Lecture 6:

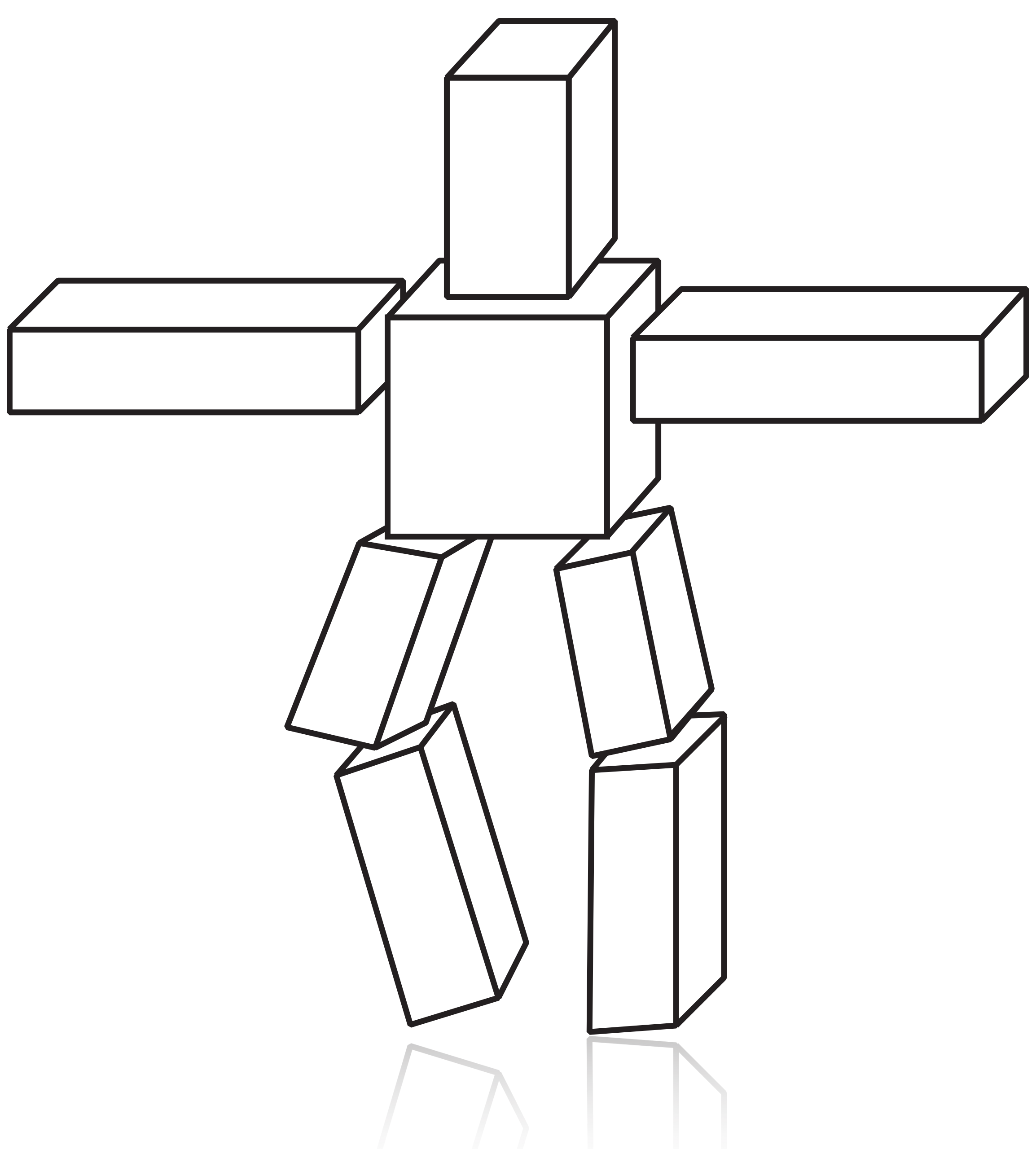
Introduction to Geometry

Computer Graphics

CMU 15-462/15-662, Fall 2015

Increasing the complexity of our models

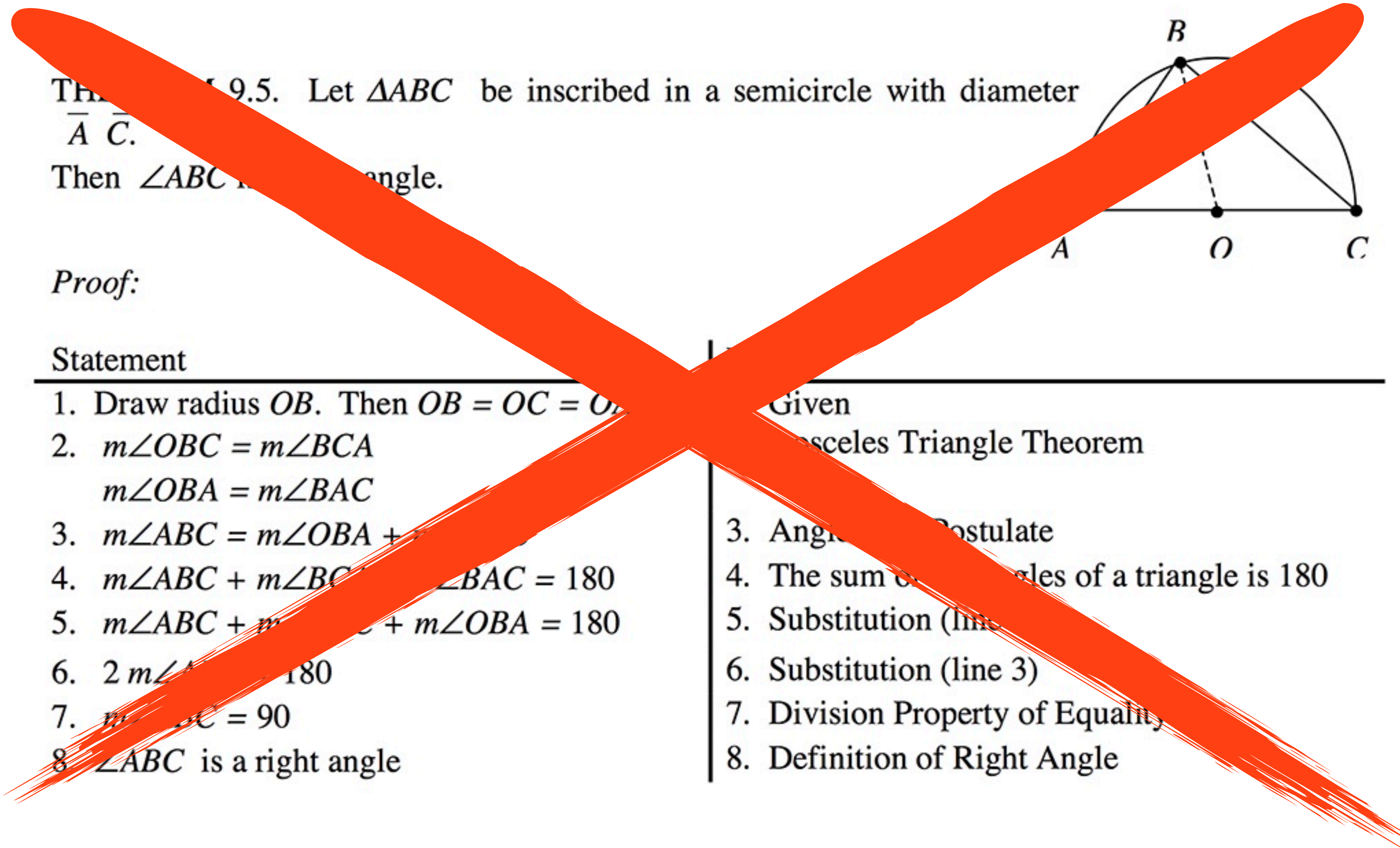
Transformations



Geometry

Materials, lighting, ...

What is geometry?



TH[illegible] 9.5. Let ΔABC be inscribed in a semicircle with diameter $\overline{A}\ \overline{C}$.
Then $\angle ABC$ [illegible] angle.

Proof:

Statement	[illegible]
1. Draw radius OB. Then $OB = OC = O$[illegible]	[illegible] Given
2. $m\angle OBC = m\angle BCA$ $m\angle OBA = m\angle BAC$	[illegible]sceles Triangle Theorem
3. $m\angle ABC = m\angle OBA +$ [illegible]	3. Angl[illegible]ostulate
4. $m\angle ABC + m\angle BC$[illegible]$BAC = 180$	4. The sum [illegible]gles of a triangle is 180
5. $m\angle ABC +$ [illegible] $+ m\angle OBA = 180$	5. Substitution (li[illegible]
6. $2\,m\angle$[illegible] 180	6. Substitution (line 3)
7. [illegible]$C = 90$	7. Division Property of Equalit[illegible]
8 $\angle ABC$ is a right angle	8. Definition of Right Angle

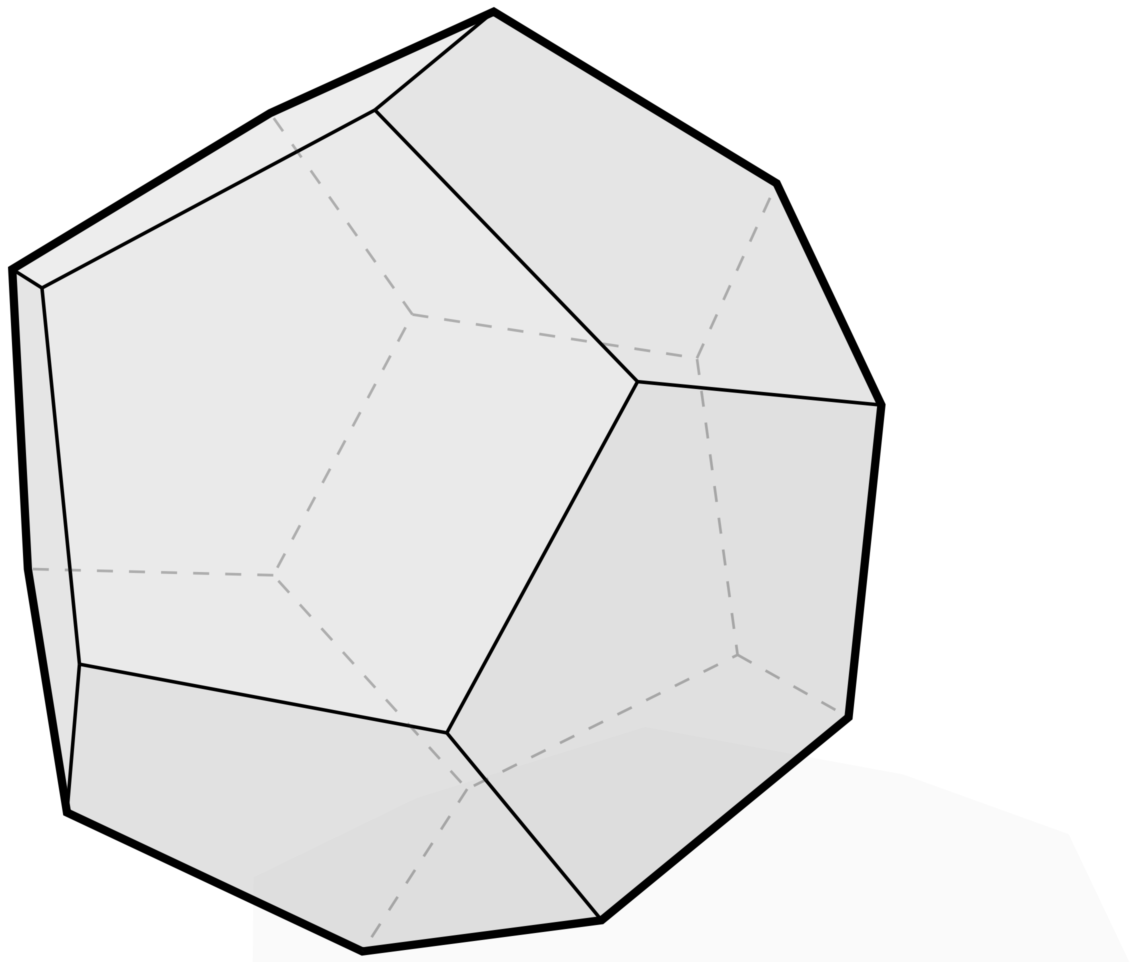
Ceci n'est pas géométrie.

What is geometry?

"Earth" "measure"

ge•om•et•ry /jēˈämətrē/ *n.*

1. The study of shapes, sizes, patterns, and positions.
2. The study of spaces where some quantity (lengths, angles, etc.) can be *measured*.

Plato: "...the earth is in appearance like one of those balls which have leather coverings in twelve pieces..."

How can we describe geometry?

IMPLICIT

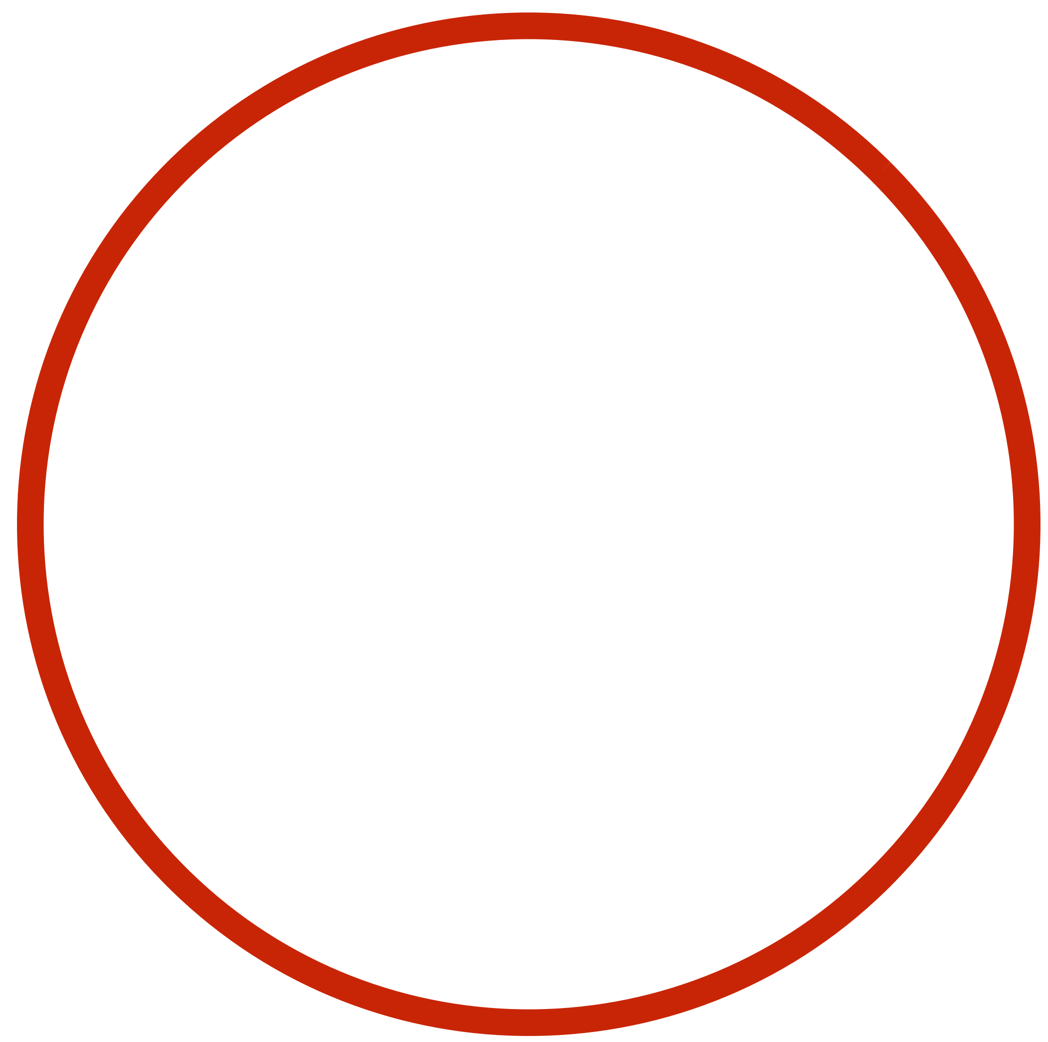
$$x^2 + y^2 = 1$$

LINGUISTIC

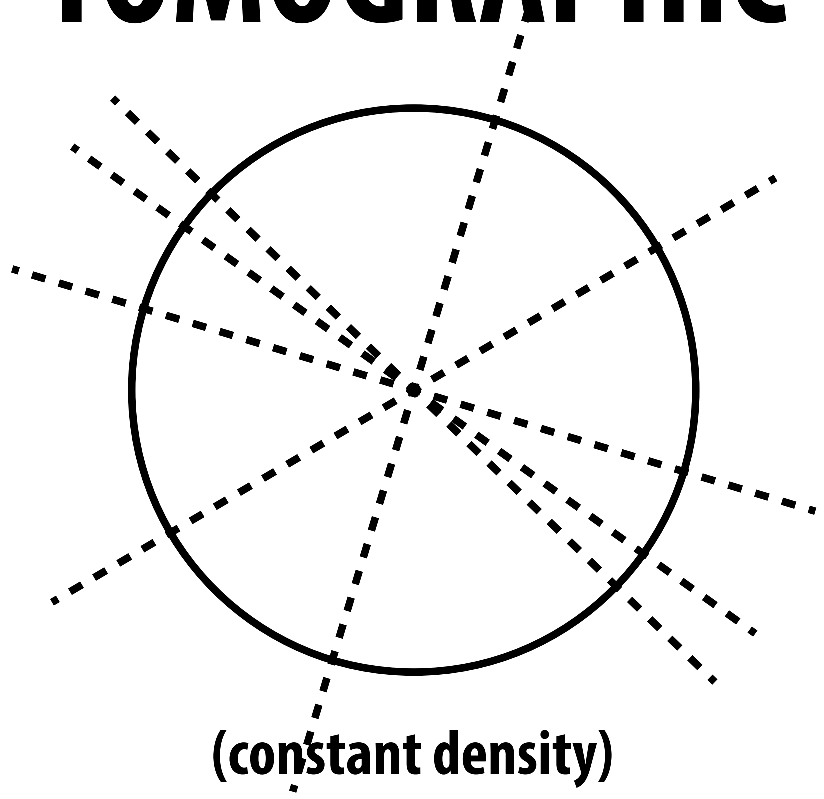
"unit circle"

EXPLICIT

$$(\underbrace{\cos\theta}_{x}, \underbrace{\sin\theta}_{y})$$

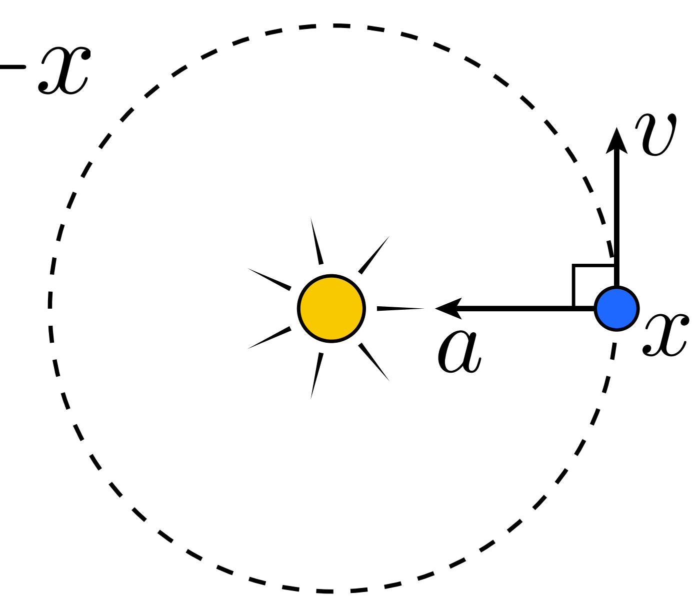


TOMOGRAPHIC

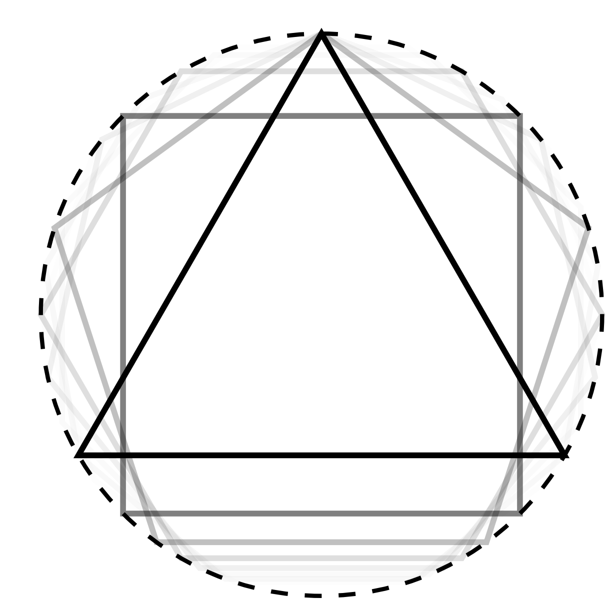


DYNAMIC

$$\frac{d^2}{dt^2}x = -x$$



DISCRETE

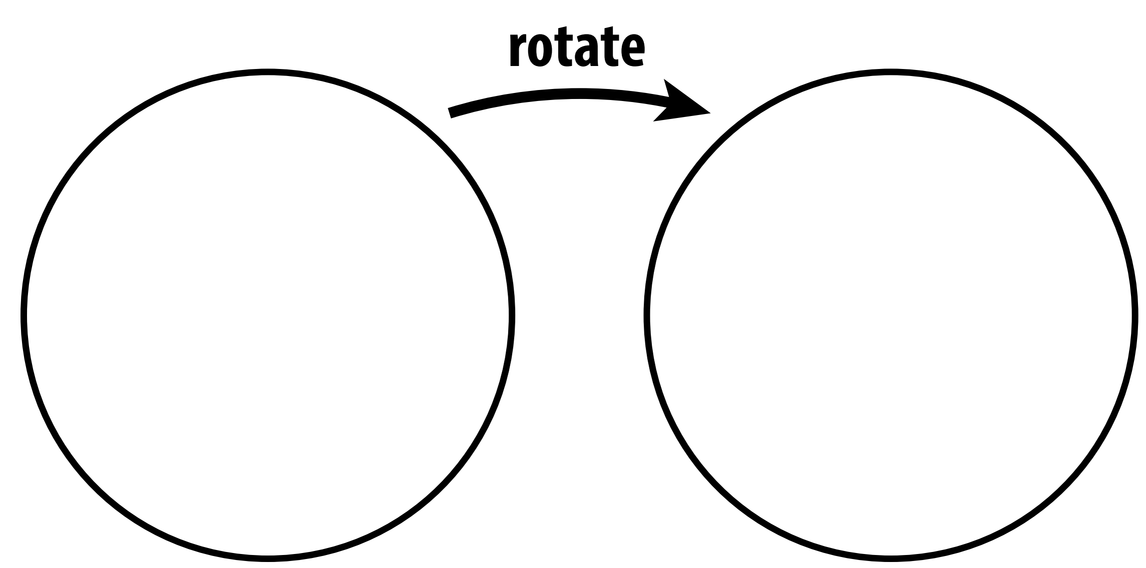


$n \to \infty$

CURVATURE

$$\kappa = 1$$

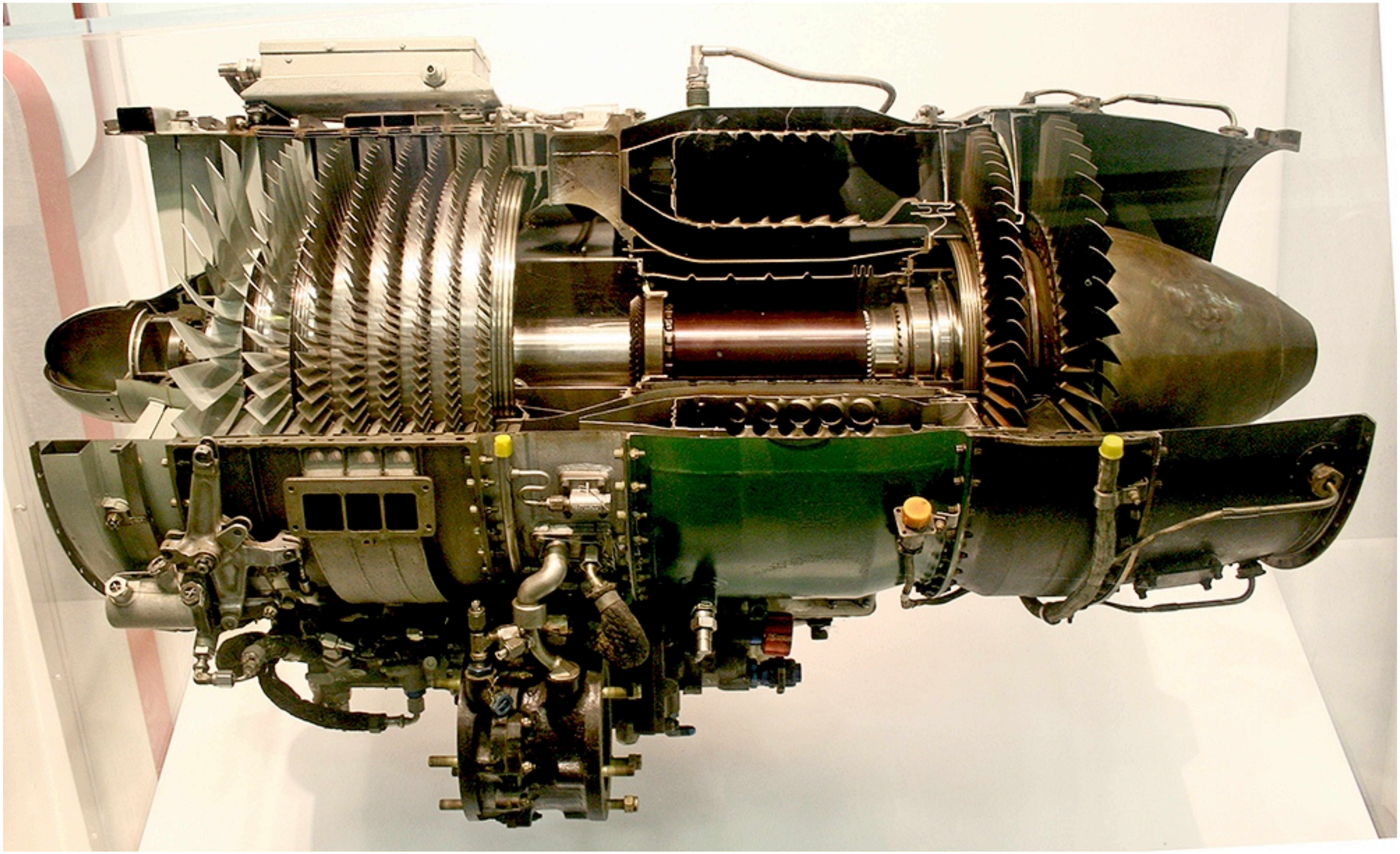
SYMMETRIC



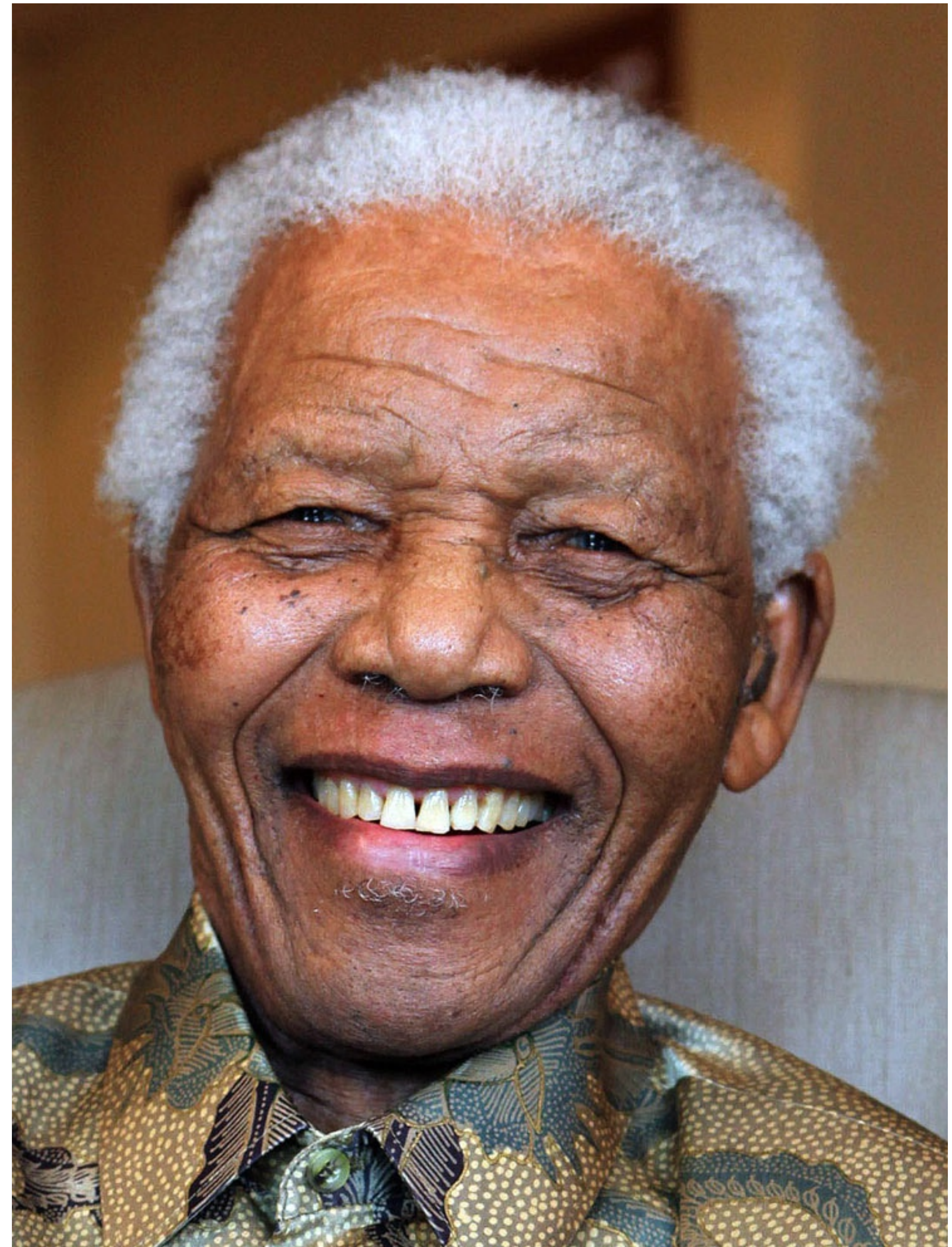
Given all these options, what's the best way to encode geometry on a computer?

Examples of geometry

Examples of geometry



Examples of geometry



Examples of geometry

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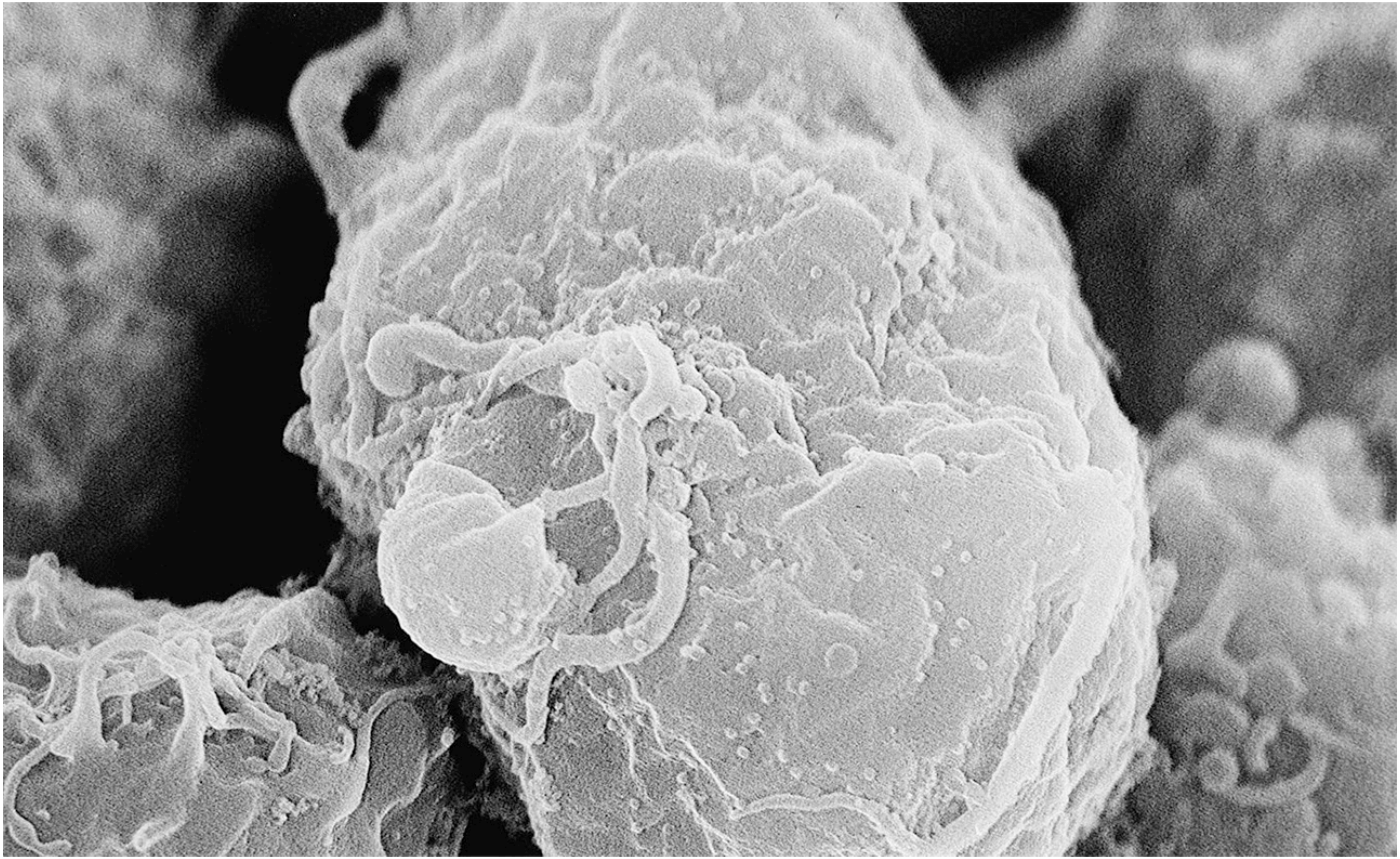
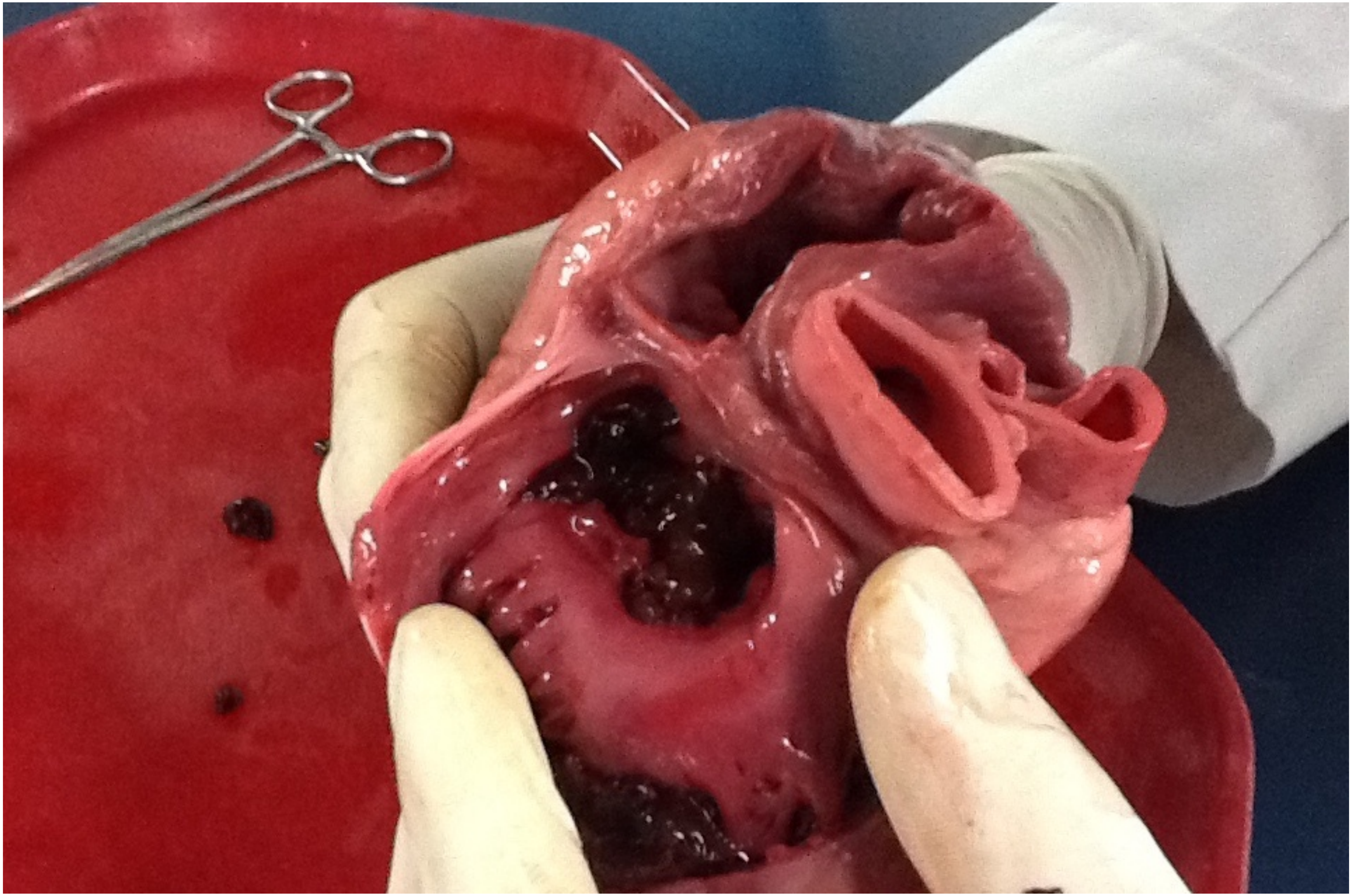


Examples of geometry



Examples of geometry

No one "best" choice—geometry is hard!

"I hate meshes.
I cannot believe how hard this is.
Geometry is hard."

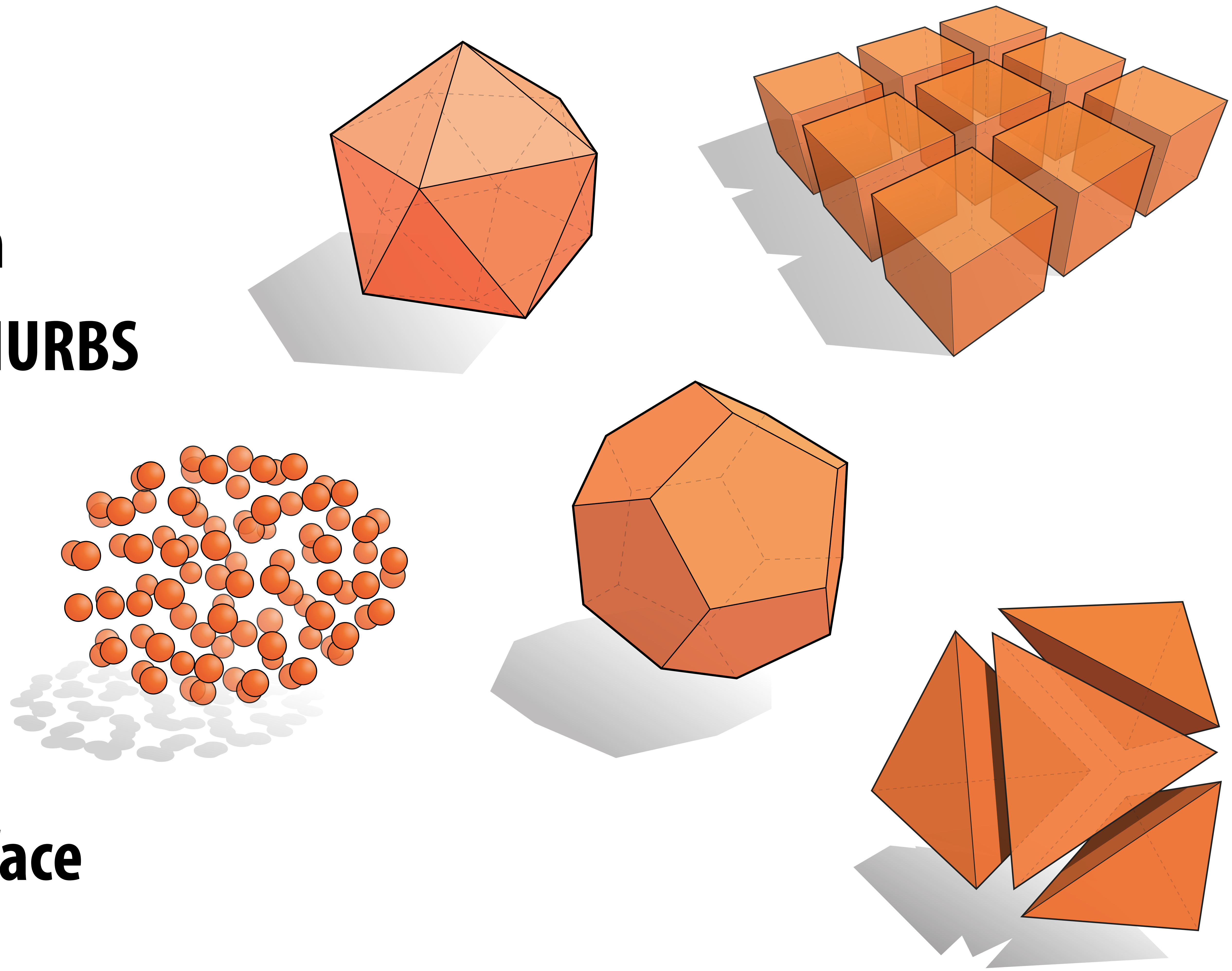
—David Baraff

Senior Research Scientist
Pixar Animation Studios

Slide cribbed from Jeff Erickson.

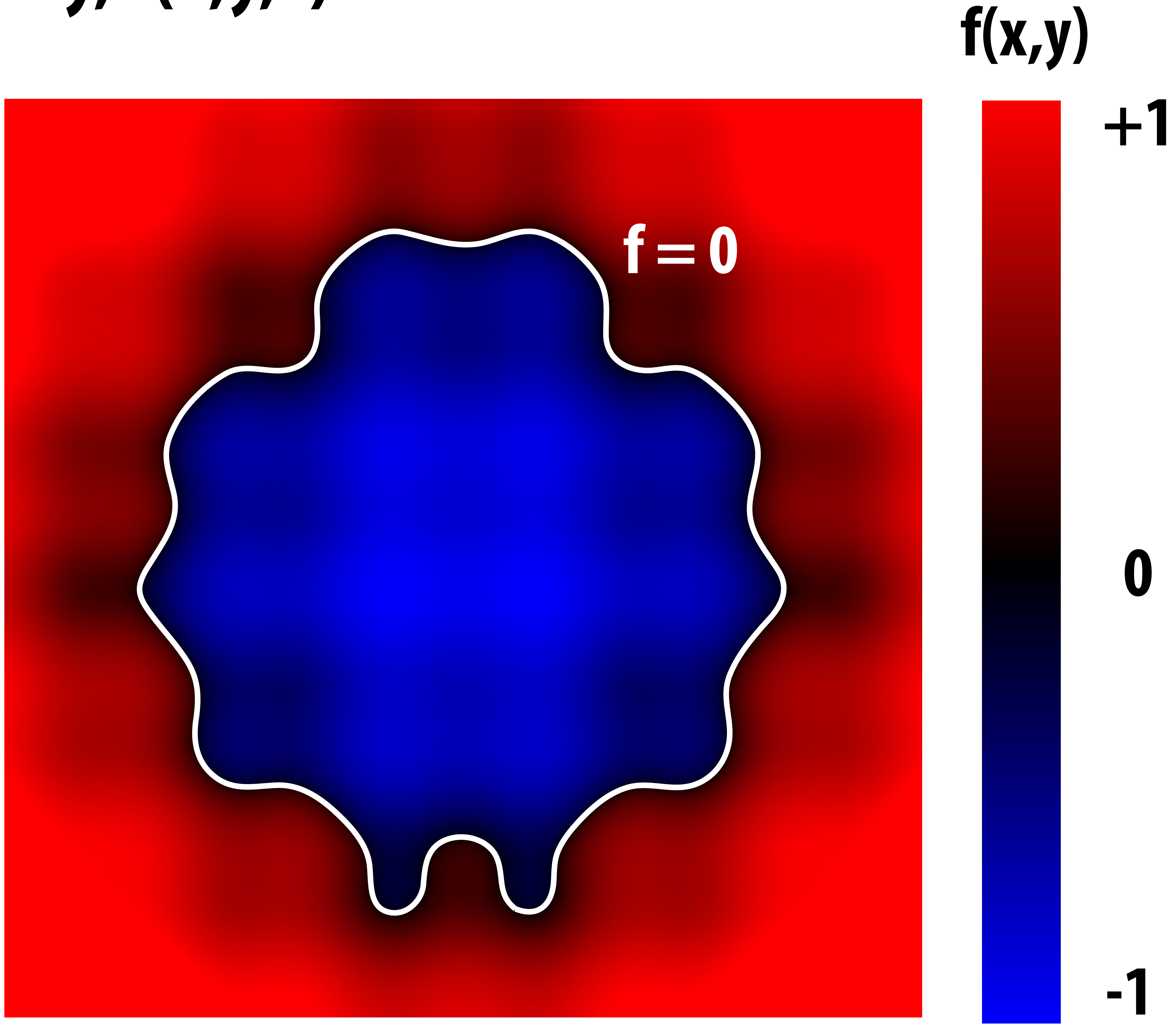
Many ways to digitally encode geometry

- **EXPLICIT**
  - point cloud
  - polygon mesh
  - subdivision, NURBS
  - L-systems
  - ...
- **IMPLICIT**
  - level set
  - algebraic surface
  - ...
- **Each choice best suited to a different task/type of geometry**



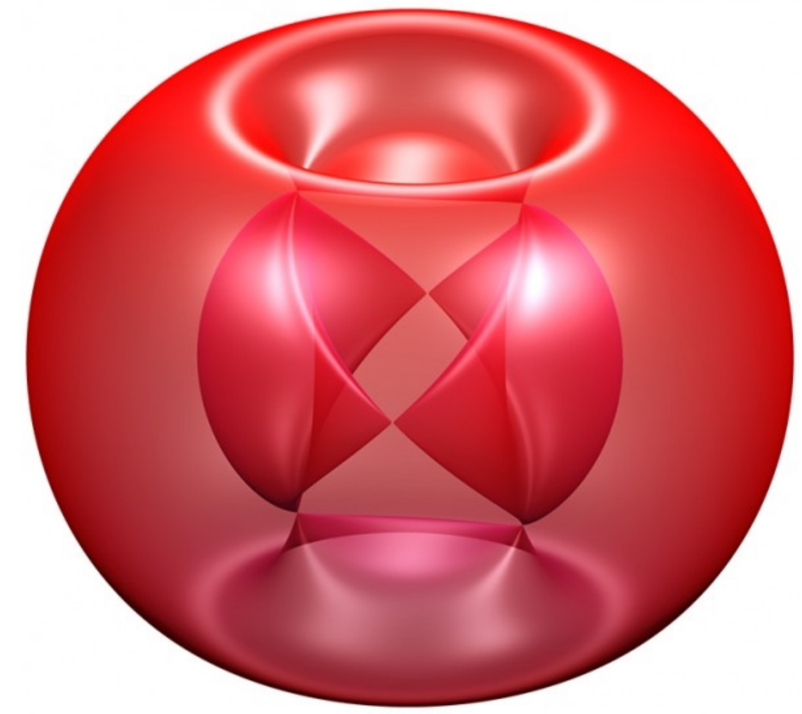
"Implicit" Representations of Geometry

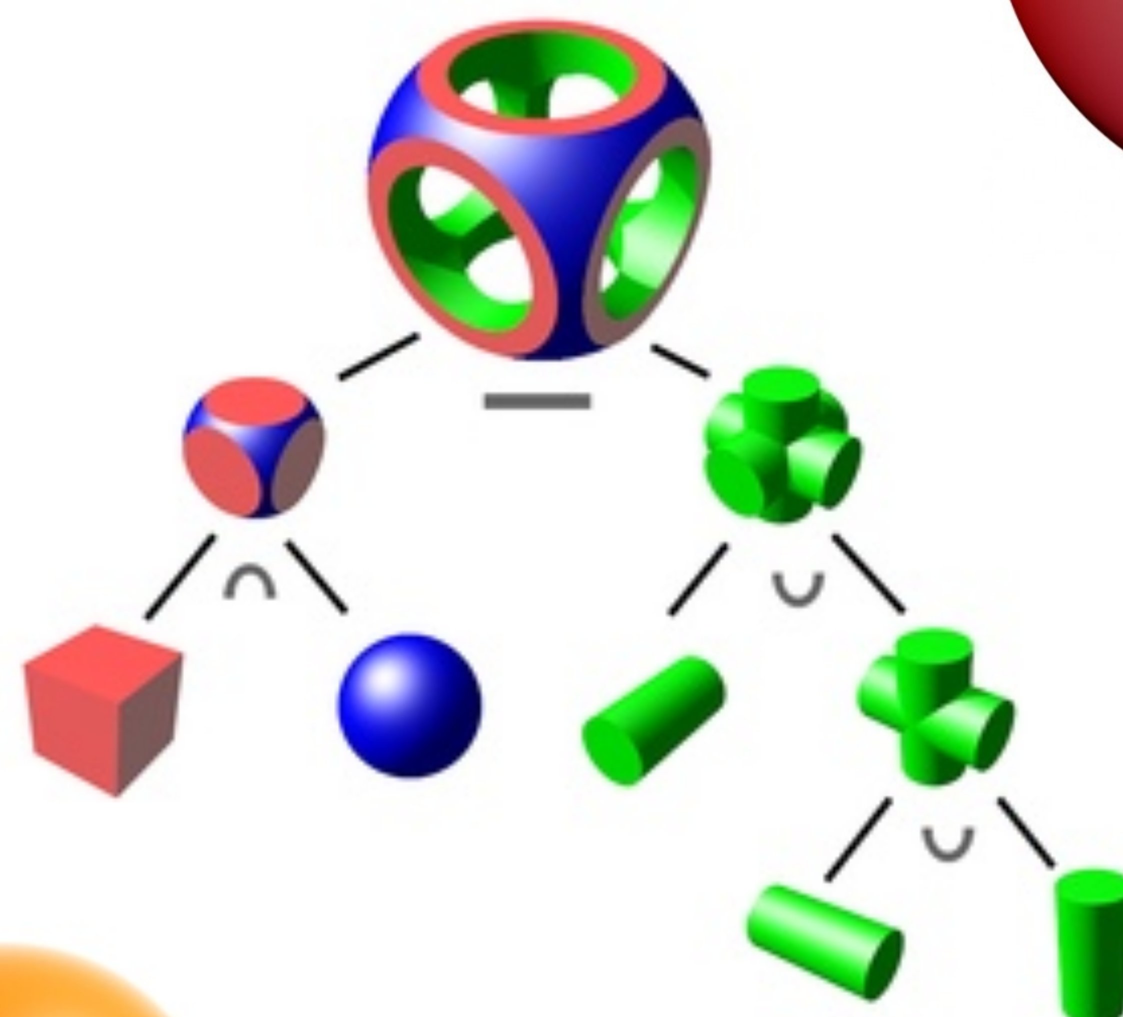
- Points aren't known directly, but satisfy some relationship
- E.g., unit sphere is all points x such that $x^2+y^2+z^2=1$
- More generally, $f(x,y,z) = 0$



Many implicit representations in graphics

- algebraic surfaces
- constructive solid geometry
- level set methods
- blobby surfaces
- fractals
- ...





(Will see some of these a bit later.)

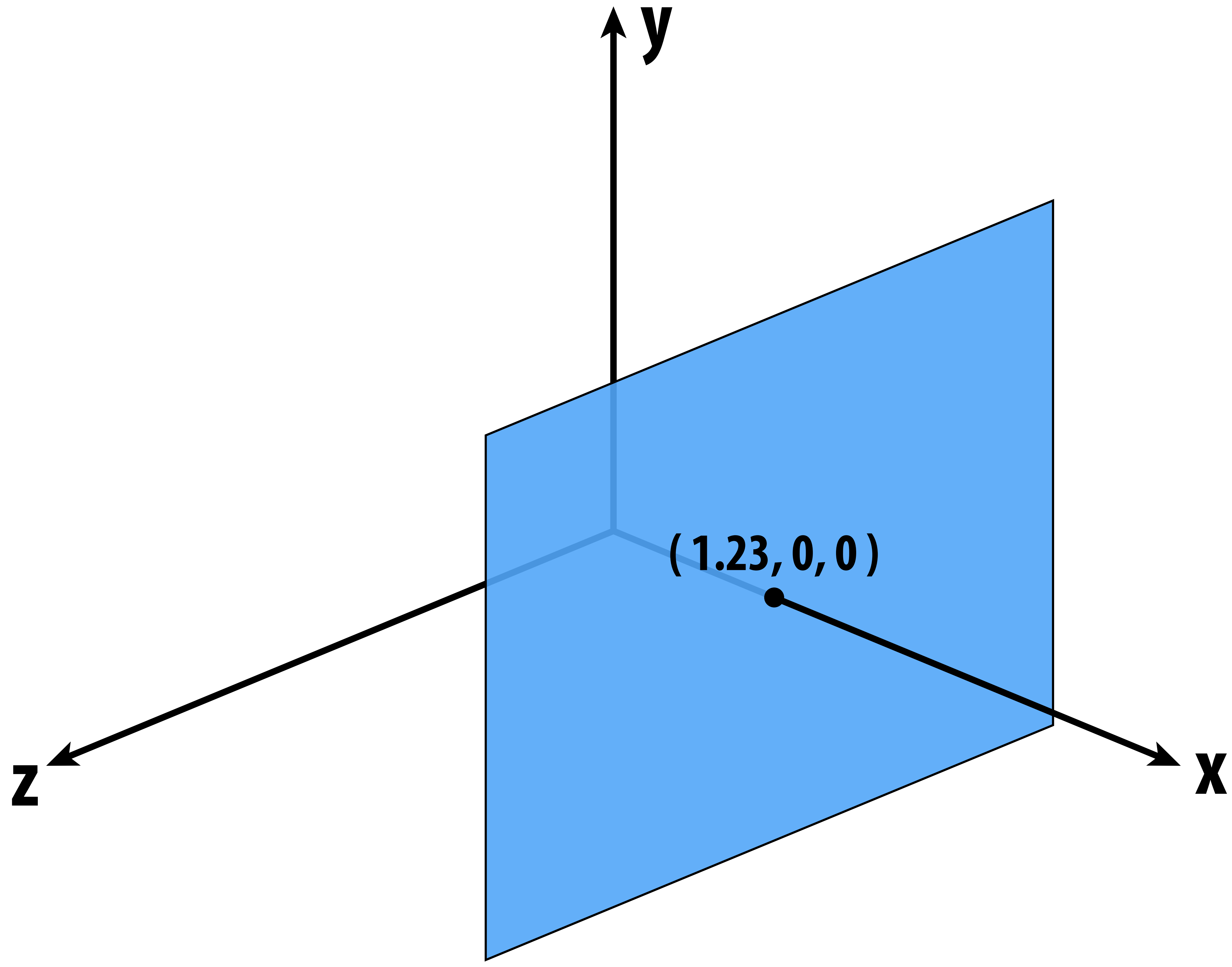
But first, let's play a game:

I'm thinking of an implicit surface $f(x,y,z)=0$.

Find *any* point on it.

Give up?

My function was f(x,y,z) = -1.23 (a plane):



Implicit surfaces make some tasks hard (like sampling).

Let's play another game.

I have a new surface $f(x,y,z) = x^2 + y^2 + z^2 - 1$

I want to see if a point is *inside* it.

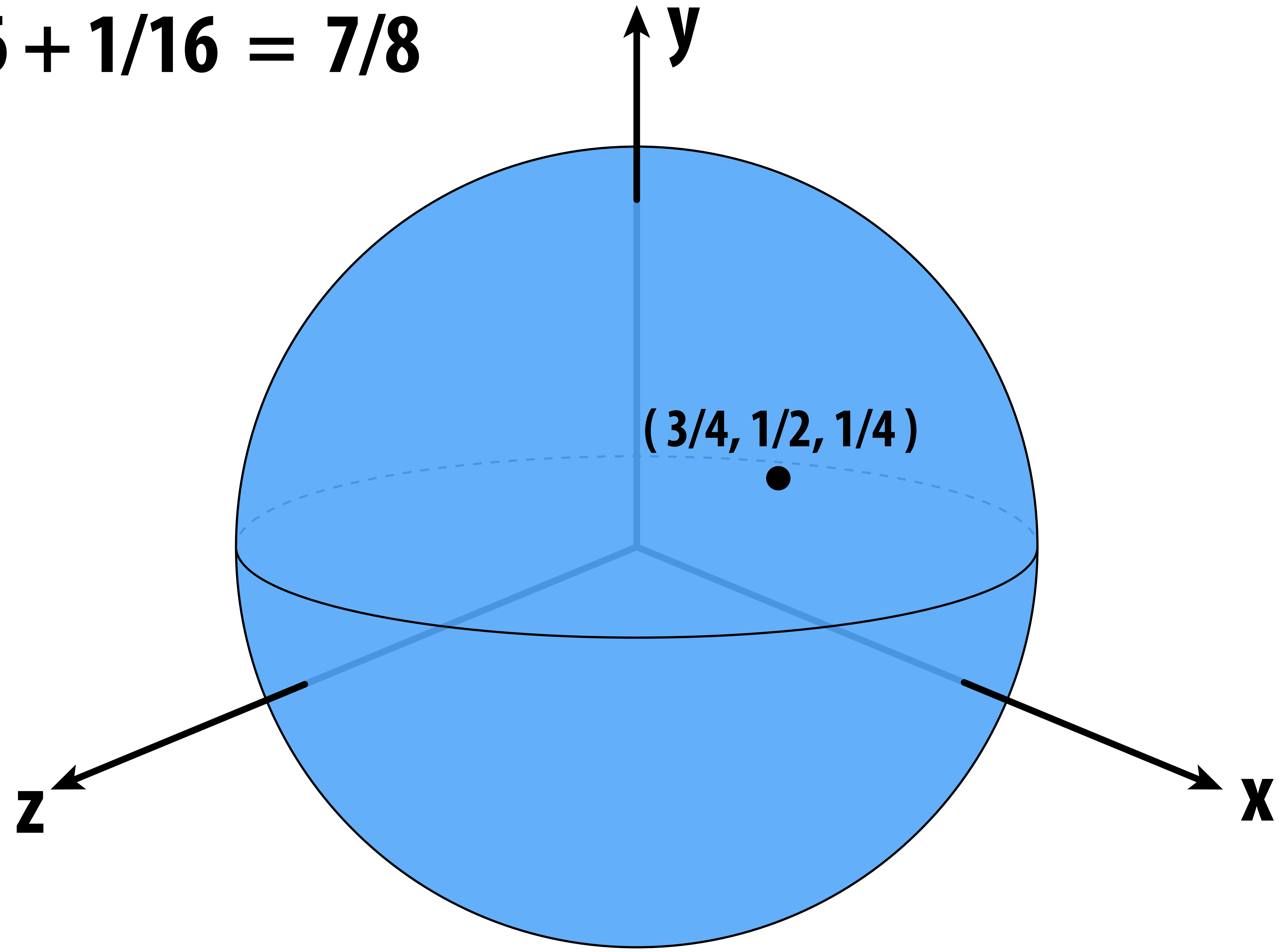
Check if this point is inside the unit sphere

How about the point (3/4, 1/2, 1/4)?

9/16 + 4/16 + 1/16 = 7/8

7/8 < 1

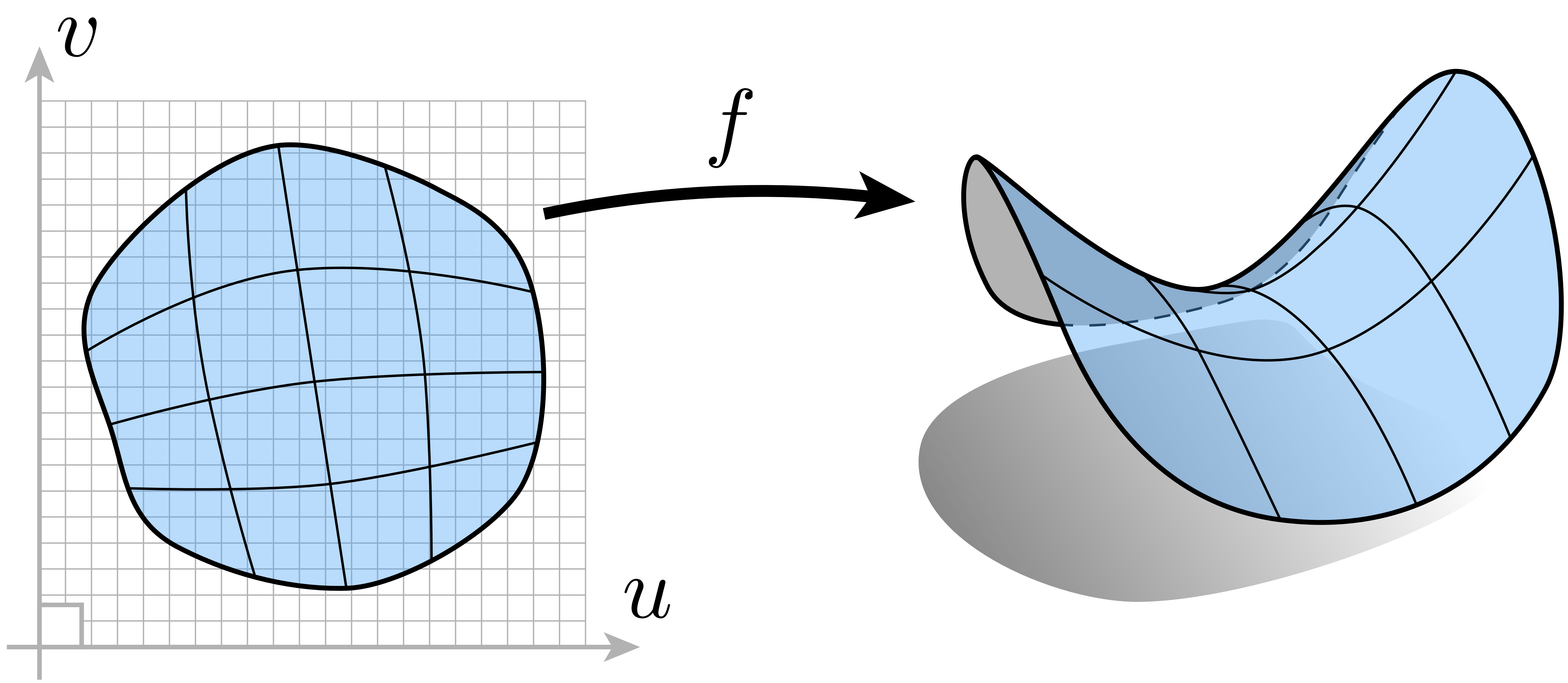
YES.



Implicit surfaces make other tasks easy (like inside/outside tests).

"Explicit" Representations of Geometry

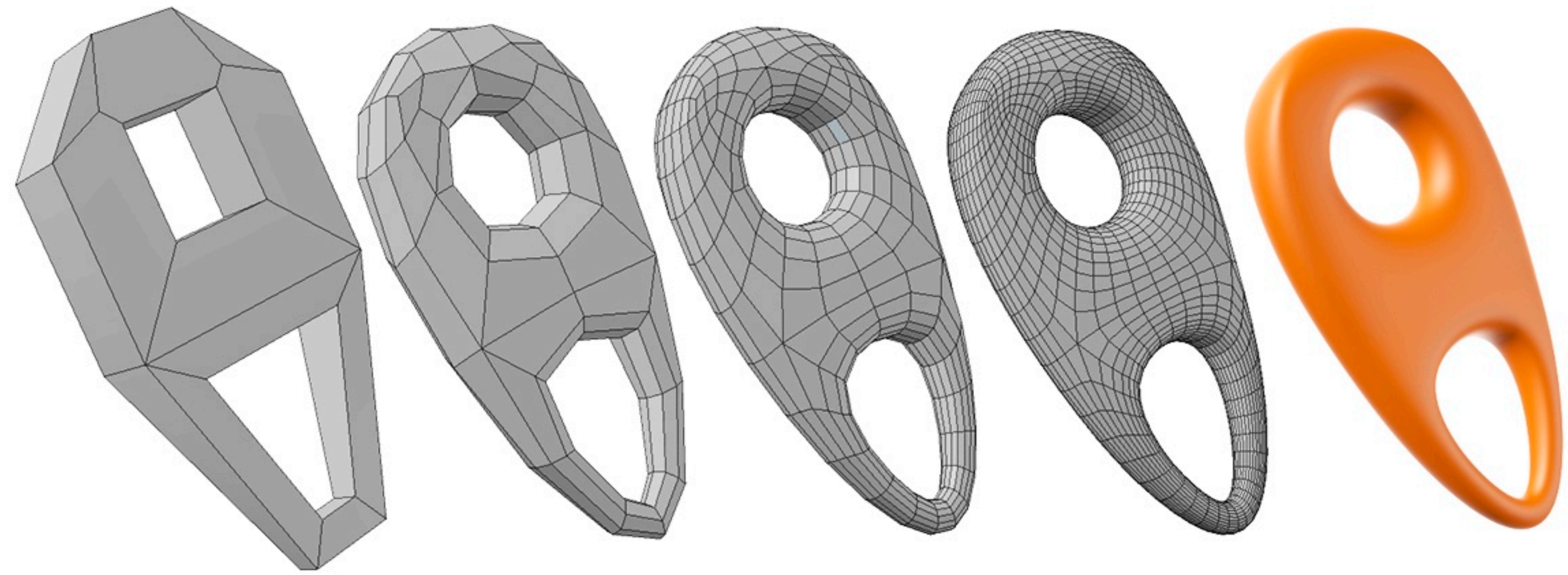
- **All points are given directly**
- **E.g., points on sphere are** $(\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)),$
  $\text{for } 0 \leq u < 2\pi \text{ and } 0 \leq v \leq \pi$
- **More generally:** $f : \mathbb{R}^2 \to \mathbb{R}^3; (u, v) \mapsto (x, y, z)$

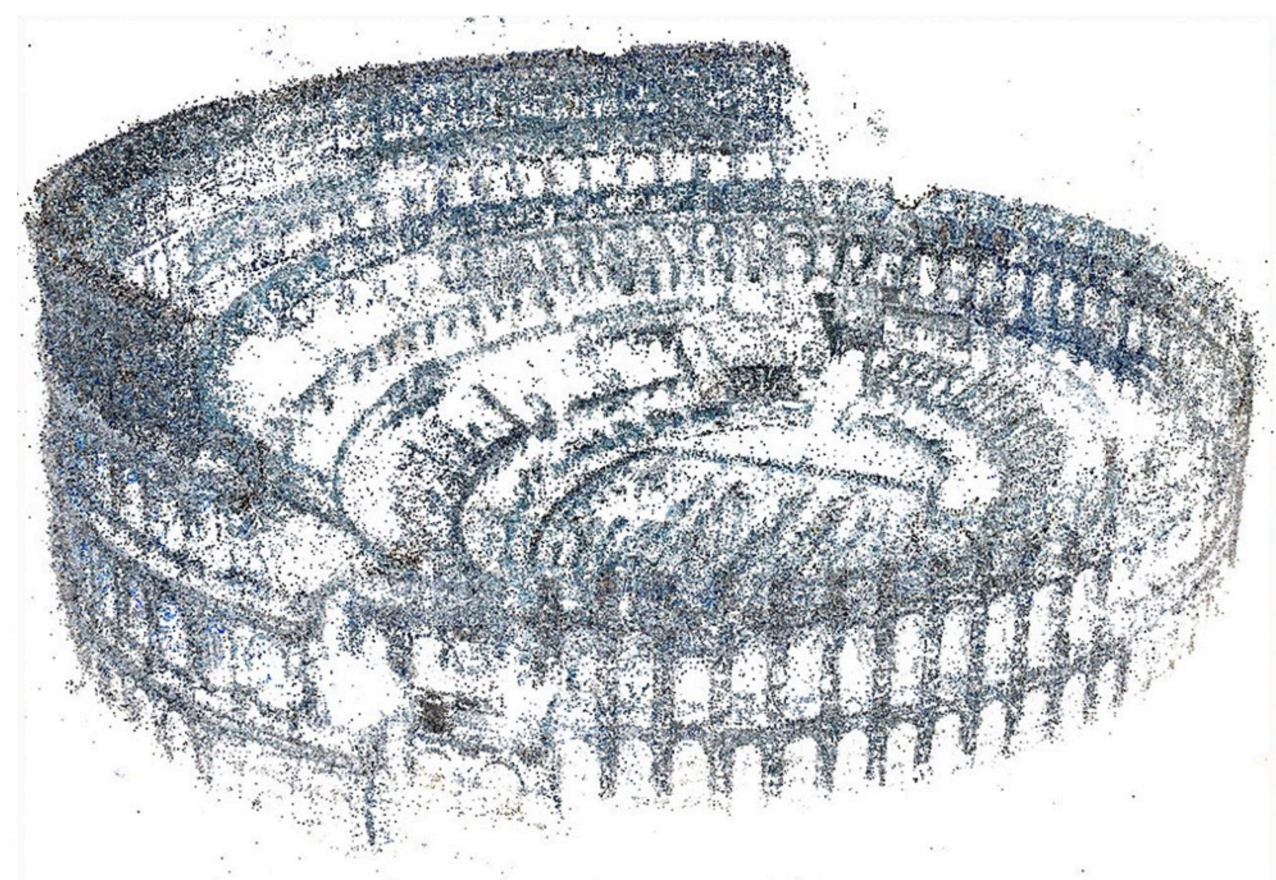


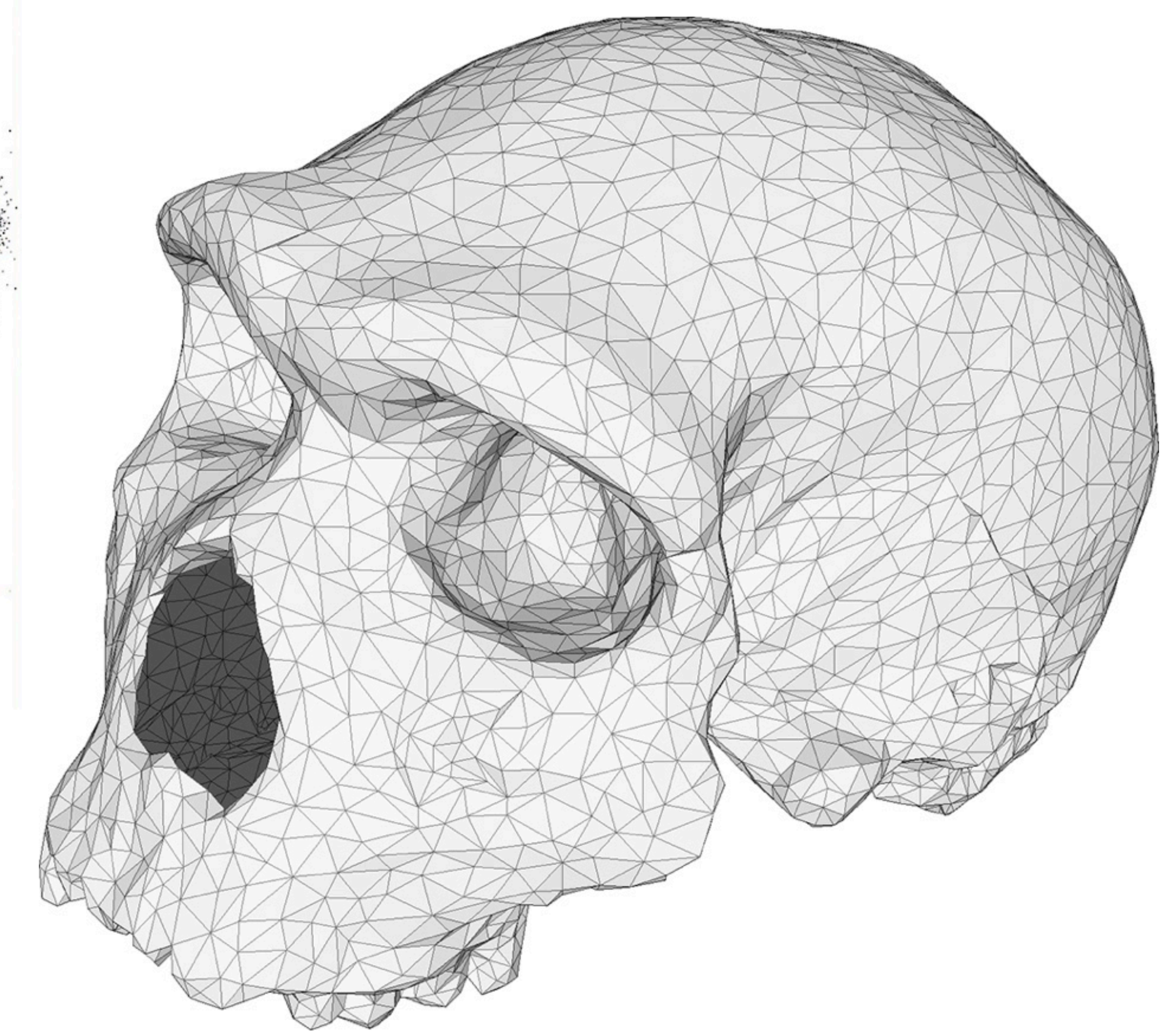
- **(Might have a bunch of these maps, e.g., one per triangle.)**

Many explicit representations in graphics

- triangle meshes
- polygon meshes
- subdivision surfaces
- NURBS
- point clouds
- ...







(Will see some of these a bit later.)

But first, let's play a game:

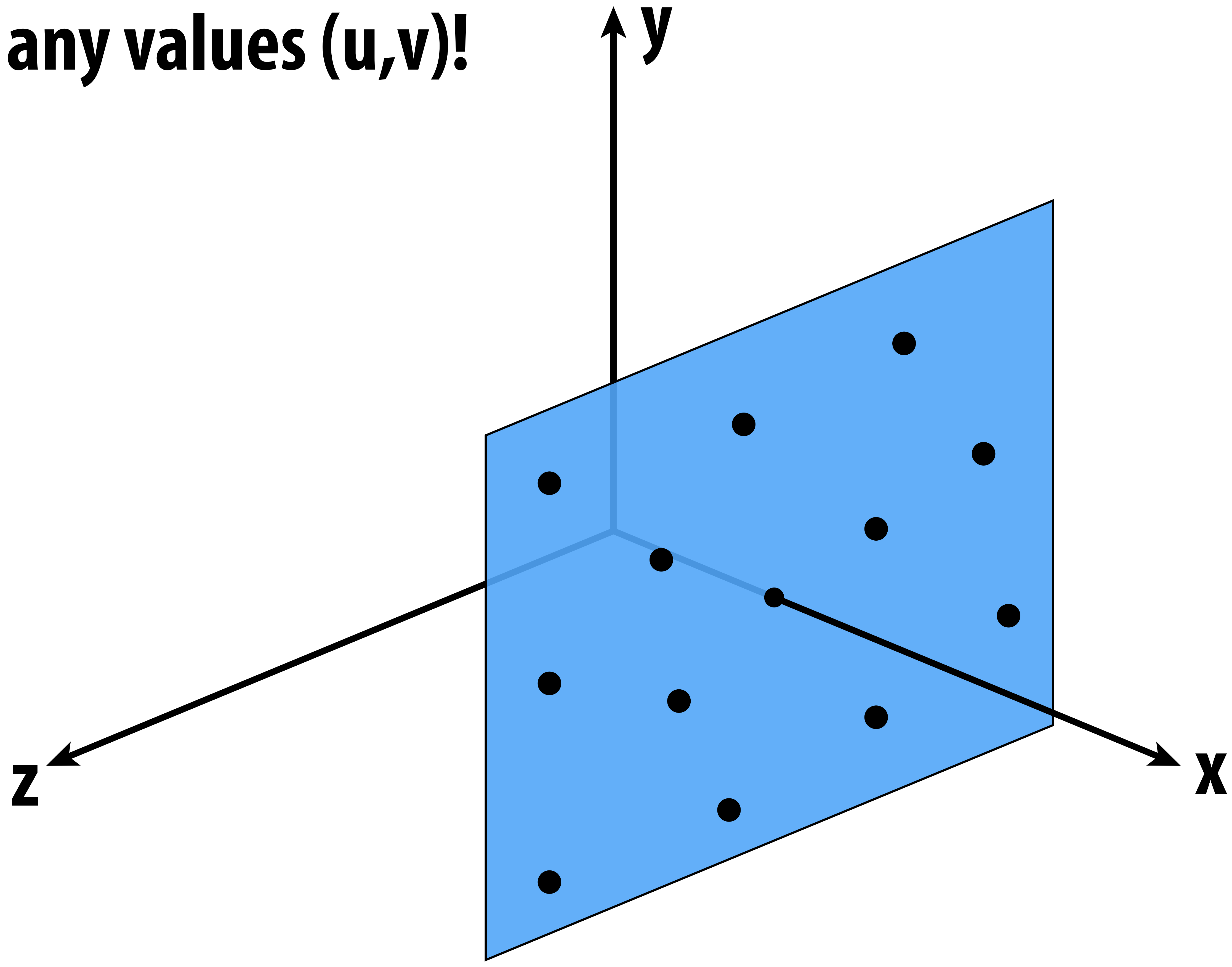
I'll give you an explicit surface.

You give me some points on it.

Sampling an explicit surface

My surface is f(u, v) = (1.23, u, v).

Just plug in any values (u,v)!



Explicit surfaces make some tasks easy (like sampling).

Let's play another game.

I have a new surface f(u,v).

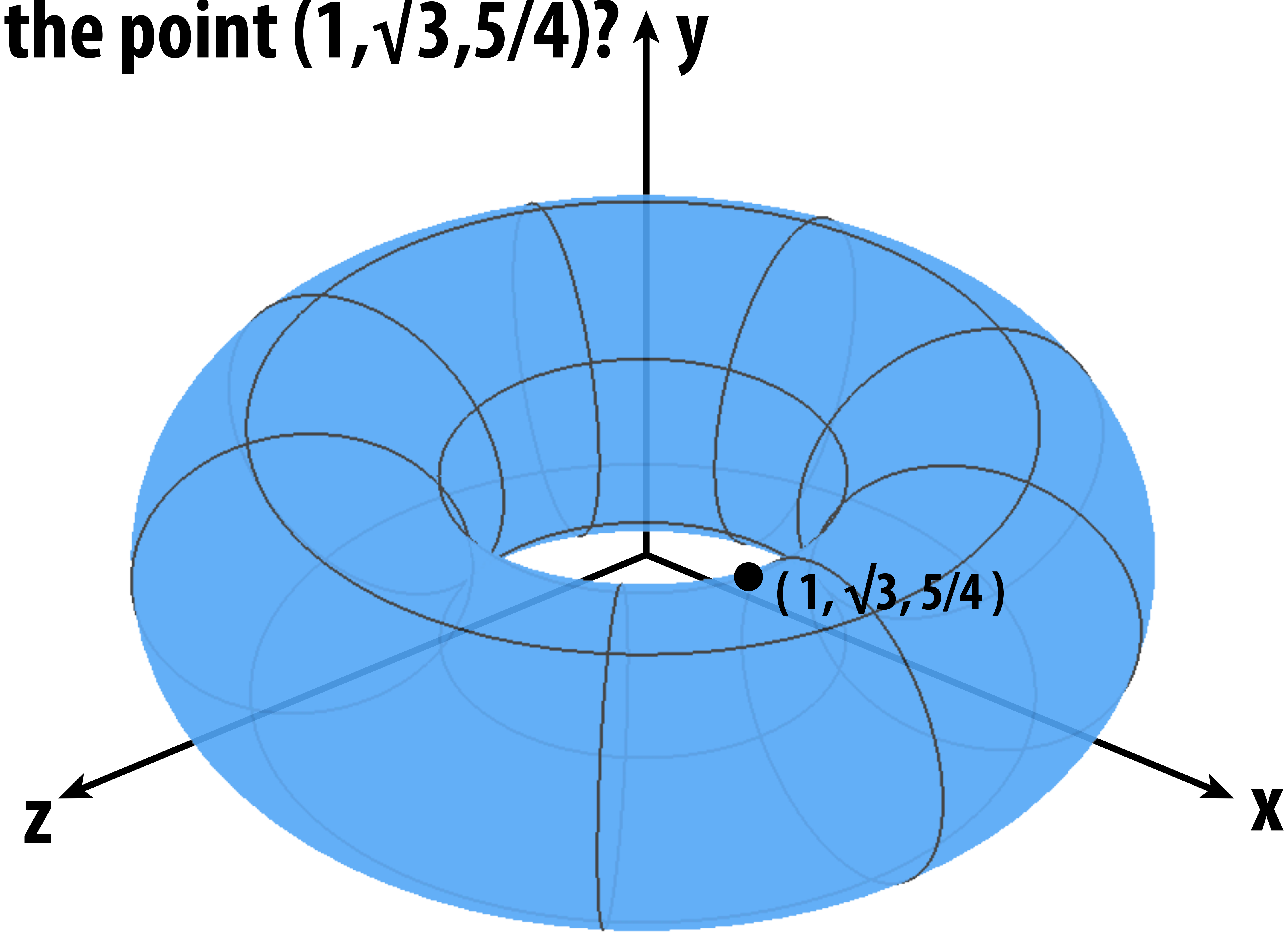
I want to see if a point is *inside* it.

Check if this point is inside the torus

My surface is $f(u,v) = (\,2+\cos(u))\cos(v),\, 2+\cos(u))\sin(v),\, \sin(u)\,)$

How about the point $(1,\sqrt{3},5/4)$?

...NO!



Explicit surfaces make other tasks hard (like inside/outside tests).

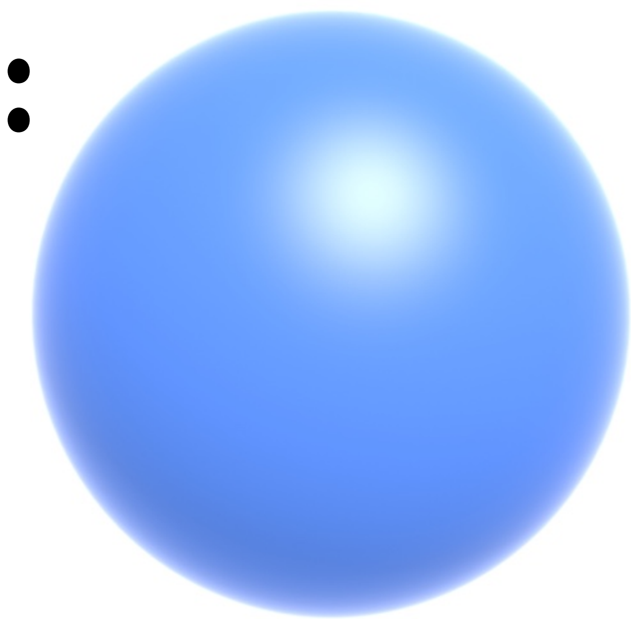
CONCLUSION:
Some representations work better than others—depends on the task!

Different representations will also be better suited to different types of geometry.

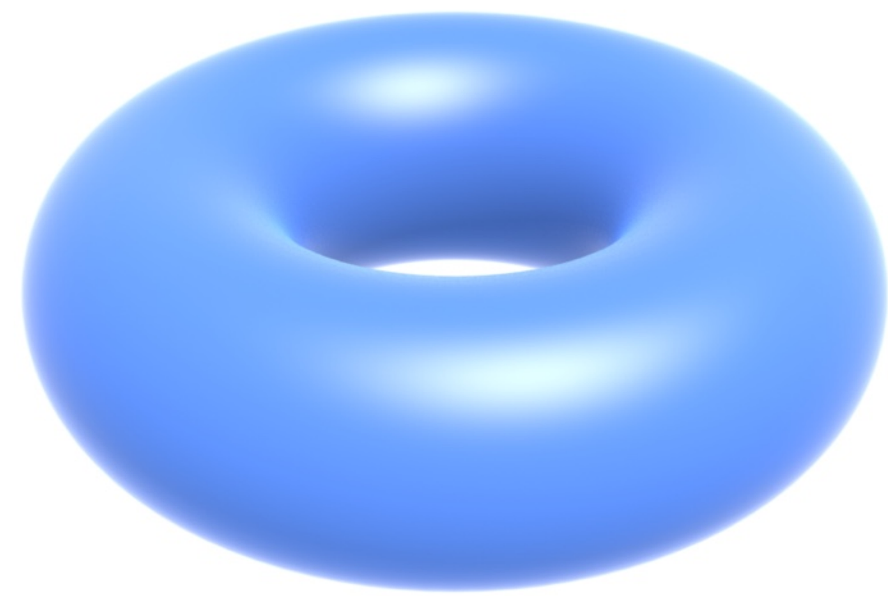
Let's take a look at some common representations used in computer graphics.

Algebraic Surfaces (Implicit)

- **Surface is zero set of a polynomial in x, y, z ("algebraic variety")**
- **Examples:**



$$x^2+y^2+z^2=1$$



$$(R-\sqrt{x^2+y^2})^2+z^2=r^2$$

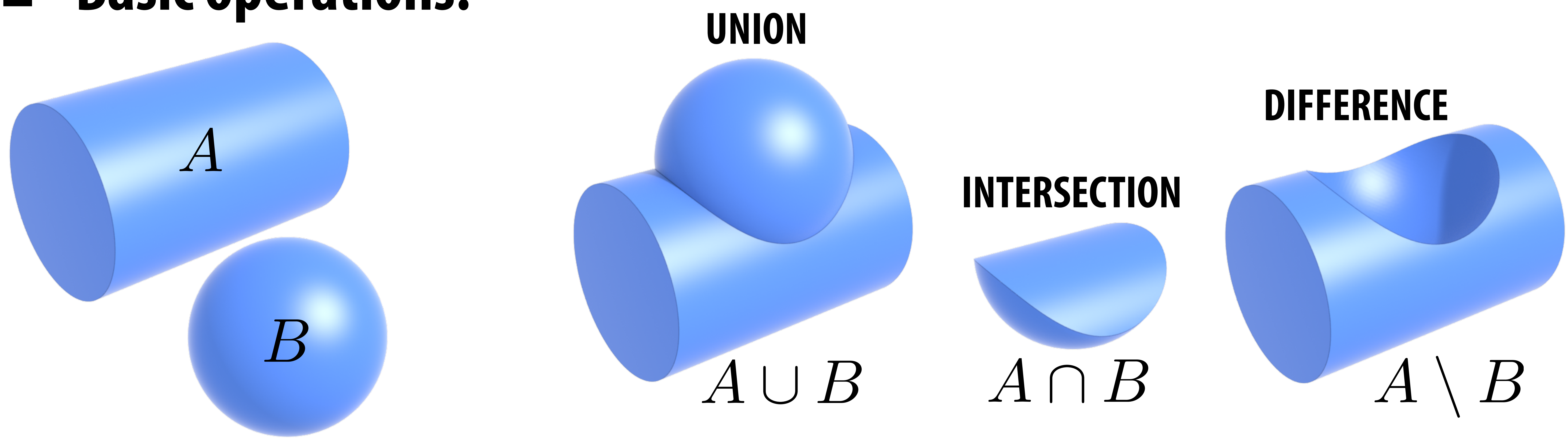
$$(x^2+\frac{9y^2}{4}+z^2-1)^3 = x^2z^3+\frac{9y^2z^3}{80}$$

- **What about more complicated shapes?**

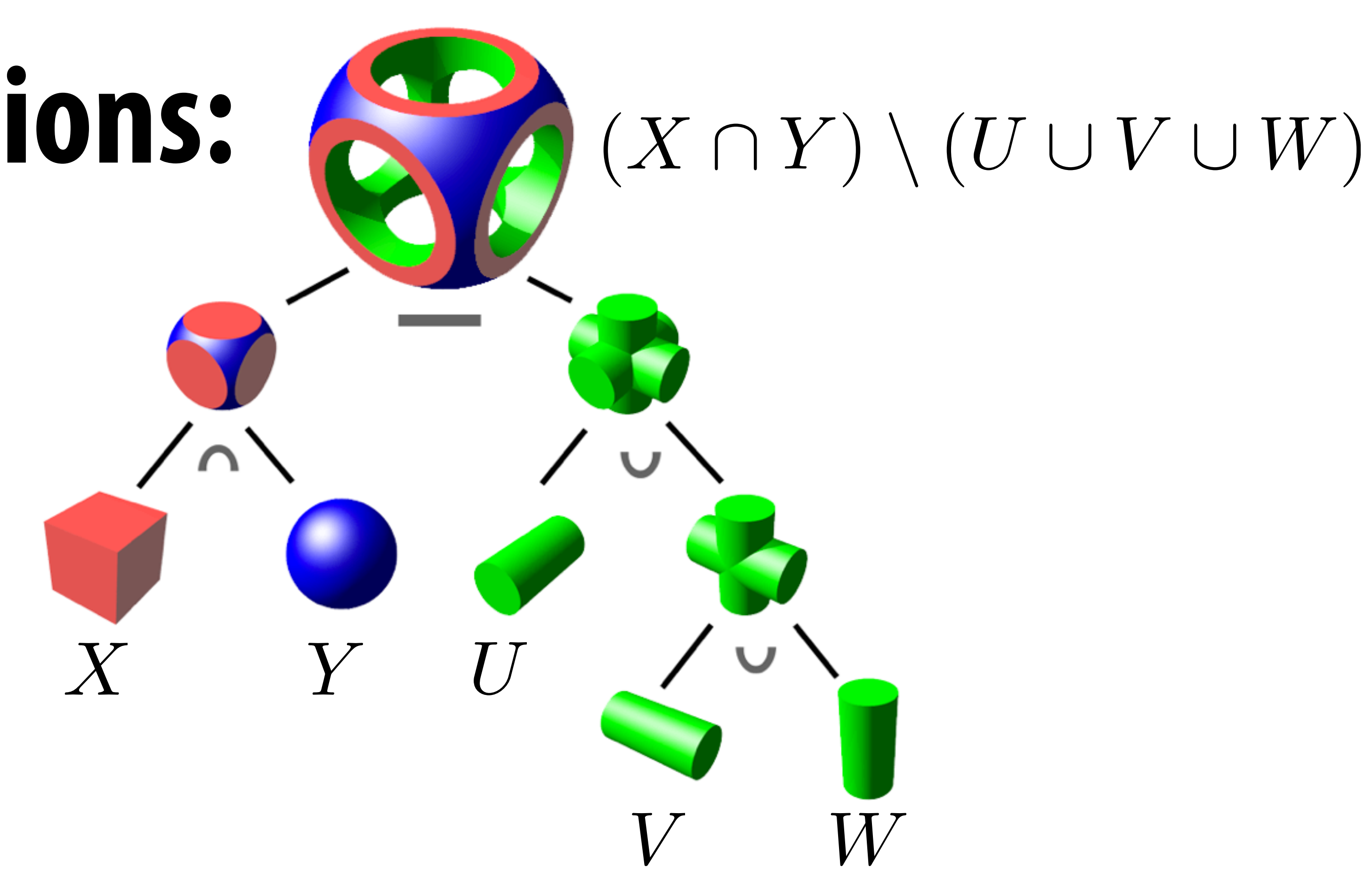
- **Very hard to come up with polynomials!**

Constructive Solid Geometry (Implicit)

- Build more complicated shapes via Boolean operations
- Basic operations:

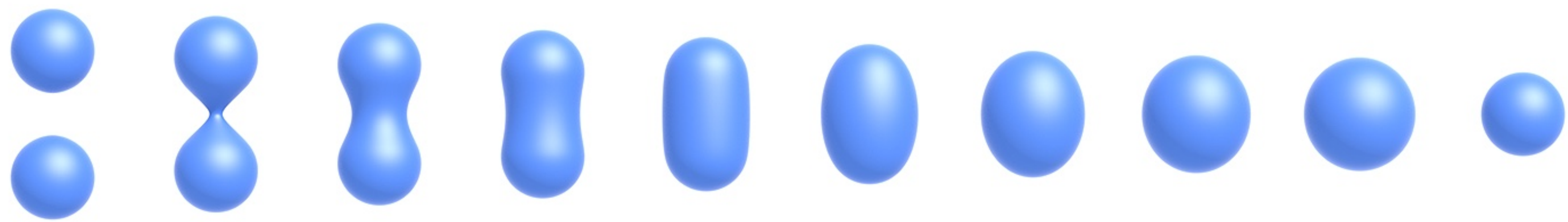


- Then chain together expressions:

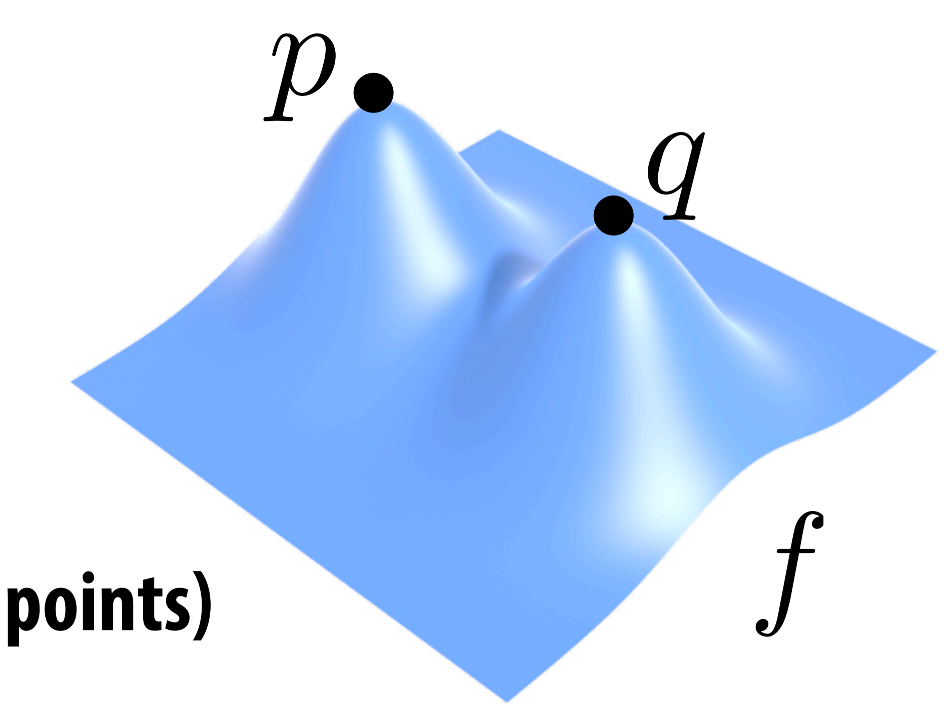


Blobby Surfaces (Implicit)

- **Instead of Booleans, gradually blend surfaces together:**

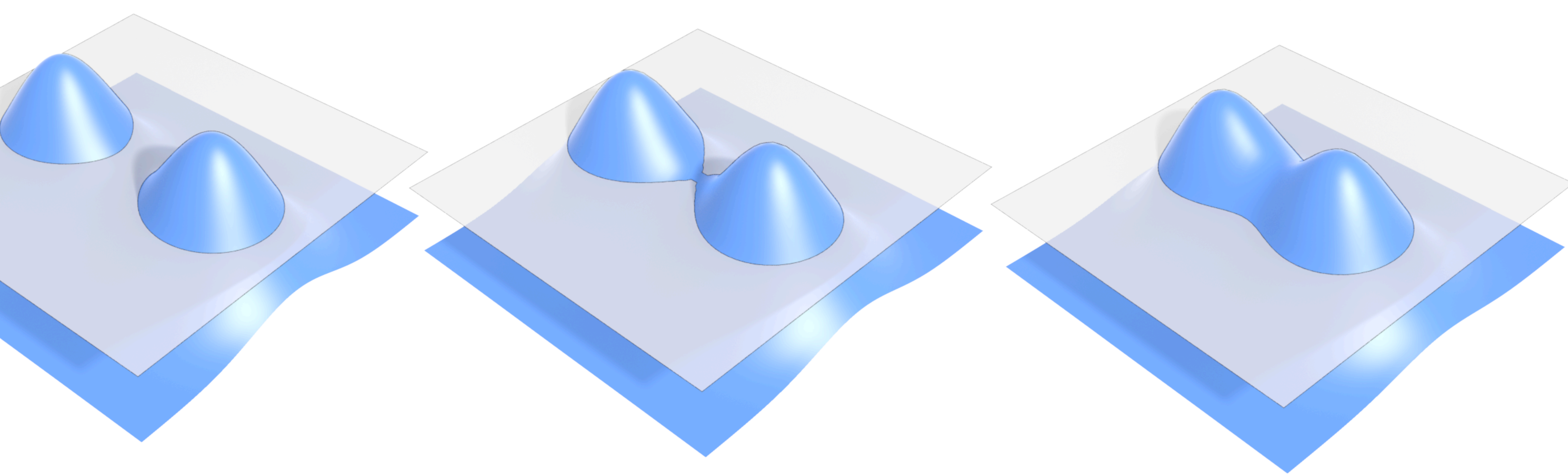


- **Easier to understand in 2D:**



$\phi_p(x) := e^{-|x-p|^2}$ **(Gaussian centered at p)**

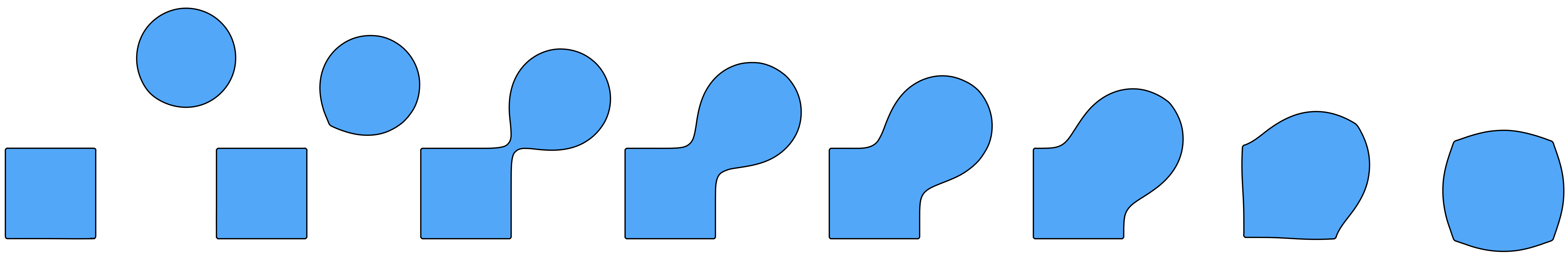
$f := \phi_p + \phi_q$ **(Sum of Gaussians centered at different points)**



$f = \frac{1}{2}$

Blending Distance Functions (Implicit)

- A *distance function* gives distance to closest point on object
- Can blend any two distance functions d_1, d_2:



- Similar strategy to points, though many possibilities. E.g.,

$$f(x) := e^{-d_1(x)^2} + e^{-d_2(x)^2} - \frac{1}{2}$$

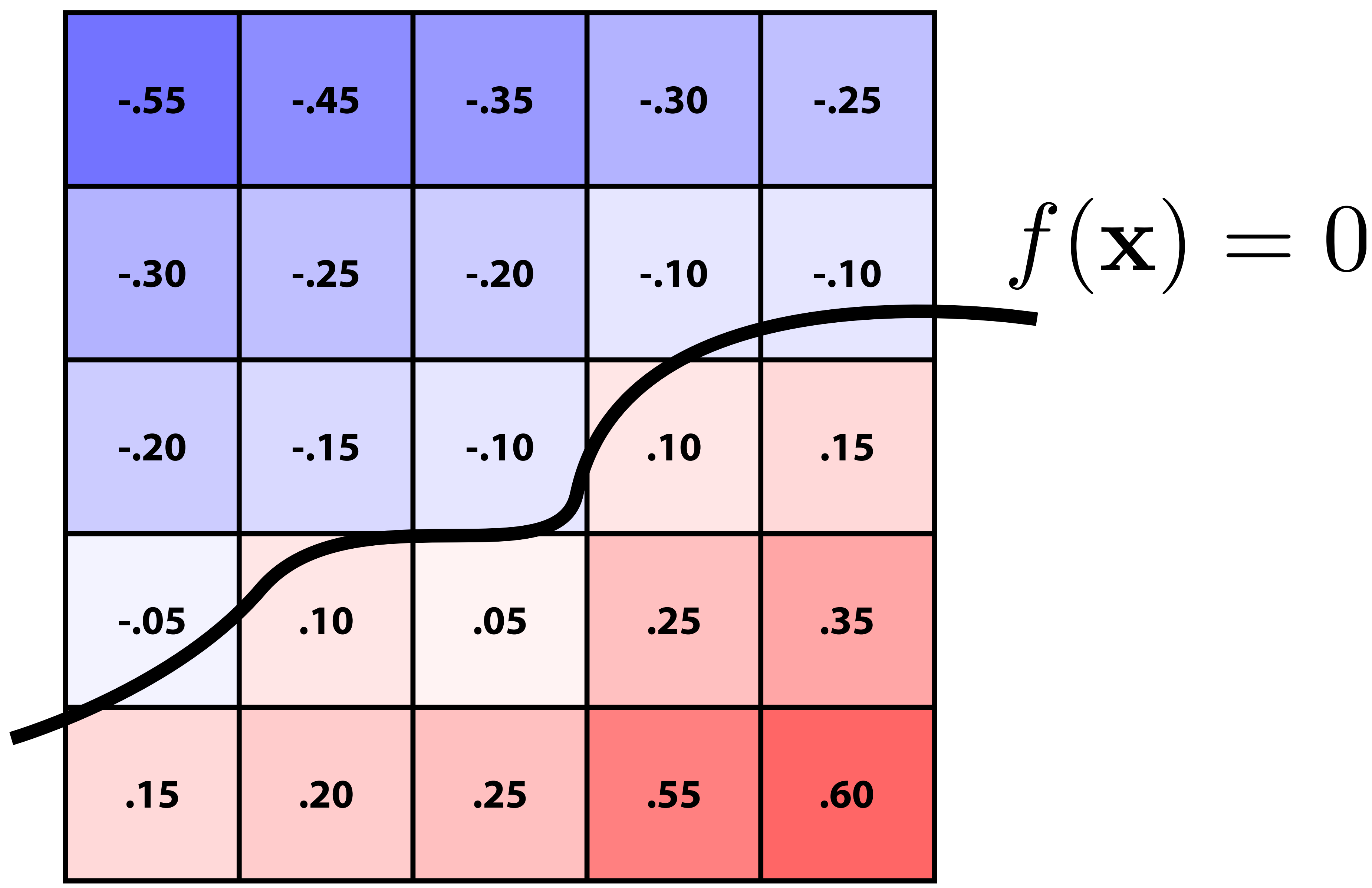
- Appearance depends on exactly how we combine functions
- Q: How do we implement a simple Boolean union?
- A: Just take the product: $f(x) := d_1(x)d_2(x)$

Scene of pure distance functions (not easy!)

See http://iquilezles.org/www/material/nvscene2008/nvscene2008.htm

Level Set Methods (Implicit)

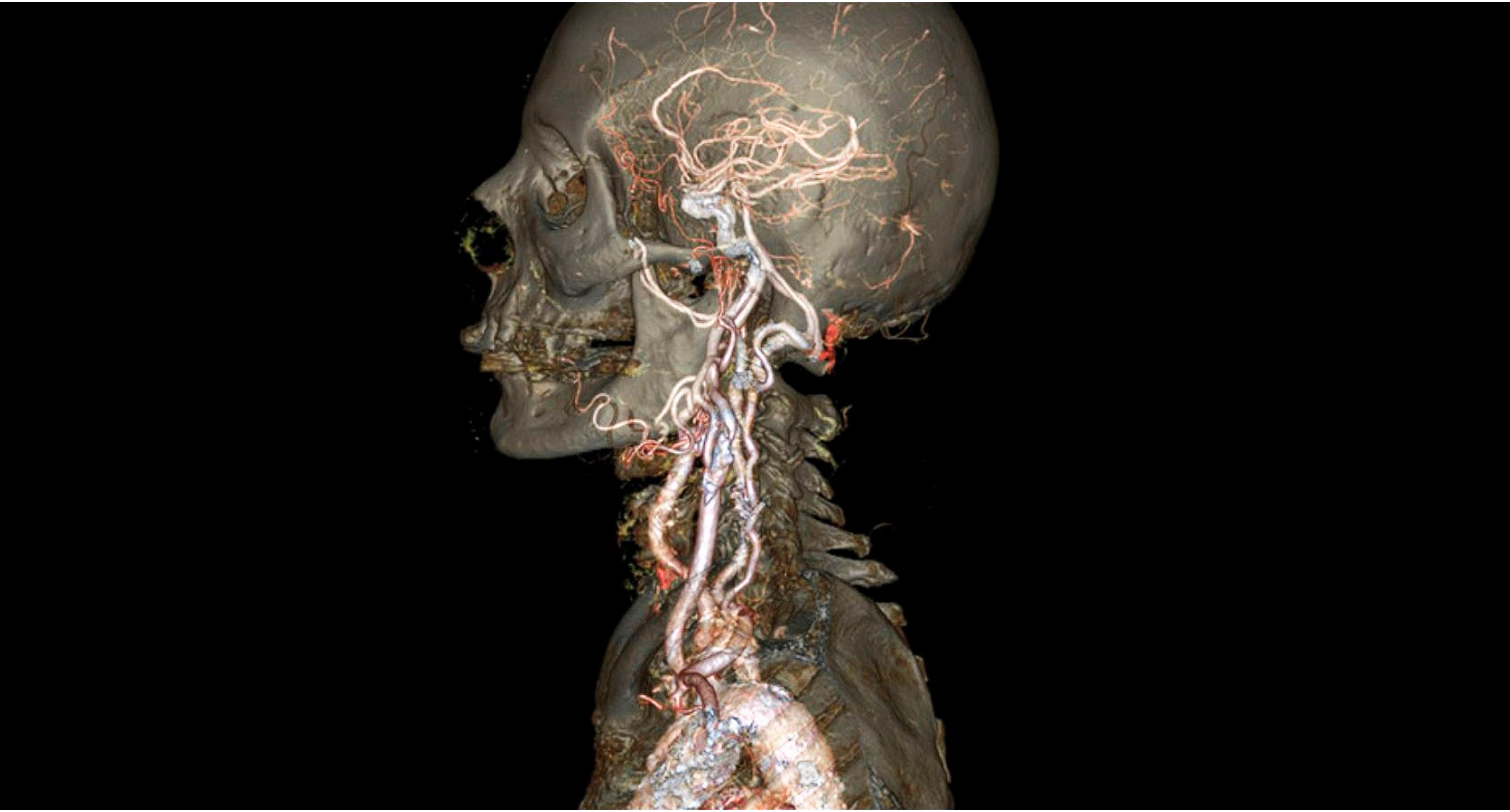
- Implicit surfaces have some nice features (e.g., merging/splitting)
- But, hard to describe complex shapes in closed form
- Alternative: store a grid of values approximating function



- Surface is found where *interpolated* values equal zero
- Provides much more explicit control over shape (like a texture)

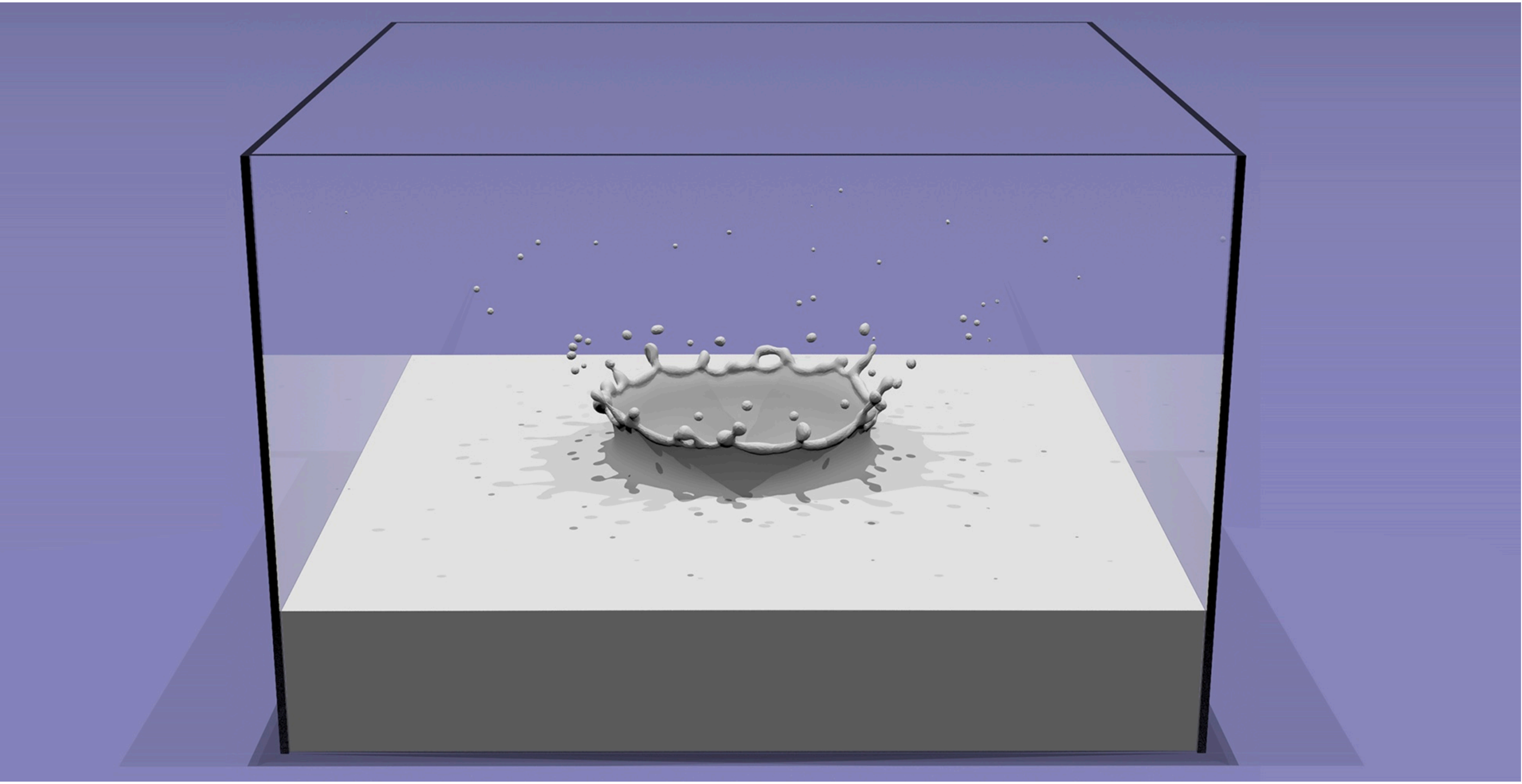
Level Sets from Medical Data (CT, MRI, etc.)

- Level sets encode, e.g., constant tissue density



Level Sets in Physical Simulation

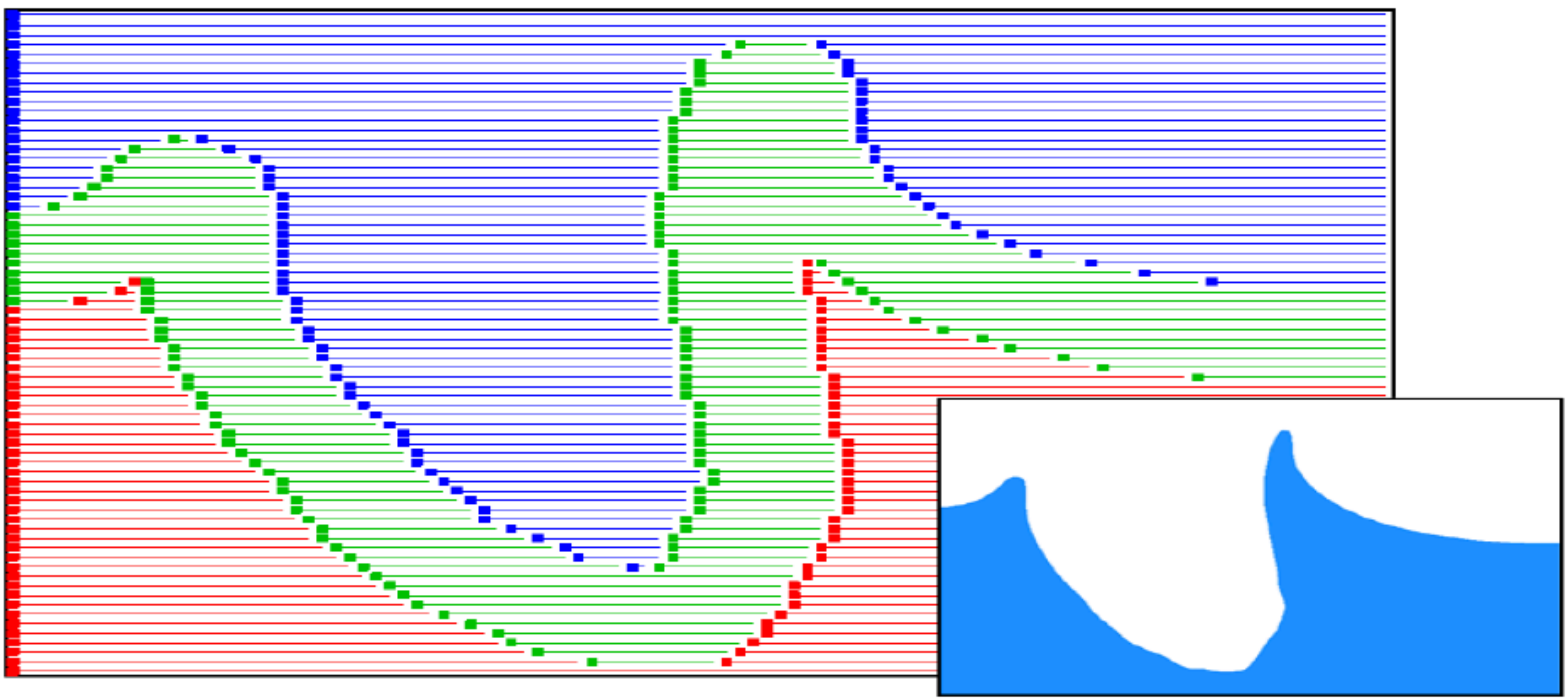
- Level set encodes distance to air-liquid boundary



See http://physbam.stanford.edu

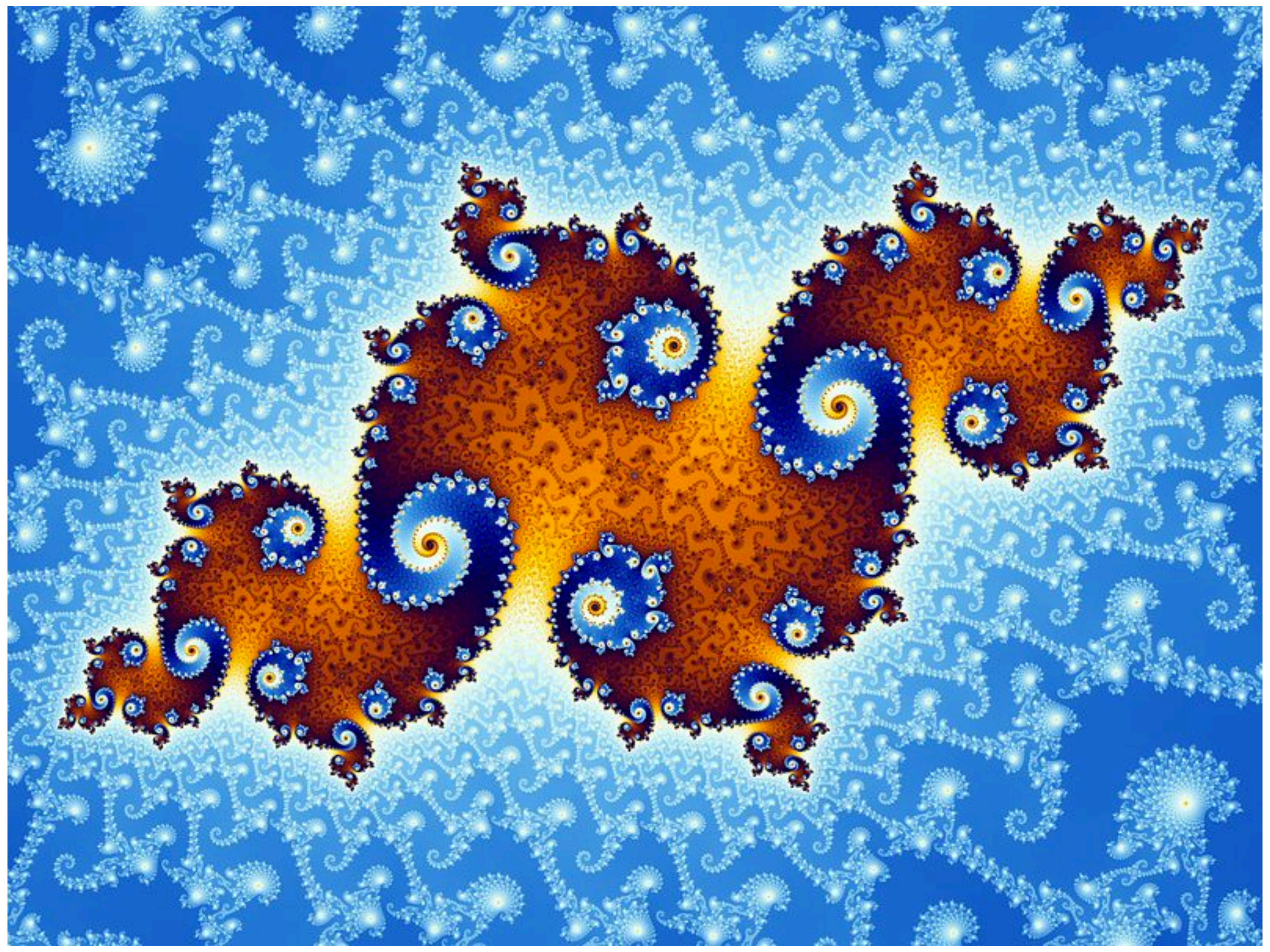
Level Set Storage

- Drawback: storage for 2D surface is now $O(n^3)$
- Can reduce cost by storing only a narrow band around surface:



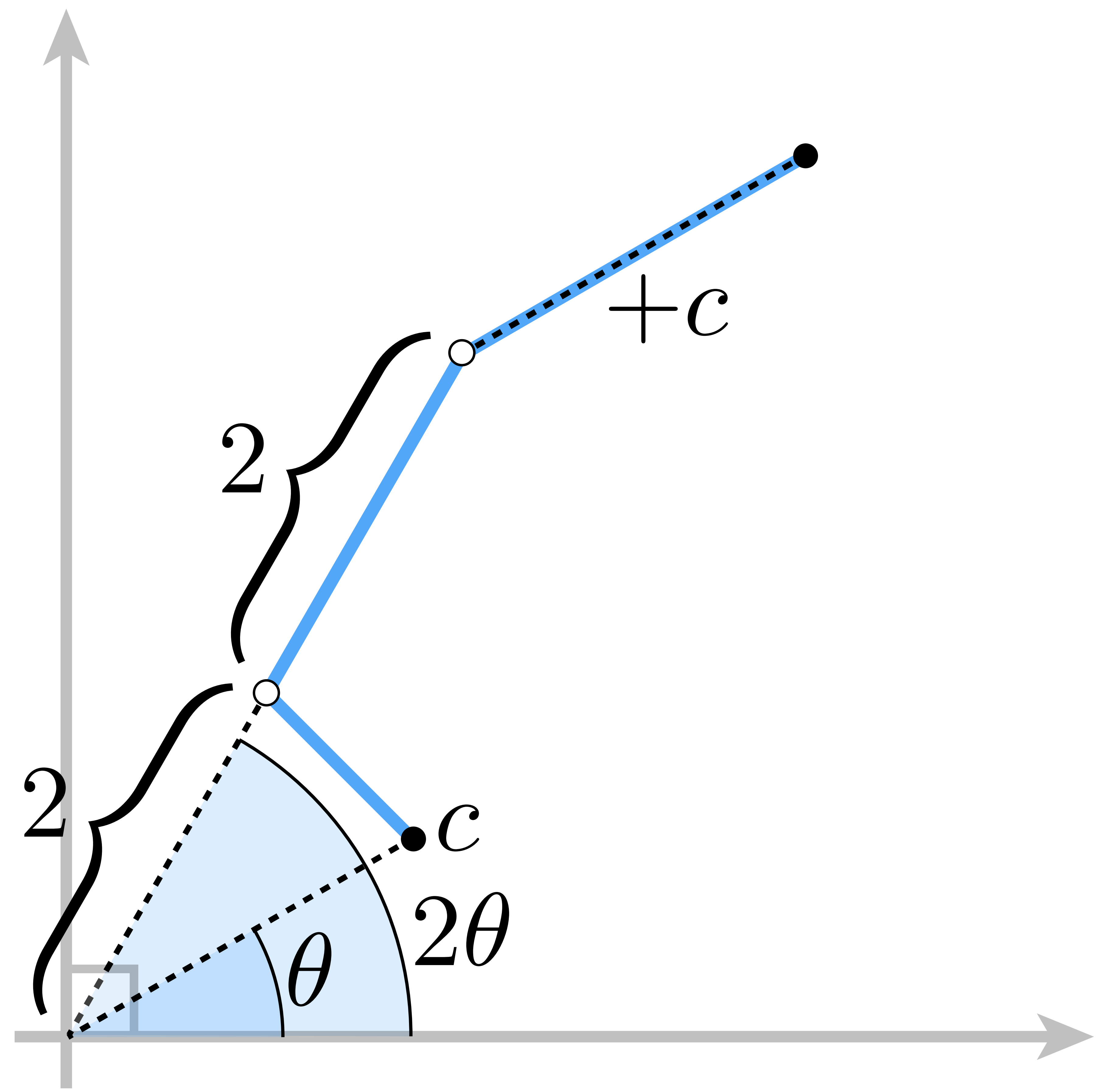
Fractals (Implicit)

- No precise definition; exhibit self-similarity, detail at all scales
- New "language" for describing natural phenomena
- Hard to control shape!



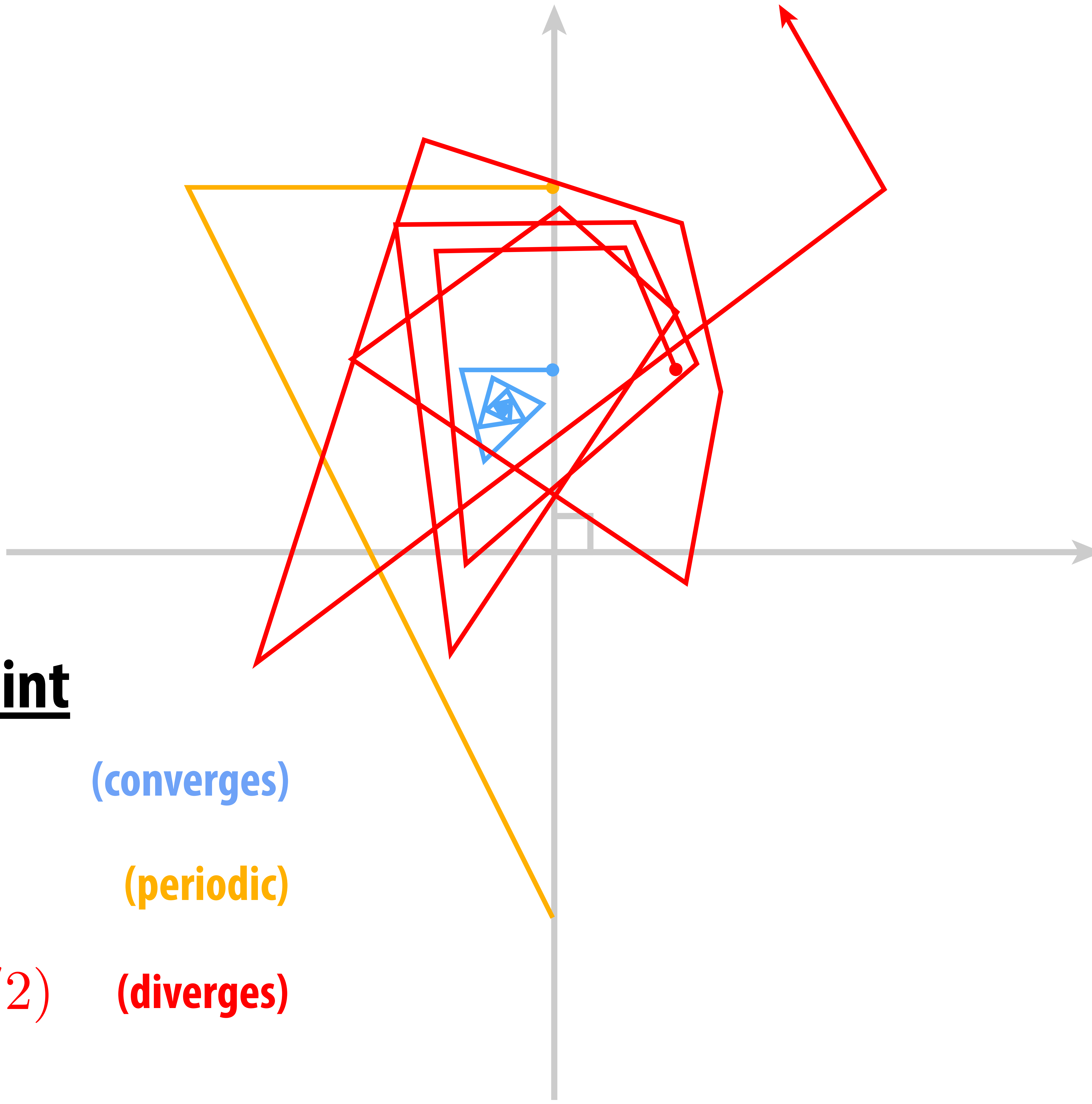
Mandelbrot Set - Definition

- For each point c in the plane:
  - double the angle
  - square the magnitude
  - add the original point c
  - repeat



If the point remains bounded (never goes to ∞), it's in the set.

Mandelbrot Set - Examples



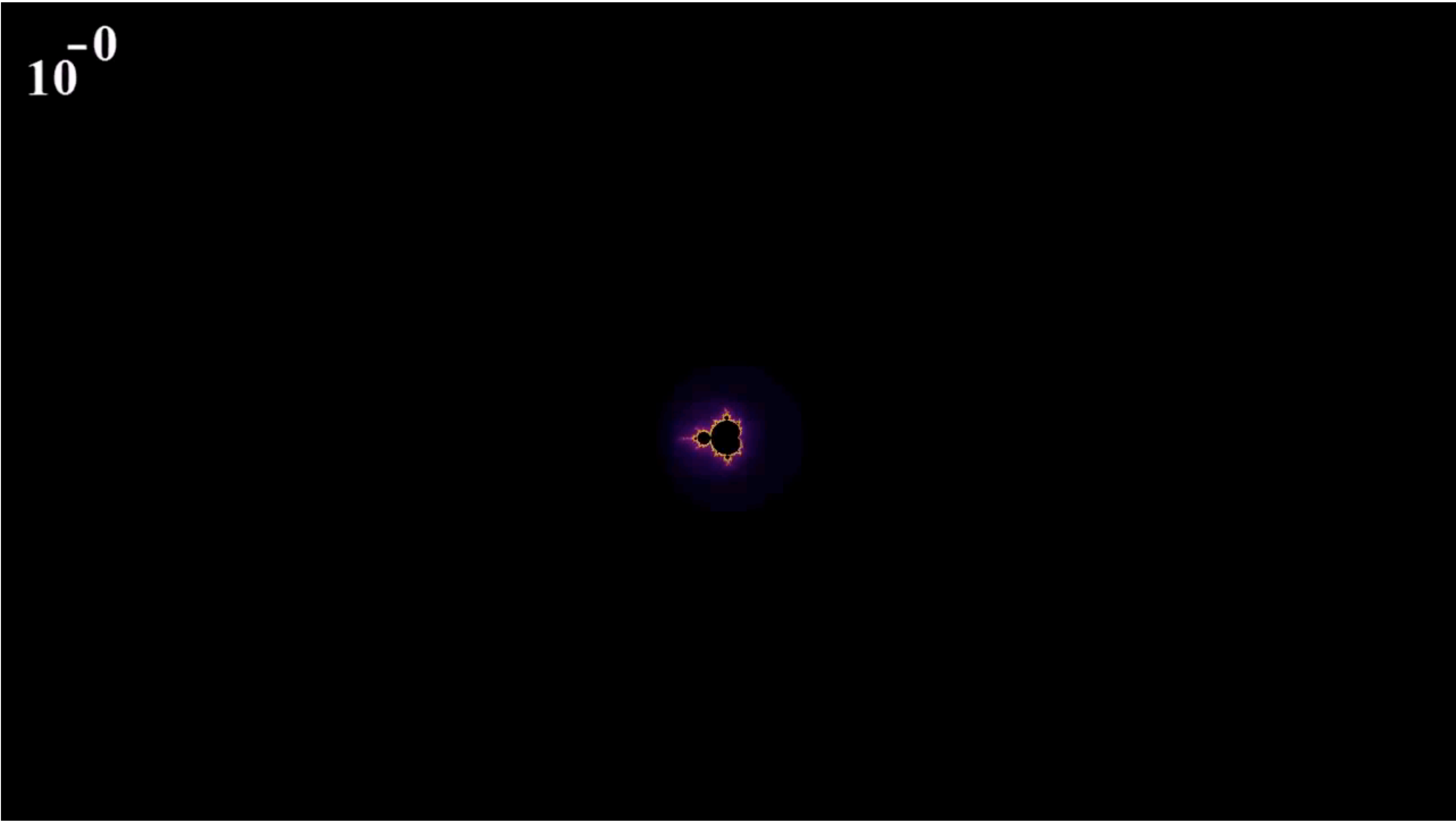
starting point

$(0, 1/2)$ **(converges)**

$(0, 1)$ **(periodic)**

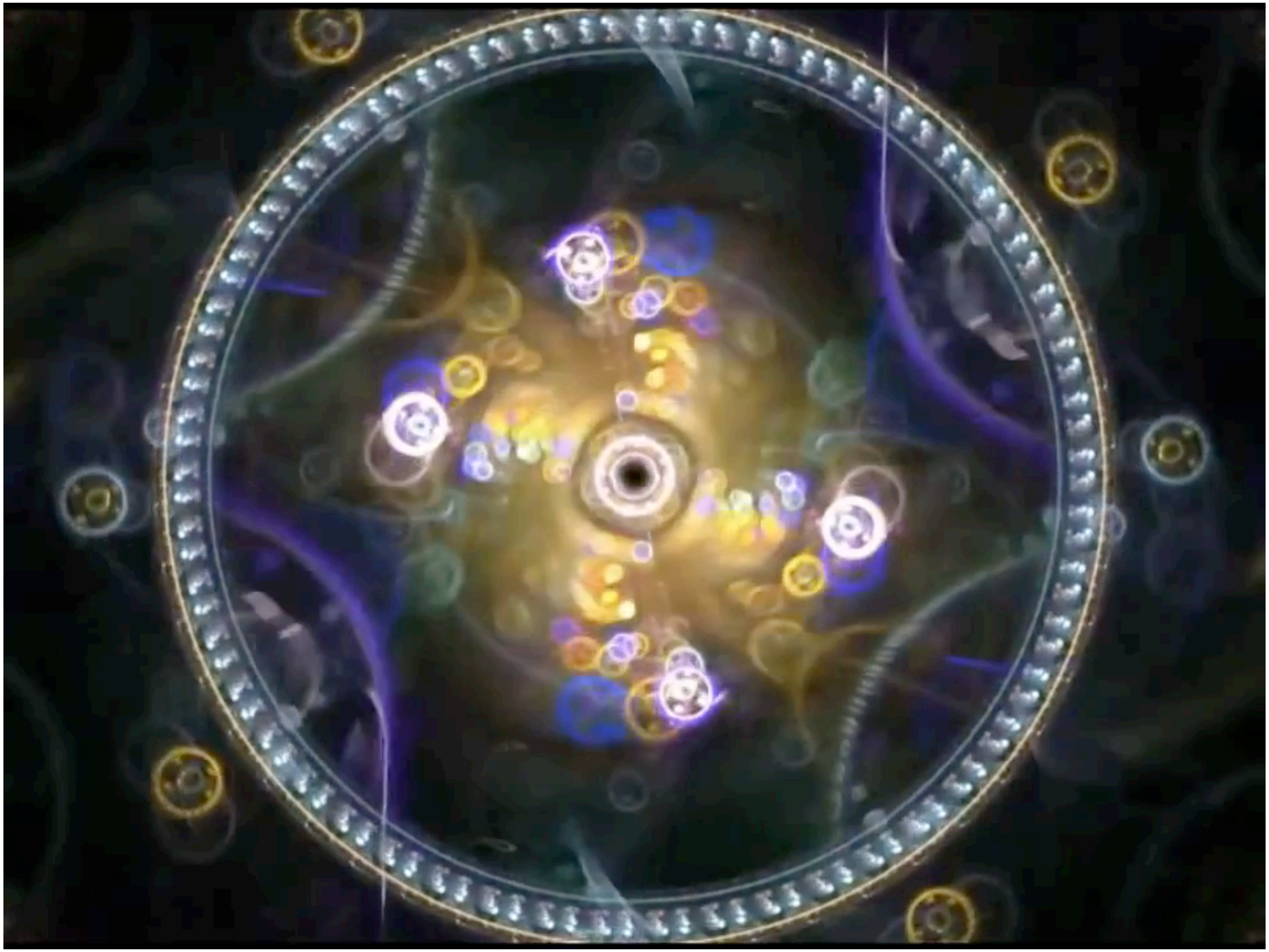
$(1/3, 1/2)$ **(diverges)**

Mandelbrot Set - Zooming In



(Colored according to how quickly each point diverges/converges.)

Iterated Function Systems



Scott Draves (CMU Alumn) - see http://electricsheep.org

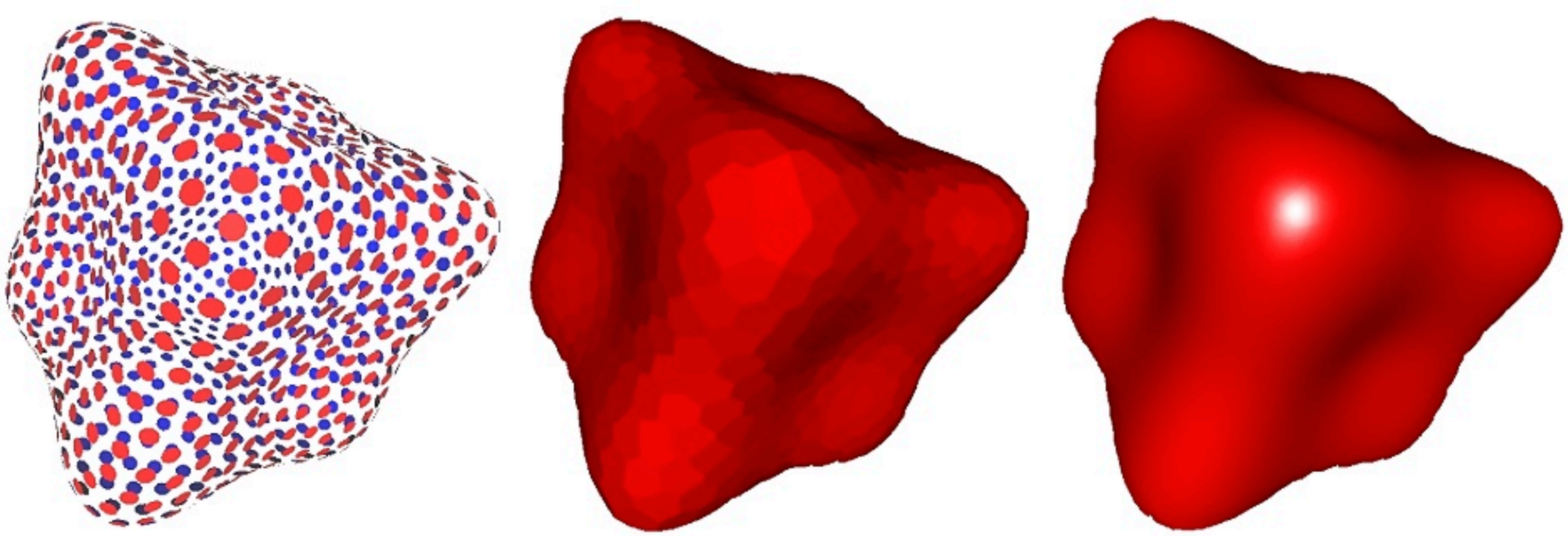
Implicit Representations - Pros & Cons

- **Pros:**
  - **description can be very compact (e.g., a polynomial)**
  - **easy to determine if a point is in our shape (just plug it in!)**
  - **other queries may also be easy (e.g., distance to surface)**
  - **for simple shapes, exact description/no sampling error**
  - **easy to handle changes in topology (e.g., fluid)**
- **Cons:**
  - **expensive to find all points in the shape (e.g., for drawing)**
  - ***very difficult to model complex shapes***

What about explicit representations?

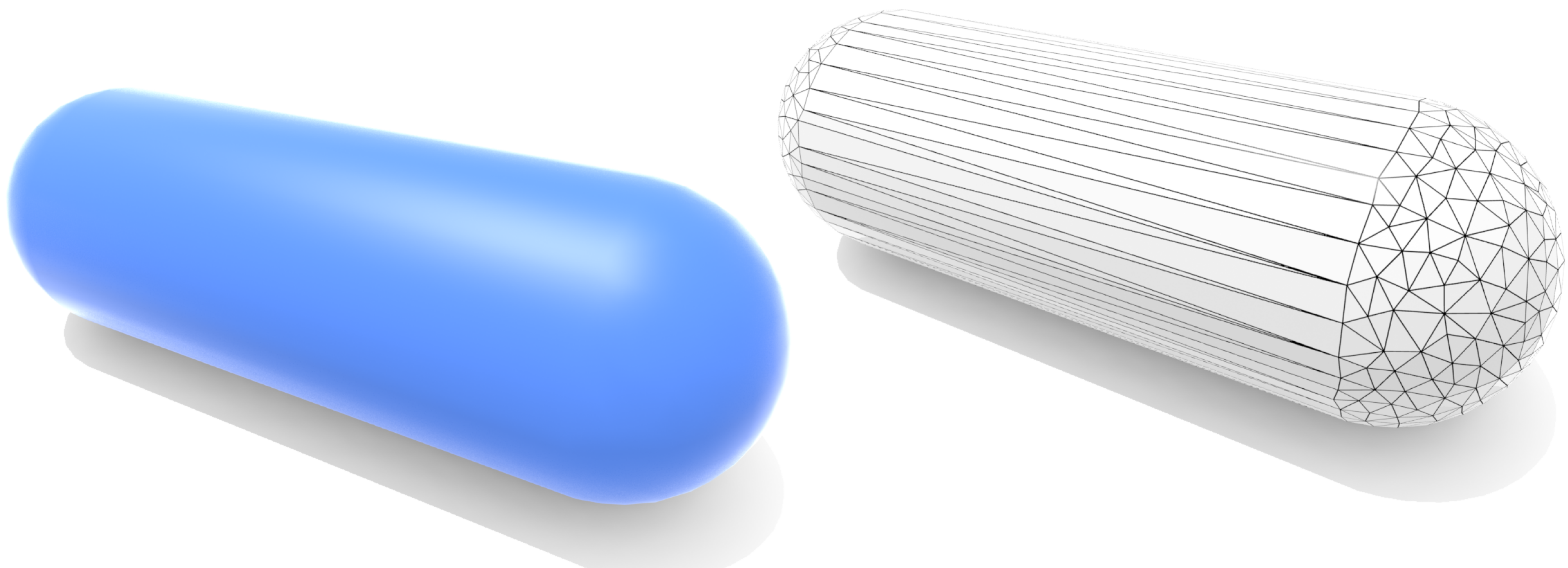
Point Cloud (Explicit)

- Easiest representation: list of points (x,y,z)
- Often augmented with *normals*
- Easily represent any kind of geometry
- Useful for LARGE datasets (>>1 point/pixel)
- Difficult to draw in undersampled regions
- Hard to do processing / simulation



Polygon Mesh (Explicit)

- Store vertices *and* polygons (most often triangles or quads)
- Easier to do processing/simulation, adaptive sampling
- More complicated data structures
- Perhaps most common representation in graphics

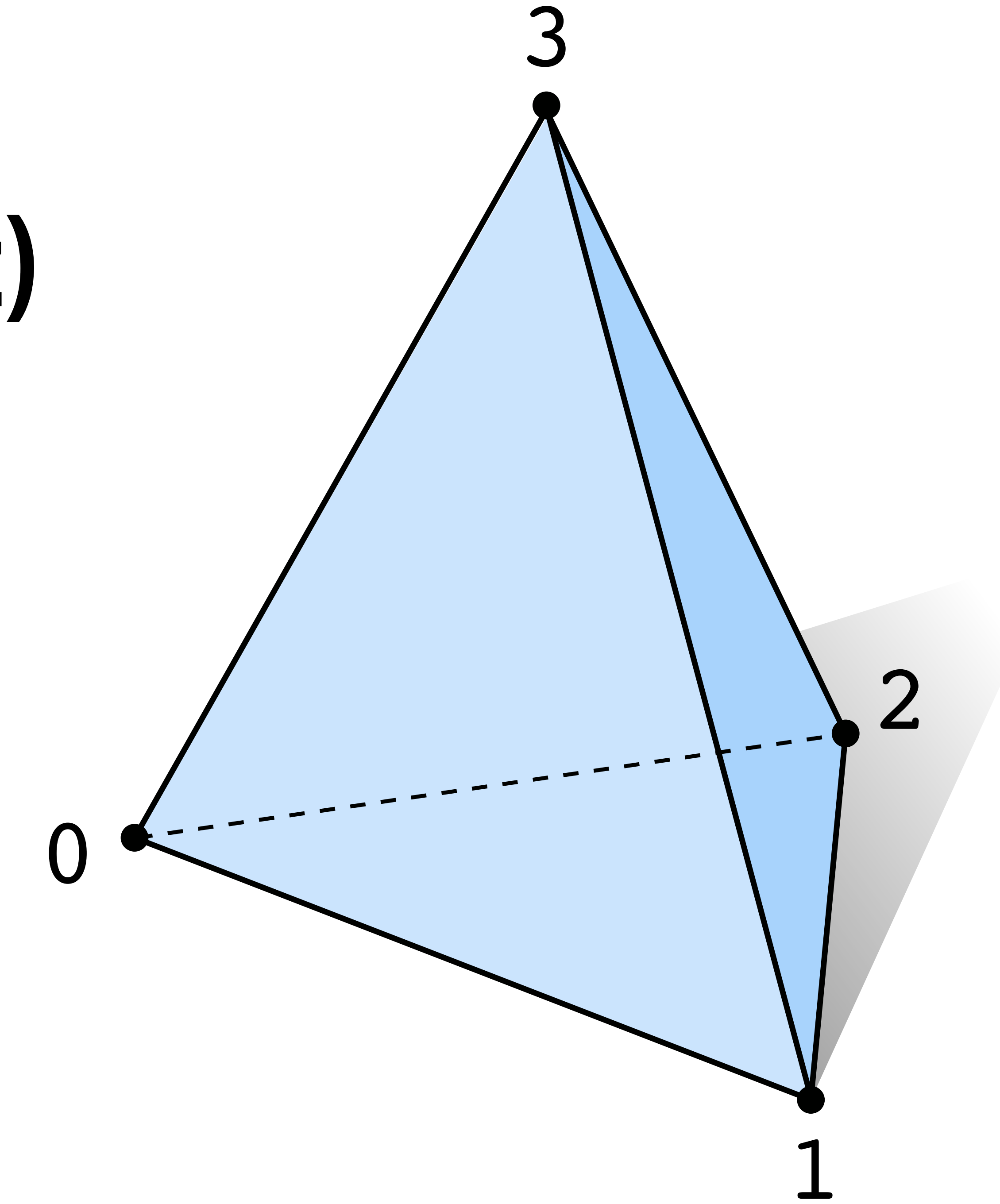


(Much more about polygon meshes in upcoming lectures!)

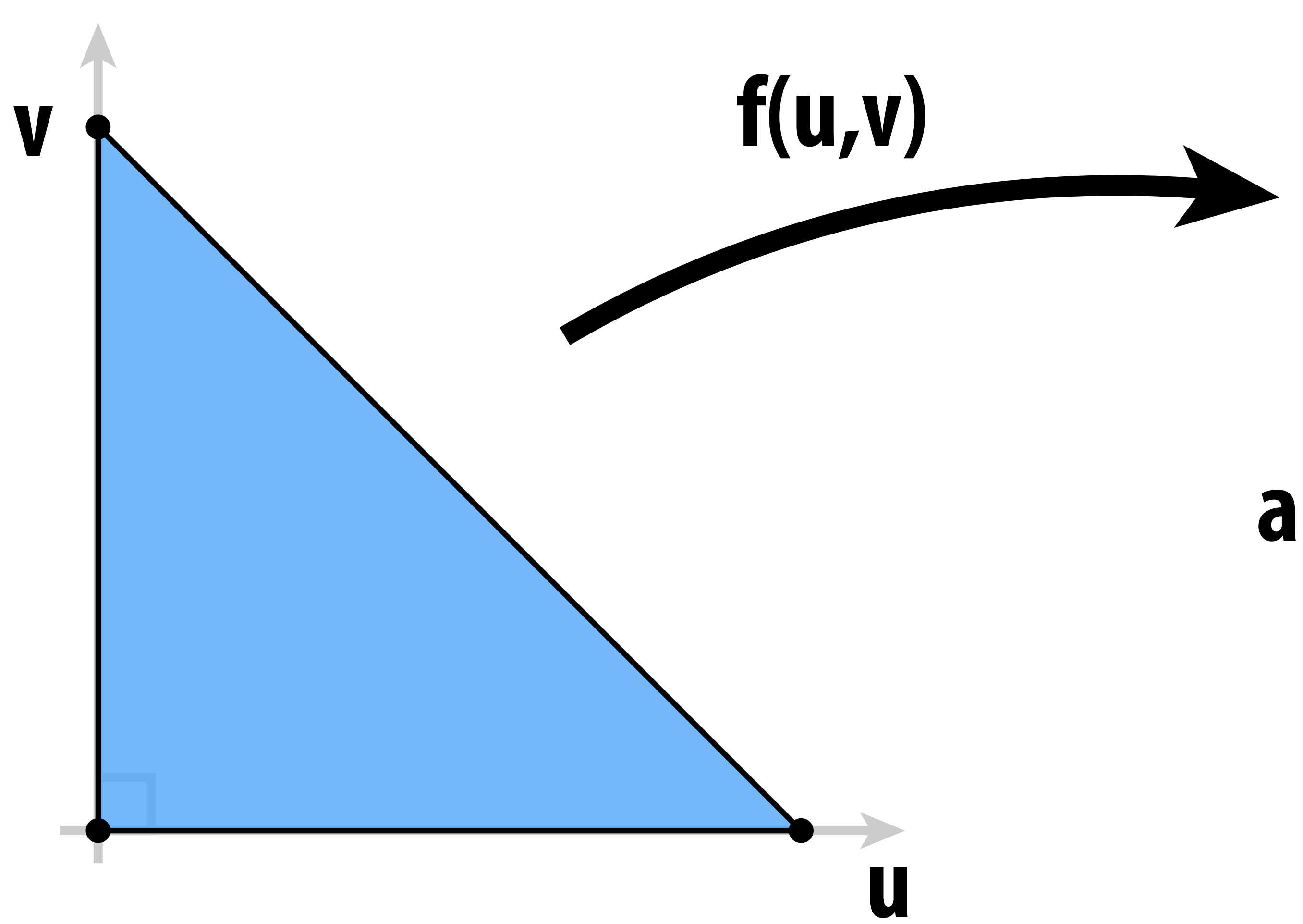
Triangle Mesh (Explicit)

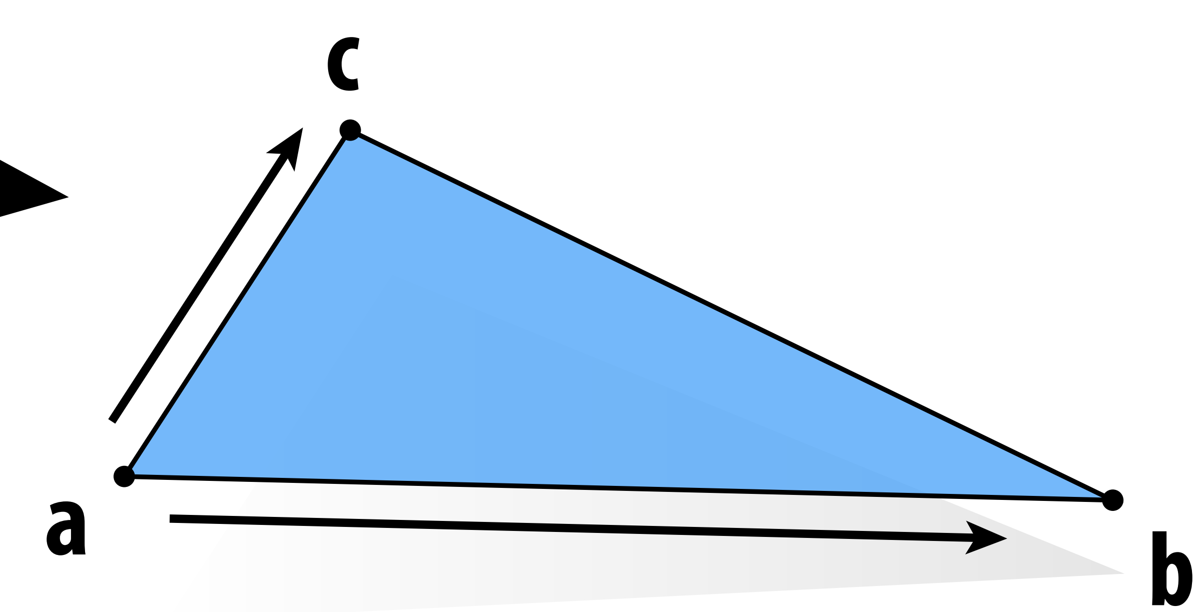
- Store vertices as triples of coordinates (x,y,z)
- Store triangles as triples of indices (i,j,k)
- E.g., tetrahedron:

	VERTICES			TRIANGLES		
	x	y	z	i	j	k
0:	−1	−1	−1	0	2	1
1:	1	−1	1	0	3	2
2:	1	1	−1	3	0	1
3:	−1	1	1	3	1	2



- Can think of triangle as *affine* map from plane into space:





$$f(u,v) = a + u(b-a) + v(c-a)$$

Bernstein Basis

- Why limit ourselves to just affine functions?
- More flexibility by using higher-order polynomials
- Instead of usual basis (1, x, x^2, x^3, ...), use Bernstein basis:

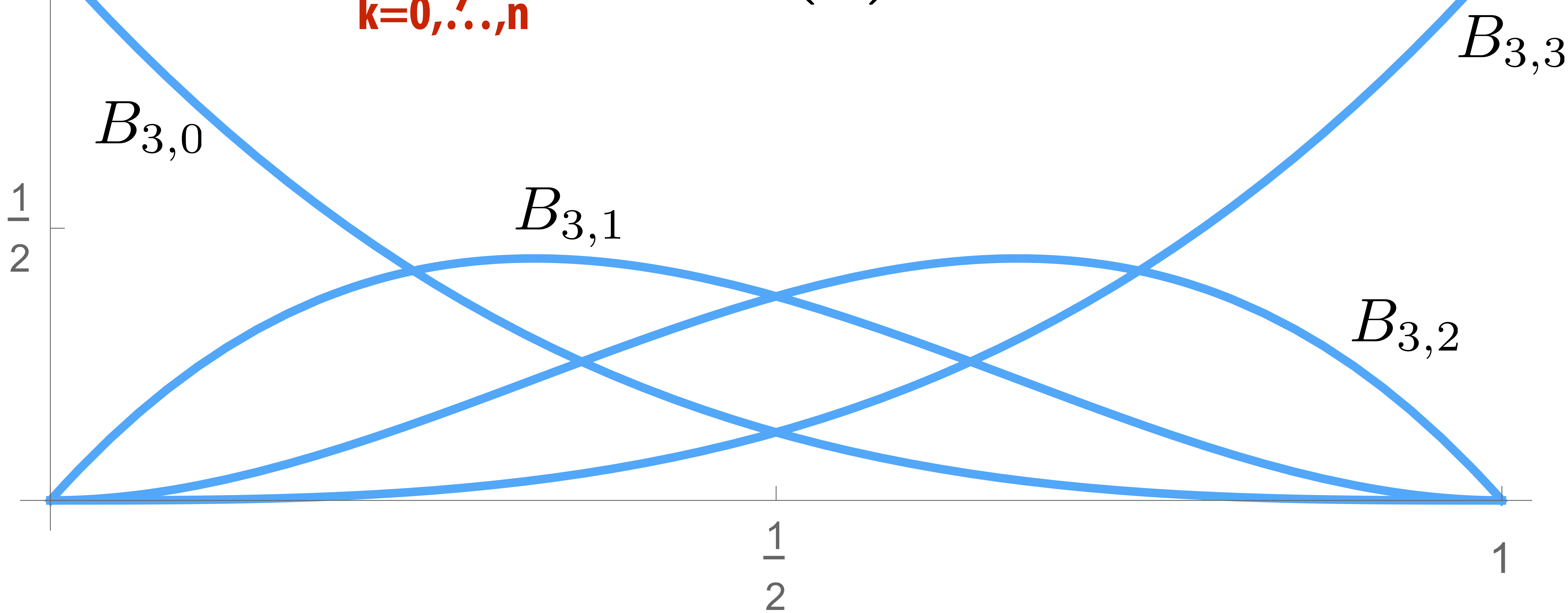
degree

0≤x≤1

"n choose k"

$$B_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$$

k=0,...,n



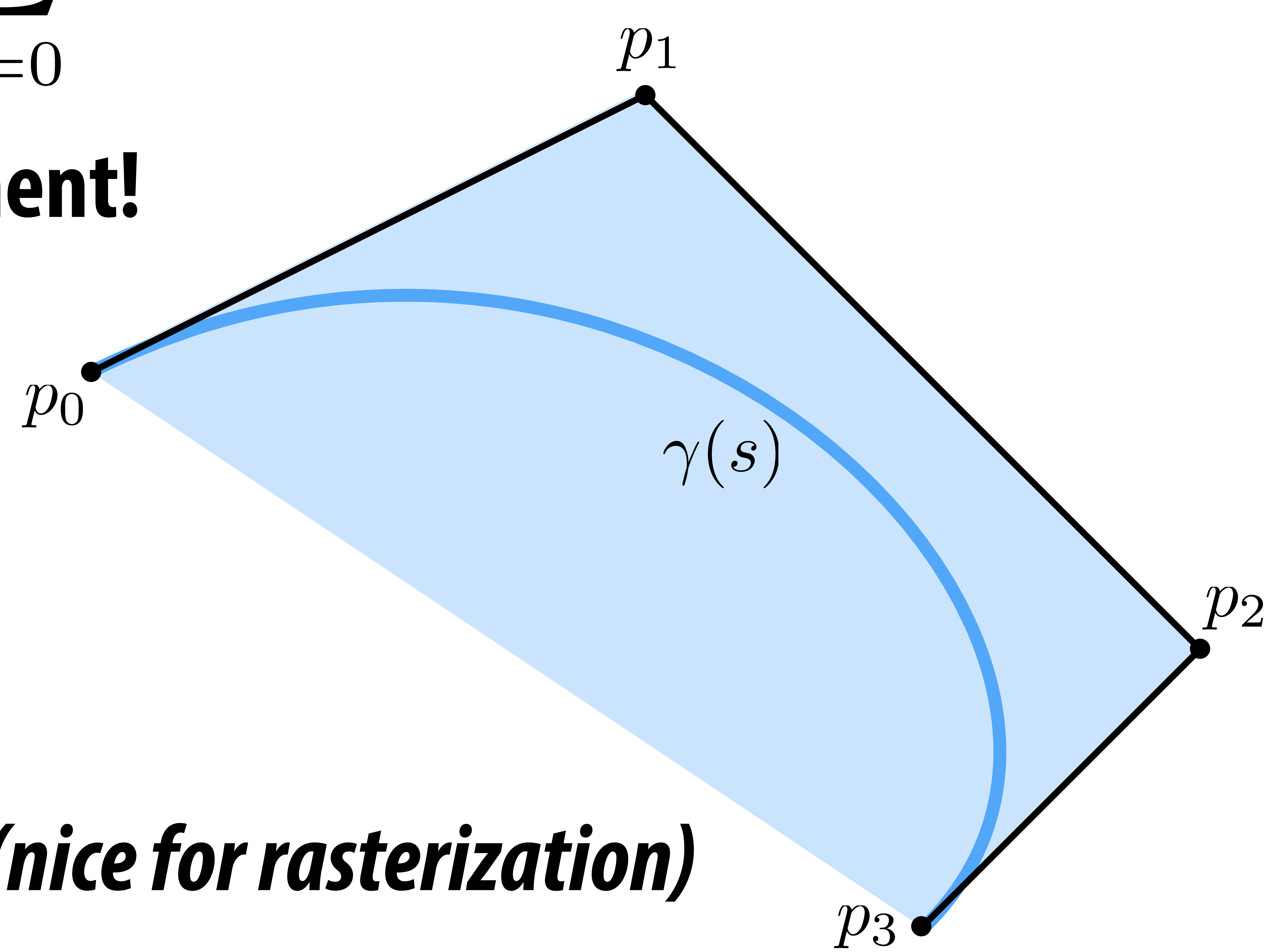
Bézier Curves (Explicit)

- **A Bézier curve is a curve expressed in the Bernstein basis:**

$$\gamma(s) := \sum_{k=0}^{n} B_{n,k}(s)p_k$$

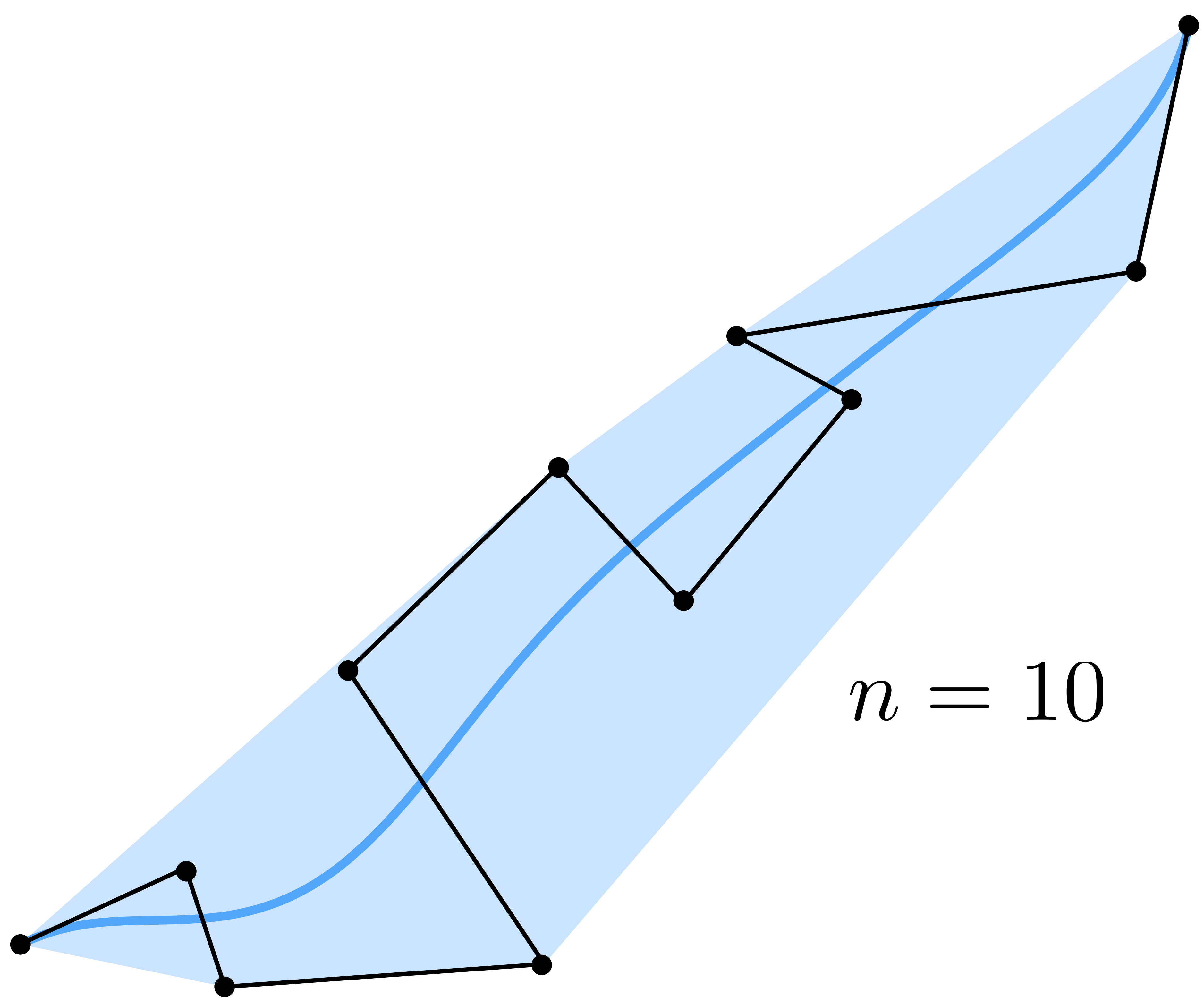
control points

- **For n=1, just get a line segment!**
- **For n=3, get "cubic Bézier":**
- **Important features:**
  1. **interpolates endpoints**
  2. **tangent to end segments**
  3. **contained in convex hull *(nice for rasterization)***



Higher-order Bézier Curves?

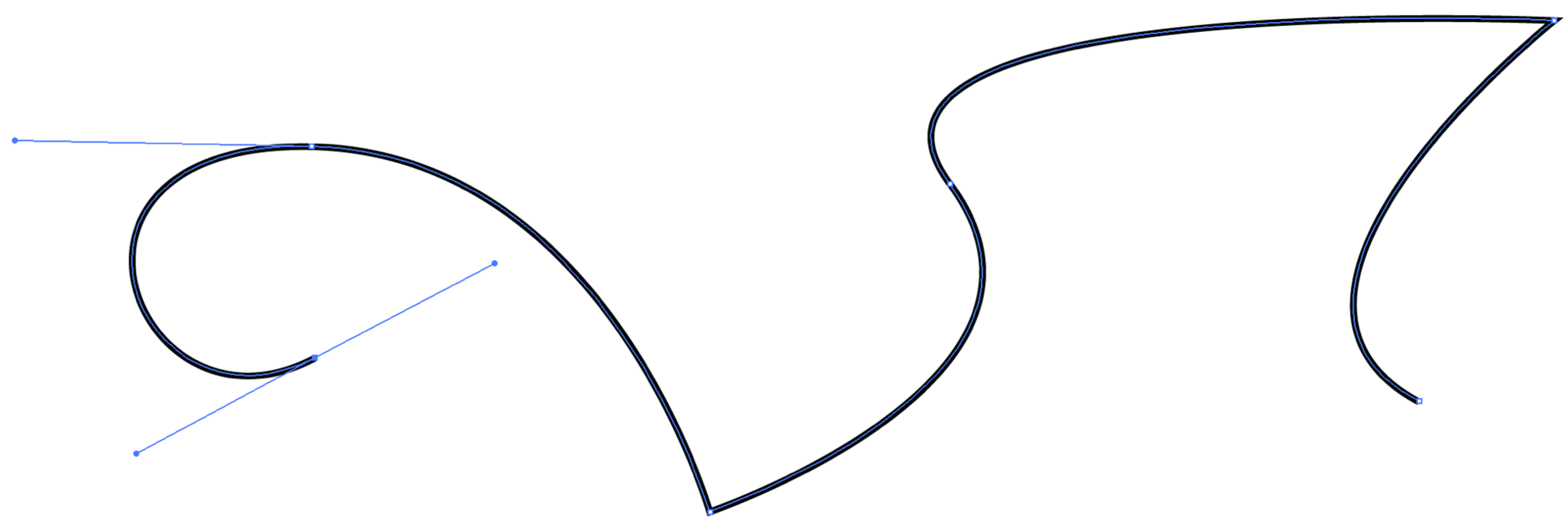
- What if we want a more interesting curve?
- High-degree Bernstein polynomials don't interpolate well:



Very hard to control!

B-Spline Curves (Explicit)

- **Instead, use many low-order Bézier curve (B-spline)**
- **Widely-used technique in software (Illustrator, Inkscape, etc.)**



- **Formally, piecewise Bézier curve:**

B-spline

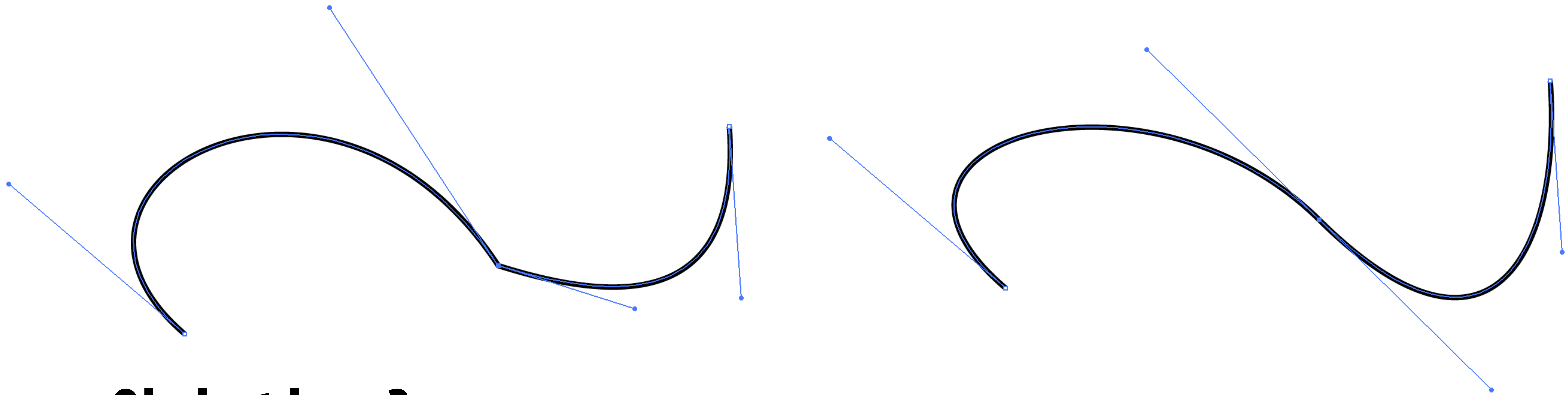
$$\gamma(u) := \gamma_i\left(\frac{u-u_i}{u_{i+1}-u_i}\right), \qquad u_i \leq u < u_{i+1}$$

Bézier

- **Location of u_i parameters are called "knots"**

B-Splines — tangent continuity

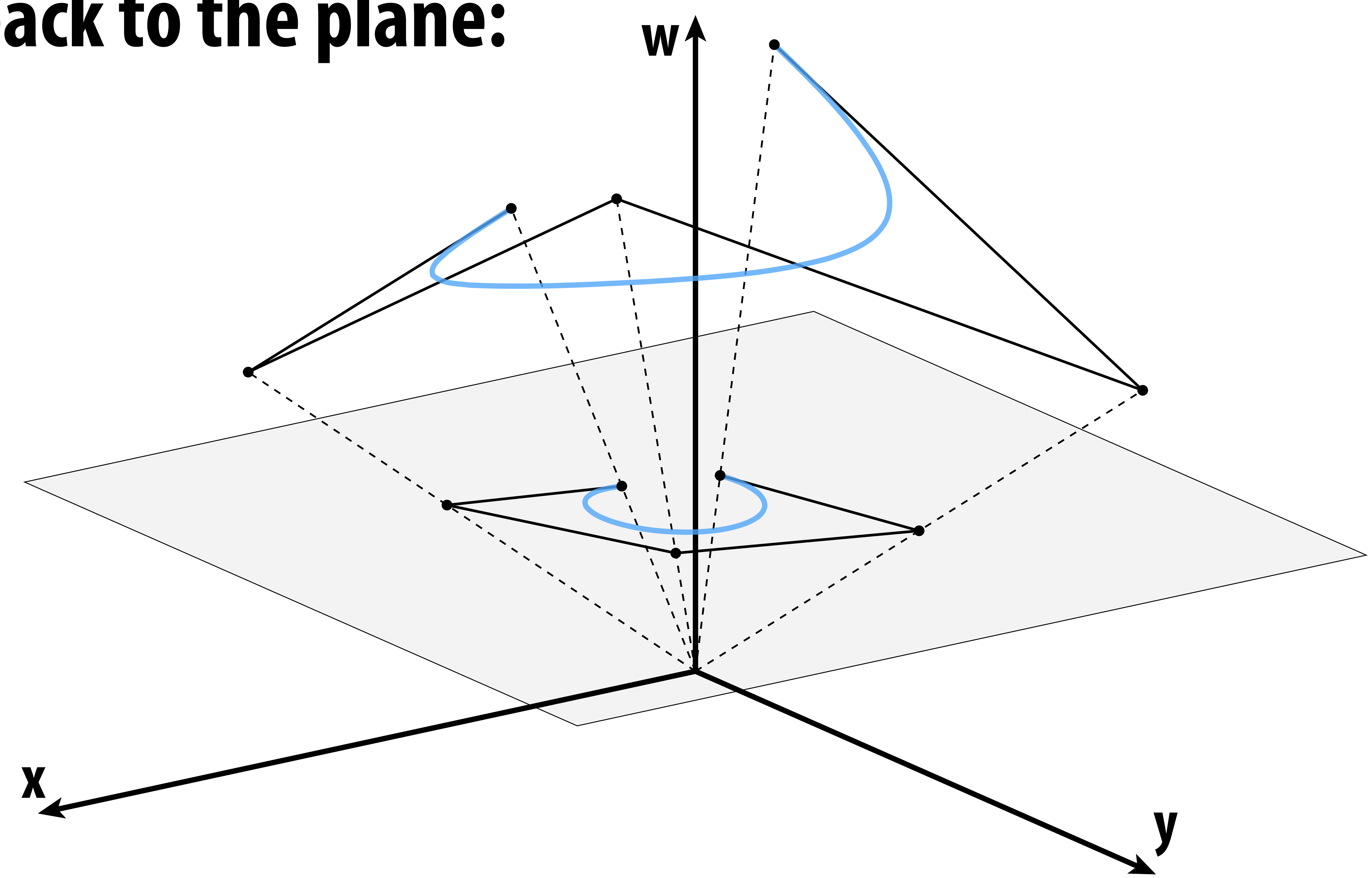
- To get "seamless" curves, want *tangents* to line up:



- Ok, but how?
- Each curve is cubic: $u^3p_0 + 3u^2(1-u)p_1 + 3u(1-u)^2p_2 + (1-u)^3p_3$
- Q: How many degrees of freedom in a single cubic Bézier
- Tangents are difference between first two & last two points
- Q: How many degrees of freedom per B-spline segment?
- Q: Could you do this with quadratic Bézier? Linear Bézier?

Rational B-Splines (Explicit)

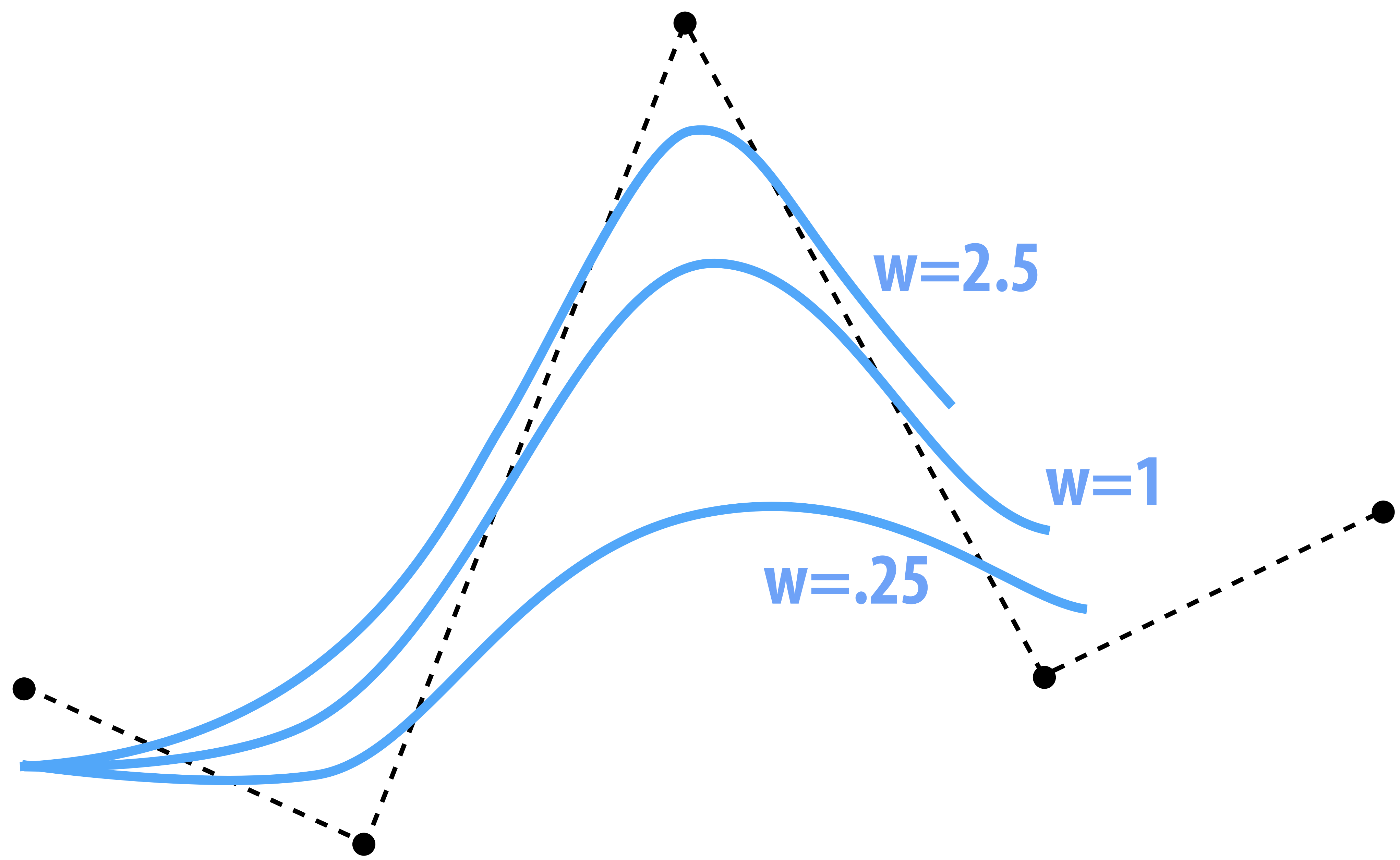
- B-Splines can't exactly represent *conics*—not even the circle!
- Solution: interpolate in homogeneous coordinates, then project back to the plane:



Result is called a *rational* B-spline.

NURBS (Explicit)

- (N)on-(U)niform (R)ational (B)-(S)pline
  - knots at arbitrary locations (non-uniform)
  - expressed in homogeneous coordinates (rational)
  - piecewise polynomial curve (B-Spline)
- Homogeneous coordinate w controls "strength" of a vertex:

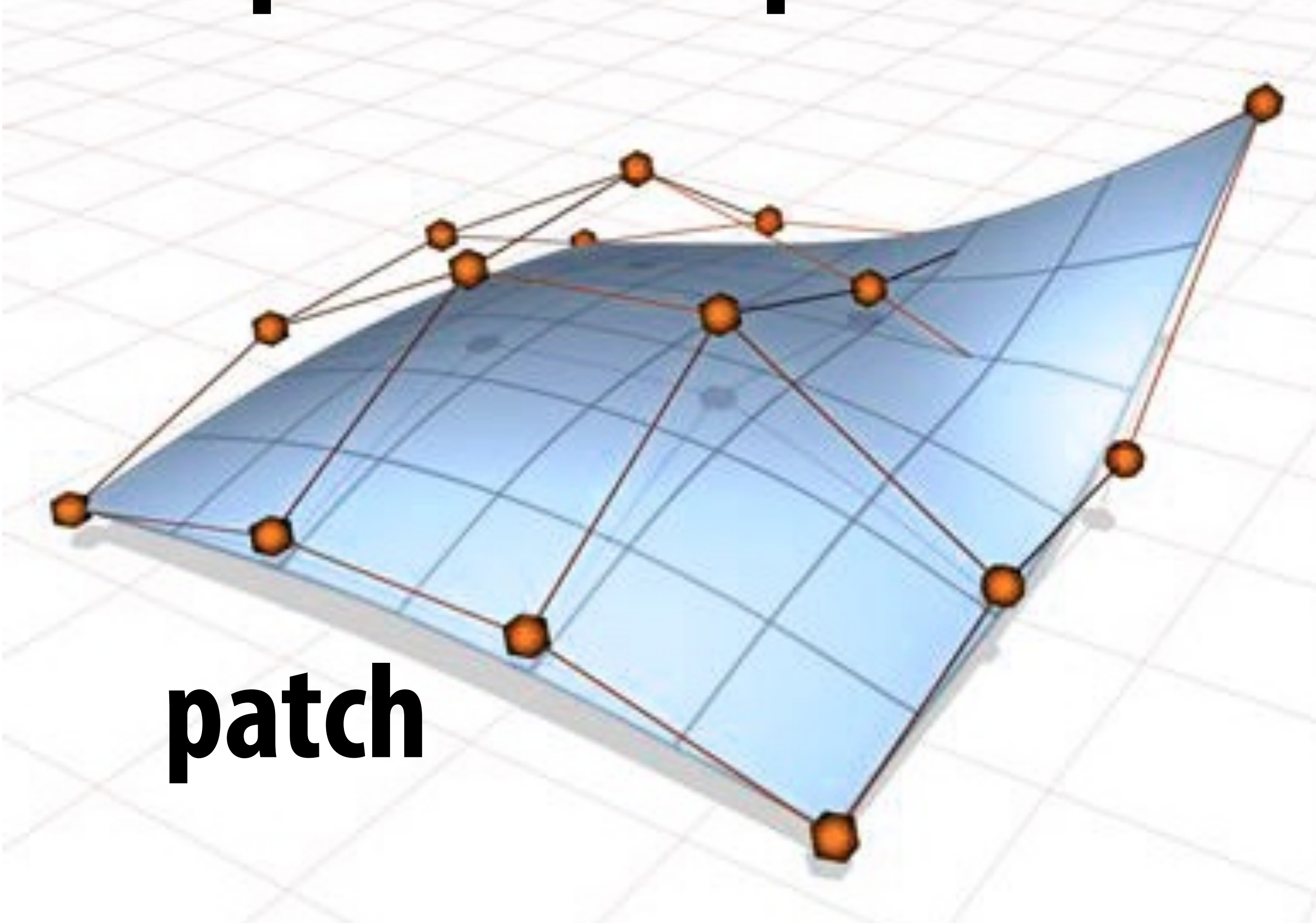


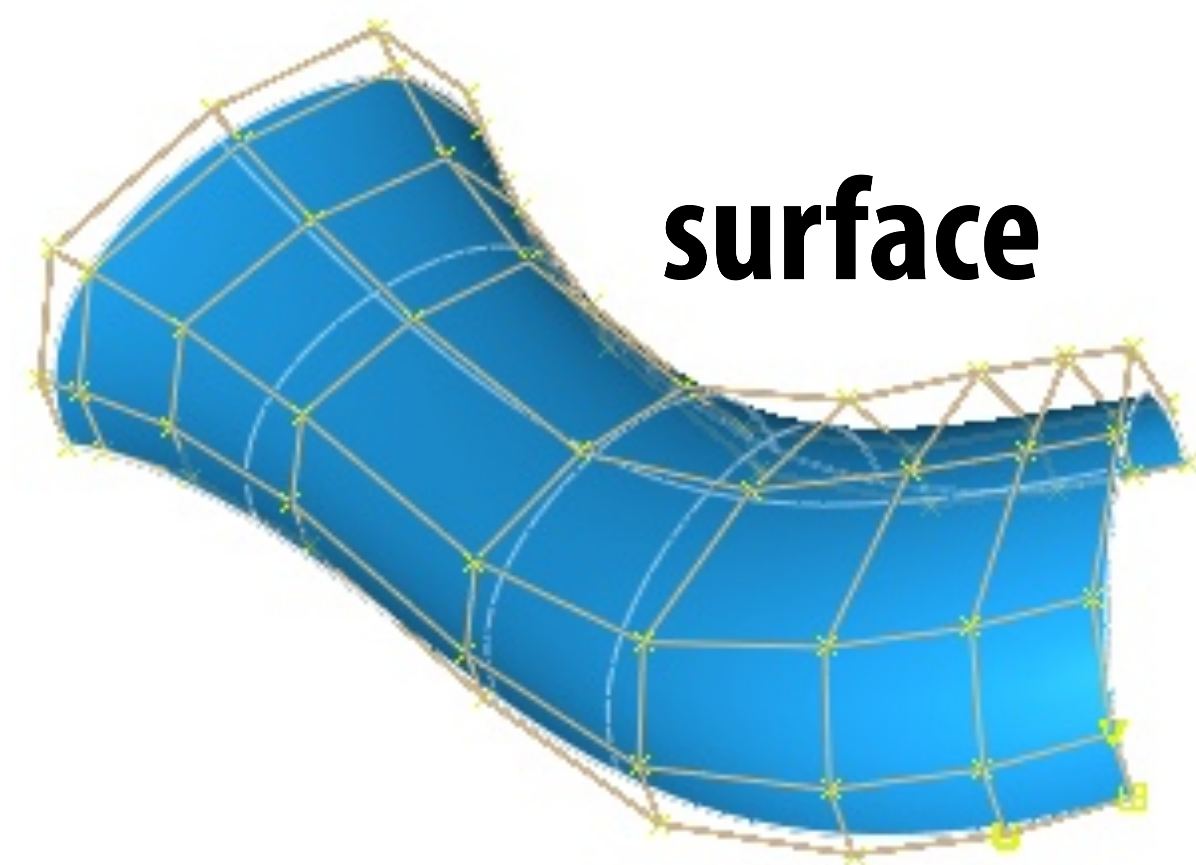
NURBS Surface (Explicit)

- Much more common than using NURBS for curves
- Use *tensor product* of scalar NURBS curves to get a patch:

$$S(u,v) := N_i(u)N_j(v)p_{ij}$$

- Multiple NURBS patches form a surface

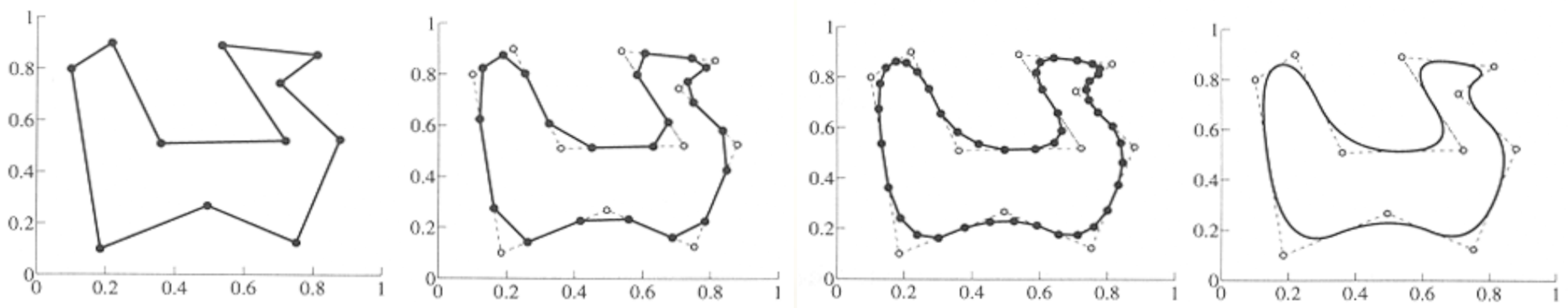




- Pros: easy to evaluate, exact conics, high degree of continuity
- Cons: Hard to piece together patches, hard to edit (many DOFs)

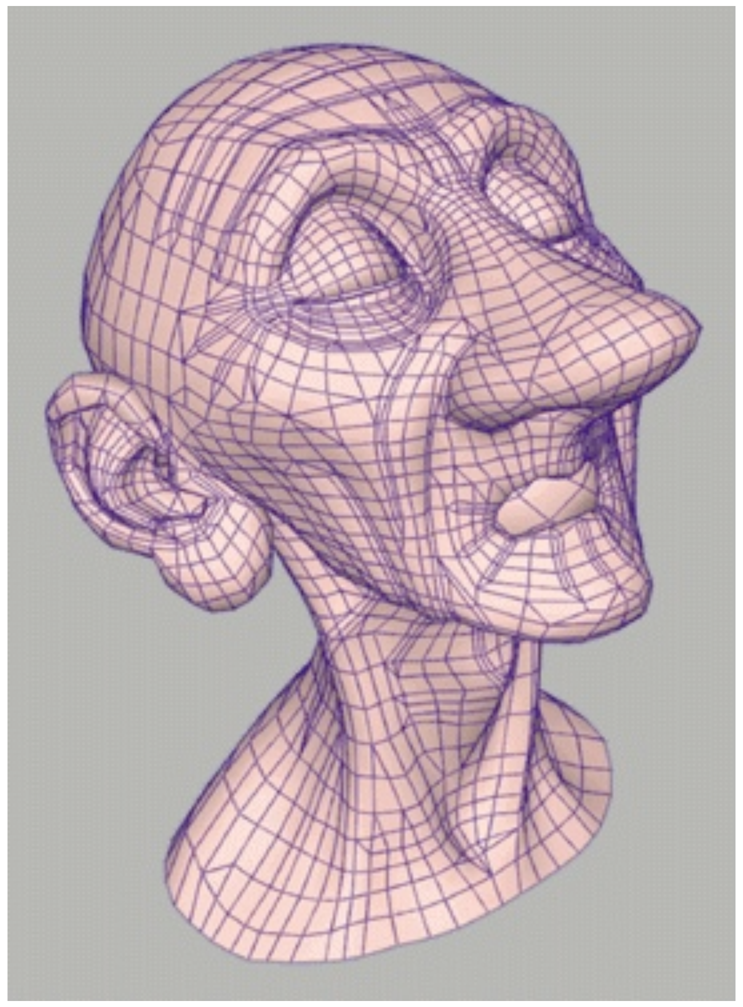
Subdivision (Explicit)

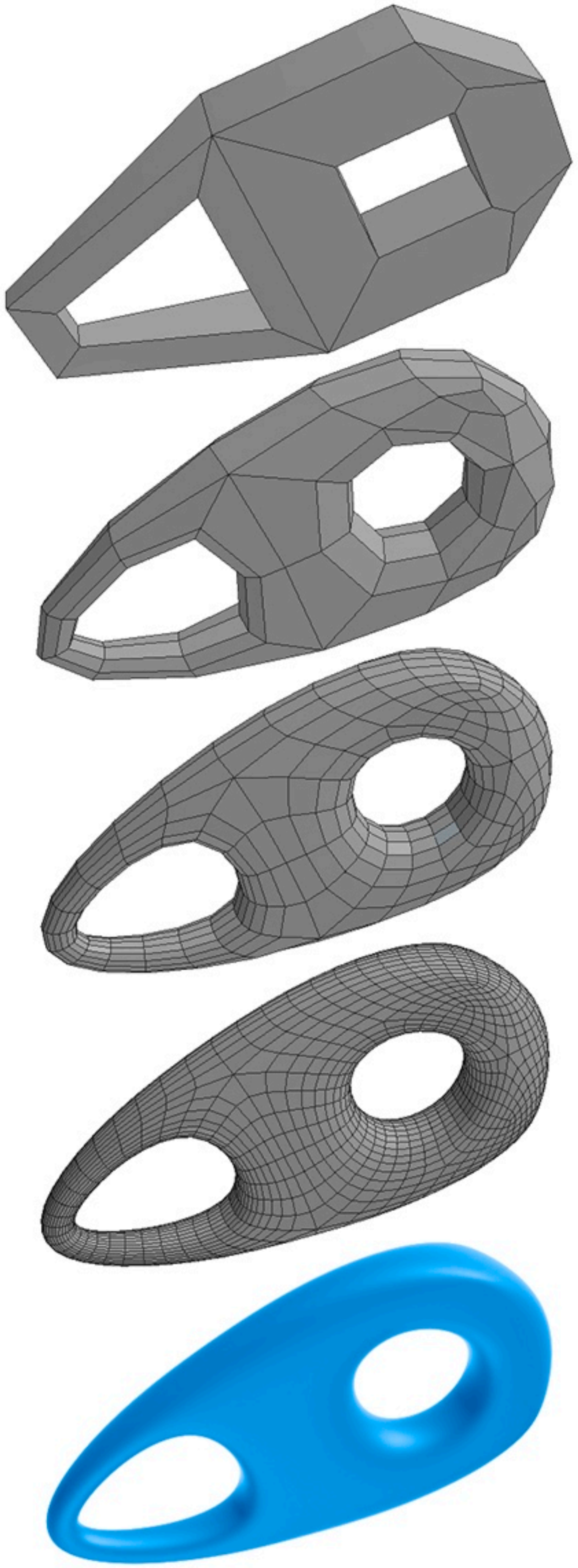
- **Alternative starting point for B-spline curves: *subdivision***
- **Start with control curve**
- **Insert new vertex at each edge midpoint**
- **Update vertex positions according to fixed rule**
- **For careful choice of averaging rule, yields smooth curve**
  - **Average with "next" neighbor (Chaikin): quadratic B-spline**
  - **Lane-Riesenfeld: B-spline curve of any degree**



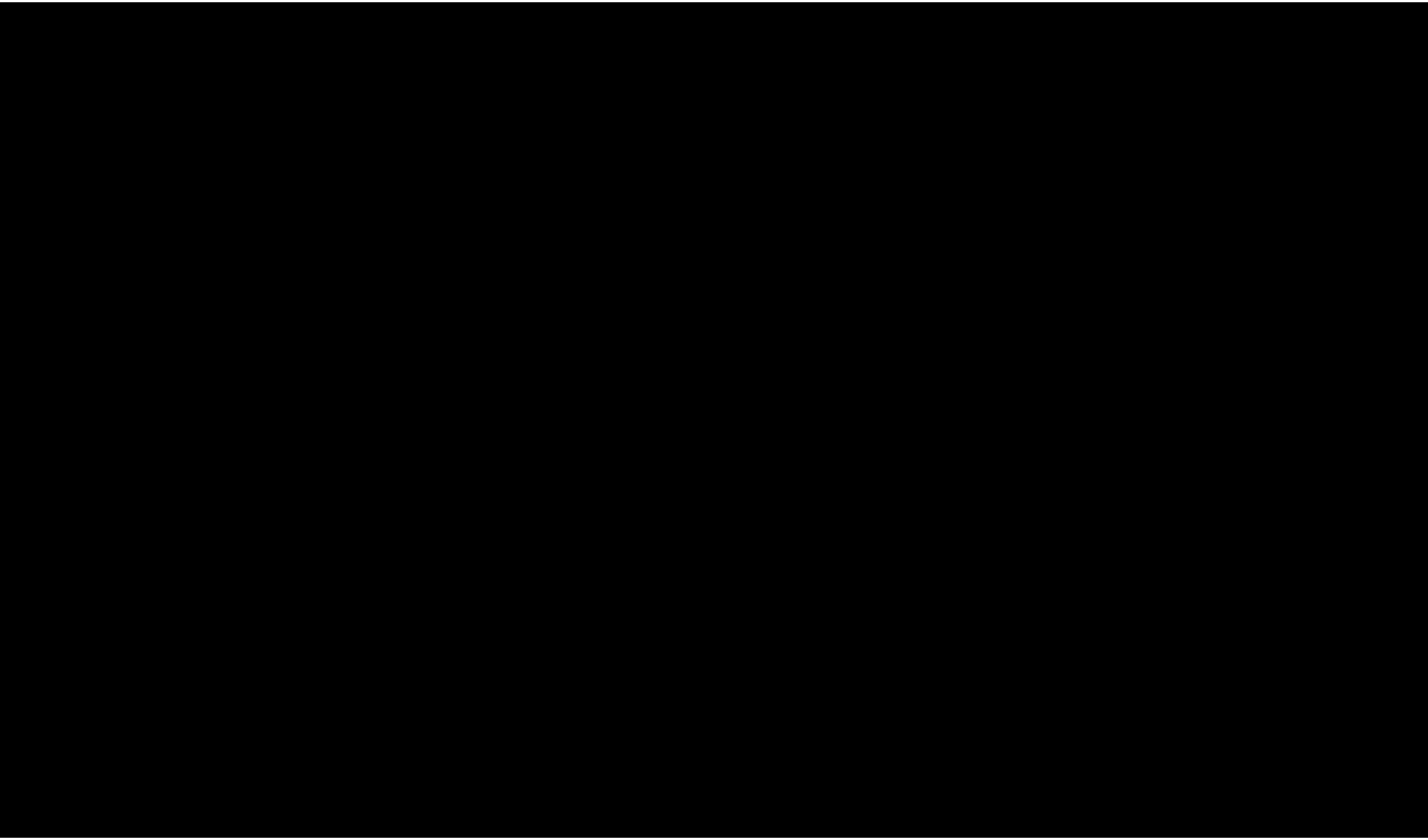
Subdivision Surfaces (Explicit)

- Start with coarse polygon mesh ("control cage")
- Subdivide each element
- Update vertices via local averaging
- Many possible rule:
  - Catmull-Clark (quads)
  - Loop (triangles)
  - ...
- Common issues:
  - interpolating or approximating?
  - continuity at vertices?
- Easier than NURBS for modeling; harder to guarantee continuity





Subdivision in Action (Pixar's "Geri's Game")



Next time: Curves, Surfaces, & Meshes