Necessary conditions for Ternary Algebras

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Abstract

Ternary algebras, constructed from ternary commutators, or as we call them, ternutators, defined as the alternating sum of products of three operators, have been shown to satisfy cubic identities as necessary conditions for their existence. Here we examine the situation where we permit identities not solely constructed from ternutators or nested ternutators and we find that in general, these impose additional restrictions; for example, the anti-commutators or commutators of the operators must obey some linear relations among themselves.

1 Introduction

The subject of ternary algebras, a special case of n-Lie algebras is generally attributed to Fillipov [1], but Filippov was following up on earlier studies that had appeared in the mathematics literature, primarily by Kurosh [2] (as remarked in [3]). Its appearance in Physics was due to the pioneering work of Nambu [4], and more recently, the work of Bagger and Lambert [5] renewed interest in ternary algebras in the Theoretical Physics community (see also the review article [6]). In general, a Ternary Bracket is a composition law for three operators, which is completely antisymmetric in the three operators; as for example Nambu Brackets, which are an extension of the idea of Poisson Brackets to three functions. In this article we study the purely algebraic structure of the algebra, with a product structure for the operators clearly in mind, and we refer to such brackets as ternutators.

2 Ternutator basics

The ternutator bracket is a completely antisymmetrised trilinear composition law for three associative operators, just as the commutator is for two operators

$$[A,B,C] \equiv ABC + BCA + CAB - ACB - CBA - BAC \qquad (1)$$

$$\equiv \frac{1}{2} \left([[A, B], C]_{+} + [[B, C], A]_{+} + [[C, A], B]_{+} \right). \tag{2}$$

Note the appearance of anti-commutators; if instead all the brackets in (2) are commutators, the right hand side is the Jacobi identity for Lie Brackets, and is zero. Corresponding operator algebras would read

$$[A_i, A_j, A_k] = f_{ijk}^{\ m} A_m, \tag{3}$$

where the structure constants $f_{ijk}^{\ \ m}$ are completely antisymmetric in i, j, k. For a simplified notation let us write

$$(ijkl...) \equiv A_i A_j A_k A_l...$$
 (4)

$$[ij]_{\pm} \equiv [A_i, A_j]_{\pm}$$

$$[ijk] \equiv [A_i, A_j, A_k]$$

$$(5)$$

$$(6)$$

$$[ijk] \equiv [A_i, A_j, A_k] \tag{6}$$

respectively for the product of n arbitrary operators (4), for the anticommutator or commutator (5), and for the ternutator (6).

3 Normal and non-normal order

We associate to each operator a label L equal to its index $L(A_{j_1}) = j_1$. For a product of three operators $A_{j_1}A_{j_2}A_{j_3}$, we say that they are in normal order if at least one of the set of indices $\{j_1, j_2\}$ and $\{j_2, j_3\}$ is in increasing order. For three given operators, five of their products are in "'normal"' order and one is in "non-normal" order. Example

$$(321)$$
 non-normal order (7)

$$(123), (132), (213), (231), (312)$$
 normal order. (8)

One clearly has

$$(321) \equiv [321] + (123) - (132) - (213) + (231) + (312). \tag{9}$$

In other words, the triple non-normal product (7) can be written as a sum of normal triple products up to a ternutator which is, through (3), of degree one in the operators (a decrease by two degrees). More generally, any product of three operators in non-normal order can be written as the sum of operators in normal order up to a ternutator.

For
$$i_3 > i_2 > i_1$$

$$(i_3i_2i_1) \equiv -[i_1i_2i_3] + (i_1i_2i_3) + (i_2i_3i_1) + (i_3i_1i_2) - (i_1i_3i_2) - (i_2i_1i_3). (10)$$

To simplify, we consider products which involve only operators with different indices

$$A_{j_1}A_{j_2}A_{j_3}\dots$$
 with $A_{j_k} \neq A_{j_m}$. for $k \neq m$. (11)

At the next ternutator level we have to define the non-normal product of one ternutator and two operators or of two ternutators and one operator leading to the appearance of a ternutator of ternutators. At higher levels, one obtains nested ternutators of ternutators.

Let us associate as follows a label L with a nested ternutator. Take all the indices i_1, \ldots, i_n of the operators which enter in the nested ternutator and define

$$L = \min\{i_1, \dots, i_n\}. \tag{12}$$

It is then easy to define the non-normal product of three nested ternutators T_3, T_2 and T_1 with label L_3, L_2, L_1 respectively. They are those with $L_3 > L_2 > L_1$. They are transformed into normal products by the obvious

$$T_3T_2T_1 = -[T_1, T_2, T_3] + T_3T_1T_2 + T_1T_2T_3 + T_2T_3T_1 + T_3T_1T_2 - T_2T_1T_3 - T_1T_3T_2.$$
(13)

The ternutator $[T_1, T_2, T_3]$ of higher nesting has label $L = L_1$.

Using these definitions, any product of different operators can be transformed in a sum of normal products.

Starting from a given product of operators, there are often many different paths which can be followed to transform them in normal order. The difference between two results when they are different lead what we call ternutator identities.

4 General considerations about identities

It is known that ternutators enjoy the seven Bremner-Nuyts identities among seven operators. These identities, [7][8], play the rôle of the Jacobi identity for ternary algebras and generate cubic necessary conditions on the structure constants of these algebras. These identities are also satisfied by Nambu Brackets,[4] a trilinear antisymmetric composition law for three operators, which associates with three functions f(x, y, z), g(x, y, z), h(x, y, z) a ternary bracket of the form

$$[f,g,h] = \det \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} \end{bmatrix}, \tag{14}$$

just as the Poisson Bracket of two functions obeys the Jacobi identity of Lie brackets. This is discussed further in [9]. They are also satisfied by other trilinear composition laws, such as that of Awata et al [10];

$$[A_i, A_j, A_k]_{Aw} = [A_i, A_j] < A_k > + [A_j, A_k] < A_i > + [A_k, A_i] < A_j >,$$
 (15)

where $\langle A_k \rangle$ denotes the trace of the operator A_k . An important question is related to the sufficiency of these conditions. For Ternary algebras depending upon a composition law which intrinsically requires the composition of three operators, only ternutators of ternutators are allowed in the search for identities. However, since we also have a product at our disposal in terms of our definition (2), we can search for more general identical relations among the operators of the

algebra. In this article we show that, much to our surprise, there exist identities involving four and six operators which lead also to necessary requirements. If we assume linear independence of the anti-commutators (or commutators) of the operators of the algebra, the identities for four (or six) operators severely restrict the allowable form of the structure constants. Conversely, the four operator and six operator identities may be interpreted as linear relations among the anti-commutators or commutators of operators of the algebra.

5 Products of Operators. Identities

In this section, we discuss the identities which can be obtained starting with products of a certain number of operators.

Quite generally, we suppose that the basic operators conveniently labeled A_1, A_2, \ldots (in some fixed order of the indices) are defined to be linearly independent.

The identities among products of operators are of two kinds.

- 1. First, we have the sui generis identities which involve nested products of termutators where all the operators in the products appear in the form of ternutators only. It implies in particular that the products must involve an odd number of generators. In an earlier article [8], we showed that there are no sui generis identities for products of five operators and seven identities for products of seven operators. For ternutator algebras (3), since the operators themselves are independent, they lead to cubic necessary conditions for the structure constants.
- Identities where some operators may appear in an unnested positions. In this section, we concentrate on such identities for products of four, five and six operators. There are obviously no such identities for products of two or three operators.

5.1 Products of four operators

The non-normal products of four operators are of three forms

• The exceptional product

$$(4321)$$
 (16)

where both the three first operators and the three last operators are not in normal form.

• The products

$$(\underline{4312})$$
, $(\underline{4213})$, $(\underline{3214})$ (17)

$$(1432)$$
, (2431) , (3421) . (18)

In the sets, we have underlined the operators which are not in normal form. They are the three first operators in (17) and the three last operators in (18).

It is easy to see that, using (9), the products in (17) and in (18) can be brought to normal form by following a path which is unique. This is not the case for the product (4321) in (16) where two different paths can be followed depending on which set of three operators is used first the set (432) or the set (321). Let us follow the two paths. For path 1, one finds

$$\frac{(4321)}{(4321)} = ([432]1) + (4231) + (3421) + (3241) - (2341) - (2431)
= ...
= ([432]1) + (3[421]) - (2[431]) - ([321]4)
+ (4231) + (3412) - (3142) - (2413)
+ (2143) - (1234) + (1324).$$
(19)

Following the same pattern, one finds for path 2

$$(4\underline{321}) = (4[321]) - ([421]3) + ([431]2) - (1[432]) + (4231) + (3412) - (3142) - (2413) + (2143) - (1234) + (1324).$$
(20)

Subtracting the results of (19) and (20), one finds the identity

$$I_4(4,3,2,1) = [[432]1]_+ - [[431]2]_+ + [[421]3]_+ - [[321]4]_+ = 0.$$
 (21)

Note the appearance of anti-commutators. This identity has also appeared in Curtright and Zachos [9], in equation (84) of that paper, but these authors did not pursue its implications. The identity can be written as

$$I_4(i_4, i_3, i_2, i_1) = \sum_{\substack{\{\mu_1, \mu_2, \mu_3, \mu_4\} \in S_4 \\ \mu_1 < \mu_2 < \mu_3}} \operatorname{sign}(P) \left[\left[A_{\mu_1}, A_{\mu_2}, A_{\mu_3} \right], A_{\mu_4} \right]_+ = 0,$$
 (22)

where S_4 are permutations of $\{i_1, i_2, i_3, i_4\}$, and sign(P) their sign.

These relations imply new necessary conditions on ternutator algebras (see below).

Indeed there also exist analogues to the identity of degree 4 for Nambu Brackets, three of them to be precise. These follow simply from the following argument. Consider the expansion of the determinant

$$\det \begin{vmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial g}{\partial \alpha} & \frac{\partial h}{\partial \alpha} & \frac{\partial k}{\partial \alpha} \\ \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} & \frac{\partial k}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} & \frac{\partial k}{\partial y} \\ \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} & \frac{\partial k}{\partial z} \end{vmatrix} \equiv 0, \tag{23}$$

for $\alpha = x, y, \text{ or } z$. Expanding the determinant on the first row we have

$$\frac{\partial f}{\partial \alpha} \big[g, h, k \big]_{\scriptscriptstyle NB} - \frac{\partial g}{\partial \alpha} \big[h, k, f \big]_{\scriptscriptstyle NB} + \frac{\partial h}{\partial \alpha} \big[k, j, g \big]_{\scriptscriptstyle NB} - \frac{\partial k}{\partial \alpha} \big[f, g, h \big]_{\scriptscriptstyle NB} = 0. \tag{24}$$

These equations, for the various choices of α , are the analogues of the identity (21), for Nambu Brackets.

5.2 Products of five operators

Take five independent operators A_1, \ldots, A_5 , a REDUCE computation shows explicitly that there are ten independent identities involving the product of two of the operators and one ternutator (made of the three remaining operators), namely $[A_i, A_j, A_k]A_lA_m$, $A_l[A_i, A_j, A_k]A_m$ and $A_lA_m[A_i, A_j, A_k]$. There are a priori 60 such products and hence 60 arbitrary coefficients; (5 choices for l, and 4 for m in each of the three classes). Grouping these terms as $A_lA_m[A_i, A_j, A_k] + A_l[A_i, A_j, A_k]A_m$ and $[A_i, A_j, A_k]A_lA_m + A_l[A_i, A_j, A_k]A_m$, we see that they fall into ten sets which are identities in consequence of the four operator identity.

There are 50 independent relations among these coefficients. Through RE-DUCE, it has been checked explicitly that a basis for the ten identities is given by the ten products

$$A_5I_4(4,3,2,1), A_4I_4(5,3,2,1), A_3I_4(5,4,2,1), A_2I_4(5,4,3,1), A_1I_4(5,4,3,2)$$

 $I_4(4,3,2,1)A_5, I_4(5,3,2,1)A_4, I_4(5,4,2,1)A_3, I_4(5,4,3,1)A_2, I_4(5,4,3,2)A_1$ (25)

They all involve four operator identities. Hence, no new identity exists at this degree. This has also been checked explicitly by hand. Remember that there are no sui generis identities for five operators.

5.3 Products of six operators

There are 21 identities involving the product of six operators A_1, \ldots, A_6 and which are quadratic in ternutators.

• There are $C_6^3 = 20$ identities which are simple consequences of the four identity. They are indexed by the choice of separating the six operators in two non overlapping sets $\{i_1 < i_2 < i_3\}$ and $\{i_4 < i_5 < i_6\}$. They are conveniently written

$$\sum_{\substack{\{j_{4},j_{5},j_{6}\}\in P_{3}(\{i_{4},i_{5},i_{6}\})\\j_{4}< j_{5}}} \operatorname{sign}(P) \left[\left[\left[A_{i_{1}},A_{i_{2}},A_{i_{3}} \right],A_{j_{4}},A_{j_{5}} \right] A_{j_{6}} \right]_{+} \\ -\sum_{\substack{\{j_{1},j_{2},j_{3}\}\in P_{3}(\{i_{1},i_{2},i_{3}\})\\j_{1}< j_{2}}} \operatorname{sign}(P) \left[\left[\left[A_{i_{4}},A_{i_{5}},A_{i_{6}} \right],A_{j_{1}},A_{j_{2}} \right] A_{j_{3}} \right]_{+} = 0. \tag{26}$$

• The remaining identity can be written democratically as

$$6 \sum_{\substack{\mu \in S_6 \\ \mu_1 < \mu_2 < \mu_3, \ \mu_4 < \mu_5 < \mu_6, \ \mu_1 < \mu_4}} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} \left[\left[A_{\mu_1}, A_{\mu_2}, A_{\mu_3} \right], \left[A_{\mu_4}, A_{\mu_5}, A_{\mu_6} \right] \right]_{-} \\
- \sum_{\substack{\mu \in S_6 \\ \mu_1 < \mu_2 < \mu_3, \ \mu_4 < \mu_5}} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} \left[\left[\left[A_{\mu_1}, A_{\mu_2}, A_{\mu_3} \right], A_{\mu_4}, A_{\mu_5} \right], A_{\mu_6} \right]_{-} = 0. (27)$$

This identity can be written in many apparently different forms adding terms which are zero when using the four-identity.

We have first obtained the exceptional identity (27) by using Reduce to built all the possible identities. Another approach to this identity is to use the Bremner-Nuyts [7][8] seven identity in the form of equation (13) of [8] singling out a A_7 operator on the right. Then the coefficient of A_7 constitutes the identity at level 6. Another way is to use the path analysis for six operators, one obtains an identity by comparing for the initial configuration (654321), the path starting with (654321) with the path starting for example by (654321). Any other path starting with (654321) or (654321) and where 6 is a spectator will give the same identity as all these three paths are equivalent as they involve five operators known to be equivalent through I4. This is another proof that the 6-identity is unique.

There seem to be no identities for six operators linear in the structure constants, i.e. of first degree in the ternutators of the basic operators, except those which follow from the 4-identity. This was essentially proved using Reduce.

6 Conditions on the structure constants from the identities of degree four

The identity (21) implies the following type of conditions on the structure constants

$$f_{432}^{\ m}[A_m, A_1]_+ - f_{431}^{\ m}[A_m, A_2]_+ + f_{421}^{\ m}[A_m, A_3]_+ - f_{321}^{\ m}[A_m, A_4]_+ = 0.$$
 (28)

If we assume that the anti-commutators are linearly independent, we find the conditions

From
$$A_1^2 \longrightarrow f_{432}^{-1} = 0.$$
 (29)

• From
$$A_1 A_2 \longrightarrow f_{432}^2 - f_{431}^1 = 0$$
 (30)

• Using $m \notin \{1, 2, 3, 4\}$, we recover a condition of the form (29)

From
$$A_1 A_5 \longrightarrow f_{432}^{5} = 0.$$
 (31)

Summarizing the results, we find

$$\begin{array}{lcl} f_{ijk}^{\ m} & = & 0 & \text{for } m \notin \{i, j, k\} \\ f_{ijm}^{\ m} & = & f_{ijk}^{\ k} \ \forall \ m, k \notin \{i, j\}. \end{array} \tag{32}$$

These results severely restrict the possible ternary algebras unless we drop the assumption of independence, and re-interpret (32) as a set of linear relations among the anti-commutators of the operators. This is a surprising result.

7 Alternative iterative approach

There is another way of looking at the identities which we have found. Consider the following iterative situation;

$$[[A_1, A_2], A_3] + [[A_2, A_3], A_1] + [[A_3, A_2], A_1] = 0,$$
 (33)

$$\left[[A_1,A_2],A_3]_+ + [[A_2,A_3],A_1]_+ + [[A_3,A_2],A_1]_+ \right. \\ = \left. 2[A_1,A_2,A_3]. \right. (34)$$

The first is the Jacobi identity, the second is twice the ternutator $[A_1, A_2, A_3]$ Now add one more operator

$$\begin{aligned} & [[A_1, A_2, A_3], A_4]_+ - [[A_2, A_3, A_4], A_1]_+ \\ & + [[A_3, A_4, A_1], A_2]_+ - [[A_4, A_1, A_2], A_3]_+ & = 0. \end{aligned} \tag{35}$$

$$[[A_1, A_2, A_3], A_4] - [[A_2, A_3, A_4], A_1] + [[A_3, A_4, A_1], A_2] - [[A_4, A_1, A_2], A_3] = 2[A_1, A_2, A_3, A_4].$$
(36)

The first equation is the 4-identity $I_4(4321) = 0$ and the second equation is the definition of the antisymmetric 4-bracket $[A_1, A_2, A_3, A_4]$ with an additional factor of 2.

This pattern of repeated nested alternating commutators and anti-commutators persists. Writing the antisymmetric n-bracket in a generalized notation (5), (6) as

$$[i_1 i_2 \dots i_n] = [A_{i_1}, A_{i_2}, \dots, A_{i_n}],$$
 (37)

we find

• For an even number of operators

$$[i_{1}i_{2}...i_{2n}] = \frac{1}{2} \sum_{\text{cyclic}} \operatorname{sign}(C) [[i_{1}i_{2}...i_{2n-1}] i_{2n}]_{-}$$

$$0 = \sum_{\text{cyclic}} \operatorname{sign}(C) [[i_{1}i_{2}...i_{2n-1}] i_{2n}]_{+}$$
(38)

where the summation is over the cyclic permutation of $\{i_1, i_2, \dots, i_{2n}\}$ and sign(C) is the sign of the permutation.

• For an odd number of operators

$$0 = \sum_{\text{cyclic}} \operatorname{sign}(C) \left[[i_1 i_2 \dots i_{2n}] i_{2n+1} \right]_{-}$$
$$[i_1 i_2 \dots i_{2n+1}] = \frac{1}{2} \sum_{\text{cyclic}} \operatorname{sign}(C) \left[[i_1 i_2 \dots i_{2n}] i_{2n+1} \right]_{+}$$
(39)

with summation over the cyclic permutations of $\{i_1, i_2, \dots, i_{2n+1}\}$.

As has been demonstrated, the identity at level 5 is not new. The level 6 identities contain the 20 identities (26) in a democratic fashion.

8 Discussion and Conclusion

For a pure mathematician, the only relevant identities are those composed of only ternary operations, i.e operations sui generis, of nested ternutators, just as for Lie algebras, the relevant identities are those composed of iterated or nested commutators, and SuperJacobi identities are ignored. It is known that in this case the Jacobi identity is both necessary and sufficient. In the case of ternary algebras, we have demonstrated an identity at the level of four operators (and one identity at the level of six operators), which involves anti-commutators (or commutators) and which can be interpreted in various ways. Either we assume that anti-commutators (or commutators) of all the operators involved are independent of one another, in which case there are very few viable ternary algebras, or we assert that having found a representation of the ternutator algebra, linear relations amongst the anti-commutators (or commutators) must be automatically satisfied. Perhaps we may exploit these relations in the search for representations of ternutators.

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