MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2008; **31**:1545–1567
Published online 5 February 2008 in Wiley InterScience
(www.interscience.wiley.com) DOI: 10.1002/mma.983
MOS subject classification: 35 Q 60; 47 J 40

On a model for electromagnetic processes inside and outside a ferromagnetic body

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Communicated by P. Colli

SUMMARY

One-dimensional Maxwell's equations are considered in a ferromagnetic body surrounded by vacuum. Existence and uniqueness of solution for the resulting system of partial differential equations withhysteresis on the whole real line is proved under suitable constitutive hypotheses. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: electromagnetic processes; ferromagnetic materials; hysteresis

1. INTRODUCTION

The aim of this paper is to establish existence and uniqueness of a solution for the following model system:

$$\begin{cases} \dfrac{\partial E}{\partial t} + \chi_\Omega (E + E_{\text{app}}) + (1-\chi_\Omega) J_{\text{ext}} + \dfrac{\partial H}{\partial x} = 0 \\ \dfrac{\partial B}{\partial t} + \dfrac{\partial E}{\partial x} = 0 \\ H = \chi_\Omega \left(\overline{\mathscr{G}}(B) + \gamma \dfrac{\partial B}{\partial t} \right) + (1-\chi_\Omega) B \end{cases} \quad \text{in } \mathbb{R} \times (0,T) \tag{1}$$

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Received 11 June 2007

where Ω is an open-bounded interval of the real line, χ_Ω is the characteristic function of the set Ω, $\overline{\mathscr{G}}$ is a suitable scalar hysteresis operator, γ is a given positive constant while E_{app} and J_{ext} are known functions.

This system arises in the context of electromagnetic processes. We show in the following section how it can be obtained by coupling in a suitable way the Maxwell equations, the Ohm law and a constitutive relation between the magnetic field and the magnetic induction.

The fully hyperbolic case with $\gamma=0$ and canonical boundary conditions have been considered in [1] in the case of regular small amplitude oscillations inside the convex hysteresis domain. Existence of weak large amplitude solutions without the convexity hypothesis was recently proved in [2]. More about scalar hyperbolic equations with hysteresis can be found in the monographs [3, 4]. For $\gamma>0$, a system similar to (1) with prescribed boundary conditions was investigated in [5] (see also [6, Chapter 2]).

Here, we propose a problem that is closer to a realistic situation, where the electromagnetic field inside a body is determined by the external field. We thus consider the evolution in the whole space, replacing the boundary conditions by the continuity of E and H across the boundary.

We restrict ourselves to the one-dimensional case. There exist vector hysteresis models of ferromagnetism, see, e.g. [7–12] but they do not have the degree of regularity that is needed for our analysis.

Some details of the physical motivation for (1) are contained in Section 2; in Section 3 we recall some basic facts about hysteresis and hysteresis operators; Section 4 contains the statement of the main results which will be proved finally in Sections 5 and 6.

2. PHYSICAL MOTIVATION

Consider an electromagnetic process in a ferromagnetic material that occupies a Euclidean domain $Q\subset\mathbb{R}^3$ in a time interval $(0,T)$. For more details on these topics we refer, for example, to [13]. From now on we set

$$Q_T:=Q\times(0,T),\quad \mathbb{R}^3_T:=\mathbb{R}^3\times(0,T)$$

We suppose for simplicity that the electric displacement $\mathbf{D}$ is proportional to the electric field $\mathbf{E}$, that is $\mathbf{D}=\varepsilon\mathbf{E}$, where ε is the electric permittivity. We assume that ε is a scalar constant and moreover we introduce the electric conductivity σ which is supposed to vanish outside Q. We denote by $\mathbf{E}_{\mathrm{app}}$ a prescribed applied electromotive force; then the Ohm law for the electric current $\mathbf{J}$ is given by the following relation:

$$\mathbf{J}=\begin{cases}\sigma(\mathbf{E}+\mathbf{E}_{\mathrm{app}}) & \text{in } Q_T\\ \mathbf{J}_{\mathrm{ext}} & \text{in } [\mathbb{R}^3\setminus Q]\times(0,T)\end{cases}\tag{2}$$

Using the characteristic function χ_Q of the set Q, i.e. $\chi_Q=1$ inside Q and $\chi_Q=0$ outside Q, we rewrite (2) as

$$\mathbf{J}=\chi_Q\sigma(\mathbf{E}+\mathbf{E}_{\mathrm{app}})+(1-\chi_Q)\mathbf{J}_{\mathrm{ext}}\quad\text{in } \mathbb{R}^3_T\tag{3}$$

DOI: 10.1002/mma

Now we recall the Ampère and the Faraday laws

$$c\nabla\times\mathbf{H}=4\pi\mathbf{J}+\varepsilon\frac{\partial\mathbf{E}}{\partial t}\quad\text{in }\mathbb{R}^3_T \tag{4}$$

$$c\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\quad\text{in }\mathbb{R}^3_T \tag{5}$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the magnetic induction and c is the speed of light in vacuum.

Coupling (4) and (5) we obtain the following system:

$$\begin{cases}\nabla\times\mathbf{H}=\chi_Q(\mathbf{E}+\mathbf{E}_{\text{app}})+(1-\chi_Q)\mathbf{J}_{\text{ext}}+\dfrac{\partial\mathbf{E}}{\partial t}\\[2ex] \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}\end{cases}\quad\text{in }\mathbb{R}^3_T \tag{6}$$

where we neglected $\sigma, c, \varepsilon, \pi$, as our analysis does not depend on the exact value of these constants.

We further simplify our model system (6) by dealing with planar waves only. More precisely, we consider an electromagnetic wave moving in a plane, which we suppose to be orthogonal to the x-axis, in a domain $Q=\{(x,y,z): x\in\Omega, (y,z)\in\mathbb{R}^2\}$, with Ω being a bounded interval of the real line.

We assume that $\mathbf{E}$ is parallel to the y-axis, i.e.

$$\mathbf{E}(x,t)=(0,E(x,t),0),\quad (x,t)\in\mathbb{R}\times(0,T)$$

This in turn implies the following restriction on $\mathbf{B}$:

$$\mathbf{B}(x,t)=(0,0,B(x,t)),\quad (x,t)\in\mathbb{R}\times(0,T)$$

and therefore also $\mathbf{H}(x,t)=(0,0,H(x,t))$ for all $(x,t)\in\mathbb{R}\times(0,T)$. Thus, we have

$$\nabla\times\mathbf{H}=\left(0,-\frac{\partial H}{\partial x},0\right),\quad \nabla\times\mathbf{E}=\left(0,0,\frac{\partial E}{\partial x}\right)$$

and (6) reduces to a one-dimensional problem

$$\begin{cases}\dfrac{\partial E}{\partial t}+\chi_\Omega(E+E_{\text{app}})+(1-\chi_\Omega)J_{\text{ext}}+\dfrac{\partial H}{\partial x}=0\\[2ex] \dfrac{\partial B}{\partial t}+\dfrac{\partial E}{\partial x}=0\end{cases}\quad\text{in }\mathbb{R}\times(0,T) \tag{7}$$

where we also set $\mathbf{E}_{\text{app}}(x,t)=(0,E_{\text{app}}(x,t),0)$ and $\mathbf{J}_{\text{ext}}(x,t)=(0,J_{\text{ext}}(x,t),0)$.

We couple this system with an appropriate constitutive relation. We choose to relate B and H by means of a constitutive law with hysteresis inside Ω and to set $B=H$ outside Ω.

The constitutive law between B and H inside Ω will be chosen according to the 'rheological' circuit model F–L as in [4, pp. 54–55], where a ferromagnetic element

$$F: B^F=H^F+4\pi M=(I+\overline{\mathscr{W}})(H^F)$$

DOI: 10.1002/mma

where M is the magnetization and $\overline{\mathscr{W}}$ is a scalar Preisach operator, is coupled in series with an induction element

$$L: H^L = \gamma \frac{\partial B^L}{\partial t}$$

The general rheological rule for series combinations yields

$$B := B^F = B^L, \quad H := H^F + H^L$$

where B is the total induction and H is the total field. Summing up we obtain

$$H = \chi_\Omega \left(\overline{\mathscr{G}}(B) + \gamma \frac{\partial B}{\partial t} \right) + (1 - \chi_\Omega) B \tag{8}$$

where we set $\overline{\mathscr{G}} := (I + \overline{\mathscr{W}})^{-1}$. By coupling (7) and (8) we finally obtain (1).

3. HYSTERESIS OPERATORS

The theory of hysteresis has a long history. A hundred years ago, Madelung [14] proposed probably the first axiomatic approach to hysteresis by defining three experimental laws of what we call nowadays return point memory hysteresis (or 'wiping-out property', cf. [10]). The model for ferromagnetic hysteresis proposed by Preisach [15] is a prominent representative that possesses the return point memory property. Only recently, Brokate and Sprekels proved (see [16, Theorem 2.7.7]) that every return point memory hysteresis operator, which admits a specific initial memory configuration, has necessarily a Preisach-type memory structure. A basic mathematical theory of hysteresis operators has been developed by Krasnosel'skiĭ and his collaborators. The results of this group are summarized in the monograph [17], which constitutes until now the main source of reference on hysteresis. Our presentation here is based on more recent results from [3, 18] that are needed here, in particular the alternative one-parametric formulation of the Preisach model based on variational inequalities.

3.1. The Preisach operator

We describe the ferromagnetic behaviour using the Preisach model (see [15]). Mathematical aspects of this model were investigated by Krasnosel'skiĭ and Pokrovskiĭ (see [17, 19, 20]). The model has been also studied in connection with partial differential equations (PDEs) by Visintin (see, for example, [4, 21]). The monograph of Mayergoyz [10] is mainly devoted to its modelling aspects.

Here we use the one-parametric representation of the Preisach operator, which goes back to [1]. The starting point of our theory is the so-called *play operator*. This operator constitutes the simplest example of continuous hysteresis operator in the space of continuous functions; it has been introduced in [17] but we can also find equivalent definitions in [4, 16]; for its extension to less regular inputs, see also [22, 23].

Let $r > 0$ be a given parameter. For a given input function $u \in \mathscr{C}^0([0, T])$ and initial condition $x^0 \in [-r, r]$, we define the output $\xi = \mathscr{P}_r(x^0, u) \in \mathscr{C}^0([0, T]) \cap BV(0, T)$ of the *play operator*

$$\mathscr{P}_r : [-r, r] \times \mathscr{C}^0([0, T]) \to \mathscr{C}^0([0, T]) \cap BV(0, T)$$

DOI: 10.1002/mma

as the solution of the variational inequality in Stieltjes integral form

$$\begin{cases} \displaystyle\int_0^T (u(t)-\xi(t)-y(t))\,\mathrm{d}\xi(t)\geqslant 0 & \forall y\in\mathscr{C}^0([0,T]),\ \ \max_{0\leqslant t\leqslant T}|y(t)|\leqslant r \\ |u(t)-\xi(t)|\leqslant r & \forall t\in[0,T] \\ \xi(0)=u(0)-x^0 \end{cases} \tag{9}$$

Let us consider now the whole family of play operators $\mathscr{P}_r$ parameterized by $r>0$, which can be interpreted as a *memory variable*. Accordingly, we introduce the *hysteresis memory state space*

$$\Lambda:=\left\{\lambda:\mathbb{R}_+\to\mathbb{R}: |\lambda(r)-\lambda(s)|\leqslant|r-s|\ \forall r,s\in\mathbb{R}_+,\ \lim_{r\to+\infty}\lambda(r)=0\right\}$$

together with its subspaces

$$\Lambda_{\tilde{R}}=\{\lambda\in\Lambda:\lambda(r)=0 \text{ for } r\geqslant\tilde{R}\},\quad \Lambda_\infty=\bigcup_{\tilde{R}>0}\Lambda_{\tilde{R}} \tag{10}$$

For $\lambda\in\Lambda$, $u\in\mathscr{C}^0([0,T])$ and $r>0$ we set

$$\wp_r[\lambda,u]:=\mathscr{P}_r(x_r^0,u),\quad \wp_0[\lambda,u]:=u$$

where x_r^0 is given by the formula

$$x_r^0:=\min\{r,\max\{-r,u(0)-\lambda(r)\}\}$$

We set

$$A_\lambda(r,u(0)):=u(0)-x_r^0 \tag{11}$$

It turns out that

$$\wp_r:\Lambda\times\mathscr{C}^0([0,T])\to\mathscr{C}^0([0,T])$$

is Lipschitz continuous in the sense that, for every $u,v\in\mathscr{C}^0([0,T])$, $\lambda,\mu\in\Lambda$ and $r>0$ we have

$$\|\wp_r[\lambda,u]-\wp_r[\mu,v]\|_{\mathscr{C}^0([0,T])}\leqslant\max\{|\lambda(r)-\mu(r)|,\|u-v\|_{\mathscr{C}^0([0,T])}\} \tag{12}$$

Moreover, if $\lambda\in\Lambda_{\tilde{R}}$ and $\|u\|_{\mathscr{C}^0([0,T])}\leqslant\tilde{R}$, then $\wp_r[\lambda,u](t)=0$ for all $r\geqslant\tilde{R}$ and $t\in[0,T]$. For more details, see Sections II.3, II.4 of [3].

Now we introduce the *Preisach plane* as follows:

$$\mathscr{P}:=\{(r,v)\in\mathbb{R}^2:r>0\}$$

and consider a function $\varphi\in L^1_{\mathrm{loc}}(\mathscr{P})$ such that there exists $\beta_1\in L^1_{\mathrm{loc}}(0,\infty)$ with

$$0\leqslant\varphi(r,v)\leqslant\beta_1(r)\quad\text{for a.e. } (r,v)\in\mathscr{P}$$

We set

$$g(r,v):=\int_0^v\varphi(r,z)\,\mathrm{d}z\quad\text{for } (r,v)\in\mathscr{P}$$

and for $\tilde{R}>0$, we put $b_1(\tilde{R}):=\int_0^{\tilde{R}}\beta_1(r)\,\mathrm{d}r$.

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Then the *Preisach operator*

$$\mathscr{W}:\Lambda_\infty \times \mathscr{C}^0([0,T]) \to \mathscr{C}^0([0,T])$$

generated by the function g is defined by the formula

$$\mathscr{W}[\lambda,u](t) := \int_0^\infty g(r, \wp_r[\lambda,u](t))\,\mathrm{d}r \tag{13}$$

for any given $\lambda\in\Lambda_\infty$, $u\in\mathscr{C}^0([0,T])$ and $t\in[0,T]$. The equivalence of this definition and the classical one in [4, 10], e.g. is proved in [1].

The function $A_\lambda(r,\cdot)$ in (11) is nondecreasing and Lipschitz continuous. Hence, the mapping

$$\mathscr{A}_\lambda:\mathbb{R}\to\mathbb{R}: \qquad u(0)\mapsto \int_0^\infty g(r, A_\lambda(r,u(0)))\,\mathrm{d}r \tag{14}$$

with which the initial input value $u(0)$ associates the initial output value $\mathscr{W}[\lambda,u](0)$ is nondecreasing and locally Lipschitz continuous.

Using the Lipschitz continuity (12) of the operator $\wp_r$, it is easy to prove that also $\mathscr{W}$ is locally Lipschitz continuous, in the sense that, for any given $\tilde{R}>0$, for every $\lambda,\mu\in\Lambda_{\tilde{R}}$ and $u,v\in\mathscr{C}^0([0,T])$ with $\|u\|_{\mathscr{C}^0([0,T])}, \|v\|_{\mathscr{C}^0([0,T])}\leqslant\tilde{R}$, we have

$$\|\mathscr{W}[\lambda,u]-\mathscr{W}[\mu,v]\|_{\mathscr{C}^0([0,T])}\leqslant\int_0^{\tilde{R}} |\lambda(r)-\mu(r)|\beta_1(r)\,\mathrm{d}r + b_1(\tilde{R})\|u-v\|_{\mathscr{C}^0([0,T])}$$

The first result on the inverse Preisach operator was proved in [24]. We make use of the following formulation proved in [3, Section II.3].

Theorem 3.1
Let $\lambda\in\Lambda_\infty$ and $b>0$ be given. Then the operator $\mathscr{F}=bI+\mathscr{W}[\lambda,\cdot]:\mathscr{C}^0([0,T])\to\mathscr{C}^0([0,T])$ is invertible and its inverse $\mathscr{G}=\mathscr{F}^{-1}$ is Lipschitz continuous with Lipschitz constant $L_{\mathscr{G}}=2/b$. If moreover $u\in W^{1,1}(0,T)$, then for a.e. $t\in(0,T)$ we have

$$0\leqslant\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{G}[\lambda,u](t)\dot{u}(t)\leqslant\frac{1}{b}\dot{u}^2(t) \tag{15}$$

In particular, for any $\lambda\in\Lambda$, the initial value mapping $bI+\mathscr{A}_\lambda$ (see (14)) is increasing, locally Lipschitz and its inverse is Lipschitz.

As we are dealing with PDEs, we should consider both the input and the initial memory configuration λ that additionally depend on x. If, for instance, $\lambda(x,\cdot)$ belongs to Λ_∞ and $u(x,\cdot)$ belongs to $\mathscr{C}^0([0,T])$ for (almost) every x, then we define

$$\overline{\mathscr{W}}[\lambda,u](x,t) := \mathscr{W}[\lambda(x,\cdot),u(x,\cdot)](t) := \int_0^\infty g(r,\wp_r[\lambda(x,\cdot),u(x,\cdot)](t))\,\mathrm{d}r$$

DOI: 10.1002/mma

4. STATEMENT OF THE MAIN RESULTS

Let Ω be an open-bounded interval of $\mathbb{R}$ and set $\Omega_T := \Omega \times (0, T)$; let us fix an initial memory configuration

$$\lambda \in L^2(\Omega; \Lambda_{\tilde{R}}) \quad \text{for some } \tilde{R} > 0 \tag{16}$$

where $\Lambda_{\tilde{R}}$ is introduced in (10).

Let $\mathscr{M}(\Omega; \mathscr{C}^0([0, T]))$ be the space of strongly measurable functions $\Omega \to \mathscr{C}^0([0, T])$, i.e. the space of functions $v : \Omega \to \mathscr{C}^0([0, T])$ such that there exists a sequence v_n of simple functions with $v_n \to v$ in $\mathscr{C}^0([0, T])$ a.e. in Ω.

We fix a constant $b_{\mathscr{F}} > 0$ and introduce the operator $\overline{\mathscr{F}} : \mathscr{M}(\Omega; \mathscr{C}^0([0, T])) \to \mathscr{M}(\Omega; \mathscr{C}^0([0, T]))$ in the following way:

$$\overline{\mathscr{F}}(u)(x, t) := \mathscr{F}(u(x, \cdot))(t) := b_{\mathscr{F}} u(x, t) + \mathscr{W}[\lambda(x, \cdot), u(x, \cdot)](t) \tag{17}$$

where $\mathscr{W}$ is the scalar Preisach operator defined in (13).

Now Theorem 3.1 yields that $\mathscr{F}$ is invertible and its inverse $\mathscr{G} = \mathscr{F}^{-1}$ is a Lipschitz continuous operator in $\mathscr{C}^0([0, T])$ with Lipschitz constant $L_{\mathscr{G}} = 2/b_{\mathscr{F}}$.

At this point we introduce the operator

$$\overline{\mathscr{G}} : \mathscr{M}(\Omega; \mathscr{C}^0([0, T])) \to \mathscr{M}(\Omega; \mathscr{C}^0([0, T])), \quad \overline{\mathscr{G}} := \overline{\mathscr{F}}^{-1} \tag{18}$$

It turns out that

$$\overline{\mathscr{G}}(w)(x, t) := \mathscr{G}(w(x, \cdot))(t) \quad \forall w \in \mathscr{M}(\Omega; \mathscr{C}^0([0, T])) \tag{19}$$

it follows from Theorem 3.1 that $\overline{\mathscr{G}}$ is Lipschitz continuous in the following sense:

$$\|\overline{\mathscr{G}}(u_1)(x, \cdot) - \overline{\mathscr{G}}(u_2)(x, \cdot)\|_{\mathscr{C}^0([0,T])} \leqslant L_{\mathscr{G}} \|u_1(x, \cdot) - u_2(x, \cdot)\|_{\mathscr{C}^0([0,T])}$$
$$\text{for any } u_1, u_2 \in \mathscr{M}(\Omega; \mathscr{C}^0([0, T])) \text{ a.e. in } \Omega \tag{20}$$

The initial conditions for Problem (7)

$$\begin{aligned} E(x, 0) &:= E_0(x) \quad \text{a.e. in } \mathbb{R} \\ B(x, 0) &:= B_0(x) \quad \text{a.e. in } \mathbb{R} \end{aligned} \tag{21}$$

are assumed in the form

$$E_0(x) := \chi_\Omega E_0^1(x) + (1 - \chi_\Omega) E_0^2(x)$$

$$B_0(x) := \chi_\Omega B_0^1(x) + (1 - \chi_\Omega) B_0^2(x)$$

as in the following we will assume different regularities for the initial data inside and outside Ω.

The full PDE system for unknown functions E, B and H reads as follows:

$$\begin{cases} \dfrac{\partial E}{\partial t} + \chi_\Omega(E + E_{\text{app}}) + (1-\chi_\Omega) J_{\text{ext}} + \dfrac{\partial H}{\partial x} = 0 \\ \dfrac{\partial B}{\partial t} + \dfrac{\partial E}{\partial x} = 0 & \text{a.e. in } \mathbb{R} \times (0,T) \\ H = \chi_\Omega\left(\overline{\mathscr{G}}(B) + \gamma \dfrac{\partial B}{\partial t}\right) + (1-\chi_\Omega) B \end{cases} \tag{22}$$

with initial conditions (21), where E_{app} and J_{ext} are given functions.

For the sake of definiteness, we assume that $\Omega = (-1,1)$, fix some $r > 1$, and set $K = (-r, r)$.

We first distinguish the case in which the data have compact support; in this case, the solutions remain compactly supported as well due to finite speed of propagation. In Section 5 we prove the following existence result.

Theorem 4.1

Consider the following assumptions on the initial data

$$\begin{aligned} E_0^1 &\in H^2(\Omega), \quad B_0^1 \in H^1(\Omega) \\ E_0^2, B_0^2 &\in H^1(\mathbb{R}\setminus\overline{\Omega}), \quad E_0^2(x) = B_0^2(x) = 0 \quad \text{for } |x| \geqslant r \end{aligned} \tag{23}$$

together with the following compatibility conditions:

$$\begin{cases} \left(\tilde{\mathscr{A}}(B_0^1) - \gamma \dfrac{\partial E_0^1}{\partial x}\right)(-1^+) = B_0^2(-1^-) & \text{(24a)} \\ \left(\tilde{\mathscr{A}}(B_0^1) - \gamma \dfrac{\partial E_0^1}{\partial x}\right)(1^-) = B_0^2(1^+) & \text{(24b)} \\ E_0^1(-1^+) = E_0^2(-1^-), \quad E_0^1(1^-) = E_0^2(1^+) & \text{(24c)} \end{cases}$$

where

$$\tilde{\mathscr{A}} = (b_{\mathscr{F}} I + \mathscr{A}_\lambda)^{-1} \tag{25}$$

is the initial value mapping associated with $\overline{\mathscr{G}}$, see Theorem 3.1. Moreover, assume that

$$E_{\text{app}} \in H^1(0,T; L^2(\Omega)), \quad J_{\text{ext}} \in H^1(0,T; L^2(K \setminus \overline{\Omega}))$$

Then Problem (22) has a unique solution

$$\begin{aligned} E &\in W^{1,\infty}(0,T; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R})) \\ B_{|\Omega} &\in H^2(0,T; L^2(\Omega)) \\ B_{|\mathbb{R}\setminus\overline{\Omega}} &\in W^{1,\infty}(0,T; L^2(\mathbb{R}\setminus\overline{\Omega})) \\ H &\in H^1(0,T; L^2(\mathbb{R})) \cap L^\infty(0,T; H^1(\mathbb{R})) \end{aligned} \tag{26}$$

Finally, in Section 6 we deal with the case of more general data, not necessarily with compact support, and the result we are able to prove is the following.

DOI: 10.1002/mma

Theorem 4.2
Consider the following assumptions on the initial data:

$$E_0^1 \in H^2(\Omega), \quad B_0^1 \in H^1(\Omega)$$
$$E_0^2 \in H^1_{\mathrm{loc}}(\mathbb{R}\backslash\overline{\Omega}), \quad B_0^2 \in H^1_{\mathrm{loc}}(\mathbb{R}\backslash\overline{\Omega})$$

together with the compatibility conditions (24a), (24b) and (24c); moreover, assume that

$$E_{\mathrm{app}} \in H^1(0,T;L^2(\Omega)), \quad J_{\mathrm{ext}} \in H^1(0,T;L^2_{\mathrm{loc}}(\mathbb{R}\backslash\overline{\Omega}))$$

Then Problem (22) has a unique solution such that

$$\begin{aligned} E &\in W^{1,\infty}(0,T;L^2_{\mathrm{loc}}(\mathbb{R}))\cap L^2(0,T;H^1_{\mathrm{loc}}(\mathbb{R})) \\ B_{|\Omega} &\in H^2(0,T;L^2(\Omega)) \\ B_{|\mathbb{R}\backslash\overline{\Omega}} &\in W^{1,\infty}(0,T;L^2_{\mathrm{loc}}(\mathbb{R}\backslash\overline{\Omega})) \\ H &\in H^1(0,T;L^2_{\mathrm{loc}}(\mathbb{R}))\cap L^\infty(0,T;H^1_{\mathrm{loc}}(\mathbb{R})) \end{aligned} \tag{27}$$

5. PROOF OF THEOREM 4.1

The main idea in proving Theorem 4.1 is to use a space discretization scheme together with a fixed point argument.

Let us fix $R > r+T$; for simplicity (this will be useful in the space discretization procedure) we take $R \in \mathbb{N}$. This parameter gives a bound for the spatial domain where the evolution takes place. Outside the ferromagnetic body, the system has finite speed of propagation and the solution will be shown to vanish outside the space–time rectangle $[-R,R]\times[0,T]$.

We prescribe the following 'boundary conditions':

$$H(-R,t)=0, \quad E(R,t)=0 \tag{28}$$

Step 1: *Freezing*. First of all we fix some Z such that

$$Z \in H^1(0,T;L^2(\Omega)) \quad \text{with } Z(x,0)=B_0^1(x) \text{ a.e. in } \Omega \tag{29}$$

and consider (22) with $\overline{\mathscr{G}}(B)$ replaced by $\overline{\mathscr{G}}(Z)$, i.e. we look for three functions E,B,H with the regularity outlined in (26) and initial conditions (21) such that the following holds:

$$\begin{cases} \dfrac{\partial E}{\partial t}+\chi_\Omega(E+E_{\mathrm{app}})+(1-\chi_\Omega)J_{\mathrm{ext}}+\dfrac{\partial H}{\partial x}=0 \\ \dfrac{\partial B}{\partial t}+\dfrac{\partial E}{\partial x}=0 \\ H=\chi_\Omega\left(\overline{\mathscr{G}}(Z)+\gamma\dfrac{\partial B}{\partial t}\right)+(1-\chi_\Omega)B \end{cases} \quad \text{a.e. in } (-R,R)\times(0,T) \tag{30}$$

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Step 2: *Existence of Solutions for* (*30*): *Space Discretization Scheme*. We now fix $n\in\mathbb{N}$ and consider the equidistant partition of the interval $[-R,R]$

$$x_k := \frac{k}{n}, \quad k=-Rn, -Rn+1, \ldots, Rn$$

The characteristic function χ_Ω reduces to

$$\chi_k = \begin{cases} 1, & k\in\{-n+1,\ldots,n\} \\ 0 & \text{else} \end{cases}$$

We introduce the function

$$F(x,t) := \chi_\Omega E_{\text{app}}(x,t) + (1-\chi_\Omega) J_{\text{ext}}(x,t)$$

It turns out that

$$F\in H^1(0,T;L^2(\mathbb{R}))\subset L^2(\mathbb{R};\mathscr{C}^0([0,T])) \tag{31}$$

Now we set

$$F_k(t) = n\int_{(k-1)/n}^{k/n} F(x,t)\,\mathrm{d}x, \quad k=-Rn+1,\ldots,Rn, \ \ t\in[0,T]$$

$$G_k(t) = n\int_{(k-1)/n}^{k/n} \overline{\mathscr{G}}(Z)(x,t)\,\mathrm{d}x, \quad k=-n+1,\ldots,n, \ \ t\in[0,T]$$

and also

$$\mathscr{D}_n := \{-Rn+1,\ldots,-n\}\cup\{n+1,\ldots,Rn-1\}$$

We now approximate (30) by a system of ordinary differential equations, where the dot indicates the time derivative.

Our aim is to find unknown functions E_k, B_k, H_k such that the following holds, for $k=-Rn+1,\ldots,Rn-1$:

$$\begin{cases} \dot{E}_k + \chi_k E_k + n(H_k - H_{k-1}) + F_k = 0 & \text{(32a)} \\ \dot{B}_k + n\,(E_{k+1} - E_k) = 0 & \text{(32b)} \\ H_k = \chi_k[G_k + \gamma \dot{B}_k] + (1-\chi_k) B_k & \text{(32c)} \end{cases}$$

This is coupled with the boundary conditions

$$H_{-Rn}(t) = 0, \quad E_{Rn}(t) = 0, \ \ t\in[0,T] \tag{33}$$

and initial conditions

$$\begin{aligned} E_k(0) &= E_0^k := E_0\left(\frac{k-1}{n}\right) \\ B_k(0) &= B_0^k := n\int_{(k-1)/n}^{k/n} B_0(x)\,\mathrm{d}x \end{aligned} \tag{34}$$

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We choose the averages for $B_k(0)$ in order to avoid difficulties related to the fact that B_0 may be discontinuous.

Eliminating $\dot{B}_k$ from (32c) and H_k, H_{k-1} from (32a), we rewrite (32a)–(32b) as a system of $2(2Rn-1)$ equations for $2(2Rn-1)$ unknown functions

$$\dot{V}=\Phi V+\tilde{F},\quad V=(E_{-Rn+1},\ldots,E_{Rn-1},B_{-Rn+1},\ldots,B_{Rn-1})$$

where Φ is a matrix and $\tilde{F}\in W^{1,2}(0,T;\mathbb{R}^{2(2Rn-1)})$. This is enough to conclude that the system, coupled with (33) and (34) admits a unique global solution.

In the following, for the sake of simplicity, we denote by $C_1,C_2,\ldots$ any constant depending possibly on the data but independent on the discretization parameter n.

We now differentiate (32a)–(32c) in time, obtaining

$$\begin{cases}\ddot{E}_k+\chi_k\dot{E}_k+n(\dot{H}_k-\dot{H}_{k-1})+\dot{F}_k=0 & (35a)\\ \ddot{B}_k+n(\dot{E}_{k+1}-\dot{E}_k)=0 & (35b)\\ \dot{H}_k=\chi_k[\dot{G}_k+\gamma\ddot{B}_k]+(1-\chi_k)\dot{B}_k & (35c)\end{cases}$$

Now we test (35a) by $\dot{E}_k$ and (35b) by $\dot{H}_k$, sum the result for $k=-Rn+1,\ldots,Rn-1$ and divide by n. We have

$$\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k|^2+\frac{1}{n}\sum_{k=-n+1}^{n}|\dot{E}_k|^2+\sum_{k=-Rn+1}^{Rn-1}(\dot{H}_k-\dot{H}_{k-1})\dot{E}_k$$
$$+\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}\dot{F}_k\dot{E}_k+\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}\ddot{B}_k\dot{H}_k+\sum_{k=-Rn+1}^{Rn-1}(\dot{E}_{k+1}-\dot{E}_k)\dot{H}_k=0$$

We remark that

$$\sum_{k=-Rn+1}^{Rn-1}[(\dot{H}_k-\dot{H}_{k-1})\dot{E}_k+(\dot{E}_{k+1}-\dot{E}_k)\dot{H}_k]$$
$$=\sum_{k=-Rn+1}^{Rn-1}[\dot{E}_{k+1}\dot{H}_k-\dot{H}_{k-1}\dot{E}_k]=[\dot{E}_{Rn}\dot{H}_{Rn-1}-\dot{H}_{-Rn}\dot{E}_{-Rn+1}]\overset{(33)}{=}0$$

Therefore, using (35c), we deduce

$$\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k|^2+\frac{1}{n}\sum_{k=-n+1}^{n}|\dot{E}_k|^2+\frac{\gamma}{n}\sum_{k=-n+1}^{n}|\ddot{B}_k|^2+\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\mathscr{D}_n}|\dot{B}_k|^2$$
$$=-\frac{1}{n}\sum_{k=-n+1}^{n}\ddot{B}_k\dot{G}_k-\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}\dot{F}_k\dot{E}_k \tag{36}$$
$$\leqslant\frac{\gamma}{2n}\sum_{k=-n+1}^{n}|\ddot{B}_k|^2+\frac{1}{2\gamma n}\sum_{k=-n+1}^{n}|\dot{G}_k|^2+\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|\dot{F}_k|^2+\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k|^2$$

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This in particular entails

$$\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k|^2+\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\mathscr{D}_n}|\dot{B}_k|^2$$

$$\leqslant\frac{1}{2\gamma}\int_{\Omega}\left|\frac{\partial\overline{\mathscr{G}}(Z)}{\partial t}\right|^2(x,t)\,\mathrm{d}x+\int_{-R}^{R}\left|\frac{\partial F}{\partial t}\right|^2(x,t)\,\mathrm{d}x+\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k|^2$$

The Gronwall lemma, (31), (15) and (29) then yield

$$\frac{1}{n}\left(\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k|^2+\sum_{\mathscr{D}_n}|\dot{B}_k|^2\right)\leqslant\frac{C_1}{n}\left[\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k(0)|^2+\sum_{\mathscr{D}_n}|\dot{B}_k(0)|^2\right]+C_2$$

We have now to show that, due to our assumptions on the data, the term

$$\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k(0)|^2+\frac{1}{n}\sum_{\mathscr{D}_n}|\dot{B}_k(0)|^2 \tag{37}$$

can be controlled by a constant independent of the discretization parameter n.

First of all, by comparison and using (32a) we have that

$$\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k(0)|^2\leqslant\frac{1}{n}\sum_{k=-n+1}^{n}|E_k(0)|^2+n\sum_{k=-Rn+1}^{Rn-1}|H_k(0)-H_{k-1}(0)|^2$$

$$+\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|F_k(0)|^2 \tag{38}$$

As $E_0^1\in H^2(\Omega)$, the first term on the right-hand side of the previous inequality can be controlled while the third one is controlled due to (31).

To estimate the second term, we first note that for $k\in\mathscr{D}_n$, we have

$$H_k(0)=B_k(0)$$

while for $k'\in\{-n+1,\dots,n\}$, Equations (32b) and (32c) yield

$$H_{k'}(0)=G_{k'}(0)+\gamma\,\dot{B}_{k'}(0)=G_{k'}(0)-\gamma n\,(E_{k'+1}(0)-E_{k'}(0))$$

We have by Theorem 3.1, (25) and (29) for $x\in\Omega$ that

$$\overline{\mathscr{G}}(Z)(x,0)=\tilde{\mathscr{A}}(B_0^1(x))$$

hence

$$G_{k'}(0)=n\int_{(k'-1)/n}^{k'/n}\tilde{\mathscr{A}}(B_0^1(x))\,\mathrm{d}x,\quad k'\in\{-n+1,\dots,n\}$$

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This yields

$$n\sum_{k=-Rn+1}^{Rn-1}|H_k(0)-H_{k-1}(0)|^2$$

$$=n\sum_{k=-Rn+1}^{-n}|B_k(0)-B_{k-1}(0)|^2+n|H_{-n+1}(0)-H_{-n}(0)|^2$$

$$+n\sum_{k=-n+2}^{n}|G_k(0)-G_{k-1}(0)-\gamma n(E_{k+1}(0)-2E_k(0)+E_{k-1}(0))|^2$$

$$+n|H_{n+1}(0)-H_n(0)|^2+n\sum_{k=n+2}^{Rn-1}|B_k(0)-B_{k-1}(0)|^2 \tag{39}$$

Two terms deserve special attention. First, by the compatibility condition (24c) and by (34)

$$H_{-n+1}(0)-H_{-n}(0)=G_{-n+1}(0)-\gamma n(E_{-n+2}(0)-E_{-n+1}(0))-B_{-n}(0)$$

$$=n\int_{-1}^{-1+1/n}\tilde{\mathscr{A}}(B_0^1(x))\,\mathrm{d}x-\gamma n\left(E_0^1\left(-1+\frac{1}{n}\right)-E_0^1(-1^+)\right)$$

$$-n\int_{-1-1/n}^{-1}B_0^2(x)\,\mathrm{d}x$$

$$=n\int_{-1}^{-1+1/n}\tilde{\mathscr{A}}(B_0^1(x))\,\mathrm{d}x-\gamma n\int_{-1}^{-1+1/n}\frac{\partial E_0^1}{\partial x}(x)\,\mathrm{d}x-n\int_{-1-1/n}^{-1}B_0^2(x)\,\mathrm{d}x$$

$$=n\int_{-1}^{-1+1/n}[\tilde{\mathscr{A}}(B_0^1(x))-\tilde{\mathscr{A}}(B_0^1(-1^+))]\,\mathrm{d}x$$

$$-\gamma n\int_{-1}^{-1+1/n}\left[\frac{\partial E_0^1}{\partial x}(x)-\frac{\partial E_0^1}{\partial x}(-1^+)\right]\mathrm{d}x$$

$$-n\int_{-1-1/n}^{-1}[B_0^2(x)-B_0^2(-1^-)]\,\mathrm{d}x+\tilde{\mathscr{A}}(B_0^1(-1^+))$$

$$-\gamma\frac{\partial E_0^1}{\partial x}(-1^+)-B_0^2(-1^-)$$

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Using the compatibility condition (24a), we obtain

$$|H_{-n+1}(0)-H_{-n}(0)| \leqslant n\int_{-1}^{-1+1/n} |\tilde{\mathscr{A}}(B_0^1(x))-\tilde{\mathscr{A}}(B_0^1(-1^+))|\,\mathrm{d}x$$

$$+\gamma n\int_{-1}^{-1+1/n}\left|\frac{\partial E_0^1}{\partial x}(x)-\frac{\partial E_0^1}{\partial x}(-1^+)\right|\mathrm{d}x$$

$$+n\int_{-1-1/n}^{-1}|B_0^2(x)-B_0^2(-1^-)|\,\mathrm{d}x$$

The initial value mapping $\tilde{\mathscr{A}}$ is Lipschitz continuous. Hence, by Fubini's theorem

$$n\int_{-1}^{-1+1/n}|\tilde{\mathscr{A}}(B_0^1(x))-\tilde{\mathscr{A}}(B_0^1(-1^+))|\,\mathrm{d}x$$

$$\leqslant C_3 n\int_{-1}^{-1+1/n}\int_{-1}^{x}\left|\frac{\partial B_0^1}{\partial \xi}(\xi)\right|\mathrm{d}\xi\,\mathrm{d}x$$

$$=C_3 n\int_{-1}^{-1+1/n}\left|\frac{\partial B_0^1}{\partial \xi}(\xi)\right|\left(\int_{\xi}^{-1+1/n}\mathrm{d}x\right)\mathrm{d}\xi$$

$$\leqslant C_3\int_{-1}^{-1+1/n}\left|\frac{\partial B_0^1}{\partial \xi}(\xi)\right|\mathrm{d}\xi\leqslant\frac{C_3}{\sqrt{n}}\left(\int_{-1}^{-1+1/n}\left|\frac{\partial B_0^1}{\partial x}\right|^2\mathrm{d}x\right)^{1/2}$$

The other terms are treated similarly, so that

$$|H_{-n+1}(0)-H_{-n}(0)|\leqslant\frac{C_4}{\sqrt{n}}\left[\int_{-1}^{-1+1/n}\left(\left|\frac{\partial B_0^1}{\partial x}\right|^2+\left|\frac{\partial^2 E_0^1}{\partial x^2}\right|^2\right)\mathrm{d}x+\int_{-1-1/n}^{-1}\left|\frac{\partial B_0^2}{\partial x}\right|^2\mathrm{d}x\right]^{1/2}$$

We can estimate in a similar way the term $n|H_{n+1}(0)-H_n(0)|^2$, using this time the compatibility condition (24b).

The remaining terms in formula (39) can be estimated in a more standard way using the regularity of the data (23), and going back to (38) we obtain

$$\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k(0)|^2\leqslant C_5$$

On the other hand, by comparison, using (32b)

$$\frac{1}{n}\sum_{\mathscr{D}_n}|\dot{B}_k(0)|^2\leqslant n\sum_{\mathscr{D}_n}|E_{k+1}(0)-E_k(0)|^2$$

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and this can be controlled using (23), which concludes the estimate of the term in (37). Therefore, integrating (36) in time we deduce

$$\max_{0\leqslant t\leqslant T}\left[\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|\dot{E}_k(t)|^2+\frac{1}{n}\sum_{\mathscr{D}_n}|\dot{B}_k(t)|^2\right]+\frac{\gamma}{n}\int_0^T\left(\sum_{k=-n+1}^{n}|\ddot{B}_k(t)|^2\right)\mathrm{d}t\leqslant C_6 \tag{40}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=-n+1}^{n}|\dot{B}_k(t)|^2\right)^{1/2}\leqslant\left(\frac{1}{n}\sum_{k=-n+1}^{n}|\ddot{B}_k(t)|^2\right)^{1/2} \tag{41}$$

from this we deduce

$$\left(\frac{1}{n}\sum_{k=-n+1}^{n}|\dot{B}_k(t)|^2\right)^{1/2}\leqslant\left(\frac{1}{n}\sum_{k=-n+1}^{n}|\dot{B}_k(0)|^2\right)^{1/2}+\int_0^t\left(\frac{1}{n}\sum_{k=-n+1}^{n}|\ddot{B}_k(\tau)|^2\right)^{1/2}\mathrm{d}\tau \tag{42}$$

To estimate the first term on the right-hand side of (42), we use (32b) and the compatibility conditions (24c). From (40)–(42) we obtain

$$\max_{0\leqslant t\leqslant T}\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1}|\dot{B}_k(t)|^2\leqslant C_7 \tag{43}$$

and by comparison,

$$\max_{0\leqslant t\leqslant T} n\sum_{k=-Rn+1}^{Rn-1}|E_{k+1}(t)-E_k(t)|^2\leqslant C_8 \tag{44}$$

At this point, with the intention to let n tend to ∞, we define the following interpolates:

$$\begin{aligned}
E^{(n)}(x,t)&=E_k(t)+n\left(x-\frac{(k-1)}{n}\right)[E_{k+1}(t)-E_k(t)]\\
H^{(n)}(x,t)&=H_{k-1}(t)+n\left(x-\frac{(k-1)}{n}\right)[H_k(t)-H_{k-1}(t)]\\
\bar{E}^{(n)}(x,t)&=E_k(t)\\
\bar{H}^{(n)}(x,t)&=H_k(t)\\
\bar{B}^{(n)}(x,t)&=B_k(t)\\
\bar{G}^{(n)}(x,t)&=G_k(t)\\
\bar{\chi}^{(n)}(x)&=\chi_k\\
\bar{F}^{(n)}(x,t)&=F_k(t)
\end{aligned}$$

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for $x\in((k-1)/n,k/n]$, $k=-Rn+1,\ldots,Rn$ and $t\in[0,T]$. Therefore, Problem (32a)–(32c) can be rewritten as

$$
\begin{cases}
\dfrac{\partial\bar{E}^{(n)}}{\partial t}+\bar{\chi}^{(n)}\bar{E}^{(n)}+\dfrac{\partial H^{(n)}}{\partial x}+\bar{F}^{(n)}=0 & \text{(45a)}\\[2ex]
\dfrac{\partial\bar{B}^{(n)}}{\partial t}+\dfrac{\partial E^{(n)}}{\partial x}=0 & \text{(45b)}\\[2ex]
\bar{H}^{(n)}=\bar{\chi}^{(n)}\left(\bar{G}^{(n)}+\gamma\dfrac{\partial\bar{B}^{(n)}}{\partial t}\right)+(1-\bar{\chi}^{(n)})\bar{B}^{(n)} & \text{(45c)}
\end{cases}
$$

The *a priori* estimate (40) gives

$$\max_{0\leqslant t\leqslant T}\left(\left\|\frac{\partial E^{(n)}}{\partial t}(t)\right\|^2_{L^2(-R,R)}+\left\|\frac{\partial\bar{B}^{(n)}}{\partial t}(t)\right\|^2_{L^2((-R,R)\setminus\bar{\Omega})}\right)+\left\|\frac{\partial^2\bar{B}^{(n)}}{\partial t^2}\right\|^2_{L^2(0,T;L^2(\Omega))}\leqslant C_6 \tag{46}$$

At this point, by comparison, (31), (45a) and (46) give

$$\max_{0\leqslant t\leqslant T}\left\|\frac{\partial H^{(n)}}{\partial x}(t)\right\|^2_{L^2(-R,R)}\leqslant C_9$$

On the other hand, (43) entails

$$\left\|\frac{\partial\bar{B}^{(n)}}{\partial t}\right\|^2_{L^\infty(0,T;L^2(-R,R))}\leqslant C_7$$

while (44) gives

$$\left\|\frac{\partial E^{(n)}}{\partial x}\right\|^2_{L^\infty(0,T;L^2(-R,R))}\leqslant C_8$$

Combining the above estimates and possibly selecting a suitable subsequence of $n\to\infty$, we find that there exist functions E, B, H in the appropriate Sobolev spaces such that the following convergences take place:

$$\left.\frac{\partial E^{(n)}}{\partial t}\to\frac{\partial E}{\partial t},\quad \frac{\partial H^{(n)}}{\partial x}\to\frac{\partial H}{\partial x}\right\}\quad \text{weakly star in } L^\infty(0,T;L^2(-R,R))$$

$$\left.\frac{\partial^2\bar{B}^{(n)}}{\partial t^2}\to\frac{\partial^2 B}{\partial t^2}\right\}\quad \text{weakly in } L^2(0,T;L^2(\Omega))$$

$$\left.\frac{\partial\bar{B}^{(n)}}{\partial t}\to\frac{\partial B}{\partial t},\quad \frac{\partial E^{(n)}}{\partial x}\to\frac{\partial E}{\partial x}\right\}\quad \text{weakly star in } L^\infty(0,T;L^2(-R,R))$$

$$E^{(n)}\to E\quad \text{uniformly in } \mathscr{C}([-R,R]\times[0,T])$$

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Note that

$$|E^{(n)}(x,t)-\bar{E}^{(n)}(x,t)|^2 \leqslant \sum_{k=-Rn+1}^{Rn-1} |E_{k+1}(t)-E_k(t)|^2 \overset{(44)}{\leqslant} \frac{C_8}{n}$$

hence

$$\frac{\partial \bar{E}^{(n)}}{\partial t} \to \frac{\partial E}{\partial t} \quad \text{weakly star in } L^\infty(0,T;L^2(-R,R))$$

Therefore, we can pass to the limit in (45a)–(45c) to see that (30) is satisfied a.e. in $(-R,R)\times(0,T)$ and the solution has the regularity outlined in (26).

As a consequence of considerations below in Step 4, we will see that the solution to Problem (30) is unique.

Step 3: *Finite speed of propagation*.

In order to find solutions to (30), we used a space discretization scheme; by doing that, we actually solved a boundary value problem in the domain $(-R,R)\times(0,T)$. We now show that system (30) is satisfied in the whole strip $\mathbb{R}\times(0,T)$.

Our argument is illustrated in Figure 1. We define the sets

$$A_T^+ = \{(x,t)\in\mathbb{R}\times(0,T);\, t\in(0,T),\, r+t<x<R\}$$

$$A_T^- = \{(x,t)\in\mathbb{R}\times(0,T);\, t\in(0,T),\, -R<x<-r-t\}$$

with the intention to prove that $E\equiv H\equiv 0$ in $A_T^+\cup A_T^-$. In A_T^+, E and H are solutions of the linear wave equation

$$\begin{cases} \dfrac{\partial E}{\partial t}+\dfrac{\partial H}{\partial x}=0 \\ \dfrac{\partial H}{\partial t}+\dfrac{\partial E}{\partial x}=0 \end{cases} \tag{47}$$

hence, satisfy the energy balance

$$\frac{\partial}{\partial t}\left(\frac{1}{2}(E^2+H^2)\right)+\frac{\partial}{\partial x}(EH)=0 \tag{48}$$

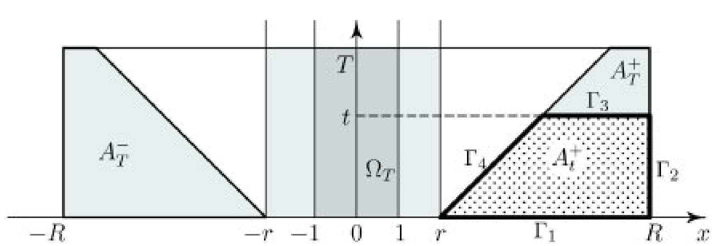

Figure 1. Finite speed of propagation outside Ω.

DOI: 10.1002/mma

We now proceed as in [25] and integrate (48) over the set $A_t^+ = A_T^+ \cap (\mathbb{R} \times (0,t))$ with some fixed $t \in (0,T)$. Using Green's formula, we obtain

$$\int_{A_t^+} \frac{\partial}{\partial t}\left(\frac{1}{2}(E^2+H^2)\right) + \frac{\partial}{\partial x}(EH)\,\mathrm{d}x\,\mathrm{d}t = \int_{\partial A_t^+}\left(EHn_1 + \frac{1}{2}(E^2+H^2)n_2\right)\mathrm{d}s \tag{49}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit outward normal vector to ∂A_t^+. The boundary of A_t^+ consists of four parts: $\partial A_t^+ = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, with $\mathbf{n} = (0,-1)$ on Γ_1, $\mathbf{n} = (1,0)$ on Γ_2, $\mathbf{n} = (0,1)$ on Γ_3 and $\mathbf{n} = (-1/\sqrt{2}, 1/\sqrt{2})$ on Γ_4. We have

$$\int_{\Gamma_1}\left(EHn_1 + \frac{1}{2}(E^2+H^2)n_2\right)\mathrm{d}s = -\int_r^R \frac{1}{2}[(E_0^2)^2(x) + (B_0^2)^2(x)]\,\mathrm{d}x = 0$$

$$\int_{\Gamma_2}\left(EHn_1 + \frac{1}{2}(E^2+H^2)n_2\right)\mathrm{d}s = \int_0^t E(R,\tau)H(R,\tau)\,\mathrm{d}\tau = 0$$

$$\int_{\Gamma_3}\left(EHn_1 + \frac{1}{2}(E^2+H^2)n_2\right)\mathrm{d}s = \int_{r+t}^R \frac{1}{2}(E^2(x,t) + H^2(x,t))\,\mathrm{d}x \geqslant 0$$

$$\int_{\Gamma_4}\left(EHn_1 + \frac{1}{2}(E^2+H^2)n_2\right)\mathrm{d}s = \frac{1}{2\sqrt{2}}\int_0^t (E-H)^2(r+\tau,\tau)\,\mathrm{d}\tau \geqslant 0$$

Comparing these identities with (48) and (49), we see that $E = H = 0$ on Γ_3. As $t \in (0,T)$ has been arbitrary, we obtain the desired result $E = H = 0$ in A_T^+.

The argument in A_T^- is fully analogous.

Step 4: *Fixed point*. Let us come back to our original problem. For any $Z \in H^1(0,T;L^2(\Omega))$ we found a solution (E,B,H) to Problem (30) with the regularity outlined in (26). We introduce the following closed subspace S of $H^1(0,T;L^2(\Omega))$

$$S := \{Z \in H^1(0,T;L^2(\Omega)) : Z(x,0) = B_0^1(x)\}$$

We take two different data $Z_1, Z_2 \in S$ and consider corresponding solutions to (30), (E_1, B_1, H_1) and (E_2, B_2, H_2), respectively, associated with Z_1 and Z_2. We obtain

$$\begin{cases} \dfrac{\partial}{\partial t}(E_1 - E_2) + \chi_\Omega(E_1 - E_2) + \dfrac{\partial}{\partial x}(H_1 - H_2) = 0 & \text{(50a)} \\ \dfrac{\partial}{\partial t}(B_1 - B_2) + \dfrac{\partial}{\partial x}(E_1 - E_2) = 0 & \text{(50b)} \\ H_1 - H_2 = \chi_\Omega\left[\overline{\mathscr{G}}(Z_1) - \overline{\mathscr{G}}(Z_2) + \gamma\dfrac{\partial}{\partial t}(B_1 - B_2)\right] + (1-\chi_\Omega)(B_1 - B_2) & \text{(50c)} \end{cases}$$

We test (50a) by $(E_1 - E_2)$ and (50b) by $(H_1 - H_2)$ and then sum the result, taking into account that the terms

$$\int_{-R}^R \frac{\partial}{\partial x}(H_1 - H_2)(E_1 - E_2)\,\mathrm{d}x, \quad \int_{-R}^R \frac{\partial}{\partial x}(E_1 - E_2)(H_1 - H_2)\,\mathrm{d}x$$

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cancel out, we deduce, using (50c)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-R}^{R}|E_1-E_2|^2\,\mathrm{d}x+\int_{\Omega}|E_1-E_2|^2\,\mathrm{d}x+\frac{3}{4}\gamma\int_{\Omega}\left|\frac{\partial}{\partial t}(B_1-B_2)\right|^2\mathrm{d}x$$

$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{(-R,R)\backslash\overline{\Omega}}|B_1-B_2|^2\,\mathrm{d}x\leqslant\frac{1}{\gamma}\int_{\Omega}|\overline{\mathscr{G}}(Z_1)-\overline{\mathscr{G}}(Z_2)|^2\,\mathrm{d}x$$

We remark that this last estimate together with the causality of the operator $\overline{\mathscr{G}}$ entails that (30) admits a unique solution, for any fixed $Z\in H^1(0,T;L^2(\Omega))$. This enables us to define the operator $J:S\to S$, which associates with every fixed datum $Z\in S$ the component $B\in S$ of the corresponding solution (E,B,H).

We now set

$$D(t):=\frac{1}{2}\left[\int_{\mathbb{R}}|E_1-E_2|^2\,\mathrm{d}x+\int_{\mathbb{R}\backslash\overline{\Omega}}|B_1-B_2|^2\,\mathrm{d}x\right]$$

and we remark that $D(0)=0$. Thus, in particular we deduce

$$\frac{\mathrm{d}D}{\mathrm{d}t}+\frac{3}{4}\gamma\int_{\Omega}\left|\frac{\partial}{\partial t}(B_1-B_2)\right|^2\mathrm{d}x\leqslant\frac{1}{\gamma}\int_{\Omega}|\overline{\mathscr{G}}(Z_1)-\overline{\mathscr{G}}(Z_2)|^2\,\mathrm{d}x \tag{51}$$

On the other hand,

$$\begin{aligned}\int_{\Omega}[\overline{\mathscr{G}}(Z_1)(x,t)-\overline{\mathscr{G}}(Z_2)(x,t)]^2\,\mathrm{d}x &\overset{(20)}{\leqslant} L_{\mathscr{G}}^2\int_{\Omega}\|Z_1(x,\cdot)-Z_2(x,\cdot)\|^2_{\mathscr{C}^0([0,t])}\,\mathrm{d}x\\ &\leqslant L_{\mathscr{G}}^2\int_{\Omega}\left(\int_0^t\left|\frac{\partial}{\partial\tau}(Z_1-Z_2)\right|(x,\tau)\,\mathrm{d}\tau\right)^2\mathrm{d}x\\ &\leqslant L_{\mathscr{G}}^2 t\int_0^t\int_{\Omega}\left|\frac{\partial}{\partial\tau}(Z_1-Z_2)\right|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned} \tag{52}$$

where we used the fact that $Z_1(0,x)=Z_2(0,x)$, a.e. in Ω, as $Z_1,Z_2\in S$.

We now introduce an equivalent norm on $H^1(0,T;L^2(\Omega))$: for all $z\in H^1(0,T;L^2(\Omega))$

$$|||z|||:=\left(\|z(0)\|^2_{L^2(\Omega)}+\int_0^T\exp\left(-\frac{2L_{\mathscr{G}}^2t^2}{\gamma^2}\right)\left\|\frac{\partial z}{\partial t}\right\|^2_{L^2(\Omega)}(t)\,\mathrm{d}t\right)^{1/2}$$

At this point we divide (51) by γ, multiply the result by $\exp(-2L_{\mathscr{G}}^2t^2/\gamma^2)$ and integrate in time, for $t\in(0,T)$. We first remark that

$$\int_0^T\exp\left(-\frac{2L_{\mathscr{G}}^2t^2}{\gamma^2}\right)\frac{\mathrm{d}D}{\mathrm{d}t}(t)\,\mathrm{d}t=\exp\left(-\frac{2L_{\mathscr{G}}^2T^2}{\gamma^2}\right)D(T)+\int_0^T D(t)\exp\left(-\frac{2L_{\mathscr{G}}^2t^2}{\gamma^2}\right)\frac{4L_{\mathscr{G}}^2t}{\gamma^2}\,\mathrm{d}t\geqslant 0$$

DOI: 10.1002/mma

and therefore we have

$$\begin{aligned}\frac{3}{4}|||J(Z_1)-J(Z_2)|||^2 &\leqslant \int_0^T \exp\left(-\frac{2L_{\mathscr{G}}^2 t^2}{\gamma^2}\right)\frac{L_{\mathscr{G}}^2 t}{\gamma^2}\int_0^t\int_\Omega\left|\frac{\partial}{\partial\tau}(Z_1-Z_2)\right|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\ &= -\frac{1}{4}\exp\left(-\frac{2L_{\mathscr{G}}^2 T^2}{\gamma^2}\right)\int_0^T\int_\Omega\left|\frac{\partial}{\partial t}(Z_1-Z_2)\right|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad +\frac{1}{4}\int_0^T\exp\left(-\frac{2L_{\mathscr{G}}^2 t^2}{\gamma^2}\right)\int_\Omega\left|\frac{\partial}{\partial t}(Z_1-Z_2)\right|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

which in turn gives

$$|||J(Z_1)-J(Z_2)|||^2\leqslant\tfrac{1}{3}|||Z_1-Z_2|||^2$$

Hence, J is a contraction on the closed subset S of $H^1(0,T;L^2(\Omega))$, which finishes the proof. □

6. PROOF OF THEOREM 4.2

We now deal with the general case.

For data as in Theorem 4.2, we define sequences indexed by $n\in\mathbb{N}$ of truncated data with supports in $(-2n,2n)$ for $n\to\infty$ as

$$J_{\mathrm{ext}}^n(x,t)=\chi_{(-n,n)}(x)J_{\mathrm{ext}}(x,t)$$

$$(E_0^2)^n(x)=\begin{cases}E_0^2(x) & \text{for } x\in(-n,n)\\ \left(2-\dfrac{x}{n}\right)E_0^2(n) & \text{for } x\in[n,2n)\\ \left(2+\dfrac{x}{n}\right)E_0^2(-n) & \text{for } x\in(-2n,-n]\\ 0 & \text{for } |x|\geqslant 2n\end{cases}$$

$$(B_0^2)^n(x)=\begin{cases}B_0^2(x) & \text{for } x\in(-n,n)\\ \left(2-\dfrac{x}{n}\right)B_0^2(n) & \text{for } x\in[n,2n)\\ \left(2+\dfrac{x}{n}\right)B_0^2(-n) & \text{for } x\in(-2n,-n]\\ 0 & \text{for } |x|\geqslant 2n\end{cases}$$

By Theorem 4.1, there exists a sequence of solutions $\{(E^n,B^n,H^n)\}_{n\in\mathbb{N}}$ associated with these data. We now refer again to the Courant–Hilbert trick and show that on every domain

$$U_r=\{(x,t)\in\mathbb{R}\times(0,T);\ 0\leqslant t\leqslant T,\ -r-2T+t\leqslant x\leqslant r+2T-t\}$$

(see Figure 2) with an arbitrarily fixed parameter $r>1$, all these solutions with $n>r+2T$ coincide.

DOI: 10.1002/mma

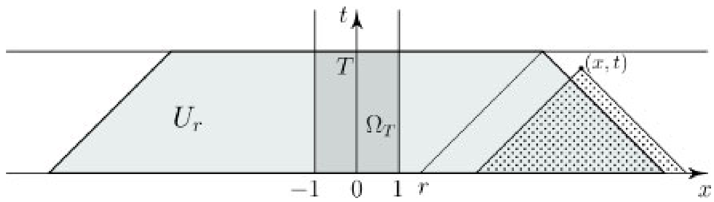

Figure 2. Set U_r and domain of dependence for (x,t) outside U_r.

Let us take two solutions (E^j, B^j, H^j) and (E^k, B^k, H^k), with $j\neq k$, $j,k>r+2T$. We set $\bar{E}:=E^j-E^k$, $\bar{B}:=B^j-B^k$, $\bar{H}:=H^j-H^k$, $\bar{G}:=\overline{\mathscr{G}}(B^j)-\overline{\mathscr{G}}(B^k)$. Then we obtain

$$\begin{cases} \dfrac{\partial\bar{E}}{\partial t}+\chi_\Omega\bar{E}+\dfrac{\partial\bar{H}}{\partial x}=0 & (53a)\\ \dfrac{\partial\bar{B}}{\partial t}+\dfrac{\partial\bar{E}}{\partial x}=0 & (53b)\\ \bar{H}=\chi_\Omega\left(\bar{G}+\gamma\dfrac{\partial\bar{B}}{\partial t}\right)+(1-\chi_\Omega)\bar{B} & (53c)\end{cases}$$

We test (53a) by $\bar{E}$, (53b) by $\bar{H}$ and integrate over U_r. We obtain

$$\int_{\Omega_T}\left(\bar{E}^2+\gamma\left(\frac{\partial\bar{B}}{\partial t}\right)^2+\bar{G}\frac{\partial\bar{B}}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t + \int_{U_r}\left[\frac{\partial}{\partial t}\left(\frac{1}{2}(\bar{E}^2+(1-\chi_\Omega)\bar{B}^2)\right)+\frac{\partial}{\partial x}(\bar{E}\bar{H})\right]\mathrm{d}x\,\mathrm{d}t=0 \tag{54}$$

The set ∂U_r consists of four straight segments, $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3, \tilde{\Gamma}_4$, with endpoints, respectively, $(-r-2T,0)$ and $(r+2T,0)$, $(r+2T,0)$ and $(r+T,T)$, $(r+T,T)$ and $(-r-T,T)$, $(-r-T,T)$ and $(-r-2T,0)$, see Figure 2.

Now, the Green formula yields

$$\begin{aligned}\int_{U_r}&\left[\frac{\partial}{\partial t}\left(\frac{1}{2}(\bar{E}^2+(1-\chi_\Omega)\bar{B}^2)\right)+\frac{\partial}{\partial x}(\bar{E}\bar{H})\right]\mathrm{d}x\,\mathrm{d}t\\ &=-\int_{\tilde{\Gamma}_1}\frac{1}{2}(\bar{E}^2+(1-\chi_\Omega)\bar{B}^2)\,\mathrm{d}s+\int_{\tilde{\Gamma}_2}\left[\frac{1}{\sqrt{2}}(\bar{E}\bar{H})+\frac{1}{2\sqrt{2}}(\bar{E}^2+(1-\chi_\Omega)\bar{B}^2)\right]\mathrm{d}s\\ &\quad+\int_{\tilde{\Gamma}_3}\frac{1}{2}(\bar{E}^2+(1-\chi_\Omega)\bar{B}^2)\,\mathrm{d}s+\int_{\tilde{\Gamma}_4}\left[-\frac{1}{\sqrt{2}}(\bar{E}\bar{H})+\frac{1}{2\sqrt{2}}(\bar{E}^2+(1-\chi_\Omega)\bar{B}^2)\right]\mathrm{d}s\end{aligned}$$

The integral over $\tilde{\Gamma}_1$ vanishes as the initial data for (E^j, B^j, H^j) and (E^k, B^k, H^k) coincide while the integral over $\tilde{\Gamma}_3$ yields a nonnegative contribution. Finally, taking into account that

$$\tilde{\Gamma}_2\cap\Omega_T=\emptyset,\quad \tilde{\Gamma}_4\cap\Omega_T=\emptyset$$

we have that the integrals over $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_4$ give a nonnegative contribution, as $\bar{H}=\bar{B}$.

DOI: 10.1002/mma

At this point (54) entails in particular

$$\int_{\Omega_T}\left(\gamma\left(\frac{\partial\bar{B}}{\partial t}\right)^2+\bar{G}\frac{\partial\bar{B}}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t\leqslant 0$$

This in turns gives, working as in (52)

$$\frac{\gamma}{2}\int_{\Omega_T}\left|\frac{\partial\bar{B}}{\partial t}\right|^2\mathrm{d}x\,\mathrm{d}t\leqslant\frac{1}{2\gamma}\int_0^T\int_\Omega|\bar{G}|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\int_0^T\frac{L_{\mathscr{G}}^2}{2\gamma}t\int_0^t\int_\Omega\left|\frac{\partial\bar{B}}{\partial\tau}\right|^2\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t$$

From the Gronwall lemma, we finally obtain $\bar{B}=0$ and therefore, by comparison, $\bar{E}=\bar{H}=0$. Hence, all solutions coincide on U_r, independently on the way we constructed the sequence of data with compact support.

Outside U_r the solution of (22) is given by the explicit formula for the solution of the linear wave equation. More precisely, outside U_r (22) reduces to the following:

$$\begin{cases}\dfrac{\partial E}{\partial t}+J_{\mathrm{ext}}+\dfrac{\partial B}{\partial x}=0\\[2mm] \dfrac{\partial B}{\partial t}+\dfrac{\partial E}{\partial x}=0\end{cases}$$

This means that

$$\frac{\partial^2 E}{\partial t^2}-\frac{\partial^2 E}{\partial x^2}=-\frac{\partial J_{\mathrm{ext}}}{\partial t}=:\tilde{J}$$

and the initial conditions are

$$E(x,0)=E_0^2,\quad \frac{\partial E}{\partial t}(x,0)=-\frac{\partial B}{\partial x}(x,0)-J_{\mathrm{ext}}(x,0)=:\tilde{E}$$

The solution for the linear wave equation can be represented as

$$E(x,t)=\frac{1}{2}[E_0^2(x+t)-E_0^2(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}\tilde{E}(\xi)\,\mathrm{d}\xi+\frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau}\tilde{J}(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

for $(x,t)\in[\mathbb{R}\times(0,T)]\setminus U_r$, and the assertion follows. $\square$

ACKNOWLEDGEMENTS

The authors wish to thank Gary Friedman for many useful discussions concerning the physical motivation of the model problem.

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DOI: 10.1002/mma