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ON THE APPLICATION OF HIGHER RANK SPINORS IN GENERAL RELATIVITY

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ABSTRACT

The formulation of two component spinors in the context of General Relativity has been extended to $(2j + 1)$*-component spinors* $j = 0,1/2,1,3/2,\ldots$ *which includes the decomposition of the Riemann tensor into Petrov types, covariant differentiation and generalised spin coefficients. The formalism is developed along parallel lines with that of quantised fields.*

§(1): *INTRODUCTION*

The scope of this work is strictly kinematic. It is a reformulation of the spinorial representation in the work of Penrose and Newman [*1*] and that of Witten [*2*].

The central difference from the formulation in references [*1*,*2*] is that throughout, irreducible representations (IR) of $SL(2,C)$ are used to characterize the spinorial quantities in question, while in references [*1*,*2*] most spinorial quantities transform like a direct product (and hence reducible) of basic two-component spinors.

The advantage of our formulation is, apart from the fact that its systematic nature gives all results a formal simplicity, that it provides a basis whereby a deeper insight into the use of spinors in General Relativity can be gained.

In section 2 we introduce a necessary amount of the technical framework, mainly the spin matrices that transform according to all finite dimensional IR's of $SL(2,C)$ as a generalisation of the basic spin matrices, namely the Pauli matrices.

Section 3 is devoted to the decomposition of the Riemann tensor $R_{\mu\nu\rho\sigma}$ with respect to a set of irreducible spinor bases, hence introducing the spinorial counterparts of the Riemann tensor, composed in all, of the twenty independent components of the $R_{\mu\nu\rho\sigma}$. This decomposition gives rise to a considerable simplification in classifying the Petrov types, provided we can put the spinors in question into a generalised version of the dyad basis [1], which are in fact introduced in section 4, and called base-covariants.

In section 5 the spinor connections defining the covariant differentiation of the spinor of arbitrary rank j is introduced, obviously to satisfy a necessity created by our introduction of higher rank spinors in our decomposition of $R_{\mu\nu\rho\sigma}$ in section 3. For the same reason, in section 6 we extend the definition of spin coefficients [1] to the case of $(2j+1)$-component orthonormal base-covariants. In the appendix some useful spinor identities are listed as well as some properties of the classical Maxwell and Riemann fields.

§(2): *THE SPIN MATRICES*

The spin matrices have been extensively discussed in the literature [3] and we only give their definition here for completeness and standardisation of notation.

The higher rank spin matrices are the generalisations of the Pauli spin matrices $\sigma_\mu = (\sigma_0, \sigma_i)$ and $\tilde{\sigma}_\mu = (\sigma_0, -\sigma_i)$, $i = 1,2,3$, which in our notation will be denoted as the spin $\frac{1}{2}$ matrices

$$\rho_\mu(\tfrac{1}{2},\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\,\sigma_\mu, \qquad \tilde{\rho}_\mu(\tfrac{1}{2},\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\,\tilde{\sigma}_\mu, \tag{1}$$

which transform under the $(\frac{1}{2},\frac{1}{2}) \sim (\frac{1}{2},0)\otimes(\frac{1}{2},0)^*$ irreducible representation [4] (IR) of $SL(2,C)$,

$$D^{(\frac{1}{2},0)}(A)_a{}^b D^{(\frac{1}{2},0)}(A^*)_{\dot{c}}{}^{\dot{d}} \rho^\mu(\tfrac{1}{2},\tfrac{1}{2})_{b\dot{d}} = \rho^\nu(\tfrac{1}{2},\tfrac{1}{2})_{a\dot{c}}\Lambda_\nu{}^\mu, \tag{2a}$$

$$D^{(0,\frac{1}{2})}(A)^{\dot{a}}{}_{\dot{b}} D^{(0,\frac{1}{2})}(A^*)^c{}_d \tilde{\rho}_\mu(\tfrac{1}{2},\tfrac{1}{2})^{\dot{b}d} = \tilde{\rho}_\nu(\tfrac{1}{2},\tfrac{1}{2})^{\dot{a}c}\Lambda^\nu{}_\mu, \tag{2b}$$

where $D^{(\frac{1}{2},0)}(A) = D^{(0,\frac{1}{2})}(A^{\dagger-1}) = A$ is an element of $SL(2,C)$ and Λ an element of the homogeneous Lorentz group. Equations (2) fix our notation of spinor indices (upper/lower, dotted undotted). In the following we shall always use lower case Latin indices for two component spinors and Greek indices (with the exception of $\alpha,\beta,\gamma,\delta$) for Space-Time. Our space-time metric $g_{\mu\nu}$ will have signature -2.

A higher (s_1,s_2) rank spin matrix with $s_1 \geq s_2$ will satisfy the analogous transformation equation to (2a)

$$D^{(s_1,0)}(A)_{M_1}{}^{N_1} D^{(s_2,0)}(A^*)_{\dot{M}_2}{}^{\dot{N}_2} \rho^{\mu_1\mu_2\cdots\mu_{2s_1}}(s_1,s_2)_{N_1\dot{N}_2}$$

$$= \rho^{\nu_1\nu_2\cdots\nu_{2s_1}}(s_1,s_2)_{M_1\dot{M}_2}\Lambda_{\nu_1}{}^{\mu_1}\cdots\Lambda_{\nu_{2s_1}}{}^{\mu_{2s_1}}, \tag{3}$$

and similarly for (2b), with $\tilde{\rho}_{\mu_1 \ldots \mu_{2s_1}}(s_1,s_2)^{\dot{M}_1 M_2}$.

A higher ($j > \frac{1}{2}$) rank, j-spinor index is then one which labels the basis for the $(j,0)$ IR of $SL(2,C)$. The spinor metric is

$$C^{M_1M_2} = (-1)^{j-M_1}\delta_{M_2,-M_1} = \sqrt{2j+1}\ [jj0]^{M_1M_2}{}_0 = (-1)^{2j}C^{M_2M_1}$$

$$C_{M_1M_2}{}^{-1} = (-1)^{j+M_1}\delta_{M_2,-M_1} = \sqrt{2j+1}\ (-1)^{2j}[jj0]_{M_1M_2}{}^0 = (-1)^{2j}C_{M_2M_1}{}^{-1} \tag{4}$$

where $[j_1j_2j]^{m_1m_2}{}_m$ is a Clebsch-Gordon coefficient (CGC) of $SU(2)$. Given that the representation matrices for the $(j,0)$ IR of $SL(2,C)$ are given in terms of lower rank representation functions of the IR's $(j_1,0)$ and $(j_2,0)$ by

$$D^{(j,0)}(A)_M{}^N = \sum_{M_1M_2N_1N_2} D^{(j_1,0)}(A)_{M_1}{}^{N_1}D^{(j_2,0)}(A)_{M_2}{}^{N_2} \cdot [j_1j_2j]^{M_1M_2}{}_M[j_1j_2j]_{N_1N_2}{}^N \tag{5}$$

and the orthogonality relations [5] of the CGC's, we find that the inductive definition of the (s_1,s_2) spin matrix in (3) is

$$\rho^{\mu_1\cdots\mu_{2s_1}}(s_1,s_2)^{(L)}{}_{M\dot{N}} = \sum_{M_1m\dot{N}_1\dot{n}} \rho^{\mu_1\cdots\mu_{2s_1-1}}(s_1-\tfrac{1}{2},s_2-\tfrac{1}{2})^{(L_1)}{}_{M_1\dot{N}_1} \cdot\rho^{\mu_{2s_1}}(\tfrac{1}{2},\tfrac{1}{2})_{m\dot{n}}[(s_1-\tfrac{1}{2})\ \tfrac{1}{2}\ s_1]^{M_1m}{}_M \cdot[(s_2-\tfrac{1}{2})\ \tfrac{1}{2}\ s_2]^{\dot{N}_1\dot{n}}{}_{\dot{N}} \tag{6}$$

and a similar one for the $\tilde{\rho}(s_1,s_2)$. The labels (L) and (L_1) denote the mode of recouplings [5] employed in the construction of an $\rho(s_1,s_2)$ spin matrix from the elementary spin matrices $\rho(\frac{1}{2},\frac{1}{2})$. Needless to say, the mode of recoupling decides the type of symmetries of the spin matrix in its space-time indices.

The orthogonality relations of the spin matrices, as well as some other spinor identities are given in the appendix.

§(3): *DECOMPOSITION OF THE RIEMANN TENSOR*

Subject to the symmetries of the Riemann tensor

$$R_{\mu\nu\rho\sigma} = - R_{\nu\mu\rho\sigma} = - R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu},$$
$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0, \tag{7}$$

we obtain the following reduction of $R_{\mu\nu\rho\sigma}$ into IR's of $SL(2,C)$:

$$[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})]\otimes[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})]$$
$$\sim [(1,0)\oplus(0,1)]\otimes[(1,0)\oplus(0,1)]$$
$$\sim [(2,0)\oplus(0,2)]\oplus(1,1)_S\oplus(0,0) \tag{8}$$

where $(1,1)_S$ is the symmetrised $[(1,0)\otimes(0,1)]\oplus[(0,1)\otimes(1,0)]$, and all IR's not possessing the symmetries in (7) are rejected.

It remains simply to construct the spin matrices with the appropriate $SL(2,C)$ transformation characters given in (8) and the required symmetry properties in their space-time indices given by (7). These are

$$\rho_{(5)}{}^{\mu\nu\rho\sigma}(2,0)_A = \sum_{\alpha,\beta} \rho^{\mu\nu}(1,0)_\alpha \rho^{\rho\sigma}(1,0)_\beta [112]^{\alpha\beta}{}_A, \tag{9a}$$

$$\rho_{(9)}{}^{\mu\nu\rho\sigma}(1,1)_{\alpha\dot\beta} = \rho^{\mu\nu}(1,0)_\alpha \rho^{\rho\sigma}(1,0)_{\dot\beta}{}^{*} + \rho^{\rho\sigma}(1,0)_\alpha \rho^{\mu\nu}(1,0)_{\dot\beta}{}^{*}, \tag{9b}$$

$$\rho_{(1)}{}^{\mu\nu\rho\sigma}(0,0) = g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}, \tag{9c}$$

where $\alpha,\dot\beta = 0,\pm1$ and $A = 0,\pm1,\pm2$. In what follows we shall consistently use the Greek letters $\alpha,\beta,\gamma,\delta$ for 1-spinor indices, lower-case Latin indices for $\frac{1}{2}$-spinors and upper-case Latin indices for 2-spinors. The labels (5), (9) and (1) in (9a-c) respectively signify the number of elements in each. The spin matrices (9) do posess the $SL(2,C)$ transformation characters set out in (8). It is trivial to check that (9c) satisfies the symmetries in (7). That the symmetries in (7) are satisfied by (9b) can be readily seen by use of the identity (A.5) in the appendix, while in the case of (9a) this can be checked straighforwardly by use of recoupling [5] coefficients.

The Riemann tensor can then be decomposed in a notation close to the previously existing [1,2] one, as

$$R^{\mu\nu\rho\sigma} = (\Psi^A \rho_{(5)}{}^{\mu\nu\rho\sigma}{}_A + \bar\Psi^{\dot A} \delta_{(5)}{}^{\mu\nu\rho\sigma}{}_{\dot A}{}^{*})$$
$$+ \Phi^{\dot\beta\alpha} \rho_{(9)}{}^{\mu\nu\rho\sigma}{}_{\alpha\dot\beta} + \Lambda \rho_{(1)}{}^{\mu\nu\rho\sigma}. \tag{10}$$

The second term in (10) is the complex conjugate of the first, which guarantees the reality of the Riemann tensor, at the same time giving it positive space-reflection signature, cf. equation (8). This is a property [6] of even rank ($2j$ = even) spinors. We shall call the (2,0)-spinor Ψ_A the Weyl Spinor, and it has five ($A = \pm 2, \pm 1, 0$) complex components, in all ten real quantities. $\Phi_{\alpha\dot{\beta}}$, the Ricci Spinor is a nine-component real (1,1)-spinor. Its reality follows from the reality of the Riemann tensor and the hermiticity of $\rho(9)$, cf. equation (A.5) in the appendix. Finally the scalar (a(0,0)-spinor) Λ is real and proportional to the Ricci scalar. The components of Ψ, Φ and Λ spinors add up to twenty independent real quantities.

It is perhaps interesting to remark at this stage that here we have the Riemann tensor playing the role of a spin-2 field just like the Maxwell tensor $F_{\mu\nu}$ does for a spin-1 field [6]. Both fields are equivalent to fields transforming respectively according to the IR's (2,0) and (1,0) (or (0,2) and (0,1)) of $SL(2,C)$. These relations are given at the end of the appendix by equations (A,9-10,12-13).

The inverse relations to (10), giving the spinors Ψ, Φ and Λ in terms of the Riemann tensor are readily computed by successively contracting the space-time indices of the Riemann tensor using $g_{\mu\sigma}$. $\rho(5)^{\mu\nu\rho\sigma} = 0$ and the identity (A.5). They are, firstly

$$\Lambda = \frac{1}{12} R \tag{11a}$$

$$\Phi_{\alpha\dot{\beta}} = \frac{1}{4} \rho^{\mu\nu}(1,1)_{\alpha\dot{\beta}}(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu}R), \tag{11b}$$

where $R_{\mu\nu}$ is the Ricci tensor and R the Ricci scalar. Substituting (11b) back into (10) and using the identity (A.6), we find

$$\begin{aligned}\Psi^A\rho(5)^{\mu\nu\rho\sigma}{}_A + \bar{\Psi}^{\dot{A}}\rho(5)^{\mu\nu\rho\sigma}{}_{\dot{A}}{}^{*} \\ = R^{\mu\nu\rho\sigma} - \frac{1}{2}(g^{\mu\sigma}R^{\nu\rho} - g^{\mu\rho}R^{\nu\sigma} - g^{\nu\sigma}R^{\mu\rho} + g^{\nu\rho}R^{\mu\sigma}) \\ + \frac{1}{6} R(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}),\end{aligned} \tag{11'}$$

which is the Weyl tensor, from which follows by use of (A.3), our definition of the Weyl spinor

$$\begin{aligned}\Psi_A &= R_{\mu\nu\rho\sigma}\rho(5)^{\mu\nu\rho\sigma}{}_A, \\ \bar{\Psi}_{\dot{A}} &= R_{\mu\nu\rho\sigma}\rho(5)^{\mu\nu\rho\sigma}{}_{\dot{A}}{}^{*}.\end{aligned} \tag{11c}$$

This completes our decomposition of the Riemann tensor, which is manifestly in exact analogy with the standard formulation [1,2]. To classify the Weyl spinor according to Petrov types we need the quantities corresponding to the dyad of the standard formulation [1,2], which we introduce in the next section.

§(4): *ORTHONORMAL BASE-COVARIANTS: CLASSIFICATION*

What we call base-covariants are related to the orthonormal tetrad $Z_{\mu\mu'}$ which, in particular, are tensors with μ the space-time index and μ' the tetrad, or base index. These satisfy the orthogonality relations.

$$Z_{\mu\mu'}Z_{\nu\nu'}g^{\mu\nu} = \eta_{\mu'\nu'} \quad \text{and} \quad Z_{\mu\mu'}Z_{\nu\nu'}\eta^{\mu'\nu'} = g_{\mu\nu}, \tag{12}$$

where $\eta_{\mu'\nu'}$ is the flat space Minowski metric. Henceforth, we shall denote 'base' indices by primes.

The base covariants we shall define are different from the above in that they are defined in spinor space and are not therefore tensors, but spinor covariants. They are in fact the direct generalisations of the dyad bases [*1*,*7*]. The dyad base-covariants operate in a two-dimensional complex space and we extend their definition to base-covariants operating in $(2j + 1)$-dimensional complex space. In other words, whereas a dyad index labels the basis for an $SL(2,C)$ IR $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$, the j-th rank base covariant will carry indices similarly pertaining to the IR $(j,0)$ or $(0,j)$.

Denoting the j-th rank base-covariants by $\zeta(j,0)_M$, $\eta(0,j)^{\dot{M}}$ in conformity with the notation of [*2*], we can express their orthonormality properties by

$$\zeta(j,0)_M{}^{M'}\bar{\eta}(0,j)^M{}_{N'} = \delta^{M'}{}_{N'}. \tag{13}$$

We recall† that for integral j the following spinorial relation holds between the $(j,0)$ and $(0,j)$ bases of the $SL(2,C)$ IR's

$$\zeta(j,0)_M = C_{MN}{}^{-1}\bar{\eta}(0,j)^N \tag{14}$$

where we have suppressed the base indices.

It follows from (14) that for integral j, we can replace the orthogonality relations (13) by the simpler ones

$$C^{MN}\zeta_M{}^{M'}\zeta_{NN'} = \delta^{M'}{}_{N'}, \tag{15}$$

which is useful in view of the fact that the spinors that concern us immediately in the expansion (10) are of even rank. Clearly (15) would not serve as an orthonormality relation of half-integral j, e.g. $j = \frac{1}{2}$ for dyads, since by the symmetry properties (4) of $C^{\dot{M}\dot{N}}$, the left hand side of (15) would vanish.

With these base covariants at our disposal, the classification of the spinors occuring in the decomposition (10) becomes a trivial

† See for example reference [*6*]

matter of picking out their appropriate components:

$$\psi^{A'} = \psi^{A}\zeta(2,0)_{A}{}^{A'}; \qquad A,A' = \pm 2, \pm 1, 0 \tag{16}$$

$$\phi^{\dot\beta'\alpha'} = \phi^{\dot\beta\alpha}\bar\zeta(1,0)_{\dot\beta}{}^{\dot\beta'}\zeta(1,0)_{\alpha}{}^{\alpha'}; \qquad \alpha,\dot\beta,\alpha',\dot\beta' = \pm 1, 0. \tag{17}$$

The five quantities $\Psi^{A'}$ on this orthonormal spinor covariant base correspond to the five Petrov types of the Weyl spinor.

Note that the Petrov type corresponding to gravitational radiation is the classical spinor field Ψ^{2} or $\bar\Psi^{-2}$, which satisfy the condition required by Weinberg's Theorem [*6*] on quantum fields of massless particles, i.e. radiation; which for a quantum field transforming according to the IR (A,B) of $SL(2,C)$ is

$$-A + B = \lambda .$$

where λ is the helicity of the radiation; provided we identify the spinor index A on the Weyl spinor with the helicity.

§(5): *COVARIANT DIFFERENTATION: SPINOR CONNECTIONS*

The content of this section is a direct generalisation of the definition of the spinor connection for a half-spinor† given in references [*1,7*], to the case of a j-spinor†. We give below the spinor connection for j-th rank one-spinors, and require that the covariant differentiation operation of a spinor with more than one index has the same properties as that of spacetime tensors [*7*].

Following references [*1,7*] we attribute the following properties to the covariant derivative:

$$\nabla_{\mu}\rho^{\nu}{}_{a\dot b} = \nabla_{\mu}\tilde\rho_{\nu}{}^{\dot b a} = 0, \tag{18}$$

$$\nabla_{\mu}C_{ab}{}^{-1} = \nabla_{\mu}C^{ab} = 0. \tag{19}$$

Further, because of our wish to define higher rank spinor connections, we endow the covariant derivative with the property

$$\nabla_{\mu}\rho^{\mu_1\cdots\mu_{2s}}(s,s')_{M\dot N} = 0, \tag{20}$$

which appears to be an extension of (18) simply. More precisely,

† The authors of references [*1,7*] use the terminology one-spinor for a $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$spinor which we simply call a $\frac{1}{2}$-spinor, thus calling $(j,0)$ and $(0,j)$spinors j-spinors. We use the term one-spinor only to mean that the spinor is labelled with one index, e.g. Ψ_{A}.

however, it follows from the definition equation (6) that (20) implies

$$\nabla_\mu [s_1 s_2 s]_{m_1 m_2}{}^{m} = 0. \tag{20'}$$

Indeed (19) is the special case $s_1 = s_2 = \frac{1}{2}$, $s = 0$ of (20'), cf. equations (4). The additional property (20) or (20') seems to be a very natural one to take and it will be shown later in this section that it does not give rise to any constraints over and above those arising from (18) and (19).

We first reintroduce [1,7] the spinor connection for a $\frac{1}{2}$-spinor, which is obtained from the two equations (18,19) directly. Thus from (19) we have

$$\Gamma_{\lambda a}{}^{a} = -\tfrac{1}{2} C_{ab}{}^{-1} \partial_\lambda C^{ba} = -\tfrac{1}{2}(C^{-1}\partial_\lambda C)_a{}^{a} \tag{21}$$

whence it follows from (18) and (A.1) that

$$2\Gamma_{\lambda a}{}^{b} = (\rho^\mu \tilde{\rho}_\nu)_a{}^{b}\Gamma_{\lambda\mu}{}^{\nu} - (\rho^\mu \partial_\lambda \tilde{\rho}_\mu)_a{}^{b} + \tfrac{1}{2}\mathrm{Tr}(C^{-1}\partial_\lambda C)^* \delta_a{}^{b}, \tag{22}$$

which gives the $\frac{1}{2}$-spinor connection $\Gamma_{\lambda a}{}^{b}$ in terms of the Christoffel symbol $\Gamma_{\lambda\mu}{}^{\nu}$. Further, by setting $a = b$ in (22) and summing we get the relationship

$$\Gamma_{\lambda\mu}{}^{\mu} = \mathrm{Tr}(\rho^\mu \partial_\lambda \tilde{\rho}_\mu) - \mathrm{Tr}((C^{-1}\partial_\lambda C) + (C^{-1}\partial_\lambda C)^*). \tag{23}$$

Before writing down the j-spinor connection, we examine the 1-spinor connection in some detail. It follows immediately from (20) for $s = 1$, $s' = 0$; and (A.3) that the 1-spinor connection is

$$\Gamma_{\lambda\alpha}{}^{\beta} = 2(\rho^{\mu\tau}(1,0)_\alpha \rho_{\nu\tau}{}^*(0,1)^\beta)\Gamma_{\lambda\mu}{}^{\nu} - \rho^{\mu\nu}(1,0)_\alpha \partial_\lambda \rho_{\mu\nu}{}^*(0,1)^\alpha, \tag{24}$$

where $\lambda,\mu,\nu,\tau = 0,\ldots,3$ and $\alpha,\beta = \pm 1, 0$. Another form into which (24) may be cast is

$$\Gamma_{\lambda\alpha}{}^{\beta} = [\tfrac{1}{2}\tfrac{1}{2}1]^{ma}{}_\alpha [\tfrac{1}{2}\tfrac{1}{2}1]_{mb}{}^{\beta}\Gamma_{\lambda a}{}^{b} - [\tfrac{1}{2}\tfrac{1}{2}1]^{mn}{}_\alpha \partial_\lambda [\tfrac{1}{2}\tfrac{1}{2}1]_{mn}{}^{\beta}. \tag{24'}$$

At this point we carry out a test to check that our additional assumption (20) does not introduce any additional constraints into the problem. Equation (23) which is calculated from (22) relates to basic spinor quantities. Now if (20) is to be of a purely kinematic nature, $\Gamma_{\lambda\mu}{}^{\mu}$ calculated from (24) should also give us (23), and for example, not involve the CGC $[\frac{1}{2}\frac{1}{2}1]^{ab}{}_\alpha$. This is in fact obtained very straightforwardly by calculating $\Gamma_{\lambda\alpha}{}^{\alpha}$ from (24) and further using the fact that

$$\Gamma_{\lambda\alpha}{}^{\alpha} = -\tfrac{1}{2}C_{\alpha\beta}{}^{-1}\partial\ C_{\lambda}{}^{\beta\alpha}; \qquad \alpha,\beta = \pm 1, 0, \tag{25}$$

itself obtained directly from (20'), when $s_1 = s_2 = 1$ and $s = 0$, and (4). The result for $\Gamma_{\lambda\mu}{}^{\mu}$ thus calculated is found to be (23), which therefore completely justifies our assumption of (20) or (20').

We are now in a position to generalise (22) and (24) to any rank-j spinor connection. These are obtained by use of (A.3) and (20). For j integral, say $j = \ell$, we get, from (20), by setting $s = \ell$ and $s' = 0$:

$$\Gamma_{\lambda M}{}^{N} = 2\ell\rho^{\mu\tau_1\cdots\tau_{2\ell-1}}(\ell,0)_M\rho^*{}_{\nu\tau_1\cdots\tau_{2\ell-1}}(0,\ell)^N\Gamma_{\lambda\mu}{}^{\nu}$$

$$- \rho^{\mu_1\cdots\mu_{2\ell}}(\ell,0)_M\partial_\lambda\rho^*{}_{\mu_1\cdots\mu_{2\ell}}(0,\ell)^N, \tag{26}$$

and for j half integral, say $j = \ell + \frac{1}{2}$, again from (20), by setting $s = \ell + \frac{1}{2}$ and $s' = \frac{1}{2}$:

$$2\Gamma_{\lambda M}{}^{N} = (2\ell + 1)\rho^{\mu\tau_1\cdots\tau_{2\ell}}(\ell + \tfrac{1}{2},\tfrac{1}{2})_{M\dot{m}}\rho_{\nu\tau_1\cdots\tau_{2\ell}}(\tfrac{1}{2},\ell + \tfrac{1}{2})^{\dot{m}N}\Gamma_{\lambda\mu}{}^{\nu}$$

$$- \rho^{\mu_1\cdots\mu_{2\ell+1}}(\ell + \tfrac{1}{2},\tfrac{1}{2})_{M\dot{m}}\partial_\lambda\rho_{\mu_1\cdots\mu_{2\ell+1}}(\tfrac{1}{2},\ell + \tfrac{1}{2})^{\dot{m}N} + \tfrac{1}{2}\mathrm{Tr}(C^{-1}\partial_\lambda C)^*. \tag{27}$$

These are the formal expressions for the rank-j spinor connections. The reduction carried out from (24) to (24') in the case of $j = 1$ is not given here because that would involve rather lengthy recursion relations which are not very instructive.

§(6): *GENERALISED SPIN COEFFICIENTS*

As in the last section, we carry over the definition of references [*1*,*7*] for spin coefficients to the case of spinors of any IR $(j,0)$ or $(0,j)$. The contents of section 4 provide a framework for our extension of the definition of spin coefficients for $j = \frac{1}{2}$, since these will be defined in terms of the base covariants of the appropriate $(2j + 1)$-component orthonormal basis.

We first recall the definition of the Ricci rotation coefficients in terms of the orthonormal tetrad $Z_{\mu\mu'}$, where the index μ is the tensor index and the primed index μ' the tetrad index:

$$\gamma_{\mu'}{}^{\nu'\rho'} = Z_{\nu}{}^{\nu'}Z^{\mu\rho'}\nabla_\mu Z_{\mu'}{}^{\nu}. \tag{28}$$

It follows from $\nabla_\mu g_{\nu\rho} = 0$ and the symmetry of the metric tensor that

$$\gamma^{\mu'\nu'\rho'} = -\gamma^{\nu'\mu'\rho'}. \tag{29}$$

In analogy with (28) the definition [*1*,*7*] of a $\frac{1}{2}$-spin coefficient is

$$\Gamma_{a'b'c'\dot{d}'} = \zeta_{b'}{}^{a}\zeta_{c'}{}^{c}\bar{\zeta}_{\dot{d}'}{}^{\dot{d}}\nabla_{c\dot{d}}\zeta_{a'a}, \tag{30}$$

and it follows from (19) and the anti-symmetry of the spinor metric for $j = \frac{1}{2}$, that

$$\Gamma_{a'b'c'\dot{d}'} = \Gamma_{b'a'c'\dot{d}'}. \tag{31}$$

Having got the orthonormal base covariants of section 4, corresponding to any IR $(j,0)$ of $SL(2,C)$ at our disposal, we can immediately extend the definition of (30) to j-spin coefficients by utilizing the base covariant $\zeta(j,0)_{M'}{}^{M}$:

$$\Gamma_{M'N'a'\dot{b}'} = \zeta_{M'}{}^{M}\zeta_{a'}{}^{a}\bar{\zeta}_{\dot{b}'}{}^{\dot{b}}\nabla_{a\dot{b}}\zeta_{N'M};$$

$$M,M',N' = \pm(2j+1),\pm 2j,\cdots,0\,\mathrm{or}\pm\tfrac{1}{2} \tag{32}$$

$$a,a',\dot{b},\dot{b}' = \pm\tfrac{1}{2}.$$

The corresponding symmetry property to (31) then is found, by use of (20') and (4), to be

$$\Gamma_{M'N'a'\dot{b}'} = (-)^{2j+1}\Gamma_{N'M'a'\dot{b}'}. \tag{33}$$

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APPENDIX

(i) *Some Spinor Identities*

Using the basic spinor identities

$$\tilde{\rho}_{\mu}{}^{\dot{b}a}\rho^{\mu}{}_{c\dot{d}} = \delta^{\dot{b}}{}_{\dot{d}}\delta_{c}{}^{a}, \qquad \mathrm{Tr}(\rho_{\mu}\tilde{\rho}_{\nu}) = g_{\mu\nu}, \tag{A.1}$$

$$\rho_{\rho}\tilde{\rho}_{\sigma}\rho_{\tau} = \tfrac{1}{2}(i\varepsilon_{\mu\rho\sigma\tau}\rho^{\mu} + g_{\rho\sigma}\rho_{\tau} - g_{\rho\tau}\rho_{\sigma} + g_{\sigma\tau}\rho_{\rho}), \tag{A.2}$$

and the orthogonality relations of the CGC [5] of $SU(2)$, the following spinor identities, which have been useful in the text above, can be derived:

$$\rho^{\mu_1\cdots\mu_n}(s_1,s_2)(L)_{M\dot{N}}\tilde{\rho}_{\mu_1\cdots\mu_n}(s_1{}',s_2{}')(L')^{\dot{N}'\dot{M}'}$$
$$= \delta_{s_1s_1{}'}\delta_{s_2s_2{}'}\delta_{(L)(L')}\delta_{M}{}^{M'}\delta_{\dot{N}}{}^{\dot{N}'} \tag{A.3}$$

$$\sum_{s_1,s_2} \rho^{\mu_1\cdots\mu_n}(s_1,s_2)(L)_{M\dot{N}}\tilde{\rho}^{\nu_1\cdots\nu_n}(s_2,s_1)(L)^{\dot{N}M}$$
$$= g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_n\nu_n}. \tag{A.4}$$

The following special cases are used frequently in the text:

$$\rho^{\mu_1\mu_2}(1,0)_{\alpha}\rho^{\nu_1\nu_2}(1,0)_{\dot{\beta}}{}^{*} + \rho^{\nu_1\nu_2}(1,0)_{\alpha}\rho^{\mu_1\mu_2}(1,0)_{\dot{\beta}}{}^{*}$$
$$= 2(g^{\mu_1\nu_2}\rho^{\nu_1\mu_2}(1,1)_{\alpha\dot{\beta}} - g^{\mu_1\nu_1}\rho^{\mu_2\nu_2}(1,1)_{\alpha\dot{\beta}}$$
$$- g^{\mu_2\nu_2}\rho^{\mu_1\nu_1}(1,1)_{\alpha\dot{\beta}} + g^{\mu_2\nu_1}\rho^{\mu_1\nu_2}(1,1)_{\alpha\dot{\beta}}) \tag{A.5}$$

$$\tilde{\rho}_{\mu\nu}(1,1)^{\dot{\beta}\alpha}\rho^{\rho\sigma}(1,1)_{\alpha\dot{\beta}} = \tfrac{1}{2}(\delta_{\mu}{}^{\rho}\delta_{\nu}{}^{\sigma} + \delta_{\mu}{}^{\sigma}\delta_{\nu}{}^{\rho} - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}), \tag{A.6}$$

$$\rho^{\mu\nu}(1,0)_{\alpha}\rho_{\rho\sigma}{}^{*}(0,1)^{\alpha} = \tfrac{1}{4}(\delta_{\rho}{}^{\mu}\delta_{\sigma}{}^{\nu} - \delta_{\sigma}{}^{\mu}\delta_{\rho}{}^{\nu} + i\varepsilon^{\mu\nu}{}_{\rho\sigma}). \tag{A.7}$$

(ii) *Maxwell and Riemann Tensors*

The Maxwell tensor $F_{\mu\nu}$, antisymmetric in μ,ν, can be related to a (1,0) spinor via

$$\phi_{\alpha}(1,0) = F_{\mu\nu}\rho^{\mu\nu}(1,0)_{\alpha}, \tag{A.8}$$

which is not however simply invertible†, and it is seen by use of the appropriate special case of (A.6) that the tensor

$$\mathcal{F}_{\mu\nu} = F_{\rho\sigma}\rho^{\rho\sigma}(1,0)_{\alpha}\tilde{\rho}_{\mu\nu}{}^{*}(0,1)^{\alpha} \tag{A.9}$$

is invertible, i.e.

$$\mathcal{F}_{\mu\nu} = \phi_{\alpha}(1,0)\tilde{\rho}_{\mu\nu}{}^{*}(0,1)^{\alpha},$$
$$\phi_{\alpha}(1,0) = \mathcal{F}_{\mu\nu}\rho^{\mu\nu}(1,0)_{\alpha}. \tag{A.9'}$$

It is immediately seen from (A.7) that

$$\mathcal{F}_{\mu\nu} = \tfrac{1}{2}(F_{\mu\nu} + \tilde{F}_{\mu\nu}) \tag{A.10}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}i\varepsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$ is the dual of $F_{\mu\nu}$.

Similarly for the Riemann tensor $R_{\mu\nu\rho\sigma}$, whose symmetries are given by (7), we can define a (2,0)-spinor by means of the spin matrix $\rho_{(5)}{}^{\mu\nu\rho\sigma}{}_{A}$ in (9a),

$$\phi_{A}(2,0) = R_{\mu\nu\rho\sigma}\rho_{(5)}{}^{\mu\nu\rho\sigma}{}_{A} \tag{A.11}$$

which is of course not invertible in the sense of the above mentioned arguments. Now exactly as above we can define the invertible tensor

$$\mathcal{R}_{\mu\nu\rho\sigma} = \phi_{A}(2,0)\rho^{*}(5)_{\mu\nu\rho\sigma}{}^{A} = R_{\tau\lambda\theta\phi}\rho(5)^{\tau\lambda\theta\phi}{}_{A}\rho^{*}(5)_{\mu\nu\rho\sigma}{}^{A}, \tag{A.12}$$

which yields, by use of identities (A.1-7) and the orthogonality of the CGC and standard tensor identities:

$$\mathcal{R}_{\mu\nu\rho\sigma} = \frac{1}{4}\,(R_{\mu\nu\rho\sigma} + \tilde{R}^{L}{}_{\mu\nu\rho\sigma} + \tilde{R}^{R}{}_{\mu\nu\rho\sigma} + \tilde{\tilde{R}}_{\mu\nu\rho\sigma})$$
$$- \frac{1}{24}\,(R + \tilde{R})(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} + i\varepsilon_{\mu\nu\rho\sigma}), \tag{A.13}$$

where the duals $\tilde{R}^{L}$, $\tilde{R}^{R}$, $\tilde{\tilde{R}}$ are

$$\tilde{R}^{L}{}_{\mu\nu\rho\sigma} = \tfrac{1}{2}i\varepsilon_{\mu\nu}{}^{\tau\lambda}R_{\tau\lambda\rho\sigma} \qquad \text{(Contd)}$$

† This is because $F_{\mu\nu}$ has got definite signature for space-inversion while $\phi_{\alpha}(1,0)$ does not, and goes into $\chi(0,1)^{\dot{\alpha}}$ under space-inversion.

(Contd) $$\tilde{R}^{R}{}_{\mu\nu\rho\sigma} = \tfrac{1}{2}i\varepsilon_{\rho\sigma}{}^{\tau\lambda}R_{\mu\nu\tau\lambda}$$

$$\tilde{\tilde{R}}_{\mu\nu\rho\sigma} = -\tfrac{1}{4}\varepsilon_{\mu\nu}{}^{\tau\lambda}\varepsilon_{\rho\sigma}{}^{\theta\phi}R_{\tau\lambda\theta\phi} \qquad \text{(A.14)}$$

and the Ricci scalars R and $\tilde{R}$

$$R = R_{\mu\nu}{}^{\mu\nu}, \qquad \tilde{R} = \tilde{R}^{L}{}_{\mu\nu}{}^{\mu\nu} = \tilde{R}^{R}{}_{\mu\nu}{}^{\mu\nu}. \qquad \text{(A.15)}$$