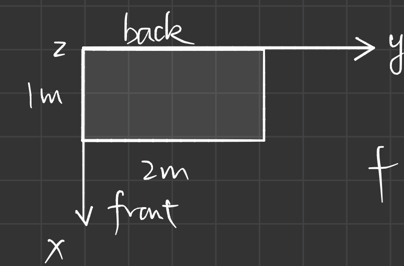
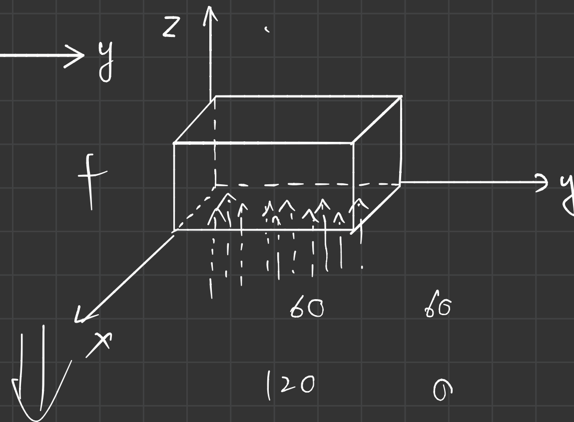
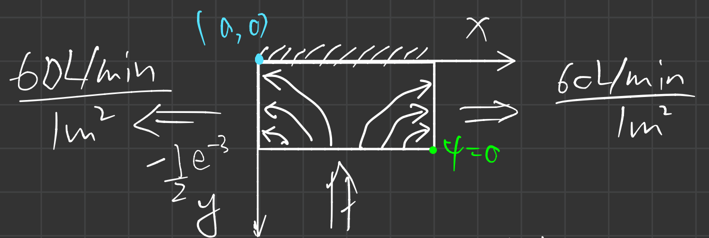
A porous bed of activated carbon is used to remove impurities from water. We can assume that Darcy's law holds for the flow of liquid within the block, which means that it can be solved as a potential flow problem.

We will assume that there is no flow out of the front or back of the bed, resulting in a 2-dimensional flow pattern, with the bed being 1 m long in this direction. The bed is 2 m wide and 1 m deep. Liquid is forced into the bottom of the bed at a flowrate of 120 l/min. We can assume that the vertical flux is constant over the bottom of the bed. The flow then splits evenly out of the 2 sides of the bed. You can assume that on the open sides the liquid flows out of the entire height of bed and that the horizontal flux is constant over these boundaries.







$\Rightarrow \frac{120L/min}{2m\cdot 1m} = 1\times10^{-3}\ m/s$

$F_y = \frac{-\partial\psi}{\partial x} = 60 \Rightarrow \psi_B = -60x + A$

$F_{xL} = \frac{-\partial\psi}{\partial y} = -60 \Rightarrow \psi_L = 60y + B$

$F_{xR} = \frac{-\partial\psi}{\partial y} = 60 \Rightarrow \psi_R = -60y + C$

$\psi_T = D$, then we assume at $(\Delta x, \Delta y)$ (i.e. (2,1))

$\psi(2,1) = 0 \Rightarrow \psi_B(2,1) = -120 + A \Rightarrow A = 120$

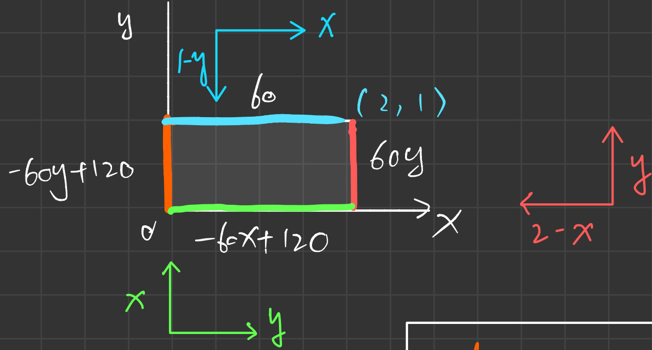
$\Rightarrow \psi_B(0,1) = 120$

and $\psi_L(0,1) = 60 + B = 120 \Rightarrow B = 60$

and $\psi_R(2,1) = -60 + C = 0 \Rightarrow C = 60.$

$A = B, \quad A - 60 = D, \quad -120 + A = C$

$$\Rightarrow \begin{cases} \psi_B = -60x + A \\ \psi_L = -60y + B = -60y + A \\ \psi_R = 60y + C = 60y + (A-120) \\ \psi_T = D = A - 60 \end{cases}$$



$$\psi_B = -60x + A$$
$$\psi_L = -60y + B$$
$$\psi_R = 60y + C$$
$$\psi_T = D$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \psi_B &= -60x+120 \\ \psi_L &= -60y+120 \\ \psi_T &= 60 \\ \psi_R &= 60y \end{aligned}}$$

Assume $\varphi(2,0)=0$:

$\psi_B(0,0) = \psi_L(0,0) \Rightarrow A = B$.

$\psi_B(0,0) = 120 \Rightarrow A = B = 120$

$\psi_L(0,1) = 60 = \psi_T = D$

$\psi_L(2,1) = 60 = \psi_R(2,1) = 60 + C$

$\psi_B(2,0) = -120+120 = 0$.

$= \psi_R(2,0) = C$

$\Rightarrow C = 0$.

1.d:

To convert the $\varphi(x,y)$ into ODE:

Let $\varphi = X(x)Y(y)$

then: $\dfrac{\partial^2\varphi}{\partial x^2} = \dfrac{d^2X(x)}{dx^2}Y(y)$

and $\dfrac{\partial^2\varphi}{\partial y^2} = \dfrac{d^2Y(y)}{dy^2}X(x)$

As $\nabla^2\varphi = 0 \Rightarrow$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0$$

let above be:

$$\lambda + (-\lambda) = 0$$

$$\Rightarrow \quad \frac{d^2X(x)}{dx^2} = \lambda^2 X$$

$$\frac{d^2Y(y)}{dy^2} = -\lambda^2 Y$$

which can be converted by:

$$X = \gamma\cosh(\lambda x) + \delta\sinh(\lambda x)$$

$$Y = \alpha\cos(\lambda y) + \beta\sin(\lambda y)$$

$$\varphi = (\gamma\cosh(\lambda x) + \delta\sinh(\lambda x))(\alpha\cos(\lambda y) + \beta\sin(\lambda y))$$

To work out the left boundary, $\varphi_L = 0$:

$$\begin{cases} \varphi_B = \varphi(x,0) = \alpha(\gamma\cosh(\lambda x) + \delta\sinh(\lambda x)) = 0 \\ \varphi_L = \varphi(0,y) = \gamma(\alpha\cos(\lambda y) + \beta\sin(\lambda y)) \neq 0 \\ \varphi_T = \varphi(x,H) = (\gamma\cosh(\lambda x) + \delta\sinh(\lambda x)) \\ \qquad (\alpha\cos(\lambda H) + \beta\sin(\lambda H)) = 0 \\ \varphi_R = \varphi(W,y) = (\gamma\cosh(\lambda W) + \delta\sinh(\lambda W)) \\ \qquad (\alpha\cos(\lambda y) + \beta\sin(\lambda y)) = 0 \end{cases}$$

$$\Rightarrow \varphi_1(x,0) = \alpha(\gamma\cosh(\lambda x) + \delta\sinh(\lambda x)) = 0$$

$\hookrightarrow \gamma = \delta = 0$ or $\alpha = 0$

As $\gamma = \delta = 0$ is a trivial solution, then $\alpha = 0$

$$\therefore \varphi_1(x,H) = 0 = \beta\sin(\lambda H)(\gamma\cosh(\lambda x) + \delta\sinh(\lambda x))$$

$\hookrightarrow \beta\sin(\lambda H) = 0 \Longrightarrow \lambda = \frac{\pi n}{H}$ (n is int)

and $\varphi_1(W,y) = \beta\sin\left(\frac{n\pi y}{H}\right)\left[\gamma\cos\left(\frac{n\pi W}{H}\right) + \delta\sinh\left(\frac{n\pi W}{H}\right)\right]$

$\hookrightarrow$ as $\gamma \neq 0$ & $\delta \neq 0 \Longrightarrow \delta = -\gamma\cos\left(\frac{n\pi W}{H}\right)$

$$\therefore \varphi_1(0,y) = \gamma\beta\sin\left(\frac{n\pi y}{H}\right) = \varphi_L$$

Use the Fourier series expression:

$$\varphi_1(0,y) = \sum_{n=1}^{\infty} \varphi_L \sin\left(\frac{n\pi y}{H}\right) \Rightarrow \varphi_n = \frac{2}{H}\int_0^H \varphi_L \sin\left(\frac{n\pi y}{H}\right) dy$$

With previous derivations:

$$\varphi_1 = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi y}{H}\right)\left[\cosh\left(\frac{n\pi x}{H}\right) - \sinh\left(\frac{n\pi x}{H}\right) / \tanh\left(\frac{n\pi W}{H}\right)\right]$$

$$r_n = \frac{2}{H}\int_0^H \varphi_L \sin\left(\frac{n\pi y}{H}\right) dy$$

For Left : $\varphi_L = -60y+120$, $H = 1m$

$$r_{n,0} = \frac{2}{\cancel{H}}\int_0^H (-60y+120)\sin\left(\frac{n\pi y}{\cancel{H}}\right) dy$$

$$= \frac{120}{\pi n}\left(-\cos(\pi n) + 2\right)$$

For Top : $\varphi_L = 60$, $H = 2m$

$$r_{n,1} = \frac{2}{H}\int_0^H 60\sin\left(\frac{n\pi y}{H}\right) dy$$

$$= \frac{120}{n\pi}\left(-\cos(\pi n) + 1\right)$$

For Right : $\varphi_L = 60y$, $H = 1m$

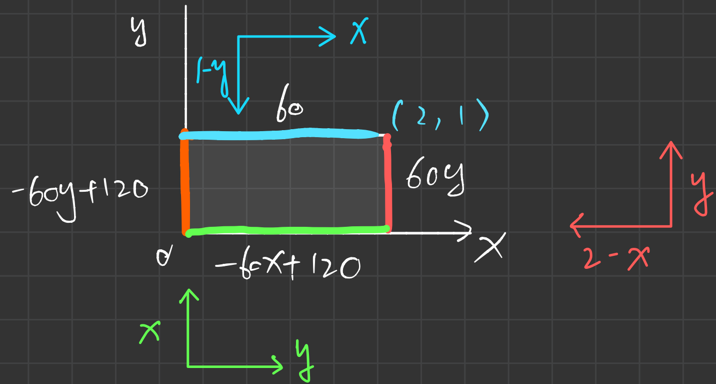
$$r_{n,2} = \frac{2}{H}\int_0^H 60y\sin\left(\frac{n\pi y}{H}\right) dy$$

$$= \frac{120}{\pi n}\left(-\cos(\pi n)\right)$$

For Botton : $\varphi_L = -60x+120$, when switched axis, it is equivalent to $\varphi_L = -60y+120$, $H = 2m$

$$r_{n,3} = \frac{2}{H}\int_0^H (-60y+120)\sin\left(\frac{n\pi y}{H}\right) dy$$

$$= \frac{240}{\pi n}$$



For left: x, y stay the same, φ_1 stays the same;

For top: y became x, x becomes $H-y \Rightarrow 1-y$

$$\Rightarrow \varphi_2 = \sum_{n=0}^{\infty} r_n \sin\left(\frac{n\pi x}{2}\right)\left(\cosh\left(\frac{n\pi(1-y)}{2}\right) - \sinh\left(\frac{n\pi(1-y)}{2}\right) / \tanh\left(\frac{n\pi}{2}\right)\right)$$

For right: y stays the same, x becomes $W-x \Rightarrow 2-x$

$$\Rightarrow \varphi_3 = \sum_{n=0}^{\infty} r_n \sin(n\pi y)\left(\cosh(n\pi(2-x)) - \sinh(n\pi(2-x)) / \tanh(2n\pi)\right)$$

For bottom: y becomes x, x becomes y

$$\Rightarrow \varphi_4 = \sum_{n=0}^{\infty} r_n \sin\left(\frac{n\pi x}{2}\right)\left(\cosh\left(\frac{n\pi y}{2}\right) - \sinh\left(\frac{n\pi y}{2}\right) / \tanh\left(\frac{n\pi}{2}\right)\right)$$