25.04.2023

Seminar 7

→ Reflections and projections (in hyper planes)

E^2	hyper plane / lines
E^3	plane

hyper plane = object described by one linear eq. ! const. term needs to be 0

$$H: \underbrace{\overbrace{a_1x_1 + \dots + a_mx_m}^{\text{lin } \phi} + a_{m+1}}_{Q(x_1, \dots x_m)} = 0$$

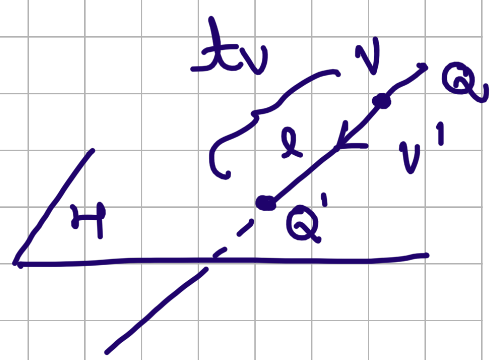
$H = \phi^{-1}(0)$ (maps to 0)

ϕ, lin $\phi: \mathbb{R}^m \to \mathbb{R}$

* H through matrix multiplication

$$H: \underbrace{[a_1 \dots a_m]}_{a^t} \left\| \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}}_{x} \right\| = -a_{m+1}$$

$\iff Q^t \cdot X = -Q_{m+1}$



VECTOR FORM

$$\ell = \{Q + t\cdot v : t \in \mathbb{R}\} = \left\{ \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} + t\cdot \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}, t \in \mathbb{R} \right\}$$

↓ can be done in cart. eq, but for more dim. use vector form

$\ell \cap H$ = set of points that satisfy the H eq. are on the line

↓

$$\begin{bmatrix} q_1 + tv_1 \\ - \quad - \quad - \\ q_m + tv_m \end{bmatrix} \rightarrow$$ plug these in the eq. of the plane

$$a_1(q_1 + tv_1) + \dots + a_m(q_m + tv_m) + a_{m+1} = 0$$

Solution:

$$t = -\frac{a_1 q_1 + \dots + a_m q_m + a_{m+1}}{a_1 v_1 + \dots + a_m v_m}$$

$$= \frac{-\phi(Q)}{(\partial \ln \phi)(v)} = \frac{-Q^t \cdot [Q]}{Q^t \cdot v} - \frac{a_{n+1}}{Q^t \cdot v}$$

$$\Rightarrow Q' = Q - \frac{\phi(Q)}{(\partial \ln \phi)(v)} \cdot v$$

$H \parallel t \Rightarrow$ denominator $= 0$

$Q \cdot V = 0 \Rightarrow Q \perp V$

$\underline{t}$: we took one point on the line

Tensor product

Def: Let $v(v_1 \dots v_n)$, $w(w_1 \dots w_m)$

Exterior product

$$v \otimes w = v \cdot w^t = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} [w_1 \dots w_m]$$

P_1: $\_ \otimes \_ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow Mat_{n \times m}(\mathbb{R})$ is linear

$\downarrow$

linear matrices

$\Rightarrow$ bc. it is def by the product of 2 matrices

P_2: $(v \otimes w)\cdot w = \overbrace{(v\cdot w)}^{\text{coefficient}}\, u$

↓

scalar product

$$\underbrace{\left(\begin{array}{c|c} u & \overline{\;v\;} \end{array} \right)}_{m\times m} \underbrace{w}_{m\times 1} = \underset{u}{\Big|}\left(\overline{\;v\;} \,\middle|\, w \right)$$

distributivity

P_3: not commutative, but:

$v \otimes w \neq w \otimes v$ $\qquad (v\cdot w^t)^t = (w^t)^t \cdot v^t$

$(v \otimes w)^t = w \otimes v$ $\qquad = w \cdot v^t$

(skew commutativity)

Interior product

$$v^t \cdot w = [v_1 \ldots v_m] \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \text{scalar}$$

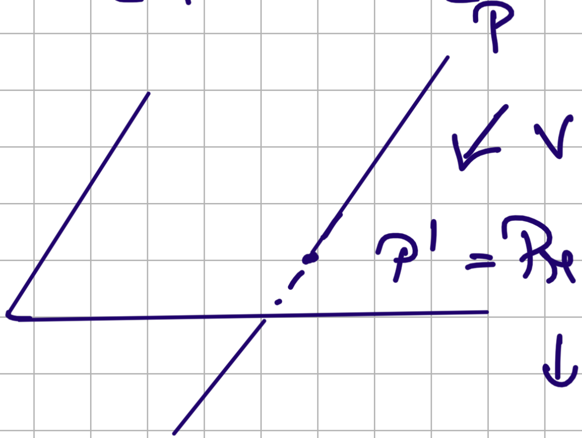
dot product

Projections :

Task : describe the map entirely

$Pr_{H,V} : \mathbb{E}^m \longrightarrow \mathbb{E}^m \supseteq H$

$$p' : \begin{bmatrix} p_1' \\ \vdots \\ p_m' \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix} + \mu \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$



$p' = Pr_{H,V}(P)$

$\downarrow$

projection of P on H plane

$$p_1' = p_1 - \frac{a_1 p_1 v_1 + \dots a_m p_m v_1}{a_1 v_1 + \dots a_m v_m} - \frac{a_{m+1} v_1}{a^t \cdot v}$$

$$\begin{bmatrix} p_1' \\ \vdots \\ p_m' \end{bmatrix} = I_m \cdot \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix} - \frac{1}{a^t \cdot v} \, v \otimes a \begin{bmatrix} v_1 a_1 \dots v_1 a_m \\ \\ \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix}$$

$(v \times a)$

$$- \frac{a_{m+1}}{a^t v} \cdot v$$

$$[P'] = \underbrace{\left(I_m - \frac{v \otimes a}{v \cdot a} \right)}_{\text{lin } Pr_{H,v}} [P] - \frac{a_{m+1}}{v \cdot a} \cdot v$$

↳ coord. of P column matrix

$$\operatorname{Im}(Pr_{H,v}) = H$$

↳ image

$f : A \to B$

$f(A) \neq B$

$v(2,1,1)$

a) Det matrix form of $Pr_{H,v}$ where $H \cdot z = 0$

$H : a_1 x + a_2 y + a_3 z + a_4 = 0$

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[Pr_{H,v}(P)] = \left(I_3 - \frac{1}{1} v \otimes a \right) [P] - \frac{0}{v \cdot a} [v]$$

loc. hyper plane is passing through the origin

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

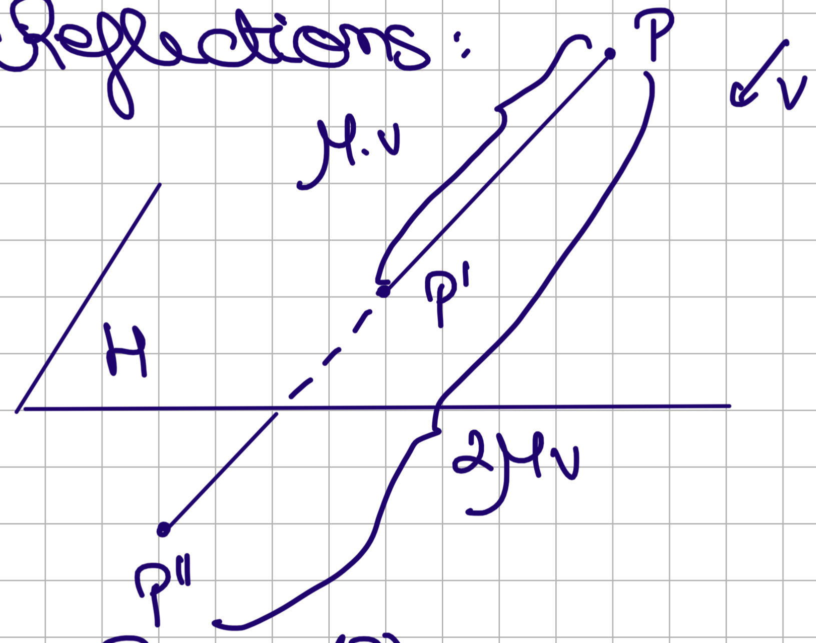
$$[Pr_{H,v}(P)] = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}$$

$$Pr_{H,v}(P) = \ell_P \cap H \quad : \quad z_P + t = 0 \Rightarrow t = -z_P$$

$$\ell_P : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_P - 2z_P \\ y_P - z_P \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} = \begin{bmatrix} x_P + 2t \\ y_P + t \\ z_P + t \end{bmatrix}$$

Reflections:



$= Ref_{H,v}(P)$

P' midpoint of PP''

$$[Pr_{H,v}(P)] = \frac{[P]\ [Ref_{H,v}(P)]}{2}$$

$$Ref_{H,v}(P) = \left[2Pr_{H,v}(P)\right] - [P]$$

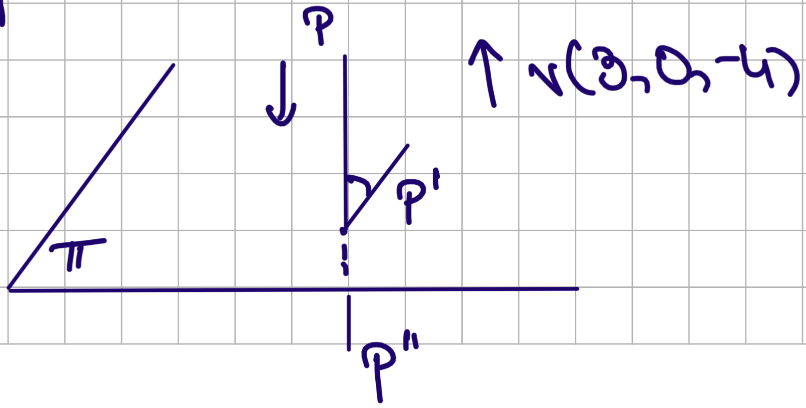
$$= 2\cdot\left(I_n - \frac{v \otimes a}{v\cdot a}\right)[P] - \frac{a_{n+1}}{v\cdot a}v - [P]$$

$$= \underbrace{\left(I_n - 2\frac{v\otimes a}{v\cdot a}\right)}_{\text{linear part}}[P] - 2\frac{a_{n+1}}{v\cdot a}v$$

$$Ref_{H,v} : E^n \to E^n$$

$$\begin{bmatrix} p_1'' \\ \vdots \\ p_n'' \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} + 2\lambda\cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

12. Give the orth. refl. in the plane $\pi: 3x - 4z = -1$



$$v = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$a_1 x + a_2 y + a_3 z + a_4 = 0 \qquad a = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$\mathrm{Ref}_{H,v}(P) = \left[2\,\mathrm{Pr}_{H,v}(P) \right] - [P]$$

$$\mathrm{Ref}_{H,v}(P) = 2\left(I_m - 2\frac{v \otimes a}{v \cdot a} \right)[P] - 2 \cdot \frac{a_{m+1}}{v \cdot a} \cdot v$$

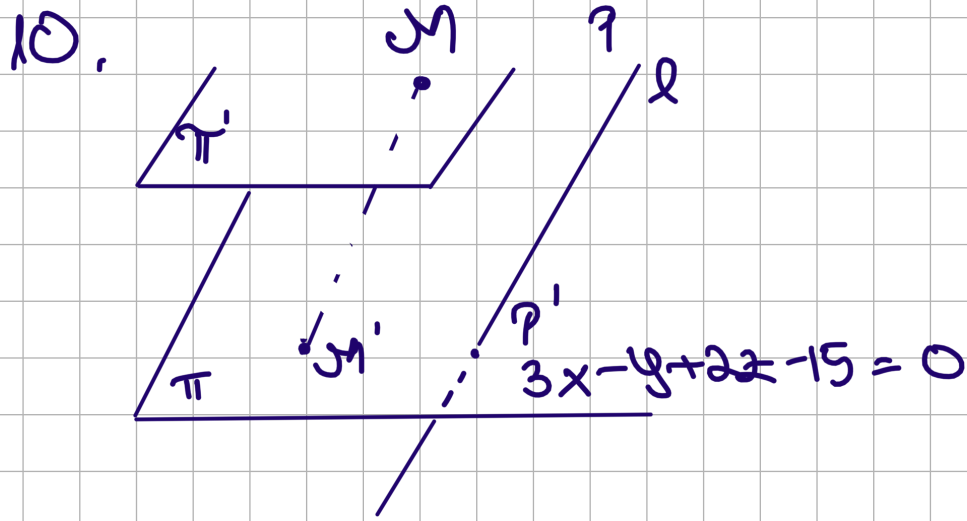
$$= \left(I_3 - 2 \cdot \frac{(3,0,-4) \otimes (3,0,-4)}{(3,0,-4)\cdot(3,0,-4)} \right)[P] - 2 \cdot \frac{1 (3,0,-4)}{(3,0,-4)(3,0,-4)}$$

$$= \left[I_3 - 2 \cdot \frac{1}{25} \cdot \begin{pmatrix} 9 & 0 & -12 \\ 0 & 0 & 0 \\ -12 & 0 & 16 \end{pmatrix} \right][P] - 2 \cdot \frac{1}{25} \cdot \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$= \frac{1}{25} \cdot \begin{pmatrix} 7 & 0 & 24 \\ 0 & 25 & 0 \\ 24 & 0 & -7 \end{pmatrix} [P] - 2 \cdot \frac{1}{25} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$= \frac{1}{25} \begin{pmatrix} 7 & 0 & 24 \\ 0 & 25 & 0 \\ 24 & 0 & -7 \end{pmatrix} \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix} - \frac{2}{25} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

10.



Det. l' s.t. it passes through

$M(1,0,7)$

$l' \parallel \pi$

$l' \cap l \neq \emptyset$

$$\frac{x-1}{4}=\frac{y-3}{2}=\frac{z}{1}$$

$P = \pi' \cap l$

$\pi': a_1x+a_2y+a_3z+a_4=0$

$$a=\begin{bmatrix}3\\-1\\2\end{bmatrix}$$

$3\cdot 1+0+2\cdot 7-c=0 \Rightarrow c=17 \Rightarrow$

$\pi': 3x-y+2z-17=0$

$l' = \pi' \cap \tilde{\pi}$ $\quad$ $\tilde{\pi}$ contains l and passes through M

$\tilde{\pi} \supseteq l \quad \tilde{\pi} \ni M$

$S = M' + (\overrightarrow{M'P'} + \overrightarrow{M'M})$