Physics Letters A 168 (1992) 313–318
North-Holland

PHYSICS LETTERS A

Electrostatic boundary-value problems of nonlinear media: a perturbation approach

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Received 24 January 1992; revised manuscript received 3 July 1992; accepted for publication 7 July 1992
Communicated by A.A. Maradudin

We develop perturbation expansions to solve nonlinear partial differential equations pertaining to the electrostatic boundary-value problems of nonlinear media. As an example in two dimensions, we apply the method to deal with a cylindrical inclusion in a host, both of either linear or nonlinear current–voltage characteristics, and derive the zeroth, first and second order series in the nonlinear conductivity coefficient.

1. Introduction

Many problems in electrostatics involve boundary surfaces on which either the potential or the surface charge density is specified. The studies of linear dielectric and conducting media are well known textbook examples. In this paper, we consider the case of nonlinear conducting media at zero frequency, serving as a typical example for nonlinear suceptibilities of all kinds [1–6]. We assume that in some regions (either in the inclusion or in the host), the current density $\boldsymbol{J}$ is related to the local electric field $\boldsymbol{E}$ by the nonlinear equation

$$\boldsymbol{J}=\sigma\boldsymbol{E}+\chi|\boldsymbol{E}|^2\boldsymbol{E}\,, \tag{1}$$

where σ and χ are first and third conductivity coefficients of the medium. χ is often called the nonlinear susceptibility. In what follows, we use the index m (respectively i) for the host (inclusion) material. That is, we denote σ_m, χ_m as the coefficients in the host and σ_i, χ_i those in the inclusion. Both σ and χ will in general take different values in the inclusion and in the host.

The above equation must be supplemented by the usual electrostatic equations, namely,

$$\nabla\cdot\boldsymbol{J}=0\,, \tag{2}$$

$$\nabla\times\boldsymbol{E}=0\,. \tag{3}$$

From eq. (3), there exists a potential φ such that

$$\boldsymbol{E}=-\nabla\varphi\,. \tag{4}$$

The boundary conditions for the continuity of the potential φ and the current $\boldsymbol{J}$ must be applied on the surfaces of inclusions:

$$\varphi^\mathrm{m}=\varphi^\mathrm{i}\quad\text{on } \partial\Omega_\mathrm{i}\,, \tag{5}$$

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{J}^\mathrm{m}=\hat{\boldsymbol{n}}\cdot\boldsymbol{J}^\mathrm{i}\quad\text{on } \partial\Omega_\mathrm{i}\ (\text{from } \nabla\cdot\boldsymbol{J}=0)\,, \tag{6}$$

where the superscripts m and i denote respectively, the quantities in the host region and in the inclusion region and $\partial\Omega_\mathrm{i}$ denotes the surface of the inclusion.

2. Perturbation expansion method

The perturbation expansion method was originally developed for solving nonlinear harmonic oscillator problems [7]. Here we extend the method to solve more complicated electrostatic problems, which is valid if the nonlinearities are small. We choose the nonlinear susceptibility of the host χ_m as the expansion parameter. The expansions (in χ_m) for the electrostatic potential read

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$$\varphi^{\rm i}=\varphi_0^{\rm i}+\chi_{\rm m}\varphi_1^{\rm i}+\chi_{\rm m}^2\varphi_2^{\rm i}+\ldots \quad \text{in } \Omega_{\rm i}\,, \tag{7}$$

$$\varphi^{\rm m}=\varphi_0^{\rm m}+\chi_{\rm m}\varphi_1^{\rm m}+\chi_{\rm m}^2\varphi_2^{\rm m}+\ldots \quad \text{in } \Omega_{\rm m}\,. \tag{8}$$

Again, the superscripts i and m denote respectively the quantities in the inclusion ($\Omega_{\rm i}$) and in the host regions ($\Omega_{\rm m}$). As eq. (1) contains the factor $|\boldsymbol{E}|^2$, it is convenient to define the quantity

$$G^\alpha=|\boldsymbol{E}^\alpha|^2=\nabla\varphi^\alpha\cdot\nabla\varphi^\alpha\,, \quad \alpha={\rm m,i}\,. \tag{9}$$

Taking the gradient of eqs. (7) and (8), we may also write the electric field as an expansion in $\chi_{\rm m}$,

$$\boldsymbol{E}^\alpha=\boldsymbol{E}_0^\alpha+\chi_{\rm m}\boldsymbol{E}_1^\alpha+\chi_{\rm m}^2\boldsymbol{E}_2^\alpha+\ldots \quad \text{in } \Omega_\alpha\,. \tag{10}$$

We write G^α as an expansion in $\chi_{\rm m}$,

$$\begin{aligned} G^\alpha&=G_0^\alpha+\chi_{\rm m}G_1^\alpha+\chi_{\rm m}^2G_2^\alpha+\ldots\\ &=(\nabla\varphi_0^\alpha)^2+2\chi_{\rm m}\nabla\varphi_0^\alpha\cdot\nabla\varphi_1^\alpha\\ &\quad+\chi_{\rm m}^2[(\nabla\varphi_1^\alpha)^2+2\nabla\varphi_0^\alpha\cdot\nabla\varphi_2^\alpha]+\ldots\,. \end{aligned} \tag{11}$$

Thus the coefficients G_j^α are given by

$$G_0^\alpha=(\nabla\varphi_0^\alpha)^2\,, \tag{12a}$$

$$G_1^\alpha=2\nabla\varphi_0^\alpha\cdot\nabla\varphi_1^\alpha\,, \tag{12b}$$

$$G_2^\alpha=(\nabla\varphi_1^\alpha)^2+2\nabla\varphi_0^\alpha\cdot\nabla\varphi_2^\alpha\,. \tag{12c}$$

The expansion for the current density in each region is given by

$$\begin{aligned} \boldsymbol{J}^\alpha&=-\sigma_\alpha\nabla\varphi_0^\alpha-\chi_{\rm m}(\sigma_\alpha\nabla\varphi_1^\alpha+\beta_\alpha G_0^\alpha\nabla\varphi_0^\alpha)\\ &\quad-\chi_{\rm m}^2[\sigma_\alpha\nabla\varphi_2^\alpha+\beta_\alpha(G_1^\alpha\nabla\varphi_0^\alpha+G_0^\alpha\nabla\varphi_1^\alpha)]+\ldots\\ &=\boldsymbol{J}_0^\alpha+\chi_{\rm m}\boldsymbol{J}_1^\alpha+\chi_{\rm m}^2\boldsymbol{J}_2^\alpha+\ldots\,, \end{aligned} \tag{13}$$

where

$$\beta_\alpha=\frac{\chi_\alpha}{\chi_{\rm m}} \quad (\alpha={\rm m,i}) \tag{14}$$

is the ratio of χ_α to the expansion parameter $\chi_{\rm m}$. Thus $\beta_{\rm m}=1$ and we denote $\beta=\beta_{\rm i}=\chi_{\rm i}/\chi_{\rm m}$. The coefficients $\boldsymbol{J}_j^\alpha$ are given by

$$\boldsymbol{J}_0^\alpha=-\sigma_\alpha\nabla\varphi_0^\alpha\,, \tag{15a}$$

$$\boldsymbol{J}_1^\alpha=-\sigma_\alpha\nabla\varphi_1^\alpha-\beta_\alpha G_0^\alpha\nabla\varphi_0^\alpha\,, \tag{15b}$$

$$\boldsymbol{J}_2^\alpha=-\sigma_\alpha\nabla\varphi_2^\alpha-\beta_\alpha(G_1^\alpha\nabla\varphi_0^\alpha+G_0^\alpha\nabla\varphi_1^\alpha)\,. \tag{15c}$$

For electrostatic problems, the electric field $\boldsymbol{E}$ and the current $\boldsymbol{J}$ must satisfy eqs. (2) and (3). Substituting eq. (15) into eq. (2), we obtain the perturbation expansions in each region,

$$\sigma_\alpha\nabla^2\varphi_0^\alpha=0\,, \tag{16a}$$

$$\sigma_\alpha\nabla^2\varphi_1^\alpha+\beta_\alpha(\nabla\varphi_0^\alpha\cdot\nabla G_0^\alpha+G_0^\alpha\nabla^2\varphi_0^\alpha)=0\,, \tag{16b}$$

$$\begin{aligned} &\sigma_\alpha\nabla^2\varphi_2^\alpha+\beta_\alpha(\nabla\varphi_1^\alpha\cdot\nabla G_0^\alpha+\nabla\varphi_0^\alpha\cdot\nabla G_1^\alpha\\ &\quad+G_1^\alpha\nabla^2\varphi_0^\alpha+G_0^\alpha\nabla^2\varphi_1^\alpha)=0\,, \quad \ldots,\\ &\quad\alpha={\rm m,i}\,. \end{aligned} \tag{16c}$$

At the boundary, the continuity of φ and $\boldsymbol{J}$ must be satisfied. Substituting eqs. (7), (8) into eq. (5), we obtain

$$\varphi_j^{\rm m}=\varphi_j^{\rm i} \quad \text{on } \partial\Omega_{\rm i}\,, \quad j=0,1,2,\ldots\,. \tag{17}$$

Similarly, using eq. (6), we have

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{J}_j^{\rm m}=\hat{\boldsymbol{n}}\cdot\boldsymbol{J}_j^{\rm i} \quad \text{on } \partial\Omega_{\rm i}\,, \quad j=0,1,2,\ldots\,. \tag{18}$$

Equations (15)–(18) summarize the necessary ingredients for solving nonlinear electrostatic boundary-value problems using the perturbation expansion method. In addition, we require the electrostatic potential to yield a uniform field at infinity. We shall discuss below a simple two-dimensional example to illustrate the use of the present method.

3. Boundary-value problems of a cylindrical inclusion

We consider the simple case of a cylindrical inclusion of radius ρ embedded in a host, subject to a uniform external electric field E_0. We identify three cases:

(a) a linear inclusion in a linear host,
(b) a nonlinear inclusion in a linear host,
(c) a linear inclusion in a nonlinear host.

For convenience of the present study, we write the corresponding boundary conditions in cylindrical coordinates. On the cylindrical surface of radius ρ, we have

$$\sigma_{\rm m}\nabla_r\varphi_0^{\rm m}|_\rho=\sigma_{\rm i}\nabla_r\varphi_0^{\rm i}|_\rho\,, \tag{19a}$$

$$\begin{aligned} &\sigma_{\rm m}\nabla_r\varphi_1^{\rm m}+G_0^{\rm m}\nabla_r\varphi_0^{\rm m}|_\rho\\ &\quad=\sigma_{\rm i}\nabla_r\varphi_1^{\rm i}+\beta G_0^{\rm i}\nabla_r\varphi_0^{\rm i}|_\rho\,, \end{aligned} \tag{19b}$$

$$\begin{aligned} &\sigma_{\rm m}\nabla_r\varphi_2^{\rm m}+G_1^{\rm m}\nabla_r\varphi_0^{\rm m}+G_0^{\rm m}\nabla_r\varphi_1^{\rm m}|_\rho\\ &\quad=\sigma_{\rm i}\nabla_r\varphi_2^{\rm i}+\beta(G_1^{\rm i}\nabla_r\varphi_0^{\rm i}+G_0^{\rm i}\nabla_r\varphi_1^{\rm i})|_\rho\,. \end{aligned} \tag{19c}$$

We require the solutions to be nonsingular at the origin and the electric field at infinity to coincide with the external applied field. Let us discuss the solutions of the three cases.

Case (a): A linear inclusion in a linear host. Here $\chi_m = \chi_i = 0$ and we have linear conducting media. We want to solve

$$\nabla^2 \varphi^m = 0 \quad \text{in } \Omega_m , \quad \nabla^2 \varphi^i = 0 \quad \text{in } \Omega_i ,$$

subject to the boundary conditions at $r=\rho$,

$$\varphi^m |_\rho = \varphi^i |_\rho , \quad \sigma_m \nabla_r \varphi^m |_\rho = \sigma_i \nabla_r \varphi^i |_\rho ,$$

where ρ is the radius of the cylindrical inclusion. We find the solutions [8]

$$\varphi^m = -E_0 (r - br^{-1}) \cos\theta , \tag{20}$$

$$\varphi^i = -cE_0 r \cos\theta , \tag{21}$$

where

$$b = (\sigma_i - \sigma_m)\rho^2/\sigma , \quad c = 2\sigma_m/\sigma , \quad \sigma = \sigma_i + \sigma_m .$$

The solution is well known [8]. We shall see in the following, that the linear part of the solution coincides with eqs. (20) and (21).

Case (b): A nonlinear inclusion in a linear host. In this case $\chi_m = 0$ and the electrostatic potential can also be solved exactly. The potential for the host is

$$\varphi^m = -(E_0 r - Br^{-1}) \cos\theta , \tag{22}$$

which automatically satisfies the boundary condition at infinity. It is easy to show that

$$\varphi^i = -Cr \cos\theta , \tag{23}$$

satisfies the nonlinear field equation. The constants B and C can be determined from the boundary conditions, we find

$$B\rho^{-1} + C\rho = E_0 \rho ,$$

$$\sigma_m B\rho^{-2} - \sigma_i C - \chi_i C^3 = -\sigma_m E_0 .$$

Eliminating B, we obtain the equation for C:

$$\chi_i C^3 + \sigma C = 2\sigma_m E_0 ,$$

again $\sigma = \sigma_i + \sigma_m$. We therefore conclude that the results for the electrostatic potential are essentially the same as those of a linear composite. They differ only by the integration constants. The solution for C can be obtained by iteration starting with $C = 2\sigma_m E_0/\sigma = cE_0$ which holds when $\chi_i = 0$. We find

$$C = cE_0 - \frac{\chi_i}{\sigma} c^3 E_0^3 + 3\left(\frac{\chi_i}{\sigma}\right)^2 c^5 E_0^5$$
$$- 12\left(\frac{\chi_i}{\sigma}\right)^3 c^7 E_0^7 + \dots . \tag{24}$$

We also find that $B = (E_0 - C)\rho^2$ which reduces to $B = bE_0$ when $\chi_i = 0$.

Case (c): A linear conclusion in a nonlinear host. Here $\chi_i = 0$ and we have a nonlinear host medium. We use the formulae of the previous section. To zeroth order,

$$\nabla^2 \varphi_0^m = 0 \quad \text{in } \Omega_m , \quad \nabla^2 \varphi_0^i = 0 \quad \text{in } \Omega_i ,$$

and the boundary conditions at $r=\rho$,

$$\varphi_0^m |_\rho = \varphi_0^i |_\rho , \quad \sigma_m \nabla_r \varphi_0^m |_\rho = \sigma_i \nabla_r \varphi_0^i |_\rho ,$$

being exactly the same as the linear case. The solutions are well known:

$$\varphi_0^m = -E_0 (r - br^{-1}) \cos\theta , \tag{20$'$}$$

$$\varphi_0^i = -cE_0 r \cos\theta . \tag{21$'$}$$

To first order, we have

$$\sigma_m \nabla^2 \varphi_1^m = -(\nabla\varphi_0^m \cdot \nabla G_0^m + G_0^m \nabla^2 \varphi_0^m)$$
$$\text{in } \Omega_m , \tag{25}$$

$$\nabla^2 \varphi_1^i = 0 \quad \text{in } \Omega_i . \tag{26}$$

Equation (25) in the host is a nonhomogeneous linear differential equation; the homogeneous solution is known. For the particular solution, we need to compute the right hand side of eq. (25). We omit the lengthy derivations and give the results:

$$(\nabla\varphi_0^m \cdot \nabla G_0^m + G_0^m \nabla^2 \varphi_0^m)$$
$$= [(8b^2 r^{-5} + 4b^3 r^{-7}) \cos\theta + 4br^{-3} \cos 3\theta] E_0^3 .$$

The particular solution of φ_1^m is

$$\bar{\varphi}_1^m = -\frac{1}{\sigma_m} [(b^2 r^{-3} + \tfrac{1}{6} b^3 r^{-5}) \cos\theta$$
$$- \tfrac{1}{2} br^{-1} \cos 3\theta] E_0^3 .$$

From

$$\frac{\partial}{\partial x}\varphi^{\mathrm{m}}(\infty)=E_0\,, \quad \frac{\partial}{\partial x}\varphi_0^{\mathrm{m}}(\infty)=E_0\,,$$

we obtain

$$\frac{\partial}{\partial x}\varphi_j^{\mathrm{m}}(\infty)=0\,, \quad j=1,2,3,\dots\,. \tag{27}$$

That is, the electrostatic potential is required to yield a uniform field at infinity. From boundary conditions (27) and nonsingular solutions at the origin, we find the general solution of φ_1^{m} and φ_1^{i}:

$$\varphi_1^{\mathrm{m}}=-\frac{1}{\sigma_{\mathrm{m}}}\,[(b_1r^{-1}+b^2r^{-3}+b^3r^{-5})\cos\theta + (b_2r^{-3}-br^{-1})\cos 3\theta]E_0^3\,, \tag{28}$$

$$\varphi_1^{\mathrm{i}}=-\frac{1}{\sigma_{\mathrm{m}}}\,(b_3r\cos\theta+b_4r^3\cos 3\theta)E_0^3\,. \tag{29}$$

The boundary conditions (17) and (19b) determine the integration constants. Note that $\beta=0$ for a linear inclusion:

$$b_1=\rho^2[\sigma_{\mathrm{m}}(1+2b_0-b_0^2+\tfrac{1}{6}b_0^3)-\sigma_{\mathrm{i}}(1+\tfrac{1}{6}b_0)b_0^2]/\sigma\,,$$

$$b_3=\sigma_{\mathrm{m}}(1+2b_0+\tfrac{1}{3}b_0^3)/\sigma\,,$$

$$b_2=\rho^4[\tfrac{1}{2}\sigma_{\mathrm{i}}b_0+\sigma_{\mathrm{m}}(\tfrac{1}{2}b_0+\tfrac{1}{3}b_0^2)]/\sigma\,,$$

$$b_4=\tfrac{1}{3}\sigma_{\mathrm{m}}\rho^{-2}b_0^2/\sigma\,,$$

where

$$b_0=b\rho^{-2}=(\sigma_{\mathrm{i}}-\sigma_{\mathrm{m}})/\sigma\,, \quad \sigma=\sigma_{\mathrm{i}}+\sigma_{\mathrm{m}}\,.$$

The second order equation

$$\sigma_{\mathrm{m}}\nabla^2\varphi_2^{\mathrm{m}}=-(\nabla\varphi_1^{\mathrm{m}}\cdot\nabla G_0^{\mathrm{m}}+\nabla\varphi_0^{\mathrm{m}}\cdot\nabla G_1^{\mathrm{m}} + G_1^{\mathrm{m}}\nabla^2\varphi_0^{\mathrm{m}}+G_0^{\mathrm{m}}\nabla^2\varphi_1^{\mathrm{m}})\,, \tag{30}$$

$$\nabla^2\varphi_2^{\mathrm{i}}=0\,. \tag{31}$$

The computation of φ_2^{m} is far more tedious, but similar to that of φ_1^{m}. We have

$$\varphi_2^{\mathrm{m}}=-\frac{1}{\sigma_{\mathrm{m}}}\,[(c_1r^{-1}+b_5r^{-3}+b_6r^{-5} + b_7r^{-7}+b_8r^{-9})\cos\theta + (c_2r^{-3}+b_9r^{-1}+b_{10}r^{-5}+b_{11}r^{-7} + b_{12}r^{-9})\cos 3\theta + (c_3r^{-5}+b_{13}r^{-1}+b_{14}r^{-3})\cos 5\theta]E_0^5\,, \tag{32}$$

and

$$\varphi_2^{\mathrm{i}}=-\frac{1}{\sigma_{\mathrm{m}}}\,(c_4r\cos\theta+c_5r^3\cos 3\theta + c_6r^5\cos 5\theta)E_0^5\,. \tag{33}$$

Where

$$b_5=-(\tfrac{5}{2}b^2+2b_1b)\,, \quad b_6=\tfrac{7}{3}b^3-b_2b-\tfrac{1}{2}b_1b^2\,,$$

$$b_7=-(\tfrac{3}{4}b^4+\tfrac{1}{2}b_2b^2)\,, \quad b_8=\tfrac{2}{15}b^5\,, \quad b_9=-\tfrac{1}{2}b_1\,,$$

$$b_{10}=-(\tfrac{3}{2}b^3+3b_2b)\,, \quad b_{11}=-(\tfrac{1}{30}b^4+\tfrac{3}{5}b_2b^2)\,,$$

$$b_{12}=\tfrac{1}{36}b^5\,, \quad b_{13}=-\tfrac{1}{4}b\,, \quad b_{14}=\tfrac{3}{4}b^2+\tfrac{3}{2}b_2\,.$$

From the boundary conditions, we determine the integration constants

$$c_1=\rho(C_1\sigma_{\mathrm{i}}-D_1\rho)/\sigma\,,$$

$$c_4=-\rho^{-1}(D_1\rho+C_1\sigma_{\mathrm{m}})/\sigma\,,$$

$$c_2=\rho^3(C_3\sigma_{\mathrm{i}}-\tfrac{1}{3}D_3\rho)/\sigma\,,$$

$$c_5=-\rho^{-3}(\tfrac{1}{3}D_3\rho+C_3\sigma_{\mathrm{m}})/\sigma\,,$$

$$c_3=\rho^5(C_5\sigma_{\mathrm{i}}-\tfrac{1}{5}D_5\rho)/\sigma\,,$$

$$c_6=-\rho^{-5}(\tfrac{1}{5}D_5\rho+G_5\sigma_{\mathrm{m}})/\sigma\,,$$

where

$$C_1=-(b_5\rho^{-3}+b_6\rho^{-5}+b_7\rho^{-7}+b_8\rho^{-9})\,,$$

$$D_1=\sigma_{\mathrm{m}}(3b_5\rho^{-4}+5b_6\rho^{-6}+7b_7\rho^{-8}+9b_8\rho^{-10}) + (2b_1+\tfrac{1}{2}b)\rho^{-2}+(4b_1b+6b^2)\rho^{-4} + (6b_2b+3b_1b^2-9b^3)\rho^{-6}+(3b_2b^2+\tfrac{14}{3}b^4)\rho^{-8} - \tfrac{13}{6}b^5\rho^{-10}\,.$$

$$C_3=-(b_9\rho^{-1}+b_{10}\rho^{-5}+b_{11}\rho^{-7}+b_{12}\rho^{-9})\,,$$

$$D_3=\sigma_{\mathrm{m}}(b_9\rho^{-2}+5b_{10}\rho^{-6}+7b_{11}\rho^{-8}+9b_{12}\rho^{-10}) + (b_1-b)\rho^{-2}+(6b_2+2b_1b+b^2)\rho^{-4} + (6b_2b+\tfrac{9}{2}b^3)\rho^{-6}+(6b_2b^2-\tfrac{2}{3}b^4)\rho^{-8} - \tfrac{1}{3}b^5\rho^{-10}\,,$$

$$C_5=-(b_{13}\rho^{-1}+b_{14}\rho^{-3})\,,$$

$$D_5=\sigma_{\mathrm{m}}(b_{13}\rho^{-2}+3b_{14}\rho^{-4})-b\rho^{-2} + (3b_2-b^2)\rho^{-4}+(6b_2b+\tfrac{1}{2}b^3)\rho^{-6}\,.$$

We learn through the calculations that higher-order terms of the potential can be solved by the same procedure and analytic formulae for them are all available. There will be no problems if we use computer algebra (algebraic manipulation programs) to derive higher-order terms of the potential. However, the analytic formulae of the potential corresponding to the zeroth, first and second orders of χ_m are sufficient from the viewpoint of practical applications and for the main purposes of our present work. The calculations reveal also some regularities in the formulae: The jth term of χ_m, φ_j^m and φ_j^i are proportional to E_0^{1+2j}, which is consistent with the expansion for C and we shall use this property in the following discussions. Only φ_j^m, φ_j^i contain terms with $\cos[(1+2j)\theta]$.

4. Discussions and conclusions

In the present work, the local electric field is obtained in terms of the external uniform field by the perturbation expansion method which should be valid in the limit of small nonlinearities. The present method offers us the opportunity to discuss the macroscopic behaviors of nonlinear composite media. For example, we can extend slightly the method of Landau and Lifshitz [9] to deal with nonlinear composite media, which is valid for low inclusion concentration:

$$\frac{1}{V}\int_{\Omega_i} [(\sigma_i-\sigma_m)\boldsymbol{E}+(\chi_i-\chi_m)|\boldsymbol{E}|^2\boldsymbol{E} + (\eta_i-\eta_m)|\boldsymbol{E}|^4\boldsymbol{E}+\dots]\,\mathrm{d}V \equiv (\sigma_e-\sigma_m)\bar{\boldsymbol{E}}+(\chi_e-\chi_m)|\bar{\boldsymbol{E}}|^2\bar{\boldsymbol{E}} + (\eta_e-\eta_m)|\bar{\boldsymbol{E}}|^4\bar{\boldsymbol{E}}+\dots, \qquad (34)$$

where $\bar{\boldsymbol{E}}$ denotes the average electric field and η_m is the fifth order coefficient of conductivity. The present method reveals also the important fact that if two composite media have *the same* microstructure and have *the same* linear part of conductivity both for the inclusion and the host, then they should have *the same* linear part of the effective conductivity. Let us consider a composite of cylindrical inclusions of density p_i. Substituting the electric field along the x-direction in the inclusion into eq. (34), we obtain

$$\sigma_e=\sigma_m+2\sigma_m p_i\frac{\sigma_i-\sigma_m}{\sigma_i+\sigma_m}, \qquad (35)$$

where p_i is the concentration of the inclusions. For case (b), the third order coefficient of the effective conductivity is

$$\chi_e=\chi_i p_i c^4, \qquad (36)$$

which coincides with the result of Zeng et al. [6]. Stroud and Hui [5] made basically the same assumptions for a spherical inclusion in the nonlinear problem, they found a result (eq. (3.2) in ref. [5]) identical to eq. (36) with the ratio of the local field to the external field $c=3\sigma_m/(\sigma_i+2\sigma_m)$ for a spherical inclusion. Their result is valid to the first order only, while we have gone further to the second order. To this end, we have also performed similar calculations for a spherical inclusion in the nonlinear problems and results identical to those of ref. [5] are obtained. The fifth order coefficient is found to be

$$\eta_e=-3\chi_i\frac{\chi_i}{\sigma}p_i c^6. \qquad (37)$$

For case (c), the third and the fifth order coefficients are respectively

$$\chi_e=\chi_m+\chi_m p_i[(\sigma_i-\sigma_m)b_3-c^3], \qquad (38)$$

$$\eta_e=\chi_m^2 p_i[(\sigma_i-\sigma_m)c_4-3c^2b_3]. \qquad (39)$$

The perturbation expansion method is often applied to evolution behaviors of nonlinear ordinary differential systems (for example, the anharmonic oscillator). In this work, we extend this method to solve the nonlinear partial differential equations pertaining to the electrostatic boundary-value problem. We give a formalism of the perturbation expansion method which is suitable to deal with field equations and boundary conditions of nonlinear composite media in the limit of small nonlinearities. As an example, we apply the perturbation expansion method to compute the potential distribution in nonlinear composite media of a two-dimensional system and obtain analytic formulae of the potential to the zeroth, first and the second order in χ_m. The method can also be used to derive analytic formulae for higher-order terms of the potential.

Acknowledgement

This work was supported by a University Direct Grant under project number 220-600-040. KWY acknowledges useful discussions with Dr. P.M. Hui.

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