Solution Manual to *Linear Algebra Done Right*, 4th Edition by Sheldon Axler

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CHAPTER 1

VECTOR SPACES

1 \mathbb{R}^n and \mathbb{C}^n

We skip this section.

2 Difinition of Vector Space

Exercise 2.1: (1B-1)

Prove that -(-v) = v for every $v \in V$.

Solution:

For $v \in V$, we have

$$-(-v) = -(-v) + (-v) + v = v. (1.1)$$

Thus we know the additive inverse of the additive inverse of v is itself.

Exercise 2.2: (1B-2)

Suppose $a \in \mathbf{F}$, $v \in V$, and av = 0. Prove that a = 0 or v = 0.

Solution:

If a = 0, then we are done.

If $a \neq 0$, then

$$v = (\frac{1}{a} \cdot a)v = \frac{1}{a}(av) = 0.$$
 (1.2)

Exercise 2.3: (1B-3)

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that v + 3x = w.

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Solution:

Let
$$x = \frac{w-v}{3}$$
, then
$$v + 3x = w. \tag{1.3}$$

This show existence. Now we show the uniqueness. Suppose there is an x' that satisfies v + 3x' = w, then

$$3(x - x') = 3x - 3x' = (w - v) - (w - v) = 0.$$
(1.4)

By Exercise 2.2, we must have x - x' = 0, thus x = x'.

Exercise 2.4: (1B-4)

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution:

Additive identity. In an empty set \emptyset , there does not exist an element 0 that v + 0 = v for all $v \in \emptyset$.

Exercise 2.4 shows that the additive identity condition can be replaced with the condition that the set is not empty(because then taking $u \in U$ and multiplying it by 0 would imply that $0 \in U$).

Exercise 2.5: (1B-6)

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in **R**. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$t + \infty = \infty + t = \infty + \infty = \infty,$$

$$t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty,$$

$$\infty + (-\infty) = (-\infty) + \infty = 0.$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over \mathbf{R} ? Explain.

Solution:

We can notice that

$$\infty = (2 + (-1))\infty = 2\infty + (-1)\infty = \infty + (-\infty) = 0.$$
 (1.5)

For $\infty \neq 0$, the set doesn't follow the distributive property. Thus $\mathbf{R} \cup \{\infty, -\infty\}$ is not a vector space.

Exercise 2.6: (1B-8)

Suppose V is a real vector space.

- The complexification of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as u + iv.
- Addition on VC is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.

• Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a+bi)(u+iv) = (au-bv) + i(av+bu)$$

for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Solution:

Just verify the six properties of vector spaces. For example:

commutativity

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

$$= (u_2 + u_1) + i(v_2 + v_1)$$

$$= (u_2 + iv_2) + (u_1 + iv_1)$$
(1.6)

for all $u_1, u_2, v_1, v_2 \in V$. The remaining five properties are the same. Thus we have the complex vector space $V_{\mathbf{C}}$.

3 Subspaces

Exercise 3.1: (1C-5)

Is \mathbb{R}^2 a subspace of the complex vector space \mathbb{C}^2 ?

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Solution:

Notice that subspaces of \mathbb{C}^2 are closed under scalar multiplication in \mathbb{C} , then

$$i(1,1) = (i,i) \notin \mathbf{R}^2$$
.

Thus \mathbb{R}^2 is not a subspace of \mathbb{C}^2 .

Exercise 3.2: (1C-6)

- (a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of \mathbf{R}^3 ?
- (b) Is $\{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of \mathbb{C}^3 ?

Solution:

(a)

The equation $a^3 = b^3$ has the only solution a = b in **R**, hence

$$\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\} = \{(a,b,c) \in \mathbf{R}^3 : a = b\}$$
(1.7)

is obviously a subspace of \mathbb{R}^3 .

(b)

In \mathbb{C}^3 , we have

$$\left(1, \frac{-1+\sqrt{3}i}{2}, 0\right) \in \{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$$
(1.8)

and

$$\left(1, \frac{-1 - \sqrt{3}i}{2}, 0\right) \in \{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}.$$
(1.9)

However.

$$\left(1, \frac{-1+\sqrt{3}i}{2}, 0\right) + \left(1, \frac{-1-\sqrt{3}i}{2}, 0\right) = (2, -1, 0) \notin \{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}.$$

Hence $\{(a,b,c)\in {\bf C}^3: a^3=b^3\}$ is not a vector space.

Exercise 3.3: (1C-11)

Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution:

Assume U_i is a subspace of V for $i \in I$, then prove $\bigcap_{i \in I} U_i$ is a subspace of V.

Additive identity

$$0 \in U_i \quad \forall i \in I \Rightarrow 0 \in \cap_{i \in I} U_i. \tag{1.10}$$

Closed under addition For all $u, v \in \cap_{i \in I} U_i$, $u, v \in U_i$, $\forall i \in I$, thus

$$u + v \in U_i, \ \forall i \in I \Rightarrow u + v \in \cap_{i \in I} U_i$$
 (1.11)

which shows $\cap_{i \in I} U_i$ is closed under addition.

Closed under scalar multiplication For all $w \in \cap_{i \in I} U_i$, $w \in U_i$, $\forall i \in I$, thus

$$\lambda w \in U_i, \ \forall i \in I \Rightarrow \lambda w \in \cap_{i \in I} U_i \tag{1.12}$$

which shows $\cap_{i \in I} U_i$ is closed under scalar multiplication.

Exercise 3.4: (1C-12)

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution:

Let U_1, U_2 be subspaces of V. Suppose $u \in U_1$ and $u \notin U_2$ while $v \in U_2$ and $v \notin U_1$, then we have $u, v \in U_1 \cup U_2$ but $u + v \notin U_1$ or U_2 , thus not in $U_1 \cup U_2$. Hence $U_1 \cup U_2$ is not closed under addition, and we get a contradiction. So either $\forall u \in U_1, u \in U_2$ or $\forall v \in U_2, v \in U_1$, or $U_1 = U_2$.

Exercise 3.5: (1C-13)

Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution:

Let U_1, U_2, U_3 be subspaces of V, consider $u \in U_1$ and $v \in U_2$. If $U \cup V \neq U$ or V, then we can assume there exist $u \notin U_2$ and $v \notin U_1$. From the assumption we know $u + v \in \bigcup_{i=1}^3 U_i$ and u + v not in U_1 or U_2 , thus $u + v \in U_3$. For the same reason we know u + 2v and 2u + v are in U_3 , and because U_3 is closed under addition and additive inverse, we have $u, v \in U_3$. Hence $C_{U_1}U_1 \cap U_2$ and $C_{U_2}U_1 \cap U_2$ are contained by U_3 .

For $w \in U_1 \cap U_2$, taking $u \in \mathcal{C}_{U_1}U_1 \cap U_2$, then u + w in U_1 but not in U_2 , and we come back to the above condition. Thus we can prove $U_1 \cap U_2 \subset U_3$, which means U_3 contains U_1 and U_2 .

Now consider the consition that $U_1 \subset U_2$. If $U_2 \subset U_3$, then we are done. If $U_1 \subset U_3$, then consider $u \in \mathcal{C}_{U_2}U_1$ and $v \in \mathcal{C}_{U_3}U_1$, because the union of U_1, U_2, U_3 is closed under addition, $u + v \in \bigcup_{i=1}^3 U_i$. Evidently $u + v \notin U_1$, if $u + v \in U_2$, then we can prove $U_2 = U_3$; the other side is the same.

From the conditions above, we prove the proposition in the question.

Vector Spaces

Remark.

Problem 15-19 in Exercises 1C prove some properties of the addition in the set of subspaces of V, including additive identity, commutativity, associativity. From these propositions we know the group of subspaces of V is not an abel group.

Exercise 3.6: (1C-21)

Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of \mathbf{F}^5 such that $\mathbf{F}^5 = U \oplus W$.

Solution:

Let $W = \{(0,0,u,v,w) \in \mathbf{F}^5 : u,v,w \in \mathbf{F}\}$. Consider $(x,y,x+y,x-y,2x) \in U \cap W$, then we have x=y=0, which means $U \cap W = \{0\}$. Hence $\mathbf{F}^5 = U \oplus W$ by 1.45.

Problem 20-22 in Exercises 1C are the same type. To solve these problems, we first write a subspace W that the union of W and the given subspaces is the vector space asked for. Then prove the intersection of the subspaces is $\{0\}$.

Exercise 3.7: (1C-23)

Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U$$
 and $V = V_2 \oplus U$,

then $V_1 = V_2$.

Solution:

Consider the example in Exercise 3.6. We change W into $W' = \{(0, z, 0, v, w) \in \mathbf{F}^5 : u, v, w \in \mathbf{F}\}$, and it is easy to verify $\mathbf{F}^5 = U \oplus W'$. Hence we give a counterexample.

Exercise 3.8: (1C-24)

A function $f: \mathbf{R} \to \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f: \mathbf{R} \to \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in R$. Let V_e denote the set of real-valued even functions on \mathbf{R} and let V_o denote the set of real-valued odd functions on \mathbf{R} . Show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

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Solution:

For $h: \mathbf{R} \to \mathbf{R} \in \mathbf{R}^{\mathbf{R}}$, suppose h(x) = f(x) + g(x), where $f \in V_e$ and $g \in V_o$. Then we must verify h(-x) = f(-x) + g(-x) = f(x) - g(x). From the above discription, let

$$f(x) = \frac{h(x) + h(-x)}{2} \tag{1.13}$$

and

$$g(x) = \frac{h(x) - h(-x)}{2}. (1.14)$$

Then we have $\mathbf{R}^{\mathbf{R}} = V_e + V_o$. Consider $f_0 \in V_e \cap V_o$,

$$f_0(x) = f_0(-x) = -f_0(x), \ \forall x \in \mathbf{R}.$$
 (1.15)

Thus
$$f_0(x) \equiv 0$$
, $V_e \cap V_o = \{0\}$. By 1.45, $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

From Exercise 3.8 we know every real-valued function can be divided into the sum of an even function and an odd function.