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THE POSTULATE OF REALIZABILITY: FORMULATION AND APPLICATIONS TO THE POST-BIFURCATION BEHAVIOUR AND PHASE TRANSITIONS IN ELASTOPLASTIC MATERIALS—I

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Abstract—The governing extremum principle for the description of stable post-bifurcation processes in elastoplastic materials with associated and nonassociated flow rules is substantiated. To derive it, a new thermomechanical postulate, called the postulate of realizability is introduced. The postulate of realizability was applied to obtain some known results (associated flow rule in classical plasticity, relations between dissipative forces and the rates for time-dependent behaviour, some extremum principles for a finite volume of perfect rigid-plastic material), as well as some new ones (nonassociated flow rules and more complex relations between dissipative forces and the rates for more complicated models, some extremum principles for a finite volume of elastoplastic material). This means that the postulate of realizability is a quite powerful and flexible tool in the theory of dissipative processes and the derived extremum principles can be considered as well grounded. Indeed, the concept of stability following from these principles has a clear physical meaning.

1. INTRODUCTION

A quasi-static instability in an inelastic region can appear in various forms: for instance as buckling, necking, shear band formation or phase transition. In contrast to purely elastic materials, for which the instability is connected with the loss of stability of the equilibrium state (at constant loading), the loss of stability of the deformation process (at varying loading) is of main importance in the elastoplastic region (Klushnikov [1]; Petryk [2]; Bažant [3]). If more than one solution is possible at some point in the deformation process under a given increment of the prescribed external quantities (forces, displacements), it is necessary (a) to determine the first time, when bifurcation takes place; (b) to find all possible solutions of the boundary value problem; (c) to choose the stable solution which includes the definition of the corresponding concept; (d) to describe the whole stable post-bifurcation process (it is possible that items (ii) and (iii) may have to be fulfilled at each point of the post-bifurcation path). Phase transitions (PTs) represent typical post-bifurcational phenomena. PT starts when some crystal lattice or, in continuum description, deformation process loses its stability under thermomechanical loading. For elastic materials the principle of minima of the free energy is used to describe the process of equilibrium PT as a consequence of equilibrium states [4]. For PT in elastoplastic materials or in elastic materials if we take into account dissipation related with PT [5, 6] this principle is not applicable and corresponding principle is lacking. It is natural to try to describe PT as the stable post-bifurcation process in dissipative materials.

Let us consider some known approaches for the description of the post-bifurcation behaviour (a comprehensive survey is not a goal of this paper).

Hill's [7, 8] bifurcation theory was extended to the initial post-bifurcation range by Hutchinson [9, 10]. Nguyen and Triantafyllidis [11] considered an arbitrary high-order rate problem. A number of results for the post-bifurcation behaviour of materials and structures were presented by Needleman and Tvergaard [12]. Some extremum principles for choosing the post-bifurcation path were suggested by Bažant [13] and Petryk and Thermann [14]. They will be considered in Part II.

At PT the jump of the deformation gradient and (for noncoherent PT) the position vector takes place and the moving discontinuity surface (interface) appears. Complete description of PT should include PT criteria, relation for jumps of all thermomechanical parameters across the interface and equation of interface motion at each stage of PT. That is why not one of existing approaches could be applied directly for materials with PT.

PTs in inelastic solids were considered in several contributions at small [5], [15–21] and large strain [20, 22, 23], but they were not considered from the aspect of instability.

The goal of this paper is to develop a general thermomechanical approach for the description of the post-bifurcational behaviour in time-independent inelastic materials without and with PT (some preliminary results were published in [23]). The aim of this Part is to substantiate the governing extremum principles for the choice of the stable post-bifurcation process among a number of possible (unstable) ones for materials without PT. The paper is based on a new proposed thermomechanical postulate called the postulate of realizability. The main essence of this postulate is as follows: if only some dissipative process (plastic flow, PT) can occur, it will occur, i.e. the first fulfilment of the necessary energetic condition is sufficient for the beginning of the dissipative process. The strict formulation of the postulate of realizability is given in Section 2. Using this postulate, the relation between dissipative stress and the rate on inelastic deformation gradient is derived along with corresponding extremum principles for rigid-plastic and elastoplastic materials at small and finite strain. From these principles the associated flow rule for classical material models follows (without additional postulates, e.g. by Drucker, Il'yushin). The nonassociated flow rule follows for materials with kinematic constraints, when the dissipation function depends on reactions of these constraints, as well as for material models with structural changes and concave yield surface (Levitas [24–26]). In Section 3, the extremum principles for a finite volume of rigid–plastic and elastoplastic materials is derived. In Section 4, the governing extremum principles for description of the stable post-bifurcation processes in finite elastoplasticity is derived using the postulate of realizability (materials with nonassociated flow rule are included). The simple examples are considered. The relation between the dissipative forces and the rates (fluxes) for time-dependent (viscous) media and the corresponding extremum principles are derived in the Appendix using the postulate of realizability. It shows that the above postulate can give some known and new relations for arbitrary dissipative systems, i.e. it is a quite general one.

Let us apply the direct tensor notation: $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}:\mathbf{B}$ mean the contraction and double contraction of the tensors $\mathbf{A}$ and $\mathbf{B}$; the transpose and inverse operations will be denoted by superscripts t and -1, respectively; $(\mathbf{A})_s = (\mathbf{A}+\mathbf{A}^t)/2$ and $\mathbf{I}$ is the unit tensor. The values at the current configuration V_t will be labelled with subscript t, and the values at the reference configuration V_τ will be without a subscript. All notations are the same through both parts of the paper.

2. THE POSTULATE OF REALIZABILITY FOR UNIFORMLY DEFORMED VOLUME OF ELASTOPLASTIC MATERIALS

In this section a new postulate called the postulate of realizability will be introduced. Using it, the relation between dissipative stress and the rate of the inelastic deformation gradient will be derived as well as the corresponding extremum principles. In the simplest case, the associative flow rule follows from this principle, but there are several ways of obtaining a nonassociated flow rule.

2.1 Rigid–plastic material

Consider a uniformly deformed representative volume of the rigid-plastic material. Let $\mathbf{F}$ and $\mathbf{P}$ be the deformation gradient and the first Piola–Kirchhoff nonsymmetric stress tensor with respect to the reference configuration V_τ. The stress power $\mathbf{P}^t:\dot{\mathbf{F}}$ for small strain regimes reads $\mathbf{T}:\dot{\boldsymbol{\varepsilon}}_p$, where $\mathbf{T}$ is the Cauchy stress tensor and $\boldsymbol{\varepsilon}_p$ is the plastic deformation, $\boldsymbol{\varepsilon}_p \ll \mathbf{I}$. For further generalization it is convenient to use instead of $\boldsymbol{\varepsilon}_p$ the plastic deformation gradient $\mathbf{F}_p = \mathbf{I} + \boldsymbol{\varepsilon}_p$, and to introduce the dissipative stresses $\mathbf{X} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$, using which we can obtain the rate of dissipation $\mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) := \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p):\dot{\mathbf{F}}_p \geq 0$. For rigid-plastic materials with the free energy function ψ which is independent of $\mathbf{F}_p$ it follows from the second law of thermodynamics that $\mathbf{T} = \mathbf{X}$, and for convenience of further generalization we will use $\mathbf{X}$ more often than $\mathbf{T}$. By definition, for time independent plastic materials $\mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$ is a homogeneous function of degree zero in $\dot{\mathbf{F}}_p$, so $\mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p) = \mathbf{X}(\boldsymbol{\kappa}, \mathbf{F}_p)$, where $\boldsymbol{\kappa} = \dot{\mathbf{F}}_p/|\dot{\mathbf{F}}_p|$ is the directing unit tensor and $|\dot{\mathbf{F}}_p| = (\dot{\mathbf{F}}_p:\dot{\mathbf{F}}_p)^{(1/2)}$ is the modulus of $\dot{\mathbf{F}}_p$. Consequently, $\mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) := \mathbf{X}(\boldsymbol{\kappa}, \mathbf{F}_p):\dot{\mathbf{F}}_p = |\dot{\mathbf{F}}_p|\, \mathbf{X}(\boldsymbol{\kappa}, \mathbf{F}_p):\boldsymbol{\kappa} = |\dot{\mathbf{F}}_p|\, \mathscr{D}(\boldsymbol{\kappa}, \dot{\mathbf{F}})$ is a homogeneous function of degree one in $\dot{\mathbf{F}}_p$. To make a geometrical interpretation convenient, we set up a correspondence between the symmetrical second-rank tensors and the six-dimensional vectors in $\mathscr{R}^6$ and will call it both a tensor and a vector.

When varying all possible tensors $\boldsymbol{\kappa} \in \mathscr{R}^6$ at fixed $\mathbf{F}_p$, the ends of vectors $\mathbf{X}(\boldsymbol{\kappa}, \mathbf{F}_p)$, corresponding to them, describe the yield surface $\varphi(\mathbf{X}, \mathbf{F}_p) = 0$ in $\mathbf{X}$-space. For all $\mathbf{X}$ for which $\varphi(\mathbf{X}, \mathbf{F}_p) < 0$, assume that $\dot{\mathbf{F}}_p = \mathbf{0}$ (at $\dot{\mathbf{F}}_p = 0$ vector $\boldsymbol{\kappa}$ and function $\mathbf{X}(\boldsymbol{\kappa}, \mathbf{F}_p)$ are undetermined). So we have $\mathbf{X} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$ and the set of vectors $\mathbf{X}(\mathbf{0}, \mathbf{F}_p)$, which are not related to any $\dot{\mathbf{F}}_p$.

It is evident that if for a given $\mathbf{T} = \mathbf{X}$ at arbitrary fixed $\mathbf{F}_p$ an inequality

$$\mathbf{X}:\dot{\mathbf{F}}_p^* - \mathscr{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p) < 0, \qquad \forall \dot{\mathbf{F}}_p^* \neq \mathbf{0} \tag{2.1}$$

is valid, then $\dot{\mathbf{F}}_p = \mathbf{0}$. Indeed, if $\dot{\mathbf{F}}_p \neq \mathbf{0}$ then $\mathbf{X} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$ for some $\dot{\mathbf{F}}_p$, and $\mathbf{X}:\dot{\mathbf{F}}_p = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p):\dot{\mathbf{F}}_p = \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$ for these $\mathbf{X}$ and $\dot{\mathbf{F}}_p$, which is in contradiction with inequality (2.1). As $\mathbf{X}:\dot{\mathbf{F}}_p^* - \mathscr{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p) = |\dot{\mathbf{F}}_p^*|\,(\mathbf{T}:\boldsymbol{\kappa}^* - \mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p))$, then inequality (2.1) admits an equivalent presentation $\mathbf{X}:\boldsymbol{\kappa}^* - \mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p) < 0$, $\forall \boldsymbol{\kappa}^* \in \mathscr{R}^6$. In a geometrical interpretation this means that the sphere $\mathbf{X}:\boldsymbol{\kappa}^*$, plotted on vector $\mathbf{X}$ as on the diameter, is inside the surface $\mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p)$ (Fig. 1). Condition $\mathbf{X}:\dot{\mathbf{F}}_p = \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$ could be met when sphere $\mathbf{X}:\boldsymbol{\kappa}^*$ and surface $\mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p)$ have the common points, i.e. at their intersection or touching. Touching is the first possibility of the beginning of plastic flow and we assume, that this possibility is realized. As main postulate we suggest

The postulate of realizability: *Let us start from the plastic equilibrium state* $\dot{\mathbf{F}}_p = \mathbf{0}$ *and vary* $\mathbf{X}$*-vector. If in the course of this variation* $\mathbf{X}$ *the condition*

$$\mathbf{X}:\dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) = 0 \tag{2.2}$$

is fulfilled the first time for some $\dot{\mathbf{F}}_p \neq \mathbf{0}$*, then plastic flow will occur with this* $\dot{\mathbf{F}}_p$ (*if condition* (2.2) *is not violated in the course of this plastic flow*).

If, in the course of $\mathbf{X}$-variation, condition (2.2) is satisfied for one or simultaneously for several tensors $\dot{\mathbf{F}}_p$, then for arbitrary other $\dot{\mathbf{F}}_p^*$ the inequality (2.1) should be held, as in the opposite case for this $\dot{\mathbf{F}}_p^*$ condition (2.2) had to be met before it was satisfied for $\dot{\mathbf{F}}_p$. Taking into account that for $\dot{\mathbf{F}}_p \neq \mathbf{0}$ $\mathbf{X} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$, we obtain the extremum principle

$$\mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p):\dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) = 0 < \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p):\dot{\mathbf{F}}_p^* - \mathscr{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p). \tag{2.3}$$

From principle (2.3) we get the normality rule

$$\mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p) = \frac{\partial \mathscr{D}}{\partial \dot{\mathbf{F}}_p'}. \tag{2.4}$$

In a geometrical interpretation, the postulate of realizability means that if in the course of the variation of $\mathbf{X}$ the sphere $\mathbf{X}:\boldsymbol{\kappa}^*$ touches the surface $\mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p)$, then $\dot{\mathbf{F}}_p \neq \mathbf{0}$ (if touching is not violated in the course of this plastic flow). Vector $\dot{\mathbf{F}}_p$ may be directed to the touching point only, because for any other direction $\boldsymbol{\kappa}^*$ we have $\mathbf{X}:\boldsymbol{\kappa}^* - \mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p) < 0$ (Fig. 1). Due to the fact that from both inequalities $\varphi(\mathbf{X}, \mathbf{F}_p) < 0$ and $\mathbf{X}:\dot{\mathbf{F}}_p^* - \mathscr{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p) < 0\; \forall \dot{\mathbf{F}}_p^* \neq \mathbf{0}$ it follows that $\dot{\mathbf{F}}_p = \mathbf{0}$, and from both conditions $\varphi(\mathbf{X}, \mathbf{F}_p) = 0$ and $\mathbf{X}:\dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) = 0$ for some $\dot{\mathbf{F}}_p$ it follows that $\dot{\mathbf{F}}_p \neq \mathbf{0}$ (if these conditions are not violated in the course of plastic flow), they are equivalent and determine the same yield surface. Let us invert the dependence $\mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$. If

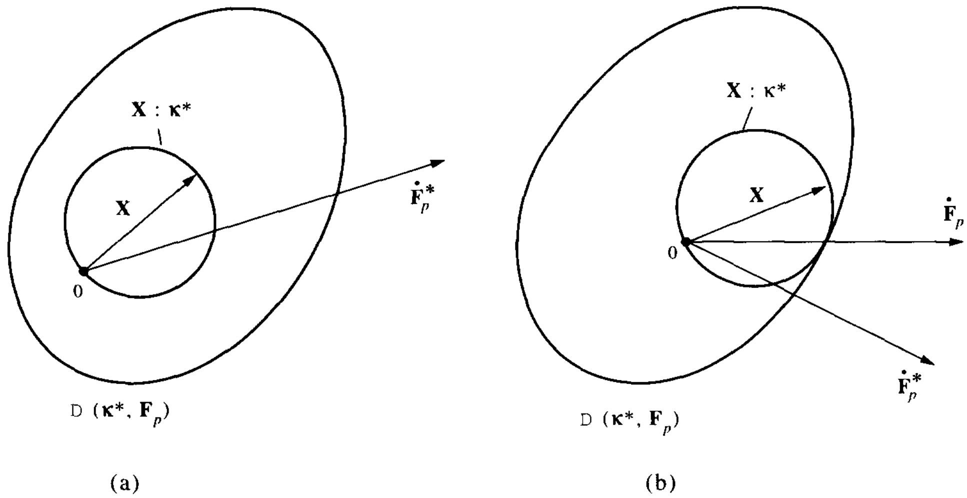

Fig. 1. On the formulation of the postulate of realizability: (a) sphere $\mathbf{X}:\boldsymbol{\kappa}^*$ is inside the surface $\mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p)$, $\dot{\mathbf{F}}_p = 0$; (b) sphere $\mathbf{X}:\boldsymbol{\kappa}^*$ touches surface $\mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p)$, $\dot{\mathbf{F}}_p \neq 0$.

$\varphi(X^*, F_p) \leq 0$, then for this $\mathbf{X}^*\, \forall \dot{\mathbf{F}}_p \in \mathscr{R}^6$ $X^*:\dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) \leq 0$, due to the fact that these conditions are equivalent. But for $\mathbf{X} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$ for some $\dot{\mathbf{F}}_p$ we have $\mathbf{X}:\dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) = 0$. Combining these conditions we obtain

$$\mathbf{X}:\dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) = 0 \geq \mathbf{X}^*:\dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) \quad \text{at} \quad \varphi(\mathbf{X}, \mathbf{F}_p) = 0 \geq \varphi(\mathbf{X}^*, \mathbf{F}_p), \tag{2.5}$$

or

$$\mathbf{X}:\dot{\mathbf{F}}_p \geq \mathbf{X}^*:\dot{\mathbf{F}}_p, \quad \text{at} \quad \varphi(\mathbf{X}, \mathbf{F}_p) = 0 \geq \varphi(\mathbf{X}^*, \mathbf{F}_p), \tag{2.6}$$

and the associated flow rule $\dot{\mathbf{F}}_p = h\, \partial\varphi/\partial\mathbf{X}'$. Here h is a scalar-valued function which is determined from the consistency condition $\dot{\varphi} = 0$. This condition is necessary and sufficient that the equality (2.2) is not violated in the course of the plastic flow, as mentioned in the realizability postulate.

The consistency condition: Consider this problem in more detail in term of $\mathscr{D}$. According to the postulate of the realizability, condition (2.2) has to be valid at time $t + \Delta t$, i.e.

$$\mathbf{X}_\Delta:\dot{\mathbf{F}}_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}) = 0, \tag{2.7}$$

where subscript Δ means that the parameter is determined at $t + \Delta t$. But for all other $\dot{\mathbf{F}}^*_{p\Delta} \neq \mathbf{0}$ we have $\mathbf{X}_\Delta:\dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_{p\Delta}) < 0$, as in the opposite case for this $\dot{\mathbf{F}}^*_{p\Delta}$ condition (2.7) had to be met before it was satisfied for $\dot{\mathbf{F}}_{p\Delta}$. Consequently, the counterpart of the extremum principle (2.3) at $t + \Delta t$ is valid:

$$\mathbf{X}_\Delta:\dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_{p\Delta}) < 0 = \mathbf{X}_\Delta:\dot{\mathbf{F}}_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}). \tag{2.8}$$

For infinitesimal Δt the right side of equation (2.8) reads

$$\begin{aligned} (\mathbf{X} + \dot{\mathbf{X}}\,\Delta t):(\dot{\mathbf{F}}_p + \ddot{\mathbf{F}}_p\,\Delta t) - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) - \frac{\partial\mathscr{D}}{\partial\dot{\mathbf{F}}'_p}:\ddot{\mathbf{F}}_p\,\Delta t - \frac{\partial\mathscr{D}}{\partial\mathbf{F}'_p}:\dot{\mathbf{F}}_p\,\Delta t \\ = (\mathbf{X}:\dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)) + \left(\mathbf{X}:\ddot{\mathbf{F}}_p - \frac{\partial\mathscr{D}}{\partial\dot{\mathbf{F}}'_p}:\ddot{\mathbf{F}}_p\right)\Delta t \\ + \left(\dot{\mathbf{X}}:\dot{\mathbf{F}}_p - \frac{\partial\mathscr{D}}{\partial\mathbf{F}'_p}:\dot{\mathbf{F}}_p\right)\Delta t + \dot{\mathbf{X}}:\ddot{\mathbf{F}}_p(\Delta t)^2 = 0. \end{aligned} \tag{2.9}$$

We assume that all derivatives exist. Neglecting the term with $(\Delta t)^2$ and taking into account equations (2.2) and (2.4) we obtain

$$\left(\dot{\mathbf{X}} - \frac{\partial\mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)}{\partial\mathbf{F}'_p}\right):\dot{\mathbf{F}}_p = 0. \tag{2.10}$$

This is the counterpart of the consistency condition $\dot{\varphi}=0$. From equation (2.10) we could obtain an expression for $|\dot{\mathbf{F}}_p|$

$$\dot{\mathbf{X}}:\boldsymbol{\kappa} - |\dot{\mathbf{F}}_p|\frac{\partial\mathscr{D}(\boldsymbol{\kappa},\mathbf{F}_p)}{\partial\mathbf{F}'_p}:\boldsymbol{\kappa}=0 \quad \text{or} \quad |\dot{\mathbf{F}}_p| = \frac{\dot{\mathbf{X}}:\boldsymbol{\kappa}}{\dfrac{\partial\mathscr{D}(\boldsymbol{\kappa},\mathbf{F}_p)}{\partial\mathbf{F}'_p}:\boldsymbol{\kappa}}. \tag{2.11}$$

Transforming the left side of equation (2.8) we get

$$\mathbf{X}:\dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta},\mathbf{F}_p) + \left(\dot{\mathbf{X}}:\dot{\mathbf{F}}^*_{p\Delta} - \frac{\partial\mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta},\mathbf{F}_p)}{\partial\mathbf{F}'_p}:\dot{\mathbf{F}}_p\right)\Delta t < 0 = \left(\dot{\mathbf{X}}:\dot{\mathbf{F}}_p - \frac{\partial\mathscr{D}}{\partial\mathbf{F}'_p}:\dot{\mathbf{F}}_p\right)\Delta t. \tag{2.12}$$

Note, that in the last term of the inequality (2.12) $\dot{\mathbf{F}}_p$ is not varied, because in equation (2.8) $\mathbf{F}_{p\Delta} = \mathbf{F}_p + \dot{\mathbf{F}}_p\,\Delta t$ and $\dot{\mathbf{F}}_p$ and $\mathbf{F}_{p\Delta}$ are determined uniquely.

Non-unique solution: Consider now the case when the extremum principle (2.3) admits more than one $\dot{\mathbf{F}}_p$. In Fig. 2(a) this fact corresponds to two touching points of the curves $\mathbf{X}:\boldsymbol{\kappa}^*$ and

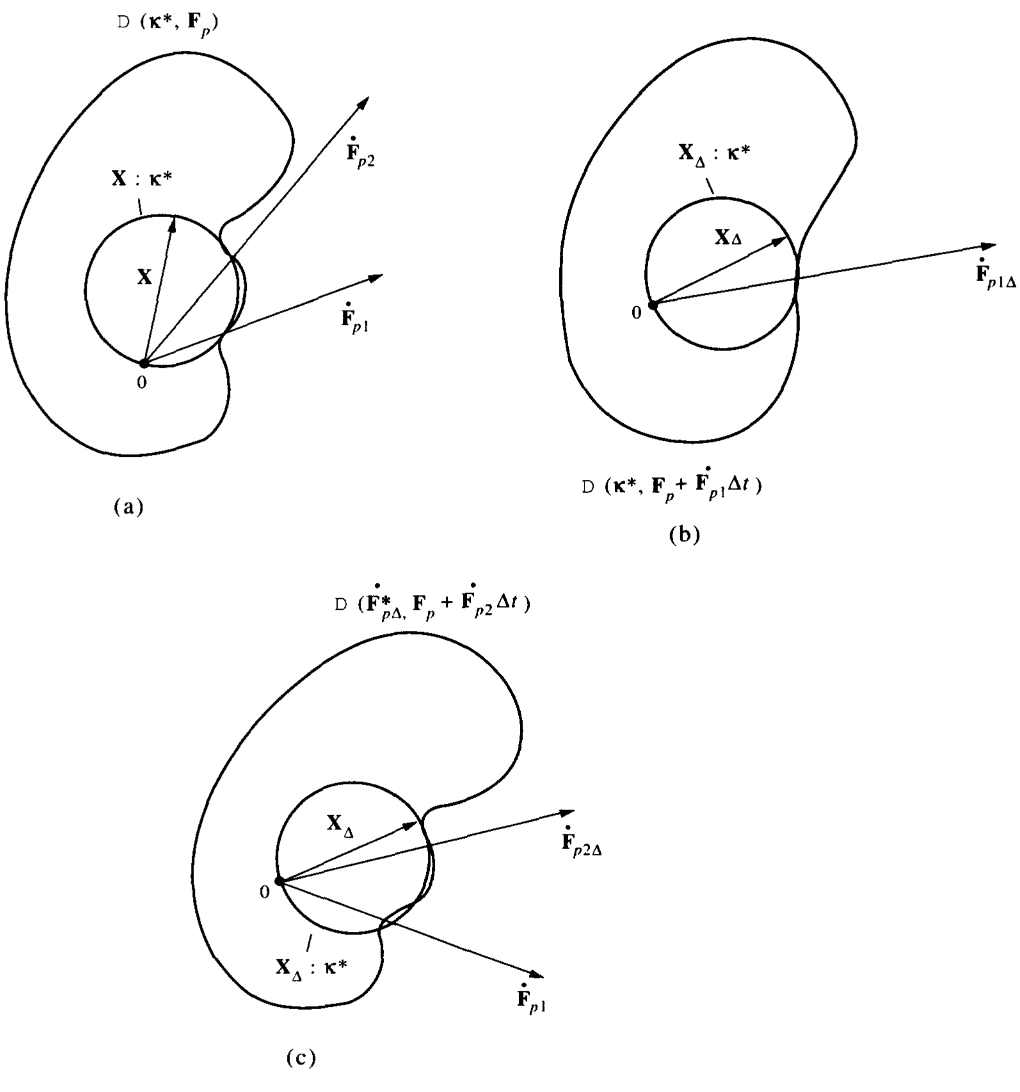

Fig. 2. (a) The cases when the postulate of realizability admits two solutions $\dot{\mathbf{F}}_{p1}$ and $\dot{\mathbf{F}}_{p2}$ at time t; (b) if the stable solution $\dot{\mathbf{F}}_{p1}$ is realized, the postulate of realizability is satisfied at the time $t+\Delta t$; (c) if the unstable solution $\dot{\mathbf{F}}_{p2}$ is realized, the postulate of realizability is violated at the time $t+\Delta t$.

$\mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}_p)$. Let the solution $\dot{\mathbf{F}}_{p1}$ be realized and at time $t + \Delta t$ we have the situation shown in Fig. 2(b). It means that

$$\mathbf{X}_\Delta : \dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_p + \dot{\mathbf{F}}_{p1}\,\Delta t) < 0 \quad \forall \dot{\mathbf{F}}^*_{p\Delta} \neq \mathbf{0}, \dot{\mathbf{F}}^*_{p\Delta} \neq \dot{\mathbf{F}}_{p1\Delta}, \tag{2.13}$$

i.e. the postulate of realizability is met at $t + \Delta t$. When the solution $\dot{\mathbf{F}}_{p2}$ is realized we assume the situation shown in Fig. 2(c). It means that

$$\exists \dot{\mathbf{F}}^*_{p\Delta} \quad \text{for which} \quad \mathbf{X}_\Delta : \dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_p + \dot{\mathbf{F}}_{p2}\,\Delta t) > 0, \tag{2.14}$$

i.e. the postulate of realizability is violated. Indeed, condition (2.7) will be satisfied the first time for one of these $\dot{\mathbf{F}}^*_{p\Delta}$ and at $\Delta t \neq 0$ the jump from $\dot{\mathbf{F}}_{p2}$ to this $\dot{\mathbf{F}}^*_{p\Delta}$ will occur. Consequently, the solution $\dot{\mathbf{F}}_{p2}$ is unstable and using the postulate of realizability we could choose a unique stable solution among several possible ones.

REMARK 1. Here and from now we will exclude the cases when condition (2.13) is met for several solutions $\dot{\mathbf{F}}_{pi}$ at $t + \Delta t$ or is not met for even one solution. For example, the postulate of realizability could not choose the unique $\dot{\mathbf{F}}_p$ for perfectly plastic media with the singular point on the yield surface.

REMARK 2. We will assume that tensors $\dot{\mathbf{F}}^*_{p\Delta}$, for which the inequality (2.14) is valid, include the real solution $\dot{\mathbf{F}}_{p1}$ [Fig. 2(c)].

These assumptions limit the possible dependence $\mathscr{D}$ on $\dot{\mathbf{F}}_p$ and $\mathbf{F}_p$ by cases where the unique solution could be found.

In the general case let us designate all the possible solutions at time t, excluding the unique one $\dot{\mathbf{F}}_p$, by $\dot{\mathbf{F}}^0_p$. Then for $\mathbf{F}_{p\Delta} = \mathbf{F}_p + \dot{\mathbf{F}}_p\,\Delta t$ and for $\mathbf{F}^0_{p\Delta} = \mathbf{F}_p + \dot{\mathbf{F}}^0_p\,\Delta t$

$$\mathbf{X}_\Delta : \dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_{p\Delta}) < 0 = \mathbf{X}_\Delta : \dot{\mathbf{F}}_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}) \quad \forall \dot{\mathbf{F}}^*_{p\Delta} \neq \mathbf{0}, \quad \dot{\mathbf{F}}^*_{p\Delta} \neq \dot{\mathbf{F}}_{p\Delta}, \tag{2.15}$$

$$\exists \dot{\mathbf{F}}^*_{p\Delta} \quad \text{for which} \quad \mathbf{X}_\Delta : \dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}^0_{p\Delta}) > 0. \tag{2.16}$$

If the solution may be found for infinitesimal Δt, then principles (2.15) and (2.16) read

$$\mathbf{X} : \dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_p) + \left(\dot{\mathbf{X}} : \dot{\mathbf{F}}^*_{p\Delta} - \frac{\partial \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_p)}{\partial \mathbf{F}^t_p} : \dot{\mathbf{F}}_p\right) \Delta t < 0$$

$$= \left(\dot{\mathbf{X}} : \dot{\mathbf{F}}_p - \frac{\partial \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}_p\right) \Delta t \quad \forall \dot{\mathbf{F}}^*_{p\Delta} \neq \mathbf{0}, \quad \dot{\mathbf{F}}^*_{p\Delta} \neq \dot{\mathbf{F}}_{p\Delta}, \tag{2.17}$$

$$\exists \dot{\mathbf{F}}^*_{p\Delta} \quad \text{for which} \quad \mathbf{X} : \dot{\mathbf{F}}^*_{p\Delta} - \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_p) + \left(\dot{\mathbf{X}} : \dot{\mathbf{F}}^*_{p\Delta} - \frac{\partial \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}^0_{p\Delta}\right) \Delta t > 0. \tag{2.18}$$

We could consider $\dot{\mathbf{F}}^*_{p\Delta} = \dot{\mathbf{F}}^0_p$ only, but designate if $\dot{\mathbf{F}}^\diamond_p$ [to differ them from the $\dot{\mathbf{F}}^0_p$ in the last term in equation (2.18)]. Then from the principles (2.17) and (2.18) it follows

$$\dot{\mathbf{X}} : \dot{\mathbf{F}}^\diamond_p - \frac{\partial \mathscr{D}(\dot{\mathbf{F}}^\diamond_p, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}_p < 0 = \dot{\mathbf{X}} : \dot{\mathbf{F}}_p - \frac{\partial \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}_p \tag{2.19}$$

$$\exists \dot{\mathbf{F}}^\diamond_p \quad \text{for which} \quad \dot{\mathbf{X}} : \dot{\mathbf{F}}^\diamond_p - \frac{\partial \mathscr{D}(\dot{\mathbf{F}}^\diamond_p, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}^0_p > 0, \tag{2.20}$$

because the first term in equations (2.17) and (2.18) is equal to zero for all the solutions $\dot{\mathbf{F}}^\diamond_p$. The consideration of $\dot{\mathbf{F}}^*_{p\Delta} \neq \dot{\mathbf{F}}^0_p$ is senseless, due to the fact that for this case the first term in inequalities (2.17) and (2.18) is equal to a finite value and the second term is infinitesimal for arbitrary possible $\dot{\mathbf{F}}^0_p$ and the result will be negative independently of $\dot{\mathbf{F}}^0_p$.

Let us describe the procedure of the application of principle (2.19). If we have an infinite number of solutions $\dot{\mathbf{F}}_p^0$ and the maximum in equation (2.19) is analytical, then

$$\dot{\mathbf{X}} = \frac{\partial^2 \mathscr{D}}{\partial \dot{\mathbf{F}}'_p \, \partial \dot{\mathbf{F}}'_p} : \ddot{\mathbf{F}}_p = \frac{\partial \mathbf{X}}{\partial \dot{\mathbf{F}}'_p} : \ddot{\mathbf{F}}_p. \tag{2.21}$$

If we have a finite number of solutions $\dot{\mathbf{F}}_p^0 = \dot{\mathbf{F}}_{pi}$, $i = 1, 2, \ldots, m$, assume that one of them, e.g. $\dot{\mathbf{F}}_{pj}$, satisfies principle (2.19). If we designate $\mathscr{H}(\boldsymbol{\kappa}, \mathbf{F}_p) = \partial \mathscr{D}(\boldsymbol{\kappa}, \mathbf{F}_p)/\partial \dot{\mathbf{F}}'_p$, then from equation (2.11)

$$|\dot{\mathbf{F}}_{pj}| = \frac{\dot{\mathbf{X}} : \boldsymbol{\kappa}_j}{\mathscr{H}(\boldsymbol{\kappa}_j, \mathbf{F}_p) : \boldsymbol{\kappa}_j}. \tag{2.22}$$

Substitution of $\dot{\mathbf{F}}_p^\diamond = \dot{\mathbf{F}}_{pi} = |\dot{\mathbf{F}}_{pi}| \, \boldsymbol{\kappa}_i$, $i \neq j$ and equation (2.22) in principle (2.19) results in

$$\dot{\mathbf{X}} : \left(\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j \frac{\mathscr{H}(\boldsymbol{\kappa}_i, \mathbf{F}_p) : \boldsymbol{\kappa}_j}{\mathscr{H}(\boldsymbol{\kappa}_j, \mathbf{F}_p) : \boldsymbol{\kappa}_j} \right) < 0, \quad \forall i \neq j. \tag{2.23}$$

If at any fixed j inequality (2.23) is met for all $i \neq j$, then $\dot{\mathbf{F}}_{pj}$ is really a solution, if not—we have to check the next j. The same procedure is valid when we use principle (2.15). Similar procedure will be used in Section 4 for the derivation of the governing extremum principle for description of the stable post-bifurcation processes in finite body and the above vivid geometrical interpretation (Fig. 2) is useful. Despite the fact that equations (2.3)–(2.6) are known, equations (2.8), (2.9)–(2.23) seem new to us.

2.2 Nonassociated flow rule

It would be a mistake to identify the realizability postulate with the associative flow rule: it is true for given simple models only. For models of materials with structural changes (Levitas [24–26]) the postulate of realizability will admit significant concavity of the yield function and nonassociated flow rule. It is easy to show that results of its applications will coincide with the ones obtained in [24–26], using the dissipation postulate.

Another possibility exists to obtain the nonassociated flow rule for systems with kinematic constraints, when $\mathbf{X}$ depends on the reactions of these constraints. Let us have one scalar constraint equation $q(\dot{\mathbf{F}}_p) = 0$. Using the realizability postulate, we obtain principle (2.3) for $\dot{\mathbf{F}}_p^* \in q(\dot{\mathbf{F}}_p^*) = 0$ and $\dot{\mathbf{F}}_p \in q(\dot{\mathbf{F}}_p) = 0$, i.e.

$$\mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p) : \dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) = 0 > \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p) : \dot{\mathbf{F}}_p^* - \mathscr{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p);$$
$$q(\dot{\mathbf{F}}_p^*) = q(\dot{\mathbf{F}}_p) = 0, \tag{2.24}$$

and

$$\mathbf{X} = \frac{\partial \mathscr{D}}{\partial \dot{\mathbf{F}}'_p} + \xi \frac{\partial q}{\partial \dot{\mathbf{F}}'_p} = \mathbf{X}_d + \mathbf{X}_\xi, \qquad \mathbf{X}_\xi = \xi \frac{\partial q}{\partial \dot{\mathbf{F}}'_p}, \tag{2.25}$$

where ξ is the Lagrange multiplier and $\mathbf{X}_\xi$ is the constraint reaction. For time independent materials, function q has to be homogeneous of degree one. From equation (2.25) $\mathbf{X}:\dot{\mathbf{F}}_p = \partial \mathscr{D}/\partial \dot{\mathbf{F}}'_p : \dot{\mathbf{F}}_p + \xi \, \partial q/\partial \dot{\mathbf{F}}'_p : \dot{\mathbf{F}}_p = \mathscr{D} + \xi q = \mathscr{D}$, where the property $\partial q/\partial \dot{\mathbf{F}}'_p : \dot{\mathbf{F}}_p = q$ (the same for $\mathscr{D}$) of the homogeneous functions is used. Identity $\mathbf{X}_\xi : \dot{\mathbf{F}}_p = 0$ means that $q(\dot{\mathbf{F}}_p) = 0$ is an ideal constraint. We could assume the dependence of $\mathbf{X}_d$ and $\mathscr{D}$ on $\mathbf{X}_\xi$—we get equations (2.24), (2.25) without any changes:

$$\mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p, \mathbf{X}_\xi) : \dot{\mathbf{F}}_p - \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p, \mathbf{X}_\xi) = 0 > \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p, \mathbf{X}_\xi) : \dot{\mathbf{F}}_p^* - \mathscr{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p, \mathbf{X}_\xi) \tag{2.26}$$

at $q(\dot{\mathbf{F}}_p^*) = q(\dot{\mathbf{F}}_p) = 0$. We show from the simple example that this assumption could lead to non-associativity, e.g. to the model

$$\tau = k + b\sigma_n, \qquad q = \dot{\varepsilon}_n + a\dot{\gamma} = 0, \tag{2.27}$$

where $\dot{\varepsilon}_n$ and σ_n are the deformation rate and the Cauchy stress components which are normal to the shear plane, respectively, and $a \neq b$ are scalars. Let us define $\mathscr{D} = (k + (b - a)\sigma_n)\dot{\gamma}$. If we include $\dot{\mathbf{F}}_p$ in $\dot{\mathbf{F}}_p^*$ the principle (2.26) results in $\tau\dot{\gamma}^* + \sigma_n\dot{\varepsilon}_n^* - \mathscr{D}(\dot{\gamma}^*, \sigma_n) \leq 0$ at $\dot{\varepsilon}_n^* + a\dot{\gamma}^* = 0$, whence

$$\sigma_n = \xi; \qquad \tau = \frac{\partial \mathscr{D}}{\partial \dot{\gamma}} + \xi a = k + b\sigma_n. \tag{2.28}$$

In the given case σ_n is a constraint reaction. An associated flow rule with a yield surface (2.28) is $\dot{\gamma} = h$, $\dot{\varepsilon}_n = -hb = -b\dot{\gamma}$; it does not coincide with the constraint (2.27).

Consequently, using the postulate of realizability, the constraint equation and the dissipative function, dependent on the constraint reaction, we obtain a model with a non-associated flow rule. The model (2.27) is very often used for the description of materials with internal friction, fissured rock masses and granular materials. The aim of this section was to show that these materials are included in the given description.

2.3 *Elastoplastic materials at small strains*

Let us assume $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p$; $\psi = \psi(\boldsymbol{\varepsilon}_e, \boldsymbol{\varepsilon}_p)$; $\mathbf{d} = \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_e + \dot{\boldsymbol{\varepsilon}}_p$, where $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}_e$ and $\boldsymbol{\varepsilon}_p$ are the total, elastic and plastic strain tensors, respectively. As in Section 2.1 we will use $\mathbf{F}_p = \mathbf{I} + \boldsymbol{\varepsilon}_p$ instead of $\boldsymbol{\varepsilon}_p$. The rate of dissipation

$$\mathscr{D} = \mathbf{T}:\mathbf{d} - \rho\dot{\psi} = \left(\mathbf{T} - \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}_e}\right):\dot{\boldsymbol{\varepsilon}}_e + \left(\mathbf{T} - \rho\frac{\partial\psi}{\partial\mathbf{F}_p'}\right):\dot{\mathbf{F}}_p \geq 0$$

(the temperature is fixed in a paper), and due to the independence $\mathscr{D}$ on $\dot{\boldsymbol{\varepsilon}}_e$ we have

$$\mathbf{T} = \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}_e}; \qquad \mathscr{D} = \left(\mathbf{T} - \rho\frac{\partial\psi}{\partial\mathbf{F}_p'}\right):\dot{\mathbf{F}}_p = \mathbf{X}:\dot{\mathbf{F}}_p \geq 0; \qquad \mathbf{X} = \mathbf{T} - \rho\frac{\partial\psi}{\partial\mathbf{F}_p'}. \tag{2.29}$$

We can repeat all the results which we obtain by the application of postulate of realizability and which led to equations (2.1)–(2.28). But now $\mathbf{X} \neq \mathbf{T}$ and the associated flow rule in $\mathbf{X}$-space does not lead (in the general case) to an associated flow rule in the $\mathbf{T}$-space.

2.4 *Rigid-plastic materials at finite strains*

In this case the stress power and (if ψ does not depend on $\mathbf{F}_p$) the rate of dissipation per unit volume in some reference configuration V_τ are equal to $\mathbf{P}':\dot{\mathbf{F}}_p$. Designating $\mathbf{X} = \mathbf{P}'$, we can repeat all the derivations and results (2.1)–(2.28) of the Sections 2.1 and 2.2, but it is necessary to make some remarks related to the correct account for finite rotations.

REMARK 1. Tensors $\mathbf{X} = \mathbf{P}'$ and $\dot{\mathbf{F}}_p$ have nine components, but only six of them contribute to the rate of dissipation and the other three are connected with rigid body rotations. Using the polar decomposition $\mathbf{F}_p = \mathbf{R}_p \cdot \mathbf{U}_p$, where $\mathbf{R}_p^{-1} = \mathbf{R}_p'$ and $\mathbf{U}_p' = \mathbf{U}_p$ are the orthogonal rotation and the right-stretch tensors respectively, we have $\mathbf{P}':\dot{\mathbf{F}}_p = \mathbf{P}':(\dot{\mathbf{R}}_p \cdot \mathbf{U}_p + \mathbf{R}_p \cdot \dot{\mathbf{U}}_p) = \mathbf{P}':\dot{\mathbf{R}}_p \cdot \mathbf{R}_p' \cdot \mathbf{F}_p + \mathbf{P}' \cdot \mathbf{R}_p:\dot{\mathbf{U}}_p = \mathbf{F}_p \cdot \mathbf{P}':\dot{\mathbf{R}}_p \cdot \mathbf{R}_p' + (\mathbf{P}' \cdot \mathbf{R}_p)_s:\dot{\mathbf{U}}_p = (\mathbf{P}' \cdot \mathbf{R}_p)_s:\dot{\mathbf{U}}_p$, as $\mathbf{F}_p \cdot \mathbf{P}' = (\rho/\rho_t)\mathbf{T}$ is a symmetric tensor and $\dot{\mathbf{R}}_p \cdot \mathbf{R}_p'$ is a skew-symmetric one. It means that if in all equations of Section 2.1 the tensors $\mathbf{X} = (\mathbf{P}' \cdot \mathbf{R}_p)_s$ and $\mathbf{F}_p = \mathbf{U}_p$ are used, all the results of this section will be valid. In this case the principle of material frame indifference will be met in explicit form.

REMARK 2. It is possible to use the expressions $\mathbf{X} = \mathbf{P}'$; $\mathbf{X} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p)$; $\mathscr{D} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p):\dot{\mathbf{F}}_p$, for the non-symmetric tensors, assuming the fulfilment of the principle of material frame-indifference. All the results (2.1)–(2.26) are valid with only additional limitations $\dot{\mathbf{F}}_{p\Delta} \neq \dot{\mathbf{R}}_{p\Delta} \cdot \mathbf{U}_{p\Delta}$;

$\dot{\mathbf{F}}^*_{p\Delta} \neq \dot{\mathbf{R}}^*_{p\Delta} \cdot \mathbf{U}_{p\Delta}$; $\dot{\mathbf{F}}^0_p \neq \dot{\mathbf{R}}^0_p \cdot \mathbf{U}_p$, i.e. we have to exclude the pure rotation without plastic strain, for which the left sides of the inequalities, e.g. (2.8), are zero. The usage of nine degrees of freedom is more convenient for the application of the approach considered above to a non-uniformly deformed volume. A similar situation occurs in nonlinear elasticity, where $\psi = \psi(\mathbf{F}_e) = \psi(\mathbf{U}_e)$ and one can get an equation for six stress components (e.g. the Cauchy stress tensor $\mathbf{T}$ or the second Piola–Kirchhoff stress tensor) or for the nine components of the stress tensor $\mathbf{P}$.

2.5 Elastoplastic materials at finite strains

We will use the Lee [27] decomposition $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p$, due to the fact that it contains the "best" measures of the elastic and plastic strains (see Levitas [25, 26, 28]). The tensor $\mathbf{F}_p$ represents in the given case the plastic deformation gradient for plastic strain without rotations (see [25, 26, 28]). For example, for polycrystalline materials $\mathbf{F}_p = \mathbf{U}_p$, for monocrystals $\mathbf{F}_p$ is the deformation at a fixed crystal lattice. Consequently, the decomposition $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p$ is invariant with respect to the rigid body rotation in the stress-free intermediate configuration. Assuming $\psi = \psi(\mathbf{F}_e, \mathbf{F}_p)$ we have

$$\mathscr{D} = \mathbf{P}' : \dot{\mathbf{F}} - \rho\dot{\psi} = \mathbf{P}' : (\dot{\mathbf{F}}_e \cdot \mathbf{F}_p + \mathbf{F}_e \cdot \dot{\mathbf{F}}_p) - \rho \frac{\partial \psi}{\partial \mathbf{F}'_e} : \dot{\mathbf{F}}_e - \rho \frac{\partial \psi}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}_p$$

$$= \left(\mathbf{F}_p \cdot \mathbf{P}' - \rho \frac{\partial \psi}{\partial \mathbf{F}'_e}\right) : \dot{\mathbf{F}}_e + \left(\mathbf{P}' \cdot \mathbf{F}_e - \rho \frac{\partial \psi}{\partial \mathbf{F}'_p}\right) : \dot{\mathbf{F}}_p, \tag{2.30}$$

whence

$$\mathbf{F}_p \cdot \mathbf{P}' = \rho \frac{\partial \psi}{\partial \mathbf{F}_e}; \qquad \mathscr{D} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p) : \dot{\mathbf{F}}_p; \qquad \mathbf{X} = \mathbf{P}' \cdot \mathbf{F}_e - \rho \frac{\partial \psi}{\partial \mathbf{F}'_p}.$$

When $\mathbf{F}_p = \mathbf{U}_p$, then $\mathscr{D} = \mathbf{X}(\dot{\mathbf{U}}_p, \mathbf{U}_p) : \dot{\mathbf{U}}_p$; $\mathbf{X} = (\mathbf{F}_e \cdot \mathbf{P}')_s - \rho\, \partial\psi/\partial\mathbf{U}'_p$. All the results of Section 2.1 are valid for these conjugated pairs $\mathbf{X}$ and $\mathbf{F}_p$ or $\mathbf{X}$ and $\mathbf{U}_p$.

REMARK. For rigid perfectly plastic and softening in the $\mathbf{X}$-space materials some problems arise concerning the implementation of the postulate of realizability (despite the fact that all the relations obtained are correct). This lies in the fact that in the formulation of the postulate of realizability we use a stress-controlled "experiment", i.e. vary the stresses. For perfectly plastic materials in stress-controlled experiments the modulus $|\dot{\mathbf{F}}_p|$ is undetermined and, in a particular case, could be equal to zero. The postulate of realizability excludes $|\dot{\mathbf{F}}_p| = 0$, but does not exclude infinitesimal $|\dot{\mathbf{F}}_p|$.

For softening materials, the modulus of the stress vector decreases in the course of plastic flow, but in this situation not only plastic flow, but unloading is also possible. In reality we assume that plastic flow will occur and determine the corresponding stress variation.

To avoid the above problems let us give a "strain-controlled" formulation of the postulate of realizability. For elastoplastic materials it has the same form as the "stress-controlled" one, with only one distinction: instead "in the course of variation $\mathbf{X}$" it is necessary to use "in the course of variation $\mathbf{F}$", because in the elastic region $\mathbf{X}$ and $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p$ at $\mathbf{F}_p = const$ connected by the elasticity law.

3. EXTREMUM PRINCIPLES FOR A FINITE VOLUME OF ELASTOPLASTIC MEDIA

In this section we derive some known extremum principles for a finite volume of rigid-plastic and some new ones for elastoplastic materials using the postulate of realizability. They will serve as a basis for the description of stable post-bifurcation behaviour in Section 4.

3.1 Rigid-plastic materials

Consider a volume v of a rigid-plastic material with a boundary S in the reference configuration V_τ. The energy balance principle results in equation

$$\int_S \mathbf{p}\cdot\mathbf{v}\,\mathrm{d}S = \int_v \mathbf{P}'(\dot{\mathbf{F}},\mathbf{F}):\dot{\mathbf{F}}\,\mathrm{d}v = \int_v \mathscr{D}(\dot{\mathbf{F}},\mathbf{F})\,\mathrm{d}v, \tag{3.1}$$

where $\mathbf{p}$ is the external traction, $\mathbf{v}$ is the velocity field. We will omit subscript p in this section. Assume that on part S_p the stress vector $\mathbf{p}$ is prescribed and on part S_v the velocity vector $\mathbf{v}=\mathbf{0}$ is given, $S=S_\mathrm{p}\cup S_\mathrm{v}$; but a mixed formulation is also possible. Here and later we exclude rigid body motions. Consequently conditions $\dot{\mathbf{F}}=\mathbf{0}$ and $\mathbf{v}=\mathbf{0}$, if they are valid for all points of the volume v, are equivalent.

Consider an arbitrary velocity field $\mathbf{v}^*\neq\mathbf{0}$, satisfying the boundary condition on S_v and corresponding to it the velocity gradient field $\dot{\mathbf{F}}^*=(\nabla\mathbf{v}^*)_s\neq\mathbf{0}$. Evidently, if for a given $\mathbf{p}$-distribution on S_p for all admissible velocity fields $\mathbf{v}^*$

$$\int_S \mathbf{p}\cdot\mathbf{v}^*\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}^*,\mathbf{F})\,\mathrm{d}v<0, \tag{3.2}$$

then $\mathbf{v}=\mathbf{0}$ and $\dot{\mathbf{F}}=\mathbf{0}$. The proof of this statement is very simple: if $\mathbf{v}\neq\mathbf{0}$ and $\dot{\mathbf{F}}\neq\mathbf{0}$ then equation (3.1) has to be met, which is in contradiction with inequality (3.2). Let us apply

The postulate of realizability: *If, starting from the plastic equilibrium* [*equation* (3.2) *is valid*] *in the course of variation of* $\mathbf{p}$ *the condition* (3.1) *is fulfilled the first time for some field* $\mathbf{v}$ (*and* $\dot{\mathbf{F}}$), *then the plastic flow will occur with this* $\mathbf{v}$ (*if the condition* (3.1) *is not violated in the course of the plastic flow*).

If in the course of $\mathbf{p}$-changes condition (3.1) is satisfied the first time for one or simultaneously for several fields $\mathbf{v}$, then for other $\mathbf{v}^*$ inequality (3.2) should be held, as in the opposite case for this $\mathbf{v}^*$ condition (3.1) had to be met before it was satisfied for $\mathbf{v}$. Thus, we get the extremum principle

$$\int_S \mathbf{p}\cdot\mathbf{v}\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}},\mathbf{F})\,\mathrm{d}v = 0 > \int_S \mathbf{p}\cdot\mathbf{v}^*\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}^*,\mathbf{F})\,\mathrm{d}v. \tag{3.3}$$

Here and later we will assume $\mathbf{v}^*\neq\mathbf{0}$, $\mathbf{v}^*\neq\mathbf{v}$, $\dot{\mathbf{F}}^*\neq\mathbf{0}$, $\dot{\mathbf{F}}^*\neq\dot{\mathbf{F}}$. Equation (3.3) is a well known extremum principle for perfect plastic materials, but it is also valid for hardening or softening materials. To fulfil condition (3.1) in the course of plastic flow at the next time instant $t+\Delta t$ the stress vectors $\mathbf{p}$ have to be changed in accordance with equation

$$\int_S \mathbf{p}_\Delta\cdot\mathbf{v}_\Delta\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}_\Delta,\mathbf{F}_\Delta)\,\mathrm{d}v = 0, \tag{3.4}$$

where $\mathbf{F}_\Delta=\mathbf{F}(t)+\dot{\mathbf{F}}(t)\,\Delta t$. Otherwise, when

$$\int_S \mathbf{p}_\Delta\cdot\mathbf{v}^*_\Delta\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}^*_\Delta,\mathbf{F}_\Delta)\,\mathrm{d}v<0 \tag{3.5}$$

for all admissible fields $\mathbf{v}^*$ and $\dot{\mathbf{F}}^*$ and for arbitrary infinitesimal Δt we obtain $\mathbf{v}_\Delta=\mathbf{0}$ and $\dot{\mathbf{F}}_\Delta=\mathbf{0}$ and at $\Delta t\to 0$ the conditions $\mathbf{v}=\mathbf{0}$ and $\dot{\mathbf{F}}=\mathbf{0}$. Combining equations (3.4) and (3.5) we obtain at time $t+\Delta t$ the same extremum principle as in equation (3.3) for time t

$$\int_S \mathbf{p}_\Delta\cdot\mathbf{v}_\Delta\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}_\Delta,\mathbf{F}_\Delta)\,\mathrm{d}v = 0 > \int_S \mathbf{p}_\Delta\cdot\mathbf{v}^*_\Delta\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}^*_\Delta,\mathbf{F}_\Delta)\,\mathrm{d}v. \tag{3.6}$$

Note that $\mathbf{F}_\Delta$ is fixed in principle (3.6). Certainly, principle (3.3) as well as (3.6) could be derived using equations (3.1) and (2.3), but it is important for future generalization to show the possibility of the application of the postulate of realizability for a finite volume of materials. For infinitesimal Δt the right side of equation (3.6) reads

$$0 > \left(\int_S \mathbf{p}\cdot\mathbf{v}_\Delta^*\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}_\Delta^*, \mathbf{F})\,\mathrm{d}v\right) + \left(\int_S \dot{\mathbf{p}}\cdot\mathbf{v}_\Delta^*\,\mathrm{d}S - \int_v \frac{\partial\mathscr{D}(\dot{\mathbf{F}}_\Delta^*, \mathbf{F})}{\partial\mathbf{F}'}:\dot{\mathbf{F}}\,\mathrm{d}v\right)\Delta t. \tag{3.7}$$

Here and later we will assume that all the derivatives used do exist. The term in the first brackets of inequality (3.7) for $\mathbf{v}_\Delta^* \neq \mathbf{v}$ and $\mathbf{v}_\Delta^* \neq \mathbf{0}$ is negative [according to equation (3.3)], but we could not confirm the same for the term in the second brackets. As the first step let us find all possible solutions of the principle (3.7) at $\Delta t = 0$. In this case, the principle (3.7) coincides with (3.3). If we label all solutions of this principle with superscript $\diamond$ then

$$\int_S \dot{\mathbf{p}}\cdot\mathbf{v}\,\mathrm{d}S - \int_v \frac{\partial\mathscr{D}(\dot{\mathbf{F}}, \mathbf{F})}{\partial\mathbf{F}'}:\dot{\mathbf{F}}\,\mathrm{d}v = 0 > \int_S \dot{\mathbf{p}}\cdot\mathbf{v}^\diamond\,\mathrm{d}S - \int_v \frac{\partial\mathscr{D}(\dot{\mathbf{F}}^\diamond, \mathbf{F})}{\partial\mathbf{F}'}:\dot{\mathbf{F}}\,\mathrm{d}v, \tag{3.8}$$

because $\int_S \mathbf{p}\cdot\mathbf{v}^\diamond\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}^\diamond, \mathbf{F})\,\mathrm{d}v = 0$ according to principle (3.3). Consideration of $\mathbf{v}_\Delta^* \neq \mathbf{v}^\diamond$ is senseless, because for the first term in inequality (3.7) equals a finite negative value and the second term is infinitesimal and the result will be always negative. The solutions $\mathbf{v}^\diamond$ corresponds to heterogeneous instantaneously perfect plastic materials, because $\mathbf{F}$ is fixed in principle (3.3).

In principle (3.8) the fields $\mathbf{v}^\diamond$ are determined at time t, but in principle (3.7) at time $t + \Delta t$. But if we use $\mathbf{v}_\Delta^* = \mathbf{v}^* + \dot{\mathbf{v}}^*\,\Delta t$ in the second term of the inequality (3.7) and neglect the terms with $(\Delta t)^2$, we obtain the same result. If we have a continuum of solutions $\mathbf{v}^\diamond$ and the maximum in principle (3.8) is analytical, then using equation (2.21) we get

$$\int_S \dot{\mathbf{p}}\cdot\delta\mathbf{v}\,\mathrm{d}S - \int_v \left(\frac{\partial\mathbf{X}}{\partial\mathbf{F}'}:\dot{\mathbf{F}}\right):\delta\dot{\mathbf{F}}\,\mathrm{d}v = 0. \tag{3.9}$$

All extremum principles stated above are obtained at the primitive boundary condition on S_v $\mathbf{v} = \mathbf{0}$. To extend this principle for arbitrary $\mathbf{v}$ on S_v do the following. Prescribe on the part S_p of the surface S instead of the vector $\mathbf{p}$ the velocity, obtained as a solution of principle (3.3)—the solution in the volume v will be the same. Then we will use in principle (3.3) only those reduced admissible fields $\mathbf{v}^*$, which meet this new boundary condition—principle (3.3) remains valid. If we have a non-unique solution for $\mathbf{v}$ on S_p, we could use each of them in principle (3.3).

If we have kinematic constraints $q_i(\dot{\mathbf{F}}) = 0$ at each point of the material, then, making use of the realizability postulate, we obtain principle (3.3) with additional terms

$$\begin{aligned}&\int_S \mathbf{p}\cdot\mathbf{v}\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}, \mathbf{F}, \mathbf{X}_{\xi i})\,\mathrm{d}v - \int_v \xi_i q_i(\dot{\mathbf{F}})\,\mathrm{d}v \\ &\quad = 0 > \int_S \mathbf{p}\cdot\mathbf{v}^*\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}^*, \mathbf{F}, \mathbf{X}_{\xi i})\,\mathrm{d}v - \int_v \xi_i q_i(\dot{\mathbf{F}}^*)\,\mathrm{d}v, \qquad \mathbf{X}_{\xi i} = \xi_i \frac{\partial q_i}{\partial \dot{\mathbf{F}}'}.\end{aligned} \tag{3.10}$$

Principle (3.10) is valid for materials with the non-associated flow rule.

3.2 *Elastoplastic materials at small strains*

Equation (2.29) will be used. The global form of the second law of thermodynamics reads

$$\int_S \mathbf{p}\cdot\mathbf{v}\,\mathrm{d}S - \int_v \rho\dot{\psi}\,\mathrm{d}v = \int_v \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)\,\mathrm{d}v \geq 0, \tag{3.11}$$

where is the given case $\mathbf{p}$ is the Cauchy stress vector and $\mathbf{F}_p = \mathbf{I} + \boldsymbol{\varepsilon}_p$, $\boldsymbol{\varepsilon}_p \ll \mathbf{I}$. For all admissible fields $\mathbf{v}^*$ equation

$$\int_S \mathbf{p}\cdot\mathbf{v}^*\,\mathrm{d}S - \int_v \rho\dot{\psi}^*\,\mathrm{d}v - \int_v \mathbf{X}:\dot{\mathbf{F}}_p^*\,\mathrm{d}v = 0 \tag{3.12}$$

where

$$\dot{\psi}^* = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}_e} : \dot{\boldsymbol{\varepsilon}}_e^* + \frac{\partial \psi}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}_p^*, \qquad \mathbf{T} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}_e}, \qquad \varphi(\mathbf{T}, \mathbf{F}_p) \le 0 \quad \text{and} \quad \nabla \cdot \mathbf{T} = \mathbf{0},$$

can be proved:

$$\int_S \mathbf{p} \cdot \mathbf{v}^* \, dS - \int_v \rho \dot{\psi}^* \, dv = \int_v (\mathbf{T} : \dot{\boldsymbol{\varepsilon}}^* - \rho \dot{\psi}^*) \, dv$$

$$= \int_v \left[\left(\mathbf{T} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}_e} \right) : \dot{\boldsymbol{\varepsilon}}_e^* + \left(\mathbf{T} - \rho \frac{\partial \psi}{\partial \mathbf{F}'_p} \right) : \dot{\mathbf{F}}_p^* \right] dv = \int_v \mathbf{X} : \dot{\mathbf{F}}_p^* \, dv.$$

As $\mathbf{X} : \dot{\mathbf{F}}_p^* < \mathcal{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p) \; \forall \dot{\mathbf{F}}_p^* \neq \dot{\mathbf{F}}_p$, $\dot{\mathbf{F}}_p^* \neq \mathbf{0}$ [see equation (2.3)], then

$$\int_S \mathbf{p} \cdot \mathbf{v}^* \, dS - \int_v \rho \dot{\psi}^* \, dv - \int_v \mathcal{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p) \, dv < 0 \tag{3.13}$$

and

$$\int_S \mathbf{p} \cdot \mathbf{v} \, dS - \int_v \rho \dot{\psi} \, dv - \int_v \mathcal{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p) \, dv = 0 > \int_S \mathbf{p} \cdot \mathbf{v}^* \, dS - \int_v \rho \dot{\psi}^* \, dv - \int_v \mathcal{D}(\dot{\mathbf{F}}_p^*, \mathbf{F}_p) \, dv. \tag{3.14}$$

The extremum principle (3.14) can be obtained using the postulate of realizability. In the first step, we can prove that if inequality (3.13) is valid for all admissible $\mathbf{v}^*$, $\dot{\mathbf{F}}^*$, $\dot{\boldsymbol{\varepsilon}}_e^*$ and $\dot{\mathbf{F}}_p^*$, which are nonzero and nonequal to its real value, then field $\dot{\mathbf{F}}_p = \mathbf{0}$, because for $\dot{\mathbf{F}}_p \neq \mathbf{0}$ we have equation (3.11). Then apply

The postulate of realizability: *Let us assume that at prescribed boundary data* $\mathbf{p}$ *on* S_p and $\mathbf{v}$ *on* S_v *inequality* (3.13) *is valid and* $\dot{\mathbf{F}}_p = \mathbf{0}$. *If in the course of boundary data variation condition* (3.11) *is met the first time for some fields* $\mathbf{v} \neq \mathbf{0}$ *and* $\dot{\mathbf{F}}_p \neq \mathbf{0}$ *then the plastic flow with these* $\mathbf{v}$ *and* $\dot{\mathbf{F}}_p$ *will occur* (*if condition* (3.11) *is not violated in the course of this plastic flow*).

Note that we did not use any constitutive equations for $\dot{\mathbf{F}}_p^*$ and $\dot{\boldsymbol{\varepsilon}}_e^*$; they satisfy only the equation $\dot{\boldsymbol{\varepsilon}}^* = \dot{\boldsymbol{\varepsilon}}_e^* + \dot{\mathbf{F}}_p^*$. The counterpart of principle (3.14) at time $t + \Delta t$ reads

$$\int_S \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, dS - \int_v \rho \dot{\psi}(\boldsymbol{\varepsilon}_{e\Delta}, \mathbf{F}_{p\Delta}) \, dv - \int_v \mathcal{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}) \, dv$$

$$= 0 > \int_S \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta^* \, dS - \int_v \rho \dot{\psi}^*(\boldsymbol{\varepsilon}_{e\Delta}, \mathbf{F}_{p\Delta}) \, dv - \int_v \mathcal{D}(\dot{\mathbf{F}}_{p\Delta}^*, \mathbf{F}_{p\Delta}) \, dv, \tag{3.15}$$

where

$$\dot{\psi}^*(\boldsymbol{\varepsilon}_{e\Delta}, \mathbf{F}_{p\Delta}) = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}_{e\Delta}} : \dot{\boldsymbol{\varepsilon}}_{e\Delta}^* + \frac{\partial \psi}{\partial \mathbf{F}'_{p\Delta}} : \dot{\mathbf{F}}_{p\Delta}^*, \qquad \mathbf{T}_\Delta = \rho \frac{\partial \psi(\boldsymbol{\varepsilon}_{e\Delta}, \mathbf{F}_{p\Delta})}{\partial \boldsymbol{\varepsilon}_{e\Delta}},$$

$$\varphi(\mathbf{T}_\Delta, \mathbf{F}_{p\Delta}) \le 0 \quad \text{and} \quad \nabla \cdot \mathbf{T}_\Delta = \mathbf{0}.$$

3.3 Elastoplastic materials at finite strains

Equations of Section 2.5 will be used in this section. The second law of thermodynamics results in equation (3.11), where

$$\mathcal{D} = \mathbf{X}(\dot{\mathbf{F}}_p, \mathbf{F}_p) : \dot{\mathbf{F}}_p, \qquad \mathbf{X} = \mathbf{P}' \cdot \mathbf{F}_e - \rho \frac{\partial \psi}{\partial \mathbf{F}'_p}.$$

For all admissible fields $\mathbf{v}^*$

$$\int_S \mathbf{p} \cdot \mathbf{v}^* \, dS - \int_v \rho \dot{\psi}^* \, dv - \int_v \mathbf{X} : \dot{\mathbf{F}}_p^* \, dv = 0, \tag{3.16}$$

where

$$\dot{\psi}^* = \frac{\partial \psi}{\partial \mathbf{F}'_e} : \dot{\mathbf{F}}_e^* + \mathbf{K} : \dot{\mathbf{F}}_p^*, \qquad \mathbf{F}_p \cdot \mathbf{P}' = \rho \frac{\partial \psi}{\partial \mathbf{F}_e}, \qquad \varphi(\mathbf{P}, \mathbf{F}_p) \le 0 \quad \text{and} \quad \nabla \cdot \mathbf{P} = \mathbf{0}.$$

Indeed, using the Gauss theorem

$$\int_S \mathbf{p}\cdot\mathbf{v}^*\,\mathrm{d}S - \int_v \rho\dot{\psi}^*\,\mathrm{d}v = \int_v (\mathbf{P}':\dot{\mathbf{F}}^* - \rho\dot{\psi}^*)\,\mathrm{d}v$$

$$= \int_v \left[\left(\mathbf{F}_\mathrm{p}\cdot\mathbf{P}' - \rho\frac{\partial\psi}{\partial\mathbf{F}'_\mathrm{e}}\right):\dot{\mathbf{F}}^*_\mathrm{e} + \left(\mathbf{P}'\cdot\mathbf{F}_\mathrm{e} - \rho\frac{\partial\psi}{\partial\mathbf{F}'_\mathrm{p}}\right):\dot{\mathbf{F}}^*_\mathrm{p}\right]\mathrm{d}v$$

$$= \int_v \mathbf{X}:\dot{\mathbf{F}}^*_\mathrm{p}\,\mathrm{d}v. \tag{3.17}$$

As $\mathbf{X}:\dot{\mathbf{F}}^*_\mathrm{p} < \mathscr{D}(\dot{\mathbf{F}}^*_\mathrm{p}, \mathbf{F}_\mathrm{p})\ \forall\dot{\mathbf{F}}^*_\mathrm{p} \neq \mathbf{F}_\mathrm{p}$, $\dot{\mathbf{F}}^*_\mathrm{p} \neq \mathbf{0}$, then

$$\int_S \mathbf{p}\cdot\mathbf{v}\,\mathrm{d}S - \int_v \rho\dot{\psi}\,\mathrm{d}v - \int_v \mathscr{D}(\dot{\mathbf{F}}_\mathrm{p}, \mathbf{F}_\mathrm{p})\,\mathrm{d}v = 0 > \int_S \mathbf{p}\cdot\mathbf{v}^*\,\mathrm{d}S - \int_v \rho\dot{\psi}^*\,\mathrm{d}v - \int_v \mathscr{D}(\dot{\mathbf{F}}^*_\mathrm{p}, \mathbf{F}_\mathrm{p})\,\mathrm{d}v. \tag{3.18}$$

This principle could be obtained using the realizability postulate for the finite volume v by the same way as in Section 3.2. We also did not use any assumption about $\dot{\mathbf{F}}^*_\mathrm{p}$ and $\dot{\mathbf{F}}^*_\mathrm{e}$, only $\dot{\mathbf{F}}^* = \dot{\mathbf{F}}^*_\mathrm{e}\cdot\mathbf{F}_\mathrm{p} + \mathbf{F}_\mathrm{e}\cdot\dot{\mathbf{F}}^*_\mathrm{p}$. The counterpart of the principle (3.18) at time $t+\Delta t$ reads

$$\int_S \mathbf{p}_\Delta\cdot\mathbf{v}_\Delta\,\mathrm{d}S - \int_v \rho\psi(\mathbf{F}_{\mathrm{e}\Delta}, \mathbf{F}_{\mathrm{p}\Delta})\,\mathrm{d}v - \int_v \mathscr{D}(\dot{\mathbf{F}}_{\mathrm{p}\Delta}, \mathbf{F}_{\mathrm{p}\Delta})\,\mathrm{d}v$$

$$= 0 > \int_S \mathbf{p}_\Delta\cdot\mathbf{v}^*_\Delta\,\mathrm{d}S - \int_v \rho\psi^*(\mathbf{F}_{\mathrm{e}\Delta}, \mathbf{F}_{\mathrm{p}\Delta})\,\mathrm{d}v - \int_v \mathscr{D}(\dot{\mathbf{F}}^*_{\mathrm{p}\Delta}, \mathbf{F}_{\mathrm{p}\Delta})\,\mathrm{d}v \tag{3.19}$$

where

$$\mathbf{F}_{\mathrm{p}\Delta}\cdot\mathbf{P}'_\Delta = \rho\frac{\partial\psi(\mathbf{F}_{\mathrm{e}\Delta}, \mathbf{F}_{\mathrm{p}\Delta})}{\partial\mathbf{F}_{\mathrm{e}\Delta}}, \qquad \varphi(\mathbf{P}_\Delta, \mathbf{F}_{\mathrm{p}\Delta}) \le 0 \quad \text{and} \quad \nabla\cdot\mathbf{P}_\Delta = \mathbf{0}.$$

4. THE GOVERNING EXTREMUM PRINCIPLES FOR THE DESCRIPTION OF THE STABLE POST-BIFURCATION PROCESS

Let the extremum principles (3.3), (3.6) or (3.18), (3.19) admit more than one solution at time t; consequently they indicate possible bifurcation. We will label all solutions of these principles with superscript 0—$\mathbf{v}^0$, $\dot{\mathbf{F}}^0$, $\dot{\mathbf{F}}^0_\mathrm{p}$, $\dot{\mathbf{F}}^0_\mathrm{e}$. There is an important difference between $\mathbf{v}^*$ and $\mathbf{v}^0$ fields: $\mathbf{v}^*$ are the kinematically admissible ones, which do not satisfy any principle, but all fields $\mathbf{v}^0$ satisfy the corresponding extremum principle. To choose one solution $\mathbf{v}$ among all admissible $\mathbf{v}^0$ we will use the postulate of realizability.

4.1 Rigid-plastic materials

Assume that only one solution $\mathbf{v}$ and $\dot{\mathbf{F}}_\mathrm{p}$ among all possible $\mathbf{v}^0$, $\dot{\mathbf{F}}^0_\mathrm{p}$ meet the postulate of realizability at $t+\Delta t$. Figure 2 gives a vivid geometrical interpretation. It means that the extremum principle (3.6)

$$\int_S \mathbf{p}_\Delta\cdot\mathbf{v}_\Delta\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}_{\mathrm{p}\Delta}, \mathbf{F}_{\mathrm{p}\Delta})\,\mathrm{d}v = 0 > \int_S \mathbf{p}_\Delta\cdot\mathbf{v}^*_\Delta\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}^*_{\mathrm{p}\Delta}, \mathbf{F}_{\mathrm{p}\Delta})\,\mathrm{d}v \tag{4.1}$$

for $\mathbf{F}_{\mathrm{p}\Delta} = \mathbf{F}_\mathrm{p} + \dot{\mathbf{F}}_\mathrm{p}\,\Delta t$, $\dot{\mathbf{F}}^*_\mathrm{p} \neq \dot{\mathbf{F}}_\mathrm{p}$, $\dot{\mathbf{F}}^*_\mathrm{p} \neq \mathbf{0}$ is valid. If other solutions $\mathbf{v}^0$, $\dot{\mathbf{F}}^0_\mathrm{p}$ and $\dot{\mathbf{F}}^0$ do not satisfy the postulate of realizability, then at $\mathbf{p}^0_\Delta = \mathbf{p}(t) + \dot{\mathbf{p}}^0\,\Delta t$; $\mathbf{F}^0_{\mathrm{p}\Delta} = \mathbf{F}_\mathrm{p}(t) + \dot{\mathbf{F}}^0_\mathrm{p}\,\Delta t$.

$$\exists\mathbf{v}^*, \dot{\mathbf{F}}^*_\mathrm{p} \quad \text{for which} \quad \int_{S_v} \mathbf{p}^0_\Delta\cdot\mathbf{v}_\Delta\,\mathrm{d}S + \int_{S_p} \mathbf{p}_\Delta\cdot\mathbf{v}^*_\Delta\,\mathrm{d}S - \int_v \mathscr{D}(\dot{\mathbf{F}}^*_{\mathrm{p}\Delta}, \mathbf{F}^0_{\mathrm{p}\Delta})\,\mathrm{d}v > 0. \tag{4.2}$$

In the first step we obtain fields $\mathbf{F}_{\mathrm{p}\Delta}, \mathbf{p}_\Delta$ in principle (4.1) and $\mathbf{F}^0_{\mathrm{p}\Delta}$ and $\mathbf{p}^0_\Delta$ in principle (4.2) at

$t + \Delta t$. And then, at $t + \Delta t$ and the same boundary conditions, we consider all possible fields $\mathbf{v}^*_\Delta$ and $\dot{\mathbf{F}}^*_{p\Delta}$.

Let us consider what may happen from the physical point of view when the postulate of realizability is violated. The solution $\mathbf{v}^0$ may be realized quasistatically (by definition). But if some perturbations give field $\mathbf{v}^*_\Delta \neq \mathbf{v}^0_\Delta$, for which inequality (4.2) is valid, then the power of the external forces will exceed the dissipated power and a positive increment of the kinetic energy will occur. Consequently, under perturbations the dynamical jump from $\mathbf{v}^0$ to some $\mathbf{v}^*$ is possible and the solutions $\mathbf{v}^0$ are unstable in this sense.

If the postulate of realizability is met, then according to principle (4.1), the jump from $\mathbf{v}_\Delta$ to $\mathbf{v}^*_\Delta \neq \mathbf{v}_\Delta$ will be accompanied by the negative increment of the kinetic energy. As the initial value of the kinetic energy is zero (the process is a quasi-equilibrium one), it is impossible. Thus, using the postulate of realizability we could choose the unique stable solution $\mathbf{v}$ among the possible unstable $\mathbf{v}^0$. Making use of this procedure at each time instant, we can find the stable post-bifurcation process.

It is reasonable to assume that inequality (4.2) is valid also for $\mathbf{v}^*_\Delta = \mathbf{v}_\Delta$, i.e. from all possible solutions $\mathbf{v}^0$ we will have a jump to the unique stable one. Then equations (4.1) and (4.2) could be used in the form

$$\begin{aligned} &\int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, dS + \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta \, dS - \int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}) \, dv > 0 \\ &= \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, dS + \int_{S_v} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, dS - \int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}) \, dv \\ &= \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}^0_\Delta \, dS + \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta \, dS - \int_v \mathscr{D}(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}^0_{p\Delta}) \, dv \\ &> \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}^0_\Delta \, dS + \int_{S_v} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, dS - \int_v \mathscr{D}(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}_{p\Delta}) \, dv. \end{aligned} \tag{4.3}$$

The first line in equation (4.3) shows that if we choose a solution $(\mathbf{F}^0_{p\Delta}, \mathbf{p}^0_\Delta)$, it will be unstable under perturbations, which gives $\mathbf{v}_\Delta$; the second and third lines in equation (4.3) show that the solutions $\mathbf{v}_\Delta$ and $\mathbf{v}^0_\Delta$ are admissible; the fourth line of equation (4.3) reveals that if we choose a unique solution $\mathbf{v}_\Delta$, it will be stable under perturbations which gives $\mathbf{v}^0_\Delta$.

If $S = S_v$ or $\mathbf{p}_\Delta = \mathbf{0}$ on S_p, then it follows from the principle (4.3)

$$\begin{aligned} &\int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}) \, dv < \int_v \mathscr{D}(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}_{p\Delta}) \, dv, \\ &\int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}) \, dv < \int_v \mathscr{D}(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}^0_{p\Delta}) \, dv. \end{aligned} \tag{4.4}$$

If $S = S_p$ or $\mathbf{v}_\Delta = \mathbf{0}$ on S_v, then

$$\begin{aligned} &\int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}) \, dv < \int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}) \, dv, \\ &\int_v \mathscr{D}(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}^0_{p\Delta}) \, dv < \int_v \mathscr{D}(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}_{p\Delta}) \, dv. \end{aligned} \tag{4.5}$$

Consider the question of the value of Δt. At $\Delta t = 0$ and perhaps for positive Δt, the extremum principle (4.3) admits more than one solution $\mathbf{v}^0$. We have to increase Δt up to a value which is sufficient to choose the unique solution from the principle (4.3). The solution of the problem (4.3) may depend on Δt, but the solution which can be realized the first time will be real (according the postulate of realizability). If a unique solution cannot be found for infinitesimal Δt, we have to consider finite Δt.

We consider a fixed volume in the reference configuration, but consideration of time $t + \Delta t$ and variation of $\mathbf{v}^0$ includes the variation of the geometry of the current configuration.

If a stable unique solution can be found for infinitesimal Δt, then from principles (4.3) using the similar procedure as in Section 3.1 we have

$$\begin{aligned}
&\int_{S_p} \dot{\mathbf{p}} \cdot \mathbf{v}\, dS + \int_{S_v} \dot{\mathbf{p}}^0 \cdot \mathbf{v}\, dS - \int_v \frac{\partial \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}^0_p\, dv > 0 \\
&= \int_{S_p} \dot{\mathbf{p}} \cdot \mathbf{v}\, dS + \int_{S_v} \dot{\mathbf{p}} \cdot \mathbf{v}\, dS - \int_v \frac{\partial \mathscr{D}(\dot{\mathbf{F}}_p, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}_p\, dv \\
&= \int_{S_p} \dot{\mathbf{p}} \cdot \mathbf{v}^0\, dS + \int_{S_v} \dot{\mathbf{p}}^0 \cdot \mathbf{v}\, dS - \int_v \frac{\partial \mathscr{D}(\dot{\mathbf{F}}^0_p, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}^0_p\, dv \\
&> \int_{S_p} \dot{\mathbf{p}} \cdot \mathbf{v}^0\, dS + \int_{S_v} \dot{\mathbf{p}} \cdot \mathbf{v}\, dS - \int_v \frac{\partial \mathscr{D}(\dot{\mathbf{F}}^0_p, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}_p\, dv.
\end{aligned} \tag{4.6}$$

It is easy to obtain the counterparts of principles (4.4), (4.5).

Let us have a finite number of solutions $\mathbf{v}^0 = \mathbf{v}_i$, $i = 1, 2, \ldots, m$. Assume that one of them, $\mathbf{v}_j$, is the stable solution. Then

$$0 > \int_S \dot{\mathbf{p}} \cdot \mathbf{v}_i\, dS - \int_v \frac{\partial \mathscr{D}(\dot{\mathbf{F}}_{pi}, \mathbf{F}_p)}{\partial \mathbf{F}'_p} : \dot{\mathbf{F}}_{pj}\, dv \quad \forall i \neq j. \tag{4.7}$$

If at any fixed j inequality (4.7) is valid for all $i \neq j$, then $\dot{\mathbf{F}}_{pj}$ is really a unique stable solution of the problem; if not, we have to check the next j. The same procedure can be applied when we use principle (4.3).

4.2 Elastoplastic materials at finite strains

Let the extremum principles (3.18) or (3.19) admit more than one solution. Using the same approach based on the postulate of realizability as in Section 4.1, we obtain the generalizations of the above principles for finding the unique stable solution:

$$\begin{aligned}
&\int_S \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta\, dS - \int_v \rho\dot{\psi}(\mathbf{F}_{e\Delta}, \mathbf{F}_{p\Delta})\, dv - \int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta})\, dv \\
&\qquad = 0 > \int_S \mathbf{p}_\Delta \cdot \mathbf{v}^*_\Delta\, dS - \int_v \rho\dot{\psi}^*(\mathbf{F}_{e\Delta}, \mathbf{F}_{p\Delta})\, dv - \int_v \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_{p\Delta})\, dv,
\end{aligned} \tag{4.8}$$

$\exists \mathbf{v}^*_\Delta, \dot{\mathbf{F}}^*_\Delta, \dot{\mathbf{F}}^*_{e\Delta}, \dot{\mathbf{F}}^*_{p\Delta}$ for which

$$\int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta\, dS + \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}^*_\Delta\, dS - \int_v \rho\dot{\psi}^*(\mathbf{F}^0_{e\Delta}, \mathbf{F}^0_{p\Delta})\, dv - \int_v \mathscr{D}(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}^0_{p\Delta})\, dv > 0, \tag{4.9}$$

where

$$\rho\dot{\psi}^*(\mathbf{F}^0_{e\Delta}, \mathbf{F}^0_{p\Delta}) = \rho\left(\frac{\partial \psi}{\partial \mathbf{F}'_e}\right)^0_\Delta : \dot{\mathbf{F}}^*_{e\Delta} + \rho\left(\frac{\partial \psi}{\partial \mathbf{F}'_p}\right)^0_\Delta : \dot{\mathbf{F}}^*_{p\Delta}. \tag{4.10}$$

The counterpart of principle (4.3) looks

$$\begin{aligned}
&\int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta\, dS + \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta\, dS - \int_v \rho\dot{\psi}(\mathbf{F}^0_{e\Delta}, \mathbf{F}^0_{p\Delta})\, dv - \int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta})\, dv > 0 \\
&= \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta\, dS + \int_{S_v} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta\, dS - \int_v \rho\dot{\psi}(\mathbf{F}_{e\Delta}, \mathbf{F}_{p\Delta})\, dv - \int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta})\, dv \\
&= \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}^0_\Delta\, dS + \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta\, dS - \int_v \rho\dot{\psi}^0(\mathbf{F}^0_{e\Delta}, \mathbf{F}^0_{p\Delta})\, dv - \int_v \mathscr{D}(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}^0_{p\Delta})\, dv \\
&> \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}^0_\Delta\, dS + \int_{S_v} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta\, dS - \int_v \rho\dot{\psi}^0(\mathbf{F}_{e\Delta}, \mathbf{F}_{p\Delta})\, dv - \int_v \mathscr{D}(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}_{p\Delta})\, dv.
\end{aligned} \tag{4.11}$$

The counterparts of the principles (4.3), (4.5) can be obtained in a similar way.

REMARK. All possible solutions $\mathbf{v}^0, \dot{\mathbf{F}}^0, \ldots$ at time t can be obtained without the principles of Section 3 using arbitrary known methods. This does not affect the applicability of the principles of this section for the choice of a unique stable solution.

4.3 Alternative governing principle

In principle (4.3) for rigid-plastic materials, all the volume integrals represent the strain power of the corresponding stress and strain rates, e.g. $\mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}):\dot{\mathbf{F}}_{p\Delta}$ or $\mathbf{P}^{0\prime}_\Delta(\dot{\mathbf{F}}^0_{p\Delta}):\dot{\mathbf{F}}^0_{p\Delta}$, calculated after some of the processes in time $[t, t+\Delta t]$. In principle (3.15) for elastoplastic material for $\mathbf{K}=\mathbf{0}$ we have

$$\int_S \mathbf{p}_\Delta \cdot \mathbf{v}^*_\Delta \, \mathrm{d}S - \int_v (\mathbf{T}_\Delta(\boldsymbol{\varepsilon}_{e\Delta}, \mathbf{F}_{p\Delta}):\dot{\boldsymbol{\varepsilon}}^*_{e\Delta} + \mathbf{T}^*_\Delta(\dot{\mathbf{F}}^*_{p\Delta}, \mathbf{F}_{p\Delta}):\dot{\mathbf{F}}^*_{p\Delta}) \, \mathrm{d}v < 0, \tag{4.12}$$

i.e. two unequal stresses $\mathbf{T}_\Delta$ and $\mathbf{T}^*_\Delta$ are used. But $\dot{\mathbf{F}}^*_{p\Delta}$ and $\mathbf{T}^*_\Delta$ are virtual fields only, in the particular case, $\nabla \cdot \mathbf{T}^*_\Delta \neq \mathbf{0}$, and the condition $\mathbf{T}_\Delta \neq \mathbf{T}^*_\Delta$ is natural.

The same inequality takes place in principle (4.11). But in these principles we consider the possibility of the realization not of virtual solutions, but of solutions which satisfy all the equations of continuum mechanics. Therefore we can assume additional conditions: we will use in the expression for $\dot{\psi}$ the same tensor $\mathbf{P}_\Delta$ and corresponding to it $\mathbf{F}_{e\Delta}$ as in the expression for $\mathscr{D}$. Consequently, instead of the first line of principle (4.11) we have

$$\int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S + \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S - \int_v \rho\dot{\psi}(\mathbf{F}_{e\Delta}, \mathbf{F}^0_{p\Delta}) \, \mathrm{d}v - \int_v \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}) \, \mathrm{d}v > 0. \tag{4.13}$$

But $\rho\dot{\psi}(\mathbf{F}_{e\Delta}, \mathbf{F}^0_{p\Delta}) + \mathscr{D}(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}) = \mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}):\dot{\mathbf{F}}_\Delta$, and from equation (4.13) we have

$$\int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S + \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S - \int_v \mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}):\dot{\mathbf{F}}_\Delta \, \mathrm{d}v > 0. \tag{4.14}$$

Consequently, instead of the principle (4.11) we obtain

$$\begin{aligned}
&\int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S + \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S - \int_v \mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}):\dot{\mathbf{F}}_\Delta \, \mathrm{d}v > 0 \\
&= \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S + \int_{S_v} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S - \int_v \mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}):\dot{\mathbf{F}}_\Delta \, \mathrm{d}v \\
&= \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}^0_\Delta \, \mathrm{d}S + \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S - \int_v \mathbf{P}^{0\prime}_\Delta(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}^0_{p\Delta}):\dot{\mathbf{F}}^0_\Delta \, \mathrm{d}v \\
&> \int_{S_p} \mathbf{p}_\Delta \mathbf{v}^0_\Delta \, \mathrm{d}S + \int_{S_v} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, \mathrm{d}S - \int_v \mathbf{P}^{0\prime}_\Delta(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}_{p\Delta}):\dot{\mathbf{F}}^0_\Delta \, \mathrm{d}v,
\end{aligned} \tag{4.15}$$

where a similar assumption was used for the last line of the principle (4.11). Principle (4.15) is valid for rigid and elastoplastic materials at small and finite strains. It is obtained from the postulate of realizability [from principle (4.11)] under an additional assumption about the possibility of jumps of $\mathbf{F}_{e\Delta}$ and $\mathbf{F}^0_{e\Delta}$ tensors.

Remember that in the first line of principle (4.15) we realize in time $[t, t+\Delta t]$ an unstable solution (the symbols $\mathbf{p}^0_\Delta$ and $\mathbf{F}^0_{p\Delta}$ show this), and then check the possibility of the realization of stable solutions $\mathbf{v}_\Delta, \dot{\mathbf{F}}_\Delta, \dot{\mathbf{F}}_{p\Delta}, \ldots$. As the power of the external forces exceeds the power of the internal stresses, this difference can be realized as kinetic energy, when we have a jump from an unstable solution to a stable one.

In the last line of the principle (4.15) we realize in time $[t, t+\Delta t]$ a stable solution (the symbols $\mathbf{p}_\Delta$ and $\mathbf{F}_{p\Delta}$ show this), and then check the possibility of the realization of the unstable solutions $\mathbf{v}^0_\Delta, \dot{\mathbf{F}}^0_\Delta, \dot{\mathbf{F}}^0_{p\Delta}, \ldots$. As the power of the internal stresses exceeds the power of the external forces, we can have a negative increment of kinetic energy. This is impossible, because the initial value of the kinetic energy for quasi-static processes is zero.

If $S = S_v$ or $\mathbf{p}_\Delta = \mathbf{0}$ on S_p, then it follows from the principle (4.15)

$$\int_v \mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}_{p\Delta}) : \dot{\mathbf{F}}_\Delta \, dv < \int_v \mathbf{P}^{0\prime}_\Delta(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}_{p\Delta}) : \dot{\mathbf{F}}^0_\Delta \, dv, \tag{4.16}$$

$$\int_v \mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}) : \dot{\mathbf{F}}_\Delta \, dv < \int_v \mathbf{P}^{0\prime}_\Delta(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}^0_{p\Delta}) : \dot{\mathbf{F}}^0_\Delta \, dv. \tag{4.17}$$

Let us define on S_v stress vector distributions $\bar{\mathbf{p}}^0_\Delta(\mathbf{r}_\tau)$ and $\bar{\mathbf{p}}_\Delta(\mathbf{r}_\tau)$ with the formulas

$$\int_{S_v} \bar{\mathbf{p}}^0_\Delta \cdot \mathbf{v}_\Delta \, dS = \int_v \mathbf{P}^{0\prime}_\Delta(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}_{p\Delta}) : \dot{\mathbf{F}}^0_\Delta \, dv. \tag{4.18}$$

$$\int_{S_v} \bar{\mathbf{p}}_\Delta \cdot \mathbf{v}_\Delta \, dS = \int_v \mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}) : \dot{\mathbf{F}}_\Delta \, dv. \tag{4.19}$$

Then using the principles (4.15)–(4.17) we obtain

$$\int_{S_v} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, dS < \int_{S_v} \bar{\mathbf{p}}^0_\Delta \cdot \mathbf{v}_\Delta \, dS; \qquad \int_{S_v} \bar{\mathbf{p}}_\Delta \cdot \mathbf{v}_\Delta \, dS < \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{v}_\Delta \, dS. \tag{4.20}$$

If on surface S_v the velocity vector $\mathbf{v}_\Delta = |\mathbf{v}_\Delta| \, \mathbf{m}$ does not depend on $\mathbf{r}_\tau$, then it follows from equation (4.20) the relations between the corresponding force components in direction $\mathbf{m}$

$$\int_{S_v} \mathbf{p}_\Delta \cdot \mathbf{m} \, dS < \int_{S_v} \bar{\mathbf{p}}^0_\Delta \cdot \mathbf{m} \, dS; \qquad \int_{S_v} \bar{\mathbf{p}}_\Delta \cdot \mathbf{m} \, dS < \int_{S_v} \mathbf{p}^0_\Delta \cdot \mathbf{m} \, dS. \tag{4.21}$$

Let us comment on equations (4.18)–(4.21). Let us realize the stable solution $\dot{\mathbf{F}}$ at the time t and the unstable one $\dot{\mathbf{F}}^0_\Delta$ at the time $t+\Delta t$, respectively. We could calculate the stress power at the time $t+\Delta t$ and the stress vector $\bar{\mathbf{p}}^0_\Delta$ distribution for the realization of the solution $\dot{\mathbf{F}}^0$ [equation (4.18)]. Then according to equation (4.20) the power of $\bar{\mathbf{p}}^0_\Delta$—distribution exceeds the power of stress vector $\mathbf{p}_\Delta$—distribution which corresponds to the stable solution. Equation (4.20) gives the corresponding relation for forces. The similar situation occurs for equations (4.19), $(4.20)_2$ and $(4.21)_2$. The difference is that at the time t we realize the unstable solution $\dot{\mathbf{F}}^0$ and compare the stress power and the force, which are necessary for realization at the time $t+\Delta t$ the stable and the unstable solutions.

Figure 3 gives interpretation of equation (4.21) in a one-dimensional case. In Fig. 3(a) at the time t stable 1 or unstable 2 solutions are realized. In Fig. 3(b) at the time t the stable solution is realized and for the realization of the unstable solution at the time $t+\Delta t$ the higher force is required. In Fig. 3(c) at the time t the unstable solution is realized and for the realization of the stable solution at the time $t+\Delta t$ the smaller force is needed.

From the principles (4.21) do not strictly follows $p^0_\Delta > p_\Delta$ [as shown in line 2 in Fig. 3(a)], because $p^0_\Delta \neq \bar{p}^0_\Delta$ and $p_\Delta \neq \bar{p}_\Delta$. It is possible to imagine that for very complex strain history dependence of the system the inequality $p_\Delta > p^0_\Delta$ is possible. This inequality does not mean that difference $p_\Delta - p^0_\Delta$ could produce positive increment of the kinetic energy at the realization of the unstable solution. After realization at the time t of the stable solution we need for the realization of the unstable one at the time $t+\Delta t$ the force $\bar{p}^0_\Delta$, which is greater than p_Δ due to

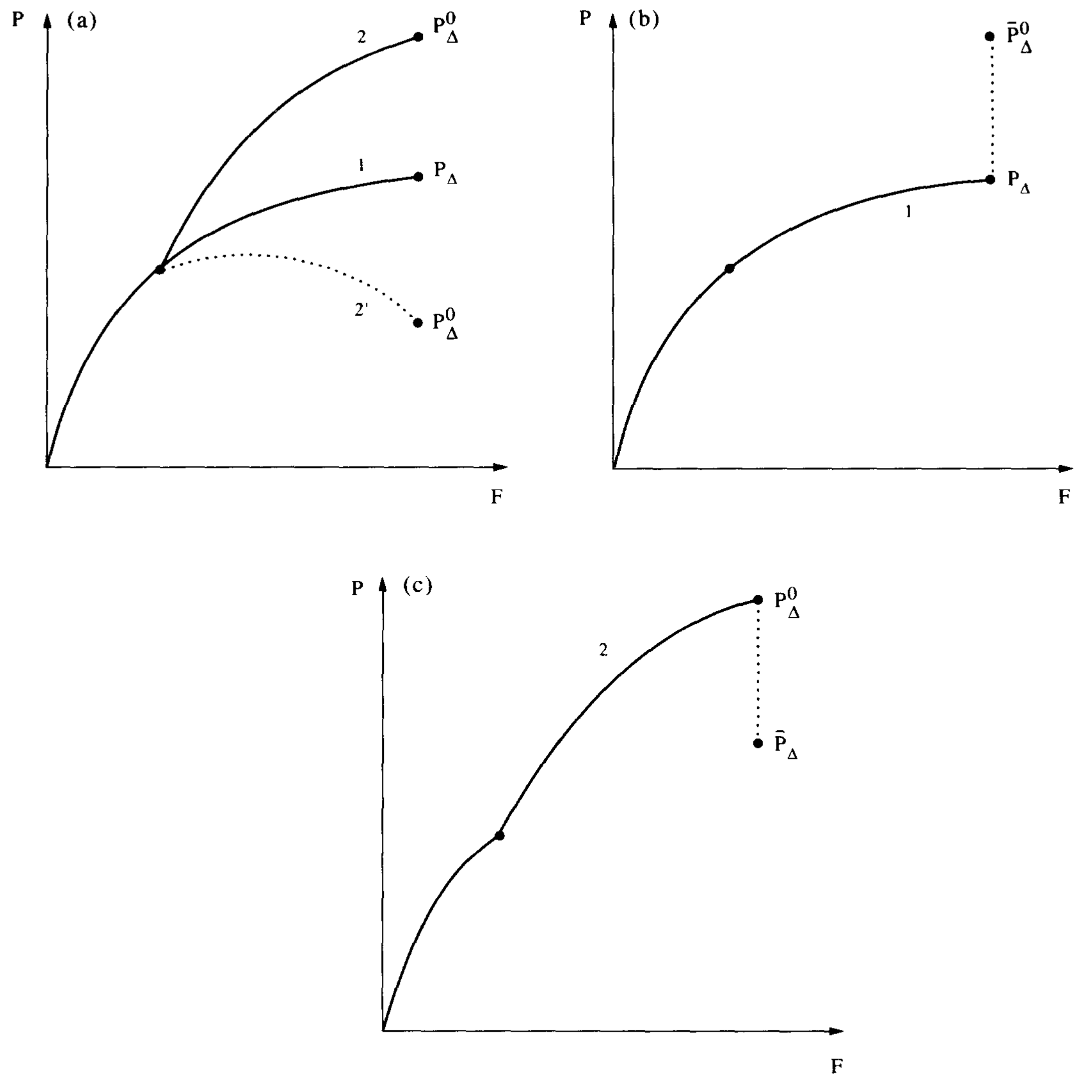

Fig. 3. Interpretation of equation (4.21) in a one-dimensional case.

specific history dependence. If such a specific history dependence does not exist, then $p^0_\Delta > p_\Delta$, and we obtain for determination of the stable solution the principle of minima of the second order work, which was suggested in [13].

Let us consider stress controlled loading, $S = S_p$ or $\mathbf{v}_\Delta = 0$ on S_v. We could not derive the principles similar to (4.16), (4.17), because after realization the same (stable or unstable) solutions at the time t we have different velocity field on S_p in corresponding line of equation (4.15). But we could define on S_p the stress vector distributions $\tilde{\mathbf{p}}^0_\Delta(\mathbf{r}_\tau)$ and $\tilde{\mathbf{p}}_\Delta(\mathbf{r}_\tau)$ using the formulas

$$\int_{S_p} \tilde{\mathbf{p}}^0_\Delta \cdot \mathbf{v}^0_\Delta \, dS = \int_v \mathbf{P}^{0\prime}_\Delta(\dot{\mathbf{F}}^0_{p\Delta}, \mathbf{F}_{p\Delta}) : \dot{\mathbf{F}}^0_\Delta \, dv. \tag{4.22}$$

$$\int_{S_p} \tilde{\mathbf{p}}_\Delta \cdot \mathbf{v}_\Delta \, dS = \int_v \mathbf{P}'_\Delta(\dot{\mathbf{F}}_{p\Delta}, \mathbf{F}^0_{p\Delta}) : \dot{\mathbf{F}}_\Delta \, dv. \tag{4.23}$$

Then using the first and the fourth lines in principle (4.15) we obtain

$$\int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}_\Delta \, dS > \int_{S_p} \tilde{\mathbf{p}}_\Delta \cdot \mathbf{v}_\Delta \, dS; \qquad \int_{S_p} \mathbf{p}_\Delta \cdot \mathbf{v}^0_\Delta \, dS < \int_{S_p} \tilde{\mathbf{p}}^0_\Delta \cdot \mathbf{v}^0_\Delta \, dS, \tag{4.24}$$

or at uniformly distributed $\mathbf{p}_\Delta$, $\tilde{\mathbf{p}}_\Delta$ and $\tilde{\mathbf{p}}^0_\Delta$ on S_p

$$\mathbf{p}_\Delta \cdot \bar{\mathbf{m}} > \tilde{\mathbf{p}}_\Delta \cdot \bar{\mathbf{m}}; \qquad \mathbf{p}_\Delta \cdot \bar{\mathbf{m}}^0 < \tilde{\mathbf{p}}^0_\Delta \cdot \bar{\mathbf{m}}^0; \tag{4.25}$$

where

$$\bar{\mathbf{m}} = \int_{S_p} \mathbf{m}\, dS, \qquad \bar{\mathbf{m}}^0 = \int_{S_p} \mathbf{m}^0\, dS, \qquad \mathbf{v}^0_\Delta = |\mathbf{v}^0_\Delta|\, \mathbf{m}^0.$$

Let us realize at the time t the stable solution $\dot{\mathbf{F}}$. If we calculate stress vector $\tilde{\mathbf{p}}^0_\Delta$-distribution and its power, which are necessary for the realization of the unstable solution at the time $t + \Delta t$ [equation (4.22)], then this power will exceed the power of the prescribed stress vector on the same velocity field $(4.24)_2$. Similar comments are valid for equations (4.23), $(4.24)_1$ and (4.25). The interpretation of (4.25) in a one-dimensional case is given in Fig. 4.

It is necessary to notice the following. We have prescribed stresses on S_p and different displacements and actual configurations after the realization of the stable and unstable solutions [Fig. 4(a)]. Despite of the above in the principles (4.24), (4.25) we fixed configuration and velocity fields on S_p and compare real and some calculated stress vectors [Figs 4(b), (c)].

REMARK 1. The situation is possible when for stable $\mathbf{v}_\Delta$ and one of the unstable solutions $\mathbf{v}^0_\Delta$ we

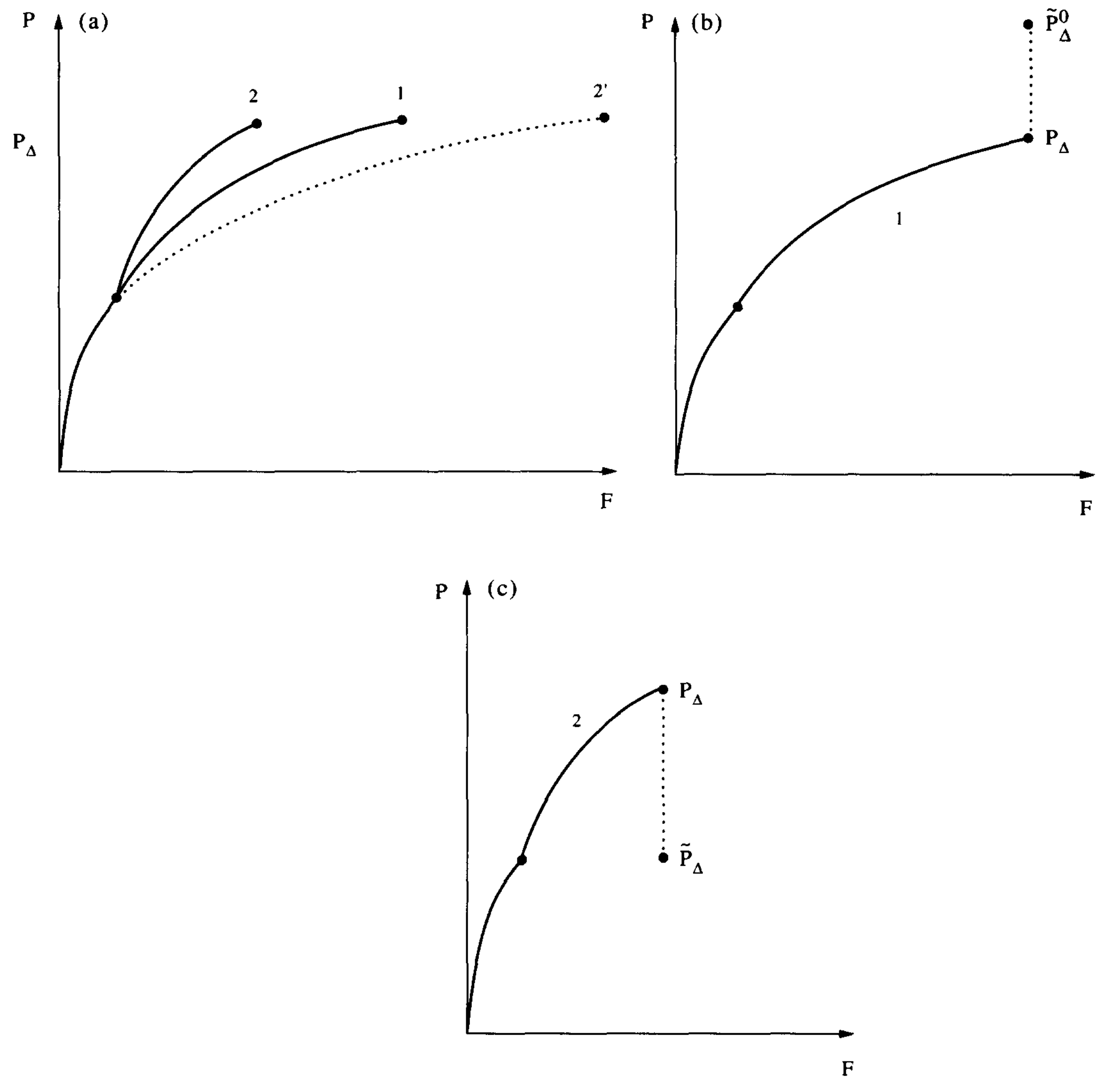

Fig. 4. Interpretation of equation (4.25) in a one-dimensional case.

cannot get two inequalities in principle (4.15), but one inequality and one equality (examples see below). For instance, in the first line of equations (4.15) we have an inequality, but in the last line we have an equality or vice versa. But even one inequality is enough to choose the unique stable solution. The equality in the last line of the principle (4.15) means, that the solutions $\mathbf{v}_\Delta$ and $\mathbf{v}_\Delta^0$ are equiprobable. But even in this case if the solution $\mathbf{v}_\Delta^0$ is realized, then inequality in the first line of the principle (4.15) means, that the jump from $\mathbf{v}_\Delta^0$ to $\mathbf{v}_\Delta$ will take place. The algorithm of the application of principle (4.15) is the following one:

(i) *Assume that one of the solutions* $\mathbf{v}_j$ *is stable. Let us realize it and consider the last inequality in the principle* (4.15) $\forall i \neq j$:

$$N(i,j) = \int_{S_p} \mathbf{p}_{\Delta j} \cdot \mathbf{v}_{\Delta i}\, dS + \int_{S_v} \mathbf{p}_{\Delta j} \cdot \mathbf{v}_{\Delta j}\, dS - \int_v \mathbf{P}'_{\Delta i}(\dot{\mathbf{F}}_{p\Delta i}, \mathbf{F}_{p\Delta j}) : \dot{\mathbf{F}}_{\Delta i}\, dv < 0. \tag{4.26}$$

(ii) *If* $N(i,j) < 0\ \forall i \neq j$, *then* $\mathbf{v}_j$ *is a stable solution.*

(iii) *If for some* $i = k$ $N(k,j) > 0$, *then we have to realize* $\mathbf{v}_k$ *in time* $[t, t + \Delta t]$ (*instead of* $\mathbf{v}_j$), *consider inequality* $N(i,k) < 0\ \forall i \neq k$ *and repeat items* (*ii*) *and* (*iii*).

(iv) *If for some* $i = m$ $N(m,j) = 0$, *then we have to realize the field* $\mathbf{v}_m$ *in time* $[t, t + \Delta t]$ (*instead of* $\mathbf{v}_j$) *and consider* $N(j,m)$. *If* $N(j,m) < 0$, *then* $\mathbf{v}_m$ *is a stable solution, if* $N(j,m) > 0$, *then* $\mathbf{v}_j$ is stable.

REMARK 2. If we have kinematic constraints equations, they can be taken into account using Lagrange multipliers. But now we compare real (not only kinematically admissible) solutions $\mathbf{v}^0$ and each of them has to meet the constraints. Consequently, principle (4.15) is valid for materials with kinematic constraints and non-associated flow rule.

5. EXAMPLES

5.1 Necking

Consider a circular bar with initial cross-sectional area S and the length $2l$. The displacement or the load are prescribed at the ends of the bar (Fig. 5). The deformation gradient and the stresses are homogeneous. The onset of necking will be considered as homogeneous strain in the region v_b with length $2b$ and the absence of a strain increment in the volume v_a, $2a = 2l - 2b$. Let us show how the principles suggested in this paper work in the simplest case.

Let the velocity v be prescribed at the ends of the bar. We will consider only axial components of the velocity vector and the deformation gradient, because other ones do not produce the mechanical work at our model. Assume that at time t and at the deformation gradient F the velocity field in volume v_b corresponding to necking is realized, $\dot{F}_p^b = \mathrm{v}/b$, $\dot{F}_p^a = 0$. Consider its stability under a "homogenizing" velocity field with $\dot{F}_{p\Delta}^a = \mathrm{v}_\Delta/a$, $\dot{F}_{p\Delta}^b = 0$. This field

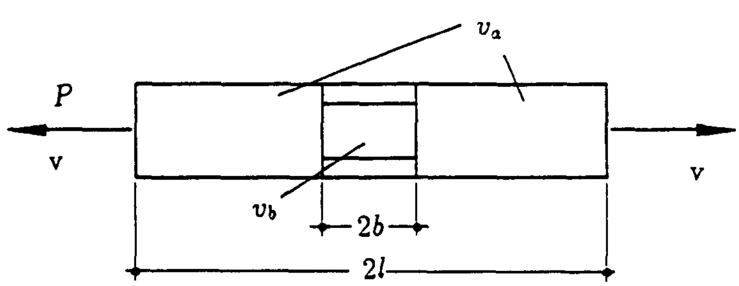

Fig. 5. Necking problem.

makes the strains in the bar homogeneous and represents the homogeneous straining in the region v_a. In this case

$$p_\Delta = P^b_\Delta = P\left(F + \frac{\mathrm{v}}{b}\Delta t\right); \qquad \int_S p_\Delta \mathrm{v}_\Delta \, \mathrm{d}S = P\left(F + \frac{\mathrm{v}}{b}\Delta t\right)\mathrm{v}_\Delta S; \tag{5.1}$$

$$\int_v P^0_\Delta(\dot{F}^0_\Delta, F_\Delta)\dot{F}^0_\Delta \, \mathrm{d}v = \int_{v_a} P^a_\Delta \dot{F}^a_\Delta \, \mathrm{d}v = P(F)\frac{\mathrm{v}_\Delta}{a} aS = P(F)\mathrm{v}_\Delta S \tag{5.2}$$

and

$$\int_S p_\Delta \mathrm{v}_\Delta \, \mathrm{d}S - \int_v P^0_\Delta(\dot{F}^0_\Delta, F_\Delta)\dot{F}^0 \, \mathrm{d}v = \left(P\left(F + \frac{\mathrm{v}}{b}\Delta t\right) - P(F)\right)\mathrm{v}_\Delta S. \tag{5.3}$$

If $P(F + \mathrm{v}\,\Delta t/b) < P(F)$, i.e. $P(F)$ is a monotonic decreasing function of F, then according to principle (4.15) the assumption about the stability of the solution with necking is valid. If $P(F + \mathrm{v}\,\Delta t/b) > P(F)$, than according to principle (4.15) this solution is unstable, because for a "homogenizing" velocity field we can get a positive increment of the kinetic energy $\Delta K = (P(F + \mathrm{v}\,\Delta t/b) - P(F))\mathrm{v}_\Delta S\,\Delta t$. The same results can be obtained in the load-controlled experiment.

Assume now that at time t the homogeneous strain and stress fields in a volume v are realized, $\dot{F} = \mathrm{v}/l$. Consider its stability under the possible necking fields: $\dot{F}^b_\Delta = \mathrm{v}_\Delta/b$, $\dot{F}^a_\Delta = 0$. In this case

$$p^0_\Delta = P_\Delta = P\left(F + \frac{\mathrm{v}}{l}\Delta t\right); \qquad \int_S p^0_\Delta \mathrm{v}_\Delta \, \mathrm{d}S = P\left(F + \frac{\mathrm{v}}{l}\Delta t\right)\mathrm{v}_\Delta S; \tag{5.4}$$

$$\int_S P_\Delta(\dot{F}_\Delta, F^0_\Delta)\dot{F}_\Delta \, \mathrm{d}v = \int_{v_a} P\left(F + \frac{\mathrm{v}}{l}\Delta t\right)\dot{F}^a_\Delta \, \mathrm{d}v = P\left(F + \frac{\mathrm{v}}{l}\Delta t\right)\mathrm{v}_\Delta S \tag{5.5}$$

and

$$\int_S p^0_\Delta \mathrm{v}_\Delta \, \mathrm{d}S - \int_v P_\Delta(\dot{F}_\Delta, F^0_\Delta)\dot{F}_\Delta \, \mathrm{d}v = 0. \tag{5.6}$$

Equation (5.6) means that under homogeneous strain and stress in the volume v in time $[t, t + \Delta t]$ we cannot choose a unique solution independently of $P(F)$ and F. It is related to the fact that the necking represents a homogeneous strain field in the volume v_a and at time $t + \Delta t$ (while we do not take into account the geometry variations) gives the same total stress power as the homogeneous solution in the volume v.

As was described in Section 4.4, if two solutions meet principle (4.15), then it is necessary to change their place: to assume that in $[t, t + \Delta t]$ the necking solution is realized and to check the "homogenizing" field. This was done above.

5.2 *Simple shear*

Consider a slab of elastoplastic material with volume v under simple shear with prescribed velocity v or shear stress τ on one edge. Let us analyse the possibility of a shear band formation with width b (volume v_b) at strain γ [Fig. 6(a)].

Assume that the homogeneous strain field in a volume v in the interval $[t, t + \Delta t]$ is realized, $F_\Delta = 1 + \gamma + \mathrm{v}\,\Delta t/l$. Consider its stability under a possible shear band formation, $\dot{F}^b_\Delta = \mathrm{v}_\Delta/b$, $\dot{F}^a_\Delta = 0$, where $v_a = v - v_b$. In this case, the power of the external stresses is equal to $\tau(\gamma + \mathrm{v}\,\Delta t/l)\mathrm{v}_\Delta$ and the power of the internal stresses is equal to $\tau(\gamma + \mathrm{v}\,\Delta t/l)\mathrm{v}_\Delta$ and we cannot choose a unique solution.

Assume further that at strain γ the strain rate field $\dot{F}^b = \mathrm{v}/b$, $\dot{F}^a = 0$ is realized, which describes the shear band formation. Consider its stability under "homogenizing" fields $\dot{F}^a_\Delta = \mathrm{v}_\Delta/a$, $\dot{F}^b_\Delta = 0$. In this case the power of the external stresses is equal to $\tau(\gamma + \mathrm{v}\,\Delta t/l)v_\Delta$, the power of the internal stresses is equal to $\tau(\gamma)\mathrm{v}_\Delta$, and their difference is $N = [\tau(\gamma + \mathrm{v}\,\Delta t/b) - \tau(\gamma)]\mathrm{v}_\Delta$.

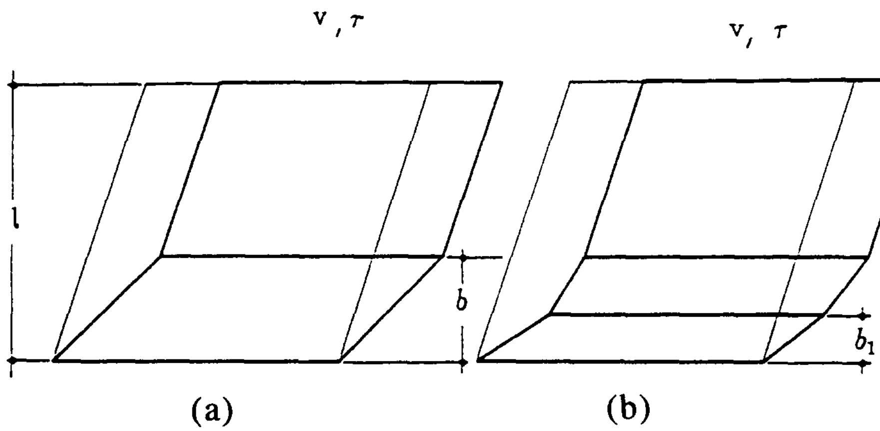

Fig. 6. Simple shearing: (a) formation of the shear band; (b) formation of a second shear band inside the first one.

If $\tau(\gamma)$ is a monotonic decreasing function of the given γ, then $N < 0$ and the shear band is stable, otherwise the *homogeneous* solution is stable.

Let $\tau(\gamma)$ be a monotonic decreasing function and a shear band with width b was formed. Consider the possible formation of a second shear band with width $b_1 < b$ inside the first one [Fig. 6(b)]. If we consider the first shear band as a whole volume v in the previous example, we obtain the same problem. Consequently, if the second band is formed in time $[t, t + \Delta t]$, then $N = [\tau(\gamma + \mathrm{v}\, \Delta t/b_1) - \tau(\gamma)]\mathrm{v}_\Delta < 0$, and the decreasing width of the shear band is in correspondence with principle (4.15). If we have a size-limiter, the band width with the minimum possible but finite b will be formed. Otherwise $b = 0$ and localization on the surface will occur. If $\tau(\gamma)$ reach 0 at some finite γ, then at $\tau = \tau_{max}$ a finite increment $\Delta\gamma$ will lead to localization on the surface, infinite strain on this surface and $\tau = 0$. Consequently, at $\tau = \tau_{max}$ the fracture will occur (Fig. 7). The same situation will be encountered in shear band formation under tension (Fig. 7).

Let us consider the following shear band width limiter at the simple shearing. Let the slab shown in Fig. 6 is the representative volume of polycrystalline materials and we decrease its height l. The assumed dependence between the overall shear stress τ and $H = l/d$ at some fixed

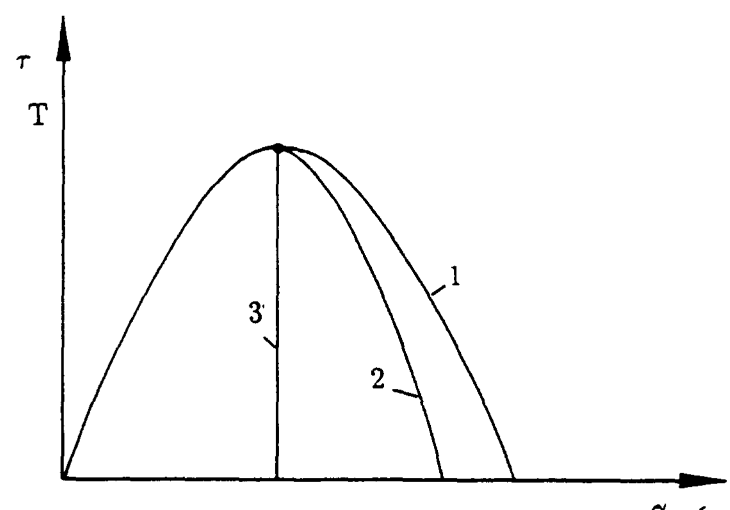

Fig. 7. Diagram of simple shearing and tension: 1—homogeneous strain; 2—strain localization with shear band size limiter; 3—strain localization with $b = 0$.

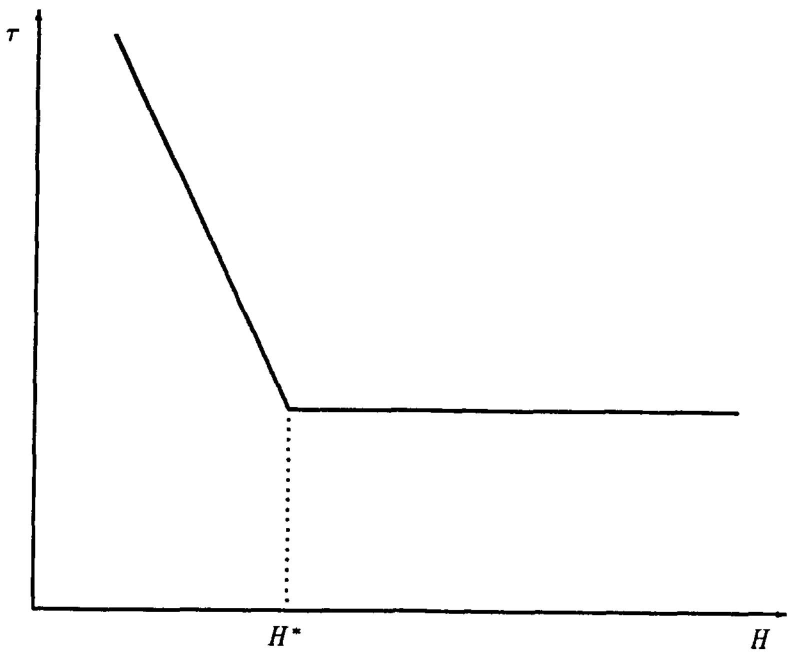

Fig. 8. Dependence of the overall shear stress on the dimensionless height of the representative volume.

γ is shown in Fig. 8, where d is the grain size. At $H > H^*$ shear stress τ does not depend on H by definition of the concept of the representative volume. At $H < H^*$ the volume under consideration is not representative and τ depends on orientations of specific grains. The probability of finding the grains with the profitable orientations with respect to shear direction is decreased with decreasing of H, so consequently τ should increase. If we have dependence $\tau(\gamma, H)$, then from the principle (4.15) it follows that after the pick point the shear band appears with the width $b = H^*d$. Both H^* and d depend on γ, the whole strain history and the texture.

6. CONCLUSIONS

The main result of the second part of this paper is the derivation of the governing extremum principle for the description of stable post-bifurcation behaviour in elastoplastic media. The media with nonassociated flow rule and finite strain regimes are included in the consideration. To derive this principle, we have introduced a new thermomechanical postulate, called the postulate of realizability. The main idea of this postulate is very simple: if only some dissipative process can occur (from the point of view of thermodynamics), it will occur, i.e. the first fulfilment of the necessary energetic condition is sufficient for the beginning of the dissipative process. The postulate of realizability was applied to obtain some known results (associated flow rule in classical plasticity, relations between dissipative forces and the rates for time-dependent behaviour, some extremum principles for a finite volume of perfect rigid-plastic material), as well as some new ones (nonassociated flow rules and more complex relations between dissipative forces and the rates for more complicated models, some extremum principles for a finite volume of elastoplastic material). This means that the postulate of realizability is quite a powerful and flexible tool in the theory of dissipative processes and the derived extremum principles can be considered as well gounded. Indeed, the concept of stability following from these principles has a clear physical meaning. If, under a prescribed increment of boundary data in time $[t, t + \Delta t]$, the stable solution for the velocity field is realized, then at time $t + \Delta t$ the power of the external stresses is less than the power of the internal stresses for all other possible solutions (velocity fields), i.e. they cannot be realized from the energetic point of view. If in the time interval $[t, t + \Delta t]$ the unstable solution is realized, then at time $t + \Delta t$ the power of the external stresses exceeds the power of the internal stresses for the velocity field, corresponding to the stable solution, i.e. the jump from the unstable solution to the stable one is dynamically (with a positive increment of the kinetic energy) possible.

REFERENCES

[1] V. D. KLUSHNIKOV, *Stability of Elastic–Plastic Systems*. Nauka, Moscow (1980) (in Russian).

[2] H. PETRYK, A consistent energy approach to defining stability of plastic deformation processes. In *Stability in the Mechanics of Continua* (Edited by F. N. SCHROEDER). Springer, Berlin (1982).

[3] Z. P. BAŽANT, Distributed cracking and nonlocal continuum. In *Finite Element Methods for Nonlinear Problems* (Edited by P. BERGAN *et al.*). Springer, Berlin (1986).

[4] Y. HUO and I. MÜLLER, *Continuum Mech. Thermodyn.* **5**, 163 (1993).

[5] V. I. LEVITAS, *Doklady AN Ukrainskoy SSR, Ser. A* **8**, 41 (1990).

[6] B. RANIECKI and O. BRUHNS, *Arch. Mech.* **43**, 343 (1991).

[7] R. HILL, *J. Mech. Phys. Solids* **6**, 236 (1958).

[8] R. HILL, Bifurcation and uniqueness in non-linear mechanics of continua. In *Problems of Continuum Mechanics.* S.I.A.M., Philadelphia (1961).

[9] J. W. HUTCHINSON, *J. Mech. Phys. Solids* **21**, 163 (1973).

[10] J. W. HUTCHINSON, Plastic buckling. In *Advances in Applied Mechanics* (Edited by C. S. YIH). Academic Press, New York (1974).

[11] Q. NGUYEN SON and N. TRIANTAFYLLIDIS, *J. Mech. Phys. Solids* **37**, 545 (1989).

[12] A. NEEDLEMAN and V. TVERGAARD, Aspects of plastic postbuckling behaviour. In *Mechanics of Solids* (Edited by H. G. HOPKINS and M. J. SEWELL). Pergamon Press, Oxford (1982).

[13] Z. P. BAŽANT, Stable states and stable paths of propagating of damage zones and interactive fractures. In *Cracking and Damage* (Edited by J. MAZARS and Z. P. BAŽANT). Elsevier, London (1989).

[14] H. PETRYK and K. THERMANN, *Int. J. Solids Structures* **29**, 745 (1992).

[15] V. G. BAR'YAKHTAR *et al.*, *Sov. Phys. Solid State* **28**, 1303 (1986).

[16] A. L. ROITBURD and D. E. TEMKIN, *Sov. Phys. Solid State* **28**, 432 (1986).

[17] I. M. KAGANOVA and A. L. ROYTBURD, *Sov. Phys. Solid State* **29**, 800 (1987).

[18] J. B. LEBLOND, J. DEVAUX and J. C. DEVAUX, *Int. J. Plasticity* **5**, 551 (1989).

[19] J. B. LEBLOND, *Int. J. Plasticity* **5**, 573 (1989).

[20] V. I. LEVITAS, *Thermomechanics of Phase Transformations and Inelastic Deformations in Microinhomogeneous Materials.* Naukova Dumka, Kiev (1992) (in Russian).

[21] F. D. FISCHER, M. BERVEILLER, K. TANAKA and E. R. OBERAIGNER, *Arch. Appl. Mech.* **64**, 54 (1994).

[22] V. I. KONDAUROV and L. V. NIKITIN, *Izv. AN SSSR, Mekh. Tverdogo Tela* **4**, 130 (1986).

[23] V. I. LEVITAS, Post-bifurcation behaviour in finite elastoplasticity. Applications to strain localization and phase transitions. Hanover University, Inst. of Structural and Computational Mechanics, Rep. No. **92/5** (1992).

[24] V. I. LEVITAS, *Strength Mater.* **12**, 1536 (1980).

[25] V. I. LEVITAS, *Large Elastoplastic Deformations of Materials at High Pressure.* Naukova Dumka, Kiev (in Russian) (1987).

[26] V. I. LEVITAS, *Large Deformation of Materials with Complex Rheological Properties at Normal and High Pressure.* Nova Science Publishers, New York (1995).

[27] E. H. LEE, *J. Appl. Mech.* **36**, 1 (1969).

[28] V. I. LEVITAS, *Strength Mater.* **18**, 1094 (1986).

[29] H. ZEIGLER, *An Introduction to Thermomechanics.* North-Holland, Amsterdam (1977).

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APPENDIX

Application of the Postulate of Realizability to Systems with an Arbitrary Dissipation Function

Our consideration was limited to systems with homogeneous dissipation functions of degree one, i.e. limited to time-independent elastoplastic materials. We will extend the approach considered above to time-dependent systems (including viscoplasticity and creep), i.e. to arbitrary dissipation functions $\mathscr{D}$. Let $\mathscr{D}(\dot{\mathbf{q}}, \dots) = \mathbf{X}(\dot{\mathbf{q}}, \dots) \cdot \dot{\mathbf{q}}$ be a dissipation function, $\mathbf{X}$ and $\dot{\mathbf{q}}$ are the work-conjugated dissipative forces and rates (fluxes), respectively. If $\mathbf{q} = \mathbf{F}_p$, the expression for $\mathbf{X}$ is given in Section 2.

Assume that dissipative function $\mathscr{D}_0(\mathbf{X}, \dots) := \mathbf{X} \cdot \dot{\mathbf{q}}(\mathbf{X}, \dots)$ exists. Let us fix an arbitrary $\mathbf{X}$ and consequently $\mathscr{D}_0(\mathbf{X}, \dots) = \mathscr{M} = \mathscr{D}(\dot{\mathbf{q}}, \dots)$, where $\mathscr{M}$ is some scalar, i.e. we consider not the equilibrium and its violation, but motion with some fixed value of $\mathscr{D}$. Let us prove the following statement: if

$$\forall \dot{\mathbf{q}} \in \mathscr{D}(\dot{\mathbf{q}}^*) = \mathscr{M} \qquad \mathbf{X} \cdot \dot{\mathbf{q}}^* - \mathscr{D}(\dot{\mathbf{q}}^*) < 0, \tag{A1}$$

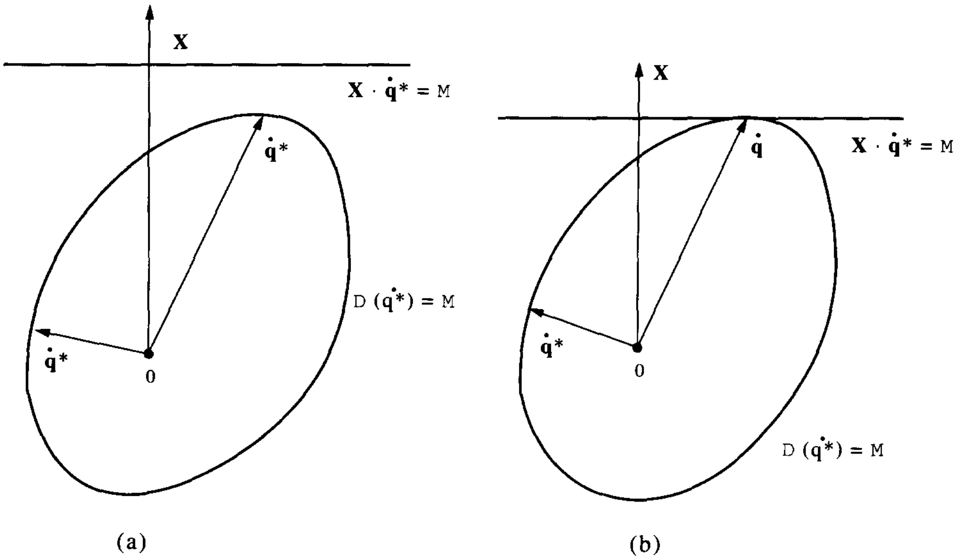

Fig. A1. Application of the postulate of realizability to systems with an arbitrary dissipation function: (a) motion with $\mathscr{D} = \mathscr{M}$ is impossible; (b) motion with $\mathscr{D} = \mathscr{M}$ occurs.

then for this $\mathbf{X}$ a motion with $\mathscr{D} = \mathscr{M}$ is impossible. The motion with $\mathscr{D} = \mathscr{M}$ is possible only for $\mathbf{X}^* = \mathbf{X}(\dot{\mathbf{q}}^*)$, if $\dot{\mathbf{q}}^* \in \mathscr{D}(\dot{\mathbf{q}}^*) = \mathscr{M}$. But for them $\mathbf{X}(\dot{\mathbf{q}}^*) \cdot \dot{\mathbf{q}}^* - \mathscr{D}(\dot{\mathbf{q}}^*) = 0$ by definition $\mathscr{D}(\dot{\mathbf{q}}^*)$, and this is in contradiction with inequality $\mathbf{X} \cdot \dot{\mathbf{q}}^* - \mathscr{D}(\dot{\mathbf{q}}^*) < 0$. Geometrically, this means that the surface $\mathscr{D}(\dot{\mathbf{q}}^*) = \mathscr{M}$ has no common points with the plane $\mathbf{X} \cdot \dot{\mathbf{q}}^* = \mathscr{M}$ (Fig. A1). A motion with $\mathscr{D} = \mathscr{M}$ is possible when the surface $\mathscr{D}(\dot{\mathbf{q}}^*) = \mathscr{M}$ and the plane $\mathbf{X} \cdot \dot{\mathbf{q}}^* = \mathscr{M}$ have common points, i.e. at their intersection or touching. Let us apply

The postulate of realizability: *Let us consider vector* $\mathbf{X}$, *which met the inequality (A1). If, in the course of* $\mathbf{X}$*-variation, the condition*

$$\mathbf{X} \cdot \dot{\mathbf{q}} - \mathscr{D}(\dot{\mathbf{q}}) = 0 \tag{A2}$$

is met the first time for some $\dot{\mathbf{q}} \in \mathscr{D}(\dot{\mathbf{q}}) = \mathscr{M}$, *then a motion with* $\mathscr{D} = \mathscr{M}$ *occurs.*

This postulate means that a motion with $\mathscr{D} = \mathscr{M}$ occurs when the surface $\mathscr{D}(\dot{\mathbf{q}}^*) = \mathscr{M}$ and the plane $\mathbf{X} \cdot \dot{\mathbf{q}}^* = \mathscr{M}$ touch each other at some point $\dot{\mathbf{q}}$, i.e. the $\mathbf{X}$ and $\dot{\mathbf{q}}$ correspond to each other (Fig. A1). For the touching point equation (A2) is valid, for all other points with $\dot{\mathbf{q}}^* \in \mathscr{D}(\dot{\mathbf{q}}^*) = \mathscr{M}$, inequality (A1). Consequently, we get the following extremum principle

$$\mathbf{X} \cdot \dot{\mathbf{q}}^* - \mathscr{D}(\dot{\mathbf{q}}^*) < 0 = \mathbf{X} \cdot \dot{\mathbf{q}} - \mathscr{D}(\dot{\mathbf{q}}) \quad \text{at} \quad \mathscr{D}(\dot{\mathbf{q}}^*) = \mathscr{D}(\dot{\mathbf{q}}), \tag{A3}$$

or

$$\mathbf{X} \cdot \dot{\mathbf{q}} < \mathbf{X} \cdot \dot{\mathbf{q}} \quad \text{at} \quad \mathscr{D}(\dot{\mathbf{q}}^*) = \mathscr{D}(\dot{\mathbf{q}}), \tag{A4}$$

whence

$$\mathbf{X} = \lambda \frac{\partial \mathscr{D}}{\partial \dot{\mathbf{q}}}; \qquad \lambda = \mathscr{D}\left(\frac{\partial \mathscr{D}}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}}\right)^{-1} \tag{A5}$$

where λ is the Lagrange multiplier, which is determined from the condition $\mathbf{X} \cdot \dot{\mathbf{q}} = \mathscr{D}(\dot{\mathbf{q}})$. Equations (A4) and (A5) are Ziegler's [29] extremum principle and relation, respectively. Consequently, using the postulate of realizability, we can prove the Ziegler relationship, but we can also obtain more general expressions, e.g. for media with structural changes (Levitas [20, 24–26]).