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Tensor models are the generalization of matrix models, and are studied as models of quantum gravity in general dimensions. In this paper, I discuss the algebraic structure in the fuzzy space interpretation of the tensor models which have a tensor with three indices as its only dynamical variable. The algebraic structure is studied mainly from the perspective of 3-ary algebras. It is shown that the tensor models have algebraic expressions, and that their symmetries are represented by 3-ary algebras. It is also shown that the 3-ary algebras of coordinates, which appear in the nonassociative fuzzy flat spacetimes corresponding to a certain class of configurations with Gaussian functions in the tensor models, form Lie triple systems, and the associated Lie algebras are shown to agree with those of the Snyder's noncommutative spacetimes. The Poincare transformations of the coordinates on the fuzzy flat spacetimes are shown to be generated by 3-ary algebras.

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I. INTRODUCTION

Tensor models have originally been introduced^{1–3} to describe the simplicial quantum gravity in more than two dimensions, with the hope to extend the successful description of the two-dimensional simplicial quantum gravity by the matrix models to other dimensions. The formulation has later been extended^{4–6} to describe spin foam and loop quantum gravities by considering Lie-group valued indices⁷. Despite various difficulties⁸ and the rather slow development since the introduction of tensor models, some interesting results have been reported recently^{9–15}. These developments strengthen the general belief that tensor models indicate the right direction to the background independent formulation of quantum gravity.

Concerning the background independence of quantum gravity, the dimensions of spacetime should be regarded as an effective dynamical quantity rather than a given constant. This viewpoint not only seems natural from the physical requirement of quantum gravity, but has also been supported by some recent results from simplicial quantum gravity¹⁶ and field theoretical treatment¹⁷. On the other hand, however, the original formulation of tensor models as well as the group field theory depend on the considering dimensions in their formalism through the rank of tensors and the choices of groups. Therefore it would be desired to find another interpretation of tensor models which singles out a tensor model that is applicable to general dimensions.

The simplest choice of a tensor model is to consider one which has a tensor with three indices as its only dynamical variable. Then, by identifying the rank-three tensor with the structure constant of an algebra charactering a fuzzy space, the tensor model can be interpreted as theory of a dynamical fuzzy space. Since one can in principle choose the values of the rank-three tensor to construct fuzzy spaces corresponding to any dimensional spaces, the rank-three tensor models can equally treat spaces in general dimensions. This idea has first been presented in Ref. 18, and the subsequent studies mainly in numerical methods have supported the validity of this basic idea^{19–26}. The purpose of the present paper is to provide a full treatment of the original incomplete presentation of the idea, and to pursue the algebraic description of the tensor models. In the sequel, it is found that 3-ary algebras^{27–29} describe the symmetries of the tensor models. 3-ary algebras have been introduced in physics by Nambu³⁰, and have recently been widely discussed in the context of M-theory^{31–33}. This unexpected common appearance of 3-ary algebras suggests the general

importance of this new way of describing symmetry in the physics of quantum spacetime.

This paper is organized as follows. In the following section, the rank-three tensor model is presented. In Section III, the structure of the algebras corresponding to the rank-three tensor models is discussed. In Section IV, the commutative case of the algebras is discussed. In Section V, the rank-three tensor model is described in terms of the algebras. In Section VI, the fuzzy flat spacetimes with the algebras corresponding to a certain class of configurations with Gaussian functions in the rank-three tensor models are discussed. In Section VII, the 3-ary algebra of the coordinates obtained in the previous section is studied, and a connection with the Snyder's noncommutative spacetime is found. In Section VIII, the Poincare symmetry of the fuzzy flat spacetimes is shown to be represented by 3-ary operations. The final section is devoted to summary, discussions and future prospects.

II. THE RANK-THREE TENSOR MODELS

The simplest generalization of the matrix models will be the tensor models which have a tensor with three indices M_{abc} (a, b, c = 1, 2, ..., N) as their only dynamical variable¹⁻³. The M_{abc} takes complex values, and the hermiticity of a matrix is generalized to

$$M_{abc} = M_{bca} = M_{cab} = M_{bac}^* = M_{acb}^* = M_{cba}^*, (1)$$

which states that the tensor takes its complex conjugate under odd permutations of the indices, while they are unchanged under even permutations.

Because of the property (1), the symmetry which can be associated to the tensor models is the real orthogonal group symmetry,

$$M_{abc} \to O_a{}^{a'} O_b{}^{b'} O_c{}^{c'} M_{a'b'c'}, \quad O \in O(N, R),$$
 (2)

instead of the unitary groups for the hermitian matrix models.

Let me define a new tensor,

$$C_{abc} = M_{abc} + M_{bac} + i(M_{abc} - M_{bac}). (3)$$

Because of the generalized hermiticity condition (1), this tensor is real, and is invariant under the cyclic permutations of the indices,

$$C_{abc} = C_{bca} = C_{cab}. (4)$$

Note that C_{abc} has no dependence under the odd permutations of the indices. The symmetry transformation for C_{abc} is given by a formula similar to (2).

In fact, C_{abc} is equivalent to M_{abc} as degrees of freedom, because, from (3), one can show that M_{abc} can also be expressed by C_{abc} as

$$M_{abc} = \frac{1}{4}(C_{abc} + C_{bac}) - \frac{i}{4}(C_{abc} - C_{bac}).$$
 (5)

Therefore, a tensor model with real C_{abc} satisfying (4) is equivalent to the one with complex M_{abc} satisfying (1). In the following discussions, real C_{abc} with (4) will be used.

III. PROPERTIES OF CORRESPONDING ALGEBRAS

In this section, I will discuss an interpretation of the tensor models in terms of the notion of fuzzy spaces.

A fuzzy space is characterized by an algebra of functions ϕ_a (a = 1, 2, ..., N) on it, which form a basis. Generally, the multiplication of such an algebra is defined by a structure constant $f_{ab}{}^c$ as

$$\phi_a \phi_b = f_{ab}{}^c \phi_c, \tag{6}$$

where f_{ab}^{c} is assumed to be real to describe a real fuzzy space. It is important to note that the algebra may not be associative; rather, nonassociativity will play essential roles in the discussions of this paper³⁴.

I also assume there exists a metric,

$$\langle \phi_a | \phi_b \rangle = h_{ab},\tag{7}$$

which is assumed to be real, symmetric, $h_{ab} = h_{ba}$, and bilinear.

The basis of functions ϕ_a can be changed to another one. A new basis may be given by functions of ϕ_a . By using (6), any nonlinear function of ϕ_a can be expressed linearly, because

$$\phi_a' = A_a{}^b \phi_b + A_a{}^{bc} \phi_b \phi_c + \cdots$$

$$= (A_a{}^d + A_a{}^{bc} f_{bc}{}^d + \cdots) \phi_d, \tag{8}$$

where A's are real numerical coefficients. Therefore, any new basis provided by functions of ϕ_a can in fact be obtained by a linear transformation of ϕ_a . Since the change of basis should be invertible, it is given by a general linear group transformation,

$$\phi_a \to M_a{}^{a'}\phi_{a'}, \quad M \in GL(N,R).$$
 (9)

Under this general linear group transformation, the $f_{ab}{}^c$ and h_{ab} are transformed by

$$f_{ab}{}^c \to M_a{}^{a'} M_b{}^{b'} M_{c'}{}^{-1c} f_{a'b'}{}^{c'},$$

$$h_{ab} \to M_a{}^{a'} M_b{}^{b'} h_{a'b'}. \tag{10}$$

The above transformations of the basis functions may be interpreted as the fuzzy space analogue to the diffeomorphisms of an ordinary space, since a diffeomorphism generates a linear transformation of a basis of functions on it¹⁸.

To relate the above algebraic structure with the rank-three tensor models, let me assume a relation,

$$C_{abc} = f_{ab}{}^{c'}h_{c'c}. (11)$$

Then the property of the cyclic symmetry (4) can be translated to the following cyclic condition on the algebra as

$$\langle \phi_a \phi_b | \phi_c \rangle = \langle \phi_a | \phi_b \phi_c \rangle = \langle \phi_b | \phi_c \phi_a \rangle. \tag{12}$$

The former equality is because

$$\langle \phi_a \phi_b | \phi_c \rangle = f_{ab}{}^d \langle \phi_d | \phi_c \rangle = f_{ab}{}^d h_{dc} = C_{abc},$$

$$\langle \phi_a | \phi_b \phi_c \rangle = f_{bc}{}^d \langle \phi_a | \phi_d \rangle = f_{bc}{}^d h_{ad} = C_{bca}.$$
(13)

The proof of the latter is similar.

Now let me define a 3-ary product by³⁵

$$[\phi_a, \phi_b; \phi_c] \equiv (\phi_a \phi_c) \phi_b - (\phi_b \phi_c) \phi_a. \tag{14}$$

The first two entries are antisymmetric,

$$[\phi_a, \phi_b; \phi_c] = -[\phi_b, \phi_a; \phi_c]. \tag{15}$$

The metric is invariant under the 3-ary operation on the third entry,

$$\langle [\phi_a, \phi_b; \phi_c] | \phi_d \rangle = -\langle \phi_c | [\phi_a, \phi_b; \phi_d] \rangle. \tag{16}$$

This can be proved by using (12) as

$$\langle [\phi_a, \phi_b; \phi_c] | \phi_d \rangle = \langle (\phi_a \phi_c) \phi_b - (\phi_b \phi_c) \phi_a | \phi_d \rangle$$

$$= \langle (\phi_a \phi_c) \phi_b | \phi_d \rangle - \langle (\phi_b \phi_c) \phi_a | \phi_d \rangle$$

$$= \langle \phi_a \phi_c | \phi_b \phi_d \rangle - \langle \phi_b \phi_c | \phi_a \phi_d \rangle$$

$$= \langle \phi_c | (\phi_b \phi_d) \phi_a \rangle - \langle \phi_c | (\phi_a \phi_d) \phi_b \rangle$$

$$= -\langle \phi_c | [\phi_a, \phi_b; \phi_d] \rangle. \tag{17}$$

This invariance can be discussed more explicitly by writing down the linear transformation associated to (14) as

$$[\phi_a, \phi_b; \phi_c] = (\phi_a \phi_c) \phi_b - (\phi_b \phi_c) \phi_a$$

$$= f_{ac}{}^d f_{db}{}^e \phi_e - f_{bc}{}^d f_{da}{}^e \phi_e$$

$$= (M_{ab})_c{}^d \phi_d, \tag{18}$$

where

$$(M_{ab})_c{}^d = f_{ac}{}^e f_{eb}{}^d - f_{bc}{}^e f_{ea}{}^d.$$
(19)

The indices of $(M_{ab})_c{}^d$ have the following anti-symmetric properties,

$$(M_{ab})_c{}^d = -(M_{ba})_c{}^d,$$

 $(M_{ab})_{cd} = -(M_{ab})_{dc},$ (20)

where $(M_{ab})_{cd} = (M_{ab})_c{}^{d'}h_{d'd}$. The latter equation can be proven by the relation (11) and the cyclic property (4).

Especially, the latter equation of (20) shows that h_{ab} is invariant under the infinitesimal transformation (18). Therefore there exist various invariant quantities under the transformation. For example, if h_{ab} is invertible,

$$h^{ab}\phi_a\phi_b \tag{21}$$

is invariant, where h^{ab} is the inverse of h_{ab} . It is clear that one can construct various invariants by contracting the lower indices of ϕ_a with h^{ab} .

Another kind of invariant is the trace defined by

$$Tr(\mathcal{O}) \equiv h^{ab} \langle \phi_a | \mathcal{O}\phi_b \rangle = h^{ab} \langle \phi_a \mathcal{O} | \phi_b \rangle. \tag{22}$$

When \mathcal{O} is an invariant, the whole expression $\text{Tr}(\mathcal{O})$ is also invariant. The latter equation is valid, because of (12).

It should be stressed that, due to the general possibility of nonassociativity of the algebra, the 3-ary operation (18) on the third entry does not satisfy the Leibnitz rule in general³⁶. Therefore, for example, the invariant (21) is not invariant under the 3-ary operation to the whole expression,

$$[\phi_a, \phi_b; h^{cd}\phi_c\phi_d] \neq 0$$
 in general, (23)

while the following equation,

$$h^{cd}[\phi_a, \phi_b; \phi_c]\phi_d + h^{cd}\phi_c[\phi_a, \phi_b; \phi_d] = 0, \tag{24}$$

holds.

The orthogonal group symmetry (2) of the tensor models and the general linear symmetry (9) of the algebra can be identified by partial gauge fixing of the latter symmetry as follows. Let me assume that the metric h_{ab} is positive definite. Then the transformation (10) allows h_{ab} to be gauge fixed to

$$h_{ab} = \delta_{ab}. (25)$$

Under the gauge fixing (25), the remaining symmetry agrees with the orthogonal group symmetry (2) of the tensor models. In this case, the Lie group generated by the infinitesimal transformation (18) is a subgroup of the orthogonal group O(N, R) in general.

With the gauge fixing (25), the structure constant of the algebra and the dynamical variable of the tensor models can be identified by

$$C_{abc} = f_{ab}{}^c. (26)$$

Thus, the degrees of freedom of fuzzy spaces and those of the tensor models coincide.

IV. IMPOSING COMMUTATIVITY OF THE ALGEBRA

In the following discussions, let me assume that the tensor C_{abc} is not only cyclic symmetric but totally symmetric,

$$C_{abc} = C_{bca} = C_{cab} = C_{bac} = C_{acb} = C_{cba}.$$
 (27)

This reduction of the degrees of freedom has been introduced previously²⁰ to simplify the analysis of the tensor models. In fact, this reduction of the degrees of freedom does not diminish the physical interests in the tensor models, since various interesting properties in relation with the general relativity have been shown^{21–26} under the reduction.

In addition to the property (12), the total symmetry requires that the algebra be commutative,

$$\phi_a \phi_b = \phi_b \phi_a. \tag{28}$$

While the commutators between ϕ_a vanish on account of the commutativity (28), the 3-ary product (14) is equivalent to an associator³⁷,

$$[\phi_a, \phi_b; \phi_c] = (\phi_a \phi_c) \phi_b - \phi_a (\phi_c \phi_b), \tag{29}$$

and takes non-vanishing values in general, reflecting the nonassociativity of the algebra.

In the commutative case, the following cyclic identity holds,

$$[\phi_a, \phi_b; \phi_c] + [\phi_b, \phi_c; \phi_a] + [\phi_c, \phi_a; \phi_b] = 0.$$
(30)

This can be shown from (28) and (29).

As in the general non-commutative case discussed in the previous section, the associator (29) does not satisfy the Leibnitz rule in general also for the commutative case. However, there exists a physically interesting 3-Leibnitz subalgebra^{28,29} which satisfies a kind of Leibnitz rule and is intimately related to the tensor models. This will be discussed in Section VII.

V. ALGEBRAIC DESCRIPTION OF THE RANK-THREE TENSOR MODELS

After the gauge fixing (25), the structure constant of the algebra and the dynamical variable of the tensor models can be identified as in (26). Therefore, it is possible to write down actions of the tensor models in terms of the algebra.

For simplicity, let me assume the total symmetry (27) of C_{abc} , which requires the commutativity (28) of the algebra. It should be straightforward to extend the following discussions also to the general cases with noncommutativity. In the quadratic order of C_{abc} , there exist two actions which are invariant under the orthogonal group symmetry (2), and they can be expressed with the algebraic language as³⁸

$$S_1^{(2)} = C_{abc}C_{abc} = \langle \phi_a \phi_b | \phi_a \phi_b \rangle,$$

$$S_2^{(2)} = C_{aac}C_{bbc} = \langle \phi_a \phi_a | \phi_b \phi_b \rangle.$$
(31)

These equations can be checked by using $\phi_a \phi_b = C_{abc} \phi_c$ under the identification (26).

More interesting forms appear in the quartic order. For example,

$$S_1^{(4)} = C_{abc}C_{abd}C_{efc}C_{efd} = \langle (\phi_a\phi_b)\phi_c | (\phi_a\phi_b)\phi_c \rangle,$$

$$S_2^{(4)} = C_{abc}C_{ade}C_{bdf}C_{cef} = \langle (\phi_a\phi_b)\phi_c | \phi_a(\phi_b\phi_c) \rangle.$$
(32)

A usage of this kind of algebraic expression is to find an action bounded from below to assure the stability of the tensor models. The action $S_1^{(4)}$ is positive definite, because it has a form of a norm square for the positive definite metric (25). On the other hand, $S_2^{(4)}$ is not so, but can be made it bounded from below by combining with $S_1^{(4)}$, as can be proven by

$$S_1^{(4)} + S_2^{(4)} = \frac{1}{2} \langle (\phi_a \phi_b) \phi_c + \phi_a (\phi_b \phi_c) | (\phi_a \phi_b) \phi_c + \phi_a (\phi_b \phi_c) \rangle \ge 0.$$
 (33)

It is obvious that one can construct various actions by considering such invariants of the algebra.

As described in Section II, the symmetry of such an action of a tensor model is given by the orthogonal group symmetry O(N, R). On the other hand, as discussed in Section III, the 3-ary product (14) can be regarded as the generators of this symmetry under the gauge fixing condition (25) and the identification (26). Therefore the symmetry of the tensor models can be written algebraically by using the 3-ary transformation (18) as

$$\delta_{ab}\phi_c = [\phi_a, \phi_b; \phi_c]. \tag{34}$$

Since, on account of (15), the number of the independent choices of ϕ_a and ϕ_b agrees with the dimension of the orthogonal group O(N, R), the 3-ary transformations (34) will span all the symmetry generators unless the configuration C_{abc} is fine-tuned not to be so³⁹. In fact, by choosing appropriate elements ϕ_a and ϕ_b , various transformations of physical interests can be constructed. In the following sections, Poincare transformations of the coordinates of fuzzy flat spacetimes will explicitly be given.

VI. THE ALGEBRA CORRESPONDING TO THE GAUSSIAN CONFIGURATIONS IN THE TENSOR MODELS

In the study of emergent general relativity from the tensor models, a certain kind of configurations in the tensor models play important roles^{21–26}. These configurations have Gaussian forms given by

$$C_{p^1 p^2 p^3} = \exp\left(-\alpha \left((p^1)^2 + (p^2)^2 + (p^3)^2\right)\right) \delta^D(p^1 + p^2 + p^3),\tag{35}$$

where the indices are *D*-dimensional momentum, $p^i = (p_1^i, p_2^i, \dots, p_D^i)$, and $\delta^D(\cdot)$ denotes the *D*-dimensional δ -function. Here $(p)^2$ denotes the square of *D*-dimensional momentum

defined by

$$(p)^2 = g^{\mu\nu} p_{\mu} p_{\nu}, \tag{36}$$

where $g^{\mu\nu}$ is a constant real symmetric two-tensor. The parameter α is redundant in the sense that it can be absorbed into the redefinition of $g^{\mu\nu}$, but it is kept there for the dimensional reason. These Gaussian configurations satisfy the totally symmetric condition (27) in the previous section, and therefore define the class of fuzzy spaces discussed so far.

From the identification (26), the algebra corresponding to (35) is given by⁴⁰,

$$\phi_{p^1}\phi_{p^2} = C_{p^1p^2p^3}h^{p^3p^4}\phi_{p^4} = \exp\left(-2\alpha\left((p^1)^2 + (p^2)^2 + p^1 \cdot p^2\right)\right)\phi_{p^1+p^2},\tag{37}$$

where $p^1 \cdot p^2$ denotes the inner product, $g^{\mu\nu}p^1_{\mu}p^2_{\nu}$. Because of the exponential factor in (37), the algebra (37) may be regarded as a deformation of the algebra of plane waves, $\phi_p \sim e^{ipx}$, in an ordinary spacetime, $\phi_{p^1}\phi_{p^2} = \phi_{p^1+p^2}$. The algebra (37) is commutative but nonassociative, and the parameter α characterizes the scale of the nonassociativity. The algebra is obviously invariant under the Poincare group in D-dimensions, and therefore it is physically interpreted as an algebra defining a D-dimensional nonassociative fuzzy flat spacetime⁴¹.

Now let me define the spacetime coordinates of this fuzzy spacetime. For the ordinary spacetime, the coordinates can be obtained by taking derivatives of the plane waves with respect to the momentum and putting the momentum to zero, $x^{\mu} = -i \frac{\partial}{\partial p_{\mu}} e^{ipx} \Big|_{p=0}$. Following this ordinary manner, let me define coordinates by

$$\hat{x}^{\mu} = -i \left. \frac{\partial \phi_p}{\partial p_{\mu}} \right|_{p=0} = i \int d^D p \, \delta^{\mu}(p) \phi_p, \tag{38}$$

where

$$\delta^{\mu}(p) \equiv \delta(p_1)\delta(p_2)\cdots\delta'(p_{\mu})\cdots\delta(p_D). \tag{39}$$

Here $\delta'(p)$ denotes the first derivative of the delta function defined by

$$\int dp \, \delta'(p)\phi_p = -\left. \frac{d\phi_p}{dp} \right|_{p=0}.$$
 (40)

Similarly, for later convenience, let me generalize this to

$$\hat{x}^{\mu_1 \mu_2 \dots \mu_n} \equiv (-i)^n \frac{\partial^n \phi_p}{\partial p_{\mu_1} \partial p_{\mu_2} \cdots \partial p_{\mu_n}} \bigg|_{p=0} = i^n \int d^D p \, \delta^{\mu_1 \mu_2 \dots \mu_n}(p) \phi_p, \tag{41}$$

where $\delta^{\mu_1\mu_2...\mu_n}(p)$ is defined similarly as (39). Then, using (37), the products of $\hat{x}^{\mu_1\mu_2...\mu_n}$ are given by

$$\hat{x}^{\mu_1 \mu_2 \dots \mu_n} \hat{x}^{\nu_1 \nu_2 \dots \nu_m} = i^{n+m} \int d^D q d^D q \, \delta^{\mu_1 \mu_2 \dots \mu_n}(p) \delta^{\nu_1 \nu_2 \dots \nu_m}(q) \, e^{-2\alpha(p^2 + q^2 + p \cdot q)} \, \phi_{p+q}. \tag{42}$$

From (42), it is easy to explicitly derive some low order products as

$$\phi_0 \hat{x}^\mu = \hat{x}^\mu, \tag{43}$$

$$\hat{x}^{\mu}\hat{x}^{\nu} = 2\alpha g^{\mu\nu}\phi_0 + \hat{x}^{\mu\nu},\tag{44}$$

$$\hat{x}^{\mu\nu}\hat{x}^{\rho} = 4\alpha g^{\mu\nu}\hat{x}^{\rho} + 2\alpha g^{\mu\rho}\hat{x}^{\nu} + 2\alpha g^{\nu\rho}\hat{x}^{\mu},\tag{45}$$

where

$$\phi_0 = \phi_p|_{p=0}. (46)$$

Especially, from (43), (44) and (45), one obtains

$$[\hat{x}^{\mu}, \hat{x}^{\nu}; \hat{x}^{\rho}] = (\hat{x}^{\mu} \hat{x}^{\rho}) \hat{x}^{\nu} - (\hat{x}^{\nu} \hat{x}^{\rho}) \hat{x}^{\mu}$$

$$= (2\alpha g^{\mu\rho} \phi_0 + \hat{x}^{\mu\rho}) \hat{x}^{\nu} - (2\alpha g^{\nu\rho} \phi_0 + \hat{x}^{\nu\rho}) \hat{x}^{\mu}$$

$$= 4\alpha (g^{\mu\rho} \hat{x}^{\nu} - g^{\nu\rho} \hat{x}^{\mu}). \tag{47}$$

VII. THE 3-ARY PRODUCT OF COORDINATES AND SNYDER'S NONCOMMUTATIVE SPACETIME

The 3-ary product defined in (14) does not satisfy the Leibnitz rule in general, but there exist the possibilities that subalgebras satisfy kinds of Leibnitz rules. In fact, one can explicitly check that the 3-ary product (47) of the coordinates satisfies the fundamental identity⁴² (or Filippov identity^{28,29}), which is given by

$$[\hat{x}^{\mu}, \hat{x}^{\nu}; [\hat{x}^{\rho}, \hat{x}^{\delta}; \hat{x}^{\epsilon}]] = [[\hat{x}^{\mu}, \hat{x}^{\nu}; \hat{x}^{\rho}], \hat{x}^{\delta}; \hat{x}^{\epsilon}] + [\hat{x}^{\rho}, [\hat{x}^{\mu}, \hat{x}^{\nu}; \hat{x}^{\delta}]; \hat{x}^{\epsilon}] + [\hat{x}^{\rho}, \hat{x}^{\delta}; [\hat{x}^{\mu}, \hat{x}^{\nu}; \hat{x}^{\epsilon}]]. \tag{48}$$

This kind of algebra is called a 3-Leibnitz algebra^{28,29}.

The antisymmetry of the first two entries (15), the cyclic identity (30), and the fundamental identity (48) shows that the coordinates \hat{x}^{μ} and the 3-ary product (14) form a Lie triple system^{27–29,43}. A Lie triple system is known to have an associated Lie algebra, and the same Lie triple system as the present case is explained in detail as an example in Ref. 43 (see

also Refs. 27–29). The construction of a Lie algebra from the Lie triple system starts with formally defining some anti-symmetric commutators and new generators $\hat{M}^{\mu\nu}$ by relations,

$$4\alpha \hat{M}^{\mu\nu} \equiv [\hat{x}^{\mu}, \hat{x}^{\nu}] = -[\hat{x}^{\nu}, \hat{x}^{\mu}],\tag{49}$$

$$[\hat{M}^{\mu\nu}, \hat{x}^{\rho}] = -[\hat{x}^{\rho}, \hat{M}^{\mu\nu}] \equiv \frac{1}{4\alpha} [\hat{x}^{\mu}, \hat{x}^{\nu}; \hat{x}^{\rho}] = g^{\mu\rho} \hat{x}^{\nu} - g^{\nu\rho} \hat{x}^{\mu}, \tag{50}$$

where the last equation is from (47). Here it is important to note that the commutator [,] is only formally defined, and is not defined by products [a,b] = ab - ba, which identically vanishes on account of the commutativity of the algebra. The consistency of the formal definitions of the commutators as a Lie algebra is guaranteed by the antisymmetry of the first two entries (15), the cyclic identity (30), and the fundamental identity (48). Especially, from consistency, one obtaines

$$[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = g^{\mu\rho} \hat{M}^{\nu\sigma} - g^{\mu\sigma} \hat{M}^{\nu\rho} - g^{\nu\rho} \hat{M}^{\mu\sigma} + g^{\nu\sigma} \hat{M}^{\mu\rho}. \tag{51}$$

Therefore $\hat{M}^{\mu\nu}$ can be interpreted as the generators of the rotational (Lorentz) transformation of the coordinates \hat{x}^{μ} of a flat space(time) with an inverse metric $g^{\mu\nu}$.

The associated Lie algebra defined by (49), (50) and (51) agrees with that of the Snyder's noncommutative spacetime⁴⁴⁴⁵. Derivation of the Snyder's noncommutative spacetime from a Lie triple system similarly as above has first been discussed in another kind of nonassociative spacetimes in Refs. 46 and 47. The two kinds of nonassociative spacetimes have different structures, but derive the same algebra of coordinates. This is probably because the coordinates defined in (38) are evaluated around p=0, and both of the nonassociative spacetimes have common infrared structures. These nonassociative spacetimes show interesting transmutation from nonassociativity to noncommutativity: the products of two \hat{x}^{μ} are commutative, but products of more than two coordinates show noncommutativity as in the case of the 3-ary product. Thus, 3-ary products and Lie triple systems open a new way of interpreting noncommutative spacetimes. As has been stressed in Refs. 46 and 47, this way of realizing noncommutative spacetimes can serve as a new solution to the problem of unwanted dimensions in noncommutative spacetimes. In Snyder's noncommutative spacetime, the algebra of the coordinates \hat{x}^{μ} does not close by themselves and $\hat{M}^{\mu\nu}$ appear as other coordinates. Therefore one has to confront a rather hard problem of making $\hat{M}^{\mu\nu}$ physically invisible. On the other hand, however, the $\hat{M}^{\mu\nu}$ introduced in (49) is just a label for representing a 3-ary operation, and does not represent any coordinates.

VIII. EXTENSION TO POINCARE SYMMETRY

In the previous section, the rotational (Lorentz) transformation of the space(time) coordinates \hat{x}^{μ} has been shown to be realized by a 3-ary operation (50). In this section, it will be shown that the translational transformation of the coordinates of the fuzzy flat spacetime can also be represented by a 3-ary operation.

What appears to be very strange in discussing the translational transformation of the spacetime coordinates \hat{x}^{μ} is that the 3-ary algebra (47) of the coordinates \hat{x}^{μ} does not apparently seem invariant under the naive translation, $\hat{x}^{\mu} \to \hat{x}^{\mu} + v^{\mu}$, where v^{μ} is a c-number vector. In fact, the left-hand side of (47) does not change under the naive translation, while the last expression of (47) is shifted. On the other hand, however, the plane wave algebra (37) is obviously invariant under the phase rotations,

$$\phi_p \to e^{i\,p\cdot v}\phi_p.$$
 (52)

This is actually the same as the phase rotation generated by the translations in an ordinary spacetime. Therefore, to find the correct translational transformation on the fuzzy flat spacetime, one has to take good care of (52).

By the replacement (52), the coordinates defined in (38) will be transformed to

$$\hat{x}^{\mu} \to -i \left. \frac{\partial e^{i \, p \cdot v} \phi_p}{\partial p_{\mu}} \right|_{p=0} = \hat{x}^{\mu} + v^{\mu} \phi_0, \tag{53}$$

where ϕ_0 is defined in (46). Therefore the shift of the coordinates under the translation is actually given by $\delta \hat{x}^{\mu} = v^{\mu}\phi_0$, and not by a simple c-number vector. In fact, ϕ_0 is not a trivial element, because, from (37),

$$\phi_0 \phi_p = e^{-2\alpha(p)^2} \phi_p. \tag{54}$$

From (54) and (42), one also has

$$\phi_0 \phi_0 = \phi_0, \tag{55}$$

$$\phi_0 \,\hat{x}^{\mu\nu} = 4\alpha g^{\mu\nu} \phi_0 + \hat{x}^{\mu\nu}. \tag{56}$$

Then the shift of the left-hand side of (47) by (53) can be computed as

$$\delta[\hat{x}^{\mu}, \hat{x}^{\nu}; \hat{x}^{\rho}] = \delta\left((\hat{x}^{\mu}\hat{x}^{\rho})\hat{x}^{\nu} - (\hat{x}^{\nu}\hat{x}^{\rho})\hat{x}^{\mu}\right)$$

$$= (v^{\mu}\phi_{0}\hat{x}^{\rho})\hat{x}^{\nu} + (\hat{x}^{\mu}v^{\rho}\phi_{0})\hat{x}^{\nu} + (\hat{x}^{\mu}\hat{x}^{\rho})v^{\nu}\phi_{0}$$

$$-(v^{\nu}\phi_{0}\hat{x}^{\rho})\hat{x}^{\mu} - (\hat{x}^{\nu}v^{\rho}\phi_{0})\hat{x}^{\mu} - (\hat{x}^{\nu}\hat{x}^{\rho})v^{\mu}\phi_{0}$$

$$= 4\alpha g^{\mu\rho}v^{\nu}\phi_{0} - 4\alpha g^{\nu\rho}v^{\mu}\phi_{0}, \tag{57}$$

where I have used (43), (44), (55) and (56). The last expression indeed agrees with the shift of the last expression of (47), and therefore (47) is invariant under the translational symmetry (53) on the fuzzy flat spacetime.

Thus, to properly incorporate the translational symmetry of the fuzzy flat spacetime, the 3-ary algebra of the coordinates \hat{x}^{μ} should be extended to include the element ϕ_0 . From (43), (44), (55) and (56), one can explicitly obtain

$$[\hat{x}^{\mu}, \phi_0; \hat{x}^{\nu}] = -[\phi_0, \hat{x}^{\mu}; \hat{x}^{\nu}] = 4\alpha g^{\mu\nu}\phi_0,$$

[The others with ϕ_0] = 0, (58)

and the extended 3-ary algebra is closed. Then one finds that the translational transformation (53) can be represented by

$$\delta \hat{x}^{\mu} = [v_{\nu} \hat{x}^{\nu}, \phi_0; \hat{x}^{\mu}]. \tag{59}$$

In fact, the 3-ary algebra extended with ϕ_0 can be regarded as the 3-ary algebra of the kind (47) extended with a new coordinate ϕ_0 with a degenerate metric,

$$g^{\mu 0} = g^{0\mu} = g^{00} = 0. ag{60}$$

This can be checked by comparing (47) with (58). Therefore, the extended algebra also forms a Lie triple system. Following the same procedure as in the previous section, the associated Lie algebra can be shown to contain the Poincare Lie algebra,

$$[\hat{T}^{\mu}, \hat{T}^{\nu}] = 0,$$

$$[\hat{M}^{\mu\nu}, \hat{T}^{\rho}] = g^{\mu\rho}\hat{T}^{\nu} - g^{\nu\rho}\hat{T}^{\mu},$$

$$[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = g^{\mu\rho}\hat{M}^{\nu\sigma} - g^{\mu\sigma}\hat{M}^{\nu\rho} - g^{\nu\rho}\hat{M}^{\mu\sigma} + g^{\nu\sigma}\hat{M}^{\mu\rho},$$
(61)

where

$$4\alpha \hat{T}^{\mu} \equiv [\hat{x}^{\mu}, \phi_0] = -[\phi_0, \hat{x}^{\mu}],$$
$$[\hat{T}^{\mu}, \hat{x}^{\nu}] = -[\hat{x}^{\nu}, \hat{T}^{\mu}] \equiv \frac{1}{4\alpha} [\hat{x}^{\mu}, \phi_0; \hat{x}^{\nu}] = g^{\mu\nu}\phi_0. \tag{62}$$

IX. SUMMARY, DISCUSSIONS AND FUTURE PROSPECTS

In this paper, I have studied the idea, first presented in Ref. 18, that the rank-three tensor models can be interpreted as theory of dynamical fuzzy spaces. The algebra of functions on a corresponding fuzzy space is shown to be constrained to have a certain cyclic property, which comes from the generalized hermiticity condition on the three-index tensor in the rank-three tensor models. The actions of the rank-three tensor models can be represented algebraically, and such algebraic description will provide a new tool to study the rank-three tensor models.

The most important implication of wide interest of this paper would be that the cyclic property of algebras may be used to define an interesting class of fuzzy spaces. As discussed in Section III, the cyclic property is the essential feature to guarantee the 3-ary operation to be a transformation which conserves the metric of the algebra. Such metric conserving transformations are of physical importance, since they should describe various physically meaningful unitary transformations on fuzzy spaces including the ones corresponding to the diffeomorphisms in usual spacetimes²⁴. It is not generally possible to obtain such metric conserving transformations in terms of products of functions as in the 3-ary operation of this paper, if the algebra of functions does not satisfy the cyclic property. Moreover, the cyclic property makes it possible to systematically generate metric conserving transformations in terms of n-ary⁴⁸ and supersymmetric⁴⁹ operations. It should be interesting to systematically study various aspects of spacetime symmetry transformations on physically motivated fuzzy spacetimes from this new perspective. It would also be possible that the algebraic structure presented in this paper may have some applications to the description of the nonassociative fuzzy spacetimes discussed in D-brane setups⁵⁰⁻⁵².

While commutation relations of coordinates can well describe some symmetric noncommutative spacetimes such as the noncommutative two-sphere⁵³, n-ary algebras of coordinates can be expected to characterize symmetric fuzzy spaces with nonassociativity^{50–52}. To pursue this expectation further, the commutative nonassociative fuzzy flat spacetimes motivated from the tensor models have been investigated, and a 3-ary relation of coordinates has been found for the fuzzy flat spacetimes. While the Lorentz symmetry is obviously satisfied in the obtained 3-ary relation, it apparently contradicts with the translational symmetry of the fuzzy flat spacetimes. But a detailed analysis of the flat spacetime algebra has shown that

the 3-ary relation is indeed consistent with a 3-ary translational transformation. This aspect is in good parallel with the fact that the apparently violated translational symmetry of a three-dimensional noncommutative space obtained from lattice gravity is indeed conserved by a Hopf algebraic translation^{54–56}.

The 3-ary relation of the coordinates of the fuzzy flat spacetimes has been shown to form a Lie triple system, and the associated Lie algebra has been shown to be identical with the Snyder's noncommutative spacetime. A similar result has been obtained in another kind of nonassociative spacetimes in Refs. 46 and 47. These examples show interesting transmutation from nonassociativity to noncommutativity, and, as stressed in Refs. 46 and 47, can provide a solution to the problem of unwanted dimensions in noncommutative spacetimes. It would be interesting to study whether other noncommutative spacetimes can also be embedded into nonassociative ones in similar manners.

What seems especially interesting is the appearance of the 3-ary product, by which the symmetry of the rank-three tensor models can be represented. This new way of describing symmetries by 3-ary algebras seems to contain a conceptual interest in origins of symmetries. In a usual framework, a symmetry is a given input rather than an emergent phenomenon. On the other hand, however, the form of the symmetry transformation by the 3-ary algebra in the tensor models suggests that the symmetry can only appear when a background is generated. This new aspect of symmetry should be explored further in the context of the general idea that spacetime and gravity are emergent phenomena⁵⁷⁻⁶⁸.

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