NORTH-HOLLAND

Triangular Blocks of Zeros in (0, 1) Matrices With Small Permanents

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ABSTRACT

Let A be a square matrix and t a positive integer. We say A is t-triangular if there exist permutation matrices P and Q such that $PAQ = B = [b_{ij}]$ has $b_{ij} = 0$ whenever $j \geqslant i + t$. We ask for which positive integers the following statement is true: If A is any square matrix with nonnegative integral entries such that $0 < \operatorname{per} A < (t+1)!$, then A is t-triangular. If $t = 1$, the statement reduces to a theorem of Brualdi. We prove the statement is true for $t = 2$ and $t = 3$, but false for $t = 6$. © *Elsevier Science Inc., 1997*

1. INTRODUCTION

If $A = [a_{ij}]$ is an $n \times n$ matrix, the permanent of A, denoted per A, is defined by

$$\operatorname{per} A = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where the sum is over all permutations σ of $\{1, 2, \dots, n\}$. We denote the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A by $A(i|j)$, with similar notation when two rows and two columns are deleted. We refer to per $A(i|j)$ as a permanental minor. The permanent of A

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can be expanded on any row, so

$$\text{per } A = \sum_{j=1}^{n} a_{ij} \text{ per } A(i|j), \qquad i = 1, 2, \ldots, n,$$

and similarly on any column. The $n \times n$ matrix A is fully indecomposable if there is no $r \times s$ submatrix of zeros where $r + s = n$, and otherwise is partly decomposable. By the Frobenius-König-Hall theorem, a matrix with nonnegative entries is fully indecomposable if and only if all permanental minors are positive. We denote the $n \times n$ matrix of ones by J_n.

If A is an $n \times n$ (0, 1) matrix with a small permanent, we may be able to say something interesting about the pattern of zeros in A. The Frobenius-König-Hall theorem is a result of this sort. We state one version of it (see, for example, [1] or [3] for a proof and generalizations).

FROBENIUS-KÖNIG-HALL THEOREM. *If A is an $n \times n$ (0, 1) matrix such that per $A = 0$, then A has an $r \times s$ submatrix of zeros for some r and s such that $r + s = n + 1$.*

Less well known is the following theorem of Brualdi [2].

THEOREM 1 (Brualdi). *If A is any square (0, 1) matrix such that* per $A = 1$, *then there exist permutation matrices P and Q such that PAQ is lower triangular with all ones on the main diagonal.*

The results of this paper grew out of an attempt to generalize Theorem 1. If A is a square matrix and t is a positive integer, we say A is t-triangular if there exist permutation matrices P and Q such that $PAQ = B = [b_{ij}]$ has $b_{ij} = 0$ whenever $j \geqslant i + t$. If A is $n \times n$ and $t < n$, then B has a "triangular" block of zeros with $n - t$ zeros along the "legs." Clearly A is 1-triangular if and only if the rows and columns of A can be permutated to get a lower triangular matrix, and every $n \times n$ matrix is trivially t-triangular if $t \geqslant n$.

We ask for which positive integers t the following statement is true:

STATEMENT. If A is any square matrix with nonnegative integral entries such that $0 <$ per $A < (t + 1)!$, then A is t-triangular.

We make several remarks about the Statement.

(1) The statement is clearly true for a given value of t if and only if it is true for all (0, 1) matrices for that value of t. If the Statement is false for some value of t, then there exists a (0, 1) matrix which is a counterexample of smallest order.

(2) The converse is false for $t > 1$ even for (0, 1) matrices. For example, the $n \times n$ matrix $B = [b_{ij}]$ which has $b_{ij} = 0$ if and only if $j \geqslant i + t$ is t-triangular, but per $B = t^{n-t}(t!)$.

(3) The Statement, if true for some value of t, is the strongest possible statement. This is because the matrix $J_{t+1} \oplus I_{n-t-1}$ has permanent equal to $(t+1)!$ but is not t-triangular.

(4) The Statement is true for $t = 1$; that is essentially Theorem 1.

(5) If the Statement is false for some value of t and if A is a counterexample of smallest order, then A is fully indecomposable. Otherwise there would exist permutation matrices P and Q such that

$$PAQ = \left[\begin{array}{c|c} B & 0 \\ \hline D & C \end{array}\right]$$

where B and C are square matrices. Since per A = (per B)(per C), both per B and per C are positive and less than or equal to per A. By the minimality of the order of A, both B and C are t-triangular and hence so is A, a contradiction.

(6) If there exist i and j such that $A(i|j)$ is $(k-1)$-triangular, then A is clearly k-triangular. Thus if the statement is true for $t = k - 1$ and false for $t = k$ with A a counterexample of smallest order, then per $A(i|j) \geqslant k!$ for each i and j [per $A(i|j)$ cannot be 0, because, as shown in remark (5), A is fully indecomposable].

It follows that each line sum of A is at most k [otherwise expanding on that line would give a permanent of at least $(k+1)!$].

In this paper we will show the statement is true for $t = 2$ and $t = 3$, but false for $t = 6$.

2. RESULTS

First we show the statement is true for $t = 2$.

THEOREM 2. *If A is a matrix with nonnegative integral entries such that $0 <$ per $A < 6$, then A is 2-triangular.*

Proof. We use induction on n, the order of A. The result is trivial for $n = 1$, 2, and 3. Assume A is a smallest order counterexample. By remark (5), A is fully indecomposable, so each line of A has at least two nonzero entries. And since the statement is true for $t = 1$, by remark (6) each line sum of A is at most 2. Hence each line of A has precisely two ones and the rest zeros. Such a matrix is clearly 2-triangular. ∎

If $A = [a_{ij}]$ is an $n \times n$ matrix with nonnegative integral entries, let $G(A)$ denote the associated bipartite multigraph with vertex bipartition $V \cup W$ where $V = \{v_1, v_2, \dots, v_n\}$, $W = \{w_1, w_2, \dots, w_n\}$ and with a_{ij} the multiplicity of the edge $[v_i, w_j]$. Suppose v_1 and w_1 are adjacent vertices of $G(A)$, each with degree 2, and that $[v_1, w_2]$ and $[v_2, w_1]$ are edges in $G(A)$. So $a_{11} = a_{12} = a_{21} = 1$ and $a_{1j} = a_{i1} = 0$ for all i, j in $\{3, 4, \dots, n\}$. Define an $(n-1) \times (n-1)$ matrix $B = [b_{ij}]$, called a contraction of A, by

$$b_{11} = a_{22} + 1,$$

$$b_{ij} = a_{i+1, j+1} \qquad \text{if} \quad i + j > 2.$$

So B is the matrix obtained from A by adding 1 to the (2, 2) entry and then deleting the first row and column. The multigraph $G(B)$, which we call a contraction of $G(A)$, can be obtained from $G(A)$ by replacing the path v_2, w_1, v_1, w_2 with the edge $[v_2, w_2]$ (so they are topologically homeomorphic). By expansion on the first row,

$$\operatorname{per} A = \operatorname{per} A(1|1) + \operatorname{per} A(12|12) = \operatorname{per} B.$$

Furthermore, $\operatorname{per} B(i|j) = \operatorname{per} A(i+1|j+1)$ for all $i, j \in \{1, \dots, n-1\}$. So contraction preserves the permanent and permanental minors of A.

If S is a subset of V, the neighborhood of S in $G(A)$ is $N(S) = \{w \in W \mid w \text{ is adjacent to some vertex in } S\}$.

LEMMA 1. *Let A be an $n \times n$ matrix with nonnegative integral entries and let $G(A)$ be the associated bipartite multigraph with vertex partition $V \cup W$. Then A is t-triangular if and only if there exists an ordering $v_{k_1}, v_{k_2}, \dots, v_{k_n}$ of the vertices in V such that each of the sets $S_i = \{v_{k_1}, v_{k_2}, \dots, v_{k_i}\}$ $(i = 1, 2, \dots, n)$ satisfies the inequality*

$$|N(S_i)| \leq i + t - 1. \tag{2.1}$$

Proof. Suppose $A = [a_{ij}]$ is t-triangular. Let P and Q be permutation matrices such that $B = PAQ = [b_{ij}]$ where $b_{ij} = 0$ whenever $j \geq i + t$. Let $(v_{k_1}, \ldots, v_{k_n})$ and $(w_{r_1}, \ldots, w_{r_n})$ be the "natural" orderings of V and W associated with $G(B)$ [$(v_{k_1}, \ldots, v_{k_n})$ and $(w_{r_1}, \ldots, w_{r_n})$ result from applying the permutations associated with P and Q to $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ respectively], so that b_{ij} is the multiplicity of $[v_{k_i}, w_{r_j}]$. If $j \geq i + t$ then $b_{ij} = 0$ and hence $w_{r_j} \notin N(S_i)$, from which the inequality (2.1) follows.

Conversely, if an ordering $v_{k_1}, v_{k_2}, \ldots, v_{k_n}$ of V satisfying (2.1) exists, choose any ordering $w_{r_1}, w_{r_2}, \ldots, w_{r_n}$ of W such that for all positive integers i, j, and m with $i < j$, if $w_{r_j} \in N(S_m)$ then $w_{r_i} \in N(S_m)$. Such an ordering of W exists because $N(S_1) \subseteq N(S_2) \subseteq \cdots \subseteq N(S_n)$.

If P and Q are the permutation matrices associated with the permutations $(v_{k_1}, \ldots, v_{k_n})$ and $(w_{r_1}, \ldots, w_{r_n})$ of $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ respectively, then $PAQ = B = [b_{ij}]$ has $b_{ij} = 0$ if $j \geq i + t$, so A is t-triangular. ∎

LEMMA 2. *If $A = [a_{ij}]$ is a $(0, 1)$ matrix which is a smallest order counterexample to the Statement for $t = 3$, then each vertex of $G(A)$, the associated bipartite graph, has degree 2 or 3. Furthermore, each component of the subgraph $H(A)$ of $G(A)$ induced by all vertices of degree two has an even number of vertices.*

Proof. By remark (5) each vertex of $G(A)$ has degree at least 2. And since the statement is true for $t = 2$, by remark (6) each vertex has degree at most 3.

Suppose $H(A)$ has a component with precisely one vertex, say v_1. Since $G(A)$ is fully indecomposable, v_1 must be adjacent to two distinct vertices, say w_1 and w_2, both of which have degree 3. Equating permanental expansions of A on the first row and first column gives

$$\operatorname{per} A(1|2) = \sum_{i=2}^{n} a_{i1} \operatorname{per} A(i|1) \geq 12,$$

since w_1 has degree 3 and $\operatorname{per} A(i|j) \geq 6$ for each i and j by remark (6). Comparing expansions on the first row and second column of A shows that $\operatorname{per} A(1|1) \geq 12$. This contradicts the assumption that $\operatorname{per} A < 24$. Hence no component of $H(A)$ contains a single vertex.

Now suppose some component $\mathscr{C}$ of $H(A)$ has precisely $2m + 1$ vertices where m is a positive integer. Suppose v_1 and w_1 are adjacent vertices in $\mathscr{C}$ and that $[v_1, w_2]$ and $[v_2, w_1]$ are edges in $G(A)$. Then $a_{11} = a_{12} = a_{21} = 1$.

Since either w_2 or v_2 has degree 2, if $a_{22} = 1$ then A is partly decomposable, contradicting the minimality of the order of A. The contracted matrix B is also a (0, 1) matrix, since $b_{11} = a_{22} + 1 = 1$. Repeated contraction eventually produces a matrix B' such that $H(B')$, the graph induced by all degree 2 vertices of $G(B')$, has a component with precisely one vertex. Since contraction preserves the permanent and permanental minors, per A = per $B' \geqslant 24$ (as argued above), a contradiction. ■

To prove the Statement for $t = 3$ we will use contraction to reduce A to a matrix B all of whose line sums are equal to 3. We will then need a lower bound on per B, perhaps as a function of the order of B. We could use the Van der Waerden–Egorycev–Falikman theorem on doubly stochastic matrices to do this, but it is easier to use the following result of Voorhoeve [4]:

LEMMA 3 (Voorhoeve). *Let $\lambda(n)$ be the smallest value of the permanent of any $n \times n$ matrix with nonnegative integral entries and all line sums equal to* 3. *Then*

$$\lambda(n+1) \geqslant \tfrac{4}{3}\lambda(n), \qquad n = 3, 4, 5, \dots,$$

and $\lambda(7) = 24$.

An incidence matrix for the projective plane of order 2 is a 7×7 matrix with permanent equal to 24.

THEOREM 3. *If A is a matrix with nonnegative integral entries such that* $0 <$ per $A < 24$, *then A is* 3-*triangular.*

Proof. Suppose A is an $n \times n$ (0, 1) matrix which is a counterexample of minimum order. By Lemma 2, all line sums of A are 2 or 3 and all components of $H(A)$, the subgraph of $G(A)$ induced by all degree 2 vertices, have an even number of vertices. If no vertex of $G(A)$ has degree 3, then A is 2-triangular. Assume $v \in V$ is a (row) vertex in $G(A)$ with degree 3. Since A is not 3-triangular, by Lemma 1 there does not exist an ordering $v_{k_1}, v_{k_2}, \dots, v_{k_n}$ of the vertices in V such that each of the sets $S_i = \{v_{k_1}, \dots, v_{k_i}\}$ $(i = 1, 2, \dots, n)$ satisfies the inequality

$$|N(S_i)| \leqslant i + 2. \tag{2.2}$$

If $v = v_{k_1}$ then (2.2) is satisfied for $i = 1$. Hence there exists a subset S of V such that $v \in S$ and

$$|N(S)| \leqslant |S| + 2, \tag{2.3}$$

but for each vertex $u \in (V \setminus S)$

$$|N(S \cup \{u\})| \geqslant |S| + 4 \tag{2.4}$$

If the inequality (2.3) is strict, then to satisfy (2.4) no vertex of $V \setminus S$ is adjacent to any vertex of $N(S)$ This creates a submatrix of zeros in A of size $(n - |S|) \times (|S| + 1)$, so per $A = 0$ by the Frobenius-König-Hall theorem. Hence equality holds in (2.3).

Each vertex of degree 2 which is adjacent to a vertex of $N(S)$ must be in S, or else the inequality (2.4) is violated. So each component of $H(A)$ which has a vertex in S has half its vertices in S and half in $N(S)$. Thus S and $N(S)$ have the same number of vertices of degree 2, and hence the sum of the degrees of the vertices in $N(S)$ is precisely 6 more than the sum of the degrees of the vertices in S. Since no vertex in $V \setminus S$ can be adjacent to more than one vertex of $N(S)$ [or else the inequality (2.4) is violated], it follows that $V \setminus S$ has at least six vertices of degree 3, so V has at least seven. Now we repeatedly contract A [and $G(A)$] until we get a matrix B all of whose line sums are equal to 3 [$G(B)$ is a cubic multigraph homeomorphic to $G(A)$]. If the end vertices of a component of $H(A)$ are adjacent to degree 3 vertices which are adjacent to each other, then B will have an entry equal to 2, but that does not impede the contraction process. Since B has order at least 7, by Lemma 3, per A = per $B \geqslant 24$. ∎

3. REMARKS

It is not known if the statement is true for $t = 4$ or $t = 5$. The techniques of this paper could probably be used to produce a proof or counterexample for $t = 4$, but there would be some complications. If contraction is used, one difficulty is that the contracted matrix would not be of doubly stochastic type (line sums could be 3 or 4), and there are no results like Voorhoeve's to get a good lower bound for the permanents of such matrices.

The Statement is false for $t = 6$. An incidence matrix for the projective plane of order 3 (a 13×13 matrix with precisely four ones in each line) has permanent equal to 3852 [3], which is less than 7!. However, this matrix is not 6-triangular, because any three rows have ones in at least nine columns.

Projective planes seem to be a likely source of examples to show the Statement is false for larger values of t. Each p rows of an incidence matrix for a projective plane of order p has ones in at least $p(p+3)/2$ columns, so such a matrix is not $p(p+1)/2$-triangular. The difficulty here is to estimate the permanent of such a matrix with sufficient accuracy to complete the argument.

CONJECTURE 1. Let $\mu(p) = \min\{\text{per } B \mid B \text{ is an incidence matrix for a projective plane of order } p\}$. If A is a matrix with nonnegative integral entries such that $0 < \text{per } A < \mu(p)$, then A is $p(p+1)/2$-triangular.

Conjecture 1 essentially says that projective plane matrices have the smallest permanents for their "triangularity." Ryser [3] suggested another manifestation of their small permanents. He felt that perhaps $\mu(p) = \min\{\text{per } A \mid A \text{ is a } (0,1)\text{-matrix of order } p^2 + p + 1 \text{ with precisely } p+1 \text{ ones in each line}\}$.

CONJECTURE 2. The Statement is true for only finitely many values of t.

In fact it may well be false for all $t \geqslant 6$.

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