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Computability of finite-dimensional linear subspaces and best approximation

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ABSTRACT

We discuss computability properties of the set $\mathcal{P}_G(x)$ of elements of best approximation of some point $x \in X$ by elements of $G \subseteq X$ in computable Banach spaces X. It turns out that for a general closed set G, given by its distance function, we can only obtain negative information about $\mathcal{P}_G(x)$ as a closed set. In the case that G is finite-dimensional, one can compute negative information on $\mathcal{P}_G(x)$ as a compact set. This implies that one can compute the point in $\mathcal{P}_G(x)$ whenever it is uniquely determined. This is also possible for a wider class of subsets G, given that one imposes additionally convexity properties on the space. If the Banach space X is computably uniformly convex and G is convex, then one can compute the uniquely determined point in $\mathcal{P}_G(x)$. We also discuss representations of finite-dimensional subspaces of Banach spaces and we show that a basis representation contains the same information as the representation via distance functions enriched by the dimension. Finally, we study computability properties of the dimension and the codimension map and we show that for finite-dimensional spaces X the dimension is computable, given the distance function of the subspace.

1. Introduction

In approximation theory the problem of best approximation is studied using methods of functional analysis [16]. In this paper we will study some parts of the problem of best approximation in Banach spaces by elements of closed convex sets and by elements of finite-dimensional linear subspaces from the viewpoint of computable analysis using the representation based approach of Weihrauch [17].

Given a normed space X the problem of best approximation is the problem of finding, for a given subset $G \subseteq X$ and a point $x \in X$, a point $g_0 \in G$ that is a nearest point to x among all elements of G, that is

$$\|x - g_0\| = \inf_{g\in G} \|x - g\| =: \mathrm{dist}(x, G) \tag{1}$$

where $\mathrm{dist}(x, G)$ denotes the distance between a point $x \in X$ and a subset $G \subseteq X$. In general the existence of such an element g_0 is not guaranteed nor has it to be unique since every element of G with the property of Eq. (1) is an *element of best approximation* of x in G. Therefore, by $\mathcal{P}_G(x)$ we denote the set of all elements of best approximation of an element $x \in X$ in a subset $G \subseteq X$, that is

$$\mathcal{P}_G(x) := \{g_0 \in G : \|x - g_0\| = \mathrm{dist}(x, G)\}.$$

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Then g_0 is an element of best approximation of x in G if and only if $g_0 \in \mathcal{P}_G(x)$. For a given subset G and an element x, the set $\mathcal{P}_G(x)$ may be empty or have more than one element. If X is a uniformly convex Banach space and G is a closed convex subset of X then $\mathcal{P}_G(x)$ consists of exactly one element for all $x \in X$. In this case a single-valued total function $P_G : X \to X$ can be defined by

$$P_G(x) = g_0 :\iff \mathcal{P}_G(x) = \{g_0\}.$$

P_G maps each element of X to its uniquely defined best approximation in G. P_G is called the *metric projection* onto G. In this paper we will present some conditions under which the mappings $G \to \mathcal{P}_G$ and $G \to P_G$ become computable in some sense defined later.

The problem of best approximation has also extensively been studied in constructive and proof-theoretic analysis. We mention, in particular, the results in [7], [8], [1, pp. 309–313], and [6, pp. 88–92] for constructive analysis. It seems that our Proposition 4.3 and Corollary 6.1 could also be derived from those results via realizability theory. However, we are not aware of any counterpart of Theorems 5.4 and 6.2 in constructive analysis. In proof-theoretic analysis classical existence proofs on best approximation have been analyzed carefully. These results even provide explicit quantitative information and some of these results could yield alternative proofs of our results, in particular of our Theorem 5.4. An excellent survey on the proof-theoretic approach is available in [9].

In the next section we briefly define some concepts from computable analysis that we will need to present our results. In Section 3 we formulate some technical results about finite linear combinations and linear independence that we use in the following section. In Section 4 we define a representation of finite-dimensional subspaces using bases of such subspaces. Furthermore, we compare our representation with further representations for finite-dimensional subspaces that can be derived from known representations for closed subsets. In Section 5 we present a first result about the computability of the metric projection in the case of closed convex subsets. In Section 6 we prove some better results about the computability of $\mathcal{P}_G(x)$ and $P_G(x)$ for finite-dimensional linear subspaces G, which are special closed convex subsets. In the last section we briefly summarize our result about the computability of the metric projection.

2. Computable Banach spaces

In this section we briefly define some concepts from computable analysis. Computability on Banach spaces has been extensively studied by Pour-El and Richards in their book [12]. We will study this subject using the representation based approach to computable analysis of Weihrauch [17]. The representation based approach is essentially compatible to the sequential approach of Pour-El and Richards, but it can more flexibly be adapted to higher degrees of uniformity. We refer the reader to [17] for all concepts that are left undefined here. In the following we assume that Banach spaces are defined over the field $\mathbb{F}$, which might either be $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 (*Computable Banach Space*)**.** A *computable normed space* $(X, \|\ \|, e)$ is a separable normed space $(X, \|\ \|)$ together with a fundamental sequence $e : \mathbb{N} \to X$ (that is the linear span of range(e) is dense in X) such that the induced metric space is a computable metric space. A *computable Banach space* is a computable normed space that is a Banach space (i.e. a complete normed space).

The induced computable metric space is the space (X, d, α_e) where d is given by $d(x, y) := \|x - y\|$ and $\alpha_e : \mathbb{N} \to X$ is defined by $\alpha_e\langle k, \langle n_0, \ldots, n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$. Here $\alpha_{\mathbb{F}}$ is a standard numbering of $\mathbb{Q}_{\mathbb{F}}$ where $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}$ in the case of $\mathbb{F} = \mathbb{R}$ and $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}[i]$ in the case of $\mathbb{F} = \mathbb{C}$. We assume that there is some $n \in \mathbb{N}$ with $\alpha_{\mathbb{F}}(n) = 0$. The linear operations (vector space addition and scalar multiplication) are automatically computable for any computable normed space.

In general, a space (X, d, α) is called a *computable metric space*, if (X, d) is a metric space with a dense sequence α such that $d \circ (\alpha \times \alpha)$ is a computable double sequence. If not mentioned otherwise, then we assume that all computable Banach spaces X are represented by their Cauchy representation δ_X (of the induced metric space). The *Cauchy representation* $\delta_X :\subseteq \Sigma^\omega \to X$ of a computable metric space X is defined such that a sequence $p \in \Sigma^\omega$ represents a point $x \in X$, if it encodes a sequence $(\alpha(n_i))_{i\in\mathbb{N}}$, which rapidly converges to x, where rapid means that $d(\alpha(n_i), \alpha(n_j)) < 2^{-j}$ for all $i > j$. Here Σ^ω denotes the set of infinite sequences over some finite set Σ (the *alphabet*) and Σ^ω is endowed with the product topology with respect to the discrete topology on Σ. A special case of a Cauchy representation is the representation of the natural numbers $\delta_{\mathbb{N}} :\subseteq \Sigma^\omega \to \mathbb{N}$, which we can also directly and equivalently define by $\delta_{\mathbb{N}}(p) := p(0)$.

In general a *representation* of a set X is a surjective map $\delta :\subseteq \Sigma^\omega \to X$. Here the inclusion symbol "$\subseteq$" indicates that the corresponding map might be partial. Given representations $\delta :\subseteq \Sigma^\omega \to X$ and $\delta' :\subseteq \Sigma^\omega \to Y$, a map $f :\subseteq X \to Y$ is called (δ, δ')-*computable*, if there exists a computable map $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta' F(p) = f\delta(p)$ for all $p \in \mathrm{dom}(f\delta)$. Analogously, one can define computability for multi-valued functions $f :\subseteq X \rightrightarrows Y$. In this case the equation above has to be replaced by the condition $\delta' F(p) \in f\delta(p)$. Here a function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is called *computable* if there exists a Turing machine which computes F. Similarly, one can define the concept of *continuity* with respect to representations, where the computable function F is replaced by a continuous function.

Cauchy representations of computable metric spaces X are known to be *admissible* and for such representations continuity with respect to representations coincides with ordinary continuity. If X, Y are computable metric spaces, then we assume that the space $\mathcal{C}(X, Y)$ of continuous functions $f : X \to Y$ is represented by $[\delta_X \to \delta_Y]$, which is a canonical function space representation. This representation satisfies two characteristic properties, evaluation and type conversion, which can be performed computably (see [17] for details). If $Y = \mathbb{F}$, then we write for short $\mathcal{C}(X) = \mathcal{C}(X, \mathbb{F})$.

We say that a representation δ is *computably reducible* to another representation δ' of the same set, in symbols $\delta \le \delta'$, if there is a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta(p) = \delta' F(p)$ for all $p \in \mathrm{dom}(\delta)$. This is equivalent to the fact that the identity $\mathrm{id} : X \to X$ is (δ, δ')-computable. Two representations are said to be *computably equivalent*, if they are mutually computably reducible to each other, in symbols $\delta \equiv \delta'$.

In the following we have to deal with closed and compact subsets of normed and metric spaces. Given a metric space X we denote the set of all closed subsets of X by $\mathcal{A}(X)$ and the set of all compact subsets of X by $\mathcal{K}(X)$. To represent these spaces we use the representations for closed and compact subsets of metric spaces that are defined and studied by Brattka and Presser in [5].

For closed subsets, we will use the representations $\delta_{\mathrm{dist}}^{>}$, $\delta_{\mathrm{dist}}^{<}$ and $\delta_{\mathrm{dist}}^{=}$, which represent closed subsets by their distance functions with negative, positive, and full information, respectively, as well as the representations δ_{range}, which represents a closed subset by a dense sequence, and δ_{fiber}, which represents a closed subset by a total function such that the set is the preimage of $\{0\}$. For compact subsets, we will use the representations δ_{cover} and $\delta_{\mathrm{mincover}}$, which represent a compact subset by all finite "rational" covers of the set and by all minimal finite "rational" covers of the set, respectively. For further information about these representations we refer the reader to [5].

If X and Y are normed spaces, we denote the set of all bounded linear operators from X to Y by $\mathcal{B}(X, Y)$ and assume that $\mathcal{B}(X, Y)$ is represented as a subset of $\mathcal{C}(X, Y)$ by the restriction of $[\delta_X \to \delta_Y]$ to $\mathcal{B}(X, Y)$. By $S_X(a, r) = S(a, r)$ we denote the sphere in X with center $a \in X$ and radius $r \ge 0$. By $B_X(a, r) = B(a, r)$ we denote the corresponding closed ball. In the case of $a = 0$ and $r = 1$, we denote the corresponding unit sphere and closed unit ball by S_X and B_X, respectively.

3. Linear combinations and linear independence

In this section we summarize some results about finite linear combinations and linear independence that we need in the following sections. To formulate our results, we first introduce two representations for finite tuples and functions on finite tuples.

Given a represented[1] space (X, δ), we denote the set of all finite tuples $(x_1, \dots, x_k)$ for some $k \in \mathbb{N}$ and $x_1, \dots, x_k \in X$ by $X^* = \bigcup_{k\in\mathbb{N}} X^k$. We equip X^* with the canonical representation $\delta^* :\subseteq \Sigma^\omega \to X^*$ that represents an element $(x_1, \dots, x_k)$ of X^* by the number k of elements and a name of the tuple as an element of X^k:

$$\delta^*\langle p, q\rangle = \bar{x} \;:\iff\; \delta_{\mathbb{N}}(p) = k, \quad \bar{x} \in X^k \quad \text{and} \quad \delta^k(q) = \bar{x}$$

where δ^k is the standard representation of the product space X^k. If not mentioned otherwise, X^k and X^* are equipped with δ^k and δ^*, respectively.

Given another represented space (Y, δ'), we denote the set of all continuous functions $f : X^k \to Y$ for some $k \in \mathbb{N}$ by $\mathcal{C}^*(X, Y) := \bigcup_{k\in\mathbb{N}} \mathcal{C}(X^k, Y)$. We equip $\mathcal{C}^*(X, Y)$ with the canonical representation $\delta_{\to}^*$ that represents an element $f : X^k \to Y$ of $\mathcal{C}^*(X, Y)$ by the dimension k of the source space X and a name of the function as element of $\mathcal{C}(X^k, Y)$:

$$\delta_{\to}^*\langle p, q\rangle = f \;:\iff\; \delta_{\mathbb{N}}(p) = k, \quad f \in \mathcal{C}(X^k, Y) \quad \text{and} \quad [\delta^k \to \delta'](q) = f.$$

If not mentioned otherwise, $\mathcal{C}(X^k, Y)$ and $\mathcal{C}^*(X, Y)$ are equipped with $[\delta^k \to \delta']$, the standard representation of the function space $\mathcal{C}(X^k, Y)$, and $\delta_{\to}^*$, respectively.

Now we are prepared to formulate the above-mentioned computability results about finite linear combinations and linear independence. Given a computable normed space X, finite linear combinations are computable in the following uniform way. By $\mathcal{B}(\mathbb{F}^k, X)$ we denote the set of all bounded linear operators from $\mathbb{F}^k$ to X equipped with the operator norm, which is defined for all $f \in \mathcal{B}(\mathbb{F}^k, X)$ by $||f|| := \max\{||f(x)|| : ||x|| = 1\}$.

Proposition 3.1. *Let X be a computable normed space. We define a mapping* $\mathrm{LC} : X^* \to \mathcal{C}^*(\mathbb{F}, X)$ *by* $\mathrm{LC}(\bar{x}) \in \mathcal{C}(\mathbb{F}^k, X) \;:\iff$ $\bar{x} \in X^k$ *and*

$$\mathrm{LC}(x_1, \dots, x_k)(\alpha_1, \dots, \alpha_k) := \sum_{i=1}^{k} \alpha_i x_i$$

for all $(x_1, \dots, x_k) \in X^*$ and $\alpha_1, \dots, \alpha_k \in \mathbb{F}$. Then

(1) LC *is $(\delta_X^*, \delta_{\to}^*)$-computable,*
(2) $\mathrm{LC}(x_1, \dots, x_k) \in \mathcal{B}(\mathbb{F}^k, X)$ *for all $(x_1, \dots, x_k) \in X^*$.*

Proof. The claims directly follow from the definition of LC and the fact that X is a computable normed space. □

Given a normed space X, by IND_X we denote the set

$$\mathrm{IND}_X := \{(x_1, \dots, x_k) \in X^* : (x_1, \dots, x_k) \text{ is linearly independent}\}$$

of all finite tuples that consist of linearly independent elements of X.[2] IND_X is an r. e. open subset of X^* (see also [12, Effective Independence Lemma] and [19, Lemma 10]). We recall that in general a subset $U \subseteq Y$ is called *r. e. open* if and only if there is a computable function $f : Y \to \mathbb{R}$ such that $f^{-1}[\{0\}] = Y \setminus U$. Complements of r. e. open sets are called *co-r. e. closed*.

1 In the following this will usually be a metric or normed space with the corresponding Cauchy representation.
2 We call a tuple $(x_1, \dots, x_k) \in X^*$ linearly independent if $\sum_{i=1}^{k} \alpha_i x_i = 0$ for $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ implies $\alpha_i = 0$ for all $i = 1, \dots, k$.

Proposition 3.2. *Let X be a computable normed space.*

(1) *There exists a $(\delta_X^*, \delta_{\mathbb{R}})$-computable function $F : X^* \to \mathbb{R}$ such that*

$$F(x_1, \ldots, x_k) = 0 \iff (x_1, \ldots, x_k) \text{ is not linearly independent}$$

or equivalently $F^{-1}[\{0\}] = X^* \setminus \mathrm{IND}_X$.

(2) IND_X *is an r. e. open subset of X^*.*

Before we prove Proposition 3.2, we prove the following technical lemma, which we will use in the proofs of Propositions 3.2 and 3.4.

Lemma 3.3. *Let X be a computable normed space. We define mappings $F_1 : X^* \to \mathbb{R}$ and $F_2 : X^* \to \mathbb{R}$ by*

(1) $F_1(x_1, \ldots, x_k) := \max\{\|\mathrm{LC}(x_1, \ldots, x_k)(\alpha_1, \ldots, \alpha_k)\| : \|(\alpha_1, \ldots, \alpha_k)\| = 1\}$,

(2) $F_2(x_1, \ldots, x_k) := \min\{\|\mathrm{LC}(x_1, \ldots, x_k)(\alpha_1, \ldots, \alpha_k)\| : \|(\alpha_1, \ldots, \alpha_k)\| = 1\}$

for all $(x_1, \ldots, x_k) \in X^*$. Then F_1 and F_2 are $(\delta_X^*, \delta_{\mathbb{R}})$-computable.

Proof. Given a δ_X^*-name of $(x_1, \ldots, x_k) \in X^*$ we can compute $k \in \mathbb{N}$. Given $k \in \mathbb{N}$ we can effectively get a $\delta_{\mathrm{mincover}}$-name of the compact unit sphere

$$S_k := \{(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}^k : \|(\alpha_1, \ldots, \alpha_k)\| = 1\}$$

of $\mathbb{F}^k$. Here by $\delta_{\mathrm{mincover}}$ we denote the representation for compact subsets by negative information defined in [5]. Since the image of a compact set under a continuous function and the minimum and maximum of a compact set of real numbers are computable (see [18,17]), the computability of F_1 and F_2 follows. □

Proof (*Proof of Proposition 3.2*). (1) We define $F := F_2$ where $F_2 : X^* \to \mathbb{R}$ is the $(\delta_X^*, \delta_{\mathbb{R}})$-computable function of Lemma 3.3. Then we have

$$\begin{aligned}
&(x_1, \ldots, x_k) \text{ is not linearly independent}\\
&\iff \exists(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}^k \setminus \{0\} : \sum_{i=1}^{k} \alpha_i x_i = 0\\
&\iff \exists(\alpha_1, \ldots, \alpha_k) \in S_k : \sum_{i=1}^{k} \alpha_i x_i = 0\\
&\iff F(x_1, \ldots, x_k) = \min\left\{\left\|\sum_{i=1}^{k} \alpha_i x_i\right\| : (\alpha_1, \ldots, \alpha_k) \in S_k\right\} = 0
\end{aligned}$$

for all $(x_1, \ldots, x_k) \in X^*$. It follows $F^{-1}[\{0\}] = X^* \setminus \mathrm{IND}_X$.

(2) As we have shown in the previous item, $X^* \setminus \mathrm{IND}_X$ is co-r. e. closed in X^*. It follows that IND_X is r. e. open in X^*. □

Given a computable normed space X, the function $\mathrm{LC}(\bar{x})$ is injective if $\bar{x}$ consists of linearly independent elements of X, that is if $\bar{x} \in \mathrm{IND}_X$. In this case, the partial inverse of $\mathrm{LC}(\bar{x})$ exists and is computable. To prove this result, we first prove that the norm of $\mathrm{LC}(\bar{x})$ is computable in a uniform way and that the norm of the partial inverse $(\mathrm{LC}(\bar{x}))^{-1}$ is computable in the same way if $\bar{x} \in \mathrm{IND}_X$.

Proposition 3.4. *Let X be a computable normed space.*

(1) *The mapping $N : X^* \to \mathbb{R}$, $N(\bar{x}) := \|\mathrm{LC}(\bar{x})\|$ is $(\delta_X^*, \delta_{\mathbb{R}})$-computable.*

(2) *The mapping $N_{\mathrm{inv}} :\subseteq X^* \to \mathbb{R}$, $N_{\mathrm{inv}}(\bar{x}) := \|(\mathrm{LC}(\bar{x}))^{-1}\|$ with $\mathrm{dom}(N_{\mathrm{inv}}) := \mathrm{IND}_X$ is $(\delta_X^*, \delta_{\mathbb{R}})$-computable.*

Proof. (1) We have $N = F_1$ where $F_1 : X^* \to \mathbb{R}$ is the $(\delta_X^*, \delta_{\mathbb{R}})$-computable function of Lemma 3.3.

(2) Given $(x_1, \ldots, x_k) \in \mathrm{dom}(N_{\mathrm{inv}}) = \mathrm{IND}_X$ we have

$$0 < F_2(x_1, \ldots, x_k) = \min\{\|\mathrm{LC}(x_1, \ldots, x_k)(\alpha_1, \ldots, \alpha_k)\| : \|(\alpha_1, \ldots, \alpha_k)\| = 1\}.$$

Here $F_2 : X^* \to \mathbb{R}$ is the $(\delta_X^*, \delta_{\mathbb{R}})$-computable function of Lemma 3.3. Furthermore, in this case the inverse operator $(\mathrm{LC}(x_1, \ldots, x_k))^{-1}$ exists as a linear bounded operator and we have

$$\begin{aligned}
\|(\mathrm{LC}(x_1, \ldots, x_k))^{-1}\| &= (\min\{\|\mathrm{LC}(x_1, \ldots, x_k)(\alpha_1, \ldots, \alpha_k)\| : \|(\alpha_1, \ldots, \alpha_k)\| = 1\})^{-1}\\
&= (F_2(x_1, \ldots, x_2))^{-1}.
\end{aligned}$$

It follows $N_{\mathrm{inv}}(x_1, \ldots, x_k) = (F_2(x_1, \ldots, x_2))^{-1}$ if $(x_1, \ldots, x_k) \in \mathrm{dom}(N_{\mathrm{inv}})$, and $N_{\mathrm{inv}}(x_1, \ldots, x_k)$ and $(F_2(x_1, \ldots, x_2))^{-1}$ are undefined if $(x_1, \ldots, x_k) \notin \mathrm{dom}(N_{\mathrm{inv}})$. We obtain $N_{\mathrm{inv}} = 1/F_2$ and N_{inv} is $(\delta_X^*, \delta_{\mathbb{R}})$-computable. □

Now we prove that the partial inverse $(\mathrm{LC}(\bar{x}))^{-1}$ is computable itself.

Proposition 3.5. *Let X be a computable normed space. We define a mapping*

$$\mathrm{LC}'_{\mathrm{inv}} :\subseteq X^* \times X \to \mathbb{F}^*,\ \mathrm{LC}'_{\mathrm{inv}}(\bar{x}, y) := (\mathrm{LC}(\bar{x}))^{-1}(y)$$

with $\mathrm{dom}(\mathrm{LC}'_{\mathrm{inv}}) := \{(\bar{x}, y) \in X^* \times X : \bar{x} \in \mathrm{IND}_X$ *and* $y \in \mathrm{range}(\mathrm{LC}(\bar{x}))\}$

(1) $\mathrm{LC}'_{\mathrm{inv}}(\bar{x}, y) \in \mathbb{F}^k$ *for all* $(\bar{x}, y) \in \mathrm{dom}(\mathrm{LC}'_{\mathrm{inv}})$ *with* $\bar{x} \in X^k$.
(2) $\mathrm{LC}'_{\mathrm{inv}}$ *is* $(\delta_X^*, \delta_X, \delta_{\mathbb{F}}^*)$*-computable.*

Proof. (1) This claim directly follows from the definition of $\mathrm{LC}'_{\mathrm{inv}}$.

(2) Given $\bar{x} = (x_1, \ldots, x_k) \in \mathrm{IND}_X$, we can compute $\mathrm{LC}(\bar{x})$ and the norm $\|(\mathrm{LC}(\bar{x}))^{-1}\| = N_{\mathrm{inv}}(\bar{x})$. Furthermore, $\mathrm{LC}(\bar{x})$ is injective in this case and we have $\mathrm{range}(\mathrm{LC}(\bar{x})) = \mathrm{span}\{x_1, \ldots, x_k\}$. Let $L \in \mathbb{N}$ such that $2^L \geq \|(\mathrm{LC}(\bar{x}))^{-1}\|$. We can effectively find such an L. Given additionally $y \in X$, for each $k \in \mathbb{N}$ we can effectively search some $(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}^*$, such that $\|\mathrm{LC}(\bar{x})(\alpha_1, \ldots, \alpha_k) - y\| \leq 2^{-k-L}$. If $y \in \mathrm{range}(\mathrm{LC}(\bar{x}))$ we will find such a tuple $(\alpha_1, \ldots, \alpha_k)$ at some time, otherwise $(\bar{x}, y) \notin \mathrm{dom}(\mathrm{LC}'_{\mathrm{inv}})$ and our search procedure will never stop. In the first case, we have

$$\begin{aligned}
\|\mathrm{LC}'_{\mathrm{inv}}(\bar{x}, y) - (\alpha_1, \ldots, \alpha_k)\| &= \|(\mathrm{LC}(\bar{x}))^{-1}(y) - (\mathrm{LC}(\bar{x}))^{-1}(\mathrm{LC}(\bar{x})(\alpha_1, \ldots, \alpha_k))\| \\
&\leq \|(\mathrm{LC}(\bar{x}))^{-1}\| \cdot \|y - \mathrm{LC}(\bar{x})(\alpha_1, \ldots, \alpha_k)\| \\
&\leq 2^L \cdot 2^{-k-L} \\
&= 2^{-k}.
\end{aligned}$$

Hence given $(\bar{x}, y) \in \mathrm{dom}(\mathrm{LC}'_{\mathrm{inv}})$ we can compute a sequence in $\mathbb{F}^*$ that converges fast to $\mathrm{LC}'_{\mathrm{inv}}(\bar{x}, y)$. □

4. Finite-dimensional linear subspaces

Given a normed space X, we denote the set of all k-dimensional linear subspaces of X by $\mathcal{L}_X^{(k)}$, the set of all finite-dimensional linear subspaces by $\mathcal{L}_X^{(*)}$, and the set of all linear subspaces by $\mathcal{L}_X$. Obviously, we have $\mathcal{L}_X^{(*)} = \bigcup_{k \in \mathbb{N}} \mathcal{L}_X^{(k)}$, and $\mathcal{L}_X = \mathcal{L}_X^{(*)}$ holds if and only if X has finite dimension. If not mentioned otherwise, X^k and X^* are equipped with δ^k, the standard representation of the product space X^k, and δ^*, respectively.

Since every finite-dimensional linear subspace of a normed space X is a closed subset of X, that is $\mathcal{L}_X^{(*)} \subseteq \mathcal{A}(X)$, we can use the representations for closed subsets of metric spaces that are defined in [5] as representations for $\mathcal{L}_X^{(*)}$. Additionally, we define the following representation for $\mathcal{L}_X^{(*)}$, which represents finite-dimensional linear subspaces by a basis of the subspace.

Definition 4.1 (*Basis Representation*). Let X be a computable normed space. We define a representation δ_{basis} of $\mathcal{L}_X^{(*)}$ by

$$\delta_{\mathrm{basis}}(p) = U :\iff \dim(U) = k,\ \delta_X^*(p) = (x_1, \ldots, x_k) \in \mathrm{IND}_X \quad \text{and} \quad \mathrm{span}\{x_1, \ldots, x_k\} = U.$$

Since a δ_{basis}-name of a finite-dimensional linear subspace U encodes a basis of U, we can compute the dimension and a basis of U from such a name.

Lemma 4.2. *Let X be a computable normed space.*

(1) *The mapping* $\dim : \mathcal{L}_X^{(*)} \to \mathbb{N},\ U \mapsto \dim(U)$ *that maps each finite-dimensional subspace to its dimension is* $(\delta_{\mathrm{basis}}, \delta_{\mathbb{N}})$*-computable.*
(2) *The multi-valued mapping* $\mathrm{basis} : \mathcal{L}_X^{(*)} \rightrightarrows X^*$ *defined by*

$$\bar{x} = (x_1, \ldots, x_k) \in \mathrm{basis}(U) :\iff (x_1, \ldots, x_k) \text{ is a basis of } U$$

is $(\delta_{\mathrm{basis}}, \delta_X^*)$*-computable.*
(3) *The single-valued right inverse*

$$\mathrm{basis}^{-1} :\subseteq X^* \to \mathcal{L}_X^{(*)},\ (x_1, \ldots, x_k) \mapsto \mathrm{span}\{x_1, \ldots, x_k\}$$

with $\mathrm{dom}(\mathrm{basis}^{-1}) = \mathrm{IND}_X$ *of* basis *is* $(\delta_X^*, \delta_{\mathrm{basis}})$*-computable.*

Proof. A δ_{basis}-name of a finite-dimensional subspace U consists of a basis of U as a tuple in X^*. The size of such a tuple is encoded in a δ_X^*-name of it. Therefore, we can get the dimension of the subspace U, which is just the size of a basis, from a δ_{basis}-name of U. The right inverse of basis is well defined as a single-valued function. Its computability follows immediately from the definition of δ_{basis}. □

Using δ_{basis} as representation, we can computably embed $\mathcal{L}_X^{(*)}$ into $\mathcal{A}(X)$. By $\mathcal{A}(X)$ we denote the set of all closed subsets of X and we equip $\mathcal{A}(X)$ with the representations $\delta_{\text{dist}}^{=}$ defined in [5]. Here $\delta_{\text{dist}}^{=}$ represents a closed set by its distance function, more precisely,

$$\delta_{\text{dist}}^{=}(p) = U :\iff [\delta_X \to \delta_{\mathbb{R}}](p) = \text{dist}_U.$$

The representation $\delta_{\text{dist}}^{=}|^{\mathcal{L}_X^{(*)}}$ is then a representation of $\mathcal{L}_X^{(*)}$. In the following we omit "$|^{\mathcal{L}_X^{(*)}}$" when we use the restricted representations of $\mathcal{A}(X)$ to simplify the presentation. It will always be clear if the representation of $\mathcal{A}(X)$ or the restriction to $\mathcal{L}_X^{(*)}$ is meant.

Proposition 4.3. *Let X be a computable normed space. Then $\delta_{\text{basis}} \le \delta_{\text{dist}}^{=}$.*

Proof. Given a δ_{basis}-name of a finite-dimensional linear subspace U, we can compute a basis $(x_1, \dots, x_k)$ of U. It remains to show that given a basis $(x_1, \dots, x_k)$, we can compute the distance function of the linear subspace $U = \text{span}\{x_1, \dots, x_k\}$ as a closed subset of X. For $x \in X$ we know that $\text{dist}(x, U) \le \|x\|$ and that there exists some $z \in U$ with $\text{dist}(x, U) = \|x - z\|$.[3] Thus there exists some $z \in U$ with $\text{dist}(x, U) = \|x - z\|$ and $\|z\| \le 2\|x\|$. By Proposition 3.4(2) given linearly independent $x_1, \dots, x_k \in X$, we can compute $\alpha := \|(\text{LC}(x_1, \dots, x_k))^{-1}\|$. It holds

$$\begin{aligned}
\|(\alpha_1, \dots, \alpha_k)\| &= \|(\text{LC}(x_1, \dots, x_k))^{-1}(\text{LC}(x_1, \dots, x_k)(\alpha_1, \dots, \alpha_k))\| \\
&\le \|(\text{LC}(x_1, \dots, x_k))^{-1}\| \cdot \left\| \sum_{i=1}^{k} \alpha_i x_i \right\| \\
&= \alpha \|y\|
\end{aligned}$$

for all $y = \sum_{i=1}^{k} \alpha_i x_i \in U$. As $z \in U \cap B_X(0, 2\|x\|)$, there exists $(\alpha_1, \dots, \alpha_k) \in \mathbb{F}^k$ with $z = \sum_{i=1}^{k} \alpha_i x_i$ and $\|(\alpha_1, \dots, \alpha_k)\| \le \alpha \cdot \|z\| \le 2\alpha\|x\|$. Thus there exists some $(\alpha_1, \dots, \alpha_k) \in B_{\mathbb{F}^k}(0, 2\alpha\|x\|)$ with $z = \text{LC}(x_1, \dots, x_k)(\alpha_1, \dots, \alpha_k)$ or equivalently

$$z \in \text{LC}(x_1, \dots, x_k)[B_{\mathbb{F}^k}(0, 2\alpha\|x\|)].$$

Given $(x_1, \dots, x_k) \in \text{IND}_X$ and x we can compute the closed ball $B_{\mathbb{F}^k}(0, 2\alpha\|x\|)$ with center 0 and radius $2\alpha\|x\|$ as a compact subset of $\mathbb{F}^k$ and thus also

$$V := \text{LC}(x_1, \dots, x_k)[B_{\mathbb{F}^k}(0, 2\alpha\|x\|)]$$

as a compact set with full information. Since we have $B_X(0, 2\|x\|) \cap U \subseteq V \subseteq U$ it follows $\text{dist}(x, U) = \|z - x\| = \text{dist}(x, V)$. The distance $\text{dist}(x, V)$ can be computed because we have V as a compact set with full information. Hence given $(x_1, \dots, x_k) \in \text{IND}_X$ and $x \in X$ we can compute $\text{dist}(x, \text{span}\{x_1, \dots, x_k\}) = \text{dist}(x, U)$. By type conversion it follows that given $(x_1, \dots, x_k) \in \text{IND}_X$ with $U = \text{span}\{x_1, \dots, x_k\}$ we can compute the distance function of U. This is equivalent to computing a $\delta_{\text{dist}}^{=}$-name of U. □

A corresponding result in constructive analysis is Proposition 2.1 in [7]. For any representation δ of $\mathcal{L}_X^{(*)}$ we define a representation $\delta_{+\dim}$ of $\mathcal{L}_X^{(*)}$ by

$$\delta_{+\dim}\langle p, q\rangle = U :\iff \delta(p) = U \quad \text{and} \quad \delta_{\mathbb{N}}(q) = \dim(U).$$

That is, $\delta_{+\dim}$ is δ enriched by the information on the dimension of the represented subset.

Theorem 4.4. *Let X be a computable normed space. Then $\delta_{\text{basis}} \equiv \delta_{\text{range}+\dim}$. If X is, additionally, a Banach space, then $\delta_{\text{basis}} \equiv \delta_{\text{dist}+\dim}^{=}$.*

Proof. Given a basis $(x_1, \dots, x_k)$ of a subspace $U \subseteq X$ we can easily generate a sequence $(y_i)_{i\in\mathbb{N}}$ that is dense in U, by systematically generating all rational linear combinations of $(x_1, \dots, x_k)$. Given, for instance a sequence $(q_i)_{i\in\mathbb{N}}$ that is dense in $\mathbb{Q}_{\mathbb{F}}^k$ we get such a sequence $(y_i)_{i\in\mathbb{N}}$ by $y_i := \text{LC}(x_1, \dots, x_k)(q_i)$. That proves $\delta_{\text{basis}} \le \delta_{\text{range}}$. It follows from Lemma 4.2 that dim is $(\delta_{\text{basis}}, \delta_{\mathbb{N}})$-computable, which implies that $\delta_{\text{basis}} \le \delta_{\text{range}+\dim}$.

For the other direction we assume that we have given a sequence $(x_i)_{i\in\mathbb{N}}$ in X such that $\overline{\{x_i : i \in \mathbb{N}\}} = U$ and we have given $\dim(U)$. We can then systematically check $(x_{i_1}, \dots, x_{i_k}) \in \text{IND}_X$ for more and more pairwise different $i_1, \dots, i_k$. By Lemma 3.3(2) the set IND_X is r.e. open and thus we can eventually find a tuple $(x_{i_1}, \dots, x_{i_k}) \in \text{IND}_X$ in this way with $k = \dim(U)$. As soon as this happens we know that the corresponding $(x_{i_1}, \dots, x_{i_k})$ is a basis of U. This implies $\delta_{\text{range}+\dim} \le \delta_{\text{basis}}$.

It follows from Proposition 4.3 and Lemma 4.2 that $\delta_{\text{basis}} \le \delta_{\text{dist}+\dim}^{=}$. If X is complete, then it follows from results in [5] that $\delta_{\text{dist}}^{=} \le \delta_{\text{range}}$ and hence $\delta_{\text{dist}+\dim}^{=} \le \delta_{\text{range}+\dim} \le \delta_{\text{basis}}$ follows. □

Next we want to show that the information on the distance function alone (without the information on the dimension) does not suffice to determine a basis. We consider an example.

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3 Finite-dimensional subspaces of normed spaces are always proximinal, that is every element of the normed space has a proximum in the subspace [16].

Example 4.5. We consider the computable Banach space ℓ_2 and the canonical standard basis $(e_1, e_2, \ldots)$, where e_i is the vector that is identical 0, except exactly in the ith position, where it is 1. Now we define a sequence of two-dimensional subspaces U_i by $U_i := \mathrm{span}\{e_1, e_{i+2}\}$. In the compact-open topology the sequence $(\mathrm{dist}_{U_i})_{i\in\mathbb{N}}$ converges to the distance function dist_U of the subspace $U := \mathrm{span}\{e_1\}$ and hence $(U_i)_{i\in\mathbb{N}}$ converges in the final topology of $\delta_{\mathrm{dist}}^{=}$ to U. Since $\dim(U) = 1$ and $\dim(U_i) = 2$ for all $i \in \mathbb{N}$, this implies that $\dim : \mathcal{L}_X^{(*)} \to \mathbb{N}$ is not $(\delta_{\mathrm{dist}}^{=}, \delta_{\mathbb{N}})$-continuous and $\delta_{\mathrm{dist}}^{=} \not\leq \delta_{\mathrm{basis}}$ for ℓ_2.

We have used the fact here that the final topology of the function space representation $[\delta_X \to \delta_{\mathbb{R}}]$ is the compact-open topology [13]. If the underlying space X itself is finite-dimensional, then the dimension $\dim(U)$ of a subspace $U \subseteq X$ can be computed, given the distance function dist_U. Before we prove this, we first study computability properties of the dimension in general. We recall that $\mathrm{codim}(U)$ denotes the codimension of U in X, i.e. the dimension of the quotient space X/U. If $(X, ||\ ||)$ is a normed space, then the quotient space $X/U := \{[x] : x \in X\}$ with $[x] := x + U$ is equipped with the norm defined by

$$||[x]||_{X/U} := \inf_{y\in U} ||x - y|| = \mathrm{dist}_U(x)$$

for all $x \in X$. For the following result we use the lower representation $\delta_{\mathbb{N}}^{<}$ of $\mathbb{N} \cup \{\infty\}$, which represents any $x \in \mathbb{N} \cup \{\infty\}$ by enumerating a sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers with $\sup_{i\in\mathbb{N}} n_i = x$.

Proposition 4.6. *Let X be a computable Banach space. Then*

(1) $\dim : \mathcal{L}_X \to \mathbb{N} \cup \{\infty\}, U \mapsto \dim(U)$ *is* $(\delta_{\mathrm{dist}}^{=}, \delta_{\mathbb{N}}^{<})$*-computable,*
(2) $\mathrm{codim} : \mathcal{L}_X \to \mathbb{N} \cup \{\infty\}, U \mapsto \mathrm{codim}(U)$ *is* $(\delta_{\mathrm{dist}}^{=}, \delta_{\mathbb{N}}^{<})$*-computable.*

Proof. We note that a $\delta_{\mathrm{dist}}^{=}$-name of a set U can be translated into a δ_{range}-name of U, since X is complete. That is we can compute a sequence $(x_i)_{i\in\mathbb{N}}$ in X such that $\overline{\{x_i : i \in \mathbb{N}\}} = U$. Having such a sequence, we can systematically check $(x_{i_1}, \ldots, x_{i_k}) \in \mathrm{IND}_X$ for more and more pairwise different $i_1, \ldots, i_k$. Any corresponding number k satisfies $k \leq \dim(U)$. By Lemma 3.3(2) IND_X is an r.e. open subset and thus we can eventually find a tuple $(x_{i_1}, \ldots, x_{i_k}) \in \mathrm{IND}_X$ in this way with $k = \dim(U)$, provided U is finite-dimensional. Otherwise, we find arbitrarily large k in this way.

Essentially, the codimension can be computed in the same way, except that one has to replace the subspace U by the quotient space X/U. We obtain a function $F_U : X^* \to \mathbb{R}$ similarly to F_2 in Lemma 3.3 by

$$F_U(x_1, \ldots, x_k) := \min\{\mathrm{dist}_U(\mathrm{LC}(x_1, \ldots, x_k)(\alpha)) : ||\alpha|| = 1\}$$

and analogously to Lemma 3.3 one obtains that $U \mapsto F_U$ is $(\delta_{\mathrm{dist}}^{=}, [\delta_X^* \to \delta_{\mathbb{R}}])$-computable and since $||[x]||_{X/U} = \mathrm{dist}_U(x)$ one obtains

$$\{(x_1, \ldots, x_k) \in X^* : ([x_1], \ldots, [x_k]) \text{ is linearly independent}\} = X^* \setminus F_U^{-1}[\{0\}].$$

Altogether, using a sequence $(x_i)_{i\in\mathbb{N}}$ of vectors dense in X, we can find tuples $(x_{i_1}, \ldots, x_{i_k}) \in X^* \setminus F_U^{-1}[\{0\}]$ and for any such tuple $k \leq \mathrm{codim}(U)$ holds and eventually we will find some tuple with $k = \mathrm{codim}(U)$, provided X/U is finite-dimensional. Otherwise, we find arbitrarily large k in this way. $\square$

In the case that X itself is finite-dimensional, the situation is simpler since $\dim(X) = \dim(U) + \mathrm{codim}(U)$ holds. In this special case the previous result allows us to compute the dimension exactly. Here we also use the fact that any finite-dimensional normed space is complete.

Corollary 4.7. *Let X be a finite-dimensional computable normed space. Then* $\dim : \mathcal{L}_X \to \mathbb{N}, U \mapsto \dim(U)$ *is* $(\delta_{\mathrm{dist}}^{=}, \delta_{\mathbb{N}})$*-computable.*

In terms of representations we can also express this result as follows.

Corollary 4.8. *Let X be a finite-dimensional computable normed space. Then* $\delta_{\mathrm{basis}} \equiv \delta_{\mathrm{dist}}^{=} \equiv \delta_{\mathrm{range+dim}}$.

It is easy to see that even in the finite-dimensional case the dimension is required as extra information for δ_{range}. One can easily construct a corresponding counterexample for $\mathbb{R}^2$ that shows $\delta_{\mathrm{range}} \not\leq \delta_{\mathrm{basis}}$. The results in this section generalize some results in [19] about representations of linear subspaces of $\mathbb{R}^n$.

5. Metric projections onto convex subsets

A subset U of a normed space X is called convex if $\lambda x + (1 - \lambda)y \in U$ holds for all $x, y \in U$ and $0 \leq \lambda \leq 1$. A normed space X is called rotund or strictly convex if $\|tx_1 + (1 - t)x_2\| < 1$ whenever $x_1, x_2 \in S_X, x_1 \neq x_2$, and $0 < t < 1$ [11]. Here by S_X we denote the unit sphere of X. If X is a normed space with a strictly convex norm, the *modulus of convexity* $\mathrm{mc} : [0, 2] \to [0, 1]$ of the norm $\|\ \|$ is defined by

$$\mathrm{mc}(\varepsilon) := \inf\left\{1 - \left\|\frac{1}{2}(x + y)\right\| : x, y \in S_X,\ \|x - y\| \geq \varepsilon\right\}.$$

The norm $\|\ \|$ is called *uniformly convex* if $\mathrm{mc}(\varepsilon) > 0$ for $0 < \varepsilon \leq 2$. By computability of the modulus of convexity mc we mean $(\delta_{\mathbb{R}}, \delta_{\mathbb{R}})$-computability.

Every finite-dimensional normed space is uniformly convex if and only if it is strictly convex [11, Prop. 5.2.14] and every uniformly convex Banach space is known to be reflexive (the Milman–Pettis theorem, see for example [11, Thm. 5.2.15]).

Given a normed space X, a subset $G \subseteq X$ of X and an element $x \in X$, we define the set

$$\mathcal{P}_G(x) := \{z \in G : \|x - z\| = \mathrm{dist}_G(x)\}$$

of all elements of best approximation of x by elements of G. The set G is called[4]

- a *proximinal set* or *set of existence* if $\mathcal{P}_G(x) \neq \emptyset$, that is $\mathcal{P}_G(x)$ contains at least one element, for all $x \in X$,
- a *semi-Chebyshev set* or *set of uniqueness* if $\mathcal{P}_G(x)$ contains at most one element for all $x \in X$,
- a *Chebyshev set* if $\mathcal{P}_G(x)$ contains exactly one element for all $x \in X$.

If a subset $G \subseteq X$ of a normed space X is a Chebyshev set we can define a total and single-valued function that maps every $x \in X$ to its uniquely defined best approximation in G.

Definition 5.1 (*Metric Projection*)**.** Let X be a normed space and $G \subseteq X$ a Chebyshev set. We define the *metric projection* $P_G : X \to X$ onto G by

$$y = P_G(x) :\iff \mathcal{P}_G(x) = \{y\}$$

for all $x \in X$.

The norm of a normed space X is strictly convex if and only if every nonempty (closed) convex subset of X is a set of uniqueness [11, Theorem 5.1.18]. If every nonempty closed convex subset of a normed space X is a set of existence, then X is reflexive. Additionally, a normed space X is reflexive and has a strictly convex norm if and only if every nonempty closed convex subset is a Chebyshev set [11]. It follows that every nonempty closed convex subset of a uniformly convex Banach space X is a Chebyshev set, that is the metric projection onto it is a total and single-valued function.

Since we have

$$\mathcal{P}_G(x) := \begin{cases} \emptyset & \text{if } x \in \overline{G} \setminus G \\ \mathcal{P}_{\overline{G}}(x) & \text{otherwise} \end{cases}$$

for every (not necessarily closed) subset G of X and element $x \in X$, a proximinal set G has to be closed.[5] Therefore in the following, we only consider best approximation by elements of closed sets. In this case the set $\mathcal{P}_G(x)$ is always closed. First, we prove that given a closed subset G of a computable normed space X and an element $x \in X$ we can compute the set $\mathcal{P}_G(x)$ of elements of best approximation of x in G as a closed set. Here we use the representation δ_{fiber} for $\mathcal{A}(X)$, which is defined by

$$\delta_{\mathrm{fiber}}(p) = A :\iff [\delta_X \to \delta_{\mathbb{R}}](p) = f \quad \text{and} \quad f^{-1}[\{0\}] = A.$$

Proposition 5.2. *Let X be a computable normed space. We define a mapping*

$$P^{\mathrm{set}} : \mathcal{A}(X) \times X \to \mathcal{A}(X),\ (G, x) \mapsto \mathcal{P}_G(x)$$

that maps every closed subset G of X and element $x \in X$ to the corresponding set of elements of best approximation. The P^{set} is $(\delta^{=}_{\mathrm{dist}}, \delta_X, \delta_{\mathrm{fiber}})$-computable.

Proof. Using the given $\delta^{=}_{\mathrm{dist}}$-name of G in combination with the given $x \in X$, we can compute a $[\delta_X \to \delta_{\mathbb{R}}]$-name of the function $f : X \to \mathbb{R}$ defined by $f(z) := |\|x - z\| - \mathrm{dist}(x, G)|$ for $z \in X$. It holds $f(z) = 0 \iff z \in S(x, \mathrm{dist}(x, G))$ for all $z \in X$. Hence f represents a δ_{fiber}-name of the sphere with center x and radius $\mathrm{dist}(x, G)$. Since every element of best approximation of x in G has the distance $\mathrm{dist}(x, G)$ from G and is an element of G it holds $\mathcal{P}_G(x) = G \cap S(x, \mathrm{dist}(x, G))$. By [5] we have $\delta^{=}_{\mathrm{dist}} \leq \delta_{\mathrm{fiber}}$. Hence we can compute a δ_{fiber}-name of G. Since the intersection of closed sets is $(\delta_{\mathrm{fiber}}, \delta_{\mathrm{fiber}}, \delta_{\mathrm{fiber}})$-computable, we can compute a δ_{fiber}-name of $\mathcal{P}_G(x) = G \cap S(x, \mathrm{dist}(x, G))$. □

The next example shows that we cannot compute positive information on the set of best approximation in general, not even in $X = \mathbb{R}^2$.

Example 5.3. We consider $X = \mathbb{R}^2$ equipped with the maximum norm

$$||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}.$$

Let $G_n := \mathrm{span}\{(2^{-n}, 1)\}$ and $G := \mathrm{span}\{(0, 1)\}$. We consider the point $x = (1, 0)$. Then $(G_n)_{n\in\mathbb{N}}$ converges to G in the Fell topology but $(\mathcal{P}_{G_n}(x))_{n\in\mathbb{N}}$ does not converge to $\mathcal{P}_G(x)$ in the lower Fell topology (see [5]). This is because any $\mathcal{P}_{G_n}(x)$ consists of a single point, whereas $\mathcal{P}_G(x)$ consists of the entire line segment from $(0, 1)$ to $(0, -1)$. Fig. 1 illustrates the situation.

A reason for the degenerate situation in this example is that $\mathbb{R}^2$ with the maximum norm is not strictly convex. As soon as we add sufficiently strong convexity conditions, the situation improves. If the modulus of convexity of the uniformly convex Banach space X is computable we can compute the metric projection of a given nonempty convex subset.

[4] For further information about these notions we refer the reader to [16,11].

[5] By $\overline{G}$ we denote the closure of G.

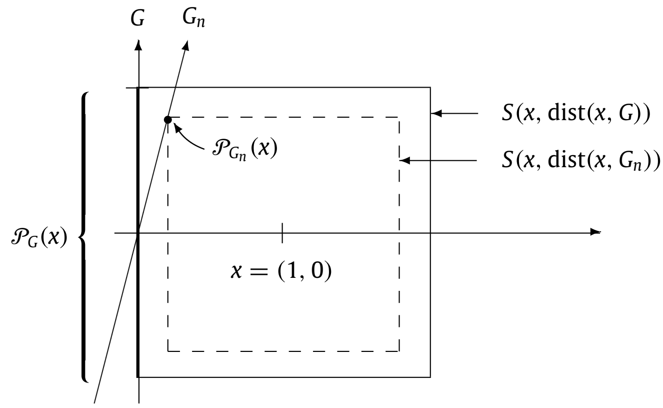

Fig. 1. The metric projections $\mathcal{P}_G(x)$ and $\mathcal{P}_{G_n}(x)$ in $\mathbb{R}^2$ with the maximum norm.

Theorem 5.4 (*Metric Projection*). *Let X be a computable Banach space with a uniformly convex norm and a computable modulus of convexity* mc. *We define a mapping*

$$\widehat{P} :\subseteq \mathcal{A}(X) \times X \to X,\ (G, x) \mapsto P_G(x)$$

with $\mathrm{dom}(\widehat{P}) := \{G \in \mathcal{A}(X) : G \neq \emptyset$ *is convex*$\} \times X$ *that maps every nonempty convex subset G of X to the corresponding metric projection. Then* $\widehat{P}$ *is* $(\delta^{=}_{\mathrm{dist}}, \delta_X, \delta_X)$*-computable.*

Proof. Since every nonempty closed convex set in a uniformly convex Banach space is a Chebyshev set, $\widehat{P}$ is well defined as a single-valued function. We have to show that given a nonempty closed convex set $G \subseteq X$, some $x \in X$, and $k \in \mathbb{N}$, we can effectively find some $z \in X$ with $\|P_G(x) - z\| < 2^{-k}$. In the following we will show even more as we can even find some $z \in G$ (and not only $z \in X$) with $\|P_G(x) - z\| < 2^{-k}$.

Since the norm of X is uniformly convex, we have $\mathrm{mc}(\varepsilon) > 0$ for $\varepsilon > 0$, hence $\mathrm{mc}(2^{-k}) > 0$ for all $k \in \mathbb{N}$. Furthermore, the mapping $k \mapsto \mathrm{mc}(2^{-k})$ is $(\delta_{\mathbb{N}}, \delta_{\mathbb{R}})$-computable. Let $G \subseteq X$ be a nonempty closed convex set and $x \in X$. Let $g_0 \in G$ be the uniquely determined best approximation of x in G, that is we have $\|x - g_0\| = \mathrm{dist}(x, G) =: r \geq 0$. Furthermore, $r = \mathrm{dist}(x, G) = \|x - g_0\|$ can be computed from the given information of G and x.

Let $k \in \mathbb{N}$. Then we have $r < 2^{-k-2}$ or $r > 2^{-k-3} > 0$. We test both inequalities simultaneously and stop if we have proved one of them to be true. Since both cases are overlapping, this can be done effectively. How we continue depends on the chosen case.

"$r < 2^{-k-2}$":

In this case the intersection of $B(x, 2^{-k-1})$ with G is nonempty and open in G. Since $\delta^{=}_{\mathrm{dist}} \leq \delta_{\mathrm{range}}$ holds for computable Banach spaces we can effectively find some $g \in G$ such that $\|x - g\| < 2^{-k-1}$, that is $g \in B(x, 2^{-k-1}) \cap G$. Then we have for g

$$\|g - g_0\| \leq \|g - x\| + \|x - g_0\| = \|g - x\| + r < 2^{-k-1} + 2^{-k-2} < 2^{-k}.$$

"$r > 2^{-k-3}$":

Let $\delta := \min\left\{2^{-k-1}, 2r \cdot \mathrm{mc}(\frac{1}{r} \cdot 2^{-k-1})\right\}$. Since mc is computable and we have already computed r, we can compute δ. As we have $\delta > 0$, hence $r + \delta > r$, the intersection between $B(x, r + \delta)$ and G is nonempty, and it is open in G. Since $\delta^{=}_{\mathrm{dist}} \leq \delta_{\mathrm{range}}$ holds for computable Banach spaces we can effectively find some $g \in G$ such that $\|x - g\| < r + \delta$. We prove that $\|g - g_0\| < 2^{-k}$ holds in this case. Therefore, let $\delta' := \|x - g\| - r < \delta$. Since $||x - g|| \geq r$ we have $\delta' \geq 0$ and since $r > 0$ it is clear that $r + \delta' > 0$. Let $g' := g - \frac{\delta'}{r+\delta'}(g - x)$. Then we have

$$\left\|g' - g\right\| = \frac{\delta'}{r + \delta'}\|g - x\| = \delta', \tag{2}$$

$$\left\|g' - x\right\| = \frac{r}{r + \delta'}\|g - x\| = r = \|x - g_0\|. \tag{3}$$

It follows

$$\begin{aligned} r &\leq \left\|\frac{1}{2}(g_0 + g) - x\right\| \quad \left(\frac{1}{2}(g_0 + g) \in G \text{ since } G \text{ is convex}\right) \\ &\leq \left\|\frac{1}{2}(g_0 + g) - \frac{1}{2}\left(g_0 + g'\right)\right\| + \left\|\frac{1}{2}\left(g_0 + g'\right) - x\right\| \\ &= \frac{1}{2}\left\|g - g'\right\| + \left\|\frac{1}{2}\left(g_0 + g'\right) - x\right\| \end{aligned}$$

$$= \frac{1}{2}\,\delta' + \left\| \frac{1}{2}\left((g_0 - x) + (g' - x)\right) \right\| \quad \text{by (2)}$$

and thus

$$\left\| \frac{1}{2}\left((g_0 - x) + (g' - x)\right) \right\| \geq r - \frac{1}{2}\,\delta' > r - \frac{1}{2}\,\delta \geq r - r \cdot \mathrm{mc}\left(\frac{1}{r} \cdot 2^{-k-1}\right)$$

so that we get with the definition of mc and Eq. (3)

$$\|g_0 - g'\| = \|(g_0 - x) - (g' - x)\| < 2^{-k-1}.$$

Now we can derive the desired result for g with Eq. (2)

$$\|g_0 - g\| \leq \|g_0 - g'\| + \|g' - g\| < \|g_0 - g'\| + \delta < 2^{-k-1} + 2^{-k-1} = 2^{-k}.$$

In both cases we can find effectively some $g \in G$ such that $\|g - g_0\| < 2^{-k}$ and our proof is complete. $\square$

In proof-theoretic analysis Proposition 17.4 in [9] provides a formula that allows us to obtain a modulus of uniqueness from a modulus of uniform convexity. This formula could be exploited for an alternative proof of the above theorem. In the next section we show that in the case of finite-dimensional linear subspaces, which are particularly nonempty closed convex sets, we can get the same result under less restrictive conditions.

6. Metric projection onto linear subspaces

Given a finite-dimensional linear subspace U of a computable Banach space X and an element x of X, we can compute the set $\mathcal{P}_U(x)$ of all elements of best approximation of x in U as a compact set with negative information. By $\mathcal{K}(X)$ we denote the set of all compact subsets of X and equip $\mathcal{K}(X)$ with the representations $\delta_{\mathrm{mincover}}$ (full information) and δ_{cover} (only negative information); see also [5].

First, we mention a computability result about metric projections onto linear subspaces that is a simple corollary of the result about nonempty closed convex sets that we have proved in the previous section.

Corollary 6.1 (*Metric Projection Onto Subspaces*)**.** *Let X be a computable Banach space with a uniformly convex norm and a computable modulus of convexity* mc. *We define a mapping*

$$P_{\mathcal{L}} : \mathcal{L}_X^{(*)} \times X \to X,\ (U, x) \mapsto P_U(x)$$

that maps every finite-dimensional linear subspace of X to the corresponding metric projection. Then $P_{\mathcal{L}}$ is $(\delta_{\mathrm{dist}}^{=}, \delta_X, \delta_X)$-computable.

Proof. Every finite-dimensional subspace U of X is a nonempty closed convex subset of X. Now we can apply Theorem 5.4. $\square$

A corresponding result in constructive analysis is Theorem 3.1 in [7].

Since finite-dimensional linear subspaces are very special nonempty closed convex subsets, it arises the question if we can for example get a result without demanding a computable modulus of convexity. In fact, this is possible and we can get some better computability results about metric projections in this case. Given a finite-dimensional linear subspace U of a Banach space X and an element x of X, we can compute the set $\mathcal{P}_U(x)$ of all best approximations of x in U as a compact set with negative information.[6]

Theorem 6.2 (*Metric Projection Onto Subspaces*)**.** *Let X be a computable Banach space. We define a mapping*

$$P_{\mathcal{L}}^{\mathrm{set}} : \mathcal{L}_X^{(*)} \times X \to \mathcal{K}(X),\ (U, x) \mapsto \mathcal{P}_U(x)$$

that maps every finite-dimensional linear subspace U and element x to the corresponding compact set of elements of best approximation. Then the mapping $P_{\mathcal{L}}^{\mathrm{set}}$ is $(\delta_{\mathrm{dist}}^{=}, \delta_X, \delta_{\mathrm{cover}})$-computable.

Proof. By Proposition 5.2 P^{set} is $(\delta_{\mathrm{dist}}^{=}, \delta_X, \delta_{\mathrm{fiber}})$-computable. Since every element of best approximation of x in U has the distance $\mathrm{dist}(x, U)$ from U it holds $\mathcal{P}_U(x) = U \cap S(x, \mathrm{dist}(x, U)) \subseteq S(x, \mathrm{dist}(x, U))$. We have $\mathrm{dist}(x, U) \leq \|x\|$ because of $0 \in U$. Thus we have $\mathcal{P}_U(x) \subseteq B_X(0, 2\|x\|) \cap U = B_U(0, 2\|x\|)$. Since $B_U(0, 2\|x\|)$ is a finite-dimensional compact subset of X, we can compute a $\delta_{\mathrm{mincover}}$-name of $B_U(0, 2\|x\|)$. Given a closed set with negative information and a compact set with negative information, we can compute the intersection of these two sets with negative information. Thus we can compute a δ_{cover}-name of $\mathcal{P}_U(x) = S(x, \mathrm{dist}(x, U)) \cap B_U(0, 2\|x\|)$. $\square$

6 Since the set of all best approximations is closed and bounded, it is compact in the case of finite-dimensional subspaces.

Solutions in a compact set (that depends computably on some input data) can be computed exactly if they are uniquely determined. This observation has been made in different forms; see for instance [10,14,15,9]. In computable analysis this observation is usually exploited in the form that it is possible to compute the unique element x of the set $\{x\}$, assuming that we have negative information on $\{x\}$ as a compact set; see for instance [3,2][7]. This fact leads us to the following result.

Theorem 6.3 (*Metric Projection*)**.** *Let X be a computable Banach space with a uniformly convex norm. We define a mapping*

$$P_{\mathcal{L}} : \mathcal{L}_X^{(*)} \times X \to X,\ (U, x) \mapsto P_U(x)$$

that maps every finite-dimensional linear subspace of X to the corresponding metric projection. Then $P_{\mathcal{L}}$ is $(\delta_{\text{dist}}^{=}, \delta_X, \delta_X)$-computable.

Proof. By Theorem 6.2 given a finite-dimensional subset U and an element x we can compute a δ_{cover}-name of $\mathcal{P}_U(x)$. If X is uniformly convex $\mathcal{P}_U(x)$ consists of the single point $P_U(x)$. We can compute a δ_X-name of $P_U(x)$ from the δ_{cover}-name of $\mathcal{P}_U(x) = \{P_U(x)\}$ as it is possible to convert the negative information of a singleton as compact set to a name of its unique element [3,2]. □

Thus, in the special case of finite-dimensional subspaces we do not need the computable modulus of convexity that we used in the more general case of convex subsets in Theorem 5.4. Given a computable Banach space X, we can also define a (partial) mapping that maps every finite-dimensional linear subspace U and element x with a unique element of best approximation in U to its best approximation.

Corollary 6.4 (*Unique Metric Projection*)**.** *Let X be a computable Banach space X. We define a (partial) mapping*

$$P_{\mathcal{L}}^{\text{unique}} :\subseteq \mathcal{L}_X^{(*)} \times X \to X,\ (U, x) \mapsto P_U(x)$$

with $\operatorname{dom}(P_{\mathcal{L}}^{\text{unique}}) := \left\{(U, x) \in \mathcal{L}_X^{(*)} \times X : |\mathcal{P}_U(x)| = 1\right\}$ *that maps every finite-dimensional linear subspace U and element x with a unique element of best approximation in U to its best approximation. Then $P_{\mathcal{L}}^{\text{unique}}$ is $(\delta_{\text{dist}}^{=}, \delta_X, \delta_X)$-computable.*

Proof. Using Theorem 6.2 and the already used results from [3,2], we obtain that the mapping $P_{\mathcal{L}}^{\text{unique}}$ is $(\delta_{\text{dist}}^{=}, \delta_X, \delta_X)$-computable. □

It seems that this result could also be derived from [8, Theorem 1] (also [1, Chapter 7, Theorem 2.12] and [6, Theorem 4.2.1]) via realizability theory.

7. Conclusions

In this paper we have studied the computability of the metric projection onto closed convex sets and finite-dimensional linear subspaces of computable Banach spaces X, which is equivalent to computing the best approximation of an element in a given subset. In uniformly convex Banach spaces with computable modulus of convexity we can compute the (unique) best approximation of a given element of X in an also given nonempty closed convex subset of X. In the case of finite-dimensional linear subspaces of Banach spaces, which are special closed convex subsets, we can omit the requirement of a computable modulus of convexity. Without the condition of uniform convexity we can compute the set of best approximations as a compact subset. If we additionally assume uniform convexity then we can compute the unique element of this compact subset, that is the metric projection.

For all results on metric projection it turned out to be convenient to represent linear subspaces via their distance functions. For finite-dimensional subspaces one can raise the question how the information contained in the distance function is related to the information contained in a basis. We have proved that if the space itself is also finite-dimensional, then these two types of information are computably equivalent. If, however, the space itself is infinite-dimensional, then one has to add the dimension of the subspace as an additional information to the distance function in order to get a representation that is equivalent to one that uses bases.

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7 See [4] for a related metatheorem in computable analysis.

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