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Haar wavelet in estimating depth profile of soil temperature

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ABSTRACT

A Haar wavelet based method for the estimation of soil temperature at different depths is described in this paper. Diurnal variation in the hourly soil temperature is estimated at different depths varying from 0 to 45 cm. This estimation is compared with observed data available for Trombay site at the depths 5, 10, and 20 cm. This Haar technique can be interpreted from incremental and multi-resolution viewpoint. The estimated values show and excellent agreement with the observed values. The Haar based estimated values are found to be more accurate than the values obtained by FDM approach. More accurate solutions can be obtained by changing the time scale in Haar wavelet; at the same time main features of the solution are preserved. Sensitivity analyses indicate that soil temperature was no so sensitive to changes of soil thermal parameters. Moreover the use of Haar wavelets is found to be accurate, simple, fast, flexible, convenient, small computation costs and computationally attractive.

1. Introduction

The application of reaction–diffusion equations is day by day increasing. Its application in determining the depth profile of soil temperature was first discussed by Carslaw and Jaeger [1] and the analytical technique for obtaining the diurnal variation of soil temperature with depth by solving one-dimensional Fickian diffusion equation is well documented in literature [13,14,16–19]. Since this method involved so many assumptions, which made the applications so restrictive, Dutta and Daoo [2] used an upwind finite difference technique, solved the same problem and some improvement in the results were reported to literature. Lepik [7,20] used the Haar wavelet method for solving nonlinear integral and differential equations. The authors used the same boundary conditions, which were used in the analytical method, and good agreements in the results with the observed data were recorded. However the mean percentage of errors were observed to be 25.1 at 5 cm depth, 15.7 at 10 cm and 9.4 at 20 cm. Dutta in his paper used 5 cm spacing for z and 1 h for Δt. In the sensitivity analysis, it has been shown a small change in soil depth $z < 5$ cm (similarly a small change in $\Delta t < 15$ min) affect the soil temperature. In this paper we applied Haar wavelet technique to solve this problem in order to (i) minimize the error propagation as soil depth increases and (ii) have computational complexity under control. We were able to arrive better results mainly because, by nature, the surface temperature and the soil temperature at any depth follow the diurnal pattern of the ground heat flux, which is of sinusoidal waveform. This paper is organized as follows. In Section 2, we present the derivation of one-dimensional unsteady state partial differential equation for heat diffusion in soil related boundary conditions and its analytic solution. In Section 3, we explain the fundamentals of Haar wavelet and the construction of operational matrices. In Section 4, we explain the solution by wavelet transform.We discuss the results by comparing with observed data in Section 5. Sensitivity analysis is carried out in Section 6. Concluding remarks are given in Section 7.

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2. Heat diffusion equation in soil

Hydraulic properties of unsaturated soils are adequately described by Van Geunuchten and Nielsen [3]. Kimball et al. [4] describes soil heat flux determination through temperature gradient method with computed thermal conductivities Messman [5] corrected errors associated with soil heat flux measurements. Dutta et al. [2] constructed a numerical model for determining depth profile of soil temperature. A brief description about the soil characteristics and methods determining soil heat flux are given in the above works. The conduction of energy per unit area in response to a temperature gradient is called soil heat flux density (G). For small change in depth (z)

$$G \cong -C_1\left(\frac{\partial u}{\partial z}\right), \tag{1}$$

where u is a measure of the heat stored (°C or K), C_1 is the ratio of the heat flux density to the temperature gradient, called thermal conductivity. Here $(\partial u/\partial z)$ is positive when the temperature decreases with depth in the soil. The negative sign is included to make G positive when heat is transferring downward. Thermal diffusivity (K) is the ratio of the thermal conductivity (C_1) to the volumetric heat capacity (C_V), i.e.,

$$K = \frac{C_1}{C_V} \quad \text{or} \quad C_1 = KC_V, \tag{2}$$

$\therefore$ (1) becomes

$$G = -KC_V\left(\frac{\partial u}{\partial z}\right), \tag{3}$$

C_1 is the thermal conductivity (Wm^{-1} min^{-1} K^{-1}), C_V is the volumetric heat capacity (MJ m^{-3} K^{-1}) and K is useful as a measure of how fast the temperature of soil layer changes. The rate at which the heat content of a layer of soil changes depends on the volumetric heat capacity (C_V) and the rate of temperature change of the soil volume per unit time. For a unit surface area, the rate of change in heat storage within the soil is

$$-C_V\left(\frac{\partial u}{\partial t}\right)\Delta z. \tag{4}$$

For unit surface area the rate of change in heat storage within the soil is also equal to the change in heat flux density through the soil layer:

$$\frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\left(-C_1\frac{\partial u}{\partial z}\right)\Delta z. \tag{5}$$

Assuming that the physical properties of the soil are constant with depth in the soil and equating expressions (4) and (5)

$$-C_V\frac{\partial u}{\partial t}\Delta z = \frac{\partial}{\partial z}\left(-C_1\frac{\partial u}{\partial z}\right)\Delta z. \tag{6}$$

Using (2) and (6), we have

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial z^2}. \tag{7}$$

The analytic solution of (7) can be obtained by method of separation of variables [21]. Assuming sinusoidal form of diurnal variation of the ground surface temperature, soil temperature $u(z,t)$ over a diurnal period for homogeneous soil conditions is given by

$$u(z,t) = u_0 + \Delta u_0 e^{\left(-z\sqrt{\frac{\omega}{2K}}\right)}\sin\left(\omega\left(t + z\sqrt{\frac{\omega}{2k}}\right)\right), \tag{8}$$

where u_0 the mean soil temperature at surface averaged over diurnal period is ω is the diurnal frequency and Δu_0 is the amplitude of the soil temperature wave at surface. Here the phase difference between the soil temperature wave at depth z and that surface is $z\sqrt{\frac{\omega}{2k}}$ and the amplitude of the soil temperature wave at depth z is $u_0 e^{\left(-z\sqrt{\frac{\omega}{2k}}\right)}$.

3. Haar wavelet preliminaries

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves [6] with magnitude +1 or -1 in some intervals and zeros elsewhere

$$h_0(t) = 1, \quad 0 \leqslant t < 1 \tag{9}$$

$$h_1(t) = \begin{cases} 1 & 0 \leqslant t < 1/2, \\ -1 & 1/2 \leqslant t < 1, \end{cases} \tag{10}$$

...

...

$$h_n(t) = h_1(2^j t - k), \quad n = 2^j + k,\ j \geqslant 0,\ 0 \leqslant k < 2^j,\ n,j,k \in Z. \tag{11}$$

Just these zeros make Haar wavelets be local, faster than other square functions such as Walsh and very useful in solving stiff systems.

Any function $y(t)$ which is square integrable in the interval $[0,1)$ can be expanded in a Haar series with an infinite number of terms

$$y(t) = \sum_{0}^{\infty} c_i h_i(t), \quad i = 2^j + k,\ j \geqslant 0,\ 0 \leqslant k < 2^j,\ t \in [0,1) \tag{12}$$

where the Haar coefficients

$$c_i = 2^j \int_0^1 y(t) h_i(t) dt \tag{13}$$

are determined in such a way that the integral square error

$$E = \int_0^1 \left[y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt \tag{14}$$

is minimum where $m = 2^j$, $j \in \{0\} \cup N$.

In general, for the function $y(t)$ to be smooth the series expansion in Eq. (11) contains an infinite number of terms. If $y(t)$ is a piecewise constant or may be approximated as piecewise constants, then the sum in Eq. (12) will be terminated after m terms, that is

$$y(t) \cong \sum_{i=0}^{m-1} c_i h_i(t) = c_m^T h_m(t), \tag{15}$$

where $t \in [0,1)$ and

$$c_m \triangleq [c_0, c_1, c_2, \ldots, c_{m-1}]^T, \tag{16}$$

where T stands for transposition, m stands for their dimension.

The first four Haar functions can be expressed as follows:

$$\left.\begin{aligned} h_0(t) &= [1 \quad 1 \quad 1 \quad 1] \\ h_1(t) &= [1 \quad 1 \quad -1 \quad -1] \\ h_2(t) &= [1 \quad -1 \quad 0 \quad 0] \\ h_3(t) &= [0 \quad 0 \quad 1 \quad -1] \end{aligned}\right\} \tag{17}$$

For instance if $y(t) = [7 \quad 1 \quad 3 \quad 0]$ is piecewise constant then,

$$y(t) = \frac{11}{4} h_0(t) + \frac{5}{4} h_1(t) + 3 h_2(t) + \frac{3}{2} h_3(t) = c^T H(t), \tag{18}$$

where

$$H(t) = H_4(t) \triangleq \begin{bmatrix} h_0(t) \\ h_1(t) \\ h_2(t) \\ h_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{19}$$

The Haar coefficients c_i can be obtained by using (5) directly. c_i can also be obtained by matrix inversion

$$c^T = y(t) H_4^{-1} = \left[\tfrac{11}{4} \quad \tfrac{5}{4} \quad 3 \quad \tfrac{3}{2} \right]. \tag{20}$$

Eq. (20) is called forward transform that is used to obtain wavelet coefficients, $y(t)$ can be recovered from the corresponding wavelet coefficients and the wavelets $h_i(t)$. Hence (18) is known as inverse transform. As H and H^{-1} contain many zeros, this phenomenon makes the Haar transform much faster than the Fourier and Walsh transforms [10–12].

3.1. Integration of Haar wavelets

In Haar wavelet analysis for a dynamical system, all functions need to be transformed into Haar series. As impulse functions are not preferred (since they are the derivatives of Haar wavelets), Integration of Haar wavelets are preferred, which is expanded into Haar series with coefficient matrix P [7,8]

$$\int_0^1 h_m(t)dt \cong P_{m\times m}\ h_m(t), \quad t\in[0,1), \tag{21}$$

where $m\times m$ square matrix P is called the operational matrix of integration, which satisfies the following recursive formula:

$$P_{1\times 1}=\left[\frac{1}{2}\right], \quad P_{m\times m}=\frac{1}{2m}\begin{bmatrix} 2mP_{\frac{m}{2}\times\frac{m}{2}} & -H_{\frac{m}{2}\times\frac{m}{2}} \\ H^{-1}_{\frac{m}{2}\times\frac{m}{2}} & O_{\frac{m}{2}\times\frac{m}{2}} \end{bmatrix}, \tag{22}$$

where $O_{\frac{m}{2}\times\frac{m}{2}}$ is a null matrix of order $\frac{m}{2}\times\frac{m}{2}$

$$H_{m\times m}\triangleq [h_m(t_0) \quad h_m(t_1) \quad \ldots h_m(t_{m-1})] \tag{23}$$

and

$$\frac{i}{m}\leqslant t< i+\frac{1}{m} \quad \text{and} \quad H^{-1}_{m\times m}=\frac{1}{m}H^T_{m\times m}diag(r), \tag{24}$$

$$r\triangleq\begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 4 & 4 & 4 & \ldots & \underbrace{\frac{m}{2}\,\frac{m}{2}\,\frac{m}{2}\cdots\frac{m}{2}}_{\frac{m}{2}elements} \end{bmatrix}^T \tag{25}$$

for $m > 2$, proof of Eq. (22) is found in [11].

4. Solution of diffusion equation by Haar wavelet

In the Haar domain, we assume that $\frac{\partial u}{\partial t}$ can be expanded as a Haar series as

$$\frac{\partial u}{\partial t}=a^t(x)H(t). \tag{26}$$

Integrating (26) and applying the integration matrix P of Section 3, we have

$$u(x,t)=a^t\int_0^t H(t)dt=a^tPH(t). \tag{27}$$

Using (26) and (27) in (7), we have

$$a^tH(t)=K(\ddot{a}^tPH(t))$$

$$\ddot{a}^t=ka^tP^{-1}, \quad \text{where } k=\frac{1}{K}, \tag{28}$$

$$\ddot{a}\triangleq\frac{d^2a}{dx^2}. \tag{29}$$

The differential equation (29) is solved using the conventional method. After dropping all the eigen values of (29) with positive real parts we have

$$a^t=a_0^te^{(-x\sqrt{\frac{1}{2k}})P^{-\frac{1}{2}}}, \tag{30}$$

$$\therefore\ u(x,t)=a_0^te^{(-x\sqrt{\frac{1}{K}})}PH(t). \tag{31}$$

Let us set

$$a_0^t=[0,0,\ldots 0,2m,\ldots,2m], \tag{32}$$

$$a_0^tP=[1,0,0,\ldots,0], \tag{33}$$

so that Eqs. (32) and (33) will satisfy the boundary conditions for $u(0,t)$. This means

$$u(0,t)=a_0^tPH(t)=h_0(t)=1. \tag{34}$$

The surface temperature $u(0,t)$ for typical values of k, u_0 and Δu_0 for the entire day was estimated using (8). $T(z,0)$ has been taken using (8) at 0th hour of the day were $0 < z < 45$ cm.

u_0 (phase difference in soil temperature) estimated using the expression $z\sqrt{\frac{1}{2\omega k}}$. The choices of m used in time discretization, construction of wavelet matrices and the constants have been fixed in accordance with these conditions. The variables involved herein are normalized so that the Figs. 1–3 for $u(x,t)$ are drawn for $x = 5$, 10, 20 in the time domain $[0,1)$.

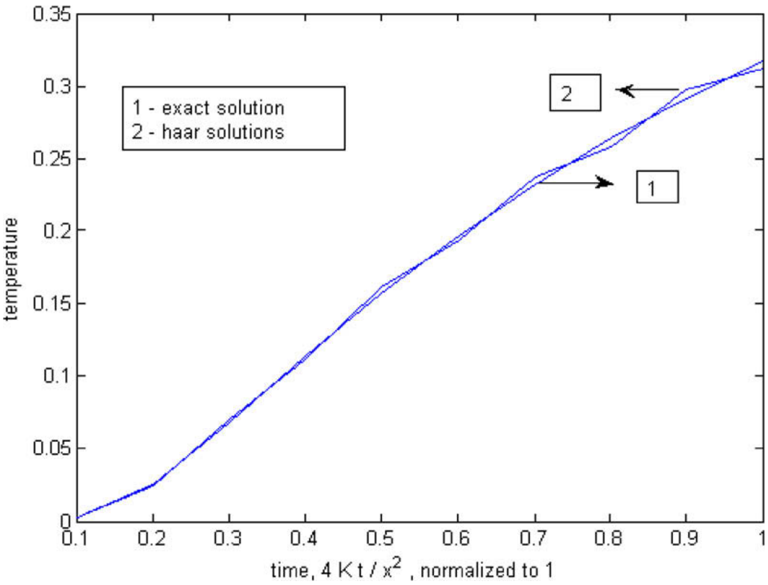

Fig. 1. $x = 20$ and $K = 12.5$.

$$\text{For a distance } x = \sqrt{\frac{k}{2}}, \quad a^t = a_0^t e^{-\frac{1}{2P}} \tag{35}$$

can be evaluated by applying MATLAB function expm($\cdot$). Then u can be obtained by matrix multiplication. The analytic solution has been described in [9] as

$$u(x,t) = erfc\left(\frac{x}{2\sqrt{kt}}\right), \tag{36}$$

where 'erfc' denotes the complementary error function.

The Haar solution and the analytical solution have been drawn together in Figs. 1–3 for $x = 5, 10, 20$ and for $t \in [0,1)$. It is worth mentioning that Haar solution provides excellent results even for small values of m (here $m = 16$). For larger values of

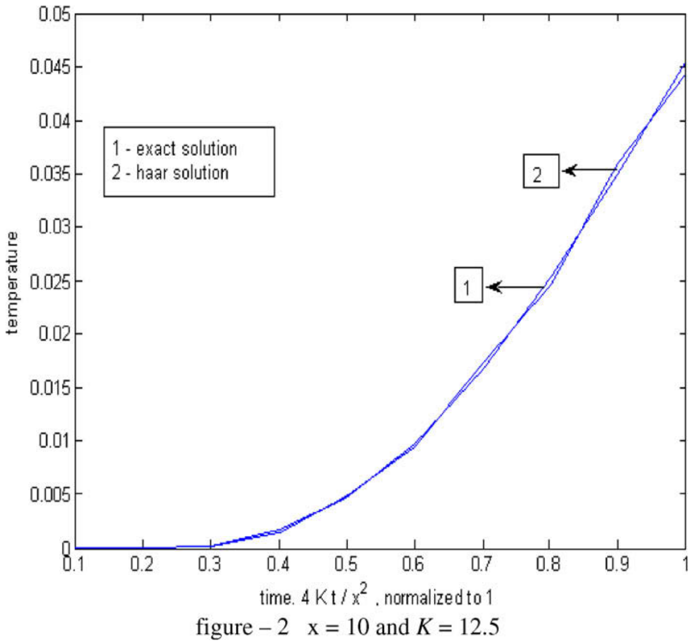

Fig. 2. $x = 10$ and $K = 12.5$.

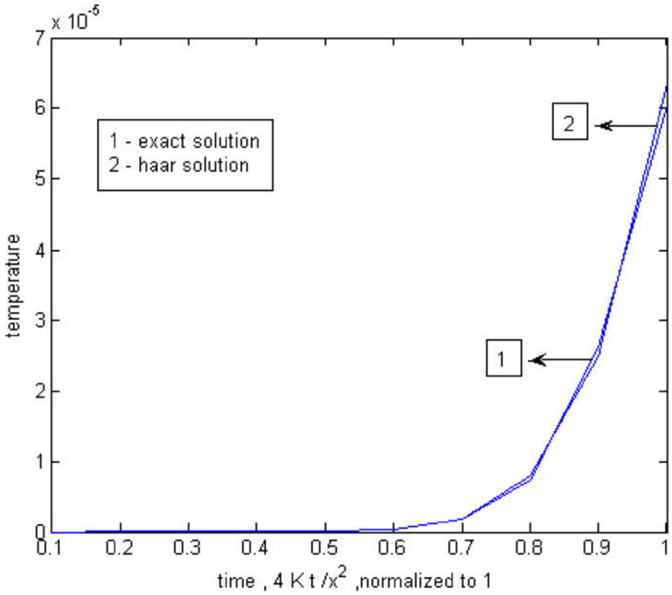

Fig. 3. $x = 5$ and $K = 12.5$.

m (*i.e.,* $m = 32$, $m = 64$), we can obtain the results closer to the real values. All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor $\times$ 86 Family 6 Model 15 Stepping 13 Genuine Intel $\sim$1596 Mhz.

5. Results and discussion

Soil temperature measurements at five specific depths namely 0, 5, 10, 20 and 40 cm at Trombay were made during January 2001 by Daoo and subsequently used by Dutta [2]. Here the same set of data is taken into account for comparison with the Haar wavelet solution for time ranging from 1 to 24 h and z values ranging from 0 to 45 cm. The values of k, u_0 and Δu_0 calculated from the data set for one day are used for comparison of the predicted soil temperature using Haar wavelet procedure at the specific depths. This comparisons along with the numerical solutions obtained by Dutta are exhibited Table 1. Errors at each stage are also compared.

Errors are shown as the ratio between the predicted and observed values. Haar wavelet based errors (E_H) and numerical method based errors (E_N) are also compared. The mean error depending upon the depth ranges between 2.72% for 20 cm depth to 9% for 5 cm depth. This is far below than the % of errors obtained by Dutta (which are 9.4 and 25.1, respectively) and the hour –wise errors are also found to be far below. This ensures that the errors are not only attributed to uncertainty in the initial conditions (ie., the depth profile of soil temperature at 0 h and time profile of soil temperature at 0 depth) but also on time discretization. Hence the Haar wavelet scheme is found to be a better tool for such speedy and accurate numerical computations [4–6]. The mean error of 2.72% may be attributed to other factors like specific heat capacity of water and water flux in soils, which factors are not considered in our modeling here. The main advantage of Haar wavelet method is its simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of signif-

Table 1
Haar values of (U) and observed (U_o) soil temperature (°C) at depths.

Hours	5 cm					10 cm					20 cm				
	U_o	U_H	U_N	% of E_H	% of E_N	U_o	U_H	U_N	% of E_H	% of E_N	U_o	U_H	U_N	% of E_H	% of E_N
2	16.0	21.1	24.2	31.8	51.0	21.4	23.6	27.7	10.3	29.3	24.3	25.3	26.5	4.1	9.0
6	14.4	13.9	21.6	−3.0	50.1	19.3	18.2	24.7	−5.7	27.8	22.2	19.8	26.5	−10.8	19.5
10	17.6	20.6	25.2	17.0	43.4	18.6	21.4	24.5	15.0	31.6	21.0	23.2	26.0	10.4	23.9
12	32.9	29.7	28.0	−9.7	−15.0	22.1	26.0	25.4	17.6	14.8	21.7	26.5	25.8	22.1	18.9
16	39.5	36.7	31.6	−7.1	−20.0	31.3	34.2	27.9	9.3	−10.9	26.2	27.4	25.9	4.6	−1.3
20	23.0	25.6	30.0	11.3	30.3	27.6	28.4	29.0	2.9	4.9	27.6	25.2	26.4	−8.7	4.3
22	20.2	26.1	27.5	29.2	35.9	25.3	27.6	28.5	9.0	12.6	26.7	26.0	26.7	−2.6	0.0
Mean of % error	–	–	–	9.9	25.1	–	–	–	8.3	15.7	–	–	–	2.7	9.4

U_o – observed temperature, U_H – Haar solution, and U_N – numerical solution.

icant wavelet coefficients. The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically.

6. Sensitivity analyses

Sensitivity analyses were used to determine the significance of: (1) the use of approximate thermal properties and (2) the use of fixed k values with respect to temperature. A sensitivity analysis of the Haar scheme was also performed with respect to the parameters k, u_0 and Δu_0 in order to substantiate their relative importance. We found that changes in thermal conductivity had little effect on u_0 but caused some differences in $u(z,t)$ in deep layers whereas changes in Δu_0 (amplitude of soil temperature wave at surface) have more effect on both u_0 and $u(z,t)$. Similar results were reported by Strathers et al. [15] that soil temperature was not sensitive to changes of soil thermal parameters.

7. Conclusion

The use of Haar wavelets in the engineering produces exciting results. The characteristics of wavelet transforms make them particularly useful for the approximation of functions with steep gradients or sharp spikes. Certainly the orthogonal and orthonormal properties of wavelet basis allow us to simplify the calculation of integrals. The Haar wavelet model described in this study provides reasonable estimates of soil temperature at any depth and time. If few more parameters like specific heat capacity of water and water fluxes in soil are also included; the model becomes a little bit complex so that further properties of wavelet transforms are required for still accurate computations of soil temperature. This work is currently ongoing.

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