infty \neq 0$, the set doesn't follow the distributive property. Thus $\mathbf{R} \cup \{\infty, -\infty\}$ is not a vector space.

Exercise 2.6: (1B-8)

Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on VC is defined by

  $$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

  $$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Solution:

Just verify the six properties of vector spaces. For example:
commutativity

$$\begin{aligned}(u_1 + iv_1) + (u_2 + iv_2) =& (u_1 + u_2) + i(v_1 + v_2) \\ =& (u_2 + u_1) + i(v_2 + v_1) \\ =& (u_2 + iv_2) + (u_1 + iv_1)\end{aligned} \tag{1.6}$$

for all $u_1, u_2, v_1, v_2 \in V$. The remaining five properties are the same. Thus we have the complex vector space $V_{\mathbf{C}}$.

3 Subspaces

Exercise 3.1: (1C-5)

Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

Solution:

Notice that subspaces of $\mathbf{C}^2$ are closed under scalar multiplication in $\mathbf{C}$, then

$$i(1,1) = (i,i) \notin \mathbf{R}^2.$$

Thus $\mathbf{R}^2$ is not a subspace of $\mathbf{C}^2$. ■