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Brief paper

Robust model predictive control using tubes☆

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Abstract

A form of feedback model predictive control (MPC) that overcomes disadvantages of conventional MPC but which has manageable computational complexity is presented. The optimal control problem, solved on-line, yields a 'tube' and an associated piecewise affine control law that maintains the controlled trajectories in the tube *despite* uncertainty; computational complexity is linear (rather than exponential) in horizon length. Asymptotic stability of the controlled system is established.

Keywords: Robustness; Model predictive control; Uncertain systems; Tubes

1. Introduction

The problem of robust model predictive control (MPC) may be tackled in several ways reviewed in Mayne, Rawlings, Rao, and Scokaert (2000; Section 4). The most obvious method is to rely on the inherent robustness of deterministic MPC (Limón Marruedo, Alamo, & Camacho, 2002); Theorem 2 and Corollary 4 establish, under certain conditions, ultimate boundedness of state trajectories of the perturbed system. The connection between robustness and smoothness of a Lyapunov function is explored in the important paper (Kellet & Teel, 2002). A second approach, called *open-loop* model predictive control in Mayne et al. (2000), is to determine the current control action by solving on-line an optimal control problem in which the uncertainty is taken into account (both in cost minimization and constraint satisfaction) and the decision variable (like that in the first approach) is a sequence of control *actions* (Zheng & Morari, 1993). Open-loop model predictive control cannot contain the 'spread' of predicted trajectories possibly rendering solutions of the uncertain optimal control problem solved on-line unduly conservative or, even, infeasible. To overcome these disadvantages, *feedback* model predictive control was advocated in, e.g. (Mayne, 1995; Kothare, Balakrishnan, & Morari, 1996; Mayne, 1997; Lee & Yu, 1997; Scokaert & Mayne, 1998; De Nicolao, Magni, & Scattolini, 2000; Magni, Nijmeijer, & van der Schaft, 2001; Magni, De Nicolao, Scattolini, & Allgöwer, 2003). Both *open-loop* and *feedback* MPC provide feedback control but, whereas in open-loop MPC the decision variable in the optimal control problem solved on-line is a sequence $\{u_0, u_1, \ldots, u_{N-1}\}$ of control *actions*, in *feedback* MPC it is a *policy* π which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ of control *laws*. Determination of a control policy is usually prohibitively difficult. Research has, therefore, focused on simplifying the feedback optimal control problem. By restricting the horizon length N to be unity in Park and Kwon (1999), the authors simplify the problem dramatically since the decision variable is now u_0, a single-control *action* (the first element of π); the price to be paid is a reduction of the domain of attraction. In Kothare et al. (1996), every feedback law $\mu_i(\cdot)$ in the decision variable (policy π) is restricted to have the form $\mu_i(x) = Kx$, $i = 0, 1, \ldots$; for each current state x, an optimal K is determined. The solution so obtained is conservative. Uncertainty in the form of a bounded additive disturbance $w \in W$ (W convex) is considered in Mayne (1995) and Scokaert and Mayne (1998); simplification is achieved (when the system is linear and the constraints polyhedral)

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by considering only those disturbances lying in the set V^N where V is the set of vertices of W. Feedback *laws* are obtained by using convex interpolation between states generated by these disturbances. The complexity of this solution is exponential in the horizon length N. A very useful proposal is made in Rossiter, Kouvaritakis, and Rice (1998) and applied to robust MPC in Kouvaritakis, Rossiter, and Schuurmans (2000), Schuurmans and Rossiter (2000) and Young Il Lee and Kouvaritakis (2000); in these papers the feedback laws $\mu_i(\cdot)$ in policy π are restricted to have the form $\mu_i(x) = v_i + Kx,\ i = 0, 1, \ldots, N-1$. This restriction transforms the decision variable from a policy to a sequence of control actions $\{v_0, v_1, \ldots, v_{N-1}\}$. The inherent feedback (via the time-invariant K) reduces the spread of trajectories and is often very effective. The optimal control problem requires, as in open-loop MPC, satisfaction of all constraints for every possible realization in W^N of the disturbance sequence.

It is the main purpose of this paper to propose a method for robust MPC of linear constrained systems with uncertainty that overcomes, to some extent, the disadvantages of the methods described above. Being multi-stage, it has a larger domain of attraction than the one-step controller of Park and Kwon (1999). Its complexity is linear in N rather than exponential as in the method proposed in Mayne (1995) and Scokaert and Mayne (1998). Finally, the policy π is more powerful than the *linear*, *time-invariant* policy of Kouvaritakis et al. (2000), Schuurmans and Rossiter (2000) and Young Il Lee and Kouvaritakis (2000) since $\mu_i(x), i=0, 1, \ldots, N-1$ is *time-varying* and *piecewise affine* and the method may be used for MPC of *time-varying* systems and systems with *model uncertainty*. The robust MPC problem for linear systems with bounded additive disturbances is specified in Section 2 where conditions that ensure that a tube controller satisfies state, control and terminal constraints are stated (Proposition 1). The main contribution of the paper is a set of tube model predictive controllers, described and analyzed in Section 3. Some illustrative simulations are presented in Section 4. An extension of tube MPC to deal with linear, parameter uncertain, systems is briefly discussed in Section 5. A few conclusions are drawn in Section 6.

2. The robust control problem

The problem that we consider first is MPC of the system

$$x^+ = f(x, u, w) := Ax + Bu + w, \tag{2.1}$$

where x, u and w are, respectively, the current state, control and disturbance (of dimensions n, m and n, respectively) and x^+ is the successor state; the disturbance w is known only to the extent that it belongs to the set W. The pair (A, B) is assumed to be reachable. Control, current state and disturbance are subject to the hard constraints

$$u \in \mathbb{U},\ x \in \mathbb{X} \quad \text{and} \quad w \in W, \tag{2.2}$$

where $\mathbb{U}$ and W are (convex, compact) polytopes and $\mathbb{X}$ is a (convex) closed polyhedron; the sets $\mathbb{U}$ and $\mathbb{X}$ contain the origin in their interiors. The set W may lie in a subspace of $\mathbb{R}^n$ and so has the form $W = \{L\psi \mid \psi \in \Psi\}$ where $\Psi \subset \mathbb{R}^p$ for some integer p; we assume that Ψ is a polytope containing the origin in its interior. Let $\mathscr{U} := \mathbb{U}^N$ ($\mathscr{W} := W^N$) denote the class of admissible control sequences $\mathbf{u} := \{u(i) \mid i = 0, 1, \ldots, N-1\}$ (admissible disturbance sequences $\mathbf{w} := \{w(i) \mid i = 0, 1, \ldots, N-1\}$); for any integer N, $\mathbb{U}^N$ denotes the Cartesian product $\mathbb{U} \times \mathbb{U} \times \cdots \times \mathbb{U}$. Let $\phi(i; x, \mathbf{u}, \mathbf{w})$ denote the solution at time i of (2.1) when the control and disturbance sequences are, respectively, $\mathbf{u}$ and $\mathbf{w}$ and the initial state is x at time 0; similarly $\phi(i; x, \pi, \mathbf{w})$ denotes the solution at time i of (2.1) when the control policy is π, the disturbance sequence is $\mathbf{w}$ and the initial state is x at time 0 (a policy π is a sequence of control laws, i.e. $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ where $\mu_i(\cdot)$ is the control law (mapping state to control) at time i). The *nominal* system is defined by

$$x^+ = Ax + Bu \tag{2.3}$$

and $\bar{\phi}(i; x, \mathbf{u})$ denotes the solution at time i of the nominal system when the input sequence is $\mathbf{u}$ and the initial state is x at time 0.

Model predictive control is defined, as usual, by specifying a finite-horizon optimal control problem that is solved on-line. In the approach adopted in this paper, the optimal control problem is the determination of a *tube* defined as a sequence $\mathbf{X} := \{X_0, X_1, \ldots, X_N\}$ of *sets* of states and an associated (time-varying piecewise affine) policy $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ satisfying

$$X_i \subset \mathbb{X} \quad \forall i \in \mathscr{I}_{N-1}, \tag{2.4}$$

$$X_N \subset X_f \subset \mathbb{X}, \tag{2.5}$$

$$\mu_i(x) \in \mathbb{U} \quad \forall (x, i) \in X_i \times \mathscr{I}_{N-1}, \tag{2.6}$$

$$f(x, \mu_i(x), w) \in X_{i+1} \quad \forall (x, w, i) \in X_i \times W \times \mathscr{I}_{N-1}, \tag{2.7}$$

where X_f is a terminal constraint set and, for any integer i, $\mathscr{I}_i := \{0, 1, \ldots, i\}$ and $\mathscr{I}_i^+ := \{1, \ldots, i\}$. An important consequence of satisfaction of these constraints is given in the next result.

Proposition 1. *Suppose that the tube* $\mathbf{X}$ *and associated policy* π *satisfy constraints* (2.4)–(2.7). *Then* $\phi(i; x, \pi, \mathbf{w}) \in X_i \subset \mathbb{X}$ *and* $\mu_i(\phi(i; x, \pi, \mathbf{w})) \in \mathbb{U}$ *for all* $i \in \mathscr{I}_{N-1}$ *and* $\phi(N; x, \pi, \mathbf{w}) \in X_f \subset \mathbb{X}$ *for every initial state* $x \in X_0$ *and every admissible disturbance sequence* $\mathbf{w} \in \mathscr{W} := W^N$ (*policy* π *steers any initial state* $x \in X_0$ *to* X_f *along a trajectory lying in the tube* $\mathbf{X}$, *therefore satisfying all state and control constraints for every admissible disturbance sequence*).

Proof. Let $x(i) := \phi(i; x, \pi, \mathbf{w})$, $u(i) := \mu_i(x(i))$. By (2.6) and (2.7), if $x(i) \in X_i$, then $u(i) \in \mathbb{U}$ and $x(i+1) \in X_{i+1}$

for every $w(i) \in W$. But $x(0) = x \in X_0$. By induction, $x(i) \in X_i \subset \mathbb{X}$ and $u(i) \in \mathbb{U}$ for all $i \in \mathcal{I}_{N-1}$; also $x(N) \in X_N \subset X_f \subset \mathbb{X}$. □

The optimal control problem is minimization of an appropriate cost function subject to constraints (2.4)–(2.7). Minimization yields an optimal policy $\pi^0 = \{\mu_0^0(\cdot), \mu_1^0(\cdot), \ldots, \mu_{N-1}^0(\cdot)\}$. The policy π^0, computed on-line, may be used in several ways, discussed below, to implement MPC. Because of the additive disturbance, convergence to a set (rather than to the origin) is the best that can be hoped for. A set Z is robustly stable if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x(0), Z) \leqslant \delta$ ($d(x, Z)$ is the distance of x from the set Z) implies $d(x(t), Z) \leqslant \varepsilon$ for all $t \geqslant 0$ for all admissible disturbance sequences. A set Z is robustly asymptotically (finite-time) attractive with domain of attraction $\mathcal{X}$ if, for all $x(0) \in \mathcal{X}$, $d(x(t), Z) \to 0$ as $t \to \infty$ ($x(j) \in Z$, $j \geqslant k$ for some finite k), for all admissible disturbance sequences. A set Z is robustly asymptotically (finite-time) stable if it is robustly stable and robustly asymptotically (finite-time) attractive. A robustly exponentially stable set Z is similarly defined.

3. Tube MPC

3.1. Disturbance invariant tube

The tube controllers presented later in this section are an extension of the simple disturbance invariant tube controller independently presented in Langson (2000), Mayne and Langson (2001) and Chisci, Rossiter, and Zappa (2001) (Chisci et al., 2001 uses a sequence $\{Z_i\}$ of uncertainty sets whereas Langson (2000), Mayne and Langson (2001) uses a single uncertainty set Z where $Z_i \subset Z_{i+1} \subset Z$ for all i and $Z_i \to Z$; Chisci et al. (2001) presents a robust controller and Mayne and Langson (2001) presents a method for robustifying any stabilizing deterministic controller). Since this controller provides a 'hot start' for our algorithms, we briefly describe its main features. In the disturbance invariant controller, the tube $\mathbf{X} := \{X_0, X_1, \ldots, X_N\}$ and associated (time-invariant, affine) policy π take the simple form $X_i = \bar{x}_i + Z$, $i \in \mathcal{I}_N$ and $\mu_i(x) = v_i + K(x - \bar{x}_i)$, $i \in \mathcal{I}_{N-1}$; here Z is a disturbance invariant set (Kolmanovsky & Gilbert, 1998) for the closed-loop system $x^+ = A_K x + w$, $A_K := A + BK$ and $\bar{x}_i := \bar{\phi}(i; \bar{x}_0, \mathbf{v})$ where $\mathbf{v} := \{v_0, v_1, \ldots, v_{N-1}\}$. The feedback matrix K is stabilizing (A_K exponentially stable). The disturbance invariant set Z (which, by definition, satisfies $A_K x + w \in Z$ for all $x \in Z$, for all $w \in W$) may be computed as described in Young Il Lee and Kouvaritakis (2000), Kolmanovsky and Gilbert (1998), Mayne and Schroeder (1997) and Raković, Kerrigan, Kouramas, and Mayne (2003). It is desirable that Z be small; the smallest disturbance invariant set (in the sense of being a subset of any disturbance invariant set) is $\sum_{i=0}^{\infty} A_K^i W$ (Kolmanovsky & Gilbert, 1998) and is polytopic if K is chosen to be a minimum time controller ($A_K^n = 0$). We recall that the Pontryagin difference $A \ominus B$ of sets A and B is defined by $A \ominus B := \{x \mid x + B \subset A\}$. We assume:

A1: The sets $\bar{\mathbb{X}} := \mathbb{X} \ominus Z$, $\bar{\mathbb{U}} := \mathbb{U} \ominus KZ$ and $\bar{X}_f := X_f \ominus Z$ exist and contain the origin.

The properties of the disturbance invariant controller are summarized in Proposition 2 that is proven in Mayne and Langson (2001) (related results appear in Chisci et al., 2001).

Proposition 2. *Suppose A1 is satisfied and the initial state $\bar{x}_0$ of the nominal system (2.3) and the nominal control sequence $\mathbf{v}$ are such that the resultant nominal state sequence $\{\bar{x}_i \mid i \in \mathcal{I}_N\}$, $\bar{x}_i := \bar{\phi}(i; \bar{x}_0, \mathbf{v})$ satisfy*

$$v_i \in \bar{\mathbb{U}}, \qquad \bar{x}_i \in \bar{\mathbb{X}}, \quad i \in \mathcal{I}_{N-1}, \quad \bar{x}_N \in \bar{X}_f \subset \bar{\mathbb{X}}. \tag{3.1}$$

Suppose also that the actual system (2.1) has initial state $x_0 \in \bar{x}_0 + Z$ and uses policy π defined above. Then the state and control sequences $\{x_i \mid i \in \mathcal{I}_N\}$ and $\{u_i \mid i \in \mathcal{I}_{N-1}\}$, $x_i := \phi(i; x_0, \mathbf{u}, \mathbf{w})$ and $u_i := v_i + K(x_i - \bar{x}_i)$, of the actual system (2.1) satisfy

$$u_i \in \mathbb{U}, \qquad x_i \in \mathbb{X} \ \forall i \in \mathcal{I}_{N-1}, \quad x_N \in X_f \subset \mathbb{X} \tag{3.2}$$

A nominal, finite-horizon, optimal control problem can now be defined. Let the (nominal) cost function $\bar{V}_N(\cdot)$ be defined by

$$\bar{V}_N(x, \mathbf{v}) := \sum_{i=0}^{N-1} \ell(\bar{x}_i, v_i) + V_f(\bar{x}_N), \tag{3.3}$$

where, for all i, $\bar{x}_i := \bar{\phi}(i; x, \mathbf{v})$ and let $\bar{\mathcal{U}}(x)$ denote the set of control sequences $\mathbf{v}$ that satisfy constraints (3.1)

$$\bar{\mathcal{U}}(x) := \{\mathbf{v} \mid v_i \in \bar{\mathbb{U}}, \bar{\phi}(i; x, \mathbf{v}) \in \bar{\mathbb{X}},\ i \in \mathcal{I}_{N-1}, \\ \bar{\phi}(N; x, \mathbf{v}) \in \bar{X}_f \subset \bar{\mathbb{X}}\}. \tag{3.4}$$

The nominal control problem is

$$\bar{P}_N(x): \quad \bar{V}_N^0(x) = \min_{\mathbf{v}} \{\bar{V}_N(x, \mathbf{v}) \mid \mathbf{v} \in \bar{\mathcal{U}}(x)\}. \tag{3.5}$$

Let $\bar{\mathcal{X}}_N := \{x \mid \bar{\mathcal{U}}(x) \neq \emptyset\}$; $\bar{\mathcal{X}}_N$ is the domain of the value function $\bar{V}_N^0(\cdot)$. The nominal control problem is a *conventional* (open-loop) optimal control problem that has constraints (3.1) that are tighter than the original constraints (3.2). The solution to this conventional problem, if the initial state is x, is the optimal control sequence $\mathbf{v}^0(x) = \{v_0^0(x), v_1^0(x), \ldots, v_{N-1}^0(x)\}$. This solution defines a policy $\pi^0(x) = \{\mu_0^0(\cdot), \mu_1^0(\cdot), \ldots, \mu_{N-1}^0(\cdot)\}$ for the actual system (2.1) where

$$\mu_i^0(x_i) := v_i^0(x) + K(x_i - \bar{x}_i), \qquad \bar{x}_i := \bar{\phi}(i; x, \mathbf{v}^0(x)). \tag{3.6}$$

By Proposition 2, this policy steers any initial state $x \in \bar{\mathcal{X}}_N$ of the actual system (2.1) to X_f in N steps for all admissible disturbance sequences.

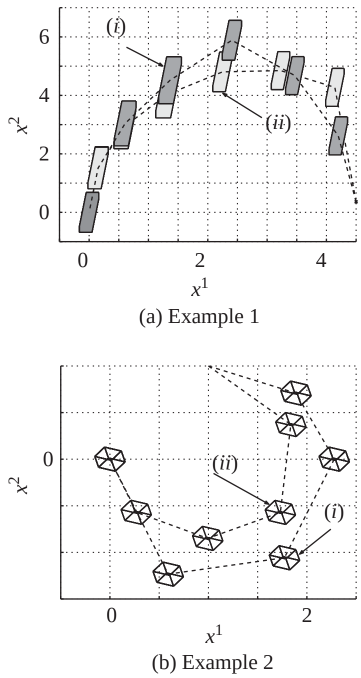


Fig. 1. MPC: (i) disturbance invariant; (ii) tube. (a) Example 1. (b) Example 2.

Disturbance invariant tubes (constructed from the solution of $\bar{P}_N(x)$) for Example 1 ($n=2$) and Example 2 ($n=3$) are shown in Fig. 1 (case (i)); in each case $Z=\sum_{i=0}^{n-1} A_K^i W$ where $A_K := A + BK$ and K is a minimum time controller. Detailed information on the examples is provided in Langson, Chryssochoos, and Mayne (2001).

The above procedure can be used, as shown in Mayne and Langson (2001), to obtain a robust receding horizon controller from *any* nominal receding horizon controller. However, it may also be used, as we show next, to obtain a simple form of MPC that will be used in a more complex tube controller described later.

3.2. Single policy disturbance invariant MPC

Although the feedback model predictive control problem is complex, its solution at the initial event (state x, time 0) provides feedback control for *all* possible disturbance sequences. Hence, in principle (if all uncertainties are adequately described by (2.1)), robust feedback control may be obtained by solving $\bar{P}_N(x)$ *once* provided that the terminal set X_f is robust control invariant (for all $x \in X_f$, there exists a $u \in U$ such that $Ax + Bu + w \in X_f$ for all $w \in W$). Under this assumption, there exists a control law $\kappa_f(\cdot)$ such that $Ax + B\kappa_f(x) + W \subset X_f$ for all $x \in X_f$. For simplicity, we choose $X_f = Z$ (which implies $\bar{X}_f = \{0\}$), $\kappa_f(x) = Kx$, $\ell(x,u) = (\frac{1}{2})|x|_Q^2 + (\frac{1}{2})|u|_R^2$ where Q and R are positive definite, $V_f(x) \equiv 0$ and $N \geqslant n$. Single policy MPC robustly steers any initial state of the actual system in $\bar{\mathcal{X}}_N$ to X_f in N steps and thereafter keeps the state in X_f using the local robust controller $\kappa_f(\cdot)$ (dual-mode control).

Single policy invariant controller:

1. At state $x(0)$, time 0, solve $\bar{P}_N(x(0))$; set $u(0) = \mu_0^0(x(0)) = v_0^0(x(0))$.
2. At state $x(k)$, time $k > 0$, set $u(k) = \mu_k^0(x(k))$ (3.6).
3. If $x(k) \in X_f$ ($k \geqslant N$), set $u(k) = Kx(k)$.

Proposition 3. *The set $X_f = Z$ is robust finite-time stable for system* (2.1) *with the single policy invariant controller. The controller satisfies the state and control constraints for all admissible disturbance sequences*; *the domain of attraction is $\bar{\mathcal{X}}_N$.*

Proof. Firstly, the origin is finite-time stable for the nominal system $\bar{x}_{i+1} = A\bar{x}_i + Bv_i^0(x)$ with initial state $\bar{x}_0 = x \in \bar{\mathcal{X}}_N$. Finite-time attractivity of the origin follows from the fact that $\mathbf{v}^0(x)$ steers any $x \in \bar{\mathcal{X}}_N$ to the origin in N steps (or less). Stability of the origin follows from the fact that the solution to $\min_{\mathbf{v}}\{\bar{V}_N(x,\mathbf{v}) \mid \bar{\phi}(N;x,\mathbf{v}) = 0\}$ is a linearly constrained linear quadratic optimal control problem so that $\mathbf{v}^0(x) = Lx$ (for some L) for all x in an output admissible set S that contains the origin in its interior; the function $x \mapsto \mathbf{v}^0(x)$ is therefore continuous in this set. Since, the associated state trajectory $\{x_0^0(x), x_1^0(x), \ldots, x_{N-1}^0(x), 0, 0, 0, \ldots\}$ is also continuous (in the initial state x) the origin is stable (for all $\delta > 0$ there exists an $\varepsilon > 0$ such that $|x| \leqslant \varepsilon$ implies $|x_i^0(x)| \leqslant \delta$ for all $i \geqslant 0$). That Z is robust finite-time stable for the actual system (2.1) follows from finite-time stability of the origin for the nominal system (which implies $\bar{x}_N = 0$) and Proposition 2 which implies that the state $x(i) = \phi(i;x,\pi) \in \bar{x}_i + Z$ for all $i \in \mathcal{I}_N$ (which implies $x(N) \in Z$). The robust controller $u = Kx$ ensures $x(j) \in Z$ for all $j \geqslant N$. □

Trajectories generated by this controller lie in the tubes marked (i) in Fig. 1; these trajectories reach the target set in N steps.

3.3. Tube MPC

The disturbance invariant tube achieves robustness with great simplicity ($\bar{P}_N(x)$ is a deterministic optimal control problem). However, the form of the tube is restricted and the feedback component K of the policy π is required to be time-invariant. Here, we remove some of these restrictions. The tube $\mathbf{X} = \{X_0, X_1, \ldots, X_N\}$ now has the more general form

$$X_0 = \{x\}, \quad X_i = z_i + \alpha_i Z, \quad \forall i \in \mathcal{I}_N^+, \tag{3.7}$$

where the sequence $\{z_i\}$ can be freely chosen (it is no longer required to satisfy the nominal difference equation (2.3)), and the sequence $\{\alpha_i\}$ permits the size of X_i to vary; we refer to z_i as the center of X_i. The set $Z = \mathrm{co}\{v^1, \ldots, v^J\}$ is a polytope and is not necessarily positively invariant. For each i, $X_i = \mathrm{co}\{x_i^1, \ldots, x_i^J\}$ where, for each j $x_i^j = z_i + \alpha_i v^j$. With each tube $\mathbf{X}$ is associated a tube control sequence $\mathbf{U} = \{U_0, U_1, \ldots, U_{N-1}\}$ where

$U_0 = \{u_0\}$ and, for each $i \in \mathscr{I}^+_{N-1}$, $U_i = \{u_i^1, \ldots, u_i^J\}$; for each j, control u_i^j is associated with vertex x_i^j. A tube pair $(\mathbf{X}, \mathbf{U})$ defines a (time-varying piecewise affine) policy $\pi = \{u_0, \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$; each control law $\mu_i(\cdot) := \mu_{X_i,U_i}(\cdot)$, where, for each $X = \mathrm{co}\{x^1, \ldots, x^J\}$, $U = \{u^1, \ldots, u^J\}$, the control law $\mu_{X,U} : X \to \mathrm{co}\{U\}$ is defined as follows: for each $x \in X$, $\lambda(x) = (\lambda^1(x), \ldots, \lambda^J(x))$ is defined to be the (unique) least-squares solution of $\sum_{j=1}^J \lambda^j x_i^j = x$ subject to the constraint $\lambda \in \Lambda := \{\lambda \mid \lambda^j \geqslant 0,\ j \in \mathscr{J},\ \sum_{j=1}^J \lambda^j = 1\}$ where $\mathscr{J} := \{1, \ldots, J\}$; the control law $\mu_{X,U}(\cdot)$ is then defined by

$$\mu_{X,U}(x) := \sum_{j=1}^{J} \lambda^j(x) u^j \quad \forall x \in X. \tag{3.8}$$

Since $x \mapsto \lambda(x)$ is piecewise affine (affine if Z is a simplex), so is $\mu_{X,U}(\cdot)$. The (time-varying piecewise affine) policy $\pi = \{u_0, \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ associated with $(\mathbf{X}, \mathbf{U})$ is defined by $\mu_i(\cdot) := \mu_{X_i,U_i}(\cdot)$, $i = 1, \ldots, N-1$. If $u_i^j \in \mathbb{U}$ for all $j \in \{1, \ldots, J\}$, then $\mu_i(x) = \mu_{X_i,U_i}(x) := \sum_{j=1}^J \lambda^j(x) u_i^j \in \mathbb{U}$ for all $x \in X_i$ since $\mathbb{U}$ is convex and $\lambda(x) \in \Lambda$; hence (2.6) is satisfied. Finally, if (X_i, U_i) satisfy

$$Ax_i^j + Bu_i^j \in X_{i+1} \ominus W \quad \forall j \in \mathscr{J} \tag{3.9}$$

for all $i \in \mathscr{I}_{N-1}$, then (2.7) holds. The decision variable for this version of tube MPC is $\theta \in \mathbb{R}^{N(1+n)+m+(N-1)mJ}$ defined by

$$\theta := \{\mathbf{a}, \mathbf{z}, \mathbf{U}\}, \tag{3.10}$$

where $\mathbf{a} := \{\alpha_1, \ldots, \alpha_N\}$, $\mathbf{z} := \{z_1, \ldots, z_N\}$ and $\mathbf{U} := \{U_0, U_1, \ldots, U_{N-1}\}$. Clearly, $\mathbf{a}$, $\mathbf{z}$ and $\mathbf{U}$ specify the tube X (since Z is fixed). For each x, let $\Theta(x)$ denote the set of θ satisfying the constraints discussed above

$$\begin{aligned}\Theta(x) = \{\theta \mid\ & \mathbf{a} \geqslant 0,\ X_i \subset \mathbb{X},\ U_i \subset \mathbb{U}_i,\ X_N \subset X_f \subset \mathbb{X}, \\ & Ax_i^j + Bu_i^j \in X_{i+1} \ominus W\ \forall (i,j) \in \mathscr{I}_{N-1} \times \mathscr{J}\},\end{aligned} \tag{3.11}$$

where $\mathbf{a} \geqslant 0$ implies $\alpha_i \geqslant 0$ for all $i \in \mathscr{I}_N^+$, $\mathbb{U}_0 := \mathbb{U}$ and $\mathbb{U}_i := \mathbb{U}^J$ for $i \geqslant 1$. Let $\mathscr{X}_N := \{x \mid \Theta(x) \neq \emptyset\}$. Since, for given x, the constraints in (3.11) are affine in $\mathbf{a}$, $\mathbf{z}$ and $\mathbf{U}$ (for example, the constraint $X_i \subset \mathbb{X}$ is equivalent to $z_i + \alpha_i v^j \in \mathbb{X}$ for all $j \in \mathscr{J}$ where $\mathbb{X}$ is a polytope), the set $\Theta(x)$ is a polyhedron for each $x \in \mathscr{X}_N$. Note that satisfaction of a difference equation is not required; the difference equation is replaced by the difference inclusion (3.9).

Proposition 4. *Suppose $x \in \mathscr{X}_N$ and $\theta \in \Theta(x)$. Let π denote the associated policy defined by (3.8). Then $\phi(i; x, \pi, \mathbf{w}) \in X_i \subset \mathbb{X}$ for all $i \in \mathscr{I}_{N-1}$, $\mu_i(\phi(i; x, \pi, \mathbf{w})) \in U$ for all $i \in \mathscr{I}_{N-1}$, and $\phi(N; x, \pi) \in X_f \subset \mathbb{X}$ for every initial state $x \in X_0$ and every admissible disturbance sequence $\mathbf{w} \in \mathscr{W}$ (policy π steers any initial state $x \in X_0$ to X_f along a trajectory lying in the tube $\mathbf{X}$, and satisfying, for each i, state and control constraints for every admissible disturbance sequence).*

Proof. Suppose $x_i \in X_i$; then $x_i = \sum_{j=1}^J \lambda_i^j(x_i) x_i^j$ and $\mu_i(x_i) = \lambda_i^j(x_i) u_i^j$. Hence $Ax + B\mu_i(x_i) = \sum_{j=1}^J \lambda_i^j(Ax_i^j + Bu_i^j)$. But since $Ax_i^j + Bu_i^j \in X_{i+1} \ominus W$, $\lambda_i^j \geqslant 0$ and $\sum_{j=1}^J \lambda_i^j = 1$, it follows that $Ax_i + B\mu_i(x_i) \in X_{i+1} \ominus W$ or $x_{i+1} = Ax_i + B\mu_i(x_i) + w \in X_{i+1}$ for all $w \in W$. But $x_0 := x \in X_0$. By induction, $x_i = \phi(i; x, \pi, \mathbf{w}) \in X_i \subset \mathbb{X}$ for all $i \in \mathscr{I}_N$ for every admissible disturbance sequence $\mathbf{w} \in \mathscr{W}$. Since $u_i^j \in \mathbb{U}$ for all $i \in \mathscr{I}^+_{N-1}$, $j \in \mathscr{J}$, $u_i = \mu_i(x_i) \in \mathbb{U}$ for all $i \in \mathscr{I}^+_{N-1}$. The remaining assertions follow. □

The cost $V_N(x, \theta)$ associated with a particular tube (particular (x, θ)) is defined by

$$V_N(x, \theta) := \ell(x, u_0) + \sum_{i=1}^{N-1} \ell(X_i, U_i) + V_f(X_N), \tag{3.12}$$

where with some abuse of notation, $\ell(X, U) := \sum_{j=1}^J \ell(x^j, u^j)$ and $V_f(X) := \sum_{j=1}^J V_f(x^j)$ where x^j is, as above, the jth vertex of the polytope X and u^j the jth element of U (u^j is the control associated with vertex x^j). We assume $\ell(x, u) = (\frac{1}{2})[|x|_Q^2 + |u|_R^2]$ and that $V_f(x) = (\frac{1}{2})|x|_P^2$ where Q, R and P are all positive definite. The tube optimal control problem $P_N(x)$ is defined by

$$P_N(x): \qquad \min_{\theta}\{V_N(x, \theta) \mid \theta \in \Theta(x)\}. \tag{3.13}$$

Since $V_N(x, \cdot)$ is quadratic and $\Theta(x)$ is polyhedral, $P_N(x)$ is a quadratic program. Examples of tubes obtained from the solution of $P_N(x)$ are shown, marked (ii), in Fig. 1 for the same two systems considered above.

3.4. Single policy tube MPC

This is similar to the single policy invariant controller; problem $P_N(x)$ is solved once, yielding the optimal decision variable $\theta^0(x) = (\mathbf{a}^0, \mathbf{z}^0, \mathbf{U}^0)(x)$ from which an optimal policy $\pi^0(x) = \{u_0^0(x), \mu_1^0(\cdot; x), \ldots, \mu_{N-1}^0(\cdot; x)\}$ is obtained using (3.8); $u_0^0(x)$ is obtained from $U_0^0(x) = \{u_0^0(x)\}$ and $\mu_i^0(\cdot; x)$ is obtained from $U_i^0(x) = \mathrm{co}\{\hat{u}_i^1(x), \ldots, \hat{u}_i^J(x)\}$ ($\hat{u}_i^j(x)$ is the jth element of $U_i^0(x)$) as shown in (3.14).

$$\mu_i^0(z; x) := \sum_{j=1}^{J} \lambda_i^j(z) \hat{u}_i^j(x) \quad \forall z \in X_i\ \forall i \in \mathscr{I}^+_{N-1}. \tag{3.14}$$

The policy $\pi^0(x)$ is then used (without updating). As before, we assume that $X_f = Z$ is robust control invariant, and that the local controller $u = Kx$ is such that Z is disturbance invariant for the system $x^+ = A_K x + w$, $A_K := A + BK$. Let *event* (x, i) denote *state x at time i*.

Single policy tube controller:

1. At event $(x(0), 0)$, solve $P_N(x(0))$; set $u(0) = u_0^0(x(0))$.
2. At event $(x(k), k)$, set $u(k) = \mu_k^0(x(k); x(0))$ (3.14).
3. If $x(k) \in X_f$ ($k \geqslant N$), set $u(k) = Kx(k)$.

Proposition 5. *The set $X_f = Z$ is robust finite-time stable for system (2.1) with the single policy tube controller. The* *controller satisfies the state and control constraints for all admissible disturbance sequences*; *the domain of attraction is* $\bar{\mathscr{X}}_N$.

The proof uses Proposition 4 and is similar to the proof of Proposition 3.

3.5. Decreasing horizon tube controller

This controller solves $P_{N_i}(x(i))$ at state x, time i yielding an optimal policy $\pi^0(x, N_i)$ where

$$\pi^0(x,N_i) = \{u_0^0(x,N_i), \mu_1^0(\cdot;x,N_i), \ldots, \mu_{N_i-1}^0(\cdot;x,N_i)\}$$

and control $\kappa(x,N_i) := u_0^0(x,N_i)$ is applied to the plant. At the next state $x^+ = Ax + B\kappa(x,N_i) + w(i)$, problem $P_{N_{i+1}}(x^+)$ is solved with $N_{i+1} = N_i - 1$ i.e. the horizon length is reduced by one at each time step. The initial value is $N_0 = N$. At time $i \geqslant N$, the disturbance invariant controller $u = Kx$ (for set $X_f = Z$) is employed. The controller $u = \kappa(x, N_i)$ is time varying. We require.

Proposition 6. *Suppose for some integer i and state x that* $P_i(x)$ *has a solution yielding a control* $\kappa(x,i)$ *that is applied to the plant. Then* $P_{i-1}(x^+)$ *has a solution for every* $x^+ \in Ax + B\kappa(x,i) + W$ *and*

$$V_{i-1}^0(x^+) \leqslant V_i^0(x) - \ell(x,\kappa(x,i)) \quad \forall x^+ \in Ax + B\kappa(x,i) + W.$$

Proof. By construction, $x^+ \in X_1^0(x,i)$ (for every $w \in W$) so that $x^+ = \sum_{j=1}^J \lambda^j x^j$ and $\mu_1^0(x^+,i) = \sum_{j=1}^J \lambda^j u^j$ for some non-negative multipliers λ^j summing to unity where x^j is the jth vertex of $X_1^0(x,i)$ and u^j is the jth element of $U_1^0(x,i)$. Then the policy π and tube $\mathbf{X}$ defined by

$$\pi := \{\mu_1^0(x^+;x,i), \mu_2^0(\cdot;x,i), \ldots, \mu_{i-1}^0(\cdot;x,i)\},$$

$$\mathbf{X} := \{x^+, X_2^0(x,i), \ldots, \ldots, X_{i-1}^0(x,i)\}$$

is feasible for problem $P_{i-1}(x^+)$. Hence,

$$V_{i-1}^0(x^+) \leqslant \ell(x^+, \mu_1^0(x^+,i)) + \sum_{k=2}^{i-1} \ell(X_k^0(x,i), U_k^0(x,i)) + V_f(X_i^0(x,i)).$$

Because $\ell(\cdot)$ is convex, it follows that $\ell(x^+, \mu_1^0(x^+,i)) \leqslant \sum_{j=1}^J \lambda^j \ell(x^j,u^j) \leqslant \ell(X_1^0(x,i), U_1^0(x,i))$ and $V_{i-1}^0(x^+) \leqslant \sum_{k=1}^{i-1} \ell(X_k^0(x,i), U_k^0(x,i)) + V_f(X_i^0(x,i)) = V_i^0(x) - \ell(x,\kappa(x,i))$ which proves the proposition. □

Proposition 7. *The decreasing horizon controller steers any initial state of* (2.1) *in* $\mathscr{X}_N$ *to* $X_f = Z$ *in time N or less and thereafter maintains the state in* X_f *for any realization* $\{w(i)\} \in \mathscr{W}$ *of the disturbance sequence* (*the set* $X_f = Z$ *is robustly finite-time attractive*).

This result follows from Proposition 4 and the disturbance invariance of $X_f = Z$ for $x^+ = A_K x + w$.

3.6. Variable horizon tube controller

This version permits repeated optimization of the horizon. The decision variable is $\phi := (\theta, N)$. The optimal control problem is

$$P(x): \quad V^0(x) = \min_{\phi} \{V(x,\phi) \mid \phi \in \Phi\},$$

where $V(x,\phi) := V_N(x,\theta)$ and $\Phi := \Theta \times [N_{\min}, N_{\max}]$. We assume, as before, that $X_f = Z$ is disturbance invariant for $x^+ = (A+BK)x + w$. The solution $\phi^0(x) = (\theta^0(x), N^0(x))$ of $P(x)$ (at state $x \in \mathscr{X}_N$) yields the value $V^0(x)$, a tube $\mathbf{X}^0(x)$ and an associated policy $\pi^0(x)$

$$\mathbf{X}^0(x) = \{x, X_1^0(x), \ldots, X_{N^0(x)}(x)\},$$

$$\pi^0(x) = \{u_0^0(x), \mu_1^0(\cdot;x), \ldots, \mu_{N^0(x)}^0(\cdot;x)\}.$$

If $x \in \mathscr{X}_{N_{\max}} \setminus X_f$, control $\kappa(x) := u_0^0(x)$ is applied to the plant; if $x \in X_f$, control $u = Kx$ is applied to the plant. The controller is time-invariant.

Proposition 8. *The variable horizon controller steers any initial state of* (2.1) *in* $\mathscr{X}_{N_{\max}}$ *to* $X_f = Z$ *in a finite time and thereafter maintains the state in* X_f *for any realization* $\{w(i)\} \in \mathscr{W}$ *of the disturbance sequence* (*the set* X_f *is robustly finite-time attractive*).

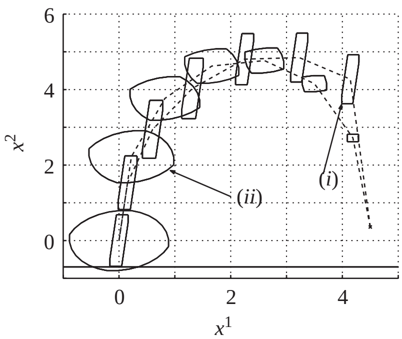
Proof. Suppose $x \in \mathscr{X}_{N_{\max}} \setminus X_f$. By Proposition 6, $V_{N^0(x)-1}^0(x^+) \leqslant V_{N^0(x)}^0(x) - \ell(x,\kappa(x))$ for all $x^+ \in Ax + B\kappa(x) + W$. Hence, $V^0(x^+) = V_{N^0(x^+)}^0(x^+) \leqslant V_{N^0(x)-1}^0(x^+) \leqslant V_{N^0(x)}^0(x) - \ell(x,\kappa(x))$. Since $V_{N^0(x)}^0(x) = V^0(x)$, it follows that

$$V^0(x^+) \leqslant V^0(x) - \ell(x,\kappa(x)) \quad \forall x^+ \in Ax + B\kappa(x) + W.$$

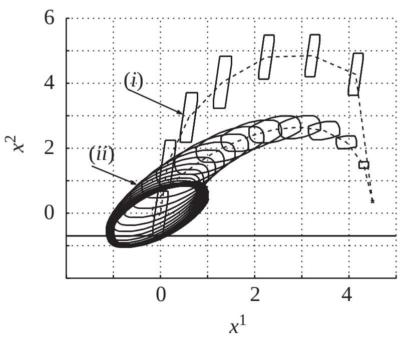
There exists a constant $c > 0$ such that $\ell(x,\kappa(x)) \geqslant c$ for all $x \in \mathscr{X}_{N_{\max}} \setminus X_f$; hence, the controller steers any initial state in $\mathscr{X}_{N_{\max}}$ to X_f in finite time. The disturbance invariant controller $u = Kx$ then keeps the state in X_f. □

4. Comparison

Here, we make a limited comparison of several approaches to robust MPC using Example 1 which is a linearized model of a flight vehicle sampled every 0.2 s. In Fig. 2a, we show predicted tubes of trajectories for (i) disturbance invariant (feedback) MPC and, (ii) open-loop MPC; as predicted the spread of trajectories is much larger for the second; the state constraint $x^2 \geqslant -0.7$ is not satisfied for all predicted trajectories for open-loop MPC indicating that the open-loop optimal control problem is infeasible. Fig. 2b shows the spread of actual trajectories for single policy disturbance invariant MPC (i) and conventional (deterministic) MPC (ii) that ignores uncertainty in the optimal control problem solved on-line; for the latter version,



(a) Predicted trajectories: (i) dist. inv.; (ii) open-loop



(b) Actual trajectories: (i) dist. inv. (a); (ii) deterministic

Fig. 2. Comparison. (a) Predicted trajectories: (i) dist. inv.; (ii) open-loop. (b) Actual trajectories: (i) dist. inv. (a); (ii) deterministic.

the terminal cost $V_f(\cdot)$ is the optimal infinite horizon value function for the unconstrained problem and X_f is the associated maximal output admissible set. The performance of the tube controller is superior; deterministic MPC has a larger spread of trajectories and the state constraint is transgressed for some disturbance sequences. Fig. 1 in Section 3 provides a comparison between the disturbance invariant controller of Section 3.1 (marked (i)) and the more general tube controller of Section 3.2 (marked (ii)). The spread of trajectories is much the same for both controllers (because the disturbance invariant set Z accurately specifies the spread of trajectories) but the paths differ since the tube controller addresses overall rather than nominal cost. The decision variable for the tube controller has dimension $N(1+n)+m+(N-1)mJ$ compared with Nm for the deterministic problem but is linear in N; the number of constraints is linear in N and much less (for large N) than the number of vertices of $\mathscr{W}$, the set of admissible disturbances. The tube controller, unlike the disturbance invariant controller, can also be effectively employed for time-varying systems and for systems with model uncertainty as we show next.

5. Model uncertainty

The tube model predictive controller may be extended to deal with model uncertainty when the system being

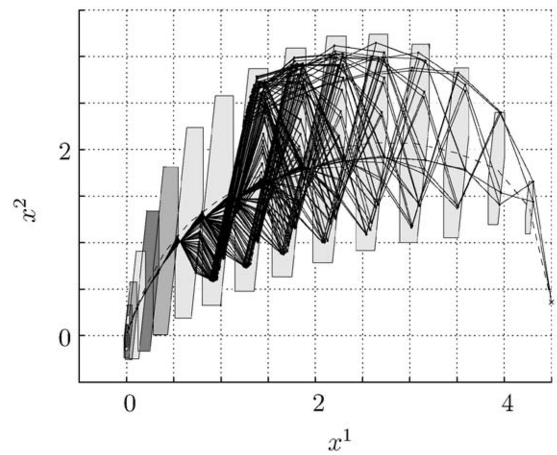


Fig. 3. Optimal tube trajectory.

controlled is described by

$$x^+ = Ax + Bu, \tag{5.1}$$

where (A,B) lies anywhere in the set $\mathrm{co}(\Gamma)$ where $\Gamma := \{(A_\ell, B_\ell) \mid \ell \in \mathscr{L}\}$. It is assumed that each pair (A_ℓ, B_ℓ), $\ell \in \mathscr{L}$, is reachable. The sequence $\mathbf{w}$ now specifies the sequence $\{A_i, B_i \mid i \in \mathscr{I}_{N-1}\}$ of actual parameter values. The tube $\mathbf{X}$ is defined as before except that (2.7) is replaced by

$$Ax + B\mu_i(x) \in X_{i+1} \quad \forall x \in X_i \quad \forall (A,B) \in \Gamma \quad \forall i \in \mathscr{I}_{n-1}.$$

The terminal cost $V_f(\cdot)$ and associated controller $u = Kx$ are determined by solving an appropriate LMI problem. Proposition 1 holds with $\phi(i,x,\pi,\mathbf{w})$ now denoting the solution of (5.1) with parameter sequence $\mathbf{w} = \{(A_i, B_i)\}$. The tube is constructed as described in Section 3.3 with (3.9) replaced by

$$Ax_i^j + Bu_i^j \in X_{i+1} \quad \forall (A,B) \in \Gamma \ \ \forall j \in \mathscr{J}$$

and (3.11) replaced by

$$\begin{aligned}\Theta(x) = \{\theta \mid \mathbf{a} \geqslant 0,\ & X_i \subset \mathbb{X},\ U_i \subset \mathbb{U}_i,\ X_N \subset X_f \subset X,\\ & Ax_i^j + Bu_i^j \in X_{i+1}\ \forall (A,B) \in \Gamma\ \forall (i,j) \in \mathscr{I}_{N-1} \times \mathscr{J}\}.\end{aligned}$$

With these modifications, Proposition 4 holds (with $\phi(i,x,\pi,\mathbf{w})$ interpreted as above) and tube model predictive controllers, similar to those described in Section 3, may be constructed. The invariant terminal set may be arbitrarily small. A modified version of Example 1 (in which $A_{22} = 0.9386$ is replaced by $A_{22} \in [0.6386, 1.2386]$) is illustrated in Fig. 3 which shows the optimal tube and 2^8 sample trajectories. If the shape of the tube were allowed to vary, the sets X_i for larger times i would be smaller; problem $P_N(x)$ remains a quadratic program if the shapes of the sets $\{X_i\}$ are pre-assigned but not otherwise. This method for dealing with parameter uncertainty approach is easily extended to cope with parameter uncertainty *and* bounded additive disturbances.

6. Conclusion

A method for achieving robust model predictive control using tubes has been presented and analyzed. The method achieves a modest improvement over the disturbance invariant controller when the system being controlled is linear and time-invariant but can also be used, unlike the disturbance invariant controller, when the system is time-varying or subject to parameter uncertainty (and bounded disturbances).

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