Enumeration of Monocyclic and Bicyclic Carbon Skeletons

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Pólya's theorem was used to enumerate the mono- and bicyclic carbon skeletons. The methods used are given as examples of the use of the theorem. The results of the enumeration agree with those found previously by systematic generation.

INTRODUCTION

Although Pólya's enumeration theory has for many years seen a great deal of use in chemistry, it has not been easily accessible to the majority of chemists. Until the publication[1b] in 1987 of the English translation of Pólya's 1937 paper,[1a] the chemist had the choice of translating a great deal of German or sorting through the literature to find articles explaining the use of the theorem. While such explanatory articles do exist,[2] they are often written in a manner that makes them inaccessible to the average chemist. In addition, these articles, when they do include an example of some relevance to chemistry, almost always use the same example, that of the enumeration of halogenated benzenes. While this system is easily visualized by the chemist and lends itself well to the application of Pólya's theorem, the paucity of other chemical examples can leave the impression that the theorem is useful only for the enumeration of substitutional isomers.

Our purpose in this discussion is twofold: to give some other examples of the use of Pólya's theorem and to present the results of the enumeration of carbon mono- and bicyclic skeletons, which have not been systematically explored until now.

DEFINITIONS

For the set of numbers (1 2 3 4 5 6) there are 6! possible orderings, or permutations, of these numbers. Each one can be represented by writing the numbers 1–6 and writing below each number k the kth element of the permutation, so if the number 3 is second in a particular permutation, you would write a 3 below 2. For example, one of the permutations of the set is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix}$$

One way to think of this permutation is as a function which maps one element of the set onto another. For this permutation, the function maps 1 onto 3, 2 onto 5, ..., and 6 onto 4; this can be represented as $f(1) = 3, f(2) = 5$, ..., and $f(6) = 4$.

A cycle of k elements is a part of a permutation which, if applied k times to those elements, restores the original ordering of those elements. The example permutation consists of two cycles: one of two elements (a cycle of order 2) and one of four elements (a cycle of order 4). The cycle of order 2 consists of the elements 2 and 5: $f(2) = 5$ and $f(5) = 2$, so $f(f(2)) = 2$ and $f(f(5)) = 5$. The cycle of order 4 consists of the elements 1, 3, 4, and 6, with the mappings 1 → 3, 3 → 6, 6 → 4, and 4 → 1. These cycles are depicted graphically in Figure 1.

A permutation can be specified by its **cycle representation**. A cycle representation is derived by putting each cycle's elements between parentheses. The cycle representation for the example permutation is (2 5)(1 3 6 4). In a cycle representation, the order of the cycles is not significant, but the order of the elements within each cycle is because that order gives the mappings. This means that the cycle representation (1 3 6 4)(2 5) is equivalent to (2 5)(1 3 6 4) but (2 5)(4 3 1 6) is not because (4 3 1 6) represents the mappings 4 → 3, 3 → 1, 1 → 6, and 6 → 4.

When information is sought about a sequence of numbers $\langle a_n \rangle = a_0, a_1, a_2, \ldots$, a **generating function** is often the method of choice. A generating function is an infinite sum set up in terms of some dummy parameter, say x, such that

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

The use of generating functions will be demonstrated throughout this paper; for now, suffice it to say that generating functions are often very useful in the derivation of numerical sequences.

Pólya's theorem deals with the permutation of objects among a number of sites, such as the permutation of halogen atoms on a benzene ring. A **figure inventory** is a polynomial or generating function representing the things to be permuted. In this paper we will be using two figure inventories. One of these, $r(x)$, is the generating function for alkyl substituents; a good discussion of the derivation of this function can be found starting on page 77 in ref 2f. This function is of the form $r(x) = 1 + x + x^2 + 2x^3 + 4x^4 + 8x^5 + 17x^6 + \ldots$ in which the coefficient of x^n is the number of alkyl substituents or **alkyl trees** with n carbons. We see from this that there is one alkyl tree of zero carbons (hydrogen), one tree of one carbon (methyl), one tree of two carbons (ethyl), two trees of three carbons (*n*-propyl and *sec*-propyl), etc. The second figure inventory we will be using is $xR(x)$, the meaning and derivation of which will be discussed in the next section.

MONOCYCLIC CARBON SKELETONS

Pólya set out the method for enumerating the C_nH_{2n} cycloalkanes in section 64 of his paper.[1a] All the C_nH_{2n} cycloalkanes are either unsubstituted rings or are derived by adding alkyl substituents to basic, simple ring cycloalkanes. For a ring of m vertices, the permutation group is $\mathcal{D}_m$, which corresponds to the cycle index

$$\frac{1}{2m}\sum_{k|m} \varphi(k) f_k^{m/k} + \begin{cases} {}^1/_2(f_1 f_2^{\mu-1}) & m = 2\mu - 1 \\ {}^1/_4(f_1^2 f_2^{\mu-1} + f_2^{\mu}) & m = 2\mu \end{cases}$$

In this expression, $\varphi(k)$ is the Euler-totient function,[3] which is the number of integers from 1 to k that have no factors in common with k, and the summation is over all integers k that are factors of m.

To show how this general expression for the cycle index works, we will use it to generate the cycle index for a five-membered ring. Here $m = 5$, so $k = 1$ and 5. For the $k = 1$ case, $\varphi(k) = 1$ and the term within the summation is therefore f_1^5; for $k = 5$, $\varphi(k) = 4$ and the summation term is

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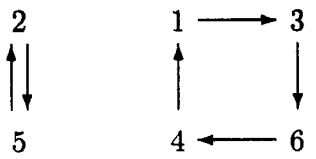

Figure 1. Cycles of example permutation.

Table I. Numbers of C_3–C_{25} Monocyclic Skeletons

no. of carbons	no. of skeletons	no. of carbons	no. of skeletons
3	1	15	60 077
4	2	16	160 629
5	5	17	430 724
6	12	18	1 158 502
7	29	19	3 122 949
8	73	20	8 437 289
9	185	21	22 836 877
10	475	22	61 918 923
11	1 231	23	168 139 339
12	3 232	24	457 225 555
13	8 506	25	1 244 935 251
14	22 565		

$4f_5^1$. Adding these terms together and multiplying by $1/2m$ gives $(f_1^5 + 4f_5^1)/10$ as the overall summation term. The μ term is easily found: for $m = 5$, $\mu = 3$, so the μ term is $1/2(f_1f_2^2)$. Adding the μ and summation terms gives

$$\frac{f_1^5 + 4f_5^1 + 5f_1f_2^2}{10}$$

as the cycle index for a five-membered ring. The first term, f_1^5, denotes five cycles of order 1; this corresponds to the identity operation under which no atom permutes with any other. The second term, $4f_5^1$, denotes four cycles of order 5; these correspond to rotations about the C_5 axis. The final term, $5f_1f_2^2$, denotes five permutations, each of which consists of one cycle of order 1 and two cycles of order 2; these correspond to rotations about the C_2 axes which pass through each vertex and its opposing side.

Each of the carbons in the ring has two sites for alkyl substitution. As we are enumerating skeletons and not stereoisomers, the two sites are considered to be equivalent. There are two permutations of two equivalent sites: (1)(2) and (1 2). The cycle index for this permutation group is $(f_1^2 + f_2)/2$. The generating function for a carbon with alkyl substituents at two equivalent sites can be found by substituting the generating function for the alkyl trees, $r(x)$, for f in the cycle index and multiplying by x to account for the substituted carbon itself; these operations give the expression

$$g(x) = x\frac{r(x)^2 + r(x^2)}{2}$$

Expanding this expression gives us the generating function $g(x) = x + x^2 + 2x^3 + 3x^4 + 7x^5 + \ldots$, which is the figure inventory for carbons with alkyl substituents on two equivalent sites. As this figure inventory is used often, we adopt Pólya's convention of referring to it as $xR(x)$.

Finally, substituting the figure inventory for carbons with two alkyl substituents, $xR(x)$, into the cycle index for a ring of m carbons gives

$$\frac{1}{2m}\sum_{k|m}\varphi(k)[x^kR(x^k)]^{m/k} + \begin{cases} {}^1/_2(xR(x)[x^2R(x^2)]^{\mu-1}) & m = 2\mu - 1 \\ {}^1/_4([xR(x)]^2[x^2R(x^2)]^{\mu-1} + [x^2R(x^2)]^{\mu}) & m = 2\mu \end{cases}$$

which, when summed over $m = 3, 4, 5, \ldots$, gives us the generating function for the C_nH_{2n} cycloalkanes, $g(x) = x^3 + 2x^4 + 5x^5 + 12x^6 + 29x^7 + 73x^8 + \ldots$. From this we see that there is one C_nH_{2n} cycloalkane skeleton of three carbons, two of four carbons, and five of five carbons. The numbers of skeletons obtained from this generating function for up to 25 carbons are given in Table I; these numbers agree with those obtained from the SKEL_GEN program for monocyclic skeletons of up to 16 carbons. Table II shows the numbers of C_3–C_{25} monocyclic skeletons with rings of from 3 to 10 carbons.

BICYCLIC CARBON SKELETONS

Pólya discussed the enumeration of the $C_nH_{2n+2-2\mu}$ alkanes in section 65 of his paper. This discussion was limited to hints for the bicyclics ($\mu = 2$) because of the unwieldiness of the expressions involved and the difficulty of expanding those expressions. The use of the computer makes the expansion of complex expressions relatively easy, so the unwieldiness of the expressions is no longer such a problem.

Table II. Numbers of Monocyclic Skeletons with Rings of up to 10 Carbons

no. of carbons	ring size							
	3	4	5	6	7	8	9	10
3	1							
4	1	1						
5	3	2	1					
6	6	4	1	1				
7	15	8	4	1	1			
8	33	24	9	5	1	1		
9	83	55	28	12	5	1	1	
10	196	147	71	40	13	6	1	1
11	491	365	198	106	47	16	6	1
12	1 214	954	521	317	136	63	18	7
13	3 068	2 431	1 418	868	428	190	73	21
14	7 754	6 327	3 773	2 462	1 252	631	240	93
15	19 834	16 369	10 153	6 778	3 716	1 923	841	319
16	50 872	42 743	27 114	18 801	10 708	5 940	2 714	1 170
17	131 423	111 595	72 705	51 561	30 823	17 706	8 758	3 931
18	340 763	292 849	194 531	141 583	87 504	52 573	27 259	13 168
19	887 839	769 805	521 447	386 865	247 438	153 442	83 847	42 365
20	2 321 193	2 030 456	1 397 482	1 056 815	694 588	444 935	252 929	134 372
21	6 090 979	5 366 844	3 749 836	2 880 894	1 942 866	1 277 737	754 737	416 765
22	16 031 341	14 222 475	10 067 417	7 850 318	5 411 640	3 648 896	2 225 129	1 275 718
23	42 319 223	37 768 154	27 057 233	21 373 466	15 034 045	10 357 898	6 505 344	3 850 137
24	112 003 765	100 510 364	72 779 710	58 182 244	41 659 417	29 278 749	18 867 696	11 501 014
25	297 164 610	267 987 501	195 963 184	158 342 918	115 231 598	82 435 343	54 384 446	34 025 545

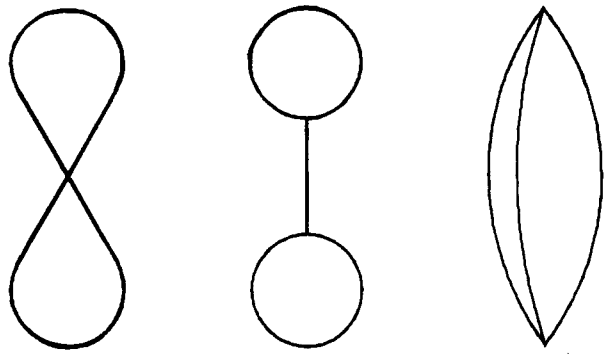

Figure 2. Three classes of bicyclic ring systems.

The first task is to find all the different classes of ring systems available for bicyclic alkanes. We may discern three classes: spiro, separated, and fused or bridged bicyclics (Figure 2).

SPIRO BICYCLICS

Having selected a class to examine, we proceed to find the cycle index for each system in that class. To find the cycle index, one must determine the permutation group of the parent system. As stated previously, the easiest way for a chemist to do this is through the use of point groups, but one must be careful about which symmetry elements are kept and which are eliminated. As an example, consider the spiro ring system in which the two rings, one of size $(k + 1)$ and the other of size $(l + 1)$ (the factor of 1 here represents the carbon common to the two rings), are of the same size $(k = l)$. The point group for this system is D_{2d}, for which the symmetry elements are E, $2S_4$, C_2, $2C'_2$, and $2\sigma_d$. These symmetry elements are equivalent to permutations which contribute the following terms to the cycle index:

$$E \Rightarrow f_1^{2k}$$

$$S_4 \Rightarrow \begin{cases} f_4^{k/2} & k \text{ even} \\ f_2^{1} f_4^{(k-1)/2} & k \text{ odd} \end{cases}$$

$$C_2 \Rightarrow \begin{cases} f_2^{k} & k \text{ even} \\ f_1^{2} f_2^{k-1} & k \text{ odd} \end{cases}$$

$$C'_2 \Rightarrow f_2^{k}$$

$$\sigma_d \Rightarrow \begin{cases} f_1^{k} f_2^{k/2} & k \text{ even} \\ f_1^{k+1} f_2^{(k-1)/2} & k \text{ odd} \end{cases}$$

Collecting all these terms will give the cycle index for the $k = l$ spiro ring system; that cycle index is

$$\frac{1}{8}\left(f_1^{2k} + 2f_2^{k} + \begin{cases} f_2^{k} + 2f_4^{k/2} + 2f_1^{k} f_2^{k/2} & k \text{ even} \\ f_1^{2} f_2^{k-1} + 2f_2^{1} f_4^{(k-1)/2} + 2f_1^{k+1} f_2^{(k-1)/2} & k \text{ odd} \end{cases}\right)$$

To find the generating function for these spiro alkanes we must multiply the cycle index by x to account for the common carbon (the cycle index deals only with the other $2k$ carbons), substitute $xR(x)$ for f, and sum over $k = 2, 3, 4, \ldots$. Performing these operations will give the generating function $g(x) = x^5 + x^6 + 5x^7 + 10x^8 + 34x^9 + \ldots$, in which, for each term cx^n, c is the number of $k = l$ spiro skeletons with n carbons. The numbers of skeletons thus obtained are given in Table III.

To convince ourselves that these figures are correct, we can draw the skeletons of the $k = l$ skeletons for the lower numbers of carbons and compare results. The $k = l$ spiro skeletons for five to eight carbons are shown in Figure 3. We see that there is one skeleton of five carbons, one of six carbons, five of seven carbons, and 10 of eight carbons, so the results of the Pólya enumeration check out, at least for the lower numbers of carbons.

Table III. Numbers of C_5–C_{25} Spiro Bicyclic Skeletons

no. of carbons	topological		enantiomeric	
	$k = l$	$k \neq l$	$k = l$	$k \neq l$
5	1		1	
6	1	1	1	1
7	5	4	6	4
8	10	16	12	17
9	34	53	43	60
10	82	175	106	209
11	240	547	322	685
12	630	1 692	864	2 204
13	1 764	5 127	2 483	6 913
14	4 752	15 408	6 829	21 387
15	13 108	45 831	19 226	65 241
16	35 703	135 500	53 293	197 104
17	98 108	398 186	148 866	590 284
18	268 553	1 165 005	413 501	1 755 804
19	737 864	3 394 880	1 151 506	5 191 723
20	2 025 779	9 861 048	3 200 184	15 276 610
21	5 572 160	28 561 403	8 901 478	44 761 350
22	15 331 017	82 525 483	24 742 852	130 682 371
23	42 230 755	237 941 827	68 800 520	380 337 474
24	116 395 955	684 778 523	191 275 736	1 103 927 291
25	321 089 489	1 967 510 639	531 888 416	3 196 494 453

Another way to check on the correctness of the enumeration is to extract the $k = l$ spiro skeletons from the files of skeletons produced by the SKEL_GEN program. The numbers of skeletons from the files match the numbers given by the Pólya enumeration for up to 15 carbons, which is as far as the SKEL_GEN program has gone for bicyclic skeletons.

This enumeration of the $k = l$ spiro skeletons serves as an example of why the chemist must be careful when using the operations of a point group to aid in finding the terms of a cycle index. We used $xR(x)$ as the figure inventory, which is the generating function for alkyl substitution at two equivalent sites on a carbon; the two sites being equivalent, we are not considering any chirality on the atoms. However, we treated the ring system as a three-dimensional object when we were finding the cycle index for the system. The skeleton that corresponds to 1,1′-dimethylspiropentane, when depicted as a three-dimensional object, should exist as a pair of enantiomers even if the chirality of individual carbons were ignored: the 1,1′-dimethylspiropentane molecule has a molecular chirality independent of the chirality of the individual atoms. How is it that we get the topological enumeration from a three-dimensional representation?

To answer this question, we must look at the symmetry operations we used to help in constructing the cycle index. Consider a $k = l$ spiro system. To perform the C_2 and C'_2 operations the skeleton needs only to be rotated in space. There is a σ_d plane in the plane of each of the rings, so performing the σ_d operation on the skeleton fixes one of the rings and exchanges the two sides of the other ring. The two σ_d operations, therefore, make the two sides of each ring equivalent and thus eliminate enantiomeric pairs from the enumeration. If you were to construct a model of a $k = l$ spiro ring system you would find that to perform the S_4 and σ_d operations you would have to take apart the model.

This suggests a method for finding a cycle index that can be used for an enantiomeric enumeration. The first step is to find the point group for the system considered as a three-dimensional object. The second step is to eliminate the symmetry operations that do not produce whole-body permutations (operations which would require disassembly of a model of the system). The third step is to find the contribution of each of the remaining operations to the cycle index, making sure to eliminate equivalent permutations, and the final step is to collect these contributions together into the cycle index. To

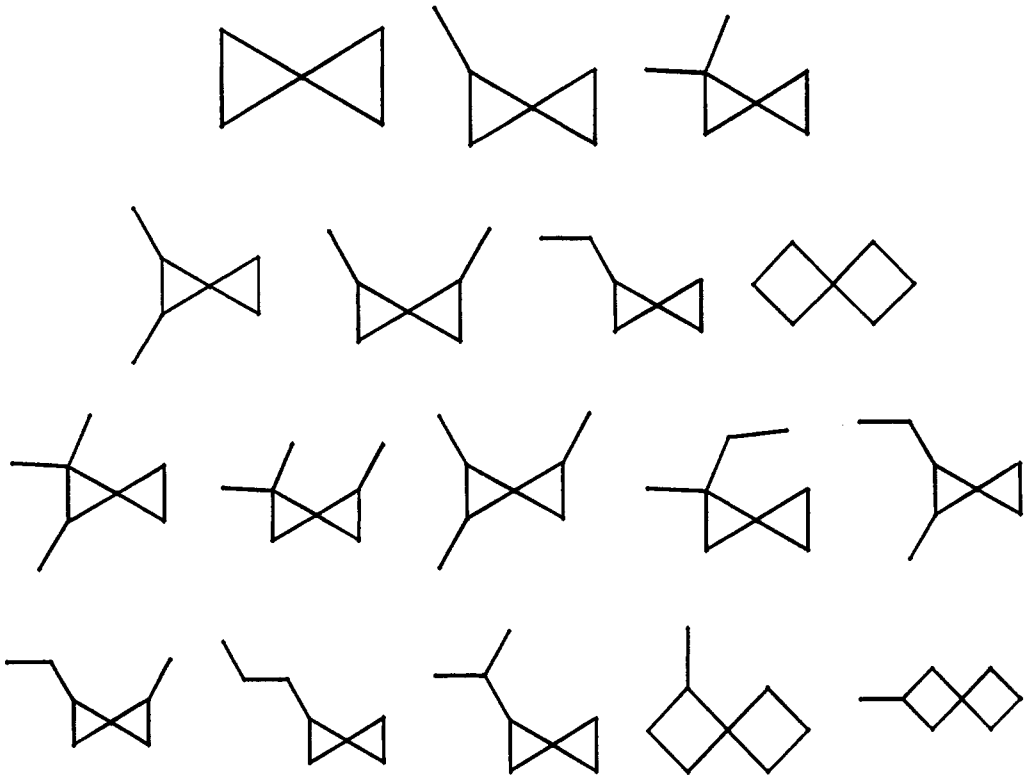

Figure 3. C_5–C_8 $k = l$ spiro skeletons.

Table IV. Numbers of C_6–C_{25} Separated Bicyclic Skeletons

no. of carbons	$k = l$	$k \neq l$
6	1	
7	3	1
8	13	7
9	41	35
10	141	147
11	440	565
12	1 391	2 042
13	4 244	7 080
14	12 913	23 799
15	38 651	78 158
16	115 082	251 994
17	339 646	800 580
18	997 709	2 512 782
19	2 915 010	7 807 698
20	8 485 573	24 053 791
21	24 612 666	73 565 545
22	71 191 458	223 576 181
23	205 393 819	675 753 702
24	591 330 506	2 032 603 573
25	1 699 226 719	6 087 798 266

demonstrate this we again use the $k = l$ spiro system. The point group is, as found above, D_{2d}, and the symmetry operations to be eliminated are S_4 and σ_d. The point group for the system without those operations is D_2. The cycle index corresponding to this point group is

$$\frac{1}{4}\left(f_1^{2k} + 2f_2^{k} + \begin{cases} f_2^{k} & k \text{ even} \\ f_1^{2}f_2^{k-1} & k \text{ odd} \end{cases}\right)$$

The corresponding generating function is obtained in the same manner as that for the D_{2d} case.

If one were to represent the spiro $k = l$ system as a flat object (both rings in the same plane), its point group would be D_{2h}; if the equivalent symmetry operations are then eliminated, the set of operations that remain are the same as those of the D_2 point group. This shows that the flat representation will also give the enantiomeric enumeration, which is to be expected. This also points out the fact that it is the *permutations* on which one's attention should be focused, not the point group. Often there will be more than one way to represent the body whose sites are to be permuted, each way with a different point group, but many or all of these will have the same permutation group. Finally, one must be careful to check that the permutation group obtained represents what it is thought to represent.

The next system to consider is that of the spiro skeletons of unequal ring sizes ($k \neq l$). The point group for the three-dimensional representation of this system is C_{2v}, the symmetry elements of which are E, C_2, σ, and σ'. Their contributions to the cycle index are

$$E \Rightarrow f_1^{k+1}$$

$$C_2 \Rightarrow \left[\begin{cases} f_2^{k/2} & k \text{ even} \\ f_1^{2}f_2^{(k-1)/2} & k \text{ odd} \end{cases}\right]\left[\begin{cases} f_2^{l/2} & l \text{ even} \\ f_1f_2^{(l-1)/2} & l \text{ odd} \end{cases}\right]$$

$$\sigma \Rightarrow f_1^{l}\left[\begin{cases} f_2^{k/2} & k \text{ even} \\ f_1f_2^{(k-1)/2} & k \text{ odd} \end{cases}\right]$$

$$\sigma' \Rightarrow f_1^{k}\left[\begin{cases} f_2^{l/2} & l \text{ even} \\ f_1f_2^{(l-1)/2} & l \text{ odd} \end{cases}\right]$$

Collecting these terms, we find that the cycle index for the topological enumeration of the $k \neq l$ spiro system is

$$\frac{1}{4}\left(f_1^{k+l} + \left[\begin{cases} f_2^{k/2} & k \text{ even} \\ f_1f_2^{(k-1)/2} & k \text{ odd} \end{cases}\right]\left[\begin{cases} f_2^{l/2} & l \text{ even} \\ f_1f_2^{(l-1)/2} & l \text{ odd} \end{cases}\right] + f_1^{l}\left[\begin{cases} f_2^{k/2} & k \text{ even} \\ f_1f_2^{(k-1)/2} & k \text{ odd} \end{cases}\right] + f_1^{k}\left[\begin{cases} f_2^{l/2} & l \text{ even} \\ f_1f_2^{(l-1)/2} & l \text{ odd} \end{cases}\right]\right)$$

The numbers of skeletons obtained by the topological and enantiomeric enumerations of both the $k = l$ and $k \neq l$ spiro

systems are given in Table III.

SEPARATED BICYCLICS

For the next class to consider, the separated ring bicyclics, it is not particularly meaningful to consider an enantiomeric enumeration: while there are separated ring systems in which the rings are fixed in a particular relative configuration, these are the exceptions and are ignored here.

Separated ring systems can be broken down into five distinct elements. There are the rings of $(k + 1)$ and $(l + 1)$ carbons, the link of m carbons between the rings, and the two junction carbons, which are the carbons common to the link and the rings. Throughout this discussion we have used f_a^b to represent b cycles of a carbons, each of these carbons having two equivalent sites for alkyl substitution. The junction carbons each have only one available site for alkyl substitution, so we now introduce the notation h_a^b to represent b cycles of a carbons, where each carbon has just one site for alkyl substitution. When deriving the generating function from the cycle index, $[x^a r(x^a)]^b$ is to be substituted for h_a^b.

The separated ring systems will have the same point groups as their spiro counterparts, so we need only factor the junction and link carbons into each symmetry operation's contribution to the cycle index. For the $k = l$ separated system, the point group is D_{2d} and the symmetry operations' contributions to the cycle index are

$$E \Rightarrow h_1^2 f_1^{2k+m}$$

$$S_4 \Rightarrow h_2^1 \begin{Bmatrix} f_2^{m/2} & m \text{ even} \\ f_1 f_2^{(m-1)/2} & m \text{ odd} \end{Bmatrix} \begin{Bmatrix} f_4^{k/2} & k \text{ even} \\ f_2^1 f_4^{(k-1)/2} & k \text{ odd} \end{Bmatrix}$$

$$C_2 \Rightarrow h_1^2 \begin{Bmatrix} f_1^m f_2^k & k \text{ even} \\ f_1^{m+2} f_2^{k-1} & k \text{ odd} \end{Bmatrix}$$

$$C'_2 \Rightarrow h_2^1 f_2^k \begin{Bmatrix} f_2^{m/2} & m \text{ even} \\ f_1 f_2^{(m-1)/2} & m \text{ odd} \end{Bmatrix}$$

$$\sigma_d \Rightarrow h_1^2 \begin{Bmatrix} f_1^{k+m} f_2^{k/2} & k \text{ even} \\ f_1^{k+m+1} f_2^{(k-1)/2} & k \text{ odd} \end{Bmatrix}$$

The cycle index for the $k = l$ separated system is therefore

$$\frac{1}{8}\left(h_1^2\left[f_1^{2k+m} + \left[\begin{Bmatrix} f_1^m f_2^k + 2f_1^{k+m} f_2^{k/2} & k \text{ even} \\ f_1^{m+2} f_2^{k-1} + 2f_1^{k+m+1} f_2^{(k-1)/2} & k \text{ odd} \end{Bmatrix}\right] + 2h_2^1\left[\left[\begin{Bmatrix} f_2^{m/2} & m \text{ even} \\ f_1 f^{(m-1)/2} & m \text{ odd} \end{Bmatrix}\right]\left[f_2^k + \left(\begin{Bmatrix} f_4^{k/2} & k \text{ even} \\ f_2 f_4^{(k-1)/2} & k \text{ odd} \end{Bmatrix}\right)\right]\right]\right)$$

For the $k \neq l$ separated system, finding the elements of the permutation group and the cycle index is relatively easy. The point group is C_{2v}, which is the same as that for the $k \neq l$ spiro system. Since none of the symmetry operators in the C_{2v} point group exchange one end of the molecule for the other, the link and endpoint carbons are all nonequivalent. This being the case, we can simply multiply the cycle index for the $k \neq l$ spiro system by $h_1^2 f_1^m$ to get the cycle index for the $k \neq l$ separated system.

The numbers of separated bicyclic skeletons are given in Table IV for both the $k = l$ and the $k \neq l$ systems.

FUSED/BRIDGED BICYCLICS

The fused/bridged bicyclics are, like the separated bicyclics, made up of five elements; these are the three arcs of k, l, and m carbons and the two endpoints, which are again represented in the cycle index as h_a^b. There are three cases to be considered: the one in which all three arcs are of the same length $(k = l = m)$, the one in which only two of the arcs are of the same length $(k = l \neq m)$, and the one in which none of the arcs are of the same length $(k \neq l \neq m)$. For each of these cases we will just give a point group and the cycle index for both the topological and the enantiomeric enumerations.

For the first case, $k = l = m$, we note that the skeletons must all be bridged, since all the arcs have the same numbers of carbons and a fused system has an arc of zero carbons. The topological enumeration of the $k = l = m$ bridged bicyclics uses the D_{3h} point group; the cycle index is

$$\frac{1}{12}\left(h_1^2\left[f_1^{3k} + 2f_3^k + 3f_1^k f_2^k\right] + h_2^1\left[\begin{Bmatrix} 3f_2^{3k/2} + f_2^{3k/2} + 2f_6^{3k/6} & k \text{ even} \\ 3f_1 f_2^{(3k-1)/2} + f_1^3 f_2^{(3k-3)/2} + 2f_3^1 f_6^{(3k-3)/6} & k \text{ odd} \end{Bmatrix}\right]\right)$$

Eliminating those symmetry operations which do not correspond to whole-body transformations (σ_d, S_3, σ_v) gives us the point group D_3 for the enantiomeric enumeration; the corresponding cycle index is

$$\frac{1}{6}\left(h_1^2\left[f_1^{3k} + 2f_3^k\right] + 3h_2^1\left[\begin{Bmatrix} f_2^{3k/2} & k \text{ even} \\ f_1 f_2^{(3k-1)/2} & k \text{ odd} \end{Bmatrix}\right]\right)$$

The topological enumeration of the $k = l \neq m$ fused/bridged bicyclics uses the C_{2v} point group; the cycle index is

$$\frac{1}{4}\left(h_1^2\left[f_1^{2k+m} + f_1^m f_2^k\right] + h_2^1\left[\begin{Bmatrix} f_2^{m/2} & m \text{ even} \\ f_1 f_2^{(m-1)/2} & m \text{ odd} \end{Bmatrix}\right]\left[\begin{Bmatrix} 2f_2^k & k \text{ even} \\ f_2^k + f_1^2 f_2^{k-1} & k \text{ odd} \end{Bmatrix}\right]\right)$$

Now, to obtain the cycle index for the enantiomeric enumeration, we must further separate the $k = l \neq m$ fused/bridged bicyclics into fused $(m = 0)$ and bridged $(m \neq 0)$. For fused bicyclics, the appropriate point group is D_2; the C'_2 and C''_2 operations of D_2 are here equivalent to the σ and σ' operations of C_{2v}, so the cycle index for the enantiomeric enumeration is the same as that found by setting $m = 0$ in the cycle index for the topological enumeration. This makes sense because the sort of enantiomers with which we have been concerned in this discussion cannot occur in these fused-ring skeletons. The point group for the bridged bicyclics is C_2 and the cycle index is

$$\frac{1}{2}\left(h_1^2 f_1^{2k+m} + h_2^1 f_2^k \begin{Bmatrix} f_2^{m/2} & m \text{ even} \\ f_1 f_2^{(m-1)/2} & m \text{ odd} \end{Bmatrix}\right)$$

The point group appropriate for topological enumeration of the $k \neq l \neq m$ fused/bridged bicyclics is C_S; the corresponding cycle index is

$$\frac{1}{2}\left(h_1^2 f_1^{k+l+m} + h_2^1\left[\begin{Bmatrix} f_2^{k/2} & k \text{ even} \\ f_1 f_2^{(k-1)/2} & k \text{ odd} \end{Bmatrix}\right]\left[\begin{Bmatrix} f_2^{l/2} & l \text{ even} \\ f_1 f_2^{(l-1)/2} & l \text{ odd} \end{Bmatrix}\right]\left[\begin{Bmatrix} f_2^{m/2} & m \text{ even} \\ f_1 f_2^{(m-1)/2} & m \text{ odd} \end{Bmatrix}\right]\right)$$

Again, to do the enantiomeric enumeration we must treat separately the $m = 0$ and $m \neq 0$ cases. For the $m = 0$ case, the point group is C_S, so once again we find that there is no enantiomeric pairing in skeletons for the fused bicyclics. For the $m \neq 0$ case, the point group is C_1, and the cycle index

Table V. Topological Enumerations of C_4–C_{25} Fused Bicyclic Skeletons

no. of carbons	$k = l \neq m$	$k \neq l \neq m$
4	1	
5	2	1
6	7	4
7	15	17
8	44	56
9	107	187
10	295	586
11	763	1827
12	2077	5562
13	5533	16811
14	15053	50225
15	40697	149135
16	111028	439818
17	302583	1290975
18	828176	3772259
19	2267939	10983684
20	6225340	31878940
21	17103834	92278466
22	47062513	266481212
23	129616014	767972142
24	357364708	2209210615
25	986110340	6345086203

Table VI. Topological Enumerations of C_5–C_{25} Bridged Bicyclic Skeletons

no. of carbons	$k = l = m$	$k = l \neq m$	$k \neq l \neq m$
5	1		
6	2	1	
7	6	5	
8	15	20	1
9	39	71	6
10	99	235	32
11	258	753	140
12	671	2345	567
13	1762	7190	2141
14	4657	21753	7731
15	12372	65188	26929
16	33036	193834	91269
17	88590	527743	302566
18	238483	1683615	985178
19	644045	4927394	3160480
20	1744542	14367845	10013151
21	4737341	41761634	31390529
22	12894158	121052806	97519183
23	35165994	350049108	300596765
24	96083192	1010123293	920275456
25	262951511	2909478000	2800635344

obtained from that group is $h_1^2 f_1^{k+l+m}$.

The numbers of skeletons obtained from the topological enumerations of the fused bicyclics are given in Table V. The numbers of bridged bicyclic skeletons obtained from the topological enumerations are given in Table VI, while those from the enantiomeric enumerations are given in Table VII.

Table VII. Enantiomeric Enumerations of C_5–C_{25} Bridged Bicyclic Skeletons

no. of carbons	$k = l = m$	$k = l \neq m$	$k \neq l \neq m$
5	1		
6	2	1	
7	6	5	
8	15	23	8
9	41	88	9
10	106	313	53
11	288	1054	251
12	777	3431	1053
13	2113	10880	4078
14	5772	33867	14945
15	15817	103836	52586
16	43422	314757	179411
17	119444	945067	597480
18	329003	2816287	1951618
19	907181	8339029	6274858
20	2503971	24562712	19912475
21	6916822	72025232	62497879
22	19120891	210399896	194330213
23	52892130	612602926	599408935
24	146399126	1778618989	1836028197
25	405437427	5151219930	5589730484

The coefficients for the generating functions of the enumerations were produced by programs written in FORTRAN on the VAX 3500. The numbers of topological mono- and bicyclic skeletons match up with those obtained previously from the SKEL_GEN and SKEL_GROW programs.[4] The enumeration programs and the numbers of skeletons (topological and enantiomeric) for higher numbers of carbons are available from the authors on request.

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REFERENCES AND NOTES

(1) (a) Pólya, G. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Math.* **1937**, *68*, 145–254. (b) Pólya, G.; Read, R. C. *Combinatorial Enumeration of Graphs, Groups, and Chemical Compounds*; Springer-Verlag: New York, 1987; pp 58–74.

(2) (a) Harary, F.; Palmer, E. M.; Robinson, R. W.; Read, R. C. Pólya's Contributions to Chemical Enumeration. In *Chemical Applications of Graph Theory*; Balaban, A. T., Ed.; Academic Press: London, 1976; pp 11–24. (b) Tucker, A. Pólya's Enumeration Formula by Example. *Math. Mag.* **1974**, *47*, 248–256. (c) Rouvray, D. H. Isomer Enumeration Methods. *Chem. Soc. Rev.* **1974**, *3*, 355–372. (d) Liu, C. L. *Introduction to Combinatorial Mathematics*; McGraw-Hill: New York, 1968; pp 126–166. (e) Hansen, P. J.; Jurs, P. C. Chemical Applications of Graph Theory. Part II: Isomer Enumeration. *J. Chem. Educ.* **1988**, *65*, 661–664. (f) Pólya, G.; Tarjan, R. E.; Woods, D. R. *Notes on Introductory Combinatorics*; Birkhäuser: Boston, 1983; pp 55–85.

(3) Ref 2f, pp 36–37.

(4) Hendrickson, J. B.; Parks, C. A. Generation and Enumeration of Carbon Skeletons. *J. Chem. Inf. Comput. Sci.* **1991**, *30*, 101–107.