The Closest-Pair Problem

Algorithmic Thinking

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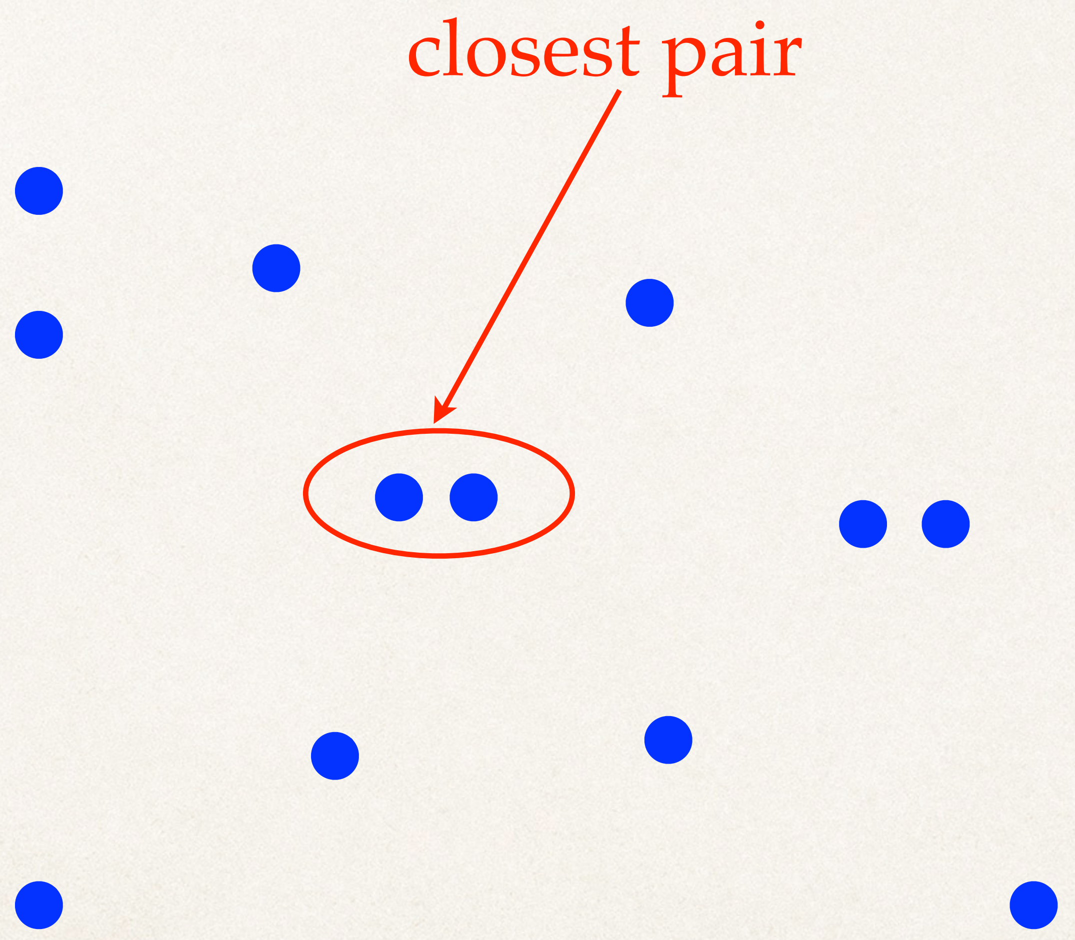
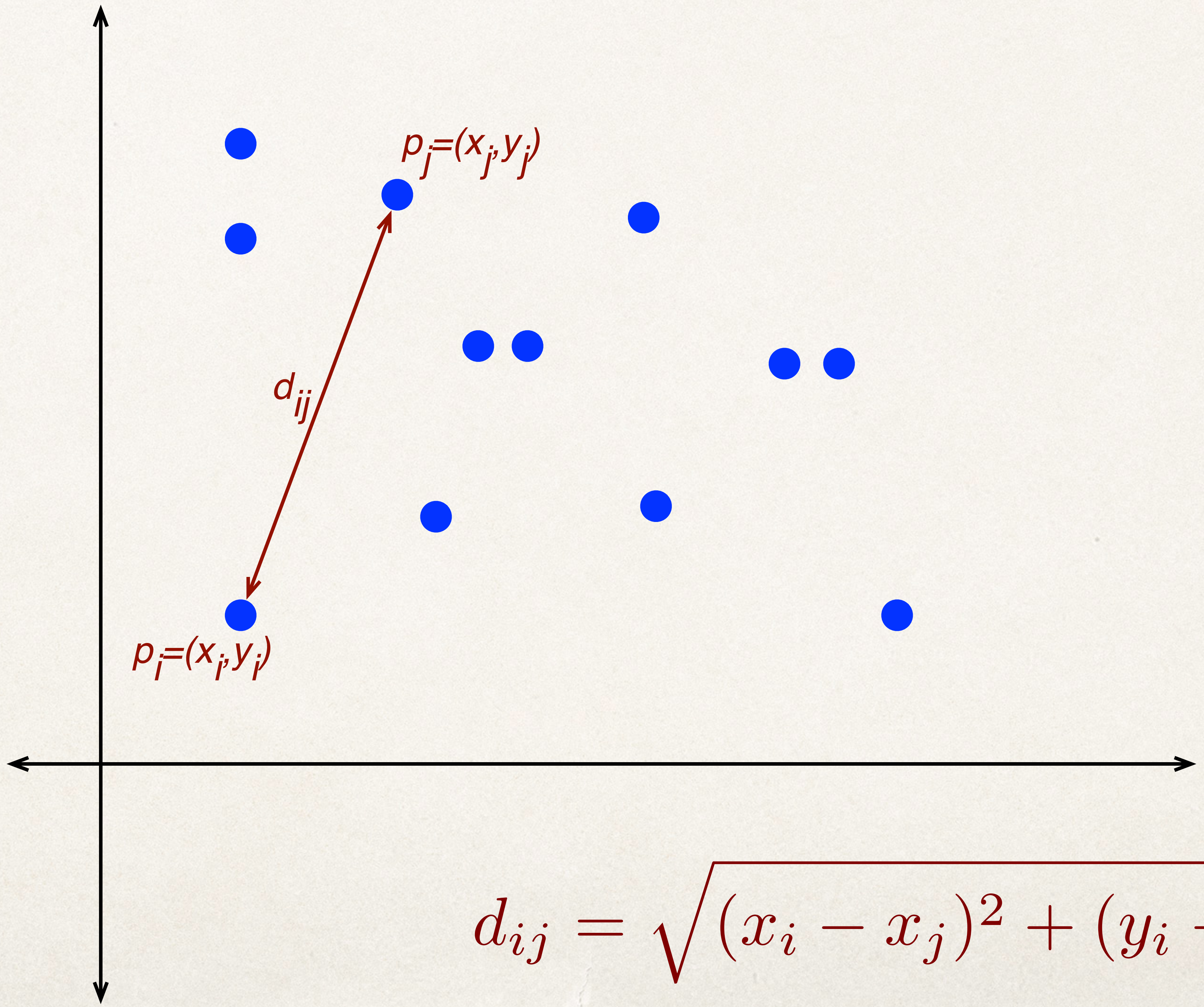
The Problem

- Input: A set P of (distinct) points in the 2D space, each given by its horizontal (x) and vertical (y) coordinates.
- Output: A pair of points $p_i, p_j \in P$ $(p_i \neq p_j)$ that are closest to each other (under the Euclidian distance).

The Problem



Distance Between Points: The Euclidian Distance



$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

First Attempt: Brute-force

- A simple brute-force algorithm for the problem:
  - Compute the distance between every two points
  - Return a pair of points with the smallest distance

Algorithm 3: SlowClosestPair.

Input: A set P of (≥ 2) points whose ith point, p_i, is a pair (x_i, y_i).
Output: A tuple (d, i, j) where d is the smallest pairwise distance of points in P, and i, j are the indices of two points whose distance is d.

```
(d, i, j) ← (∞, −1, −1);
foreach p_u ∈ P do
    foreach p_v ∈ P (u ≠ v) do
        (d, i, j) ← min{(d, i, j), (d_{p_u,p_v}, u, v)};   // min compares the first element of each tuple
return (d, i, j);
```

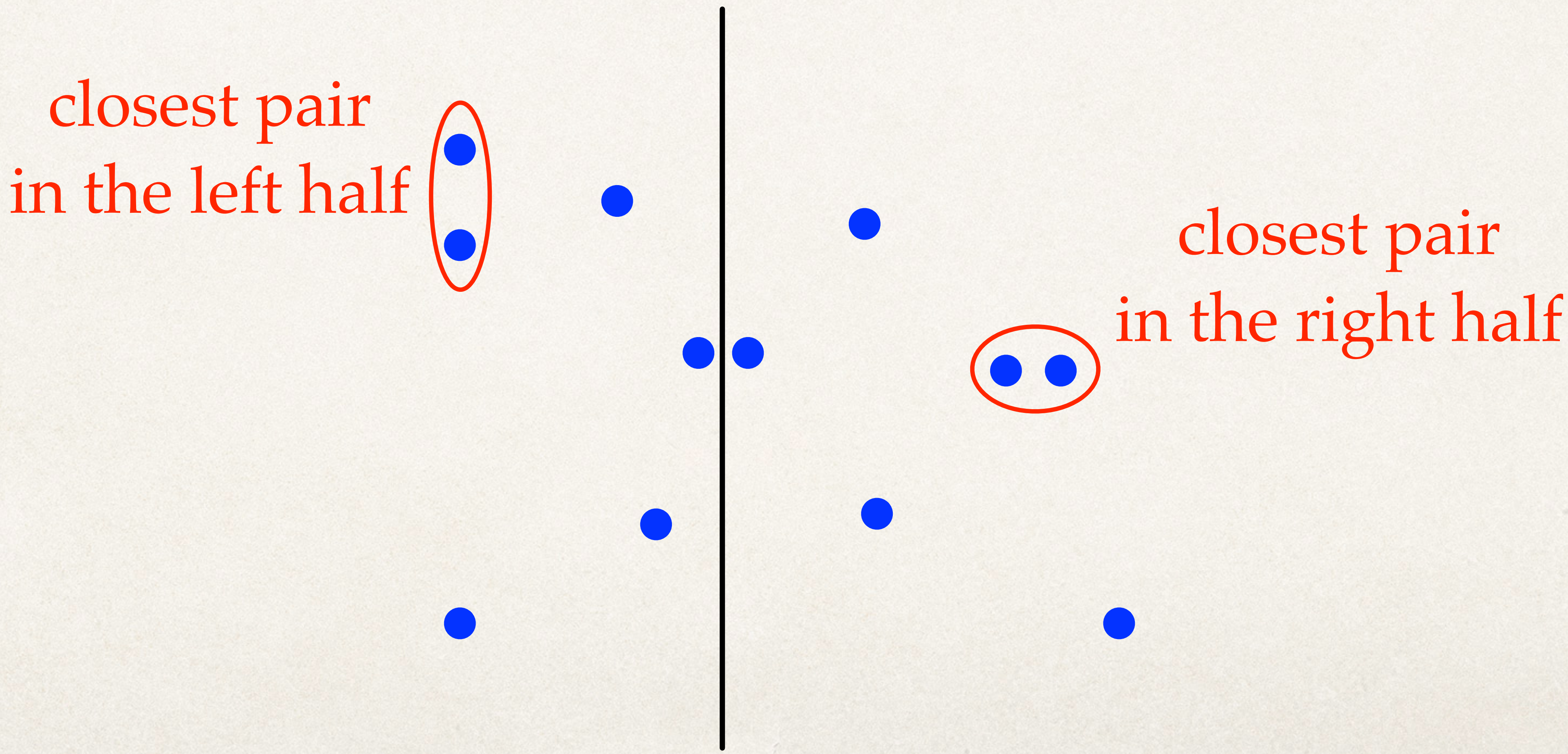
- What is the running time of the brute-force algorithm?
- Can we do better?

Divide-and-Conquer

The Divide Phase

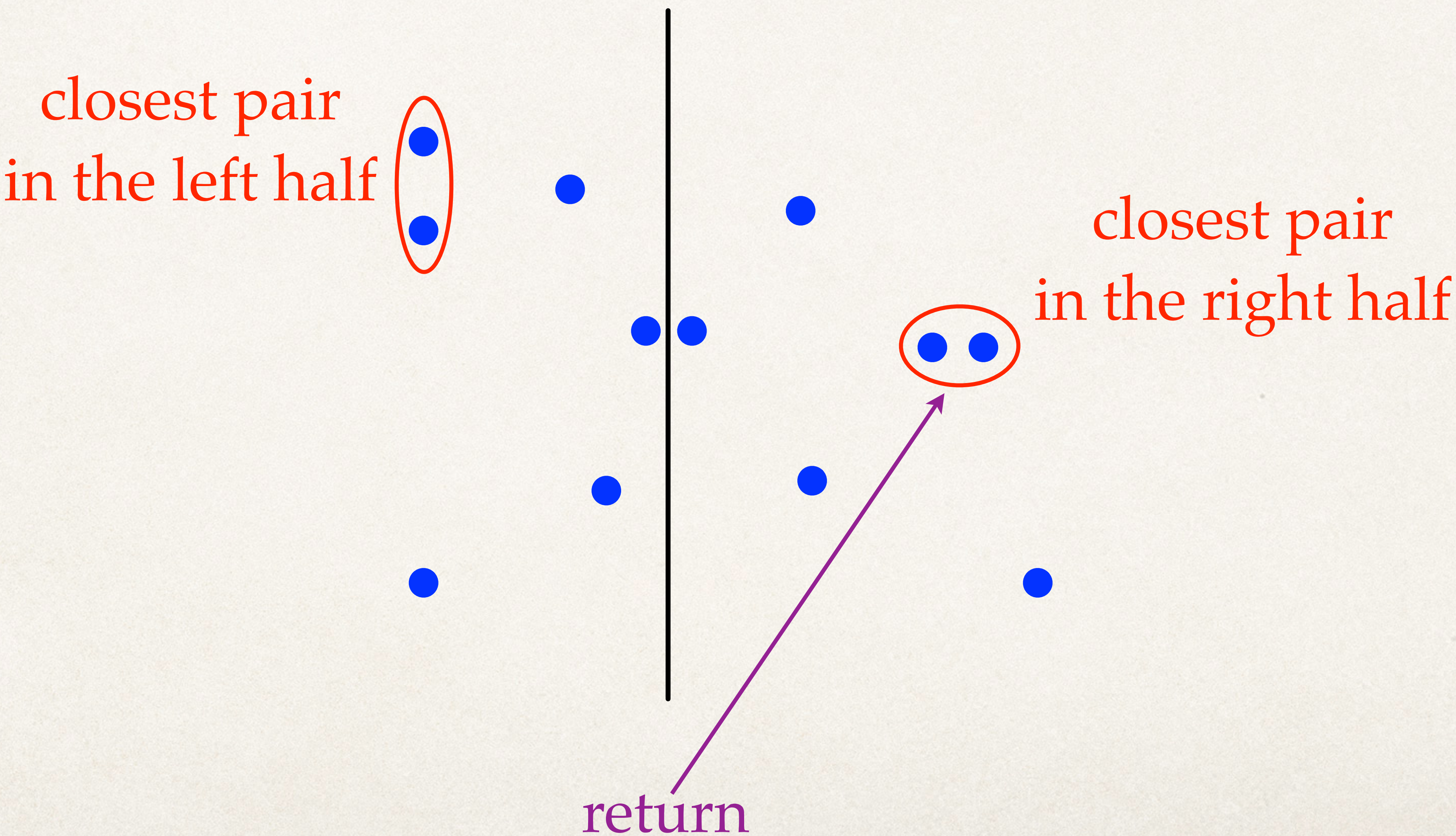
Draw a vertical line so that half of the points are to the left of the line and the other half are to the right of the line

Recursively find a closest pair in each half



The Merge Phase: A First Attempt

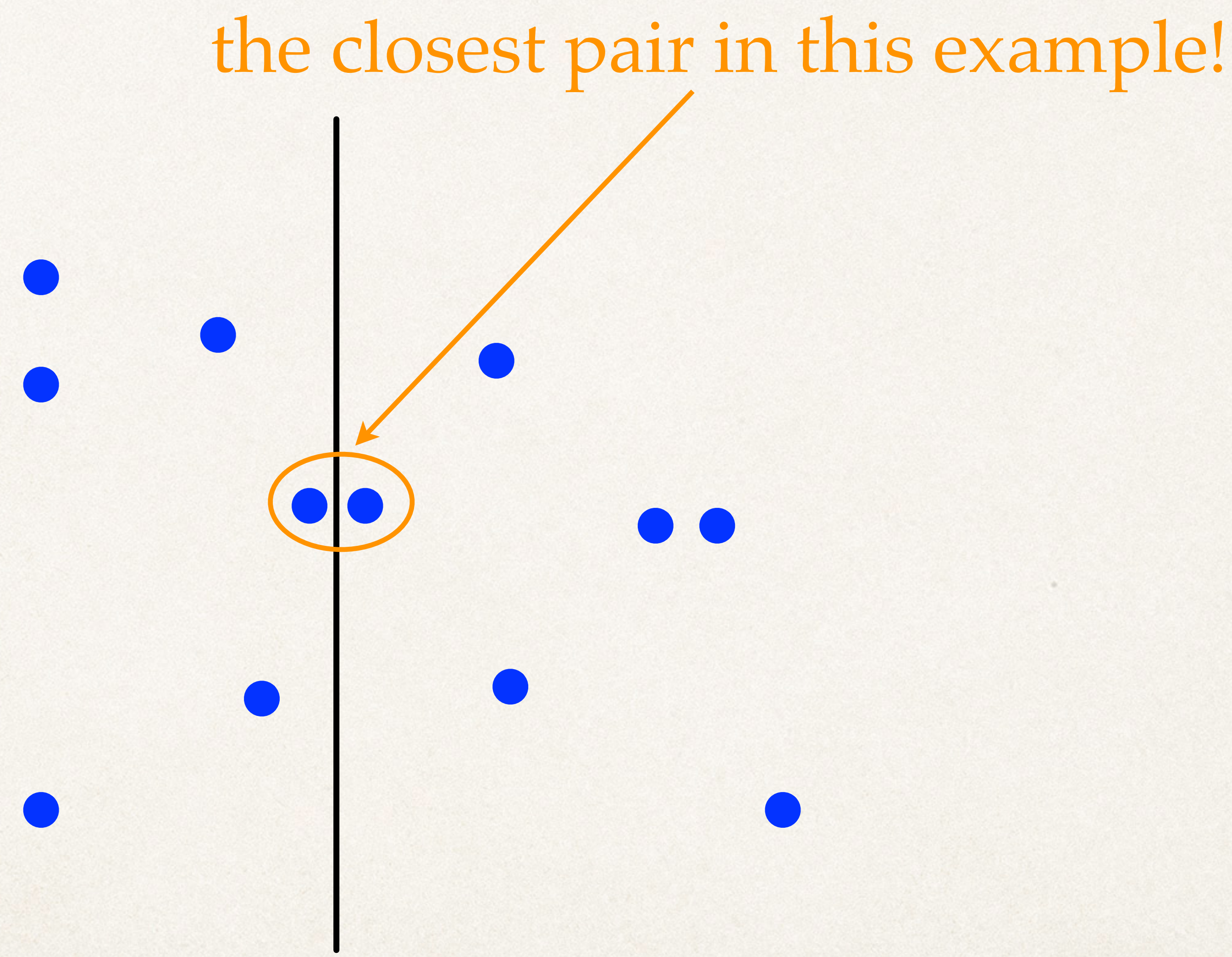
If (p_i,p_j) is a closest pair on the left, and (p_k,p_l) is a closest pair on the right, then return (p_i,p_j) if $d_{ij}<d_{kl}$, and return (p_k,p_l) otherwise.



The Merge Phase: A First Attempt

- This attempt at solving the problem doesn’t work if at any iteration of the divide phase, two closest points reside on either side of the vertical line!

The Merge Phase: A First Attempt



The Merge Phase: A Second Attempt

- Suppose we've found a closest pair (p_i,p_j) on the left, and a closest pair (p_k,p_l) on the right
- Now, compute the distance between every point on the left and every point on the right, and find a closest such pair of points, say (p_r,p_s)
- Finally, return a pair out of the three that has the smallest distance, and we're done!

The Merge Phase: A Second Attempt

- The problem with this second attempt is that it is basically the brute-force algorithm!
- Can we do the Merge phase more efficiently?
- The answer is yes!

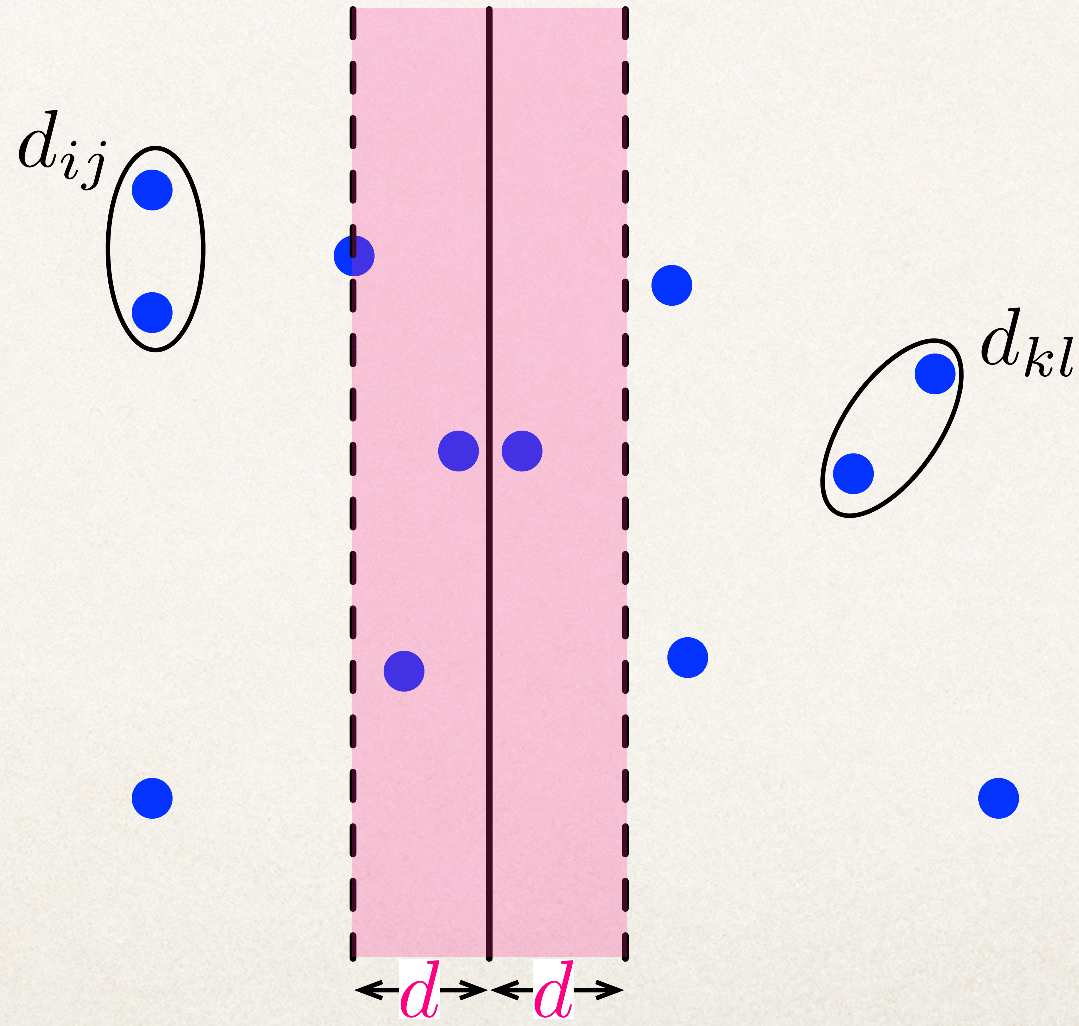
The Merge Phase: A Third Attempt

- Key observation 1:
  - If d_{ij} is the smallest pairwise distance on the left and d_{kl} is the smallest pairwise distance on the right, then we need to consider points on either side of the line that are at most $d=\min\{d_{ij},d_{kl}\}$ distance apart.

The Merge Phase: A Third Attempt

- Key observation 1:

$$d = \min\{d_{ij}, d_{kl}\}$$

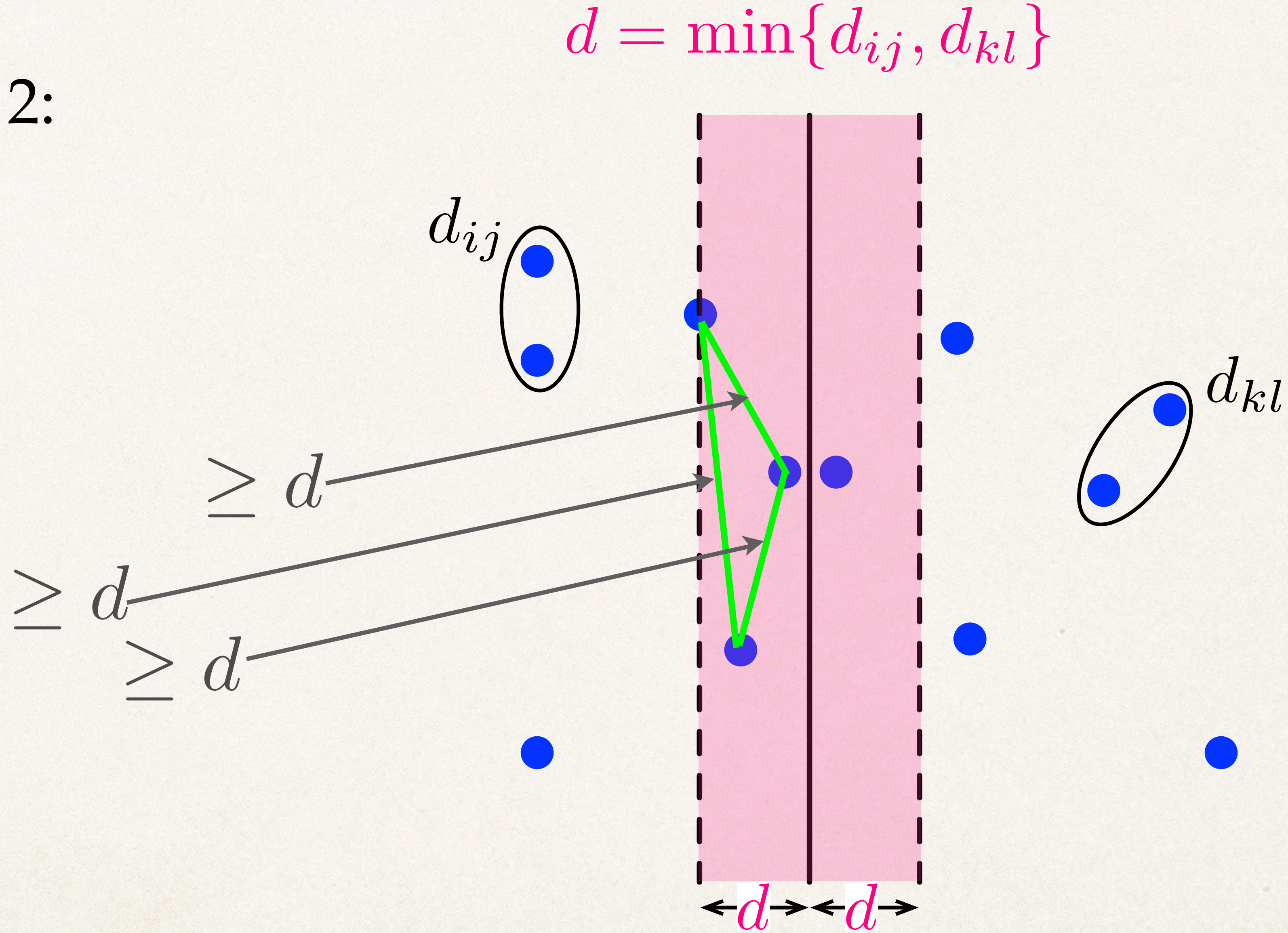


The Merge Phase: A Third Attempt

- Key observation 2:
  - In each half (left and right) of the (shaded) rectangle, every pair of points is at least distance *d* apart (why?).

The Merge Phase: A Third Attempt

- Key observation 2:



Due to this observation, non-consecutive points in each half of the rectangle must be separated by some minimum vertical distance.

The Merge Phase: A Third Attempt

- Consequence of observation 2:
  - Let $S[0..m\text{-}1]$ be an array (or a list) of all the points inside the rectangle sorted by nondecreasing order of their y-coordinates. If the distance between $S[i]$ and $S[j]$ is smaller than d, then $|i\text{-}j| < 4$.

The Merge Phase: A Third Attempt

- Putting all three observations together gives the idea for efficiently finding a closest pair across the vertical line:

  - Identify all points whose horizontal distance from the vertical line is ≤d.

  - Let $S[0..m-1]$ be an array (or a list) of all these points sorted by nondecreasing order of their y-coordinates.

  - Going through the element of S in order, for each element $S[i]$ inspect the next three ones to find the closest to $S[i]$, and record the pair of indices i and j with correspond to the closest pair thus found.

- And, now to the full algorithm...

Algorithm 4: FastClosestPair.

Input: A set P of (≥ 2) points whose ith point, p_i, is a pair (x_i, y_i), sorted in nondecreasing order of their horizontal (x) coordinates.

Output: A tuple (d, i, j) where d is the smallest pairwise distance of the points in P, and i, j are the indices of two points whose distance is d.

$n \leftarrow |P|$;
if $n \leq 3$ **then**
  $(d, i, j) \leftarrow \mathbf{SlowClosestPair}(P)$;

base case

else
  $m \leftarrow \lfloor n/2 \rfloor$;
  $P_\ell \leftarrow \{p_i : 0 \leq i \leq m-1\}$; $P_r \leftarrow \{p_i : m \leq i \leq n-1\}$; // P_ℓ and P_r are also sorted
  $(d_\ell, i_\ell, j_\ell) \leftarrow \mathbf{FastClosestPair}(P_\ell)$;
  $(d_r, i_r, j_r) \leftarrow \mathbf{FastClosestPair}(P_r)$;

divide

  $(d, i, j) \leftarrow \min\{(d_\ell, i_\ell, j_\ell), (d_r, i_r + m, j_r + m)\}$;
  $mid \leftarrow \frac{1}{2}(x_{m-1} + x_m)$; // center line of strip
  $(d, i, j) \leftarrow \min\{(d, i, j), \mathbf{ClosestPairStrip}(P, mid, d)\}$;

merge

return (d, i, j);

Algorithm 5: ClosestPairStrip.

Input: A set P of points whose ith point, p_i, is a pair (x_i, y_i); mid and w, both of which are real numbers.

Output: A tuple (d, i, j) where d is the smallest pairwise distance of points in P whose horizontal (x) coordinates are within w from mid.

Let S be a list of the set $\{i : |x_i - mid| < w\}$;
Sort the indices in S in nondecreasing order of the vertical (y) coordinates of their associated points;
$k \leftarrow |S|$;
$(d, i, j) \leftarrow (\infty, -1, -1)$;
for $u \leftarrow 0$ **to** $k - 2$ **do**
    for $v \leftarrow u + 1$ **to** $\min\{u + 3, k - 1\}$ **do**
        $(d, i, j) \leftarrow \min\{(d, i, j), (d_{p_{S[u]}, p_{S[v]}}, S[u], S[v])\}$;
return (d, i, j);