Seminar Nr.2, Classical Probability; Rules of Probability; Conditional Probability; Independent Events

<u>**Theory Review**</u>

Classical Probability: $P(A) = \dfrac{\text{nr. of favorable outcomes}}{\text{total nr. of possible outcomes}} = \dfrac{N_f}{N_t}$.

Mutually Exclusive Events: A, B **m. e. (disjoint, incompatible)** $<=> P(A \cap B) = 0$.

Rules of Probability:

$P(\overline{A}) = 1 - P(A)$;
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
$P(A \setminus B) = P(A) - P(A \cap B)$.

Conditional Probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$.

Independent Events: A, B **ind.** $<=> P(A \cap B) = P(A)P(B) <=> P(A|B) = P(A)$.

Total Probability Rule: $\{A_i\}_{i \in I}$ a partition of S, then $P(E) = \sum\limits_{i \in I} P(A_i)P(E|A_i)$.

Multiplication Rule: $P\left(\bigcap\limits_{i=1}^{n} A_i\right) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2)\ \ldots\ P\left(A_n | \bigcap\limits_{i=1}^{n-1} A_i\right)$.

1.The faces of a cube are painted each in a different color (the cube is transparent on the inside). Then the cube is broken into 1000 smaller, equally-sized cubes and one such cube is randomly picked. Find the probability of the following events:
a) A: the cube picked has exactly three colored faces;
b) B: the cube picked has exactly two colored faces;
c) C: the cube picked has exactly one colored face;
d) D: the cube picked has no colored faces.

2. (Pigeonhole Principle) A postman distributes n letters in N mailboxes. What is the probability of the event A: there are m letters in a given (fixed) mailbox ($0 \leq m \leq n$)?

3. (Breaking Passwords) An account uses 8-character passwords, consisting of letters (distinguishing between lower-case and capital letters) and digits. A spy program can check about 1 million passwords per second.
a) On the average, how long will it take the spy program to guess your password?
b) What is the probability that the spy program will break your password within a week?
c) Same questions, if capital letters are not used.

4. (System Reliability) Compute the reliability of the system in Figure 1 if each of the five components is operable with probability 0.92, independently of each other.

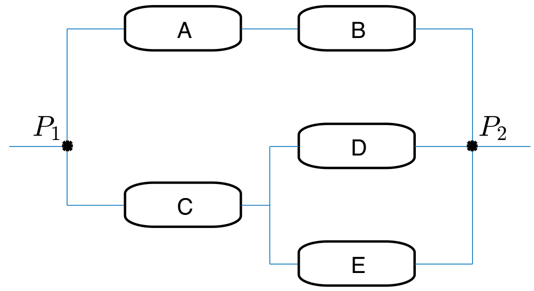


Figure 1: System Reliability

5. Among employees of a certain firm, 70% know C/C++, 60% know Fortran and 50% know both. What portion of programmers
a) does not know Fortran?
b) does not know C/C++ and does not know Fortran?
c) knows C/C++, but not Fortran?
d) Are "knowing C/C++" and "knowing Fortran" independent of each other?
e) What is the probability that someone who knows Fortran, also knows C/C++?
f) What is the probability that someone who knows C/C++, does not also know Fortran?

6. Three shooters aim at a target. The probabilities that they hit the target are 0.4, 0.5 and 0.7, respectively. Find the probability that the target is hit exactly once.

7. Under good weather conditions, 80% of flights arrive on time. During bad weather, only 50% of flights arrive on time. Tomorrow, the chance of good weather is 60%. What is the probability that your flight will arrive on time?

Bonus Problems: HW

8. There are 15 people waiting at a subway station. A train having 5 cars arrives and the passengers get in randomly. If each car has a capacity exceeding 15 persons, find the probability that at least one passenger will get into each car (event A).

9. There are two piles of used computer parts. All items in one pile are in working condition, while in the other pile, 1/4 of the items are defective. An item is chosen randomly from one of the two piles and it is in working condition. If it is put back in the same pile, what is the probability that a second item, chosen at random *from the same pile* as the first one, will turn out to be defective?

19.10.2023

Seminar 2

1. The faces of a cube are painted each in a different color (the cube is transparent on the inside). Then the cube is broken into 1000 smaller, equally-sized cubes and one such cube is randomly picked. Find the probability of the following events:
a) A: the cube picked has exactly three colored faces;
b) B: the cube picked has exactly two colored faces;
c) C: the cube picked has exactly one colored face;
d) D: the cube picked has no colored faces.

a) $P(\overbrace{\text{"3 col faces"}}^{A}) = \frac{8}{1000}$

b) $P(\overbrace{\text{"2 col faces"}}^{B}) = \frac{8 \cdot 12}{1000}$

c) $P(\overbrace{\text{"1 col faces"}}^{C}) = \frac{64 \cdot 6}{1000}$

d) $P(\underbrace{\text{"no col faces"}}_{\overline{A \cup B \cup C}}) = \frac{8^3}{1000} = 1 - P(A \cup B \cup C)$

$= 1 - (P(A) + P(B) + P(C))$

A, B events $\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

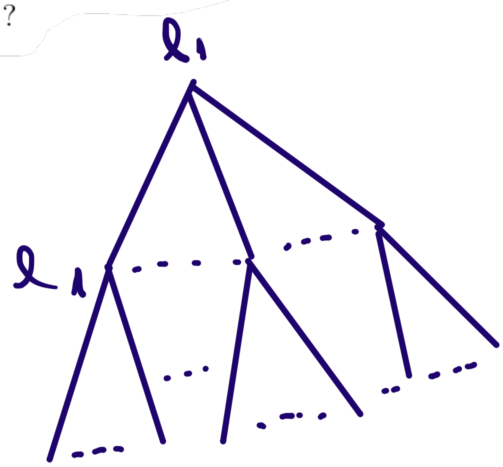
A, B events → independent if $P(A \cap B) = P(A) \cdot P(B)$

A, B events → disjoint if $A \cap B = \emptyset$ (it cannot happen at the same time)

$$\boxed{\begin{array}{l} A, B \text{ ev.}, \ P(A) > 0, \ P(B) > 0 \\ A, B \text{ independent} \iff P(A \mid B) = P(A) \\ \qquad\qquad\qquad\quad\ \iff P(B \mid A) = P(B) \end{array}}$$

2. (Pigeonhole Principle) A postman distributes n letters in N mailboxes. What is the probability of the event A: there are m letters in a given (fixed) mailbox ($0 \leq m \leq n$)?

$$\boxed{P = \frac{\text{nr. fav. cases}}{\text{nr. pos. cases}}}$$



Nr of pos. cases: N^n

$f : \{l_1 \ldots l_n\} \to \{b_1 \ldots b_N\}$

Nr of fav. cases: C_n^m (we do not care about the order) $\cdot$ $(N-1)^{n-m}$

$$\Rightarrow P = \frac{C_n^m \cdot (N-1)^{n-m}}{N^n}$$

4. (System Reliability) Compute the reliability of the system in Figure 1 if each of the five components is operable with probability 0.92, independently of each other.

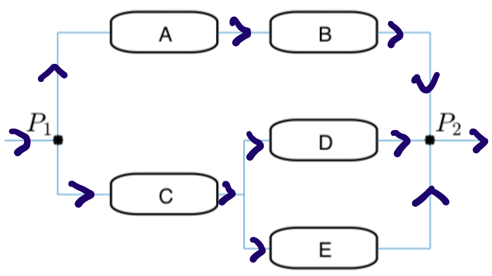


Figure 1: System Reliability

$P(\text{"comp. is on"}) = 0.92$

$$\boxed{\begin{array}{l} A, B \text{ events } \overset{\text{independent}}{\Rightarrow} \\ P(A \cap B) = P(A) \cdot P(B) \end{array}}$$

A, B, C, D, E components

$$\left.\begin{array}{ll} \boxed{A, B} & C_{AB} \\ \boxed{C, D, E} & C_{CDE} \end{array}\right\} \Rightarrow$$

$$P(C_{AB} \cup C_{CDE}) = P(C_{AB}) + P(C_{CDE}) - \underbrace{P(C_{AB} \cap C_{CDE})}_{\text{ind}}$$

$$P(C_{AB}) = P(A \cap B) \overset{\text{ind}}{=} P(A) \cdot P(B)$$

$$P(C_{CDE}) = P(C \cap (D \cup E)) \overset{\text{ind}}{=} P(C) \cdot P(D \cup E)$$

$$= P(C) \cdot [P(D) + P(E) - \underbrace{P(D \cap E)}_{P(D) \cdot P(E)}]$$

5. Among employees of a certain firm, 70% know C/C++, 60% know Fortran and 50% know both. What portion of programmers
a) does not know Fortran?
b) does not know C/C++ and does not know Fortran?
c) knows C/C++, but not Fortran?
d) Are "knowing C/C++" and "knowing Fortran" independent of each other?
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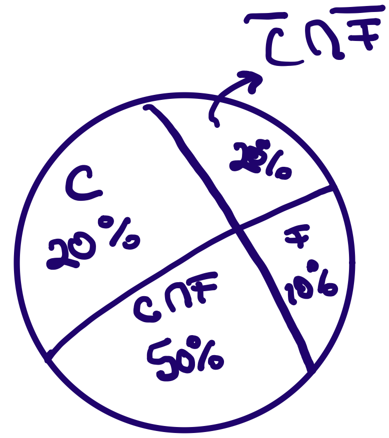

70% ... C
60% ... F
50% ... C&F

a) $P(\bar{F}) = 40\% = 0{,}4$

b) $P(\bar{C}\cap\bar{F}) = 20\% = 0{,}2$

c) $P(C\cap\bar{F}) = 20\% = 0{,}2$

d) C, F independent? $P(C\cap F) \overset{?}{=} P(C)\cdot P(F)$
$0.5 \overset{?}{=} 0.42$
NO!

e) $P(C|F) = \dfrac{P(C\cap F)}{P(F)} = \dfrac{0{,}5}{0{,}6} = \dfrac{5}{6}$

given F,
find the prob of C

$P(A \text{ given } B) = P(A|B) := \dfrac{P(A\cap B)}{P(B)}$

f) $P(\bar{F}|C) = \dfrac{P(\bar{F}\cap C)}{P(C)} = \dfrac{P(C\cap\bar{F})}{0{,}7} = \dfrac{0.2}{0.7}$

6. Three shooters aim at a target. The probabilities that they hit the target are 0.4, 0.5 and 0.7, respectively. Find the probability that the target is hit exactly once. → only independence

S_i: „shooter i hits the target"

$P(S_1) = 0.4$ $\quad P(S_2) = 0.5$ $\quad P(S_3) = 0.7$

$P($"exactly once the target is hit"$) = ?$

$$\underbrace{(S_1 \cap \overline{S_2} \cap \overline{S_3}) \cup (\overline{S_1} \cap S_2 \cap \overline{S_3}) \cup (\overline{S_1} \cap \overline{S_2} \cap S_3)}_{\text{mutually exclusive}}$$

$$= P(S_1 \cap \overline{S_2} \cap \overline{S_3}) + \ldots$$

$$\overset{\text{ind}}{=} P(S_1) \cdot P(\overline{S_2}) \cdot P(\overline{S_3})$$