PAPER

No Bel–Robinson tensor for quadratic curvature theories

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No Bel–Robinson tensor for quadratic curvature theories

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Abstract

We attempt to generalize the familiar covariantly conserved Bel–Robinson tensor $B_{\mu\nu\alpha\beta} \sim RR$ of GR and its recent topologically massive third derivative order counterpart $B \sim R\,DR$ to quadratic curvature actions. Two very different models of current interest are examined: fourth-order $D = 3$ 'new massive gravity' and second-order $D > 4$ Lanczos–Lovelock. On dimensional grounds, the candidates here become $B \sim DRDR + RRR$. For the $D = 3$ model, there indeed exist conserved $B \sim \partial R\partial R$ in the linearized limit. However, despite a plethora of available cubic terms, B cannot be extended to the full theory. The $D > 4$ models are not even linearizable about flat space, since their field equations are quadratic in curvature; they also have no viable B, a fact that persists even if one includes cosmological or Einstein terms to allow linearization about the resulting dS vacua. These results are an unexpected, if hardly unique, example of linearization instability.

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1. Introduction

The Bel–Robinson tensor $B_{\mu\nu\alpha\beta}$ was first discovered in ordinary Einstein gravity (GR) at $D = 4$, in a search for a gravitational counterpart of the usual matter stress tensor $T_{\mu\nu}$. Since there can be neither local tensorial gravitational candidates of second derivative order (because quantities $\sim \partial g_{\mu\nu}\,\partial g_{\alpha\beta}$, being frame-dependent, can be made to vanish at any point) nor any non-covariantly conserved ones, the successful candidate was indeed quadratic in the tensorial 'field strengths'—curvatures $B \sim RR$, and covariantly conserved, in analogy with the quadratic Maxwell $T_{\mu\nu} \sim F\,F$. Subsequently, conserved B were found for arbitrary dimension, and matter analogs have also been constructed (see, e.g., [1] for earlier references).

Very recently, we showed that a conserved $B \sim R\,DR$ could be defined for topologically massive gravity (TMG) [2]. Given this theory's third derivative order, we speculated there on

extending B to other gravitational systems, in particular, to the currently popular quadratic curvature models. This paper reports a negative outcome in two active, very different theories. For the fourth derivative $D = 3$ new massive gravity (NMG) [3], without an Einstein term [4] for simplicity, there is a $B \sim \partial R \partial R$ at a linearized level, but it cannot be extended to the full theory, despite an enormous number of possible cubic correction terms. The Lanczos–Lovelock (LL) [5] models' quadratic curvature field equations $\sim RR = 0$, obviously cannot even be linearized about flat space and turn out to have no conserved B either. Adding cosmological or Einstein terms to permit linearization (to effective cosmological GR) about the resulting dS vacua only allows the usual linear $B \sim RR$ of GR, but no nonlinear extension.

2. Fourth-order models

Since the machinery is considerably more complicated here than in GR, we analyze the (purely quadratic part of) NMG [3]:

$$I = \int d^3 x \sqrt{-g}(\bar{R}^{\mu\nu} \bar{R}_{\mu\nu} - \bar{R}^2), \qquad \bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R. \tag{1}$$

(The Cotton tensor of TMG is the curl of the Schouten tensor $\bar{R}_{\mu\nu}$.) The resulting field equations,

$$\Box \bar{R}_{\mu\nu} - D_\mu D_\nu \bar{R} - 4 \bar{R}_\mu{}^\sigma \bar{R}_{\nu\sigma} + \bar{R} \bar{R}_{\mu\nu} + g_{\mu\nu} \bar{R}^2 = 0, \tag{2}$$

of course obey Bianchi identities since (1) is an invariant. The field equations begin as $DD\bar{R} \sim 0$, so B cannot just imitate the RR of GR, but must instead start as $B \sim D\bar{R} D\bar{R}$, along with possible cubic, $\bar{R}\bar{R}\bar{R}$, terms since they have the same dimension as $D\bar{R} D\bar{R}$. Schematically, then, we expect that

$$B \sim D\bar{R} D\bar{R} + \bar{R}\bar{R}\bar{R}. \tag{3}$$

(One can also consider similar terms $\sim D(\bar{R}D\bar{R})$ or $\bar{R}DD\bar{R}$, but they are already irrelevant at a linear level.) We begin with the linearized (about flat space) truncation, $\partial\partial\bar{R} = 0$, of (2); the corresponding linearized B is simply given by

$$B_{\mu\nu\alpha\beta} = \bar{R}_{\alpha\beta,}{}^\sigma (\bar{R}_{\sigma\nu,\mu} - \bar{R}_{\mu\nu,\sigma}). \tag{4}$$

Its conservation

$$\partial^\mu B_{\mu\nu\alpha\beta} = 0 \tag{5}$$

is verified using the field equations, the Bianchi identity and antisymmetry of the parenthesis in (4). Conservation also holds for all permutations of B's $(\nu\alpha\beta)$ indices; for completeness, we also note the identically conserved, if irrelevant, $b_{\mu\nu\alpha\beta} = D^\gamma H_{[\gamma\mu]\nu\alpha\beta}$, H antisymmetric in $\gamma \leftrightarrow \mu$.

Thus encouraged, we turn to candidates (3) at a nonlinear level, where the now covariant derivatives D_μ no longer commute. At first sight, there are so many available independent terms cubic in $\bar{R}_{\mu\nu}$ and $g_{\mu\nu} \bar{R}$ that success seems guaranteed; however, it turns out to be unattainable. The simplest procedure is to take the covariant divergence of the initial $B \sim D\bar{R} (D\bar{R} - D\bar{R})$ ansatz and try to compensate for the resulting cubic $\sim \bar{R}\bar{R} D\bar{R}$ terms by adding suitable $\bar{R}\bar{R}\bar{R}$ to B. The procedure is straightforward, using the $D = 3$ properties

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= g_{\mu\rho} \bar{R}_{\nu\sigma} + g_{\nu\sigma} \bar{R}_{\mu\rho} - g_{\nu\rho} \bar{R}_{\mu\sigma} - g_{\mu\sigma} \bar{R}_{\nu\rho}, \\ [D_\alpha, D_\beta]V^\gamma &= \left[\delta^\gamma_\beta \bar{R}_{\sigma\alpha} - \delta^\gamma_\alpha \bar{R}_{\sigma\beta} + g_{\sigma\alpha} \bar{R}^\gamma_\beta - g_{\sigma\beta} \bar{R}^\gamma_\alpha \right] V^\sigma. \end{aligned} \tag{6}$$

We find

$$D^\mu B_{\mu\nu\alpha\beta} = \left[\bar{R}^{\rho\mu} \bar{R}_{\rho\beta} (\bar{R}_{\mu\nu;\alpha} - \bar{R}_{\alpha\nu;\mu}) + \tfrac{1}{2} \bar{R}_{\alpha\beta;\sigma} (\bar{R}^{\sigma\rho} \bar{R}_{\rho\nu} - \bar{R}^\sigma_\nu \bar{R}) \right] + (\alpha \leftrightarrow \beta), \tag{7}$$

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to which the above $(\alpha\beta\nu)$ permutations could be added. The problem is now to find, among all $(\bar{R}\,\bar{R}\,\bar{R})_{\mu\nu\alpha\beta}$, those whose μ-divergence cancels (7), or the sum of permutations. Note that (7) contains both μ- and α, β-derivatives, and that the latter must arise from cubic terms of the form $\sim g_{\mu\alpha}(\bar{R}\,\bar{R}\,\bar{R})_{\nu\beta}$, and possibly $g_{\mu\alpha}\,g_{\nu\beta}\,(\bar{R}\,\bar{R}\,\bar{R})$; these are also plentiful. This systematic approach failed: any compensating term also created a new problematic one. We then undertook the brute force approach, involving the sum of *all* dimensionally possible candidate terms $\sim D\bar{R}\,D\bar{R} + \bar{R}\,DD\bar{R} + \bar{R}\bar{R}\bar{R}$; to be sure, any $D\bar{R}\,D\bar{R}$ combinations that fail the linearized conservation test could be excluded *a priori*.

　　We sketch the procedure and counts, but omit the uninstructive, laborious, details[4]. Start with the 39 possible terms of the form $\bar{R}\bar{R}\bar{R}$, including $g\bar{R}\bar{R}\bar{R}$ terms, together with the 75 $D(\bar{R}\,D\bar{R})$ candidates, giving a total of $m = 114$ potential terms. Take the divergence of each of these, enforcing the on-shell and Bianchi and three-dimensional simplifications. That gives, for each term, a right-hand side that is a linear combination of $(DD\bar{R})\,D\bar{R}$, $\bar{R}\bar{R}D\bar{R}$ and $\bar{R}\,DDD\bar{R}$. There are $n = 231$ unique terms of this form in these sums. We can view the action of D^{μ} as a linear operator taking the m starting ingredients to a space of n outputs: assign to each candidate term a basis vector $\{\mathbf{e}_i\}_{i=1}^{m}$, and to each term appearing in the sums $D^{\mu}\,\mathbf{e}_i$, the basis vector $\{\mathbf{E}_i\}_{i=1}^{n}$, the matrix $\mathbb{D} \in \mathbb{R}^{n \times m}$ just maps $\mathbf{e}_k$ to its sum: $\mathbb{D}\,\mathbf{e}_k = \sum_{i=1}^{n}\alpha_i\,\mathbf{E}_i$. We constructed this matrix using a symbolic algebra package and found that $\mathbb{D}$ has no null space, and so no conserved Bel–Robinson tensor exists.

3. Second-order, LL, theories

At the other end of the quadratic curvature spectrum from the $D = 3$ model (1) lie the $D > 4$ LL theories, whose field equations remain of second derivative order, but do not admit any linearized expansion about flat space. The Lagrangians are, in vielbein formulation,

$$L = \epsilon^{abcdf\dots}\,\epsilon^{ABCDF\dots}\,R_{abAB}\,R_{cdCD}\,e_{fF}\,\dots. \tag{8}$$

The field equations simply consist in removing one vielbein factor,

$$E^{fF} \equiv (\epsilon\epsilon\,RR\,e)^{fF} = 0, \tag{9}$$

since the curvatures' variations vanish by the cyclic identities. Lowercase indices are local, capitals are world. Diffeomorphism invariance of the action ensures that $D_F\,E^{fF}$ vanishes identically by the cyclic identities. Indeed, this is even true in $D > 5$: if we open any second pair of indices (gG), then we obtain an identically conserved, hence uninteresting, $H^{fg\dots FG\dots}$. We cannot even start with a linearized B, since the field equations (9) are intrinsically nonlinear. That there is no true B is rather clear; for example, attaching a curvature to H above loses conservation, which no additional cubics can help, while no $DR\,DR$ term is relevant just because it is linearizable. Extending LL to include a cosmological or Einstein term does allow for linearization about a nonflat, dS, vacuum: a dS metric ansatz reduces (9) to the equation $\sim a\,\Lambda^2 = b\,\Lambda$ in either choice. There is always a flat, $\Lambda = 0$, but also now a dS, $\Lambda = b/a$, vacuum[5]. About the latter, the system linearizes to cosmological Einstein, which of course enjoys the same conserved $B \sim RR$ as $\Lambda = 0$ GR. However, just like its predecessor in NMG, this B evaporates as soon as we go to the full theory, since there is still no way to take into account the nonlinear RR part of (9).

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[4]　These may be found in the notebook at: http://people.reed.edu/~jfrankli/NullSpace.nb.
[5]　While the appearance of a second, dS, vacuum in R+GBLL—indeed in all R+R^2 models but $(D = 4)\,R$+Weyl^2—is reasonably clear [6], it may seem counter-intuitive that, unlike GR, quadratic models with cosmological term allow both flat and dS vacua.

4. Conclusion

We conclude that both, NMG and Einstein+LL, violate Fermi's rule; if the lowest order works, the conjecture is correct, while pure LL doesn't even have a lowest (here linearized) order. Clearly, there is no invariance here like that [1] underlying B-conservation in GR.

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References

[1] Deser S 1999 *Gravitation and Relativity in General* ed F Atrio-Barandela and J Martins (Singapore: World Scientific) (arXiv:gr-qc/9901007)
    Deser S and Yang Z 1990 *Class. Quantum Grav.* **7** 1491
[2] Deser S and Franklin J 2011 *Class. Quantum Grav.* **28** 032002 (arXiv:1011.4240)
[3] Bergshoeff E A, Hohm O and Townsend P K 2009 *Phys. Rev. Lett.* **102** 201301 (arXiv:hep-th/0901.1766)
[4] Deser S 2009 *Phys. Rev. Lett.* **103** 101302 (arXiv:hep-th/0904.4473)
[5] Lanczos C 1938 *Ann. Math.* **39** 842
    Lovelock D 1971 *J. Math. Phys.* **12** 498
[6] Boulware D G and Deser S 1985 *Phys. Rev. Lett.* **55** 2656