Solution Manual to *Linear Algebra Done Right*, 4th Edition by Sheldon Axler

Chenyu Tang

CONTENTS

Contents 3

1 Vector Spaces 5

1 $\mathbb{R}^n$ and $\mathbb{C}^n$ 5

2 Difinition of Vector Space 5

3 Subspaces 7

CHAPTER 1

VECTOR SPACES

1 $\mathbb{R}^n$ and $\mathbb{C}^n$

We skip this section.

2 Difinition of Vector Space

Exercise 2.1: (1B-1)

Prove that $-(-v) = v$ for every $v \in V$.

Solution:

For $v \in V$, we have

$$-(-v) = -(-v) + (-v) + v = v. \tag{1.1}$$

Thus we know the additive inverse of the additive inverse of v is itself. ■

Exercise 2.2: (1B-2)

Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution:

If $a = 0$, then we are done.
If $a \neq 0$, then

$$v = (\frac{1}{a} \cdot a)v = \frac{1}{a}(av) = 0. \tag{1.2}$$

■

Exercise 2.3: (1B-3)

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution:

Let $x = \frac{w-v}{3}$, then
$$v + 3x = w. \tag{1.3}$$
This show existence. Now we show the uniqueness. Suppose there is an x' that satisfies $v + 3x' = w$, then
$$3(x - x') = 3x - 3x' = (w - v) - (w - v) = 0. \tag{1.4}$$
By Exercise 2.2, we must have $x - x' = 0$, thus $x = x'$. ∎

Exercise 2.4: (1B-4)

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution:

Additive identity. In an empty set $\varnothing$, there does not exist an element 0 that $v + 0 = v$ for all $v \in \varnothing$. ∎

Exercise 2.4 shows that the additive identity condition can be replaced with the condition that the set is not empty(because then taking $u \in U$ and multiplying it by 0 would imply that $0 \in U$).

Exercise 2.5: (1B-6)

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define
$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$
and
$$\begin{gathered} t + \infty = \infty + t = \infty + \infty = \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty, \\ \infty + (-\infty) = (-\infty) + \infty = 0. \end{gathered}$$
With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution:

We can notice that

$$\infty = (2 + (-1))\infty = 2\infty + (-1)\infty = \infty + (-\infty) = 0. \tag{1.5}$$

For $\infty \neq 0$, the set doesn't follow the distributive property. Thus $\mathbf{R} \cup \{\infty, -\infty\}$ is not a vector space. ■

Exercise 2.6: (1B-8)

Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on VC is defined by

  $$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

  for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

  $$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

  for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Solution:

Just verify the six properties of vector spaces. For example:
commutativity

$$\begin{aligned}(u_1 + iv_1) + (u_2 + iv_2) =& (u_1 + u_2) + i(v_1 + v_2) \\ =& (u_2 + u_1) + i(v_2 + v_1) \\ =& (u_2 + iv_2) + (u_1 + iv_1)\end{aligned} \tag{1.6}$$

for all $u_1, u_2, v_1, v_2 \in V$. The remaining five properties are the same. Thus we have the complex vector space $V_{\mathbf{C}}$. ■

3 Subspaces

Exercise 3.1: (1C-5)

Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

Solution:

Notice that subspaces of $\mathbf{C}^2$ are closed under scalar multiplication in $\mathbf{C}$, then

$$i(1,1) = (i,i) \notin \mathbf{R}^2.$$

Thus $\mathbf{R}^2$ is not a subspace of $\mathbf{C}^2$. ■

Exercise 3.2: (1C-6)

(a) Is $\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?

(b) Is $\{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?

Solution:

(a)
The equation $a^3 = b^3$ has the only solution $a = b$ in $\mathbf{R}$, hence

$$\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\} = \{(a,b,c) \in \mathbf{R}^3 : a = b\} \tag{1.7}$$

is obviously a subspace of $\mathbf{R}^3$.
(b)
In $\mathbf{C}^3$, we have

$$\left(1, \frac{-1+\sqrt{3}i}{2}, 0\right) \in \{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\} \tag{1.8}$$

and

$$\left(1, \frac{-1-\sqrt{3}i}{2}, 0\right) \in \{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\}. \tag{1.9}$$

However,

$$\left(1, \frac{-1+\sqrt{3}i}{2}, 0\right) + \left(1, \frac{-1-\sqrt{3}i}{2}, 0\right) = (2,-1,0) \notin \{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\}.$$

Hence $\{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\}$ is not a vector space. ■

Exercise 3.3: (1C-11)

Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution:

Assume U_i is a subspace of V for $i \in I$, then prove $\cap_{i\in I} U_i$ is a subspace of V.

Additive identity

$$0 \in U_i \quad \forall i \in I \Rightarrow 0 \in \cap_{i\in I} U_i. \tag{1.10}$$

Closed under addition For all $u, v \in \cap_{i\in I} U_i$, $u, v \in U_i,\ \forall i \in I$, thus

$$u + v \in U_i,\ \forall i \in I \Rightarrow u + v \in \cap_{i\in I} U_i \tag{1.11}$$

which shows $\cap_{i\in I} U_i$ is closed under addition.

Closed under scalar multiplication For all $w \in \cap_{i\in I} U_i$, $w \in U_i,\ \forall i \in I$, thus

$$\lambda w \in U_i,\ \forall i \in I \Rightarrow \lambda w \in \cap_{i\in I} U_i \tag{1.12}$$

which shows $\cap_{i\in I} U_i$ is closed under scalar multiplication. ■

Exercise 3.4: (1C-12)

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution:

Let U_1, U_2 be subspaces of V. Suppose $u \in U_1$ and $u \notin U_2$ while $v \in U_2$ and $v \notin U_1$, then we have $u, v \in U_1 \cup U_2$ but $u + v \notin U_1$ or U_2, thus not in $U_1 \cup U_2$. Hence $U_1 \cup U_2$ is not closed under addition, and we get a contradiction. So either $\forall u \in U_1,\ u \in U_2$ or $\forall v \in U_2,\ v \in U_1$, or $U_1 = U_2$. ■

Exercise 3.5: (1C-13)

Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution:

Let U_1, U_2, U_3 be subspaces of V, consider $u \in U_1$ and $v \in U_2$. If $U \cup V \neq U$ or V, then we can assume there exist $u \notin U_2$ and $v \notin U_1$. From the assumption we know $u + v \in \cup_{i=1}^{3} U_i$ and $u + v$ not in U_1 or U_2, thus $u + v \in U_3$. For the same reason we know $u + 2v$ and $2u + v$ are in U_3, and because U_3 is closed under addition and additive inverse, we have $u, v \in U_3$. Hence $\complement_{U_1} U_1 \cap U_2$ and $\complement_{U_2} U_1 \cap U_2$ are contained by U_3.

For $w \in U_1 \cap U_2$, taking $u \in \complement_{U_1} U_1 \cap U_2$, then $u + w$ in U_1 but not in U_2, and we come back to the above condition. Thus we can prove $U_1 \cap U_2 \subset U_3$, which means U_3 contains U_1 and U_2.

Now consider the consition that $U_1 \subset U_2$. If $U_2 \subset U_3$, then we are done. If $U_1 \subset U_3$, then consider $u \in \complement_{U_2} U_1$ and $v \in \complement_{U_3} U_1$, because the union of U_1, U_2, U_3 is closed under addition, $u + v \in \cup_{i=1}^{3} U_i$. Evidently $u + v \notin U_1$, if $u + v \in U_2$, then we can prove $U_2 = U_3$; the other side is the same.

From the conditions above, we prove the proposition in the question. ■

Remark.

> Problem 15-19 in Exercises 1C prove some properties of the addition in the set of subspaces of V, including additive identity, commutativity, associativity. From these propositions we know the group of subspaces of V is not an abel group.

Exercise 3.6: (1C-21)

Suppose
$$U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$
Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution:

> Let $W = \{(0, 0, u, v, w) \in \mathbf{F}^5 : u, v, w \in \mathbf{F}\}$. Consider $(x, y, x+y, x-y, 2x) \in U \cap W$, then we have $x = y = 0$, which means $U \cap W = \{0\}$. Hence $\mathbf{F}^5 = U \oplus W$ by 1.45. ■

Problem 20-22 in Exercises 1C are the same type. To solve these problems, we first write a subspace W that the union of W and the given subspaces is the vector space asked for. Then prove the intersection of the subspaces is $\{0\}$.

Exercise 3.7: (1C-23)

Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that
$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$
then $V_1 = V_2$.

Solution:

> Consider the example in Exercise 3.6. We change W into $W' = \{(0, z, 0, v, w) \in \mathbf{F}^5 : u, v, w \in \mathbf{F}\}$, and it is easy to verify $\mathbf{F}^5 = U \oplus W'$. Hence we give a counterexample. ■

Exercise 3.8: (1C-24)

A function $f : \mathbf{R} \to \mathbf{R}$ is called *even* if
$$f(-x) = f(x)$$
for all $x \in \mathbf{R}$. A function $f : \mathbf{R} \to \mathbf{R}$ is called *odd* if
$$f(-x) = -f(x)$$
for all $x \in R$. Let V_e denote the set of real-valued even functions on $\mathbf{R}$ and let V_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

Solution:

For $h : \mathbf{R} \to \mathbf{R} \in \mathbf{R}^{\mathbf{R}}$, suppose $h(x) = f(x) + g(x)$, where $f \in V_e$ and $g \in V_o$. Then we must verify $h(-x) = f(-x) + g(-x) = f(x) - g(x)$. From the above discription, let

$$f(x) = \frac{h(x) + h(-x)}{2} \tag{1.13}$$

and

$$g(x) = \frac{h(x) - h(-x)}{2}. \tag{1.14}$$

Then we have $\mathbf{R}^{\mathbf{R}} = V_e + V_o$.

Consider $f_0 \in V_e \cap V_o$,

$$f_0(x) = f_0(-x) = -f_0(x),\ \forall x \in \mathbf{R}. \tag{1.15}$$

Thus $f_0(x) \equiv 0$, $V_e \cap V_o = \{0\}$. By 1.45, $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$. ■

From Exercise 3.8 we know every real-valued function can be divided into the sum of an even function and an odd function.