2.4 Dual Values and Reduced Costs in the Simplex Method (Self-Study)

Objectives

- Explain the interpretation of dual values in linear programming.
- Describe how dual values are used in the primal simplex algorithm and how pricing options influence the behavior of the simplex algorithms (Self-Study).

102

The dual value of a constraint measures the per-unit change in the optimal objective value as the limit of the constraint is changed.

Interpretation of Dual Values

- The *dual value* of a constraint is defined as follows:

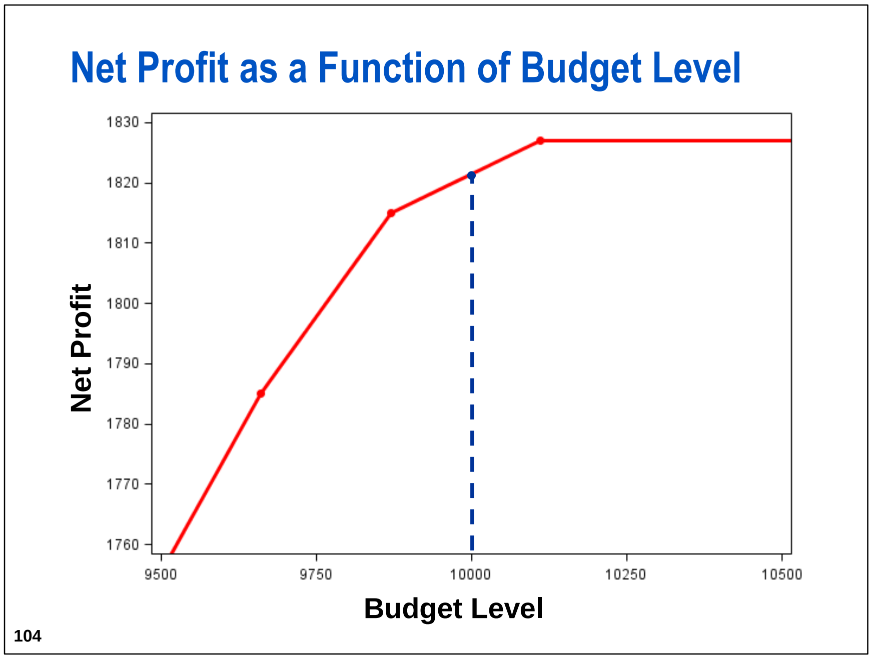
$$\textbf{Dual Value} = \frac{\textbf{Change in optimal objective}}{\textbf{Unit increase in constraint limit}}$$

- This assumes that the extreme point determining the optimal solution is not overdetermined.

overdetermined
(degenerate)

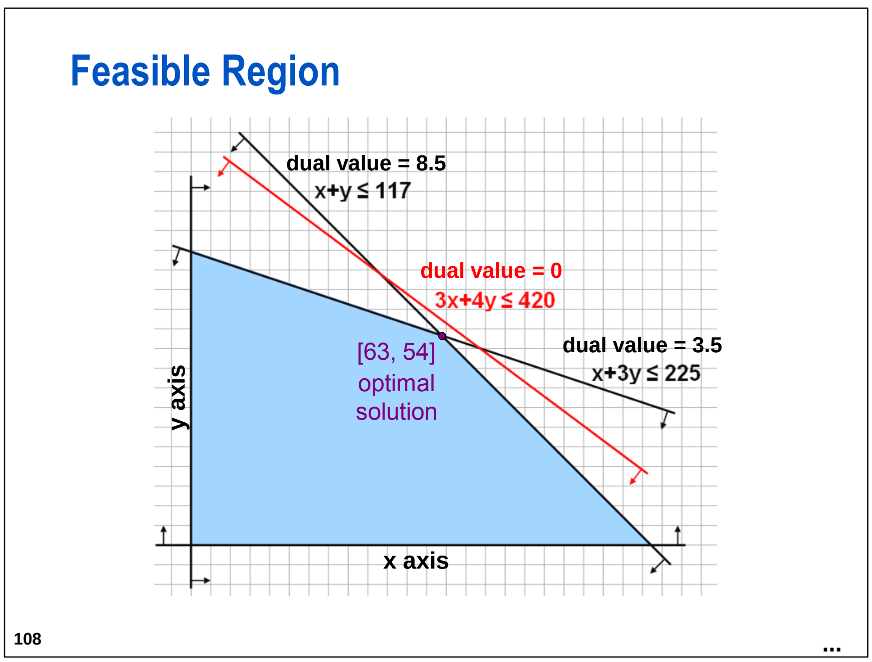
106

At the apex of the pyramid, four planes intersect in a single point (one more than is necessary).



The dual value of the budget constraint gives the slope of the graph at the current value of 10,000.

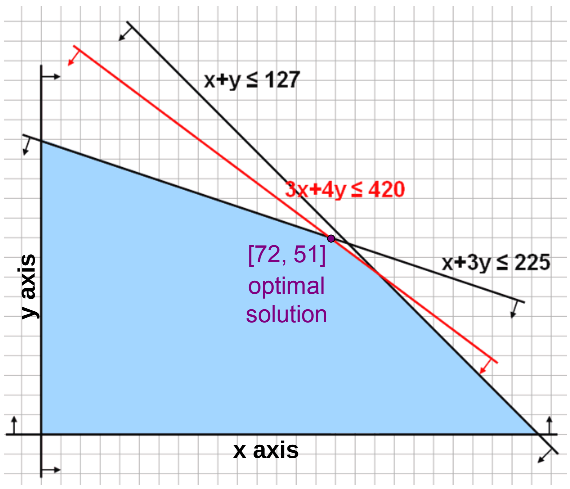
The dual values for the two-dimensional linear programming problem are pictured below.



The constraint that is not tight (satisfied with equality) at the optimal solution has a dual value of zero.

How does the dual value of the constraint $x + y \leq 117$ predict the change in the optimal objective value if the limit of the constraint is increased to 127?

Feasible Region



114

The predicted change in the objective value is accurate only in a neighborhood of the optimal solution. However, it does ***bound*** the change in the objective value:

$$12(72) + 19(51) = 1833 \leq 1782 + 8.5(10) = 1867$$

The latter quantity is the objective value of the point $[x,y] = [78,49]$, which is the intersection of the lines $x + y = 127$ and $x + 3y = 225$.

Dual Values for the Furniture-Making Problem

Dual values can be listed using the PRINT statement.

```
print Usage.dual;

print Budget.dual;
```

PROC OPTMODEL Output

```
            Usage.
[1]           DUAL

labor         1.30
metal         0.00
wood          2.45

        Budget.
          DUAL

          0.05
```

If additional overtime hours are available for \$21 (time-and-a-half), would they be used?

No. For an additional hour at \$21 = \$14 + \$7, the optimal objective value changes by (at most) \$1.30 – \$7 = –\$5.70.

118

Interpreting dual values can be tricky. Here, increasing the limit of the resource usage constraints increases only the ***availability*** of the resources at their current cost.

Making Tables: Pricing an Activity

Tables require three hours of labor, one pound of metal, and two cubic feet of wood. Tables sell for $98. Should any be produced?

Net profit for a table:

$98 – 3($14) – 1($20) – 2($11)
= $98 – $84 = $14

Net profit for a table:

$98 – 3($14) – 1($20) – 2($11)
= $98 – $84 = $14

Cost of reduced availability:

3($1.30) + 1($0) + 2($2.45)
+ 84($0.05) = $13

Cost of reduced availability:

3($1.30) + 1($0) + 2($2.45)
+ 84($0.05) = $13

$14 – $13 = $1 is the *reduced profit* of making tables.

121

The numbers in parentheses that are used to compute the cost of reduced availability are the dual values for the availability of labor, metal, and wood, and the budget constraint. For labor, the dual value $1.30 is multiplied by 3 because each table requires three hours of labor.

Making tables appears to be a profitable activity. However, the dual values themselves cannot inform the decision about the most profitable mix of tables and other products.

Due to a series of unfortunate events, the canonical form of a linear programming problem ***minimizes*** the objective function, so the computed quantity is also referred to as the *reduced cost* of making tables. Here, it is **profit** that is reduced.

For a variable in the model that is at its upper or lower bound, the reduced profit or cost is the amount that the variable's objective function coefficient must ***decrease*** before the variable can change from its current value. For example, in the furniture-making problem, the reduced profit of **NumProd['desks']** is −1, so the selling price of desks must increase by $1 (decrease by −$1) before any desks are produced.

Dual Values and Reduced Profits in the Furniture-Making Problem

```
%let budget_limit = 10000;

proc optmodel;
   /* declare sets and parameters */
   set RESOURCES = /labor metal wood/;
   set PRODUCTS = /desks chairs bookcases bedframes/;
   num selling_price {PRODUCTS} = [94 79 125 109];
   num cost {RESOURCES} = [14 20 11];
   num availability {RESOURCES} = [225 117 420];
   num required {PRODUCTS, RESOURCES} =
     [2 1 3 1 1 3 3 1 4 2 1 4];

   /* declare variables */
   var NumProd {PRODUCTS} >= 0;

   impvar Revenue = sum {p in PRODUCTS}
      selling_price[p] * NumProd[p];

   impvar AmountUsed {r in RESOURCES} =
      sum {p in PRODUCTS} NumProd[p] * required[p,r];

   impvar TotalCost = sum {r in RESOURCES}
      cost[r] * AmountUsed[r];

   /* declare constraints */
   con Usage {r in RESOURCES}:
      AmountUsed[r] <= availability[r];
      /* Note, Budget constraint from the exercise is added */
      con Budget: TotalCost <= &budget_limit;

   /* declare objective */
   max NetProfit = Revenue - TotalCost;

   solve;

   print NumProd NumProd.dual;
   print AmountUsed availability Usage.dual;
   print TotalCost Budget.ub Budget.dual;
   print _VAR_.name _VAR_.dual;
   print _CON_.name _CON_.body _CON_.ub _CON_.dual;
quit;
```

PROC OPTMODEL Output

The OPTMODEL Procedure

Solution Summary

Solver	LP
Algorithm	Dual Simplex
Objective Function	NetProfit
Solution Status	Optimal
Objective Value	1821.5
Primal Infeasibility	5.684342E-14
Dual Infeasibility	0
Bound Infeasibility	0
Iterations	6
Presolve Time	0.00
Solution Time	0.00

[1]	Num Prod	Num Prod. DUAL
bedframes	57.5	0
bookcases	28.0	0
chairs	26.0	0
desks	0.0	-1

[1]	Amount Used	availability	Usage. DUAL
labor	225.0	225	1.30
metal	111.5	117	0.00
wood	420.0	420	2.45

Total Cost	Budget.UB	Budget. DUAL
10000	10000	0.05

[1]	_VAR_.NAME	_VAR_. DUAL
1	NumProd[desks]	-1
2	NumProd[chairs]	0
3	NumProd[bookcases]	0
4	NumProd[bedframes]	0

```
                         The SAS System       11:56 Sunday, October 4, 2015   3

                       The OPTMODEL Procedure

                                 _CON_.                 _CON_.
            [1]    _CON_.NAME      BODY    _CON_.UB       DUAL

            1      Usage[labor]      225.0        225       1.30
            2      Usage[metal]      111.5        117       0.00
            3      Usage[wood]       420.0        420       2.45
            4      Budget          10000.0      10000       0.05
```