Calculus

Onkar Pandit

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Contents

1 Introduction

This branch of mathematics solves questions which are continuous in nature. It deals with rate of change of any function and other important applications on functions. In my opinion, this branch of mathematics develops tools for analyzing functions and various operations on them, because of this it is used vastly in various fields.

2 Functions

Functions map one or more variables to a single value. The general notation of function f is given as $f : A \to B$ where f maps elements from A to B. The set A is called *domain* of function f and B is called *codomain*, and the set of values attained by the function (which is a subset of B) is called its *range* or *image*. Most important property of function to remember is that output from function is always a single value.

Is it always true? Can we have a mathematical entity which produces multiple outputs can be called as mathematical function? In these kind of cases, we can consider range of functions as vectors, so in a way we get multiple values, but in reality the function is outputting only single mathematical object i.e. a vector. Therefore, in these types of functions, it is assumed that they operate on the higher dimensions. This means, they take a point in the R^m dimension as an input and transforms to R^n dimensional point. Also, suppose we do not want to consider the picture of range as vectors, then we can consider that as a *relation* which can be considered as a superset of functions i.e. function is a special type of relation.

3 Limits

In a layman's term *limit* of function tells what is the value of function when the variable value on which function depends becomes closer to some value. For example,

$$\lim_{x \to a} f(x) = L$$

This reads as x approaches to a the value of function $f(x)$ becomes closer and closer to L.

It is important to note that, the limit does not gurantee that the value is actually L when x tends to reach a, it only says that when the x value approaches a, the $f(x)$ value nears L. In some cases when function is continuous at $x = a$, the value of $f(x)$ may not be equal to L.

Now, let us see formal definition of limit of function.

Definition 3.1. We say that $\lim_{x \to a} f(x) = L$, i.e. L is limit of a function $f(x)$ as $x \to a$ if,

1. $f(x)$ need not be defined at $x = a$ but it must be defined for all other intervals which contains a.

2. for every $\epsilon > 0$ one can find $\delta > 0$ such that $\forall x \in A$ function f follows: $|x - a| < \delta \quad \rightarrow |f(x) - L| < \epsilon$

Definition 3.2 (Right limits of function)**.** We say that $\lim_{x\to a^+} f(x) = L$, i.e. L is limit of a function $f(x)$ as $x \to a$ from the right hand side of the function which means from the higher values of a if,

1. $f(x)$ need not be defined at $x = a$ but it must be defined for all other intervals which contains a from the right side of a.

2. for every $\epsilon > 0$ one can find $\delta > 0$ such that $\forall x \in A$ function f follows: $a < x < a + \delta \quad \rightarrow |f(x) - L| < \epsilon$

Definition 3.3 (Left limits of function)**.** We say that $\lim_{x\to a^-} f(x) = L$, i.e. L is limit of a function $f(x)$ as $x \to a$ from the left hand side of the function which means from the lower values of a if,

1. $f(x)$ need not be defined at $x = a$ but it must be defined for all other intervals which contains a from the left side of a.

2. for every $\epsilon > 0$ one can find $\delta > 0$ such that $\forall x \in A$ function f follows: $a - \delta < x < a \quad \rightarrow |f(x) - L| < \epsilon$

Theorem 3.1. If both-sided limits of function exist as $\lim_{x\to a^+} f(x) = L_+$ and $\lim_{x\to a^-} f(x) = L_-$ then $\lim_{x\to a} f(x) = L$ exists if and only if $L = L_+ = L_-$.

Properties:

$$\lim_{x\to a} c = c \quad \cdots \text{ where c is constant} \tag{1}$$

If,

$$\lim_{x\to a} F_1(x) = L_1 \text{ and } \lim_{x\to a} F_2(x) = L_2$$

then sum, product and division of functions is given as:

$$\lim_{x\to a}(F_1(x) + F_2(x)) = L_1 + L_2 \tag{2}$$

$$\lim_{x\to a}(F_1(x) - F_2(x)) = L_1 - L_2 \tag{3}$$

$$\lim_{x\to a}(F_1(x).F_2(x)) = L_1.L_2 \tag{4}$$

$$\lim_{x\to a}(\frac{F_1(x)}{F_2(x)}) = \frac{L_1}{L_2} \text{ if } \lim_{x\to a} F_2(x) \neq 0 \tag{5}$$

Definition 3.4. If there is no number L such that $\lim_{x\to a} f(x) = L$ then we say that limit does not exist for $f(x)$ at a.

For example, let us look at the sign function,

$$sign(x) = \begin{cases} -1 \quad if \;\; x < 0 \\ 0 \quad if \;\; x = 0 \\ 1 \quad if \quad x > 0 \end{cases}$$

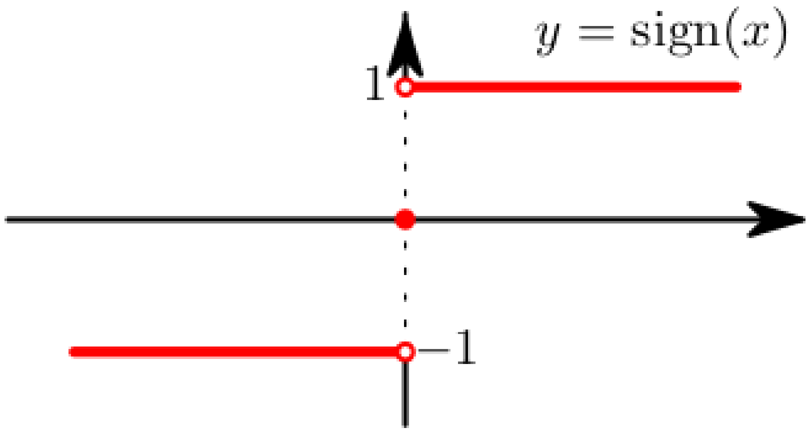

Figure 1: Limit of sign function does not exist at $x = 0$

4 Continuity of function

Definition 4.1. Suppose function $g : A \to B$, then g is continuous at a if

$$\lim_{x \to a} g(x) = g(a) \tag{6}$$

Which can be extended $\forall a \in A$, suppose:

$$\lim_{x \to a} g(x) = g(a) \quad \forall a \in A \tag{7}$$

then the function $g(x)$ is said to be continuous function.

5 Differentiation of function

Definition 5.1. Suppose function $f(x)$ is defined in some interval (c, d) and $c < a < d$ then $f(x)$ is said to be differentiable at a if the following limit exists and its value is given as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{8}$$

Derivative formulae:

$$c' = 0 \qquad \frac{d}{dx}c = 0 \quad \text{Constant rule} \tag{9}$$

$$(u+v)' = u' + v' \qquad \frac{d}{dx}(u \pm v) = \frac{d}{dx}u \pm \frac{d}{dx}v \quad \text{Sum rule} \tag{10}$$

$$(u.v)' = u'.v + v'u \qquad \frac{d}{dx}(u.v) = \frac{d}{dx}u.v + u.\frac{d}{dx}v \quad \text{Product rule} \tag{11}$$

$$(\frac{u}{v})' = \frac{u'v - v'u}{v^2} \qquad \frac{d}{dx}(\frac{u}{v}) = \frac{\frac{d}{dx}u.v - u.\frac{d}{dx}v}{v^2} \quad \text{Division rule} \tag{12}$$

$$\frac{d}{dx}(u(x))^n = n(u(x))^{n-1}\frac{du}{dx} \quad \text{Power rule} \tag{13}$$

Composition functions: Suppose f, g are two functions then,

$$f \circ g(x) = f(g(x)) \tag{14}$$

Theorem 5.1 (Chain rule)**.** The differentiation of compositions is given as:

$$(f \circ g(x))' = f'(g(x)).g'(x) \tag{15}$$

From this generalized chain rule is given as:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} \tag{16}$$

where z is function of y and y is function of x. It can be extended for any number of dependencies as

$$\frac{dv}{dx} = \frac{dv}{du_1} \times \frac{du_1}{du_2} \times \frac{du_2}{du_3} \times \cdots \times \frac{du_n}{dx} \tag{17}$$

where v is function of u_1, u_1 is function of u_2, u_2 is function of u_3, and so on and finally u_n is function of x.

Tangent vs normal line The tangent is a straight line which just touches the curve at a given point and the normal is a straight line which is perpendicular to the tangent. To get equation of these lines we make use of the equation of the line $y - y_1 = m(x - x_1)$ where m is the slope of the line which passes through points (x_1, y_1). We also make use of the fact that if two lines with gradients m_1 and m_2 respectively are perpendicular, then $m_1 m_2 = -1$.

At any given point on the curve, its derivative gives the slope of the line which is tangent to the curve at that point.

Let us see this with an example. Suppose, we wish to calculate the equation of the tangent line to the curve $f(x) = x^3 - 3x^2 + x - 1$ at $x = 3$. First we calculate the points through which the tangent line should pass, this the point which should line on the curve. Therefore, its x coordinate is 3, y=f(3)=2, i.e. point $(3, 2)$ must be on the tangent. Now, we calculate slope of the tangent, which would be $f'(3)$, for that first we get the differentiation of the function: $f'(x) = 3x^2 - 6x + 1$, hence $f'(3) = 10$. The equation of the line is given as $y - 2 = 10(x - 3)$ which is $y = 10x - 28$.

Now, let us find the equation of the normal line at $x = 3$. Remember that, normal line is perpendicular to the curve as well as to the tangent. Therefore, slope of the normal line is one over the differentiation i.e. $m_N = -\frac{1}{f'(a)}$ at point a. In our problem slope of the normal line is $-\frac{1}{10}$ and it also passes through the point (3,2), therefore the equation of the normal line is given as $y - 2 = -\frac{1}{10}(x - 3)$.

5.1 Maxima and Minima

A function has global maximum at a if $f(x) < f(a)$ for all x other than a which is also called as "absolute maxima".

A function can have multiple local maximums at a in domain for $\delta > 0$ if $f(x) < f(a)$ for all $a - \delta < x < a + \delta$ i.e. for some neighbourhood of a $f(a)$ is always greater in that neighbourhood.

Theorem 5.2. Suppose f is a differentiable function on some interval $[a, b]$.Every local maximum or minimum of f is either one of the end points of the interval $[a, b]$, or else it is a stationary point for the function f where stationary point is the point c such that $f'(c) = 0$.

Now, as this theorem states the derivative is 0 at maxima or minima but does not specify whether it is going to be minima or maxima. The following theorem states that.

Theorem 5.3. If in some small interval $(c - \delta, c + \delta)$ you have $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ then f has *local minimum* at $x = c$.

If in some small interval $(c - \delta, c + \delta)$ you have $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ then f has *local maximum* at $x = c$.

The reason is simple: if f increases then the slope is positive and if f decreases then slope is negative. Therefore, if f' is negative to to the left of c then f is decreasing and if f' is positive to the right of c then f is increasing which means it has local minimum at c.

Theorem 5.4. If c is a stationary point for a function f , and if $f''(c) < 0$ then f has a *local maximum* at $x = c$.

If c is a stationary point for a function f , and if $f''(c) > 0$ then f has a *local minimum* at $x = c$.

Note that the theorem does not say anything about $f''(c) = 0$ in that case we have fall back on the previous theorem.

5.2 Convex and Concave

By definition, a function f is **convex on some interval** $a < x < b$ if the line segment connecting any pair of points on the graph lies above the piece of the graph between those two points.

A function f is **concave on some interval** $a < x < b$ if the line segment connecting any pair of points on the graph lies below the piece of the graph between those two points.

A point on the graph of f where $f''(x)$ changes sign is called an **inflection point**.

Theorem 5.5. A function f is convex on some interval $a < x < b$ if and only if $f''(x) \geq 0 \ \ \forall x$ in that interval.

A function f is convex on some interval $a < x < b$ if and only if $f'(x)$ is non decreasing in that interval.

6 Integration

It is an opposite operation of differentiation. It is also known as anti-derivative. In general sense, it calculates the area under the curve.

Definition 6.1. If f is a function defined on an interval $[a, b]$, then we say that

$$\int_a^b f(x)\,dx = I \tag{18}$$

Then this I gives area under the curve spanned by a, b interval. Suppose, the function goes down the x-axis i.e. if its value is negative for some x then the this area is subtracted from the positive graph area.

Integration Rules:

- $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int cf(x)\,dx = c \int f(x)\,dx$
- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Method of Substitution While solving integration, it is useful to use the derivative rule so as to do the substitution to make the problem easier to solve. We know the chain rule of derivative,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

so the integration rule of substitution is given as

$$\int f'(g(x))g'(x)\,dx = f(g(x)) \tag{19}$$

Suppose, we consider $f'(\cdot) = F(\cdot)$, then the equation can be further simplified with $u = g(x)$ and $du = g'(x)dx$, hence the new integral by substitution becomes,

$$\int F(u)du \tag{20}$$

For example, solve $\int 2xsin(x^2+3)\,dx$.
First we substitute, $u = x^2 + 3$ which gives $dx = \frac{1}{2x}\,du$. Putting this into integration, we get $\int sinu\,du$ which is easily solvable.

Integration by parts Another derivative rule which is used while calculating integrals is of multiplication of two functions.

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

Integrating this equation,

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

Solve $\int x \exp^x\ dx$
Suppose, $f(x) = \exp^x$ and $g(x) = x$ then $g'(x) = 1$, which means

$$\begin{aligned}\int x \exp^x\ dx &= x \exp^x - \int \exp^x\ dx \\ &= x \exp^x - \exp^x + c\end{aligned}$$

Differentiation formulae:	**integration formulae:**
1. $\frac{d}{dx}x^n = nx^{n-1}$	1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ n \neq -1$
2. $\frac{d}{dx}sinx = cosx$	2. $\int cosx\,dx = sinx + C$
3. $\frac{d}{dx}cosx = -sinx$	3. $\int sinx\,dx = -cosx + C$
4. $\frac{d}{dx}tanx = sec^2x$	4. $\int sec^2x\,dx = tanx + C$
5. $\frac{d}{dx}cotx = -csc^2x$	5. $\int csc^2x\,dx = -cotx + C$
6. $\frac{d}{dx}secx = secx\,tanx$	6. $\int secx\,tanx\,dx = secx + C$
7. $\frac{d}{dx}cscx = -cscx\,cotx$	7. $\int cscx\,cotx\,dx = -cscx + C$
8. $\frac{d}{dx}\ln x = \frac{1}{x}$	8. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
9. $\frac{d}{dx}e^x = e^x$	9. $\int e^x\,dx = e^x + C$
10. $\frac{d}{dx}a^x = a^x\ \ln a$	10. $\int a^x\,dx = \frac{a^x}{\ln a} + C$
11. $\frac{d}{dx}sin^{-1}x = \frac{1}{\sqrt{(1-x^2)}}$	11. $\int \frac{1}{\sqrt{(1-x^2)}}\,dx = sin^{-1}x + C$
12. $\frac{d}{dx}tan^{-1}x = \frac{1}{1+x^2}$	12. $\int \frac{1}{x^2+1}\,dx = tan^{-1}x + C$
13. $\frac{d}{dx}sec^{-1}x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	13. $\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = sec^{-1}x + C$

7 Series and sequences

A series is, roughly speaking, a description of the operation of adding infinitely many quantities, one after the other, to a given starting quantity. The study of series is a major part of calculus and its generalization, mathematical analysis. Series are used in most areas of mathematics, even for studying finite structures (such as in combinatorics) through generating functions. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, computer science, statistics and finance.

The infinite sequence of additions implied by a series cannot be effectively carried on (at least in a finite amount of time). However, if the set to which the terms and their finite sums belong has a notion of limit, it is sometimes possible to assign a value to a series, called the sum of the series. This value is the limit as n tends to infinity (if the limit exists) of the finite sums of the n first terms of the series, which are called the nth partial sums of the series. That is,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

When this limit exists, one says that the series is **convergent** or summable, or that the sequence $(a_1, a_2, a_3, \ldots)$ is summable. In this case, the limit is called the sum of the series. Otherwise, the series is said to be **divergent**.

Geometric series A geometric series is the sum of an infinite number of terms that have a constant ratio between successive terms. For example, the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ as each term is multiplied with $\frac{1}{2}$.

In general, a geometric series is written as $a + ar + ar^2 + ar^3 + \cdots$, where a is the coefficient of each term and r is the common ratio between adjacent terms.

Closed form summation is given as:

$$a + ar + ar^2 + ar^3 + \cdots + ar^n = \sum_{k=0}^{n} ar^k = a\left(\frac{1 - r^{n+1}}{1 - r}\right) \tag{21}$$

7.1 Taylor Series

The Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. For most common functions, the function and the sum of its Taylor series are equal near this point. The partial sum formed by the first n + 1 terms of a Taylor series is a polynomial of degree n that is called the nth Taylor polynomial of the function. Taylor polynomials are approximations of a function, which become generally better as n increases. Taylor's theorem gives quantitative estimates on the error introduced by the use of such approximations.

The Taylor series for a function $f(x)$ is given as:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \frac{f'''(a)(x - a)^3}{3!} + \cdots \tag{22}$$

In a compact form it can be written as:

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x - a)^n}{n!} \tag{23}$$

7.2 Maclaurin series

It is a special cases of Taylor series evaluated at $a = 0$ which is given as

$$\begin{aligned} f(x) &\approx f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots \\ &\approx \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} \end{aligned}$$

1. $\exp^x \approx \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty$
2. $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad -\infty < x < \infty$
3. $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad -\infty < x < \infty$
4. $\sinh x \approx x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \qquad -\infty < x < \infty$
5. $\cosh x \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \qquad -\infty < x < \infty$

8 Parametric Equations and graphs

In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations If x and y are continuous functions of t on an interval I, then the equations: $x = x(t)$, and $y = y(t)$ are called parametric equations in parameter t. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a parametric curve or plane curve, and is denoted by C.

Parametrization of a curve We can obtain parametric equations from a given equation of a curve, it is called parametrization of a curve. For example, suppose $y = x^2 + 3$ is a simple equation of a curve. We can parametrize it as $x(t) = t$ and $y(t) = t^2 + 3$. There can be many parametrization possible for any given equation.

Cycloids Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a *cycloid*. A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations:

$$x(t) = a(1 - \cos t) \qquad y(t) = a(1 - \sin t) \tag{24}$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x-axis at a constant height equal to the radius of the wheel. If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at \qquad y(t) = a \tag{25}$$

Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parametrization of the circular motion of the ant (relative to the center of the wheel) is given by

$$x(t) = -a\cos t \qquad y(t) = -a\sin t \tag{26}$$

Then adding these equations, we get equation for cycloid as Eq. 24.

Derivative of Parametric Equations Consider the plane curve defined by the parametric equations $x = x(t)$, and $y = y(t)$. Suppose that $x'(t)$, and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \tag{27}$$

The second order derivative is given as:

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dt)}{dx/dt} \tag{28}$$

Area under the curve We have seen the approach for taking the derivative of the curves given by parametric equations. We now, give the integral equations which are used to get the area under the curve of the equation with parameters.

Consider the non-self-intersecting plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$ for $a \leq t \leq b$ and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)dt \tag{29}$$

Arc Length of a Parametric Curve In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length.

Given a plane curve defined by the functions $x = x(t)$ and $y = y(t)$ for $a \leq t \leq b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $a < t_0 < t_1 < \cdots < t_n < b$. The width of each subinterval is given by $\Delta t = \frac{b-a}{n}$. We can calculate the length of each line segment:

$$
\begin{aligned}
d_1 &= \sqrt{(x(t_0) - x(t_1))^2 + (y(t_0) - y(t_1))^2} \\
d_2 &= \sqrt{(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2} \\
&\vdots \\
d_n &= \sqrt{(x(t_{n-1}) - x(t_n))^2 + (y(t_{n-1}) - y(t_n))^2}
\end{aligned}
$$

If we sum all these distances, we can get the distance from a to b, which is given as:

$$
s \approx \sum_{k=1}^{n} d_k = \sum_{k=1}^{n} \sqrt{(x(t_{k-1}) - x(t_k))^2 + (y(t_{k-1}) - y(t_k))^2}
$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t, then the Mean Value Theorem applies, so in each subinterval $[t_k, t_{k-1}]$ there exist $\hat{t}_k$ and $\tilde{t}_k$ such that

$$
\begin{aligned}
x(t_k) - x(t_{k-1}) = x'(\hat{t}_k)((t_k) - (t_{k-1}) = x'(\hat{t}_k)\Delta t \\
y(t_k) - y(t_{k-1}) = y'(\tilde{t}_k)((t_k) - (t_{k-1}) = y'(\tilde{t}_k)\Delta t
\end{aligned}
$$

Putting this in the previous equation

$$
\begin{aligned}
s &= \sum_{k=1}^{n} \sqrt{(x(t_{k-1}) - x(t_k))^2 + (y(t_{k-1}) - y(t_k))^2} \\
&= \sum_{k=1}^{n} \sqrt{(x'(\hat{t}_k)\Delta t)^2 + (y'(\tilde{t}_k)\Delta t)^2} \\
&= \sum_{k=1}^{n} \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2}\Delta t
\end{aligned}
$$

Now suppose we take such infinite intervals we will exactly get the arc length.

$$
\begin{aligned}
d &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2}\Delta t \\
&= \int_a^b \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2}\Delta t \\
&= \int_a^b \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2}\, dt \\
&= \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt
\end{aligned}
$$

This can be further simplified as:

$$
\begin{aligned}
d &= \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt \\
&= \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dx}\frac{dx}{dt})^2}\, dt \\
&= \int_a^b \frac{dx}{dt}\sqrt{1 + (\frac{dy}{dx})^2}\, dt \\
&= \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx
\end{aligned}
$$

9 Contour Plots

To visualize 3 or more dimensional graphs in 2-dimensions it is cut at different values of z. This cutting produces different lines on the graph which are called as contour plots. These are useful to see the steepness of the graph, if the circles are closer to each other it means original graph is steep otherwise shallow. The warmer colours represent more height and lighter shades represent depth of the curve.

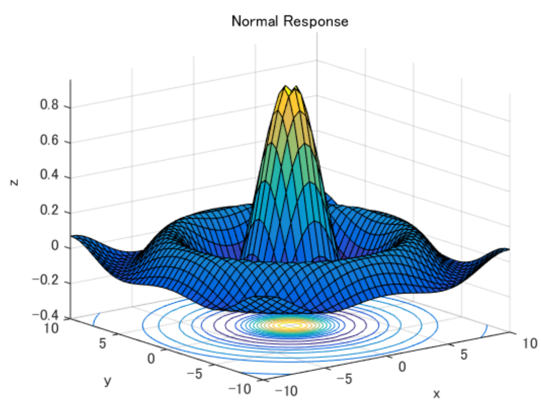

Figure 2: Contour plot from three-dimensional figure, darker shades of the color show height and lighter colors show depth.

Discussion:

1. **So far we have seen tools for continuous and well defined function. What are the similar tools for discrete functions? For example, I wanted to find out the maximum value of $\binom{n}{k}$ function, but I will not be able to apply derivative to it get the maxima.**
2. **Can I calculate rate of change in function with respect to two variables?**
3. **How does the limit and continuity extend in multi-variable case? There is a detailed analysis on differentiation of multi variable functions but have not seen anything on limits and continuity.**