Are There Signed Cospectral Graphs?

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The label dependency of the characteristic polynomials of signed graphs is discussed. It is shown that one could have two inequivalent labeled signed graphs with the same characteristic polynomials which we call signed cospectral.

The signed graphs with a given labeling are defined by the following adjacency matrices:

$$A_{ij} = \begin{cases} 1 \text{ if } i \text{ and } j \text{ are connected and } i > j \\ -1 \text{ if } i \text{ and } j \text{ are connected and } i < j \\ 0 \text{ otherwise} \end{cases}$$

Note that the adjacency matrix of a signed graph depends on the labeling of the graph in question. The characteristic polynomial of a signed graph with the above adjacency matrix is defined as

$$|\mathbf{A} - \lambda\mathbf{I}| = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_n$$

Of course, c_1 is zero for signed graphs which contain no loops. As discussed previously in refs 1 and 2, the characteristic polynomials of signed graphs are label dependent. In this paper, it is shown that the characteristic polynomials of two equivalent labeled graphs are the same while the converse is false.

Two labelings of a graph are said to be equivalent if one is "transformable" into another by a permutation of the vertices of the graph which preserves the connectivity. Mathematically, a permutation p belongs to the automorphism group of a graph if

$$\mathbf{A} = \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} \tag{1}$$

where $\mathbf{P}$ is the permutation matrix corresponding to the permutation p, while $\mathbf{P}^{\mathrm{T}}$ is the transpose of $\mathbf{P}$. All such permutations p which satisfy relation 1 constitute a group called the automorphism group of the graph. This group has been investigated in some detail (see, for example, ref 3).

Labelings of a graph can be envisaged as functions from the set of vertices (D) of a graph to a set (R) containing n different colors, where n is the number of vertices. Two labelings are said to be equivalent if one is transformable into another by an element $g \in G$, the automorphism group of the graph. Symbolically, the corresponding functions f_i and f_j from D to R are said to be equivalent if

$$F_i(d) = f_j(gd) \ \forall \ d \in D \tag{2}$$

Pólya's Theorem,[4] which has found extensive applications in the chemical literature,[5] provides a direct solution to the question of how many different ways one can label a graph. Define the cycle index of the automorphism group G as

$$P_G = \frac{1}{|G|}\sum_{g\in G} x_1^{b_1}x_2^{b_2}...x_k^{b_k} \tag{3}$$

where $x_1^{b_1}x_2^{b_2}...x_k^{b_k}$ is a representation for $g \in G$ if it generates b_1 cycles of length 1, b_2 cycles of length 2, ..., b_k cycles of length k upon its action on the set of vertices. The generating function for inequivalent labelings is given by

$$\mathrm{GF} = P_G(x_K \rightarrow \sum_i w_i^k) \tag{4}$$

where the arrow symbol is for replacing every x_k in the cycle index by $\sum_i w_i^k$ and w_1, w_2, w_3, ... w_n are the weights for different elements in the set R. The coefficient of $w_1^{b_1}w_2^{b_2}...w_n^{b_n}$ in the GF gives the number of inequivalent ways of labeling the given graph with b_1 labels of type 1, b_2 labels of type 2, ... b_n labels of type n. The present case of labeling the vertices of a graph corresponds to the coefficient of $w_1w_2...w_n$ in the GF since all the labels are different. This is readily seen to lead to a known result[6]

$$n_l = \frac{n!}{|G|} \tag{5}$$

where n_l is the number of equivalence classes of labeling and $|G|$ is the number of elements in the automorphism group. Of course, the GF is so powerful that it enumerates other possible ways of labeling the graph as well.

Characteristic polynomials of unsigned fullerene cages have been discussed before.[1,7] Two labelings in the same equivalence class are equivalent, and thus it is evident that the characteristic polynomial of the corresponding signed cage is identical. The converse is, in general, not true as we illustrate with the dodecahedral C_{20} fullerene cage. Consider three of the several possible ways of labeling the C_{20} fullerene cage shown in Figure 1. In general, the point group of the cage as a permutation group of vertices is a subgroup of the automorphism group. For simplicity if we take G to be the I_h group for the C_{20} cage, then there are 20!/120 maximum possible inequivalent labelings, 3 of which are illustrated in Figure 1.

Table 1. Characteristic Polynomials of Three Signed Graphs in Figure 1

power	coefficient		
	graph I	graph II	graph III
20	1	1	1
18	30	30	30
16	375	375	375
14	2540	2540	2540
12	10 175	10 135	10 135
10	24 782	24 206	24 206
8	36 505	33 545	33 545
6	31 400	24 840	24 840
4	14 640	8440	8440
2	3200	960	960
0	256	16	16

The characteristic polynomials of the 3 signed cages in Figure 1 are shown in Table 1. Note that the absolute values of the first six coefficients (i.e., up to power 14) are identical

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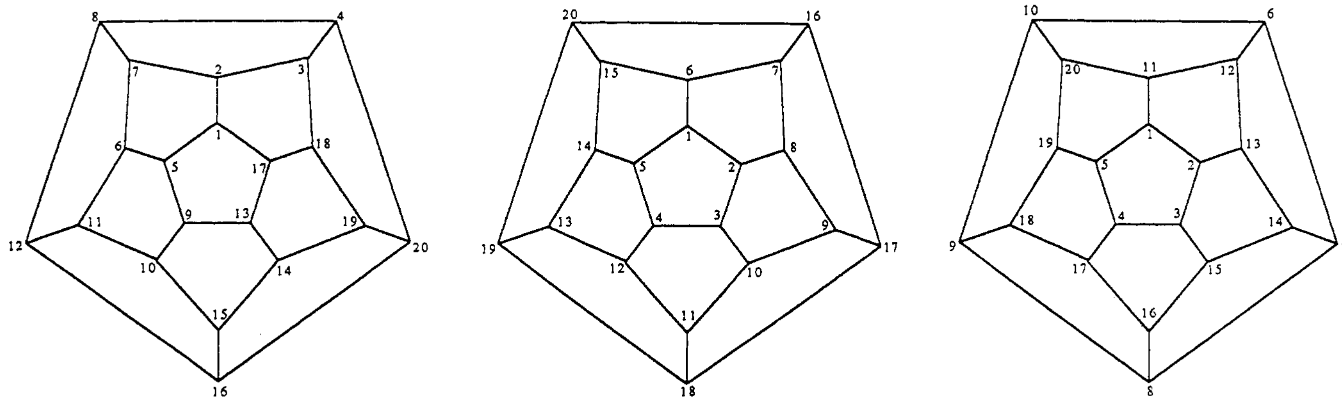

I II III

Figure 1. Three possible inequivalent labelings for the C_{20} fullerene cage.

to the corresponding coefficients of the unsigned graphs. We have verified with a computer code that all three labelings in Figure 1 are inequivalent. That is, the permutations which convert I to II, II to III, and I to III are not members of the automorphism group. While the characteristic polynomials of I and II are different, those of II and III are identical! The graphs II and III are related by the permutation (1)(2)(3)(4)(5)(6,11,16)(7,12,17)(8,13,18)(9,14,19)-(10,15,20). It is easily verified that this permutation is not in the automorphism group and yet the signed characteristic polynomials for graphs II and III are the same. Thus, it is an open problem to compute the number of different ways a graph can be labeled such that each labeling would produce a unique signed characteristic polynomial.

The fact that there are two inequivalent labeled sign graphs with the same signed characteristic polynomial is quite interesting. We call these signed-cospectral graphs. The problem of finding isospectral and cospectral graphs has a long history. However, the existence of signed cospectral graphs is relatively new to the best of my knowledge. There is an open question, namely, how many different signed characteristic polynomials exist for a given labeled graph. Is there an algorithm for the construction of cospectral signed graphs? To seek an answer to the latter question one first needs to know what makes the two inequivalent labeled graphs in Figure 1 cospectral. We hope that future investigations will be aimed at seeking answers to such unsolved questions and to develop techniques for the construction of cospectral signed graphs as defined here.

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