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Regularization of currents and entropy

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Abstract. Let T be a positive closed (p,p)-current on a compact Kähler manifold X. Then, there exist smooth positive closed (p,p)-forms T_n^+ and T_n^- such that $T_n^+ - T_n^- \to T$ weakly. Moreover, $\|T_n^\pm\| \leq c_X\|T\|$ where $c_X > 0$ is a constant independent of T. We also extend this result to positive pluriharmonic currents. Then we study the wedge product of positive closed $(1,1)$-currents having continuous potential with positive pluriharmonic currents. As an application, we give an estimate for the topological entropy of meromorphic maps on compact Kähler manifolds.

Résumé. Soit T un (p,p)-courant positif fermé sur une variété kählérienne compacte X. Alors, il existe des (p,p)-formes lisses, positives fermées T_n^+ et T_n^- telles que $T_n^+ - T_n^- \to T$ faiblement. De plus, on a $\|T_n^\pm\| \leq c_X\|T\|$ où $c_X > 0$ est une constante indépendante de T. Nous montrons aussi ce résultat pour les courants positifs pluriharmoniques. Nous étudions également le produit extérieur de $(1,1)$-courants positifs fermés à potentiel continu avec des courants pluriharmoniques positifs. Comme application, nous donnons une estimation de l'entropie topologique des applications méromorphes d'une variété kählérienne compacte.

1 Introduction

Let (X,ω) be a compact Kähler manifold of dimension k. Demailly [7] has shown that for a positive closed $(1,1)$-current T on X, there exist smooth positive closed $(1,1)$-forms T_n^+ which converge weakly (i.e. in the sense of currents) to $T + c\omega$ where $c > 0$ is a constant. Moreover, there is a constant $c_X > 0$, independent of T, such that $\|T_n^+\|$ and c are bounded by $c_X\|T\|$. We refer to Demailly's papers [6, 7] for the basics on currents on complex manifolds. Recall that the mass of a positive (p,p)-current S is defined by $\|S\| := \int_X S \wedge \omega^{k-p}$. Our main result is the following theorem where the positivity can be understood in the weak or strong sense.

Theorem 1.1 *Let (X,ω) be a compact Kähler manifold of dimension k. Then, for every positive closed (p,p)-current T on X, there exist smooth closed (p,p)-forms T_n^+ and T_n^- such that $T_n^+ - T_n^-$ converge weakly to the* *current T. Moreover, $\|T_n^\pm\| \leq c_X\|T\|$ where $c_X > 0$ is a constant independent of T.*

We deduce from this theorem the following corollary which is proved in [10] for projective manifolds.

Corollary 1.2 *Let (X, ω) be a compact Kähler manifold of dimension k. Then, for every positive closed (p,p)-current T on X, there exist smooth closed (p,p)-forms T_n^+ which converge weakly to a current T' with $T' \geq T$. Moreover, $\|T_n^+\| \leq c_X\|T\|$ and $\|T'\| \leq c_X\|T\|$ where $c_X > 0$ is a constant independent of T.*

Let (X', ω') be another compact Kähler manifold of dimension $k' \geq k$ and let $\Pi : X' \longrightarrow X$ be a surjective holomorphic map. We want to define the pull-back of the current T by the map Π. When Π is a finite map, this problem is studied in [19, 12]. In general, the map Π is a submersion only in the complement of an analytic subset C of X'. Let π denote the restriction of Π to $X' \setminus \mathrm{C}$. Then, $\pi^*(T)$ is well defined and is a positive closed (p,p)-current on $X' \setminus \mathrm{C}$. Let (T_n^+) and c_X be as in Corollary 1.2. Define $S_n := \Pi^*(T_n^+)$. The (p,p)-forms S_n are smooth and positive on X'. Their classes in $H^{p,p}(X', \mathbb{C})$ are bounded since $(\|T_n^+\|)$ is bounded. It follows that $(\|S_n\|)$ is bounded. Taking a subsequence, we can assume that S_n converge to a current S. We also have $S \geq \pi^*(T)$ on $X' \setminus \mathrm{C}$. In particular, $\pi^*(T)$ has finite mass. Following Skoda [22], the trivial extension $\widetilde{\pi^*(T)}$ of $\pi^*(T)$ on X' is a positive closed current. So, we have the following corollary.

Corollary 1.3 *Let X, X', Π, π and T be as above. Then, the positive current $\widetilde{\pi^*(T)}$ is well defined and closed. Moreover, there exists a constant $c_\Pi > 0$ independent of T such that $\|\widetilde{\pi^*(T)}\| \leq c_\Pi\|T\|$. The map $T \mapsto \widetilde{\pi^*(T)}$ is l.s.c. in the sense that if $T_n \to T$, then any cluster point τ of $\big(\widetilde{\pi^*(T_n)}\big)$ satisfies $\tau \geq \widetilde{\pi^*(T)}$.*

In [19], Méo gave an example which shows that, in general, when X and X' are not compact, the current $\pi^*(T)$ on $X \setminus \mathrm{C}$ is not always of bounded mass near C.

Consider a dominating meromorphic self-map $f : X \longrightarrow X$ of X. Define $f^n := f \circ \cdots \circ f$ (n times) the n-th iterate of f. We refer to the survey [20] for the theory of iteration of meromorphic maps. Let I_n be the indeterminacy set of f^n. Then I_n is an analytic subset of codimension ≥ 2 of X. Denote by Ω_f the set of points $x \in X \setminus I_1$ such that $f^n(x) \notin I_1$ for every $n \geq 1$. A subset $F \subset \Omega_f$ is called (n,ϵ)-*separeted*, $\epsilon > 0$, if

$$\max_{0\leq i\leq n-1} \mathrm{dist}(f^i(x), f^i(y)) \geq \epsilon \text{ for } x, y \in F \text{ distinct.}$$

The *topological entropy* $\mathrm{h}(f)$ (see [5]) is defined by

$$\mathrm{h}(f) := \sup_{\epsilon>0} \left(\limsup_{n\to\infty} \frac{1}{n} \log \max \left\{ \#F,\ F\ (n,\epsilon)\text{-separated} \right\} \right).$$

Let Γ_n be the closure in X^n of the set of points

$$(x, f(x), \ldots, f^{n-1}(x)), \quad x \in \Omega_f.$$

This is an analytic subset of dimension k of X^n. Let Π_i be the canonical projections of X^n on its factors. We consider on X^n the Kähler metric $\omega_n := \sum \Pi_i^*(\omega)$. Define following Gromov [17],

$$\mathrm{lov}(f) \quad := \quad \limsup_{n\to\infty} \frac{1}{n} \log(\mathrm{vol}(\Gamma_n)) = \limsup_{n\to\infty} \frac{1}{n} \log \left(\int_{\Gamma_n} \omega_n^k \right). \tag{1}$$

Define also *the dynamical degree of order* p of f by

$$d_p \quad := \quad \limsup_{n\to\infty} \left(\int_{X\setminus I_n} f^{n*}(\omega^p) \wedge \omega^{k-p} \right)^{1/n}. \tag{2}$$

Using an inequality of Lelong [18], Gromov [17] proved that $\mathrm{h}(f) \leq \mathrm{lov}(f)$. Following Gromov and Yomdin [23, 16, 17], we have

$$\mathrm{h}(f) = \mathrm{lov}(f) = \max_{1\leq p\leq k} \log d_p$$

when f is a holomorphic map. Using Corollary 1.2, we prove, in the same way as in [10], that the sequences in (1)(2) are convergent and that the dynamical degrees d_p are bimeromorphic invariants of f. More precisely, if $\Pi : X' \longrightarrow X$ is a bimeromorphic map between compact Kähler manifolds, the dynamical degrees of $\Pi^{-1} \circ f \circ \Pi$ are equal to d_p. Using Corollary 1.2, we also get the following result.

Theorem 1.4 *Let f be a dominating meromorphic self-map on a compact Kähler manifold X of dimension k. Let d_p be the dynamical degrees of f. Then*

$$\mathrm{h}(f) \leq \mathrm{lov}(f) = \max_{1\leq p\leq k} \log d_p.$$

This theorem gives a partial answer to a conjecture of Friedland [15] which says that $\mathrm{h}(f) = \max_{1\leq p\leq k} \log d_p$. Theorem 1.4 is already proved in [10] for rational maps on projective manifolds. Corollary 1.2 permits to extend the proof to the case of compact Kähler manifolds. One can also extend some results on meromorphic correspondences or transformations, which are proved in the projective case in [11] (see also [8]).

In the last two sections, we extend Theorem 1.1 to positive pluriharmonic currents and currents of class DSH. We also study the intersection of such currents with positive closed $(1,1)$-currents.

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2 A classical lemma

We will give here a classical lemma that we use in Section 3. Let B denote the unit ball in $\mathbb{R}^m$. Let $K(x,y)$ be a function with compact support in $\mathrm{B}\times\mathrm{B}$, smooth in $\mathrm{B}\times\mathrm{B}\setminus\Delta$ where Δ is the diagonal of $\mathrm{B}\times\mathrm{B}$. Assume that, for every (x,y)

$$|K(x,y)| \quad \leq \quad A|x-y|^{2-m} \tag{3}$$

where $A>0$ is a constant and $x=(x_1,\ldots,x_m)$ are coordinates of $\mathbb{R}^m$. Observe that for every y

$$\|K(.,y)\|_{\mathrm{L}^{1+\delta}} \quad \leq \quad A' \tag{4}$$

for some $\delta>0$ and $A'>0$. Assume also that for every x, y

$$|\nabla K(x,y)| \quad \leq \quad A|x-y|^{1-m}. \tag{5}$$

In this section, we identify ν, a current of degree 0 and of order 0, with the current of degree m, $\nu \mathrm{d}y_1\wedge\ldots\wedge\mathrm{d}y_m$. Let $\mathcal{M}$ denote the set of Radon measures on $\mathbb{R}^m$. We define a linear operator P on $\mathcal{M}$ by:

$$P\mu(x) := \int_{y\in\mathbb{R}^m} K(x,y)\mathrm{d}\mu(y).$$

Observe that the function $P\mu$ has support in B. We have the following lemma.

Lemma 2.1 *The operator P maps continuously $\mathcal{M}$ into $\mathrm{L}^{1+\delta}$. It also maps continuously L^p into L^q, L^∞ into $\mathcal{C}^0$ and $\mathcal{C}^0$ into $\mathcal{C}^1$, where $q=\infty$ if $p^{-1}+(1+\delta)^{-1}\leq 1$ and $p^{-1}+(1+\delta)^{-1}=1+q^{-1}$ otherwise.*

All the assertions are easy to deduce from (3)(4)(5) and the Hölder inequality.

3 Proof of Theorem 1.1

Let Δ denote the diagonal of $X \times X$. We first give a *weak regularization* of the current of integration $[\Delta]$. Let $\widetilde{X \times X}$ denote the blow-up of $X \times X$ along Δ. Following Blanchard [4], $\widetilde{X \times X}$ is a Kähler manifold. Let $\pi : \widetilde{X \times X} \longrightarrow X \times X$ be the canonical projection and $\widetilde{\Delta} := \pi^{-1}(\Delta)$. Then $\widetilde{\Delta}$ is a smooth hypersurface in $\widetilde{X \times X}$. If γ is a closed strictly positive $(k-1, k-1)$-form on $\widetilde{X \times X}$, then $\pi_*(\gamma \wedge [\widetilde{\Delta}])$ is a non-zero positive closed (k, k)-current on $X \times X$ supported on Δ. So, it is a multiple of $[\Delta]$. We choose γ so that $\pi_*(\gamma \wedge [\widetilde{\Delta}]) = [\Delta]$. We will use the following regularization of $[\widetilde{\Delta}]$.

Since $[\widetilde{\Delta}]$ is a positive closed $(1, 1)$-current, there exist a quasi-p.s.h. function φ and a smooth closed $(1, 1)$-form Θ' such that $\mathrm{dd^c}\varphi = [\widetilde{\Delta}] - \Theta'$. Recall that $\mathrm{d^c} := \frac{i}{2\pi}(\overline{\partial} - \partial)$. Demailly's regularization theorem [7] implies the existence of smooth functions φ_n and of a smooth positive closed $(1, 1)$-form Θ on $\widetilde{X \times X}$ such that

- $\mathrm{dd^c}\varphi_n \geq -\Theta$;
- φ_n decrease to φ.

In this case, independently of Demailly's theorem, we can construct the functions φ_n as follows. Observe that φ is smooth out of $\widetilde{\Delta}$ and $\varphi^{-1}(-\infty) = \widetilde{\Delta}$. Let $\chi : \mathbb{R} \cup \{-\infty\} \to \mathbb{R}$ be a smooth increasing convex function such that $\chi(x) = 0$ on $[-\infty, -1]$, $\chi(x) = x$ on $[1, +\infty[$ and $0 \leq \chi' \leq 1$. Define $\chi_n(x) := \chi(x + n) - n$ and $\varphi_n := \chi_n \circ \varphi$. The functions φ_n are smooth decreasing to φ and we have

$$\begin{aligned}
\mathrm{dd^c}\varphi_n &= (\chi_n'' \circ \varphi)\mathrm{d}\varphi \wedge \mathrm{d^c}\varphi + (\chi_n' \circ \varphi)\mathrm{dd^c}\varphi \\
&\geq (\chi_n' \circ \varphi)\mathrm{dd^c}\varphi = -(\chi_n' \circ \varphi)\Theta' \geq -\Theta \qquad (6)
\end{aligned}$$

where we choose the smooth positive closed form Θ big enough so that $\Theta - \Theta'$ is positive.

Define $\Theta_n^+ := \mathrm{dd^c}\varphi_n + \Theta$ and $\Theta_n^- := \Theta - \Theta'$ then $\Theta_n^+ - \Theta_n^- \to [\widetilde{\Delta}]$. We have $\|\Theta_n^\pm\| \leq c_0$ where $c_0 > 0$ is a constant. The forms $\Theta_n^\pm$ are smooth. Define

$$\widetilde{K}_n^\pm := \gamma \wedge \Theta_n^\pm \quad \text{and} \quad K_n^\pm := \pi_*(\widetilde{K}_n^\pm).$$

The (k, k)-forms $K_n^\pm$ are positive closed with coefficients in L^1 and smooth out of Δ. We also have $K_n^+ - K_n^- \to [\Delta]$ weakly and $\|K_n^\pm\| \leq c_1$, $c_1 > 0$. This is what we call a *weak regularization* of $[\Delta]$. We will use $K_n^\pm$ to regularize the current T. The following lemma shows that the coefficients of $K_n^{\pm}$ satisfy inequalities of type (3) and (5) for $m = 2k$. Then, the singularities of $K_n^{\pm}$ are the same than for the Bochner-Martinelli kernel.

Lemma 3.1 *Let $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_k)$, $|x_i| < 3$, $|y_i| < 3$, be local holomorphic coordinates of a chart of $X \times X$ such that $\Delta = (y = 0)$ in that chart. Let $H_n^{\pm}$ be a coefficient of $K_n^{\pm}$ in these coordinates. Then, there exists a constant $A_n > 0$, depending on n, such that*

$$|H_n^{\pm}(x, y)| \leq A_n |y|^{2-2k} \quad \textit{and} \quad |\nabla H_n^{\pm}| \leq A_n |y|^{1-2k}$$

for $|x_i| \leq 1$, $|y_i| \leq 1$ and $y \neq 0$.

Proof. By symmetry, it is sufficient to consider (x, y) in the open sector S defined by the inequalities $|x_i| < 3$, $|y_i| < 3$, $|y_i| < 3|y_1|$ and prove the estimates in the sector S' defined by $|x_i| < 2$, $|y_i| < 2$ and $|y_i| < 2|y_1|$ (we can assume that y_1 is the largest coordinate of the point $y \neq 0$). Let $\widetilde{S}$ and $\widetilde{S}'$ be the interiors of $\pi^{-1}(\overline{S})$ and of $\pi^{-1}(\overline{S}')$ respectively. We consider the coordinate system (x, Y) of $\widetilde{S}$ with $|x_i| < 3$, $Y_1 = y_1$ and $Y_i = y_i/y_1$, $|y_1| < 3$, $|y_i| < 3|y_1|$ for $i = 2, \ldots, k$. We have $\pi(x, Y) = (x, y)$ for $(x, y) \in S$. The equation of $\widetilde{\Delta}$ in $\widetilde{S}$ is $Y_1 = 0$.

Since $\widetilde{K}_n^{\pm}$ are smooth on $\widetilde{S}$, they are finite sums of forms of type

$$\Phi(x, Y) = L(x, Y) \mathrm{d}x_I \wedge \mathrm{d}\overline{x}_{I'} \wedge \mathrm{d}Y_J \wedge \mathrm{d}\overline{Y}_{J'}$$

where L is a smooth function, I, I', J, J' are subsequences of $\{1, \ldots, k\}$ and $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_m}$ if $I = \{i_1, \ldots, i_m\}$. Hence, in S the forms $K_n^{\pm}$ are finite sums of forms of type $\pi_*(\Phi)$.

Observe that $\pi_*(\Phi)$ is obtained from $\Phi(x, Y)$ replacing Y_1 by y_1 and Y_i by y_i/y_1. There are here at most $2k - 2$ factors of the form $\mathrm{d}(y_i/y_1) = \mathrm{d}y_i/y_1 - y_i \mathrm{d}y_1/y_1^2$ or their conjugate. Hence, the coefficients of $\pi_*(\Phi)$ on S are finite sums of

$$L(x, y_1, y_2/y_1, \ldots, y_k/y_1) P(y) y_1^{-m} \overline{y}_1^{-n}$$

where P is a homogeneous polynomial such that $\deg(P) + 2k - 2 \geq m + n$. Since $\widetilde{S}' \Subset \widetilde{S}$, L is bounded on $\widetilde{S}'$ and $L(x, y_1, y_2/y_1, \ldots, y_k/y_1)$ is bounded on S'. The first estimate of the lemma follows.

For the second estimate, it is sufficient to observe that the coefficients in the gradient of

$$L(x, y_1, y_2/y_1, \ldots, y_k/y_1) P(y) y_1^{-m} \overline{y}_1^{-n}$$

are combinations of functions of the same type with homogeneous polynomials P such that $\deg(P)+2k-1\geq m+n$.

$\square$

Define

$$T_n^\pm(x) \quad := \quad \int_{y\in X} K_n^\pm(x,y)\wedge T(y). \tag{7}$$

Let π_i denote the canonical projections of $X\times X$ on its factors. We have

$$T_n^\pm \quad := \quad (\pi_1)_*\big(K_n^\pm\wedge \pi_2^*(T)\big). \tag{8}$$

Observe that $\pi_2^*(T)$ is well defined since π_2 is a submersion. The currents $K_n^\pm\wedge\pi_2^*(T)$ are positive closed and well defined on $X\times X\setminus\Delta$. They are of finite mass since, for each n, $\|K_n^+(.,y)\|_{\mathrm{L}^1}$ is uniformly bounded. A priori, the mass depends on n. By Skoda's extension theorem [22], their trivial extensions are positive and closed. It follows that $T_n^\pm$ are well defined and are positive closed currents on X. The use of Skoda theorem can be replaced by an argument similar to the one in the proof of the following lemma.

Lemma 3.2 *The currents $T_n^+-T_n^-$ converge weakly to T when $n\to\infty$. Moreover, $\|T_n^\pm\|\leq c\|T\|$ where $c>0$ is a constant independent of n and T.*

Proof. Define $\Pi:=\pi_2\circ\pi$. Observe that Π is a submersion from $\widetilde{X\times X}$ onto X and $\Pi_{|\widetilde\Delta}$ is a submersion from $\widetilde\Delta$ onto X. Indeed, consider charts $U\Subset V'\subset X$ that we identify with open sets in $\mathbb{C}^k$. Assume that U is small enough and $0\in U$. We can, using the change of coordinates $(z,w)\mapsto(z-w,w)$ on $V'\times U$, reduce to the product situation $V\times U$, $U\Subset V\subset\mathbb{C}^k$ where Δ is identified to $\{0\}\times U$. The blow-up along $\{0\}\times U$ is still a product. So Π^* of a current is just integration on fibers. We can use this local model for the assertions below.

The potential of $\widetilde\Delta$ is integrable with respect to $\Pi^*(T)$ since its singularity is like $\log\mathrm{dist}(z,\widetilde\Delta)$ and this function has bounded integral on fibers of Π. In particular, $[\widetilde\Delta]\wedge\Pi^*(T)$ is well defined and is equal to $(\Pi_{|\widetilde\Delta})^*(T)$, and $[\widetilde\Delta]$ has no mass for $\Pi^*(T)$ nor for $\widetilde K_n^\pm\wedge\Pi^*(T)$. We then have

$$K_n^\pm\wedge\pi_2^*(T)=\pi_*(\widetilde K_n^\pm\wedge\Pi^*(T)) \tag{9}$$

since the formula is valid out of Δ. The potentials of $\widetilde K_n^+$ are decreasing and the currents $\widetilde K_n^-$ are independent of n, hence

$$\widetilde K_n^+\wedge\Pi^*(T)-\widetilde K_n^-\wedge\Pi^*(T)\to\gamma\wedge[\widetilde\Delta]\wedge\Pi^*(T)=\gamma\wedge(\Pi_{|\widetilde\Delta})^*(T). \tag{10}$$

Since $\pi_{|\widetilde{\Delta}}$ is a submersion onto Δ, we have $(\Pi_{|\widetilde{\Delta}})^*(T) = (\pi_{|\widetilde{\Delta}})^*(\pi_{2|\widetilde{\Delta}})^*(T)$. Hence

$$\pi_*\big(\gamma \wedge (\Pi_{|\widetilde{\Delta}})^*(T)\big) = (\pi_{2|\Delta})^*(T).$$

This and (9) (10) imply that

$$K_n^+ \wedge \pi_2^*(T) - K_n^- \wedge \pi_2^*(T) \to (\pi_{2|\Delta})^*(T).$$

Taking the direct image under π_1 gives $T_n^+ - T_n^- \to T$.

Since Π is a submersion, $\|\Pi^*(T)\| \leq c_2\|T\|$ where $c_2 > 0$ is independent of T. Observe that since $\widetilde{K}_n^\pm$ are smooth we can compute $\|\widetilde{K}_n^\pm \wedge \Pi^*(T)\|$ cohomologically. The cohomological classes of $\widetilde{K}_n^\pm$ are bounded, hence there exists a constant $c_3 > 0$ such that $\|\widetilde{K}_n^\pm \wedge \Pi^*(T)\| \leq c_3\|T\|$. It follows that

$$\|T_n^\pm\| = \|(\pi_1)_*\pi_*(\widetilde{K}_n^\pm \wedge \Pi^*(T))\| \leq c\|T\|$$

where $c > 0$ is independent of n and T.

□

The proof of Theorem 1.1 is completed by the following three steps.

Step 1. We show first that we can choose in Theorem 1.1 forms $T_n^\pm$ with L^1 coefficients. Define $T_n^\pm$ as in (7)(8). We use partitions of unity of X and of $X \times X$ in order to reduce the problem to the case of $\mathbb{R}^m$. Following Lemmas 2.1 and 3.1, the forms $T_n^\pm$ have L^1 coefficients. Lemma 3.2 implies that $T_n^+ - T_n^- \to T$ and $\|T_n^\pm\| \leq c\|T\|$. Of course, in general, $T_n^+ - T_n^-$ do not converge in L^1 since the constants A_n in Lemma 3.1 depend on n.

Step 2. We can now assume that T is a form with L^1 coefficients. Define $T_n^\pm$ as in (7)(8). Lemmas 2.1 and 3.1 imply that $T_n^\pm$ are forms with coefficients in $\mathrm{L}^{1+\delta}$. We also have $T_n^+ - T_n^- \to T$ and $\|T_n^\pm\| \leq c\|T\|$. Hence, we can assume that T is a form with $\mathrm{L}^{1+\delta}$ coefficients. We repeat this process N times with $N \geq \delta^{-1}$. Lemmas 2.1, 3.1 and 3.2 allow to reduce the problem to the case where T is a form with L^∞ coefficients. If we repeat this process two more times, we can assume that T is a $\mathcal{C}^1$ form.

Step 3. Now assume that T is of class $\mathcal{C}^1$. We can also assume that T is strictly positive. Let Ω be a smooth real closed (p,p)-form cohomologous to T. Using standard Hodge theory [6], there is a real $(p-1,p-1)$-form u of class $\mathcal{C}^2$ such that $T = \Omega + \mathrm{dd^c}u$. Let (u_n) be a sequence of real smooth $(p-1,p-1)$-forms such that $u_n \to u$ in $\mathcal{C}^2$ norm. The current $T_n := \Omega + \mathrm{dd^c}u_n$ converges to T in $\mathcal{C}^0$ norm. Moreover, T_n is positive for n big enough since T is strictly positive. This completes the proof of Theorem 1.1.

□

4 Pluriharmonic currents

In this section, we extend Theorem 1.1 to positive pluriharmonic currents, i.e. positive dd^c-closed currents. We have the following result which is new even for bidegree $(1,1)$ currents.

Theorem 4.1 *Let T be a positive dd^c-closed (p,p)-current on a compact Kähler manifold (X,ω). Then there exist smooth positive dd^c-closed forms $T_n^\pm$ such that $T_n^+ - T_n^- \to T$. Moreover, $\|T_n^\pm\| \leq c_X\|T\|$ where $c_X > 0$ is a constant independent of T.*

We deduce from this theorem the following corollary.

Corollary 4.2 *Let X, X', Π, π and C be as in Corollary 1.3. If T is as in Theorem 4.1, then the positive dd^c-closed current $\widetilde{\pi^*(T)}$ is well defined. Moreover the operator $T \mapsto \widetilde{\pi^*(T)}$ is l.s.c. and $\|\widetilde{\pi^*(T)}\| \leq c_\Pi\|T\|$ where $c_\Pi > 0$ is a constant independent of T.*

To prove the corollary, observe that by Theorem 4.1, the positive pluriharmonic current $\pi^*(T)$, which is well defined on $X' \setminus \mathrm{C}$, has finite mass. Following Alessandrini-Bassanelli [1], $\widetilde{\pi^*(T)}$ satisfies $dd^c\widetilde{\pi^*(T)} \leq 0$. Then, Stokes Theorem implies that $dd^c\widetilde{\pi^*(T)} = 0$.

Proof of Theorem 4.1. We use the same idea as in Section 3. Clearly $T_n^\pm$ given by (7)(8)(9) are pluriharmonic positive currents. We only need to check that $T_n^+ - T_n^- \to T$. The rest of proof is the same as in Theorem 1.1.

Let φ and φ_n be q.p.s.h. functions as in Section 3. We want to prove the analog of (10):

$$(dd^c\varphi_n + \Theta') \wedge \Pi^*(T) \to (\Pi_{|\widetilde{\Delta}})^*(T) \tag{11}$$

The problem is local. Define $S := (\Pi_{|\widetilde{\Delta}})^*(T)$. We choose as in Lemmas 3.1 and 3.2 local holomorphic coordinates $(x_1,\ldots,x_{2k})$ of an open set $W \subset \widetilde{X\times X}$, $|x_i| < 1$, so that in W

- $\widetilde{\Delta} = \{x_{2k} = 0\}$; hence $\psi := \varphi - \log|x_{2k}|$ is smooth and $dd^c\psi = -\Theta'$;
- $\Pi(x_1,\ldots,x_{2k}) = (x_1,\ldots,x_k)$.

Define $\tau(x_1,\ldots,x_{2k}) := (x_1,\ldots,x_{2k-1})$. Since $\Pi = \Pi_{|\widetilde{\Delta}}\circ\tau$, we have $\Pi^*(T) = \tau^*(S)$ in W.

Observe that $(\mathrm{dd^c}\varphi_n + \Theta') \wedge \tau^*(S)$ is supported in $(\varphi < -n+2)$ and, by (6), $(\mathrm{dd^c}\varphi_n + \Theta') \wedge \tau^*(S) \geq (1-\chi'_n \circ \varphi)\Theta' \wedge \tau^*(S)$. The definition of χ_n implies that the measures $(1-\chi'_n \circ \varphi)\Theta' \wedge \tau^*(S)$ tend to 0. Hence, every limit value of $(\mathrm{dd^c}\varphi_n + \Theta') \wedge \tau^*(S)$ is a positive $\mathrm{dd^c}$-closed current supported in $\widetilde{\Delta}$. Following Bassanelli [2], it is a current on $\widetilde{\Delta}$ (this is true for every positive current T supported in $\widetilde{\Delta}$ such that $\mathrm{dd^c}T$ is of order 0). Hence, in order to prove (11) we only have to check that

$$\int_W \Psi(x_{2k})(\mathrm{dd^c}\varphi_n + \Theta') \wedge \tau^*(\Phi \wedge S) \to \int_{\widetilde{\Delta}} \Phi \wedge S$$

for every test $(2k-p-1, 2k-p-1)$-form Φ with compact support in $\widetilde{\Delta} \cap W$ and for every function $\Psi(x_{2k})$ supported in $\{|x_{2k}| < 1\}$, such that $\Psi(0) = 1$. Observe that since $\tau^*(\Phi \wedge S)$ is proportional to $\mathrm{d}x_1 \wedge \mathrm{d}\overline{x}_1 \wedge \ldots \wedge \mathrm{d}x_{2k-1} \wedge \mathrm{d}\overline{x}_{2k-1}$ only the component of $\mathrm{dd^c}\varphi_n + \Theta'$ with respect to $\mathrm{d}x_{2k} \wedge \mathrm{d}\overline{x}_{2k}$ is relevant. When $(x_1, \ldots, x_{2k-1})$ is fixed, we have

$$\int_{x_{2k}} \Psi(\mathrm{dd}^{\mathrm{c}}_{x_{2k}}\varphi_n + \Theta') \to 1$$

since $\mathrm{dd}^{\mathrm{c}}_{x_{2k}}\varphi_n + \Theta'$ converges to the Dirac mass δ_0 and $\Psi(0) = 1$. The last integral is uniformly bounded with respect to n and $x_1, \ldots, x_{2k-1}$ because by (6) one can prove that the masses of the measures $\mathrm{dd}^{\mathrm{c}}_{x_{2k}}\varphi_n + \Theta'$ on a compact sets of $\{|x_{2k}| < 1,\ x_1, \ldots, x_{2k-1} \text{ fixed}\}$ are uniformly bounded. This implies the result.

□

Remark 4.3 Theorem 4.1 implies that on an arbitrary compact Kähler manifold (X, ω) positive pluriharmonic currents T of bidegree $(1,1)$ have finite energy. We then have $T = \Omega + \partial S + \overline{\partial S} + i\partial\overline{\partial}v$ with Ω smooth closed, S, ∂S , $\overline{\partial}S$ in L^2 and v in L^1. The energy of T is equal to $\int \overline{\partial}S \wedge \partial\overline{S} \wedge \omega^{k-2}$. The case of the projective space is treated in [14]. To extend the result to an arbitrary compact Kähler manifold, one has to use the approximation Theorem 4.1, to go from a priori estimates on smooth positive pluriharmonic forms to the estimates on positive pluriharmonic currents.

Let $\mathrm{DSH}^p(X)$ denote the space of (p,p)-currents $T = T_1 - T_2$ where T_i are negative currents, such that $\mathrm{dd^c}T_i = \Omega_i^+ - \Omega_i^-$ with $\Omega_i^\pm$ positive closed. Observe that $\|\Omega_i^+\| = \|\Omega_i^-\|$. We define the *DSH-norm* of T as

$$\|T\|_{\mathrm{DSH}} := \min\{\|T_1\| + \|T_2\| + \|\Omega_1^+\| + \|\Omega_2^+\|,\ T_i,\ \Omega_i^\pm \text{ as above}\}.$$

We say that $T_n \to T$ in $\mathrm{DSH}^p(X)$ if $T_n \to T$ weakly and $(\|T_n\|_{\mathrm{DSH}})$ is bounded.

The spaces $\mathrm{DSH}^p(X)$ are analoguous to the space generated by q.p.s.h. functions. They are useful in order to study the regularity of Green currents in dynamics [12]. The proof of the following theorem, which gives the density of smooth forms in $\mathrm{DSH}^p(X)$, follows the lines of previous approximation results and is left to the reader. In this case, for the control of the mass of $T_n^\pm$ we need to estimate the mass of $dd^c\varphi_n \wedge \Pi^*(T)$. It is sufficient to estimate the mass of $\varphi_n\Pi^*(dd^cT)$ using the definition of φ_n.

Theorem 4.4 *Let T be a current in $\mathrm{DSH}^p(X)$. Then there exist smooth real (p,p)-forms T_n such that $T_n \to T$. Moreover, $\|T_n\|_{\mathrm{DSH}} \leq c_X\|T\|_{\mathrm{DSH}}$ where $c_X > 0$ is a constant independent of T.*

Remark 4.5 We have $\widetilde{K}_n^+ - \widetilde{K}_n^- - \gamma\wedge[\widetilde{\Delta}] = \gamma\wedge dd^c(\varphi_n - \varphi)$ and $\varphi_n = \varphi$ out of the set $(\varphi < -n+2)$. Hence $\mathrm{supp}(\widetilde{K}_n^+ - \widetilde{K}_n^-)$ converge to $\widetilde{\Delta}$, $\mathrm{supp}(K_n^+ - K_n^-)$ converge to Δ and $\mathrm{supp}(T_n^+ - T_n^-)$ converge to $\mathrm{supp}(T)$.

We also have the following useful proposition.

Proposition 4.6 *Let T be a continuous form and $T_n^\pm$ be the forms defined in (7)(8). Then $T_n := T_n^+ - T_n^-$ converge uniformly to T.*

Proof. We can approximate T uniformly by smooth forms. We then assume that T is smooth (see Lemma 3.1). The form $T_n - T$ is the push-forward of $(\widetilde{K}_n^+ - \widetilde{K}_n^- - \gamma\wedge[\widetilde{\Delta}])\wedge\Pi^*(T)$ by $\Pi' := \pi_1 \circ \pi$. The last current is equal to $\widetilde{T}_n := dd^c(\varphi_n - \varphi)\wedge\gamma'$ where γ' is a smooth form. Using a partition of unity, we reduce the problem to a local situation with the coordinates $x = (x', x'') = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_{2k})$, $\Pi'(x) = x'$, $\widetilde{\Delta} = (x_{2k} = 0)$ and γ' of compact support as in the proof of Theorem 4.1. We have to check that $\Pi'_*(\widetilde{T}_n)(x') = \int_{x''}\widetilde{T}_n(x)$ converge uniformly to 0.

Observe that the last integral is taken in the neighbourhood $(\varphi < -n+2)$ of $(x_{2k} = 0)$ and the form $\widetilde{T}_n - dd^c_{x''}(\varphi_n - \varphi)\wedge\gamma'$ is of order $1/|x_{2k}|$ since in the difference we get at most one derivative with respect to x_{2k}. Hence, it is sufficient to estimate $\int_{x''} dd^c_{x''}(\varphi_n - \varphi)\wedge\gamma' = \int_{x''}(\varphi_n - \varphi)\wedge dd^c_{x''}\gamma'$. It is clear that these forms converge uniformly to 0.

$\square$

5 Intersection of currents

We want to consider a class of positive pluriharmonic currents which are of interest in some problems of complex analysis and dynamics. Some of their properties are given in [21, 1, 2, 13, 9, 14]. Given a compact Kähler manifold (X,ω) of dimension k, we want to define the intersection $S\wedge T$ of a positive closed $(1,1)$-current S with a positive pluriharmonic current T of bidegree (p,p), $1\leq p\leq k-1$. We have seen a special case of this situation in the last section.

We write $S=\alpha+\mathrm{dd^c}u$ with α smooth and u a q.p.s.h. function. We say that u is *a potential* of S.

Theorem 5.1 *Assume that u is continuous. Then $S\wedge T$ is well defined and is a positive* $\mathrm{dd^c}$*-closed current. Moreover $S\wedge T$ depends continuously on S and T in the following sense. Let T_n be positive pluriharmonic currents converging weakly to T. If $S_n=\alpha+\mathrm{dd^c}u_n$ with u_n continuous converging uniformly to u then $S_n\wedge T_n$ converges weakly to $S\wedge T$. In particular, it holds when the u_n are continuous and decrease to u.*

We first prove the following proposition for smooth potentials. We will see later that it can be extended to continuous q.p.s.h. functions $v^\pm$ and that $\mathrm{d}v^\pm\wedge R$ and $\mathrm{d^c}v^\pm\wedge R$ are well defined in this case.

Proposition 5.2 *Let $v^\pm$ and $v=v^+-v^-$ be smooth real functions on X such that $\mathrm{dd^c}v^\pm=\Theta^\pm-\alpha$ where α is a smooth closed $(1,1)$-form and $\Theta^\pm$ are positive closed $(1,1)$-currents. Let R be a positive current in $\mathrm{DSH}^p(X)$ with $\mathrm{dd^c}R=\Omega^+-\Omega^-$ where $\Omega^\pm$ are positive closed currents. Then*

$$\begin{aligned}&\int \mathrm{d}v\wedge\mathrm{d^c}v\wedge R\wedge\omega^{k-p-1}\leq\\ &\leq\quad \|v\|_{\mathrm{L}^\infty}\left(2\int\alpha\wedge R\wedge\omega^{k-p-1}+3(\|v^+\|_{\mathrm{L}^\infty}+\|v^-\|_{\mathrm{L}^\infty})\|\Omega^\pm\|\right).\end{aligned}$$

In particular, if R is positive pluriharmonic, we have

$$\int \mathrm{d}v\wedge\mathrm{d^c}v\wedge R\wedge\omega^{k-p-1}\leq 2\|v\|_{\mathrm{L}^\infty}\int[\alpha]\wedge[R]\wedge[\omega]^{k-p-1}.$$

Proof. Observe that $\|v\|_{\mathrm{L}^\infty}\leq\|v^+\|_{\mathrm{L}^\infty}+\|v^-\|_{\mathrm{L}^\infty}$, $\|\Theta^+\|=\|\Theta^-\|$ and

$\|\Omega^+\| = \|\Omega^-\|$. Hence

$$
\begin{aligned}
&\int \mathrm{d}v \wedge \mathrm{d^c}v \wedge R \wedge \omega^{k-p-1} \leq \\
&\leq \frac{1}{2}\left|\int \mathrm{dd^c}v^2 \wedge R \wedge \omega^{k-p-1}\right| + \left|\int v\mathrm{dd^c}v \wedge R \wedge \omega^{k-p-1}\right| \\
&= \frac{1}{2}\left|\int v^2 \wedge \mathrm{dd^c}R \wedge \omega^{k-p-1}\right| + \left|\int v(\Theta^+ - \Theta^-) \wedge R \wedge \omega^{k-p-1}\right| \\
&\leq \frac{1}{2}\|v\|_{\mathrm{L}^\infty}^2 \int (\Omega^+ + \Omega^-) \wedge \omega^{k-p-1} + \|v\|_{\mathrm{L}^\infty} \int (\Theta^+ + \Theta^-) \wedge R \wedge \omega^{k-p-1} \\
&= \|v\|_{\mathrm{L}^\infty}^2 \|\Omega^\pm\| + 2\|v\|_{\mathrm{L}^\infty} \int \alpha \wedge R \wedge \omega^{k-p-1} + \\
&\quad + \|v\|_{\mathrm{L}^\infty} \int \mathrm{dd^c}(v^+ + v^-) \wedge R \wedge \omega^{k-p-1} \\
&\leq \|v\|_{\mathrm{L}^\infty}(\|v^+\|_{\mathrm{L}^\infty} + \|v^-\|_{\mathrm{L}^\infty})\|\Omega^\pm\| + 2\|v\|_{\mathrm{L}^\infty} \int \alpha \wedge R \wedge \omega^{k-p-1} + \\
&\quad + \|v\|_{\mathrm{L}^\infty} \int (v^+ + v^-) \wedge (\Omega^+ - \Omega^-) \wedge \omega^{k-p-1} \\
&\leq \|v\|_{\mathrm{L}^\infty}\left(2\int \alpha \wedge R \wedge \omega^{k-p-1} + 3(\|v^+\|_{\mathrm{L}^\infty} + \|v^-\|_{\mathrm{L}^\infty})\|\Omega^\pm\|\right).
\end{aligned}
$$

□

Proof of Theorem 5.1. Observe that, when u_n decreases to u, the Hartogs lemma implies that u_n converges uniformly to u. By Demailly's regularization theorem [7], we can assume that u_n are smooth and uniformly convergent to u. So, $S_n \wedge T_n$ is well defined. We will prove that $S_n \wedge T_n$ converges. This also implies that the limit depends only on S and T.

We first consider the case where $T_n = T$. We then have

$$S_n \wedge T = \alpha \wedge T + \mathrm{d}(\mathrm{d^c}u_n \wedge T) - \mathrm{d^c}(\mathrm{d}u_n \wedge T) - \mathrm{dd^c}(u_n T). \tag{12}$$

Proposition 5.2 applied to $u_n - u_m$ and the Cauchy criterion imply that $\mathrm{d}u_n$ and $\mathrm{d^c}u_n$ converge in $\mathrm{L}^2(T \wedge \omega^{k-p-1})$. Hence $S_n \wedge T$ converges and $\mathrm{d}u \wedge T$, $\mathrm{d^c}u \wedge T$ are well defined.

To complete the proof we write

$$S_n \wedge T_n - S \wedge T = \mathrm{dd^c}(u_n - u) \wedge T_n + \mathrm{dd^c}u \wedge (T_n - T) + \alpha \wedge (T_n - T).$$

The last term tends to zero. Proposition 5.2 implies that $\int \mathrm{d}(u_n - u) \wedge \mathrm{d^c}(u_n - u) \wedge T_n \wedge \omega^{k-p-1}$ has zero limit. An identity as in (12) shows that the first term tends to 0. For the second term, we observe first that if γ is a test 1-form and v is a smooth q.p.s.h. function with $\|u-v\|_{\mathrm{L}^\infty} \leq \epsilon$, then

$$\int \mathrm{d}u \wedge \gamma \wedge (T - T_n) \wedge \omega^{k-p-1} = \int \mathrm{d}v \wedge \gamma \wedge (T - T_n) \wedge \omega^{k-p-1} + \\ + \int \mathrm{d}(u-v) \wedge \gamma \wedge (T - T_n) \wedge \omega^{k-p-1}.$$

The first integral tends to zero. Schwarz's inequality and Proposition 5.2 imply

$$\begin{aligned} &\left| \int \mathrm{d}(u-v) \wedge \gamma \wedge T_n \wedge \omega^{k-p-1} \right|^2 \leq \\ &\leq \mathrm{const} \int \mathrm{d}(u-v) \wedge \mathrm{d^c}(u-v) \wedge T_n \wedge \omega^{k-p-1} \\ &\leq \mathrm{const} \|u-v\|_{\mathrm{L}^\infty} \|T_n\| \end{aligned}$$

and similarly for T.

□

In the same way, using the full strength of Proposition 5.2, one can prove the following theorem.

Theorem 5.3 *Let T be a current in $\mathrm{DSH}^p(X)$ and S as in Theorem 5.1. Then $S \wedge T$ is well defined and belongs to $\mathrm{DSH}^{p+1}(X)$. Moreover $S \wedge T$ depends continuously on S and T. The topology on the T variable is the topology of $\mathrm{DSH}^p(X)$.*

It is enough to assume T positive and to modify (12) into

$$S_n \wedge T = \alpha \wedge T + \mathrm{d}(\mathrm{d^c} u_n \wedge T) - \mathrm{d^c}(\mathrm{d}u_n \wedge T) - \mathrm{dd^c}(u_n T) + u_n \mathrm{dd^c} T.$$

Remarks 5.4 If S_i are positive closed $(1,1)$-currents with continuous potentials, then $S_1 \wedge \ldots \wedge S_m \wedge T$ is symmetric in S_i since this is true when S_i and T are smooth. Let u be a p.s.h. function on an open ball $\Omega \subset X$. By the maximum regularization procedure as in [6], if u is continuous we can extend u to a continuous q.p.s.h. function on X. Hence, $\mathrm{dd^c} u \wedge T$ is well defined on Ω.

When T is only a (positive pluriharmonic) (p,p)-current on Ω, we don't know how to define $\mathrm{dd^c} u \wedge T$ without additional hypothesis on u. Assume that T, $\mathrm{d}T$ and $\mathrm{dd^c}T$ are of order 0 and u is locally integrable with respect to the coefficient measures of T, $\mathrm{d}T$ and $\mathrm{dd^c}T$. Then we can define

$$\mathrm{dd^c}u\wedge T := \mathrm{dd^c}(uT) + u\mathrm{dd^c}T - \mathrm{d}(u\mathrm{d^c}T) + \mathrm{d^c}(u\mathrm{d}T).$$

If u_n are p.s.h. and decrease to u or if u_n converge uniformly to u, we have $\mathrm{dd^c}u_n\wedge T\to \mathrm{dd^c}u\wedge T$. When T is positive, we also have an inequality of Chern-Levine-Nirenberg type (see [6, p.126]). More precisely, if K, L are compact sets in Ω with $L\Subset K$, then

$$\|\mathrm{dd^c}u\wedge T\|_L \leq c_{K,L}(\|uT\|_K + \|u\mathrm{d}T\|_K + \|u\mathrm{dd^c}T\|_K).$$

Note that positive harmonic currents associated to a lamination by Riemann surfaces satisfy the above hypothesis (see [3]).

If T is of bidegree $(1,1)$ we can extend Theorem 5.1 to currents S with bounded potential.

Proposition 5.5 *Let T be a positive pluriharmonic current of bidegree $(1,1)$ in (X,ω). If u is a bounded q.p.s.h. function then $\mathrm{dd^c}u\wedge T$ is well defined. If (u_n) is a bounded sequence of q.p.s.h. functions converging pointwise to u with $\mathrm{dd^c}u_n\geq -c\omega$, then $\mathrm{dd^c}u_n\wedge T\to\mathrm{dd^c}u\wedge T$.*

Proof. We can assume that u_n are smooth and positive. It is easy to check that u_n^2 are q.p.s.h. and converge to u^2. It follows that ∂u_n (resp. $\overline{\partial}u_n$) converge to ∂u (resp. $\overline{\partial}u$) weakly in $\mathrm{L}^2(X)$.

As in Theorem 5.1, we only need to show that $\partial u_n\wedge T$ (resp. $\overline{\partial}u_n\wedge T$) converges weakly. Recall that we can write $T=\Omega+\partial S+\overline{\partial S}+i\partial\overline{\partial}v$ with Ω smooth closed, ∂S, $\overline{\partial}S$ in L^2, v in L^1 [14]. If γ is a test 1-form, we have

$$\begin{aligned}\int\partial u_n\wedge T\wedge\gamma\wedge\omega^{k-1} &= -\int u_n\partial T\wedge\gamma\wedge\omega^{k-1} - \int u_n T\wedge\partial\gamma\wedge\omega^{k-1}\\ &= -\int u_n\partial\overline{\partial S}\wedge\gamma\wedge\omega^{k-1} - \int u_n T\wedge\partial\gamma\wedge\omega^{k-1}\end{aligned}$$

The second term tends to $\int uT\wedge\partial\gamma\wedge\omega^{k-1}$. The first term is equal to

$$-\int\partial u_n\wedge\overline{\partial S}\wedge\gamma\wedge\omega^{k-1}+\int u_n\overline{\partial S}\wedge\partial\gamma\wedge\omega^{k-1}$$

which converges to

$$-\int\partial u\wedge\overline{\partial S}\wedge\gamma\wedge\omega^{k-1}+\int u\overline{\partial S}\wedge\partial\gamma\wedge\omega^{k-1}$$

since $u_n\to u$ and $\partial u_n\to\partial u$ weakly in L^2.

The convergence of $\overline{\partial}u_n\wedge T$ is proved in the same way.

$\square$

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