

1 Electrostatics

1.1 Electric Field

1.1.1 Introduction

The fundamental problem that electrodynamics seeks to answer is the following: given a collection of electric charges q_1, q_2, q_3, \dots (the *source charges*), what force do they exert on another charge Q (the *test charge*)?

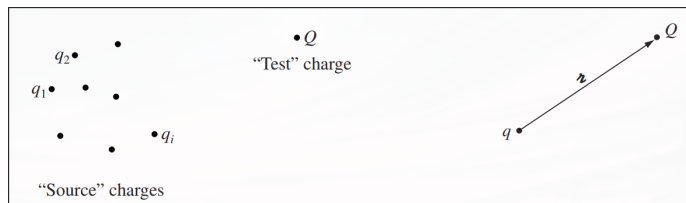


Figure 1: Source charges exerting a force on a test charge.

Electrostatics: Charges are at rest relative to one another.

Electrodynamics: Charges are in motion relative to one another.

These problems are solvable due to the **principle of superposition**, which states that the interaction between any two charges is unaffected by the presence of other charges. As a result, we can calculate the force due to each source charge independently and then add the results:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \dots$$

While this principle simplifies the physics conceptually, real problems may still be complex because the positions, velocities, and accelerations of charges (both present and past) can matter. To build intuition, we begin with electrostatics.

1.1.2 Coulomb's Law

The force on a test charge Q due to a single point charge q , at rest and separated by a distance r , is given by **Coulomb's Law**:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{\mathbf{r}}$$

Here, ϵ_0 is the **permittivity of free space**:

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2$$

It is often convenient to define the constant

$$k = \frac{1}{4\pi\epsilon_0} = 9.0 \times 10^9 \text{ Nm}^2/\text{C}^2$$

so Coulomb's Law can be written more compactly as

$$\mathbf{F} = k \frac{qQ}{r^2} \hat{\mathbf{r}}$$

1.1.3 The Electric Field

The **electric field** \mathbf{E} is a vector quantity defined at every point in space and depends on the configuration of source charges. Conceptually, it represents an entity that fills space around charges: when a test charge is placed at a given point, the electric field at that point determines the force it experiences.

If multiple point charges are present, the total force on a test charge Q is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots = k \left(\frac{q_1 Q}{r_{1Q}^2} \hat{\mathbf{r}}_{1Q} + \frac{q_2 Q}{r_{2Q}^2} \hat{\mathbf{r}}_{2Q} + \cdots \right)$$

Factoring out Q ,

$$\mathbf{F} = Q \mathbf{E}$$

where the electric field is defined as

$$\mathbf{E}(\mathbf{r}) = k \sum_{i=1}^n \frac{q_i}{r_{iQ}^2} \hat{\mathbf{r}}_{iQ}$$

Note that the electric field depends only on the source charges and the observation point, not on the test charge itself.

1.1.4 Charge Distributions

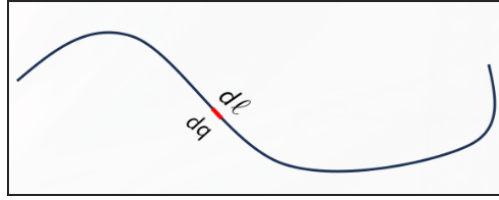
If the charge is distributed continuously over some region, the sum becomes an integral:

$$\mathbf{E}(\mathbf{r}) = k \int \frac{1}{r_{qQ}^2} \hat{\mathbf{r}}_{qQ} dq$$

In many physical situations, charge is distributed continuously rather than concentrated at a point. Such configurations are described using a **charge density**.

1. Linear Charge Distribution

Charge is distributed along a one-dimensional object (e.g., a thin wire) and is described by the **linear charge density**.



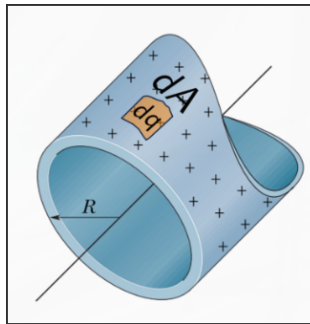
$$\lambda \equiv \frac{dq}{d\ell} \quad [\text{C m}^{-1}]$$

For an infinitesimal segment of length $d\ell$,

$$dq = \lambda \, d\ell$$

2. Surface Charge Distribution

Charge is distributed over a two-dimensional surface and is described by the **surface charge density**.



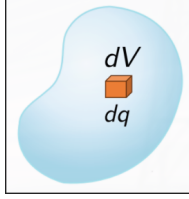
$$\sigma \equiv \frac{dq}{dA} \quad [\text{C m}^{-2}]$$

For an infinitesimal surface element dA ,

$$dq = \sigma \, dA$$

3. Volume Charge Distribution

Charge is distributed throughout a three-dimensional region and is described by the **volume charge density**.



$$\rho \equiv \frac{dq}{dV} \quad [\text{C m}^{-3}]$$

For an infinitesimal volume element dV ,

$$dq = \rho \, dV$$

Average Charge Density

In cases where the charge distribution is not uniform or when only the total charge over a finite region is known, it is useful to define an **average charge density**.

$$\bar{\lambda} = \frac{q}{L} \quad \bar{\sigma} = \frac{q}{A} \quad \bar{\rho} = \frac{q}{V}$$

where:

- q is the total charge,
- L is the total length,
- A is the total surface area,
- V is the total volume.

If there is uniform charge distribution, the charge density at any location is equivalent to the average charge density ($\rho = \bar{\rho}$).

1.2 Divergence and Curl of Electrostatic Fields

1.2.1 Field Lines, Flux, and Gauss's Law

The rest of electrostatics is assembling tools and tricks to avoid the involved integrals. The first of these tools is electric field lines. An **electric field line** is an imaginary curve drawn through a region of space so that its tangent at any point is in the direction of the electric field vector at that point.

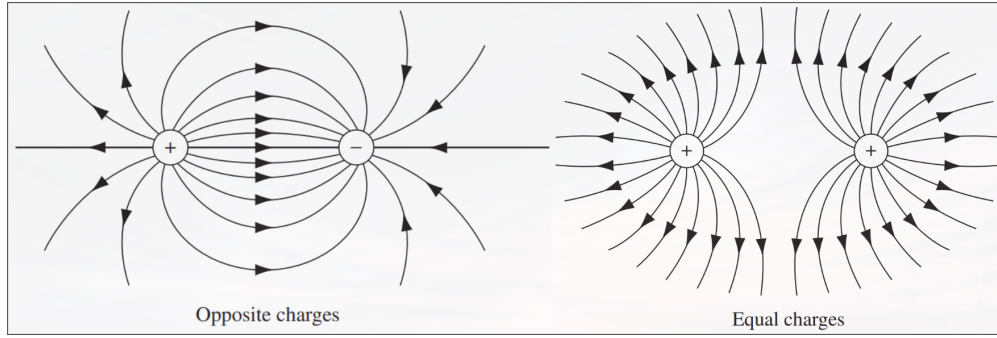


Figure 2: Field line for two oppoite charges and two equal charges.

The magnitude of the field is indicated by the density of the field lines: it's stronger near the center point charge where the field lines are closer together. Field lines never intersect, because \mathbf{E} has a unique direction at each point. In terms of direction of the arrows, field lines begin on positive charges and end on negative ones. Field lines also cannot terminate in midair, though they may extend out to infinity.

For a given (arbitrary) density of electric field lines, the **flux** of the electric field \mathbf{E} through a surface \mathcal{S} is proportional to the number of field lines passing through the surface. This follows because the field strength is proportional to the density of field lines (the number per unit area). Hence, $\mathbf{E} \cdot d\mathbf{A}$ is proportional to the number of field lines passing through the infinitesimal area $d\mathbf{A}$. The dot product selects the component of \mathbf{E} perpendicular to the surface:

$$\Phi_{\mathbf{E}} \equiv \int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{A}$$

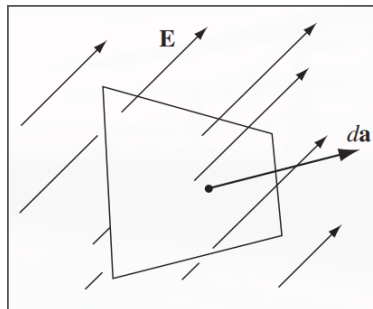


Figure 3: Flux is proportional to the number of electric field lines passing through a surface.

Since only the component of \mathbf{E} perpendicular to the surface contributes to the flux, we may write (for a field of constant magnitude over the surface)

$$\Phi_{\mathbf{E}} = E \int_{\mathcal{S}} \cos \theta \, dA$$

where θ is the angle between \mathbf{E} and the area vector $d\mathbf{A}$.

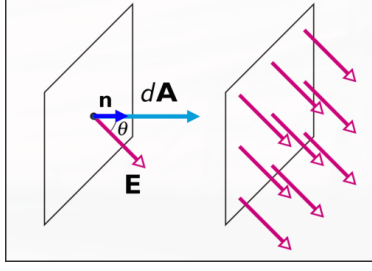


Figure 4: Angle between \mathbf{E} and the surface element $d\mathbf{A}$.

The electric flux through any closed surface reflects the total charge contained inside the surface. This is because electric field lines due to charges outside the surface pass into and out of it in equal amounts, giving no net contribution. Only field lines that originate from positive charges inside the surface or terminate on negative charges inside the surface contribute to a net flux. This is the essence of **Gauss's Law**.

To derive a quantitative explanation, consider a point charge q located at the origin. The electric field produced by the charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}.$$

The electric flux through a spherical surface of radius r centered on the charge is

$$\Phi_{\mathbf{E}} = \oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r^2} \hat{\mathbf{r}} \right) \cdot (r^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}).$$

Since $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$, this simplifies to

$$\Phi_{\mathbf{E}} = \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\phi = \frac{q}{4\pi\epsilon_0} (4\pi) = \frac{q}{\epsilon_0}.$$

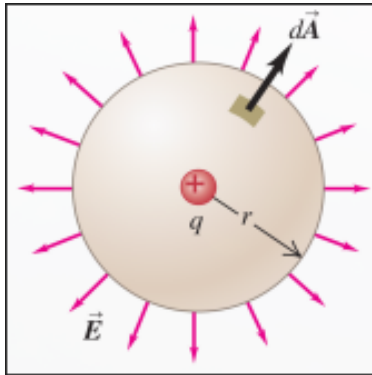


Figure 5: Point charge q at the origin within an enclosed surface.

Notice that as the surface area increases as r^2 , the magnitude of the electric field decreases as $1/r^2$. In terms of field lines, this means that the same number of field lines pass through any spherical surface enclosing the charge. Consequently, the flux through any closed surface enclosing the charge

is $\frac{q}{\epsilon_0}$, independent of the size or shape of the surface.

Now consider a collection of n point charges enclosed by a closed surface. By the principle of superposition, the total electric field is

$$\mathbf{E} = \sum_{i=1}^n \mathbf{E}_i.$$

The total flux through the surface is therefore

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^n \oint \mathbf{E}_i \cdot d\mathbf{a} = \sum_{i=1}^n \frac{q_i}{\epsilon_0} = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

Thus, for any closed surface,

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}}$$

This is **Gauss's Law**. It can be used in three primary ways:

1. If the enclosed charge Q_{enc} is known, the electric flux can be found directly.
2. If Q_{enc} is known and sufficient symmetry exists, the electric field \mathbf{E} can be determined.
3. If the electric field \mathbf{E} is known, the enclosed charge can be found by integration.

1.3 Work and Energy in Electrostatics

1.3.1 The Work it Takes to Move a Charge

If you wanted to move a test charge Q from point \mathbf{a} to point \mathbf{b} , how much work will you have to do?

$$W = \int_{\mathbf{b}}^{\mathbf{a}} \mathbf{F} \cdot d\mathbf{l}$$

Since the electric force of Q is $\mathbf{F} = Q\mathbf{E}$, the force you must exert, in opposition to the electric force, is $-Q\mathbf{E}$. Therefore

$$W = -Q \int_{\mathbf{b}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

Dividing through Q we have

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}$$

In words, *the potential difference between points \mathbf{a} and \mathbf{b} is equal to the work per unit charge required to carry a particle from \mathbf{a} to \mathbf{b} .* If you set the reference point to infinity, you see that *potential* is potential energy *per unit charge*.

$$W = Q[V(\mathbf{r}) - V(\infty)] = QV(\mathbf{r})$$

1.4 Conductors

1.4.1 Basic Properties

In an insulator, each electron is on a short leash, attached to a particular atom. However, in a metallic conductor one or more electrons per atom are free to roam. An ideal conductor is defined to have an unlimited supply of free charges. From this definition, the basic electrostatic properties of ideal conductors are:

1. **$\mathbf{E} = 0$ inside a conductor**, because free charges (electrons) immediately move in response to any field, rearranging themselves on the surface to create an opposing, equal internal field, neutralizing it.

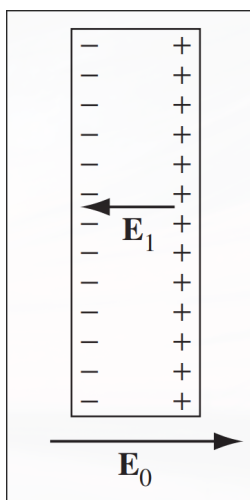


Figure 6: The electric field created by the moved charges opposes the external electric field.

2. **$\rho = 0$ inside a conductor**. This follows from the differential form of **Gauss' law**,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

If $\mathbf{E} = 0$, then $\rho = 0$. There is still charge present, but the *net* volume charge density is zero.

3. **Any net charge resides on the surface of a conductor**.
4. **A conductor is equipotential**. For any two points on a conductor, the electric potential V is the same everywhere.
5. **The electric field is perpendicular to the surface, just outside a conductor**.

1.4.2 Induced Charges

If you hold a $+q$ charge near an uncharged conductor, the two will attract. This occurs because the positive charge pulls negative charges to the nearer surface of the conductor, inducing a charge separation. Since opposite charges attract, the net force is attractive.

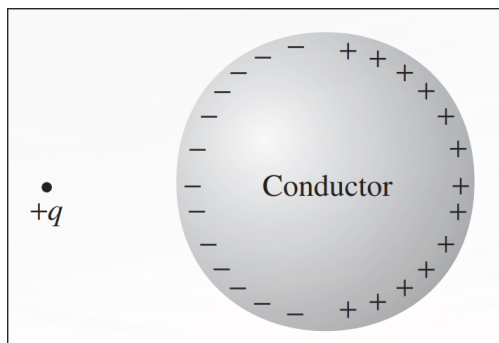


Figure 7: $+q$ inducing charge on a nearby conductor.

If instead a $+q$ charge is placed inside a cavity within a conductor, the conductor rearranges its surface charges so that the electric field inside the conducting material remains zero.

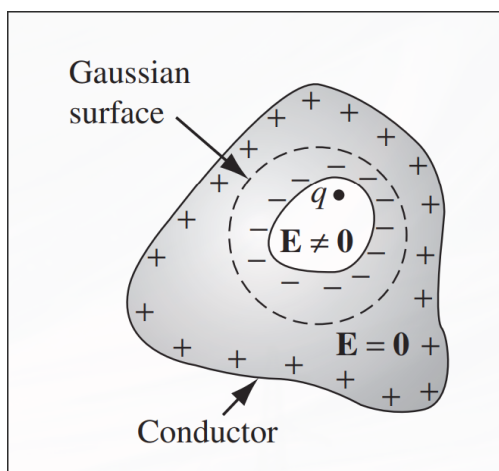


Figure 8: $+q$ inside a cavity of a conductor.

As a result, no electric field from the internal charge leaks outside the conductor, and no external electric fields penetrate the conductor. The conductor effectively isolates regions of space from one another.

1.4.3 Surface Charge

Since charge cannot exist in the bulk of a conductor, any excess charge must reside on the surface. We describe this using the **surface charge density**

$$\sigma = \frac{dq}{dA}.$$

Just outside the surface of a conductor, the electric field has two important properties:

1. **The electric field is perpendicular to the surface.** If there were a tangential component of \mathbf{E} , it would exert a force on free electrons along the surface, causing them to move. Since conductors in electrostatic equilibrium have no surface currents, the tangential component must be zero.
2. **The magnitude of the electric field just outside the surface is**

$$E = \frac{\sigma}{\epsilon_0}.$$

This result follows directly from **Gauss' law**. Consider a thin cylindrical Gaussian “pillbox” that straddles the surface of the conductor. The electric field inside the conductor is zero, and the field is perpendicular to the surface, so there is no flux through the curved side of the pillbox. Gauss' law gives

$$\oint \mathbf{E} \cdot d\mathbf{A} = EA = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}.$$

Canceling the area A yields

$$E = \frac{\sigma}{\epsilon_0}.$$

Thus, the surface charge density directly determines the strength of the electric field just outside a conductor.

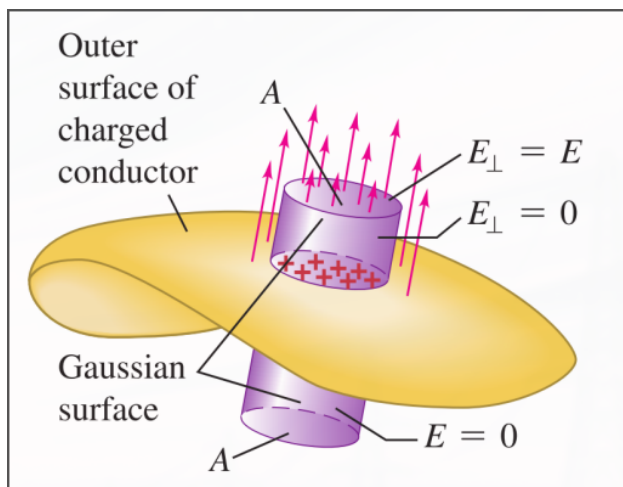


Figure 9: The field just outside a charged conductor is perpendicular to the surface.

2 Potentials

2.1 Electric Dipoles

An electric dipole consists of a pair of point charges with:

- equal magnitude of charge q ,
- opposite signs,
- separated by a distance d .

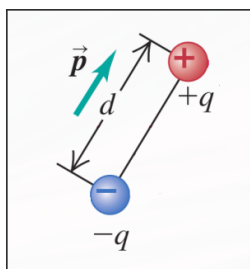


Figure 10: Diagram of an electric dipole.

An electric dipole is defined by the dipole moment vector

$$\mathbf{p} = q\mathbf{r}_{-+}$$

where

$$\mathbf{r}_{-+} = \mathbf{r}_- - \mathbf{r}_+, \quad d = |\mathbf{r}_{-+}|.$$

The magnitude of the dipole moment is therefore

$$p = qd.$$

Dipole moments are vectors and therefore add vectorially. If two dipoles have moments \mathbf{p}_1 and \mathbf{p}_2 , the total dipole moment is $\mathbf{p}_1 + \mathbf{p}_2$. This allows closely spaced charges to be approximated as dipoles, simplifying calculations.

Although a water molecule is electrically neutral overall, the arrangement of its chemical bonds causes a displacement of charge. This results in a partial negative charge near the oxygen atom and partial positive charges near the hydrogen atoms, forming an electric dipole.

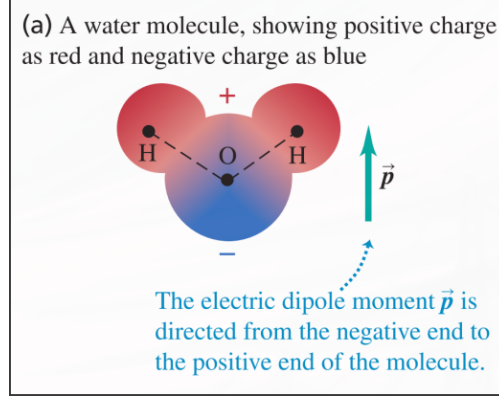


Figure 11: O–H bonds in a water molecule form an electric dipole.

2.1.1 Forces and Torque on Electric Dipoles

The forces \mathbf{F}_+ and \mathbf{F}_- acting on the charges of a dipole in an external electric field are equal in magnitude and opposite in direction, but they do not act along the same line. As a result, the net force on the dipole is zero (in a uniform field), but a **torque** acts on the dipole.

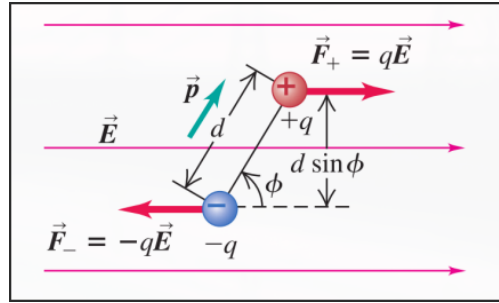


Figure 12: The net force on an electric dipole in a uniform field is zero, but a torque acts to rotate the dipole.

Using the definition of torque,

$$\begin{aligned}
 \tau &= \tau_+ + \tau_- \\
 &= \mathbf{r}_+ \times \mathbf{F}_+ + \mathbf{r}_- \times \mathbf{F}_- \\
 &= \mathbf{r}_+ \times q\mathbf{E} + \mathbf{r}_- \times (-q)\mathbf{E} \\
 &= q(\mathbf{r}_- - \mathbf{r}_+) \times \mathbf{E} \\
 &= \mathbf{p} \times \mathbf{E}.
 \end{aligned}$$

Thus,

$$\tau = \mathbf{p} \times \mathbf{E}$$

The magnitude of the torque is

$$|\tau| = pE \sin \phi,$$

where ϕ is the angle between \mathbf{p} and \mathbf{E} .

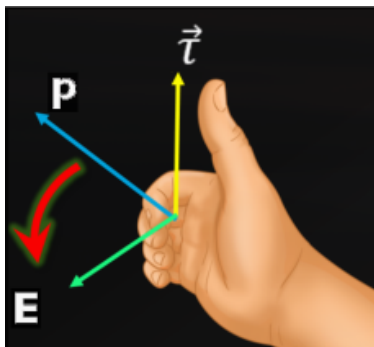


Figure 13: The direction of $\boldsymbol{\tau}$ is determined using the right-hand rule.

2.1.2 Potential Energy of Electric Dipoles

The potential energy of a dipole with moment \mathbf{p} placed in a uniform external electric field \mathbf{E} is defined as the negative of the work done by the field on the dipole:

$$dU = -dW.$$

The infinitesimal work done by the electric field is

$$\begin{aligned} dW &= dW_+ + dW_- \\ &= \mathbf{F}_+ \cdot d\mathbf{r}_+ + \mathbf{F}_- \cdot d\mathbf{r}_- \\ &= q\mathbf{E} \cdot d\mathbf{r}_+ - q\mathbf{E} \cdot d\mathbf{r}_- \\ &= q\mathbf{E} \cdot (d\mathbf{r}_+ - d\mathbf{r}_-). \end{aligned}$$

Assuming \mathbf{E} is constant,

$$\begin{aligned} W &= q\mathbf{E} \cdot (\mathbf{r}_+ - \mathbf{r}_-) \\ &= q\mathbf{E} \cdot \mathbf{r}_{-+} \\ &= \mathbf{p} \cdot \mathbf{E}. \end{aligned}$$

Therefore, the potential energy of the electric dipole is

$$\boxed{U = -\mathbf{p} \cdot \mathbf{E}.$$