Seminar Nr.1, Euler's Functions; Counting, Outcomes, Events

Theory Review

Euler's Gamma Function: $\Gamma : (0,\infty) \to (0,\infty), \Gamma(a) = \int\limits_0^\infty x^{a-1}e^{-x}dx.$

1. $\Gamma(1) = 1$;
2. $\Gamma(a+1) = a\Gamma(a)$, $\forall a > 0$;
3. $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$;
4. $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{2}\int\limits_0^\infty e^{-\frac{t^2}{2}}dt = \int\limits_{\mathbb{R}} e^{-t^2}dt = \sqrt{\pi}.$

Euler's Beta Function: $\beta : (0,\infty)\times(0,\infty) \to (0,\infty), \beta(a,b) = \int\limits_0^1 x^{a-1}(1-x)^{b-1}dx.$

1. $\beta(a,1) = \dfrac{1}{a}$, $\forall a > 0$;
2. $\beta(a,b) = \beta(b,a)$, $\forall a,b > 0$;
3. $\beta(a,b) = \dfrac{a-1}{b}\beta(a-1,b+1)$, $\forall a > 1, b > 0$;
4. $\beta(a,b) = \dfrac{b-1}{a+b-1}\beta(a,b-1) = \dfrac{a-1}{a+b-1}\beta(a-1,b)$, $\forall a > 1$, $b > 1$;
5. $\beta(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $\forall a > 0, b > 0$.

Arrangements: $A_n^k = \dfrac{n!}{(n-k)!}$;

Permutations: $P_n = A_n^n = n!$;

Combinations: $C_n^k = \dfrac{A_n^k}{P_k} = \dfrac{n!}{k!(n-k)!}$.

De Morgan's laws:

$$\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A}_i \quad \text{and} \quad \overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A}_i.$$

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1. In how many ways can 10 students be seated in a classroom with
a) 15 chairs?
b) 10 chairs?

2. Find the number of possible outcomes for the following events:
a) three dice are rolled;
b) two letters and three digits are randomly selected.

3. A firm offers a choice of 10 free software packages to buyers of their new home computer. There are 25 packages available, five of which are computer games, and three of which are anti-virus programs.
a) How many selections are possible?
b) How many selections are possible, if exactly three computer games are selected?
c) How many selections are possible, if exactly three computer games and exactly two anti-virus programs are selected?

4. A person buys n lottery tickets. For $i = \overline{1,n}$, let A_i denote the event: the i^{th} ticket is a winning one. Express the following events in terms of $A_1, ..., A_n$.
a) A: all tickets are winning;

b) B: all tickets are losing;
c) C: at least one is winning;
d) D: exactly one is winning;
e) E: exactly two are winning;
f) F: at least two are winning;
g) G: at most two are winning.

5. Three shooters aim at a target. For $i = \overline{1,3}$, let A_i denote the event: the i^{th} shooter hits the target. Express the following events in terms of A_1, A_2 and A_3.
a) A: the target is hit;
b) B: the target is not hit;
c) C: the target is hit exactly three times;
d) D: the target is hit exactly once;
e) E: the target is hit exactly twice.

Solutions:

1. In how many ways can 10 students be seated in a classroom with
a) 15 chairs?
b) 10 chairs?

a) A_{15}^{10}

b) P_{10}

2. Find the number of possible outcomes for the following events:
a) three dice are rolled;
b) two letters and three digits are randomly selected.

a) 3 dice

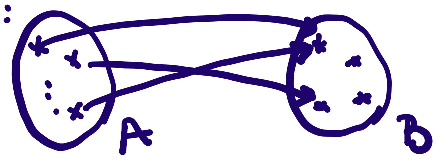
$$6 \cdot 6 \cdot 6 = 6^3$$

$d_1 \quad d_2 \quad d_3$

$f: \{d_1, d_2, d_3\} \to \{1 \dots 6\}$

Nr. functions $f: A \to B$: card $A = n$, card $B = m$

Function:



• each input → only one output

• draw an arrow from each elem in A

? how many functions $A \to B$: $\underbrace{m \cdot \dots \cdot m}_{n \text{ times}} = m^n$

b) 2 letters, 3 digits

$$26^2 \cdot 10^3$$

b') password with 2 let., 3 digits.

$$\boxed{C_n^k = C_n^{n-k}}$$

$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\_}$

$$\underbrace{C_5^2 \cdot 26^2}_{\text{letters}} \cdot \underbrace{10^3}_{\text{digits}}$$

$$\underbrace{C_5^3 \cdot 26^2 \cdot 10^3}_{\text{digits first}}$$

I. choose pos. for letters

ex: $\underset{1}{\_}\ \underset{2}{\overset{l}{\_}}\ \underset{3}{\_}\ \underset{4}{\overset{l}{\_}}\ \underset{5}{\_}$

II. put the letters on the chosen pos

$\underset{1}{\overset{l}{\_}}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\overset{l}{\_}}$

III. put the digits on the rem. pos.

b'') password with distinct 2 letters, distinct 3 digits

$$C_5^3 \cdot \underbrace{26 \cdot 25}_{A_{26}^2} \cdot \underbrace{10 \cdot 9 \cdot 8}_{A_{10}^3}$$

$$5! \, C_{26}^2 \cdot C_{10}^3$$

3. A firm offers a choice of 10 free software packages to buyers of their new home computer. There are 25 packages available, five of which are computer games, and three of which are anti-virus programs.
a) How many selections are possible?
b) How many selections are possible, if exactly three computer games are selected?
c) How many selections are possible, if exactly three computer games and exactly two anti-virus programs are selected?

10 free pack.

$\downarrow$

$$\underbrace{5g. + 3a. + 17o.}_{25 \text{ pack}}$$

a) C_{25}^{10}

b) $C_{20}^{7} \cdot C_{5}^{3}$

c) $C_5^3 \cdot C_3^2 \cdot C_{17}^{5}$

5. Three shooters aim at a target. For $i = \overline{1,3}$, let A_i denote the event: the i^{th} shooter hits the target. Express the following events in terms of A_1, A_2 and A_3.
a) A: the target is hit;
b) B: the target is not hit;
c) C: the target is hit exactly three times;
d) D: the target is hit exactly once;
e) E: the target is hit exactly twice.

$\boxed{\begin{array}{ll} \underline{\text{or}} & \cup \\ \underline{\text{and}} & \cap \\ \underline{\text{not}} & - \end{array}}$

A_i : i^{th} shooter
$i = 1, 2, 3$

a) $A = A_1 \cup A_2 \cup A_3$

b) $B = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3} = \overline{A}$

c) $C = A_1 \cap A_2 \cap A_3$

d) $D = (A_1 \cap \overline{A_2} \cap \overline{A_3}) \cup (\overline{A_1} \cap A_2 \cap A_3) \cup (\overline{A_1} \cap \overline{A_2} \cap A_3)$

e) $E = (\overline{A_1} \cap A_2 \cap A_3) \cup (A_1 \cap \overline{A_2} \cap A_3) \cup (A_1 \cap A_2 \cap \overline{A_3})$

$E = (A - C) - D$

4. A person buys n lottery tickets. For $i = \overline{1,n}$, let A_i denote the event: the i^{th} ticket is a winning one. Express the following events in terms of $A_1, ..., A_n$.

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b) B: all tickets are losing;

c) C: at least one is winning;

d) D: exactly one is winning;

e) E: exactly two are winning;

f) F: at least two are winning;

g) G: at most two are winning.

a) $A: A_1 \cap A_2 \cap \dots A_m = \bigcap_{i=1}^{m} A_i$

b) $B: \overline{A_1} \cap \overline{A_2} \cap \dots \overline{A_m} = \bigcap_{i=1}^{m} \overline{A_i}$

c) $C: A_1 \cup A_2 \cup \dots A_m = \overline{B}$

$= \overline{\bigcap_{i=1}^{m} \overline{A_i}} = \bigcup_{i=1}^{m} \overline{\overline{A_i}} = \bigcup_{i=1}^{m} A_i$

d) $D: (A_1 \cap \overline{A_2} \cap \dots \overline{A_m}) \cup$

$(\overline{A_1} \cap A_2 \dots \overline{A_m}) \cup \dots$

$\cup (\overline{A_1} \cap \overline{A_2} \dots A_m) = \bigcup_{i=1}^{m} \Big(A_i \cap \bigcap_{\substack{k=1 \\ k \neq i}}^{m} \overline{A_k} \Big)$

e) $E: \bigcup_{1 \le i < j \le m} \Big(A_i \cap A_j \cap \bigcap_{\substack{k \neq i, j \\ k=1}}^{m} \overline{A_k} \Big)$

f) $F: C \setminus D$

g) $G: \overline{F} \cup E = B \cup D \cup E$