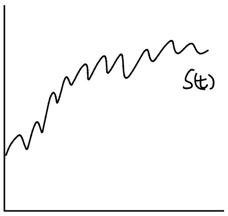
These notes contain the introduction to stochastic calculus upto the derivation of the Black and Scholes model

Now we are going to explore modelling of stock prices using stochastic Differential Equations.

$$R_i = \frac{S_{i+1} - S_i}{S_i}$$



$\delta t \rightarrow$ Time Step

$\mu \rightarrow$ mean Returns

$\sigma \rightarrow$ Std. deviation

Lets assume returns are calculated every other day

$$\hat{R}_i = \frac{S_{i+2} - S_i}{S_i}$$

notation for the same

$$(R_i + 1)(R_{i+1} + 1) - 1$$

$$= \frac{\cancel{S_{i+1}}}{S_i} \cdot \frac{S_{i+2}}{\cancel{S_{i+1}}} - 1$$

$$\therefore \hat{R}_i = (R_i + 1)(R_{i+1} + 1) - 1$$

$$\hat{R}_i = \underbrace{\cancel{R_i R_{i+1}}}_{\sim 0} + \underbrace{R_i}_{\text{Normal}} + \underbrace{R_{i+1}}_{\text{Normal (Assumed)}} + \cancel{1} - \cancel{1}$$

$$\therefore \text{Mean } \hat{\mu} = \mu + \mu = 2\mu \qquad (\text{time shift} = \delta t)$$

$$\text{Std dev } \hat{\sigma} = \sqrt{\sigma^2 + \sigma^2} = \sqrt{2} \cdot \sigma$$

So here, we have doubled the time shift, and observed that the mean was doubled and the Std Deviation was multiplied by a factor of $\sqrt{2}$.

So now we have a discrete model,

$$R_i = \underbrace{\mu}_{\text{Expected Return}} \delta t + \underbrace{\sigma}_{\text{Volatility}} \underbrace{\phi}_{\text{Standard normal parameter}} (\delta t)^{1/2}$$

Now to come up with a continuous model,

$$\frac{S_{i+1} - S_i}{S_i} = \mu \delta t + \sigma \phi (\delta t)^{1/2}$$

$$\underline{S_{i+1} - S_i} = \mu S_i \, \delta t + \sigma S_i \, \phi (\delta t)^{1/2}$$

$$\boxed{dS = \mu S_i \, dt + \sigma S_i \, \phi (dt)^{1/2}}$$

We cannot have both dt and $\sqrt{dt}$ in the same equation as $\sqrt{dt}$ will always dominate dt

Solution to this is to make use of a wiener process. We introduce a new random variable. This contains the normal variable. It has a mean of 0 and the variance is the time step ∂t.

$$\phi (\delta t)^{1/2} \longrightarrow dX$$

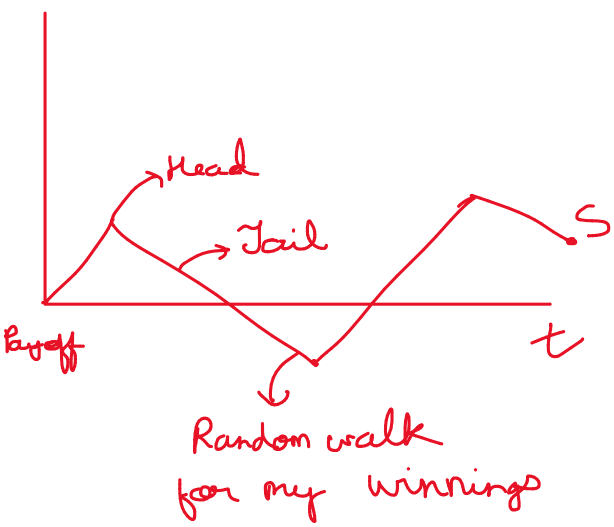
The equation becomes

$$\boxed{dS = \mu S_i \, dt + \sigma S_i \, dX}$$

continuous stochastic process

Now continuing into Brownian motion,

To understand Brownian motion, let's consider a game where if I get heads, I get 1 rupee and if I get tails, I lose a rupee



$R_i = \pm 1$ randomly

winning / lose of a time

$$S_i = \sum_{j=1}^{i} R_j$$

Sum of winning

Brownian motion follows the Markov property.

Markov property: This property dictates that The coin has no memory. This means that whatever happened in the past rounds does not affect the future. That means just because I got heads in 10 consecutive rounds before does not mean that there is higher chance of getting heads in the next round.

Martingale property: Expected winnings of the future are always what I have in hand now.

$$E[S_j \mid S_i, \; i < j] = S_i$$

Dividing into n steps over time t

Dividing into n steps over time t

$$R_i = \pm\sqrt{\frac{t}{n}}$$

$$S_i = \sum_{j=1}^{n} R_j$$

$$\Rightarrow E[S_i] = 0 \qquad \because E[R_i] = 0$$

$$\therefore E[R_i^2] = 0$$

$i = j$ $\quad E[R_i R_j] = E[R_i]E[R_j] = 0$ Due to this property

$$E[S_i^2] = E[(R_1 + \dots R_i)(R_1 + \dots R_i)]$$

$$= E[R_1^2 + \cancel{R_1 R_2}^{\,0} + \dots \cancel{R_i R_{i-1}}^{\,0} + R_i^2]$$

$$= E[R_1^2 + R_2^2 + \dots + R_i^2]$$

$$\boxed{E[S_i^2] = i \times t/n}$$

$$\boxed{E[S_n^2] = t}$$

Setting the limit to $n \to \infty$, and call it $X(t)$

$$E[X(t)] = 0$$

$$E[X(t)^2] = t$$

Thus the properties of Brownian motion are:

- It is a finite process
- It is a markov distribution
- It is continuous process
- It is a martingale (Distribution of X(t) given $\tau < t$ only depends on $X(\tau)$
- Also the distribution follows normality property. At any time $\tau < t$, $X(t) - X(\tau)$ is normal with mean 0 and Std dev $\sqrt{t - \tau}$

- We can simulate Brownian motion with discrete time steps

Stochastic Calculus and Ito's lemma

Let X(t) be a brownian motion.

(New definition) A stochastic Integral

$$w(t) = \int_0^t f(\tau)\, dX(\tau)$$

This shows that the terms are not anticipatory.

or

$$w(t) = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_{j-1}) \left[X(t)_j - X(t)_{j-1} \right]$$

We are observing what changes happens to the portfolio after making a change in the j-1. So we are pushing the limit to infinity. We are evaluating what we are doing at time Tj-1 and checking how it is affecting after that. So we are not anticipating the market

Taking derivative on both sides

$$dw = f(t)\, dX$$

$$\therefore w(t) = \underbrace{\int_0^t g(\tau)\, d\tau}_{\text{deterministic}} + \underbrace{\int_0^t f(\tau)\, d\tau}_{\text{random}}$$

$$dw = \underbrace{g(t)\, dt}_{\text{deterministic}} + \underbrace{f(t)\, dX}_{\text{Random}}$$

So now we have a definition of stochastic calculus. We can therefore presumably use the definition and solve problems. This is analogous to the derivation of the integral definition in calculus.

To explain further, let us take a look at regular calculus:

$$F(x) = x^2$$

$$dF = 2x\, dx$$

Now let us do the same thing in stochastic calculus:

Itô's lemma

Now let us do the same thing in stochastic calculus.

Itô's lemma

Let us start with a "Taylor Expansion"

$$F(x+dx) = F(x) + \frac{dF}{dx}dx + \frac{1}{2}\frac{d^2F}{dx^2} + \ldots\ldots \rightarrow 0$$

Now remember the rule of Thumb:

$$dx^2 = dt$$

This rule makes sense because of the many calculations we have been doing with variance, quadratic variance and brownian motion.

$$\boxed{dF = \frac{dF}{dx}dx + \frac{1}{2}\frac{d^2F}{dx^2}dt}$$

Itô's lemma

☆ ☆ Remember This

Let us continue the calculation in stochastic calculus

Using Itô's lemma

$$F(x) = x^2$$

$$dF = 2x\,dx + \frac{1}{2}\cdot 2x\,dt$$

$$dF = 2x\,dx + dt$$

Regular calculus

$$F(x) = x^2$$

$$dF = 2x\,dx$$

Generalized Itos lemma

We use Itô's lemma if we want to find the value of dF from a given F(x)

We use ITO ...

from a given $F(x)$

So a generalised version would be:-

for $F(s)$

$$dS = a(S)\,dt + b(S)\,dX$$

So if we want to find dF

we use Taylor Expansion

$$F(s+ds) = F(s) + \frac{dF}{ds}ds + \frac{1}{2}\frac{d^2F}{ds^2}ds^2 + \cancel{\cdots\cdots} \to 0$$

$$dF = \frac{dF}{ds} \times ds + \frac{1}{2}\frac{d^2F}{ds^2}\left(\cancel{a(s)^2 dt^2}^{\to 0} + \cancel{2a(s)b(s)\,dt\,dx}^{\to 0} + \cancel{b(s)^2 dx^2}^{\to dt}\right)$$

$$[dx^2 = dt]$$

We are left with

$$\boxed{dF = \frac{dF}{ds}ds + \frac{1}{2}b(s)^2\frac{d^2F}{ds^2}dt}$$

Example

Considering a brownian motion with drift

$$dS = \underbrace{\mu\, dt} + \underbrace{\sigma\, dX}$$

$$dS = \underbrace{\mu\, dt}_{\text{drift}} + \underbrace{\sigma\, dX}_{\text{Brownian motion}}$$

Integrating B.S

$$S(t) = S(0) + \int_0^t \mu\, dz + \int_0^t \sigma\, dX(z)$$

$$\Downarrow$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sigma \left(X(t_i) - X_i(t_i - 1) \right)$$

$$\boxed{S(t) = S(0) + \mu t + \sigma \left(X(t) - X(0) \right)}$$

Example 2 lognormal distribution

$$F(S) = \log(S)$$

If S is lognormally distributed, $\log(S)$ has a normal distribution.

We have to find an equation for S.

Sol'n: We have an equation for dS

$$dS = \mu S\, dt + \sigma S\, dX$$

$$\frac{dF}{dS} = \frac{1}{S} \qquad \frac{d^2F}{dS^2} = -\frac{1}{S^2}$$

$$dF = \frac{dF}{dS} \times dS + \frac{1}{2} \sigma^2 S^2 \frac{d^2F}{dS^2}\, dt$$ } from generalised lemma

$$dF = \frac{1}{S}\left(\mu S\,dt + \sigma S\,dX\right) + \frac{1}{2}\sigma^2 S^2 \times \frac{-1}{S^2}\,dt$$

$$dF = \underbrace{\left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\,dX}_{\text{This is result for Brownian motion with drift}}$$

Integrating

$$F(t) = F(0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\left(X(t) - X(0)\right)$$

$$F = \log(S)$$

$$\boxed{\therefore\ S = S(0)\cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\left(X(t) - X(0)\right)}}$$

Multi Dimensional Ito's lemma

Suppose we have a stochastic variable S with equation

$$dS = a(S,t)\,dt + b(S,t)\,dX$$

Suppose $V(S,t)$ is given, what is dV?

Last time, a, b, and V were only a function of S and not t. This time they are a function of both

2D Taylor Expansion

$$V(X + \delta X, Y + \delta Y)$$

2D Taylor Expansion:

$$f(x+\delta x, y+\delta y) = f(x,y) + \left(\frac{\delta F}{\delta x}\delta x + \frac{\delta F}{\delta y}\delta y\right) + \frac{1}{2}\left(\frac{\delta^2 f}{\delta x^2}\delta x^2 + 2\frac{\delta f}{\delta x \delta y}\delta x\,\delta y + \frac{\delta^2 f}{\delta y^2}\delta y^2\right) + \cdots\cdots$$

$$v(s+ds, t+dt) = v(s,t) + \frac{\partial v}{\delta s}ds + \frac{\delta v}{\delta t}dt + \frac{1}{2}\left(\frac{\delta^2 v}{\delta s^2}ds^2 + \cancel{2\frac{\delta^2 v}{\delta s \delta t}ds\,dt} \xrightarrow{} 0 + \cancel{\frac{\delta^2 v}{\delta t^2}dt^2} \xrightarrow{} 0\right)$$

$$\therefore\ dv = \frac{\delta v}{\delta s}ds + \frac{\delta v}{\delta t} + \frac{1}{2}\left(\frac{\delta^2 v}{\delta s^2}\left(\cancel{a(s,t)^2 dt^2} \xrightarrow{} 0 + \cancel{2a(s,t)b(s,t)\,dt\,dx} \xrightarrow{} 0 + b(s,t)^2 \cancel{dx^2} \xrightarrow{} dt\right)\right)$$

$$\underline{dx^2 = dt}$$

$$\therefore\ \boxed{dV = \frac{\delta v}{\delta s}ds + \frac{\delta v}{\delta t}dt + \frac{1}{2}b(s,t)^2\frac{\delta^2 V}{\delta s^2}dt}$$

Black and Scholes model

The black and scholes model is a pricing model used to determine the fair price or the theoretical value of a European call option. It is based on six variables such as volatility, type of option, underlying stock price, time, strike price, and risk-free rate.

Let $S(t) \longrightarrow$ Asset price (The "underlying")

$V(S,t) \longrightarrow$ Option price

<u>Assumption</u>: S is lognormal

(we are having long option, and shorting asset price here)

$$\Pi = V - \Delta S$$

Portfolio value

Hedging factor

$$\boxed{d\Pi = dV - \Delta dS}$$

Applying Multi-Dimensional Itô's lemma which was derived earlier:-

$$\boxed{dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt}$$

$$d\Pi = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS$$

$$d\Pi = \left(\frac{\partial V}{\partial S} - \Delta\right) dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

choosing $\Delta = \dfrac{\partial V}{\partial S}$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

$$d\Pi = \underbrace{\left(\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} \right)}_{\text{Deterministic Term}} dt$$

Since we have entirely deterministic equation, we can apply no arbitrage argument.
That means, if we have cash flows that are known, they cannot change.

$$\underbrace{d\Pi = r\Pi\, dt}_{\text{Cash flow = money in Bank}} = \underset{\text{risk free rate of interest}}{r} \left(V - \overset{\frac{\delta V}{\delta S}}{\cancel{\Delta}} S \right) dt$$

$$\therefore \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \cancel{dt}$$

$$= \left(rV - r\frac{\delta V}{\delta S} \cdot S \right) \cancel{dt}$$

$$\boxed{\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} = rV - r\frac{\delta V}{\delta S} S}$$

$$\boxed{\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} + rS\frac{\delta V}{\delta S} - rV = 0}$$

hus the B ack ch quale

We started with asset and option

We assumed asset was lognormal random walk and hence we take position to long option and short the asset.

Till now we understood how the black scholes model was derived.
Given below is some additional information about the model and application taken up from a few websites.

Assumptions of the Black Scholes Model

- **The constant risk-free rate of return**

One of the factors affecting the option prices is the risk-free rate return. The assumption is that the risk-free return will remain constant from the point we purchase the Option to its expiry date. It is also assumed that you can borrow or lend with this rate easily. The reason this is important is that if you found that all things remaining same, if the options return is equal to the risk-free rate, then people would go for the risk-free asset and not the Option.

- **Log return of risky asset's price is a random walk**

Here, we assume that the markets are efficient and the markets have a drift which is increasing and follow the geometric brownian motion. This shows that while we cannot exactly predict the price of the underlying asset, we can calculate its expected return.

- **Dividends are not taken into account**

We assume that the stock doesn't pay any dividends and hence its value is dependent on its price only.

- **No arbitrage opportunities**

If we see that there is an arbitrage opportunity when it comes to the option and the underlying asset, we would immediately use that to our advantage and won't have to worry about the option price. Hence, we assume that there are no arbitrage opportunities.

- **No limit on buying and selling of the underlying risky asset**

We assume that we can buy or sell any amount of the underlying asset to maximise our gains. In this manner, we don't have to worry about an upper limit on the number of transactions we are allowed.

- **No transactions costs**

Quite simply, this model doesn't take into account the brokerage, commissions, borrow charges or any other transaction costs we might incur while trading Options. Thus, we have to be careful of accounting for them when we are evaluating different options.

Great! Now that we have gone through the assumptions of the model, let's get to the heart of the matter, i.e. the Black-Scholes equation.

From <https://blog.quantinsti.com/black-scholes-model/>

Black Scholes formula

let's try to gain an understanding of the factors which could affect the Options price. Now, thanks to the assumptions of the Black Scholes model, we don't have to worry about the intricate details like stock dividends and fluctuating interest rates etc while deriving the options price. But what can possibly impact the options price?

The first is obviously the underlying asset's price. Moving further, we would also like to know what happens if the price increases or increases until we reach the expiration date of the Option.

This is where the expected rate of return comes into the picture.

Along with this, we also have to consider the time value of money. In simple terms, $100 right now is more valuable to $100 in one year, because you can put this $100 in a bank and get some interest after a year.

Let’s expand on this. Suppose you can get a 4% rate of interest in a bank. Thus, after one year, it will be ($100) + (4/100)*(100) = $104. You can put that in the form of a formula as (Your amount)*(1 + i%). Now, if you think that you are going to get $104 in one year, then you just have to divide it by (1+i%) to get its present value. We call this the discounting factor. The ‘i’, in this case, is the interest you could get.

For the sake of this article, we will not go into the nitty-gritty of it but when it comes to the Black Scholes Model, the discounting factor is (e-rT).

All right! So far we have realised that the option price can be affected by the underlying asset price. The time to expiry as well as the Exercise price, i.e. the Strike price.

We also have to take into account the volatility of the underlying asset. Why is that important?

Let’s say that there are two stocks, A and B. If you are buying a European call Option, you would be concerned about how far the price can go at the time of expiry, either high or lower than the strike price. This can be deduced by finding the volatility ie the standard deviation of log normal returns.

Let us list them down now.

- S = Stock price
- N() = Cumulative Standard normal distribution
- K = Strike price of the option
- t = time till the option expires
- r = risk-free rate of interest
- e= exponential term ie 2.7183
- C= option price

For the sake of simplicity, we are considering the underlying asset to be a stock and the stock option is a European Call option. The reason we are using a European Call Option is that this option can only be exercised at the time of expiry and not before.

Now we can move to the actual formula which looks like this.

Now we can move to the actual formula which looks like this.

$$C = S * N(d_1) - K * e^{-rt} * N(d_2)$$

Now what is d_1 and d_2?

Let me lay down the values before I try to explain it. Thus,

$$d_1 = \frac{ln\frac{S}{K} + (r + \frac{s^2}{2})t}{s\sqrt{t}}$$

and

$$d_1 = \frac{ln\frac{S}{K} + (r - \frac{s^2}{2})t}{s\sqrt{t}}$$

Where, s = standard deviation of log returns and

ln = natural logarithm

While the actual derivation of these terms is somewhat lengthy and entails a deep dive into statistics, we can see that we are using the same terms and more importantly we are taking the natural log of the ratio of the stock price and the exercise price.

Coming back to the main formula, we can actually divide it into two parts,

The first part, S*N(d1) is what you get ie the underlying stock if we decide to exercise our right to buy the stock.

The second part, K*e(-rt)*N(d2) is what you have to pay to receive that option. Thus the exercise price, i.e. K is multiplied by the discounting factor e(-rt) as this is the amount which we could have invested in a riskless asset instead of buying the option.

The cumulative standard normal distribution function i.e. N() gives us the probability values for the expected values. Think of it as a probability value between 0 and 1. Thus you would now understand why we subtract the second part of the equation from the first to get the Option price.

That's all there is to the option pricing model. You can simply put the values in the equation and find the Option price. And depending on different options trading strategies, you can create a risk-neutral portfolio for yourself.

All right, hold on. Sure, we can get all the values of the variables, but what about volatility. How do you gauge the volatility of the underlying asset?

Well, the first thought that came to your mind is correct, we look up the historical prices, calculate their log normal returns and easily find the volatility. Then we assume that historical volatility will be more or less similar to the future volatility and thus calculate the options price on it.

But, there is another way to go about it, which seems like a shortcut. You see, if you check the options data for any stock, you will find a dozen of them at various strike prices, option prices etc. Now, we can use the option price which the market believes is the right price and use it as our "C" in the Black-Scholes equation to find the volatility. This is called the Implied volatility. You can check out this article which goes in-depth about the concept.

Awesome! We have understood how the Black Scholes Equation works for a European Call Option. Now let's see if we can implement this in Python.

Black Scholes in Python

If you want to find the current options data using python, you can use yahoo finance module to extract the relevant options data for a company.

```
import yfinance as yf # Import yahoo finance module
tesla = yf.Ticker("TSLA") # Passing Tesla Inc. ticker

opt = tesla.option_chain('2022-06-17') #retreiving option chains data for 17 June
```

To see the option calls, you will input the following code

```
opt.calls
```

Similarly for put options, you use the following code:

```
opt.puts
```

Now, we could probably code a few lines to implement the formula in Python, but the great thing about Python is its extensive use of libraries. Thus, we have the python library, mibian which makes it extremely easy to deduce the option prices.

The python code is simply,

```
BS([underlyingPrice, strikePrice, interestRate, daysToExpiration], volatility=x, callPrice=y, putPrice=z)
```

The syntax for BS function with the input as volatility along with the list storing an underlying price, strike price, interest rate and days to expiration:

```
c = mibian.BS([427.53, 300, 0.25, 4], volatility=60)
```

Here, we have taken our example of Tesla and input the Underlying price as $427.53, the exercise or Strike price as $300, Risk-free interest rate as 0.25% and days until expiration as 4.

We have put the volatility figure as 60%.

Now, if we need to find the Option call price of Tesla, we will just write the following

```
c.callPrice
```

The output is:

127.53821909748126

From <https://blog.quantinsti.com/black-scholes-model/>

What do you think? Is the Black Scholes model right? Why don't you try finding the options call price for another stock and leave the details in the comments.

All right! We have looked at the formula as well as its implementation in Python. Now we move on to the next topic, i.e. the limitations of the model.

Limitations

Before we list down the limitations of the Black Scholes Model, we have to understand that the creators of this model had to sacrifice a few things before they could build a working model. Having said that, let us list down the limitations:

- Volatility and the risk-free rate of returns are assumed to be constant even though it is dynamic in reality
- The stock price is assumed to be a random walk and thus large price moves due to certain factors like earnings reports, mergers and acquisitions are not incorporated in the model
- In case of stocks which pay dividends during the period we have calculated the options price, the model doesn't take the dividend into account, thus not correctly pricing the option
- While the pricing of in the money and out of the money options are accurate, it tends to deviate sharply when it comes to pricing deep out of the money options
- While other factors are directly observed and calculated, volatility has to be estimated and thus, could lead to different option prices

From <https://blog.quantinsti.com/black-scholes-model/>

So we have covered the derivation and working of the Black Scholes option pricing model. There are models that overcome the limitations of this model, but they do not form part of this course

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