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Moment estimates for weak solutions to the Navier–Stokes equations in half-space

Cheng He[1,*,†] and Lizhen Wang[2]

[1]*Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, People's Republic of China*
[2]*Department of Mathematics and Center for Nonlinear Studies, Northwest University, Xi'an, People's Republic of China*

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SUMMARY

We establish the moment estimates for a class of global weak solutions to the Navier–Stokes equations in the half-space. Copyright © 2009 John Wiley & Sons, Ltd.

KEY WORDS: Navier–Stokes equations; moment estimate

1. INTRODUCTION

Let $\mathbb{R}^3_+ =: \{(x_1, x_2, x_3) : x_3 > 0\}$. We consider the viscous incompressible fluid flow moving within the three-dimensional half-space $\mathbb{R}^3_+$, which can be described by the following Navier–Stokes equations:

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u = -\nabla\pi \quad \text{in } \mathbb{R}^3_+ \times (0,\infty) \\ \nabla\cdot u = 0 \quad \text{in } \mathbb{R}^3_+ \times [0,\infty) \end{cases} \tag{1}$$

with homogeneous boundary conditions

$$u = 0 \quad \text{on } \partial\mathbb{R}^3_+ \times (0,\infty) \quad \text{and} \quad u \to 0 |x| \to \infty \tag{2}$$

*Correspondence to: Cheng He, Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, People's Republic of China.
†E-mail: chenghe@amss.ac.cn

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and the initial conditions

$$u(x,0)=u_0(x) \quad \text{in } \mathbb{R}^3_+ \tag{3}$$

Here $u=u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ and $\pi=\pi(x,t)$ denote the unknown velocity vector field and the scalar pressure, respectively. v is the viscosity. $u_0(x)$ is the initial velocity vector field. For simplicity, the viscosity v is normalized to 1.

We are interested in constructing a class of global weak solution u to the Navier–Stokes equations (1), which satisfies the moment estimate

$$\|(1+\||x|^2)^{\frac{\alpha}{2}}|u(t)\|_2^2+\int_0^t \|(1+|x|^2)^{\frac{\alpha}{2}}\nabla u(\tau)\|_2^2 \mathrm{d}\tau \leqslant C, \quad t\geqslant 0 \tag{M}$$

for some $\alpha>0$. Hereafter, $\|\cdot\|_r$ denotes the L^r-norm.

For the Cauchy problem, the following results are known.

- M. E. Schonbek and T. P. Schonbek [1] showed (M) with any $\alpha<\frac{7}{4}$ for a class of smooth solution.
- He and Xin [2] showed that (M) for weak solution with $\alpha=\frac{3}{2}$ under the assumption that $u_0\in L^1(\mathbb{R}^3)\cap L^2_\sigma(\mathbb{R}^3)$, $|x|^{\frac{3}{2}}u_0\in L^2(\mathbb{R}^3)$.
- Bae and Jin [3] showed (M) for weak solution with $1<\alpha<\frac{5}{2}$ under the assumption that $u_0\in L^2_\sigma(\mathbb{R}^3)$ and $(1+|x|)u_0\in L^1(\mathbb{R}^3)$, $(1+|x|)^\alpha u_0\in L^2(\mathbb{R}^3)$.
- Brandolese [4] constructed a local smooth solution $u\in C([0,T],\mathbb{Z}_\alpha)$ with some positive T, and for $\frac{3}{2}<\alpha<\frac{9}{2}(\alpha\neq\frac{5}{3},\frac{7}{2})$, under the assumption that $u_0\in\mathbb{Z}_\alpha$. Here $f\in\mathbb{Z}_\alpha$ if and only if f satisfies

$$(1+|x|^2)^{\alpha-2}f\in L^2(\mathbb{R}^3), \quad (1+|x|^2)^{\alpha-1}\nabla f\in L^2(\mathbb{R}^3), \quad (1+|x|^2)^{\alpha}\Delta f\in L^2(\mathbb{R}^3)$$

The situation changes in the case of the domain with boundary. The difficulty comes from the lack of the weighted estimate with respect to pressure π because of the appearance of the boundary. In the case of exterior domain, Farwig [5] constructed a class of weak solutions such that

$$\||x|^{\frac{\alpha}{2}}u(t)\|_2^2+\frac{2(1-\alpha)}{1+\alpha}\int_s^t \||x|^{\frac{\alpha}{2}}\nabla u(\tau)\|_2^2 \mathrm{d}\tau \leqslant \||x|^{\frac{\alpha}{2}}u(s)\|_2^2, \quad 0<\alpha<1$$

for $s=0$, a.e, $s>0$ and all $t\geqslant s$; and

$$\||x|^{\frac{1}{2}}u(t)\|_2^2+2\int_s^t \||x|^{\frac{1}{2}}\nabla u(\tau)\|_2^2 \mathrm{d}\tau \leqslant \||x|^{\frac{1}{2}}u(s)\|_2^2+C(u_0,\delta)|t-s|^\delta$$

for $s=0$, a.e., $s>0$ and all $t\geqslant s$, where $\delta>0$ is arbitrary. From Equation (1), it is obvious that the pressure π obeys the Poisson equation $-\Delta\pi=-\sum_{ij=1}^3 \partial^2_{ji}(u_iu_j)$; thus, the pressure can be represented by the single and double layer potential. Thus, it is possible to obtain the weighted estimate of the pressure since the boundary is compact. Recently, He and Miyakawa [6] made full use of these properties and established (M) for a class of global weak solution with $\alpha<\frac{3}{2}$, and (M) remained valid for $\frac{3}{2}\leqslant\alpha<\frac{5}{2}$ only under some strict assumptions on solution (u,π).

While in the case of half-space, these arguments cannot be applied because the boundary is non-compact. Hence, the results are still incomplete. Recently, Jun Choe and Ja Jin [7] deduced the decay rates with respect to $\|x_3u(t)\|_2$. Bae [8] and Fröhlich [9] showed the local existence of

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strong solution in weighted L^q-spaces, respectively. However, the weighted energy inequality is unknown to our knowledge. In this paper, we will show that (M) is valid for $\alpha<3/q-\frac{3}{2}$ with some $1<q<\frac{6}{5}$. In dealing with our estimates, a crucial role is played by the results of Giga and Sohr [10] (see Lemma 3.2 in Section 3) on the maximal regularity of solutions to the non-stationary Stokes equations. Another crucial role is played by the fact that a weak solution is smooth after some time T. Hence, we can obtained the decay estimates with respect to $\|\nabla u(t)\|_2$ and $\|\nabla\pi(t)\|_q$, exactly following the arguments in [11, 12]. By making full use of these decay estimates, it is possible to deduce the uniform moment estimates for weak solutions.

The moment estimates of weak solution can give us more information on the distribution of the energy of weak solution. It is also provide us the further message on the partial regularity of the weak solution as doing in [5].

2. THE MAIN RESULTS

Before stating our results, we introduce some function spaces which will be used. Let $L^p(\mathbb{R}^3_+), 1\leqslant p\leqslant\infty$, represent the usual Lesbegue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C^\infty_{0,\sigma}(\mathbb{R}^3_+)$ denote the set of all C^∞ vector functions with compact support in $\mathbb{R}^3_+$, such that $\operatorname{div}\phi=0$. $W^{k,p}(\mathbb{R}^3_+)$, $k\in\mathbb{N}$ and $1\leqslant p\leqslant\infty$, is the standard Sobolev spaces. And let $L^\beta_\sigma(\mathbb{R}^3_+)$ be the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3_+)$ with respect to the L^β-norm. Let $L^p(0,T;X), 1\leqslant p\leqslant\infty$, be the set of function $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T\|f(t)\|_X^p\,\mathrm{d}t<\infty$, for a given Banach space X with norm $\|\cdot\|_X$. Define the Leray-Hopf operator

$$\mathbb{P}:\ L^p(\mathbb{R}^3_+)\to L^p_\sigma(\mathbb{R}^3_+),\quad 1<p<\infty$$

Then the Stokes operator $A=-\mathbb{P}\Delta$ with domain

$$D_q(A)=\{u\in W^{2,q}(\mathbb{R}^3_+):u|_{\partial\mathbb{R}^3_+}=0\}\cap L^q_\sigma(\mathbb{R}^3_+)$$

We also need the Banach Space

$$D_q^{1-1/s,s}:=\left\{v\in L^q_\sigma(\mathbb{R}^3_+):\|v\|_{D_q^{1-1/s,s}}=\|v\|_q+\left(\int_0^\infty\|t^{\frac{1}{s}}Ae^{-tA}v\|_q^s\frac{\mathrm{d}t}{t}\right)^{\frac{1}{s}}<\infty\right\}$$

for $1<s,q<\infty$. We will consider the Leray-Hopf weak solutions defined as follows:

Definition 2.1
A *Leray-Hopf weak solution* of system (1)–(3) in $Q_T\equiv\mathbb{R}^3_+\times(0,T)$ is a vector field $u:Q_T\to\mathbb{R}^3$ such that

$$u\in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3_+))\cap L^2(0,T;W^{1,2}(\mathbb{R}^3_+))\tag{4}$$

$$\int_{Q_T}(u\cdot\partial_t w+u\otimes u:\nabla w-\nabla u:\nabla w)\,\mathrm{d}x\,\mathrm{d}t=0\tag{5}$$

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for any $w\in C_{0,\sigma}^{\infty}(Q_T)$; and for any $t\in[0,T]$, u satisfies the energy inequality

$$\|u(t)\|_2^2+2\int_0^t \|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\leqslant\|u_0\|_2^2 \tag{6}$$

and u takes the initial value in the following sense:

$$\|u(\cdot,t)-u_0(\cdot)\|_2\to 0 \quad \text{as } t\to 0 \tag{7}$$

The definition is meaningful also for $T=\infty$ if we replace the closed interval $[0,T]$ by $[0,\infty)$ throughout the definition.

Our main result can be stated as follows:

Theorem 2.1
Let $u_0\in L_\sigma^2(\mathbb{R}_+^3)\cap D_q^{1-1/s,s}$ with

$$4=3/q+2/s,\quad 1<q\leqslant\tfrac{6}{5},\quad 1<s<2$$

Then there exists a class of global Leray-Hopf weak solution (u,π) satisfying the energy inequality (6). In addition, there is some positive T depending the initial data u_0, such that, for any $t\geqslant T$, the solution (u,π) satisfies the following estimates:

$$\begin{aligned}\|u(t)\|_p &\leqslant Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})},\quad q\leqslant p\leqslant\infty\\ \|\nabla u(t)\|_r &\leqslant Ct^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})},\quad q\leqslant r\leqslant\infty\\ \|\partial_t u(t)\|_l+\|\partial_x^2 u(t)\|_l+\|\nabla\pi(t)\|_l &\leqslant Ct^{-1-\frac{3}{2}(\frac{1}{q}-\frac{1}{l})},\quad q\leqslant l<\infty\end{aligned} \tag{8}$$

Moreover, If $(1+|x|^2)^{\frac{\alpha}{2}}u_0\in L^2(\mathbb{R}_+^3)$ for some $1<\alpha<3/q-\frac{3}{2}$, then the weak solution u satisfies the moment estimate

$$\|(1+|x|^2)^{\frac{\alpha}{2}}u(t)\|_2^2+\int_0^t \|(1+|x|^2)^{\frac{\alpha}{2}}\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\leqslant C \tag{9}$$

for any $t>0$. Further, for any $0<\beta<\alpha$, we have

$$\|(1+|x|^2)^{\frac{\beta}{2}}u(t)\|_2^2\leqslant C(1+t)^{-\frac{3(\alpha-\beta)}{\alpha}(\frac{1}{q}-\frac{1}{2})} \tag{10}$$

for any $t\geqslant 0$. Here C depends on norms $\|u_0\|_2$ and $\|u_0\|_{D_q^{1-1/s,s}}$.

Remarks

(1) It is well known that the Navier–Stokes equation remains invariant under the translation with respect to time t and space variables x. It should be noted that, in our results, the solution u and initial velocity u_0 belong to the same weighted space.
(2) Fujigaki and Miyakawa [13] showed that the weak solution u can decay as

$$\|u(t)\|_2\leqslant C(1+t)^{-\frac{5}{4}},\quad t\geqslant 0$$

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if the initial velocity $u_0\in L^1(\mathbb{R}^3_+)$ and $x_3u_0\in L^1(\mathbb{R}^3_+)$. Then, in this case, (10) can be improved as

$$\|(1+|x|^2)^{\frac{\beta}{2}}u(t)\|_2^2\leqslant C(1+t)^{-\frac{5(\alpha-\beta)}{2\alpha}},\quad 0<\beta<\alpha,\ \ t>0$$

(3) The restrict on $\alpha<3/q-\frac{3}{2}$ is due to the lack of the estimate $\|\nabla\pi\|_1$, which is obtained in [12] in the case of exterior domain.

3. PROOF OF THEOREM 2.1

In order to construct the weak solution, we apply the approximate solutions $\{u^n\}_{n\geqslant 1}$ defined by

$$\begin{cases}\partial_t u^n-v\Delta u^n+(u^n*\phi_{1/n}\cdot\nabla)u^n=-\nabla\pi^n & \text{in } \mathbb{R}^3_+\times(0,\infty)\\ \nabla\cdot u^n=0 \quad \text{in } \mathbb{R}^3_+\times[0,\infty)\\ u^n=0 \quad \text{on } \partial\mathbb{R}^3_+\times(0,\infty) \quad \text{and} \quad u^n\to 0\ |x|\to\infty\\ u^n(x,0)=u_0^n(x) \quad \text{in } \mathbb{R}^3_+\end{cases}\tag{11}$$

Here $\phi_{1/n}(x)=n^3\phi(nt,nx)$ with the standard mollifier ϕ in $\mathbb{R}_+\times\mathbb{R}^3_+$, and

$$(u^n*\phi_{1/n})(t,x)=\int_{\mathbb{R}_+\times\mathbb{R}^3}u^n(\tau,y)\phi_{1/n}(t-\tau,x-y)\,\mathrm{d}y\,\mathrm{d}\tau$$

(u^n extend to outside of $\mathbb{R}^3_+$ by zero); and $u_0^n\in C^\infty_{0,\sigma}(\mathbb{R}^3_+)$ with

$$u_0^n\to u_0 \quad \text{in } L^2(\mathbb{R}^3_+)\cap L^q(\mathbb{R}^3_+) \quad \text{as } n\to\infty$$

for some $1<q<\frac{6}{5}$.

We note that $|(u^n*\phi_{1/n})(x)|\leqslant C(n)\|u^n\|_2$ for any fixed n. It is well known that there exist unique smooth solutions $\{u^n\}_{n\geqslant 1}$, for each $n\geqslant 1$, to system (11). As $u_0\in L^2_\sigma(\mathbb{R}^3_+)$, it is easy to see that

$$\|u^n(t)\|_2^2+2\int_0^t\|\nabla u^n(\tau)\|_2^2\,\mathrm{d}\tau\leqslant\|u_0\|_2^2$$

for any $t>0$, uniformly in $n\geqslant 1$. This implies that $\{u^n\}_{n=1}^\infty$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}^3_+))\cap L^2(0,T;W^{1,2}(\mathbb{R}^3_+))$. By interpolation inequality, we know that $\{u^n\}_{n=1}^\infty$ is uniformly bounded in $L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}(\mathbb{R}^n_+))$. As $u_0\in D_q^{1-1/s,s}$ for $4=3/q+2/s$ with $1<q<\frac{3}{2}$ and $1<s<2$, by Giga and Sohr [10], we know that $\{\pi^n\}$ is uniformly bounded in $L^s(0,T;L^q(\mathbb{R}^3_+))$. By Lions-Aubin

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lemma, we know that there is a subsequence of $\{u^n,\pi^n\}_{n=1}^{\infty}$, denoted by itself, and a pair (u,π), such that

$$\begin{cases} u^n \rightharpoonup u & \text{weak-star in } L^{\infty}(0,T;L^2(\mathbb{R}^3_+)) \\ u^n \rightharpoonup u & \text{weakly in } L^2(0,T;W^{1,2}(\mathbb{R}^3_+)) \\ u^n \to u & \text{strongly in } L^p(0,T;L^p_{\text{loc}}(\mathbb{R}^3_+)) \text{ for any } p\in(1,\tfrac{10}{3}) \\ \pi^n \rightharpoonup \pi & \text{weakly in } L^s(0,T;L^{3q/(3-q)}(\mathbb{R}^3_+)) \end{cases} \tag{12}$$

as $n\to\infty$, and (u,π) is a weak solution to Navier–Stokes equations (1) by a routine arguments (cf. [14, 15]).

First, it is obvious that

Lemma 3.1
Let $u_0\in L^2_\sigma(\mathbb{R}^3_+)$. Then

$$\|u(t)\|_2^2+2\int_0^t \|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\leqslant\|u_0\|_2^2 \tag{13}$$

for any $t>0$.

The estimate below is due to Giga and Sohr [10].

Lemma 3.2
Let $u_0\in L^2_\sigma(\mathbb{R}^3_+)\cap D_q^{1-1/s,s}$ with

$$4=3/q+2/s,\quad 1<q<\tfrac{3}{2},\quad 1<s<2$$

Then

$$\int_0^{\infty}(\|\partial_t u\|_q^s+\|\nabla^2 u\|_q^s+\|\nabla\pi\|_q^s)\,\mathrm{d}t\leqslant(\|u_0\|_2^2+\|u_0\|_{D_q^{1-1/s,s}})^s \tag{14}$$

It is well known that there is $t_0>0$ so that the weak solution (u,π) becomes a strong solution of (1) for $t\geqslant t_0$. Hence, following the arguments with minor change in [11, 12, 16, 17], we can show that

Lemma 3.3
Let $u_0\in L^2_\sigma(\mathbb{R}^3_+)\cap L^q(\mathbb{R}^3_+)$ with $1<q<\frac{6}{5}$. Then there exist some $t_0>0$ such that,

$$\begin{aligned} \|u(t)\|_2 &\leqslant C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})}, \quad t>0 \\ \|u(t)\|_p &\leqslant C(t-t_0)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}, \quad q\leqslant p\leqslant\infty,\ t>t_0 \\ \|\nabla u(t)\|_r &\leqslant C(t-t_0)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}, \quad q\leqslant r\leqslant\infty,\ t>t_0 \\ \|\partial_t u(t)\|_l+\|\partial_x^2 u(t)\|_l+\|\nabla\pi(t)\|_l &\leqslant C(t-t_0)^{-1-\frac{3}{2}(\frac{1}{q}-\frac{1}{l})}, \quad q\leqslant l<\infty,\ t>t_0 \end{aligned} \tag{15}$$

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In order to establish the uniform weighted estimates, we choose a cutoff function $\psi\in C_0^\infty(\mathbb{R}^3)$ with $\psi(x)\equiv 1$ for $|x|\leqslant R$, $\psi(x)\equiv 0$ for $|x|\geqslant 2R$ and $0\leqslant\psi\leqslant 1$ for some $0<R<\infty$. We multiply both sides of Equation (11) by $(1+|x|^2)^\alpha u^n\psi^2$, then integrate over $[0,t]\times\mathbb{R}^3_+$ and get

$$\begin{aligned}
&\|(1+|x|^2)^{\frac{\alpha}{2}}u^n(t)\psi\|_2^2+2\int_0^t\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u^n(t)\|_2^2\,\mathrm{d}\tau\\
&\quad\leqslant\int_0^t\int_{\mathbb{R}^3_+}|u^n|^2\psi^2|\Delta(1+|x|^2)^\alpha|\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}^3_+}|u^n|^2|u^n*\phi_{1/n}|\psi^2|\nabla(1+|x|^2)^\alpha|\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+2\int_0^t\int_{\mathbb{R}^3_+}|\pi^n||u^n|\psi^2|\nabla(1+|x|^2)^\alpha|\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}^3_+}|u^n|^2(1+|x|^2)^\alpha|\Delta\psi^2|\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+\int_0^t\int_{\mathbb{R}^3_+}|u^n|^2|u^n*\phi_{1/n}|(1+|x|^2)^\alpha|\nabla\psi^2|\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+2\int_0^t\int_{\mathbb{R}^3_+}|\pi^n||u^n|(1+|x|^2)^\alpha|\nabla\psi^2|\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}\tag{16}$$

Applying the convergence relations (12), we take the limit as $n\to\infty$ in (16) and deduce that

$$\begin{aligned}
&\|(1+|x|^2)^{\frac{\alpha}{2}}u(t)\psi\|_2^2+2\int_0^t\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u(t)\|_2^2\,\mathrm{d}\tau\\
&\quad\leqslant\int_0^t\int_{\mathbb{R}^3_+}|u|^2\psi^2|\Delta(1+|x|^2)^\alpha|\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}^3_+}|u|^3\psi^2|\nabla(1+|x|^2)^\alpha|\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+2\int_0^t\int_{\mathbb{R}^3_+}|\pi||u|\psi^2|\nabla(1+|x|^2)^\alpha|\,\mathrm{d}x\,\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}^3_+}|u|^2(1+|x|^2)^\alpha|\Delta\psi^2|\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+\int_0^t\int_{\mathbb{R}^3_+}|u|^3(1+|x|^2)^\alpha|\nabla\psi^2|\,\mathrm{d}x\,\mathrm{d}\tau+0.2\int_0^t\int_{\mathbb{R}^3_+}|\pi||u|(1+|x|^2)^\alpha|\nabla\psi^2|\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}$$

Note that

$$|\nabla\psi|\leqslant\frac{C}{R},\quad |\Delta\psi|\leqslant\frac{C}{R}$$

and the support of $\nabla\psi$ is contained in the subdomain $\{x\in\mathbb{R}^3: R\leqslant|x|\leqslant 2R\}$. Then

$$(1+|x|^2)^\alpha|\Delta\psi^2|\leqslant C(1+|x|^2)^{\alpha-1}$$

$$(1+|x|^2)^\alpha|\nabla\psi^2|\leqslant C\psi(1+|x|^2)^{\alpha-\frac{1}{2}}$$

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Thus,

$$
\begin{aligned}
&\|(1+|x|^2)^{\frac{\alpha}{2}}u(t)\psi\|_2^2+2\int_0^t \|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u(t)\|_2^2\,\mathrm{d}\tau\\
&\quad\leqslant C\int_0^t\int_{\mathbb{R}^3_+}|u|^2(1+|x|^2)^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}\tau+C\int_0^t\int_{\mathbb{R}^3_+}|u|^3\psi(1+|x|^2)^{\alpha-\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad+C\int_0^t\int_{\mathbb{R}^3_+}|\pi||u|\psi(1+|x|^2)^{\alpha-\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
\tag{17}
$$

In order to establish the required moment estimates, we will consider two cases: $0<\alpha\leqslant 1$ and $1<\alpha<3/q-\frac{3}{2}$ for some $1<q<\frac{6}{5}$.

Lemma 3.4
Let $u_0\in L^2_\sigma(\mathbb{R}^3_+)\cap D_q^{1-1/s,s}$ with

$$4=3/q+2/s,\quad 1<q<\tfrac{6}{5},\quad 1<s<2$$

If $(1+|x|^2)^{\frac{\alpha}{2}}u_0\in L^2(\mathbb{R}^3_+)$ for some $0<\alpha\leqslant 1$, then we hold

$$\|(1+|x|^2)^{\frac{\alpha}{2}}u(t)\|_2^2+\int_0^t \|(1+|x|^2)^{\frac{\alpha}{2}}\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\leqslant C \tag{18}$$

for any $t>0$.

Proof
If $\alpha\leqslant 1$, it is obvious that

$$(1+|x|^2)^{\alpha-1}\leqslant C$$

Hence, by Lemma 3.1, we have

$$I_1=:C\int_{\mathbb{R}^3_+}|u|^2(1+|x|^2)^{\alpha-1}\,\mathrm{d}x\leqslant C\|u\|_2^2 \tag{19}$$

Note that

$$(1+|x|^2)^{\alpha-\frac{1}{2}}\leqslant C(1+|x|^2)^{\frac{\alpha}{2}}$$

since $\alpha\leqslant 1$. By Hölder inequality, we have

$$
\begin{aligned}
I_2&=C\int_{\mathbb{R}^3_+}|u|^3\psi(1+|x|^2)^{\alpha-\frac{1}{2}}\,\mathrm{d}x\\
&\leqslant C\int_{\mathbb{R}^3_+}|u|^3\psi(1+|x|^2)^{\frac{\alpha}{2}}|\,\mathrm{d}x\\
&\leqslant C\|u\|_{12/5}^2\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_6
\end{aligned}
$$

Applying for the estimate

$$\|u\|_{12/5}\leqslant C\|u\|_2^{\frac{3}{4}}\|\nabla u\|_2^{\frac{1}{4}}$$

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we deduce

$$I_2\leqslant C\|u\|_2^{\frac{3}{2}}\|\|\nabla u\|_2^{\frac{1}{2}}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_6$$

Note that u vanishes on the boundary. Extending to outside of $\mathbb{R}^3_+$ for u by zero, we have, by interpolation inequality with weights as given in [18],

$$\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_6\leqslant C\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2+C\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2$$

Thus, by the Young inequality, we deduce that

$$\begin{aligned}I_2 &\leqslant C\|u\|_2^{\frac{3}{2}}\|\|\nabla u\|_2^{\frac{1}{2}}(\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2+\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2)\\&\leqslant \tfrac{1}{3}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2^2+C\|u\|_2^3\|\|\nabla u\|_2+\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2^2\end{aligned}\tag{20}$$

Similarly, by Sobolev's inequality, we have

$$\begin{aligned}I_3 =: {}& C\int_{\mathbb{R}^3_+}|\pi||u|\psi(1+|x|^2)^{\alpha-\frac{1}{2}}\,\mathrm{d}x\\&\leqslant C\|\pi\|_{3q/(3-q)}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_{3q/(4q-3)}\\&\leqslant C\|\nabla\pi\|_q\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_{3q/(4q-3)}\end{aligned}$$

Extend to outside of $\mathbb{R}^3_+$ for u by zero. By interpolation inequality with weights as given in [18], we have

$$\begin{aligned}&\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_{3q/(4q-3)}\\&\leqslant C\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_2^{\frac{7q-6}{2q}}(\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2+\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2)^{\frac{6-5q}{2q}}\end{aligned}$$

By Young's inequality, we get

$$\begin{aligned}I_3 \leqslant{}& \tfrac{1}{3}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2^2\\&+C\|\nabla\pi\|_q^{\frac{4q}{3(3q-2)}}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_2^{\frac{2(7q-6)}{3(3q-2)}}+\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2^2\end{aligned}\tag{21}$$

Substituting (19)–(21) into (17), we obtain that

$$\begin{aligned}&\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u(t)\|_2^2+\int_0^t\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\\&\leqslant C\int_0^t\|\nabla\pi\|_q^{\frac{4q}{3(3q-2)}}\|(1+|x|^2)^{\frac{\alpha}{2}}u\|_2^{\frac{2(7q-6)}{3(3q-2)}}\,\mathrm{d}\tau+C\int_0^t\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2^2\,\mathrm{d}\tau\\&\quad+C\int_0^t\|u\|_2^2\,\mathrm{d}\tau+C\int_0^t\|u\|_2^3\|\|\nabla u\|_2\,\mathrm{d}\tau\end{aligned}\tag{22}$$

As $0<\alpha<1$, direct calculation shows that

$$(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\psi|\leqslant C(1+|x|^2)^{\frac{\alpha-1}{2}}$$

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Then

$$\|(1+|x|^2)^{\frac{\alpha}{2}} u|\nabla\psi|\|_2^2 \leqslant C \int_{R\leqslant|x|\leqslant 2R} |u|^2 \, dx \to 0$$

as $R\to\infty$ since $u\in L^\infty(0,\infty; L^2(\mathbb{R}^3_+))$.Therefore, taking the limit $R\to\infty$ in (22), we obtain

$$\begin{aligned}
&\|(1+|x|^2)^{\frac{\alpha}{2}} u(t)\|_2^2 + \int_0^t \|(1+|x|^2)^{\frac{\alpha}{2}} \nabla u(\tau)\|_2^2 \, d\tau \\
&\leqslant C\int_0^t \|\nabla\pi\|_q^{\frac{4q}{3(3q-2)}} \|(1+|x|^2)^{\frac{\alpha}{2}} u\|_2^{\frac{2(7q-6)}{3(3q-2)}} \, d\tau \\
&\quad + C\int_0^t \|u\|_2^2 \, d\tau + C\int_0^t \|u\|_2^3 \|\|\nabla u\|_2 \, d\tau
\end{aligned} \tag{23}$$

By Lemmas 3.1 and 3.3, we have that

$$\begin{aligned}
&\int_0^\infty (\|u(\tau)\|_2^2 + \|u(\tau)\|_2^3 \|\|\nabla u(\tau)\|_2) \, d\tau \\
&= \left(\int_0^{2t_0} + \int_{2t_0}^\infty\right) (\|u(\tau)\|_2^2 + \|u(\tau)\|_2^3 \|\|\nabla u(\tau)\|_2) \, d\tau \leqslant C(u_0, t_0)
\end{aligned}$$

since $1<q\leqslant\frac{6}{5}$. By Young's inequality, we have

$$\|(1+|x|^2)^{\frac{\alpha}{2}} u\|_2^{\frac{2(7q-6)}{3(3q-2)}} \leqslant C(1+\|(1+|x|^2)^{\frac{\alpha}{2}} u\|_2^2)$$

Let $Y(t) =: 1+\|(1+|x|^2)^{\frac{\alpha}{2}} u(t)\|_2^2$. Then (23) tells us that

$$Y(t) \leqslant C\int_0^t \|\nabla\pi\|_q^{\frac{4q}{3(3q-2)}} Y(\tau) \, d\tau + C \tag{24}$$

Note that

$$\frac{4q}{3(3q-2)} < \frac{2q}{4q-3} = s$$

By Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
\int_0^\infty \|\nabla\pi(\tau)\|_q^{\frac{4q}{3(3q-2)}} \, d\tau &= \left(\int_0^{2t_0} + \int_{2t_0}^\infty\right) \|\nabla\pi(\tau)\|_q^{\frac{4q}{3(3q-2)}} \, d\tau \\
&\leqslant \left(\int_0^{2t_0} \|\nabla\pi(\tau)\|_q^s \, d\tau\right)^{\frac{2(4q-3)}{3(3q-2)}} t_0^{\frac{q}{3(3q-2)}} + C\int_{2t_0}^\infty \tau^{-\frac{4q}{3(3q-2)}} \, d\tau \\
&\leqslant C(\|u_0\|_2^2 + \|u_0\|_{D_q^{1-1/s,s}})^{\frac{4q}{3(3q-2)}} \cdot t_0^{\frac{q}{3(3q-2)}} + C t_0^{-\frac{6-5q}{3(3q-2)}}
\end{aligned}$$

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Then, by Gronwall's inequality, from this and (24), it follows that

$$Y(t)\leqslant C \quad \text{for any } t>0$$

This and (23) give us the main estimate (18). □

Next we turn to the case of $\alpha>1$.

Lemma 3.5
Let $u_0\in L^2_\sigma(\Omega)\cap D_q^{1-1/s,s}$ with

$$4=3/q+2/s, \quad 1<q<\tfrac{6}{5}, \quad 1<s<2$$

If $(1+|x|^2)^{\frac{\alpha}{2}}u_0\in L^2(\mathbb{R}^3_+)$ for some $1<\alpha<3/q-\frac{3}{2}$, then we hold

$$\|(1+|x|^2)^{\frac{\alpha}{2}}u(t)\|_2^2+\int_0^t \|(1+|x|^2)^{\frac{\alpha}{2}}\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\leqslant C \tag{25}$$

for any $t>0$.

Proof
We note that (17) remain valid for any α. Hence, we only need to estimate the terms I_i for $i=1,2,3$ by different arguments. By the Hölder inequality, we have

$$\begin{aligned}
I_1 &\leqslant C\int_{\mathbb{R}^3_+}|u|^2(1+|x|^2)^{\alpha-1}\,\mathrm{d}x\\
&=C\int_{\mathbb{R}^3_+}((1+|x|^2)^{\frac{\alpha}{2}}|u|)^{\frac{2(\alpha-1)}{\alpha}}|u|^{\frac{2}{\alpha}}\,\mathrm{d}x\\
&\leqslant C(1+t)^{-\frac{3(2-q)}{2\alpha q}}\|(1+|x|^2)^{\frac{\alpha}{2}}u\|_2^{\frac{2(\alpha-1)}{\alpha}}
\end{aligned} \tag{26}$$

Here we have used (15). Note that we consider the case of $\alpha>1$ in this case. In the following, we estimate the term I_3. Let $1<q<\frac{6}{5}$. By the Hölder and Sobolev inequalities, we deduce

$$\begin{aligned}
I_3 &\leqslant C\int_{\mathbb{R}^3_+}|\pi||u|\psi(1+|x|^2)^{\alpha-\frac{1}{2}}\,\mathrm{d}x\\
&\leqslant C\|\pi\|_{3q/(3-q)}\|(1+|x|^2)^{\alpha-\frac{1}{2}}\psi u\|_{3q/(4q-3)}\\
&\leqslant C\|\nabla\pi\|_q\|(1+|x|^2)^{\alpha-\frac{1}{2}}\psi u\|_{3q/(4q-3)}
\end{aligned} \tag{27}$$

As before, we extend u to outside of $\mathbb{R}^3_+$ by zero, apply for the interpolation inequality with weight in [18], and get

$$\begin{aligned}
&\|(1+|x|^2)^{\alpha-\frac{1}{2}}\psi u\|_{3q/(4q-3)}\\
&\quad\leqslant C\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_2^{1-\theta}(\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2+\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2)^{\theta}
\end{aligned} \tag{28}$$

with

$$\theta=\frac{6}{q}-2\alpha-3>0 \quad \text{i.e. } \alpha<\frac{3}{q}-\frac{3}{2}$$

(Note that $3/q-\frac{3}{2}>1$ when $1<q<\frac{6}{5}$). By Young's inequality, we deduce that

$$\begin{aligned} I_3 &\leqslant \tfrac{1}{2}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2^2+\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2^2 \\ &\quad +C\|\nabla\pi\|_q^{\frac{2q}{2\alpha q+5q-6}}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_2^{\frac{4(\alpha q+2q-3)}{2\alpha q+5q-6}} \end{aligned} \tag{29}$$

Similarly, we have

$$\begin{aligned} I_2 &\leqslant C\int_{\mathbb{R}^3_+}|u|^3\psi(1+|x|^2)^{\alpha-\frac{1}{2}}\,\mathrm{d}x \\ &\leqslant C\|u\|_{6q/(3-q)}^2\|(1+|x|^2)^{\alpha-\frac{1}{2}}\psi u\|_{3q/(4q-3)} \end{aligned} \tag{30}$$

Applying the interpolation and Sobolev inequalities, we get

$$\|u\|_{6q/(3-q)}\leqslant C\|u\|_2^{\frac{3-2q}{2q}}\|\nabla u\|_2^{\frac{4q-3}{2q}}\leqslant C(1+t)^{-\frac{3(2-q)(3-2q)}{8q^2}}\|\nabla u\|_2^{\frac{4q-3}{2q}}$$

Here we used (15). Thus, the last estimate together with (15) and (28), we deduce that

$$\begin{aligned} I_2 &\leqslant C(1+t)^{-\frac{3(2-q)(3-2q)}{4q^2}}\|\nabla u\|_2^{\frac{4q-3}{q}}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_2^{4+2\alpha-\frac{6}{q}} \\ &\quad \times(\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2+\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2)^{\frac{6}{q}-2\alpha-3} \\ &\leqslant \tfrac{1}{2}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u\|_2^2+\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2^2 \\ &\quad +C(1+t)^{-\frac{3(2-q)(3-2q)}{2q(2\alpha q+5q-6)}}\|\nabla u\|_2^{\frac{2(4q-3)}{2\alpha q+5q-6}}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_2^{\frac{4(\alpha q+2q-3)}{2\alpha q+5q-6}} \end{aligned} \tag{31}$$

Here the Young inequality has been used.

Substituting (26), (27) and (31) into (17), we get

$$\begin{aligned} &\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u(t)\|_2^2+\int_0^t\|(1+|x|^2)^{\frac{\alpha}{2}}\psi\nabla u(\tau)\|_2^2\,\mathrm{d}\tau \\ &\leqslant C\int_0^t\left(\|\nabla\pi\|_q^{\frac{2q}{2\alpha q+5q-6}}+(1+\tau)^{-\frac{3(2-q)(3-2q)}{2q(2\alpha q+5q-6)}}\|\nabla u\|_2^{\frac{2(4q-3)}{2\alpha q+5q-6}}\right)\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_2^{\frac{4(\alpha q+2q-3)}{2\alpha q+5q-6}}\,\mathrm{d}\tau \\ &\quad +C\int_0^t(1+\tau)^{-\frac{3(2-q)}{2\alpha q}}\|(1+|x|^2)^{\frac{\alpha}{2}}\psi u\|_2^{\frac{2(\alpha-1)}{\alpha}}\,\mathrm{d}\tau+\int_0^t\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2^2\,\mathrm{d}\tau \end{aligned} \tag{32}$$

Direct calculations show that

$$(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\psi|\leqslant C(1+|x|^2)^{\frac{1}{2}}$$

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since $1<\alpha<3/q-\frac{3}{2}$ with $1<q<\frac{6}{5}$. From Lemma 3.4, we know that

$$\int_0^t\int_{\mathbb{R}^3_+}(1+|x|^2)|u(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau$$

is well defined. Hence, we have

$$\int_0^t\|(1+|x|^2)^{\frac{\alpha}{2}}u|\nabla\psi|\|_2^2\,\mathrm{d}\tau\leqslant C\int_0^t\int_{R\leqslant|x|\leqslant 2R}(1+|x|^2)|u(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau\to 0$$

as $R\to\infty$. Taking the limit as $R\to\infty$ in (32), we deduce that

$$\begin{aligned}
&\|(1+|x|^2)^{\frac{\alpha}{2}}u(t)\|_2^2+\int_0^t\|(1+|x|^2)^{\frac{\alpha}{2}}\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\\
&\quad\leqslant C\int_0^t\left(\|\nabla\pi\|_q^{\frac{2q}{2\alpha q+5q-6}}+(1+\tau)^{-\frac{3(2-q)(3-2q)}{2q(2\alpha q+5q-6)}}\|\nabla u\|_2^{\frac{2(4q-3)}{2\alpha q+5q-6}}\right)\|(1+|x|^2)^{\frac{\alpha}{2}}u\|_2^{\frac{4(\alpha q+2q-3)}{2\alpha q+5q-6}}\,\mathrm{d}\tau\\
&\qquad+C\int_0^t(1+\tau)^{-\frac{3(2-q)}{2\alpha q}}\|(1+|x|^2)^{\frac{\alpha}{2}}u\|_2^{\frac{2(\alpha-1)}{\alpha}}\,\mathrm{d}\tau
\end{aligned}\tag{33}$$

Let $T=2t_0$. Applying Lemmas 3.1 and 3.2, direct calculation implies that

$$\frac{4(\alpha q+2q-3)}{2\alpha q+5q-6}\geqslant\frac{2(\alpha-1)}{\alpha}\quad\text{if }\alpha\geqslant\frac{6}{q}-5$$

$$\begin{aligned}
\int_0^\infty\|\nabla\pi(\tau)\|_q^{\frac{2q}{2\alpha q+5q-6}}\,\mathrm{d}\tau&=\left(\int_0^T+\int_T^\infty\right)\|\nabla\pi(\tau)\|_q^{\frac{2q}{2\alpha q+5q-6}}\,\mathrm{d}\tau\\
&\leqslant\left(\int_0^\infty\|\pi(\tau)\|_q^s\,\mathrm{d}\tau\right)^{\frac{4q-3}{2\alpha q+5q-6}}\cdot T^{\frac{2\alpha q+q-3}{2\alpha q+5q-6}}\\
&\quad+C\int_T^\infty\tau^{-\frac{2q}{2\alpha q+5q-6}}\,\mathrm{d}\tau\\
&\leqslant CT^{\frac{2\alpha q+q-3}{2\alpha q+5q-6}}+CT^{\frac{2\alpha q+3q-6}{2\alpha q+5q-6}}\leqslant C,\ \left(\alpha<\frac{3}{q}-\frac{3}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty(1+\tau)^{-\frac{3(2-q)(3-2q)}{2q(2\alpha q+5q-6)}}\|\nabla u(\tau)\|_2^{\frac{2(4q-3)}{2\alpha q+5q-6}}\,\mathrm{d}\tau&=\left(\int_0^T+\int_T^\infty\right)(1+\tau)^{-\frac{3(2-q)(3-2q)}{2q(2\alpha q+5q-6)}}\|\nabla u(\tau)\|_2^{\frac{2(4q-3)}{2\alpha q+5q-6}}\,\mathrm{d}\tau\\
&\leqslant C\left(\int_0^T(1+\tau)^{-\frac{2(2-q)(3-2q)}{2q(2\alpha q+q-3)}}\,\mathrm{d}\tau\right)^{\frac{2\alpha q+q-3}{2\alpha q+5q-6}}\\
&\quad\times\left(\int_0^T\|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau\right)^{\frac{4q-3}{2\alpha q+5a-6}}\\
&\quad+C\int_T^\infty(1+\tau)^{-\frac{3(2-q)(3-2q)}{2q(2\alpha q+5q-6)}}\tau^{-(\frac{3}{q}-\frac{1}{2})\cdot\frac{2(4q-3)}{2\alpha q+5q-6}}\,\mathrm{d}\tau
\end{aligned}$$

$$\leqslant C(T) \quad \text{if } q \in \left(0, \frac{6}{5}\right)$$

$$\int_0^t (1+\tau)^{-\frac{3(2-q)}{2\alpha q}} d\tau \leqslant C \quad \text{if } \alpha < \frac{3}{q} - \frac{3}{2}$$

Therefore, by similar arguments as before, (33) give us that

$$\|(1+|x|^2)^{\frac{\alpha}{2}} u(t)\|_2^2 \leqslant C(u_0, T)$$

for any $t>0$. This and (33) give us our estimate (25). □

By Lemma 3.5, we have

Lemma 3.6

Let the assumptions of Lemma 3.5 hold. Then for any $0<\beta<\alpha$, we have

$$\|(1+|x|^2)^{\frac{\beta}{2}} u(t)\|_2^2 \leqslant C(1+t)^{-\frac{3(\alpha-\beta)}{\alpha}(\frac{1}{q}-\frac{1}{2})} \tag{34}$$

for any $t \geqslant 0$.

Proof

By the interpolation inequality, we have

$$\begin{aligned}\|(1+|x|^2)^{\frac{\beta}{2}} u(t)\|_2^2 &= \int_{\mathbb{R}^3_+} ((1+|x|^2)^{\beta} |u(t)|^{2\beta}) |u(t)|^{2(1-\beta)}\, \mathrm{d}x \\ &\leqslant \|(1+|x|^2)^{\frac{\alpha}{2}} u(t)\|_2^{2\beta} \|u(t)\|_2^{2(1-\beta)}\end{aligned}$$

Applying (25) and (15), we have

$$\|(1+|x|^2)^{\frac{\beta}{2}} u(t)\|_2^2 \leqslant C(1+t)^{-\frac{3(\alpha-\beta)}{\alpha}(\frac{1}{q}-\frac{1}{2})}$$

This gives us our desired result (34). □

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