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Existence of a nodal solution with minimal energy for a Kirchhoff equation

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We show the existence of a nodal solution (sign-changing solution) for a Kirchhoff equation of the type

$$-M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(u) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a bounded domain in $\mathbf{R}^3$, M is a general C^1 class function and f is a superlinear C^1 class function with subcritical growth. The proof is based on a minimization argument and a quantitative deformation lemma.

1 Introduction

In this paper we study the existence of a nodal solution (sign-changing solution) for the following problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(u) & \text{in} \quad \Omega, \\ u^+ \neq 0 \text{ and } u^- \neq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbf{R}^3$ is a smooth bounded domain and

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}, \quad \text{for all} \quad x \in \Omega.$$

Notice that, in this case, $u = u^+ + u^-$ and $|u| = u^+ - u^-$. The class of problems (P) is called of Kirchhoff type because it comes of a important application of the Physic and Engineering. For instance, in 1883, Kirchhoff [18] studied the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{K}$$

which extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in Equation (K) have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

When an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time: this introduces changes of the tension in the string. This induced Kirchhoff to propose a nonlinear correction of the classical D'Alembert equation. Later on, Woinowsky-Krieger (Nash - Modeer) incorporated this correction in the classical Euler-Bernoulli equation for the beam (plate) with hinged ends. See, for example, [4], [5] and the references therein.

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Besides this important application, the presence of the term $M\big(\int_\Omega |\nabla u|^2 dx\big)$ causes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Therefore, the study on this class of problems has increased considerably in the last ten years, as can be seen in [1], [2], [6], [10], [12]–[17], [19], [20]–[22], [26]–[29] and references therein.

However, interestingly all the authors of these articles studied positive solutions or nontrivial solutions. Only the articles [23], [24], [30] consider solutions that change sign (nodal solution). In these three articles the authors show the existence of solutions which change sign to the problem

$$\begin{cases} -\big(a + b\int_\Omega |\nabla u|^2 dx\big)\Delta u = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{ZP}$$

where $a, b > 0$.

In [30] the authors showed the existence of the sign changing solutions to the problem (ZP) for the cases 4-sublinear, asymptotically 4-linear and 4-superlinear. In that paper the authors use variational methods and invariant sets of descent flow. In [23] the authors showed the same result found in [30] without considering the Ambrosetti-Rabinowitz condition. In [24] the authors study the result found in [30] considering now the case asymptotically 3-linear.

Our result completes the results found in [23], [24] and [30] because we show the existence of a nodal solution for a general class of problems that include but are not restricted to the type (ZP).

Before stating ours main results, we need the following hypotheses on the function M.

The function $M : \mathbf{R}_+ \to \mathbf{R}_+$ is C^1 class and satisfies the following conditions:

(M_1): The function M is increasing and $0 < M(0) =: m_0$.

(M_2): The function $t \mapsto \dfrac{M(t)}{t}$ is decreasing.

A typical example of a function verifying the assumptions $(M_1)-(M_2)$ is given by

$$M(t) = m_0 + bt, \quad \text{where} \quad m_0 > 0 \quad \text{and} \quad b > 0.$$

This example was considered in [18], [23], [24] and [30]. More generally, each function of the form

$$M(t) = m_0 + bt + \sum_{i=1}^{k} b_i t^{\gamma_i}$$

with $b_i \geq 0$ and $\gamma_i \in (0, 1)$ for all $i \in \{1, 2, \ldots, k\}$ verifies the hypotheses $(M_1)-(M_2)$. An another example is $M(t) = m_0 + ln(1 + t)$.

We assume that the function f is C^1 class and satisfies

(f_1):

$$\lim_{|t|\to 0^+} \frac{f(t)}{t} = 0.$$

(f_2): There is $q \in (4, 6)$ such that

$$\lim_{|t|\to\infty} \frac{f(t)}{t^{q-1}} = 0.$$

(f_3): There is $\theta \in (4, 6)$ such that

$$0 < \theta F(t) \leq f(t)t, \ \forall |t| > 0, \quad \text{where} \quad F(t) = \int_0^t f(s)\, ds.$$

(f_4): The map

$$t \longmapsto \frac{f(t)}{t^3}$$

is increasing in $|t| > 0$.

The main result of this paper is:

Theorem 1.1 *Suppose that the function M satisfies $(M_1)-(M_2)$ and the function f satisfies $(f_1)-(f_4)$. Then problem (P) possesses a nodal solution with minimal energy.*

In recent years, the study of nodal solutions for problems involving the case $M \equiv 1$ have received a special attention, as can be seen in [3], [7]–[9] and [11]. In the proof of Theorem 1.1 we use an argument that can be found in [7]. But, due to the presence of the function M, some estimates more refined are need, such as in Lemma 2.4 for instance.

The paper is organized as follows. In the Section 2 we prove some technical lemmas used in the proof of the main result. In the Section 3, we prove Theorem 1.1.

2 Variational framework and technical lemmas

We say that $u \in H_0^1(\Omega)$ is a weak nodal solution of the problem (P) if $u^+ \neq 0$, $u^- \neq 0$ in Ω and it verifies

$$M\big(\|u\|^2\big)\int_\Omega \nabla u \nabla \phi \, dx - \int_\Omega f(u)\phi \, dx = 0, \quad \text{for all} \quad \phi \in H_0^1(\Omega),$$

where

$$\|u\| := \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2}.$$

In view of $(f_1)-(f_2)$, we have that the functional $J : H_0^1(\Omega) \to \mathbf{R}$ given by

$$J(u) := \frac{1}{2}\widehat{M}\big(\|u\|^2\big) - \int_\Omega F(u)\, dx$$

is well defined, where $\widehat{M}(t) = \int_0^t M(s)\, ds$. Moreover, $J \in C^1\big(H_0^1(\Omega), \mathbf{R}\big)$ with the following derivative

$$J'(u)v = M\big(\|u\|^2\big)\int_\Omega \nabla u \nabla v \, dx - \int_\Omega f(u)v \, dx.$$

Thus, the weak solutions of (P) are precisely the critical points of J. Associated to the functional J we define the Nehari manifold

$$\mathcal{N} := \left\{u \in H_0^1(\Omega)\backslash\{0\} : J'(u)u = 0\right\}.$$

In the Theorem 1.1 we prove that there is $w \in \mathcal{M}$ such that

$$J(w) = \min_{v \in \mathcal{M}} J(v),$$

where

$$\mathcal{M} := \left\{w \in \mathcal{N} : J'(w)w^+ = 0 = J'(w)w^-\right\}.$$

From (M_1) we have that $M\big(\|w^\pm\|^2\big) \leq M\big(\|w\|^2\big)$, for $w \in \mathcal{M}$. Thus, this last inequality implies that

$$J'(w^\pm)w^\pm \leq 0, \quad \text{for all} \quad w \in \mathcal{M}. \tag{2.1}$$

Let us begin by establishing some preliminary results which will be exploited in the last section for a minimization argument.

Lemma 2.1

(a) *For all $u \in \mathcal{N}$ we have*

$$J(u) \geq \frac{(\theta - 4)}{4\theta} m_0 \|u\|^2.$$

(b) *There is* $\rho > 0$ *such that*

$$\|u\| \geq \rho, \quad \textit{for all} \quad u \in \mathcal{N}$$

and

$$\|w^{\pm}\| \geq \rho, \quad \textit{for all} \quad w \in \mathcal{M}.$$

Proof. From the definition of $\widehat{M}$ and (M_2), we get

$$\widehat{M}(t) \geq \frac{1}{2}M(t)t, \quad \text{for all} \quad t \geq 0. \tag{2.2}$$

Now using (2.2) and a direct calculation we obtain

$$\frac{1}{2}\widehat{M}(t) - \frac{1}{\theta}M(t)t \geq \frac{(\theta-4)}{4\theta}m_0 t, \quad \text{for all} \quad t \geq 0. \tag{2.3}$$

Thus, by (f_3) and (2.3) we conclude

$$J(u) = J(u) - \frac{1}{\theta}J'(u)u \geq \frac{(\theta-4)}{4\theta}m_0\|u\|^2, \quad \text{for all} \quad u \in \mathcal{N},$$

which proves (a).

For the proof of (b), notice that by (f_1) and (f_2), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$f(t)t \leq \epsilon|t|^2 + C_\epsilon|t|^q. \tag{2.4}$$

Now from the definition of $\mathcal{N}$, (M_1), (2.4) and Sobolev's embedding we have

$$0 < \rho := \left[\Big(m_0 - \frac{\epsilon C_2}{C_1}\Big)\frac{1}{C_\epsilon}\right]^{1/(q-2)} \leq \|u\|,$$

for all $u \in \mathcal{N}$ and for some $C_1, C_1 > 0$.

From (2.1) and repeating the reasoning before we obtain

$$0 < \rho \leq \|w^{\pm}\|.$$

□

We apply the next result in the last section to every bounded minimizing sequence of J on $\mathcal{M}$ in order to ensure that the candidate minimizer is different from zero.

Lemma 2.2 *If* (w_n) *is a bounded sequence in* $\mathcal{M}$*, then*

$$\liminf_{n\to\infty} \int_\Omega |w_n^{\pm}|^q \, dx > 0.$$

Proof. Using (2.1) again and repeating the reasoning before we obtain,

$$0 < m_0\rho^2 \leq M\big(\|w_n^{\pm}\|^2\big)\|w_n^{\pm}\|^2 \leq \epsilon \int_\Omega |w_n^{\pm}|^2 \, dx + C_\epsilon \int_\Omega |w_n^{\pm}|^q \, dx.$$

Since (w_n) is bounded, there is $C > 0$ such that

$$0 < m_0\rho^2 \leq \epsilon C + C_\epsilon \int_\Omega |w_n^{\pm}|^q \, dx$$

and the result follows from the last inequality. □

Next results try to infer geometrical information of J with respect to $\mathcal{M}$ in the same way that one is used to do about $\mathcal{N}$. To be more precise, note the similarity between the next result and that which states that for each $v \in H_0^1(\Omega)\backslash\{0\}$ there exists $t_v > 0$ such that $t_v v \in \mathcal{N}$.

Lemma 2.3 *If* $v \in H_0^1(\Omega)$ *with* $v^{\pm} \neq 0$*, then there are* $t, s > 0$ *such that*

$$J'(tv^+ + sv^-)v^+ = 0$$

and

$$J'(tv^+ + sv^-)v^- = 0.$$

Proof. Let $V : (0, +\infty) \times (0, +\infty) \to \mathbf{R}^2$ be a continuous function given by

$$V(t, s) = (J'(tv^+ + sv^-)(tv^+), J'(tv^+ + sv^-)(sv^-)).$$

Note that

$$J'(tv^+ + sv^-)(tv^+) = t^2 M\big(t^2\|v^+\|^2 + s^2\|v^-\|^2\big)\|v^+\|^2 - \int_\Omega f(tv^+)tv^+dx. \tag{2.5}$$

Using (M_1), (2.4) and Sobolev's embedding in (2.5), we have

$$J'(tv^+ + sv^-)(tv^+) \geq (m_0 - \epsilon C)t^2\|v^+\|^2 - t^q C_\epsilon C\|v^+\|^q,$$

for some $C > 0$. Thus, there exists $r > 0$ sufficiently small such that

$$J'(rv^+ + sv^-)(rv^+) > 0, \quad \text{for all} \quad s > 0.$$

Arguing of the same way we get

$$J'(tv^+ + rv^-)(rv^-) > 0, \quad \text{for all} \quad t > 0.$$

On the other hand, by (M_2), there exists $K_1 > 0$ such that

$$M(t) \leq M(1)t + K_1, \quad \text{for all} \quad t \geq 0, \tag{2.6}$$

and by (f_3), there are $K_2, K_3 > 0$ such that

$$F(t) \geq K_2 t^\theta - K_3. \tag{2.7}$$

Using (2.6), (2.7) and (f_3) in (2.5), we have

$$J'(tv^+ + sv^-)(tv^+) \leq t^4 M(1)\|v^+\|^4 + t^2 s^2 M(1)\|v^+\|^2\|v^-\|^2 + K_1 t^2\|v^+\|^2$$
$$-\frac{t^\theta}{\theta}K_2\int_\Omega |v^+|^\theta dx + K_3|\Omega|,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Thus, since $\theta > 4$, for $R > 0$ sufficiently large, we get

$$J'(Rv^+ + sv^-)(Rv^+) < 0, \quad \text{for all} \quad s \leq R.$$

Arguing of the same way we get

$$J'(tv^+ + Rv^-)(Rv^-) < 0, \quad \text{for all} \quad t \leq R.$$

In particular,

$$J'(rv^+ + sv^-)(rv^+) > 0 \quad \text{and} \quad J'(tv^+ + rv^-)(rv^-) > 0, \quad \text{for all} \quad t, s \in [r, R]$$

and

$$J'(Rv^+ + sv^-)(Rv^+) < 0 \quad \text{and} \quad J'(tv^+ + Rv^-)(Rv^-) < 0, \quad \text{for all} \quad t, s \in [r, R].$$

Now the lemma follows applying Miranda's theorem [25]. □

At this point, some useful remarks follow. First of all, let us observe that, from (M_2) we have

$$M'(t)t \leq M(t), \quad \text{for all} \quad t \geq 0, \tag{2.8}$$

that implies

$$t \longmapsto \frac{1}{2}\widehat{M}(t) - \frac{1}{4}M(t)t \ \text{ is increasing.} \tag{2.9}$$

Now using (f_4) we get

$$f'(t)t \geq 3f(t), \quad \text{for all} \quad |t| \geq 0, \tag{2.10}$$

that implies

$$t \mapsto \frac{1}{4} f(t)t - F(t) \text{ is increasing,} \quad \text{for all} \quad |t| > 0. \tag{2.11}$$

Besides, from (M_1) we obtain

$$\begin{aligned} \widehat{M}(t+s) &= \int_0^{t+s} M(\tau)\,d\tau = \widehat{M}(t) + \int_t^{t+s} M(\tau)\,d\tau \\ &= \widehat{M}(t) + \int_0^{s} M(\gamma + t)\,d\gamma \\ &\geq \widehat{M}(t) + \int_0^{s} M(\gamma)\,d\gamma \\ &= \widehat{M}(t) + \widehat{M}(s), \quad \text{for all} \quad t, s \in [0, +\infty). \end{aligned} \tag{2.12}$$

Now, we can define a suitable function and its gradient vector field which are related to functional J and will be involved in particular in the application of the deformation lemma. Indeed, for each $v \in H_0^1(\Omega)$ with $v^\pm \neq 0$ we consider

$$h^v : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbf{R} \quad \text{given by} \quad h^v(t, s) = J(tv^+ + sv^-)$$

and and its gradient $\Phi^v : [0, +\infty) \times [0, +\infty) \to \mathbf{R}^2$ defined by

$$\begin{aligned} \Phi^v(t, s) &= \Big(\Phi_1^v(t, s), \Phi_2^v(t, s)\Big) = \Big(\frac{\partial h^v}{\partial t}(t, s), \frac{\partial h^v}{\partial s}(t, s)\Big) \\ &= \big(J'(tv^+ + sv^-)v^+, J'(tv^+ + sv^-)v^-\big), \end{aligned}$$

for every $(t, s) \in [0, +\infty) \times [0, +\infty)$. Furthermore, we consider the Hessian matrix of h^v or the Jacobian matrix of Φ^v, i.e.

$$(\Phi^v)'(t, s) = \begin{pmatrix} \dfrac{\partial \Phi_1^v}{\partial t}(t, s) & \dfrac{\partial \Phi_1^v}{\partial s}(t, s) \\ \dfrac{\partial \Phi_2^v}{\partial t}(t, s) & \dfrac{\partial \Phi_2^v}{\partial s}(t, s) \end{pmatrix},$$

for every $(t, s) \in [0, +\infty) \times [0, +\infty)$. Indeed, in the following we aim to prove that, if $w \in \mathcal{M}$, the function h^w has a critical point and in particular a global maximum in $(t, s) = (1, 1)$,

Lemma 2.4 *If $w \in \mathcal{M}$, then*

(a) $h^w(t, s) < h^w(1, 1) = J(w)$,
for all $t, s \geq 0$ such that $(t, s) \neq (1, 1)$.
(b) $det(\Phi^w)'(1, 1) > 0$.

P r o o f . Since $w \in \mathcal{M}$, then

$$J'(w)w^\pm = J'(w^+ + w^-)w^\pm = 0.$$

Thus,

$$\Phi^w(1, 1) = \Big(\frac{\partial h^w}{\partial t}(1, 1), \frac{\partial h^w}{\partial s}(1, 1)\Big) = (0, 0).$$

Moreover, from (2.6) and (2.7) we get

$$\begin{aligned} h^w(t,s) &= J(tw^+ + sw^-) \\ &\le \frac{1}{4}M(1)\|tw^+ + sw^-\|^4 + \frac{1}{2}K_1\|tw^+ + sw^-\|^2 \\ &\quad -K_2\int_\Omega |tw^+ + sw^-|^\theta dx + K_3|\Omega|. \end{aligned}$$

Since $4 < \theta < 6$, then

$$\lim_{|(t,s)|\to+\infty} h^w(t,s) = -\infty,$$

that implies $(1,1)$ is a critical point of h^w and h^w has a global maximum point in (a,b).

Now we prove that $a, b > 0$. Suppose, by contradiction that $b = 0$. Thus, $J'(aw^+)aw^+ = 0$ implies

$$\frac{M(\|aw^+\|^2)}{a^2}\|w^+\|^2 = \int_\Omega \frac{f(aw^+)}{a^3}w^+ dx. \tag{2.13}$$

Moreover, using (2.1) we have

$$M\big(\|w^+\|^2\big)\|w^+\|^2 \le \int_\Omega f(w^+)w^+ dx. \tag{2.14}$$

Considering (2.13) and (2.14) we have

$$\left[\frac{M(\|w^+\|^2)}{\|w^+\|^2} - \frac{M(a^2\|w^+\|^2)}{a^2\|w^+\|^2}\right]\|w^+\|^4 \le \int_\Omega \left[\frac{f(w^+)}{(w^+)^3} - \frac{f(aw^+)}{(aw^+)^3}\right](w^+)^4 dx.$$

The last inequality, (M_2) and (f_4) imply $a \le 1$.

Now note that

$$\begin{aligned} h^w(a,0) &= J(aw^+) = J(aw^+) - \frac{1}{4}J'(aw^+)(aw^+) \\ &= \left[\frac{1}{2}\widehat{M}\big(\|aw^+\|^2\big) - \frac{1}{4}M\big(\|aw^+\|^2\big)\|aw^+\|^2\right] \\ &\quad + \int_\Omega \left[\frac{1}{4}f(aw^+)aw^+ - F(aw^+)\,dx\right]. \end{aligned} \tag{2.15}$$

Using (2.9) and (2.11) in (2.15) we obtain

$$\begin{aligned} h^w(a,0) &\le \left[\frac{1}{2}\widehat{M}\big(\|w^+\|^2\big) - \frac{1}{4}M\big(\|w^+\|^2\big)\|w^+\|^2\right] \\ &\quad + \int_\Omega \left[\frac{1}{4}f(w^+)w^+ - F(w^+)\,dx\right] \\ &= J(w^+) - \frac{1}{4}J'(w^+)w = J(w^+) = h^w(1,0). \end{aligned}$$

Now, our aim is to prove

$$J(w^+) = h^w(1,0) < J(w) = h^w(1,1).$$

By Lemma 2.1 we have $J(w^-) \ge 0$. Thus,

$$J(w^+) \le J(w^+) + J(w^-) = \frac{1}{2}\left[\widehat{M}\big(\|w^+\|^2\big) + \widehat{M}\big(\|w^-\|^2\big)\right] - \int_\Omega \big(F(w^+) + F(w^-)\,dx. \tag{2.16}$$

By (2.12) we get

$$J(w^+) < \frac{1}{2}\widehat{M}\big(\|w^+\|^2 + \|w^-\|^2\big) - \int_\Omega \big(F(w^+) + F(w^-)\,dx.$$

Since the supports of w^+ and w^- are disjoint, we obtain

$$h^w(1,0) = J(w^+) < J(w) = h^w(1,1),$$

which is an absurd because $(a,0)$ is a maximum point. In the same way we prove that $0 < a$.

Now we will prove that $0 < a, b \leq 1$. Since (a,b) is another critical point of h^w, we have

$$a^2 M\big(\|aw^+ + bw^-\|^2\big)\|w^+\|^2 = \int_\Omega f(aw^+)aw^+\,dx.$$

Without loss of generality, we can suppose $b \leq a$. Thus

$$\frac{M(a^2\|w^+ + w^-\|^2)}{a^2\|w^+ + w^-\|^2}\|w^+\|^2\|w^+ + w^-\|^2 \geq \int_\Omega \frac{f(aw^+)}{(aw^+)^3}(w^+)^4\,dx. \tag{2.17}$$

On the other hand, $J'(w)w^+ = 0$ implies

$$\frac{M(\|w^+ + w^-\|^2)}{\|w^+ + w^-\|^2}\|w^+\|^2\|w^+ + w^-\|^2 = \int_\Omega \frac{f(w^+)}{(w^+)^3}(w^+)^4\,dx. \tag{2.18}$$

Combining (2.17) and (2.18) we get

$$\begin{aligned}&\left[\frac{M(a^2\|w^+ + w^-\|^2)}{a^2\|w^+ + w^-\|^2} - \frac{M(\|w^+ + w^-\|^2)}{\|w^+ + w^-\|^2}\right]\|w^+\|^2\|w^+ + w^-\|^2\\ &\qquad \geq \int_\Omega \left[\frac{f(aw^+)}{(aw^+)^3} - \frac{f(w^+)}{(w^+)^3}\right](w^+)^4 dx.\end{aligned}$$

From the last inequality, (M_2) and (f_4) imply $0 < b \leq a \leq 1$.

Now we will prove that h^w does not have global maximum in $[0,1]\times[0,1]\backslash\{(1,1)\}$. We will show that

$$h^w(a,b) < h^w(1,1).$$

Note that $\|aw^+ + bw^-\|^2 = \|aw^+\|^2 + \|bw^-\|^2 \leq \|w^+\|^2 + \|w^-\|^2$ and since $\widehat{M}$ is increasing we have

$$h^w(a,b) = J(aw^+ + bw^-) \leq \frac{1}{2}\widehat{M}\big(\|w^+\|^2 + \|w^-\|^2\big) - \int_\Omega F(aw^+ + bw^-)\,dx. \tag{2.19}$$

By (f_3) we get $\frac{1}{4}\int_\Omega f(aw^+ + bw^-)(aw^+ + bw^-)\,dx \geq 0$.

Thus, put this information in (2.19) we obtain

$$\begin{aligned}h^w(a,b) = J(aw^+ + bw^-) &\leq \frac{1}{2}\widehat{M}\big(\|w^+\|^2 + \|w^-\|^2\big)\\ &\quad + \int_\Omega \left[\frac{1}{4}f(aw^+ + bw^-)(aw^+ + bw^-) - F(aw^+ + bw^-)\right]dx.\end{aligned}$$

Since w^+ and w^- have supports disjoint we get

$$\begin{aligned}h^w(a,b) = J(aw^+ + bw^-) &\leq \frac{1}{2}\widehat{M}\big(\|w^+\|^2 + \|w^-\|^2\big) + \int_\Omega \left[\frac{1}{4}f(aw^+)(aw^+) - F(aw^+)\right]dx\\ &\quad + \int_\Omega \left[\frac{1}{4}f(bw^-)(bw^-) - F(bw^-)\right]dx.\end{aligned}$$

Now, using (2.11) and the fact that w^+ and w^- have disjoint supports again, we get

$$h^w(a,b) < J(w^+ + w^-) = J(w) = h^w(1,1)$$

and item (a) is proved.

Let us prove item (b). Consider the notations

$$\Phi_1^w(t,s) = J'(tw^+ + sw^-)w^+ \quad \text{and} \quad \Phi_2^w(t,s) = J'(tw^+ + sw^-)w^-.$$

Thus,

$$\Phi_1^w(t,s) = tM\big(\|tw^+ + sw^-\|^2\big)\|w^+\|^2 - \int_\Omega f(tw^+)w^+dx$$

and

$$\Phi_2^w(t,s) = sM\big(\|tw^+ + sw^-\|^2\big)\|w^-\|^2 - \int_\Omega f(sw^-)w^-dx$$

and note that

$$\frac{\partial \Phi_1^w}{\partial t}(t,s) = M\big(\|tw^+ + sw^-\|^2\big)\|w^+\|^2 + 2t^2M'\big(\|tw^+ + sw^-\|^2\big)\|w^+\|^4 - \int_\Omega f'(tw^+)(w^+)^2dx$$

implies

$$\frac{\partial \Phi_1^w}{\partial t}(1,1) = M\big(\|w^+ + sw^-\|^2\big)\|w^+\|^2 + 2M'\big(\|w^+ + sw^-\|^2\big)\|w^+\|^4 - \int_\Omega f'(w^+)(w^+)^2dx.$$

Using (2.8) in the last equality we obtain

$$\begin{aligned}\frac{\partial \Phi_1^w}{\partial t}(1,1) &\le 3M(\|w\|^2)\|w^+\|^2 - \int_\Omega f'(w^+)(w^+)^2dx\\ &= \int_\Omega \Big[3f(w^+)w^+ - f'(w^+)(w^+)^2\Big]dx.\end{aligned}$$

From (2.10) we obtain

$$\frac{\partial \Phi_1^w}{\partial t}(1,1) < 0. \tag{2.20}$$

Arguing on the same way we conclude

$$\frac{\partial \Phi_2^w}{\partial s}(1,1) < 0. \tag{2.21}$$

Since $\frac{\partial \Phi_1^w}{\partial s}(1,1) = 0$ and $\frac{\partial \Phi_2^w}{\partial t}(1,1) = 0$ and considering (2.20) and (2.21) we conclude that $det(\Phi^w)'(1,1) = \frac{\partial \Phi_1^w}{\partial t}(1,1)\frac{\partial \Phi_2^w}{\partial s}(1,1) > 0$ and the item (b) is proved.

3 Proof of Theorem 1.1

In this section we will prove the existence of $w \in \mathcal{M}$ in which the infimum of J is attained on $\mathcal{M}$. After, following some arguments used in [7] by Bartsch, Weth and Willem (see also [3]) and, in particular, applying a deformation lemma, we find that w is a critical point of J and then a least energy nodal solution of (P). In order to complete the proof of Theorem 1.1, we conclude by showing that w has exactly two nodal domains.

First of all, by Lemma 2.1, there exists $c_0 \in \mathbf{R}$ such that

$$0 < c_0 = \inf_{v\in\mathcal{M}} J(v).$$

Thus, there exists a minimizing sequence (w_n) in $\mathcal{M}$ which is bounded from Lemma 2.1. again. Hence, by Sobolev imbedding theorem, without loss of generality, we can assume up to a subsequence that there exist $w, w_1, w_2 \in H_0^1(\Omega)$ such that

$$\begin{aligned}&w_n \rightharpoonup w,\ w_n^+ \rightharpoonup w_1,\ w_n^- \rightharpoonup w_2 \quad \text{in} \quad H_0^1(\Omega),\\ &w_n \to w,\ w_n^+ \to w_1,\ w_n^- \to w_2 \quad \text{in} \quad L^q(\Omega), q \in (1, 2^*).\end{aligned}$$

Since the transformations $w \to w^+$ and $w \to w^-$ are continuous from $L^q(\Omega)$ in $L^q(\Omega)$ (see Lemma 2.3 in [11] with suitable adaptations), we have that $w^+ = w_1 \geq 0$ and $w^- = w_2 \leq 0$. At this point, we can prove that $w \in \mathcal{M}$. Indeed, by $w_n^+ \to w^+$ and $w_n^- \to w^-$ in $L^q(\Omega)$ it is, as $n \to +\infty$

$$\int_\Omega |(w_n)^\pm|^q dx \longrightarrow \int_\Omega |w^\pm|^q dx.$$

Then, by Lemma 2.2, we conclude that $w^\pm \neq 0$ and consequently $w = w^+ + w^-$ is sign-changing. By Lemma 2.3, there exist $t, s > 0$ such that

$$\begin{aligned} &J'(tw^+ + sw^-)w^+ = 0, \\ &J'(tw^+ + sw^-)w^- = 0, \end{aligned} \tag{3.1}$$

then $tw^+ + sw^- \in \mathcal{M}$. Now, let us prove that $t, s \leq 1$. First let us observe that, since f has a subcritical growth, we get

$$\int_\Omega f((w_n)^\pm)(w_n)^\pm dx \longrightarrow \int_\Omega f(w^\pm)w^\pm dx$$

and

$$\int_\Omega F((w_n)^\pm)\, dx \longrightarrow \int_\Omega F(w^\pm)\, dx.$$

Thus, since $J'(w_n)w_n^\pm = 0$, by (M_1) we have

$$J'(w^+)w^+ \leq 0 \quad \text{and} \quad J'(w^-)w^- \leq 0. \tag{3.2}$$

Consequently, combining (3.1) and (3.2) and arguing as in the proof of Lemma 2.4 item (a), we obtain $0 < t, s \leq 1$.

In the next step we show that $J(tw^+ + sw^-) = c_0$ and $t = s = 1$ or better $J(w) = c_0$. Indeed, since $t, s \leq 1$ and $w_n \rightharpoonup w$ as $n \to +\infty$, exploiting the arguments used in the proof of Lemma 2.4 item (a) and the weak lower semicontinuity of J we get

$$c_0 \leq J(tw^+ + sw^-) = h^w(t, s) \leq h^w(1, 1) = J(w^+ + w^-) \leq \liminf_{n\to+\infty} J(w_n) = c_0.$$

At this point, by using a quantitative deformation lemma and adapting the arguments used in [7] with slight technical changes, we point out that w is a critical point of J, i.e. $J'(w) = 0$. If we reason by contradiction, we find that there exist a positive constant $\alpha > 0$ and $v_0 \in H_0^1(\Omega)$, $\|v_0\| = 1$ such that

$$J'(w)v_0 = 2\alpha > 0.$$

By the continuity of J', we can choose a radius $r > 0$ so that

$$J'(v)v_0 = \alpha > 0, \quad \text{for every} \quad v \in B_r(w) \subset H_0^1(\Omega) \text{ with } v^\pm \neq 0.$$

Let us fix $D = (\xi, \chi) \times (\xi, \chi) \subset \mathbf{R}^2$ with $0 < \xi < 1 < \chi$ such that

(i) $(1, 1) \in D$ and $\Phi^w(t, s) = (0, 0)$ in $\overline{D}$ if and only if $(t, s) = (1, 1)$;
(ii) $c_0 \notin h^w(\partial D)$;
(iii) $\left\{tw_0^+ + sw_0^- : (t, s) \in \overline{D}\right\} \subset B_r(w)$,

where h^w and Φ^w are defined as in Section 2 and satisfy Lemma 2.4. At this point, we can choose a smaller radius $r' > 0$ such that

$$\mathcal{B} = \overline{B_{r'}(w)} \subset B_r(w) \quad \text{and} \quad \mathcal{B} \cap \left\{tw^+ + sw^- : (t, s) \in \partial D\right\} = \emptyset. \tag{3.3}$$

Now define a continuous mapping $\rho : H_0^1(\Omega) \to [0, +\infty)$ such that

$$\rho(u) := \operatorname{dist}(u, \mathcal{B}^c), \quad \text{for all} \quad u \in H_0^1(\Omega),$$

then a bounded Lipschitz vector field $V : H_0^1(\Omega) \to H_0^1(\Omega)$ given by

$$V(u) = -\rho(u)v_0$$

and, for every $u \in H_0^1(\Omega)$, denoting $\eta(\tau) = \eta(\tau, u)$ we consider the following Cauchy problem

$$\begin{cases} \eta'(\tau) = V(\eta(\tau)), \text{ for all } \tau > 0, \\ \eta(0) = u. \end{cases}$$

Now, we observe that there exist a continuous deformation $\eta(\tau, u)$ and $\tau_0 > 0$ such that for all $\tau \in [0, \tau_0]$ the following properties hold:

(a) $\eta(\tau, u) = u$ for all $u \notin \mathcal{B}$;
(b) $\tau \to J(\eta(\tau, u))$ is decreasing for all $\eta(\tau, u) \in \mathcal{B}$;
(c) $J(\eta(\tau, w_0)) \le J(w) - \frac{r'\alpha}{2}\tau$.

Item (a) follows immediately from the definition of ρ. Indeed, $u \notin \mathcal{B}$ implies $\rho(u) = 0$ and the unique solution satisfying the above Cauchy problem is constant with constant value u.

As concerns as item (b), let us first observe that, since $\eta(\tau) \in \mathcal{B} \subset B_r(w)$, $J'(\eta(\tau))v_0 = \alpha > 0$ and, by the definition of ρ, it is $\rho(\eta(\tau)) > 0$. Now, differentiating J with respect to τ, for all $\eta(\tau) \in \mathcal{B}$, we have that

$$\frac{d}{d\tau}\left(J(\eta(\tau)) = J'(\eta(\tau))\eta'(\tau) = -\rho(\eta(\tau))J'(\eta(\tau))v_0 = -\rho(\eta(\tau))\alpha < 0\right.$$

thus concluding that $J(\eta(\tau, u))$ is decreasing with respect to τ.

In order to prove item (c), being $\tau_0 > 0$ such that $\eta(\tau, u) \in \mathcal{B}$ for every $0 \le \tau \le \tau_0$, we can assume without loss of generality

$$\|\eta(\tau, w) - w\| \le \frac{r'}{2} \Longleftrightarrow \eta(\tau, w) \in \overline{B_{\frac{r'}{2}}(w)}, \quad \text{for every} \quad 0 \le \tau \le \tau_0.$$

Thus, since $\rho(\eta(\tau, w)) = \operatorname{dist}(\eta(\tau, w), \mathcal{B}^c) \ge \frac{r'}{2}$ it follows that

$$\frac{d}{d\tau}\left(J(\eta(\tau, w)) = -\rho(\eta(\tau, w))\alpha \le \frac{r'\alpha}{2}\right.$$

and, integrating on $[0, \tau_0]$ we finally get

$$J(\eta(\tau, w_0)) - J(w) \le -\frac{r'\alpha}{2}\tau.$$

At this point, let us consider a suitable deformed path $\overline{\eta}_0 : \overline{D} \to X$ defined by

$$\overline{\eta}_{\tau_0}(t, s) := \eta(\tau_0, tw^+ + sw^-), \quad \text{for all} \quad (t, s) \in \overline{D}$$

so that

$$\max_{(t,s)\in\overline{D}} J(\overline{\eta}_{\tau_0}(t, s)) < c_0.$$

Indeed, by (b) and the fact that η satisfies the initial condition $\eta(0, u) = u$, for all $(t, s) \in \overline{D} - \{(1, 1)\}$ it is

$$\begin{aligned} J(\overline{\eta}_{\tau_0}(t, s)) &= J(\eta(\tau_0, tw^+ + sw^-)) \le J(\eta(0, tw^+ + sw^-)) \\ &= J(tw^+ + sw^-) = h^w(t, s) < c_0, \end{aligned}$$

and, for $(t, s) = (1, 1)$, by (c) we get

$$\begin{aligned} J(\overline{\eta}_{\tau_0}(1, 1)) &= J(\eta(\tau_0, w^+ + w^-)) = J(\eta(\tau_0, w)) \\ &\le J(w) - \frac{r'\alpha}{2}\tau_0 < J(w) < c_0. \end{aligned}$$

Then, $\overline{\eta}_{\tau_0}(\overline{D}) \cap \mathcal{M} \ne \emptyset$, i.e.

$$\overline{\eta}_{\tau_0}(t, s) \notin \mathcal{M}, \quad \text{for all} \quad (t, s) \in \overline{D}. \tag{3.4}$$

On the other side, defining $\Psi_{\tau_0} : \overline{D} \to \mathbf{R}^2$ as

$$\Psi_{\tau_0} := \left(\frac{J'(\overline{\eta}_{\tau_0}(t,s))(\overline{\eta}_{\tau_0}(t,s))^+}{t}, \frac{J'(\overline{\eta}_{\tau_0}(t,s))(\overline{\eta}_{\tau_0}(t,s))^-}{s} \right),$$

we observe that, for all $(t,s) \in \partial D$, by (3.3) and (a) for $\tau = \tau_0$, it is

$$\Psi_{\tau_0}(t,s) = \left(J'(tw^+ + sw^-)w^+, J'(tw^+ + sw^-), w^- \right) = \Phi^w(t,s).$$

Then, since by Brouwer's topological degree

$$\deg(\Psi_{\tau_0}, D, (0,0)) = \deg(\Phi^w, D, (0,0)) = \operatorname{sgn}(\det(\Phi^w)'(1,1)) = 1,$$

we get that Ψ_{τ_0} has a zero $(\overline{t}, \overline{s}) \in D$ namely

$$\Psi_{\tau_0}(\overline{t},\overline{s}) = (0,0) \iff J'(\overline{\eta}_{\tau_0}(\overline{t},\overline{s}))(\overline{\eta}_{\tau_0}(\overline{t},\overline{s}))^{\pm} = 0.$$

Consequently there exists $(\overline{t},\overline{s}) \in D$ such that $\overline{\eta}_{\tau_0}(\overline{t},\overline{s}) \in \mathcal{M}$ and we have a contradiction with (3.4). We conclude that w is a critical point of J and the proof of Theorem 1.1 is complete. □

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