

Chapter 6

Dynamic Programming



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Algorithmic Paradigms

- Weighted interval scheduling problem.
- Knapsack problem.
- Shortest path problem.

Algorithmic Paradigms

Greed. Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer. Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

Dynamic Programming History

Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

Etymology.

- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
 - "it's impossible to use dynamic in a pejorative sense"
 - "something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography*.

Dynamic Programming Applications

Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, systems,

Some famous dynamic programming algorithms.

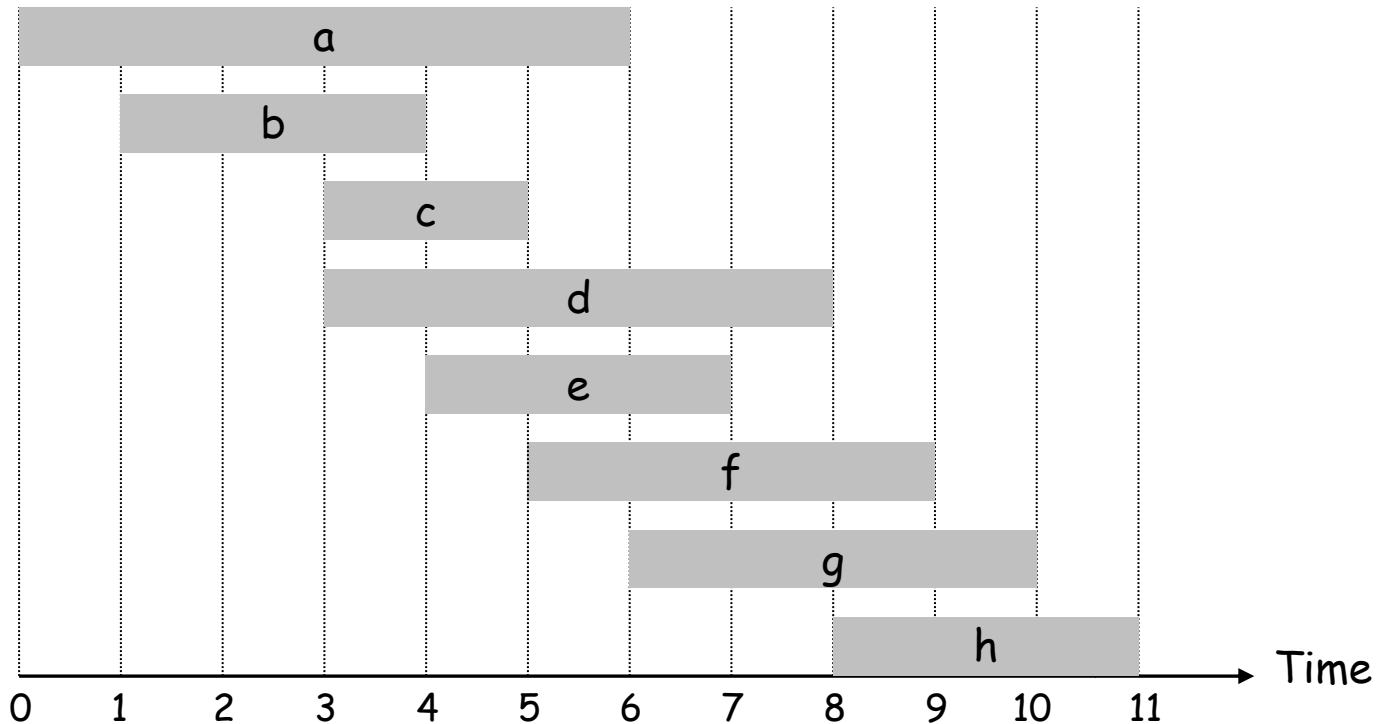
- Viterbi for hidden Markov models.
- Unix diff for comparing two files.
- Smith-Waterman for sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

6.1 Weighted Interval Scheduling

Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job j starts at s_j , finishes at f_j , and has weight or value v_j .
- Two jobs **compatible** if they don't overlap.
- Goal: find maximum **weight** subset of mutually compatible jobs.

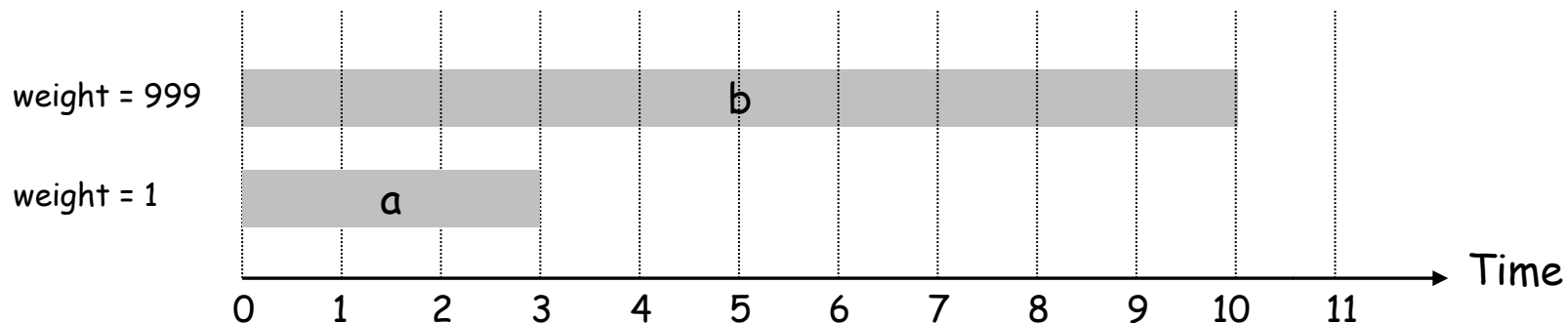


Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.

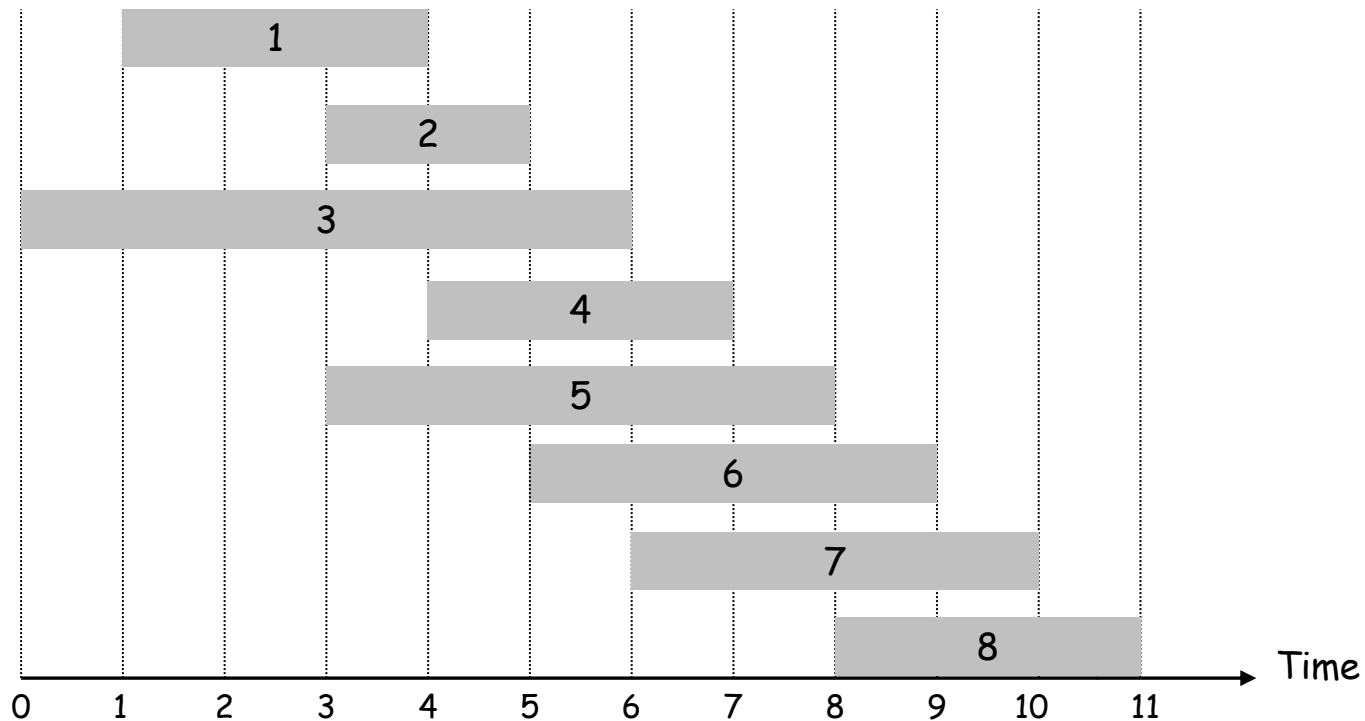


Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j .

Ex: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$.



Dynamic Programming: Binary Choice

Notation. $OPT(j)$ = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

- Case 1: OPT selects job j.
 - can't use incompatible jobs $\{ p(j) + 1, p(j) + 2, \dots, j - 1 \}$
 - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., $p(j)$
- Case 2: OPT does not select job j.
 - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., $j-1$

↖
↙
optimal substructure

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)), OPT(j-1) \} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

Brute force algorithm.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

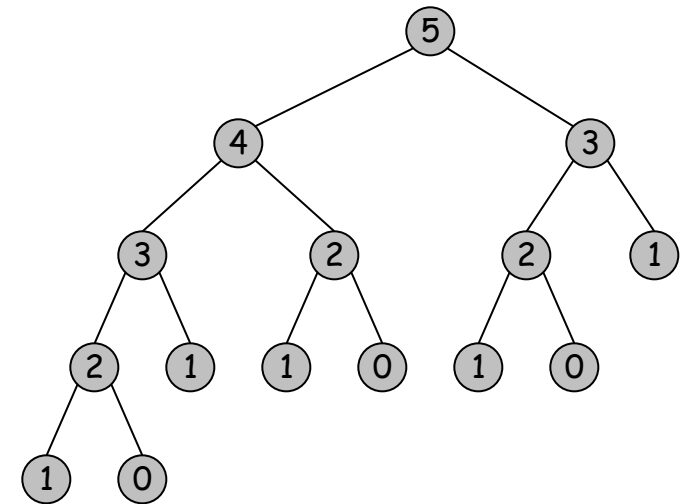
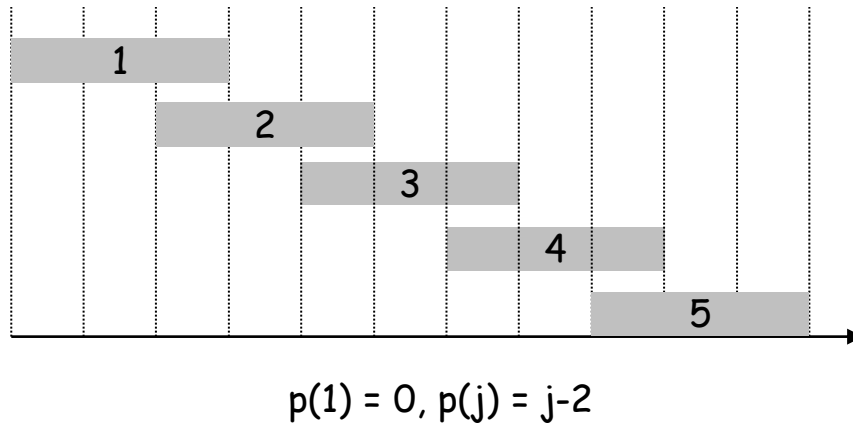
Compute $p(1), p(2), \dots, p(n)$

```
Compute-Opt(j) {  
    if (j = 0)  
        return 0  
    else  
        return max( $v_j + \text{Compute-Opt}(p(j))$ ,  $\text{Compute-Opt}(j-1)$ )  
}
```

Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems \Rightarrow exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

for $j = 1$ to n

$M[j] = \text{empty}$ \leftarrow global array

$M[j] = 0$

M-Compute-Opt(j) {

if ($M[j]$ is empty)

$M[j] = \max(w_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j-1))$

return $M[j]$

}

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

- Sort by finish time: $O(n \log n)$.
- Computing $p(\cdot)$: $O(n)$ after sorting by start time.
- $M\text{-Compute-Opt}(j)$: each invocation takes $O(1)$ time and either
 - (i) returns an existing value $M[j]$
 - (ii) fills in one new entry $M[j]$ and makes two recursive calls
- Progress measure $\Phi = \#$ nonempty entries of $M[\]$.
 - initially $\Phi = 0$, throughout $\Phi \leq n$.
 - (ii) increases Φ by 1 \Rightarrow at most $2n$ recursive calls.
- Overall running time of $M\text{-Compute-Opt}(n)$ is $O(n)$. ▀

Remark. $O(n)$ if jobs are pre-sorted by start and finish times.

Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
    if (j = 0)
        output nothing
    else if ( $v_j + M[p(j)] > M[j-1]$ )
        print j
        Find-Solution(p(j))
    else
        Find-Solution(j-1)
}
```

▫ # of recursive calls $\leq n \Rightarrow O(n)$.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

```
Iterative-Compute-Opt {  
    M[0] = 0  
    for j = 1 to n  
        M[j] = max( $v_j + M[p(j)]$ , M[j-1])  
}
```


6.4 Knapsack Problem

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

$$W = 11$$

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i .

Ex: { 5, 2, 1 } achieves only value = 35 \Rightarrow greedy not optimal.

Dynamic Programming: False Start

Def. $\text{OPT}(i)$ = max profit subset of items $1, \dots, i$.

- Case 1: OPT does not select item i .
 - OPT selects best of $\{1, 2, \dots, i-1\}$
- Case 2: OPT selects item i .
 - accepting item i does not immediately imply that we will have to reject other items
 - without knowing what other items were selected before i , we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. $OPT(i, w)$ = max profit subset of items 1, ..., i with weight limit w.

- Case 1: OPT does not select item i.
 - OPT selects best of $\{ 1, 2, \dots, i-1 \}$ using weight limit w
- Case 2: OPT selects item i.
 - new weight limit = $w - w_i$
 - OPT selects best of $\{ 1, 2, \dots, i-1 \}$ using this new weight limit

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max \{ OPT(i-1, w), v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n -by- W array.

```
Input:  $n, w_1, \dots, w_N, v_1, \dots, v_N$ 

for  $w = 0$  to  $W$ 
     $M[0, w] = 0$ 

for  $i = 1$  to  $n$ 
    for  $w = 1$  to  $W$ 
        if  $(w_i > w)$ 
             $M[i, w] = M[i-1, w]$ 
        else
             $M[i, w] = \max \{M[i-1, w], v_i + M[i-1, w-w_i]\}$ 

return  $M[n, W]$ 
```

Knapsack Algorithm

		<div> <div></div> <div>W + 1</div> <div></div> </div>											
		0	1	2	3	4	5	6	7	8	9	10	11
<div> <div>n + 1</div> <div></div> <div></div> <div></div> <div></div> <div></div> <div></div> </div>	ϕ	0	0	0	0	0	0	0	0	0	0	0	0
	{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
	{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
	{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
	{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
	{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Problem: Running Time

Running time. $\Theta(n W)$.

- Not polynomial in input size!
- "Pseudo-polynomial."
- Decision version of Knapsack is NP-complete. [Chapter 8]

Knapsack approximation algorithm. There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]

Dynamic Programming Summary

Recipe.

- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques.

- Binary choice: weighted interval scheduling.
- Multi-way choice: segmented least squares. ←
- Adding a new variable: knapsack.
- Dynamic programming over intervals: RNA secondary structure.

Viterbi algorithm for HMM also uses DP to optimize a maximum likelihood tradeoff between parsimony and accuracy

↖ CKY parsing algorithm for context-free grammar has similar structure

Top-down vs. bottom-up: different people have different intuitions.

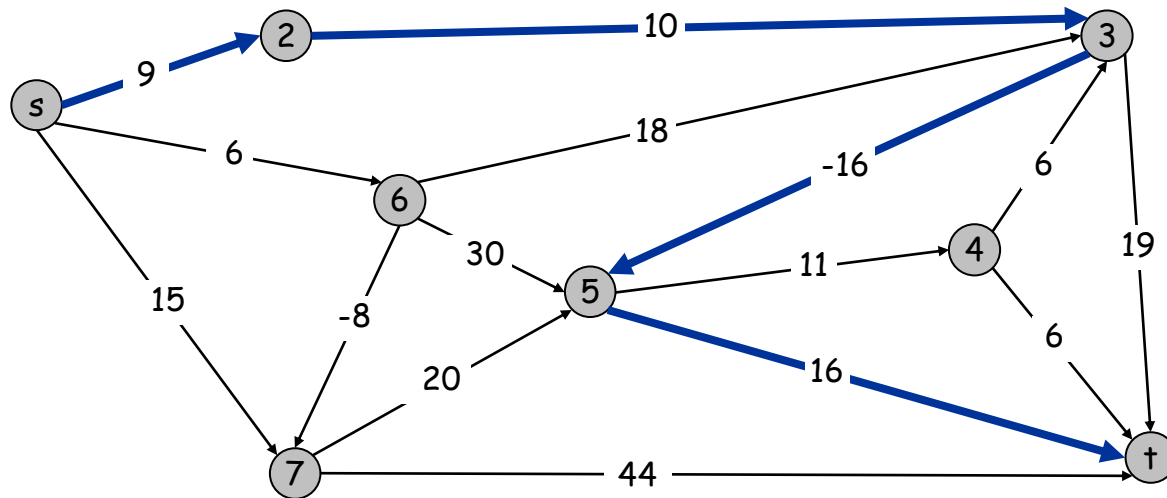
6.8 Shortest Paths

Shortest Paths

Shortest path problem. Given a directed graph $G = (V, E)$, with edge weights c_{vw} , find shortest path from node s to node t .

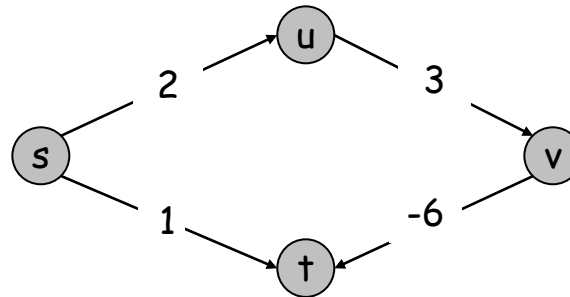
↖ allow negative weights

Ex. Nodes represent agents in a financial setting and c_{vw} is cost of transaction in which we buy from agent v and sell immediately to w .

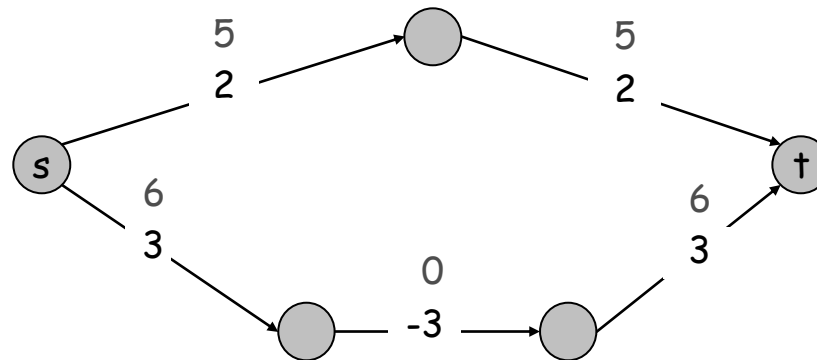


Shortest Paths: Failed Attempts

Dijkstra. Can fail if negative edge costs.

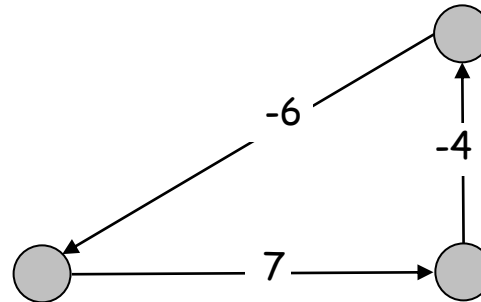


Re-weighting. Adding a constant to every edge weight can fail.

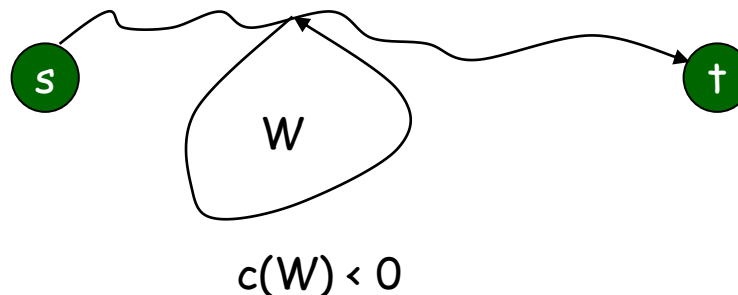


Shortest Paths: Negative Cost Cycles

Negative cost cycle.



Observation. If some path from s to t contains a negative cost cycle, there does not exist a shortest s - t path; otherwise, there exists one that is simple.



Shortest Paths: Dynamic Programming

Def. $OPT(i, v)$ = length of shortest v - t path P using at most i edges.

- Case 1: P uses at most $i-1$ edges.
 - $OPT(i, v) = OPT(i-1, v)$
- Case 2: P uses exactly i edges.
 - if (v, w) is first edge, then OPT uses (v, w) , and then selects best w - t path using at most $i-1$ edges

$$OPT(i, v) = \begin{cases} 0 & \text{if } i = 0 \\ \min \left\{ OPT(i-1, v), \min_{(v, w) \in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Remark. By previous observation, if no negative cycles, then $OPT(n-1, v)$ = length of shortest v - t path.

Shortest Paths: Implementation

```
Shortest-Path( $G, t$ ) {  
    foreach node  $v \in V$   
         $M[0, v] \leftarrow \infty$   
     $M[0, t] \leftarrow 0$   
  
    for  $i = 1$  to  $n-1$   
        foreach node  $v \in V$   
             $M[i, v] \leftarrow M[i-1, v]$   
            foreach edge  $(v, w) \in E$   
                 $M[i, v] \leftarrow \min \{ M[i, v], M[i-1, w] + c_{vw} \}$   
}
```

Analysis. $\Theta(mn)$ time, $\Theta(n^2)$ space.

Finding the shortest paths. Maintain a "successor" for each table entry.

Shortest Paths: Practical Improvements

Practical improvements.

- Maintain only one array $M[v]$ = shortest v - t path that we have found so far.
- No need to check edges of the form (v, w) unless $M[w]$ changed in previous iteration.

Theorem. Throughout the algorithm, $M[v]$ is length of some v - t path, and after i rounds of updates, the value $M[v]$ is no larger than the length of shortest v - t path using $\leq i$ edges.

Overall impact.

- Memory: $O(m + n)$.
- Running time: $O(mn)$ worst case, but substantially faster in practice.

Bellman-Ford: Efficient Implementation

```
Push-Based-Shortest-Path( $G, s, t$ ) {  
  foreach node  $v \in V$  {  
     $M[v] \leftarrow \infty$   
     $\text{successor}[v] \leftarrow \phi$   
  }  
  
   $M[t] = 0$   
  for  $i = 1$  to  $n-1$  {  
    foreach node  $w \in V$  {  
      if ( $M[w]$  has been updated in previous iteration) {  
        foreach node  $v$  such that  $(v, w) \in E$  {  
          if ( $M[v] > M[w] + c_{vw}$ ) {  
             $M[v] \leftarrow M[w] + c_{vw}$   
             $\text{successor}[v] \leftarrow w$   
          }  
        }  
      }  
    }  
    If no  $M[w]$  value changed in iteration  $i$ , stop.  
  }  
}
```


6.10 Negative Cycles in a Graph

Detecting Negative Cycles

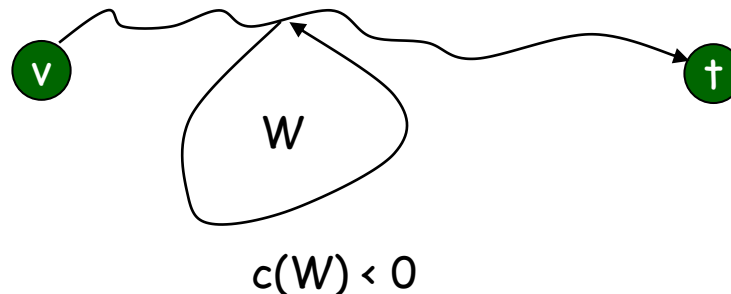
Lemma. If $\text{OPT}(n,v) = \text{OPT}(n-1,v)$ for all v , then no negative cycles.

Pf. Bellman-Ford algorithm.

Lemma. If $\text{OPT}(n,v) < \text{OPT}(n-1,v)$ for some node v , then (any) shortest path from v to t contains a cycle W . Moreover W has negative cost.

Pf. (by contradiction)

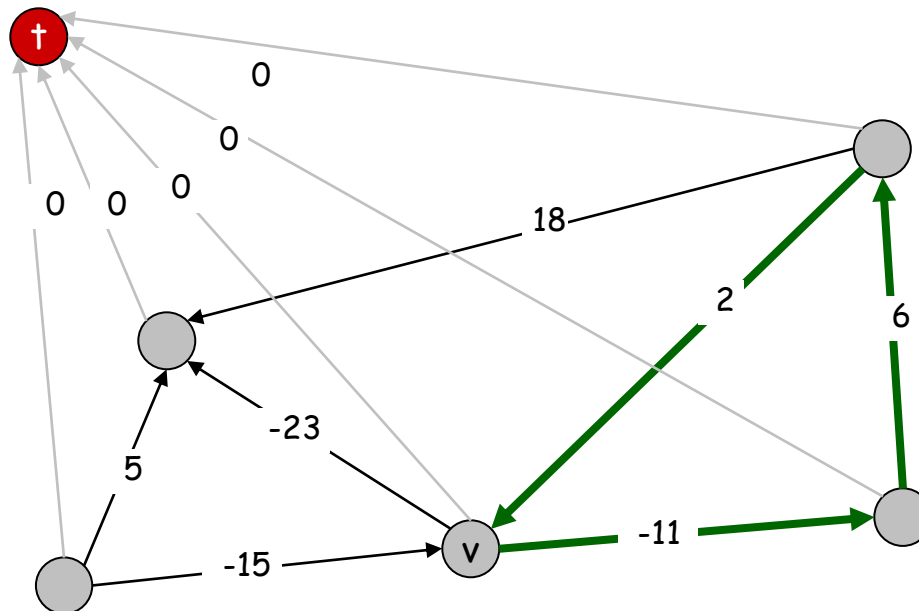
- Since $\text{OPT}(n,v) < \text{OPT}(n-1,v)$, we know P has exactly n edges.
- By pigeonhole principle, P must contain a directed cycle W .
- Deleting W yields a v - t path with $< n$ edges $\Rightarrow W$ has negative cost.



Detecting Negative Cycles

Theorem. Can detect negative cost cycle in $O(mn)$ time.

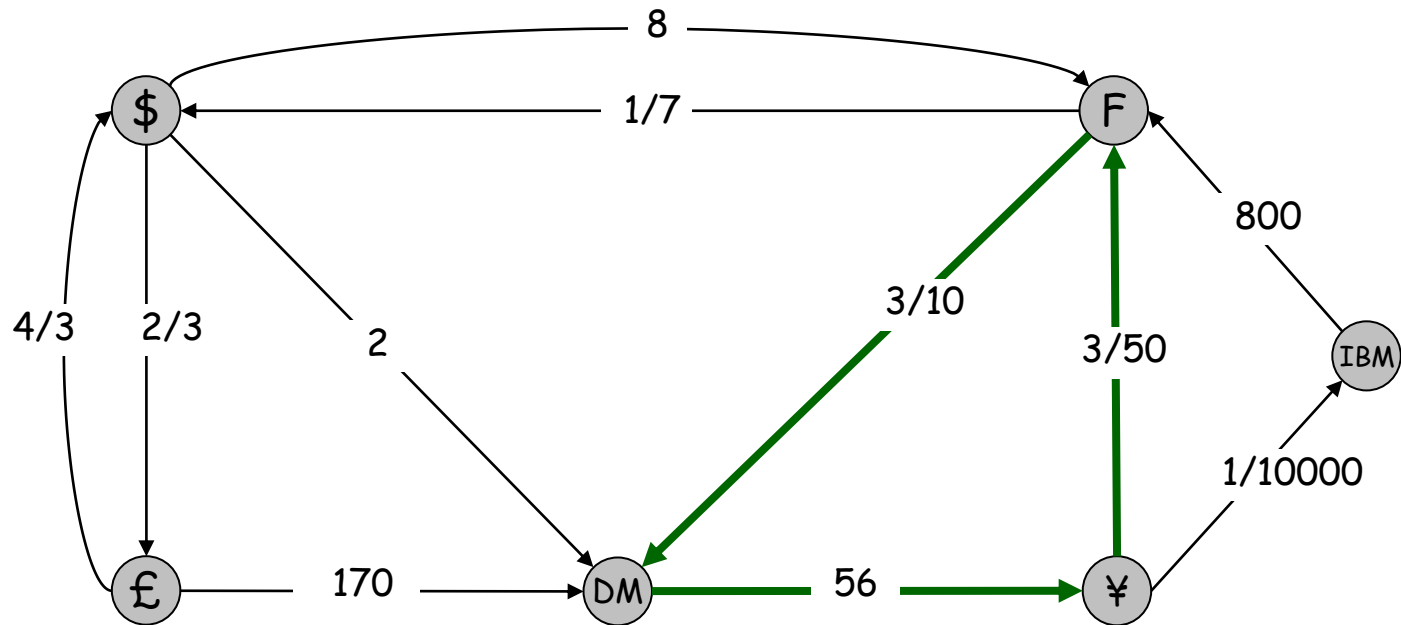
- Add new node t and connect all nodes to t with 0-cost edge.
- Check if $\text{OPT}(n, v) = \text{OPT}(n-1, v)$ for all nodes v .
 - if yes, then no negative cycles
 - if no, then extract cycle from shortest path from v to t



Detecting Negative Cycles: Application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!



Detecting Negative Cycles: Summary

Bellman-Ford. $O(mn)$ time, $O(m + n)$ space.

- Run Bellman-Ford for n iterations (instead of $n-1$).
- Upon termination, Bellman-Ford successor variables trace a negative cycle if one exists.
- See p. 288 for improved version and early termination rule.