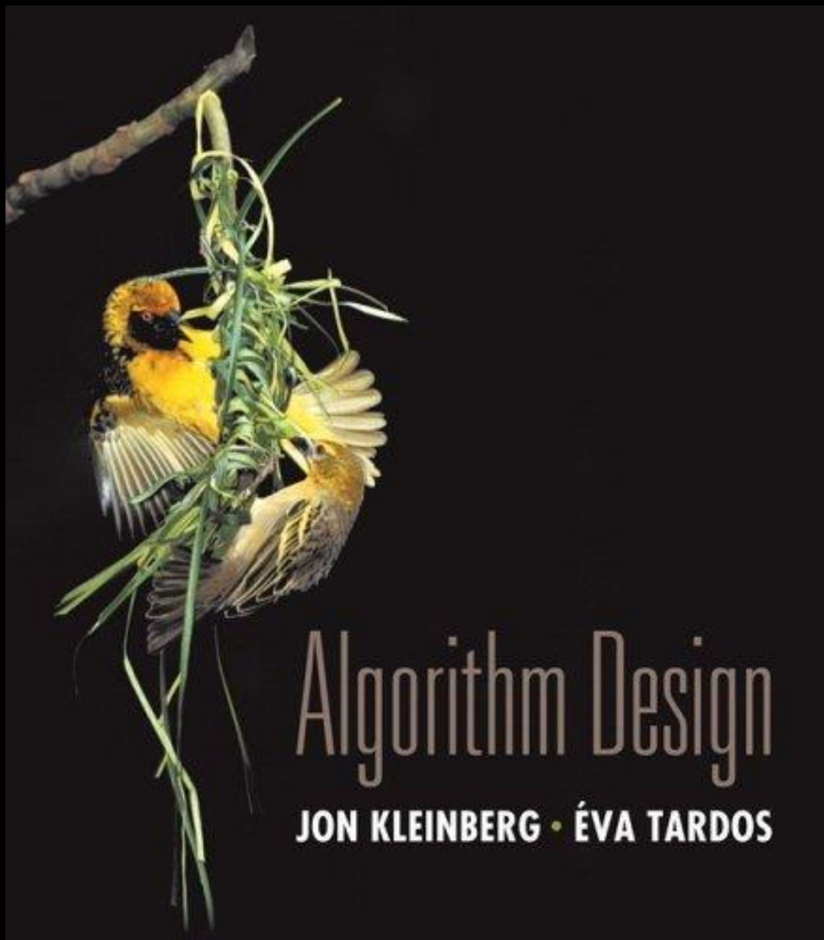


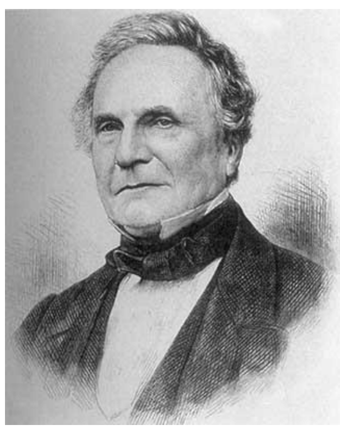
Chapter 2

Basics of Algorithm Analysis

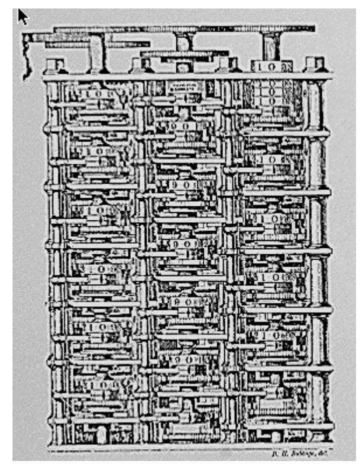
Slides by Kevin Wayne.

2.1 Computational Tractability

> As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - *Charles Babbage*



Charles Babbage (1864)



Analytic Engine (schematic)

Polynomial-Time

Brute force. For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.

- Typically takes 2^N time or worse for inputs of size N.
- Unacceptable in practice.

n! for stable matching with n men and n women

Desirable scaling property. When the input size doubles, the algorithm should only slow down by some constant factor C.

> There exists constants c > 0 and d > 0 such that on every input of size N, its running time is bounded by $c N^d$ steps.

Def. An algorithm is poly-time if the above scaling property holds.

choose $C = 2^d$

Worst-Case Analysis

Worst case running time. Obtain bound on largest possible running time of algorithm on input of a given size N.

- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

Average case running time. Obtain bound on running time of algorithm on random input as a function of input size N.

- Hard (or impossible) to accurately model real instances by random distributions.
- Algorithm tuned for a certain distribution may perform poorly on other inputs.

Worst-Case Polynomial-Time

Def. An algorithm is efficient if its running time is polynomial.

Justification: It really works in practice!

- Although $6.02 \times 10^{23} \times N^{20}$ is technically poly-time, it would be useless in practice.
- In practice, the poly-time algorithms that people develop almost always have low constants and low exponents.
- Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

Exceptions.

- Some poly-time algorithms do have high constants and/or exponents, and are useless in practice.
- Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare.

simplex method
Unix grep

Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

2.2 Asymptotic Order of Growth

Asymptotic Order of Growth

Upper bounds. $T(n)$ is $O(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $T(n) \leq c \cdot f(n)$.

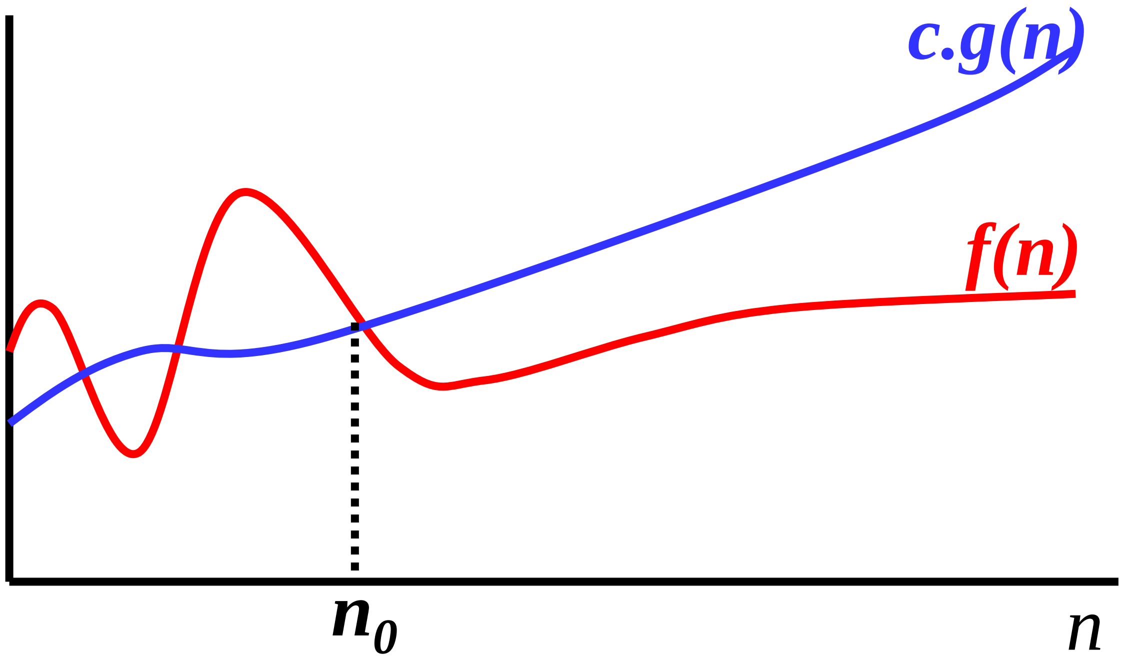
Lower bounds. $T(n)$ is $\Omega(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $T(n) \geq c \cdot f(n)$.

Tight bounds. $T(n)$ is $\Theta(f(n))$ if $T(n)$ is both $O(f(n))$ and $\Omega(f(n))$.

Ex: $T(n) = 32n^2 + 17n + 32$.

- $T(n)$ is $O(n^2)$, $O(n^3)$, $\Omega(n^2)$, $\Omega(n)$, and $\Theta(n^2)$.
- $T(n)$ is not $O(n)$, $\Omega(n^3)$, $\Theta(n)$, or $\Theta(n^3)$.

Asymptotic Order of Growth

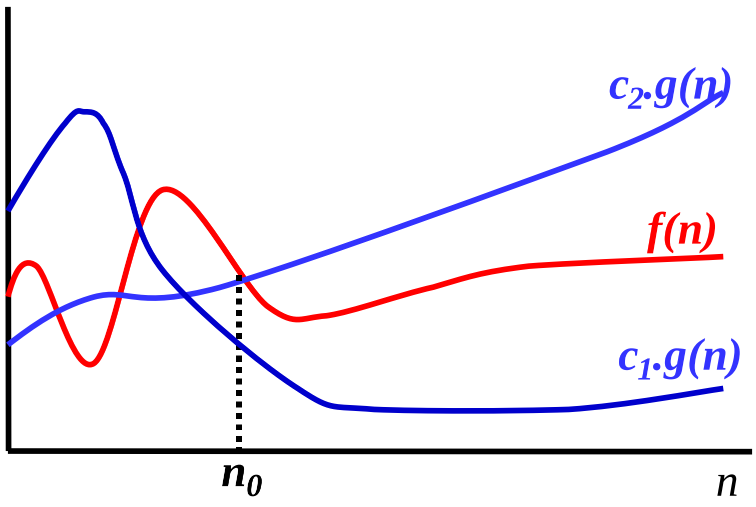


$$\boldsymbol{f(n) = O(g(n))}$$

There exist two constants ***c*** et ***n₀*** such that

$$0 \leq f(n) \leq c.g(n) \quad \text{for} \quad n \geq n_0$$

Asymptotic Order of Growth



$$f(n) = \Theta(g(n))$$

There exist three constants ***c_1, c_2 and n_0*** such that
$c_1 g(n) \le f(n) \le c_2 g(n) \quad$ *for* $n \ge n_0$

Using limits to compare order of growth of functions.

$$\lim_{n \to \infty} g(n)/f(n) = \begin{cases} 0 & \text{then } g(n) \in O(f(n)) \\ & \text{[but } g(n) \notin \Theta(f(n))\text{]} \\ c > 0 & \text{then } g(n) \in \Theta(f(n)) \\ \infty & \text{then } g(n) \in \boldsymbol{\Omega}(f(n)). \\ & \text{[but } g(n) \notin \Theta(f(n))\text{]} \end{cases}$$

Using limits to compare order of growth of functions.

L'Hôpital's Rule:

If the limit of the ratio f(x)/g(x) as x approaches 'a' exists in an indeterminate form (0/0 or $\pm\infty/\pm\infty$).

Then, $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x)$.

(L'Hôpital's Rule is only applicable when a number of conditions are met. But usually, it works for the type of functions that represent complexities of algorithms)

Notation

Slight abuse of notation. $T(n) = O(f(n))$.

- Asymmetric:
  - $f(n) = 5n^3$; $g(n) = 3n^2$
  - $f(n) = O(n^3) = g(n)$
  - but $f(n) \neq g(n)$.
- Better notation: $T(n) \in O(f(n))$.

Any comparison-based sorting algorithm requires Ω (n log n) comparisons.

Properties

Transitivity.

- If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

- If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
- If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
- If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$.

Asymptotic Bounds for Some Common Functions

Polynomials. $a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$.

Polynomial time. Running time is $O(n^d)$ for some constant d independent of the input size n.

Logarithms. $O(\log_a n) = O(\log_b n)$ for any constants a, b > 0.

can avoid specifying the base ($\log_a n = \log_b n \log_a b$)

Logarithms. For every x > 0, $\log n = O(n^x)$.

log grows slower than every polynomial

Exponentials. For every r > 1 and every d > 0, $n^d = O(r^n)$.

every exponential grows faster than every polynomial

Asymptotic Bounds for Some Common Functions

$\log n = O(n).$

$$\lim_{n\to\infty} g(n)/f(n) = \lim_{n\to\infty} (\log_2 n)/n$$

$$= \lim_{n\to\infty} (\log_e n)(\log_2 e)/n$$

$$= \lim_{n\to\infty} (1/n \log_2 e)/1$$

$$= \lim_{n\to\infty} (\log_2 e)/n$$

$$= 0$$

2.4 A Survey of Common Running Times

Linear Time: O(n)

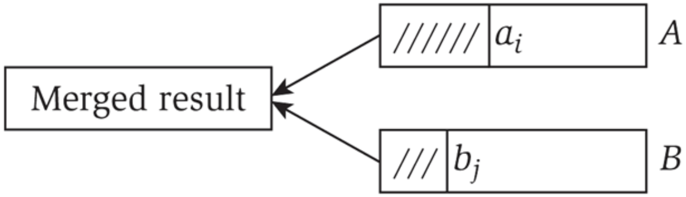
Linear time. Running time is at most a constant factor times the size of the input.

Computing the maximum. Compute maximum of n numbers $a_1, \ldots, a_n$.

```
max ← a1
for i = 2 to n {
   if (ai > max)
      max ← ai
}
```

Linear Time: O(n)

Merge. Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole.




```
i = 1, j = 1
while (both lists are nonempty) {
   if (a_i ≤ b_j) append a_i to output list and increment i
   else           append b_j to output list and increment j
}
append remainder of nonempty list to output list
```

Claim. Merging two lists of size n takes O(n) time.

Pf. After each comparison, the length of output list increases by 1.

O(n log n) Time

O(n log n) time. Arises in divide-and-conquer algorithms.

also referred to as linearithmic time

Sorting. Mergesort and heapsort are sorting algorithms that perform O(n log n) comparisons.

Largest empty interval. Given n time-stamps $x_1, \ldots, x_n$ on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

O(n log n) solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

Quadratic Time: $O(n^2)$

Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

```
min ← (x1 - x2)^2 + (y1 - y2)^2
for i = 1 to n {
   for j = i+1 to n {
      d ← (xi - xj)^2 + (yi - yj)^2
      if (d < min)
         min ← d
   }
}
```

← don't need to take square roots

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion. ← see chapter 5

Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Given n sets $S_1, \ldots, S_n$ each of which is a subset of 1, 2, …, n, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

```
foreach set Si {
   foreach other set Sj {
      foreach element p of Si {
         determine whether p also belongs to Sj
      }
      if (no element of Si belongs to Sj)
         report that Si and Sj are disjoint
   }
}
```

Polynomial Time: $O(n^k)$ Time

Independent set of size k. Given a graph, are there k nodes such that no two are joined by an edge?

k is a constant

$O(n^k)$ solution. Enumerate all subsets of k nodes.

```
foreach subset S of k nodes {
   check whether S in an independent set
   if (S is an independent set)
      report S is an independent set
   }
}
```

- Check whether S is an independent set = $O(k^2)$.
- Number of k element subsets = $\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k\,(k-1)\,(k-2)\cdots(2)\,(1)} \le \frac{n^k}{k!}$
- $O(k^2\, n^k / k!) = O(n^k)$.

poly-time for k=17,
but not practical

Exponential Time

Independent set. Given a graph, what is maximum size of an independent set?

$O(n^2\, 2^n)$ solution. Enumerate all subsets.

```
S* ← φ
foreach subset S of nodes {
   check whether S in an independent set
   if (S is largest independent set seen so far)
      update S* ← S
   }
}
```