# On the Stability of Nonisothermal Fiber Spinning—General Case


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Work reported earlier by Shah and Pearson on the stability of a nonisothermal fiber-spinning process for purely viscous fluids is extended to the general case when inertia, surface tension, and gravity forces are important. The analysis of an isothermal spinning operation shows that in most practical situations inertia will be the dominant force from the point of view of stability. The analysis of nonisothermal operation in the presence of viscous and inertia forces indicates that the critical extension ratio $E_{crit}$ can be usefully correlated with a single parameter $Sr = \bar{k}\mathrm{St}e^{-\mathrm{St}} + 4\mathrm{Re}e^{\bar{k}e^{-\mathrm{St}}}$. This involves three parameters, Reynolds number, Re; Stanton number, St; and a dimensionless temperature viscosity coefficient, $\bar{k}$. A plot of $E_{crit}^{1/2}$ vs. $Sr$ shows that for $Sr >$ about 0.6 very large values of the maximum stable extension ratio are obtained.


In a previous paper, Shah and Pearson (1972) (henceforth SP) have investigated the stability of nonisothermal fiber spinning for purely viscous fluids. In this paper, we shall extend the analysis to take into account inertia, gravity, and surface tension forces and assess the effect of these forces on the stability of the system. The viscosity of the fluid is assumed to be dependent upon temperature alone, as in SP eq 4; the stability analysis follows that of SP; the fiber is assumed to extend vertically downward.

### Equations of Motion and Energy

We use the same notation as in SP (see also Figure 1). Making the same assumptions as in MP (Matovich and Pearson, 1969) the governing equation of motion for the general case becomes (Matovich, 1966)

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = \rho g + \frac{6\eta}{a}\frac{\partial a}{\partial x}\frac{\partial v}{\partial x} + 3\eta\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial \eta}{\partial x}\frac{\partial v}{\partial x} + \frac{\sigma}{a^2}\frac{\partial a}{\partial x} \quad (1)$$

Here $g$ is the acceleration of gravity, $\rho$ is the density, and $\sigma$ is the surface tension of the fluid. The equations of continuity and energy are as before (SP eq 2 and 3)

$$v\frac{\partial a}{\partial x} + \frac{a}{2}\frac{\partial v}{\partial x} = -\frac{\partial a}{\partial t} \quad (2)$$

$$\rho C_p v\frac{\partial \theta}{\partial x} + \frac{2N\theta}{a} = -\rho C_p\frac{\partial \theta}{\partial t} \quad (3)$$

The viscosity relation is

$$\eta = \eta_a e^{-k\theta} \quad (4)$$

### "Isothermal" Spinning

Equation 1 introduces three factors previously disregarded in the stability analysis of PM (Pearson and Matovich, 1969) and SP. To illustrate their effects, we shall first consider the temperature-independent case, $k = 0$, which decouples eq 3 from 1 and 2. This proves to be useful, because it suggests that we neglect two of the factors, gravity and surface tension, when considering the full temperature-dependent system, at least for application to present-day industrial processes.

**Steady-State Equations.** The steady-state equation of motion for an isothermal operation will be the same as eq 34 of MP. Thus

$$\rho\bar{v}\bar{v}' = \rho g - 3\eta_a\frac{(\bar{v}')^2}{\bar{v}} + 3\eta_a\bar{v}'' - \sigma\pi^{1/2}\frac{\bar{v}'}{2Q^{1/2}\bar{v}^{1/2}} \quad (5)$$

Using the same dimensionless variables for velocity and distance as those used in SP, namely $\psi = \bar{v}/\bar{v}_0$ and $\xi = x/L$, $L$ being the length of the fiber between die exit and wind-up spool, the above equation can be written in the following dimensionless form

$$\psi\psi'' - (\psi')^2 = \mathrm{Re}\psi^2\psi' - (\mathrm{Re}/\mathrm{Fr})\psi + (\mathrm{Re}/\mathrm{We})\psi^{1/2}\psi' \quad (6)$$

where

$$\mathrm{Re} = \frac{\rho L\bar{v}_0}{3\eta_a};\quad \mathrm{Fr} = \frac{\bar{v}_0^2}{gL};\quad \mathrm{We} = 2\rho\left(\frac{Q}{\pi}\right)^{1/2}\frac{\bar{v}_0^{3/2}}{\sigma} = \frac{2\bar{a}_0\bar{v}_0^2\rho}{\sigma} \quad (7)$$

Equation 6 is the same as eq 41 of MP. The boundary conditions for eq 6 are the same as those given by eq 9 and 10 of MP, namely fixed velocity and flow rate at $x = 0$ and either fixed velocity or fixed tension at the wind up, $x = L$.

The solutions of eq 3 under various limiting conditions are given by MP. In the present study, however, this equation was solved numerically by a fourth-order Runge–Kutta method (Ralston, 1965).

**Perturbation Equations.** Assuming a knowledge of the steady-state velocity and radius distributions we can write the following equations for the perturbed flow (*cf.* PM)

$$v(x,t) = \bar{v}(x)[1 + \beta(x, t)] \quad (8)$$

$$a(x,t) = \bar{a}(x)[1 + \alpha(x, t)] \quad (9)$$

Substituting eq 8 and 9 into eq 1 and 2 using the dimensionless time variable

$$\tau = \bar{v}_0 t/L \quad (9a)$$

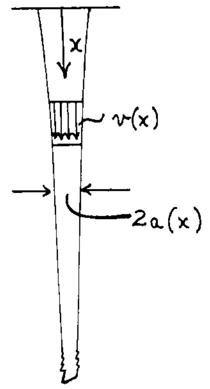


Figure 1. Coordinate system for extending filament

together with the ones described earlier, one obtains after linearization the following equations for the perturbation variables $\alpha$ and $\beta$

$$\psi \frac{\partial^2\beta}{\partial\xi^2} + \left(\frac{\partial\psi}{\partial\xi} - \mathrm{Re}\psi^2\right)\frac{\partial\beta}{\partial\xi} + \left\{\frac{\partial^2\psi}{\partial\xi^2} - \frac{1}{\psi}\left(\frac{\partial\psi}{\partial\xi}\right)^2 - 2\mathrm{Re}\psi\frac{\partial\psi}{\partial\xi}\right\}\beta + \frac{\mathrm{Re}}{\mathrm{We}}\frac{1}{\psi^{1/2}}\frac{\partial\psi}{\partial\xi}\alpha + \frac{2\mathrm{Re}}{\mathrm{We}}\psi^{1/2}\frac{\partial\alpha}{\partial\xi} + 2\frac{\partial\psi}{\partial\xi}\frac{\partial\alpha}{\partial\xi} = \mathrm{Re}\psi\frac{\partial\beta}{\partial\tau} \quad (10)$$

$$\psi\frac{\partial\alpha}{\partial\xi} + \frac{\psi}{2}\frac{\partial\beta}{\partial\xi} = -\frac{\partial\alpha}{\partial\tau} \quad (11)$$

The boundary and initial conditions to eq 10 and 11 will be the same as those discussed by SP with the omission of the temperature perturbation $\gamma$ in each case. The homogeneous nature of the equations means that the magnitude of the imposed disturbance only affects the solution by a simple overall multiplicative factor.

**Solution of Perturbation Equations.** It is obvious that eq 10 and 11 are not conveniently solvable by standard analytical techniques. Hence, in the present study these equations were solved numerically by the same finite difference method as the one used in SP, with $\Delta\tau = 0.0005$.

**Results and Discussion.** The relative importance of inertia, gravity, and surface tension effects can best be compared with that of the viscous effect by considering the three dimensionless groups, Re, Re/Fr, and Re/We, respectively. The operating conditions will determine the extension ratio $E = \bar{v}(L)/\bar{v}(0)$. The full temperature-independent problem involves the 4-space of parameters (Re, Re/Fr, Re/We, $E$), but we shall seek first the curves, in the sub spaces (Re, 0, 0, $E$), (0, Re/Fr, 0, $E$), (0, 0, Re/We, $E$), that separate the domain of stability from that of instability. It should be noted that these curves will correspond to a stability analysis using velocity profiles given by eq 39, 40, and 41 of MP.

With the same criterion for stability as the one discussed previously by SP, results were obtained for $E_{\mathrm{crit}}^{1/2}$ at various values of Re($g$, $\sigma \rightarrow 0$), Re/Fr($\rho\bar{v}_0$, $\sigma \rightarrow 0$), and Re/We ($\rho \rightarrow 0$), by trial and error calculations on the computer. The ranges of values of Re, Re/Fr, and Re/We chosen were those of the most practical interest.

The three plots shown in Figure 2 indicate that inertia has the strongest effect on the stability of the spinning process. If gravity and surface tension forces are negligible, then for Re > about 0.16, the spinning process is stable for all practical purposes. If inertia and gravity forces are negligible, an increase in the value of Re/We (or surface tension force) will make the process more and more unstable, its effect being felt for Re/We > 0.05. However, the effect of Re/We on the critical value of $E$ is not so strong as that of Re. Lastly, gravity is found to aid stability, though rather mildly and for values of Re/Fr > 10.

In an industrial fiber spinning operation the magnitudes of Re and Re/We are usually of the same order while the values of Re/Fr could be as high as 10 to 100 fold that of Re or Re/We (Moore, 1971). The results shown in Figure 2 indicate that for industrial operations inertia will be the dominant force from the point of view of stability, and so surface tension and gravity forces can perhaps be neglected in eq 1 without a significant loss of accuracy in stability calculations.

The dominant effect of the inertia force can to a certain extent be seen from an evaluation of various terms on the right-hand side of eq 6. Since, in general, $\psi$ and $\psi' \gg 1$, the contribution of the first term is expected to be considerably larger than those of the second and third terms.

This suggests that an analysis of nonisothermal fiber spinning can also be simplified by neglecting gravity and surface tension forces in eq 1. This is particularly important because

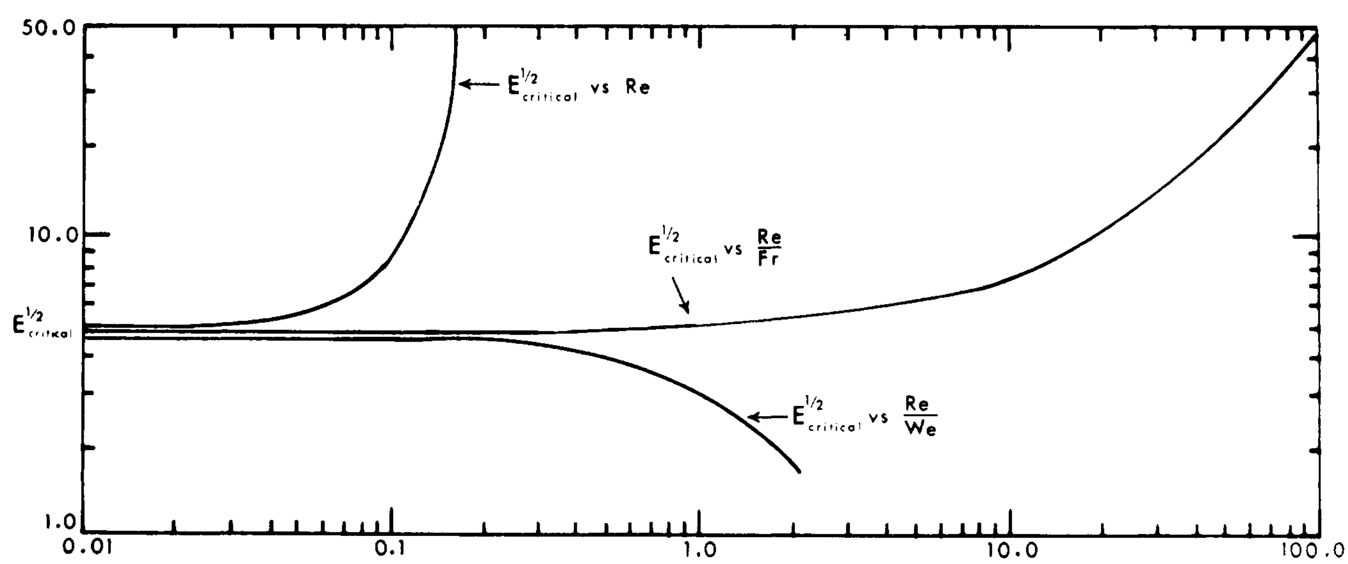


Figure 2. Plots of $E_{\mathrm{crit}}^{1/2}$ vs. Re, Re/We, and Re/Fr

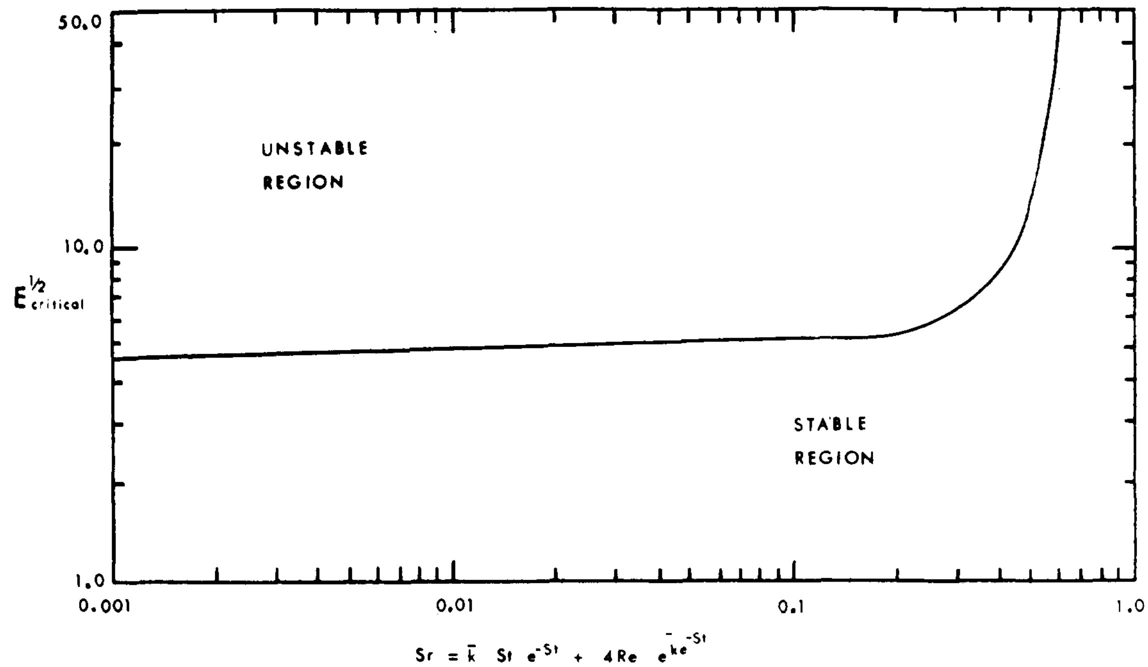


Figure 3. A plot of $E_{crit}^{1/2}$ vs. $Sr$. The curve shown is an approximate correlation; no single curve exists and the one given is only meant to give a qualitative indication of behavior to be expected

of the difficulty of presenting the results for a large number of independent parameters. Hence the analysis of the nonisothermal spinning process that is presented in the rest of the paper considers only viscous and inertia forces in the equation of motion.

### "Nonisothermal" Spinning

Omitting surface tension and gravity forces and using eq 4 above, one can write the equation of motion as

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = \frac{6}{a}\eta_a\frac{\partial a}{\partial x}\frac{\partial v}{\partial x}e^{-k\theta} + 3\eta_a\frac{\partial^2 v}{\partial x^2}e^{-k\theta} - 3k\eta_a\frac{\partial\theta}{\partial x}\frac{\partial v}{\partial x}e^{-k\theta} \quad (12)$$

The equations of continuity and energy are still given by eq 2 and 3.

**Steady-State Equations.** The steady state equation of motion in dimensionless form can now be written as

$$\psi\psi'' - \psi'^2 - \bar{k}\Phi'\psi\psi' - \mathrm{Re}e^{\bar{k}\Phi}\psi^2\psi' = 0 \quad (13)$$

The dimensionless steady-state equation of energy will be the same as eq 10 of SP, namely

$$\Phi' + \mathrm{St}\psi^{1/6}\Phi = 0 \quad (14)$$

**Perturbation Equations.** The perturbation equations for this case can be obtained by substituting eq 8 and 9 above together with

$$\theta(x, t) = \bar{\theta}(x)[1 + \gamma(x, t)] \quad (15)$$

into eq 2, 3, and 12, linearizing, and using the appropriate dimensionless variables as described earlier. Thus, one obtains for $\beta$

$$\psi\frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\psi}{\partial\xi} - \bar{k}\Phi'\psi - \mathrm{Re}e^{\bar{k}\Phi}\psi^2\right)\frac{\partial\beta}{\partial\xi} - \mathrm{Re}e^{\bar{k}\Phi}\psi\psi'\beta - \bar{k}\Phi\frac{\partial\psi}{\partial\xi}\frac{\partial\gamma}{\partial\xi} - \bar{k}\frac{\partial\Phi}{\partial\xi}\frac{\partial\psi}{\partial\xi}\gamma + 2\frac{\partial\psi}{\partial\xi}\frac{\partial\alpha}{\partial\xi} = \mathrm{Re}e^{\bar{k}\Phi}\psi\frac{\partial\beta}{\partial\tau} \quad (16)$$

The perturbation equations for $\alpha$ and $\gamma$ will the the same as eq 18 and 19 of SP, namely

$$\psi\frac{\partial\alpha}{\partial\xi} + \frac{\psi}{2}\frac{\partial\beta}{\partial\xi} = -\frac{\partial\alpha}{\partial\tau} \quad (17)$$

and

$$\psi\frac{\partial\gamma}{\partial\xi} - \mathrm{St}\psi^{1.167}\left(\frac{1}{3}\beta + \alpha\right) = -\frac{\partial\gamma}{\partial\tau} \quad (18)$$

It should be noted that in the above eq 13–18 the dimensionless quantities $\Phi$ and $\bar{k}$ are defined in the same manner as eq 8 and 11 of SP, namely $\Phi = \bar{\theta}/\bar{\theta}_0$ and $\bar{k} = k\bar{\theta}_0$, where $\bar{\theta}_0$ is the initial mean temperature of the filament (at $\xi = 0$). The boundary conditions for eq 13 and 14 are the same as those described in SP for eq 9 and 10. Similarly, the boundary and initial conditions for eq 16–18 will be the same as those described in SP for eq 17 to 19.

Both eq 13 and 14 were solved by the same fourth-order Runge–Kutta method as the one used for the solution of eq 6. Equations 16 to 18 were solved numerically by the same finite difference method as the one described in SP with $\Delta\tau = 0.0005$.

**Results and Discussion.** A comparison between the results of $E_{crit}^{1/2}$ *vs.* Re shown in Figure 2 and $E_{crit}^{1/2}$ *vs.* $\bar{k}\mathrm{St}e^{-\mathrm{St}}$ shown in Figure 3 of SP indicate that these two curves are almost identical if $\bar{k}\mathrm{St}e^{-\mathrm{St}} \simeq 4\mathrm{Re}$. This observation together with a comparison of the steady-state and perturbation eq 12 and 16 with the similar eq 9 and 17 of SP suggest that $E_{crit}^{1/2}$ for the present case may well be correlated with a single quantity, $Sr = \bar{k}\mathrm{St}e^{-\mathrm{St}} + 4\mathrm{Re}e^{\bar{k}e^{-St}}$ itself a function of the three independent parameters $\bar{k}$, St, and Re involved in the analysis.

Values of $E_{crit}^{1/2}$ were obtained by a trial and error method for the ranges: $\bar{k}$ from 1 to 5, St from 0.001 to 5, and Re from 0.01 to 0.3. These ranges are believed to be the ones of most practical interest. These values of $E_{crit}^{1/2}$ were plotted against the quantity $Sr$. Figure 3 shows the single approximate correlation curve that was obtained. As expected, the curve shown in Figure 3 reduces very closely to the one shown in Figure 2 for the case $\bar{k} \rightarrow 0$ and to the one shown in Figure 3 of SP for the case $\mathrm{Re} \rightarrow 0$.

The rather simple correlation curve that we have presented could be of considerable qualitative significance for industrial operations. One of the difficulties arises in choosing the appropriate value of $\eta_a$. If the ambient temperature is below the melting point then one is tempted to put Re = 0. However for materials that have a finite viscosity up to the melting point, it is more realistic to use the maximum melt viscosity in the choice of $\eta_a$. It should also be noted that the relevant term in $Sr$ is $4\mathrm{Re}e^{\bar{k}e^{-St}}$: for small St this is close to $4\mathrm{Re}e^{\bar{k}}$ which is the value that would be obtained for $4\mathrm{Re}_{\mathrm{char}}$ if $\eta_0 = \eta_a e^{-\bar{k}}$ were used in the definition of $\mathrm{Re}_{\mathrm{char}}$. Thus the relevant melt viscosity to be used in the calculation of the Reynolds number is effectively a characteristic viscosity rather than the maximum one. As an example, consider $v_0 = 20$ cm/sec, $L = 50$ cm, $\rho = 1$ g/cm$^3$, $\eta_{\mathrm{char}} = 10^4$ g/cm sec, all realistic values. This yields a value for $\mathrm{Re}_{\mathrm{char}} = 0.03$, which is within the range of importance of $\mathrm{Re}_{\mathrm{char}}$ in $Sr$, where now we have redefined $Sr = \bar{k}\mathrm{St}e^{-\mathrm{St}} + 4\mathrm{Re}_{\mathrm{char}}$.

## Nomenclature

$a(x,t)$ = fiber radius
$\bar{a}(x)$ = steady-state fiber radius
$C_p$ = heat capacity
$E$ = extension ratio = $\bar{v}(L)/\bar{v}(0)$
Fr = Froude number as defined by eq 7
$g$ = gravity acceleration
$k$ = viscosity–temperature coefficient as defined by eq 4
$\bar{k}$ = dimensionless viscosity–temperature coefficient as defined by eq 11 of SP
$L$ = length of fiber
$N$ = convective heat transfer coefficient
$Q$ = volumetric rate of flow of material in fiber
$r$ = radial coordinate
Re = Reynolds number as defined by eq 7
$Sr$ = $\bar{k}\mathrm{St}e^{-\mathrm{St}} + 4\mathrm{Re}e^{-\bar{k}e^{-\mathrm{St}}}$
St = Stanton number as defined by eq 12 of SP
$t$ = time
$u$ = radial velocity
$v(x,t)$ = axial velocity of fiber
$\bar{v}(x)$ = steady-state axial velocity of fiber
We = Weber number as defined by eq 7
$x$ = axial coordinate

Greek Letters

$\alpha(x,t)$ = radius perturbation
$\beta(x,t)$ = velocity perturbation
$\gamma(x,t)$ = temperature perturbation
$\eta$ = viscosity
$\eta_a$ = viscosity at ambient temperature
$\theta(x)$ = difference in temperature of fiber and ambient temperature as defined by eq 75 of MP
$\bar{\theta}(x)$ = steady-state value of $\theta(x)$
$\rho$ = fiber density
$\tau$ = dimensionless time as defined by eq 9a
$\Phi$ = dimensionless temperature as defined by eq 8 of SP
$\psi(\xi)$ = dimensionless velocity $(\bar{v}/\bar{v}_0)$
$\sigma$ = surface tension of fluid

Subscripts

0 = condition at $x = 0$

Superscripts

$'$, $''$ = first and second derivatives of the quantity with respect to $x$, respectively

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