# Incomplete Perfect Mendelsohn Designs with Block Size 4 and One Hole of Size 7

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**ABSTRACT**

**Let $v, k$, and $n$ be positive integers. An incomplete perfect Mendelsohn design, denoted by $k$-IPMD($v, n$), is a triple $(X, Y, \mathbb{B})$ where $X$ is a $v$-set (of points), $Y$ is an $n$-subset of $X$, and $\mathbb{B}$ is a collection of cyclically ordered $k$-subsets of $X$ (called *blocks*) such that every ordered pair $(a, b) \in (X \times X)\backslash(Y \times Y)$ appears $t$-apart in exactly one block of $\mathbb{B}$ and no ordered pair $(a, b) \in Y \times Y$ appears in any block of $\mathbb{B}$ for any $t$, where $1 \leq t \leq k - 1$. In this article, we obtain conclusive results regarding the existence of 4-IPMD($v, 7$) where the necessary conditions are $v \equiv 2$ or 3(mod 4) and $v \geq 22$. We also provide an application to the problem relating to coverings of PMDs with block size 4.** © 1993 John Wiley & Sons, Inc.

## 1. INTRODUCTION

Let $v$ and $k$ be positive integers. A $(v, k, 1)$-*Mendelsohn design,* briefly $(v, k, 1)$-MD, is a pair $(X, \mathbb{B})$ where $X$ is a $v$-set (of *points*) and $\mathbb{B}$ is a collection of cyclically ordered $k$-subsets of $X$ (called *blocks*) such that every ordered pair of points of $X$ are consecutive in exactly one block of $\mathbb{B}$. If for all $t = 1, 2, \ldots, k - 1$, every ordered pair of points

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of $X$ are $t$-apart in exactly one block of $\mathbb{B}$, then the $(v,k,1)$-MD is called *perfect* and is denoted by $(v,k,1)$-PMD.

In graph–theoretic terms, a $(v,k,1)$-PMD is equivalent to the decomposition of the complete directed graph $DK_v$ on $v$ vertices into $k$-circuits such that for any $r$, $1 \leq r \leq k-1$, and for any two distinct vertices $x$ and $y$, there is exactly one circuit along which the (directed) distance from $x$ to $y$ is $r$. It is easy to see that the number of blocks in a $(v,k,1)$-PMD is $v(v-1)/k$ and hence an obvious necessary condition for its existence is $v(v-1) \equiv 0 \pmod{k}$. This condition is known to be sufficient in most cases, but certainly not in all (see, for example, [3, 4, 8–10].

Let $v$, $k$, and $n$ be positive integers. An *incomplete perfect Mendelsohn design,* denoted by $k$-IPMD$(v,n)$, is a triple $(X, Y, \mathbb{B})$ where $X$ is a $v$-set (of *points*), $Y$ is an $n$-subset of $X$, and $\mathbb{B}$ is a collection of cyclically ordered $k$-subsets of $X$ (called *blocks*) such that every ordered pair $(a,b) \in (X \times X) \backslash (Y \times Y)$ appears $t$-apart in exactly one block of $\mathbb{B}$ and no ordered pair $(a,b) \in (Y \times Y)$ appears in any block of $\mathbb{B}$ for any $t$, where $1 \leq t \leq k-1$. For all practical purposes, the $k$-IPMD$(v,n)$ can be viewed as a $(v,k,1)$-PMD with a *hole* of size $n$ based on the set $Y$.

IPMDs have proved to be useful tools in the construction of PMDs and other structures (see [3, 13, 15]). The necessary conditions for the existence of a $k$-IPMD$(v,n)$ were developed in [2], namely, $(v-n)(v-(k-1)n-1) \equiv 0 \pmod{k}$ and $v \geq (k-1)n+1$. These basic necessary conditions were shown to be sufficient for the case $k=3$, with the one exception of $v=6$ and $n=1$. However, there is much work left to be done for the other values $k \geq 4$. In this article, we shall investigate the special case where $k=4$ and $n=7$. It is shown that a 4-IPMD$(v,7)$ exists if and only if $v \equiv 2$ or $3 \pmod{4}$ and $v \geq 22$.

Let $v$ and $k$ be positive integers. A $(v,k,1)$-*perfect Mendelsohn covering design* (briefly $(v,k,1)$-PMCD) is a pair $(X, \mathbb{C})$ where $X$ is a $v$-set (of *points*) and $\mathbb{C}$ is a collection of cyclically ordered $k$-subsets of $X$ (called *blocks*) such that every ordered pair of points of $X$ appears $t$-apart in at least one block of $\mathbb{C}$ for all $t = 1, 2, \ldots, k-1$. A $(v,k,1)$-PMCD is called a minimum covering if no other $(v,k,1)$-PMCD has fewer blocks, and the number of blocks in a minimum covering is called the covering number, denoted by $c(v,k)$. If we define $C(v,k) = \lceil v(v-1)/k \rceil$, where $\lceil x \rceil$ denotes the least integer $y$ such that $y \geq x$, then it is easy to see that $c(v,k) \geq C(v,k)$. As an application of our main result, we are able to show that $c(v,4) = C(v,4) = \lceil v(v-1)/4 \rceil$ for all values of $v \geq 20$.

The existence of $(v,4,1)$-PMDs will form the basis for most of our constructions. The problem of existence was initially studied by N. S. Mendelsohn [9] and it remained open for some time after. However, we now have a fairly conclusive result in the form of the following theorem (see [1, 4, 13]).

**Theorem 1.1** *A $(v,4,1)$-PMD exists for every positive integer $v \equiv 0$ or $1 \pmod{4}$ with the exception of $v=4$ and $8$ and possibly excepting $v=12$.*

## 2. PRELIMINARIES

In order to establish our main result, we shall employ both direct and recursive methods of construction. Our recursive methods will involve various notions of block designs which we briefly describe in what follows. The interested reader may wish to refer to [5, 7, 12] for more details.

Let $K$ be a set of positive integers. A *pairwise balanced design* (PBD) of index unity $B(K, 1; v)$ is a pair $(X, \mathbb{B})$ where $X$ is a $v$-set (of *points*) and $\mathbb{B}$ is a collection of subsets of $X$ (called *blocks*) with sizes from $K$ such that every pair of distinct points of $X$ is contained in exactly one block of $\mathbb{B}$. We shall denote by $B(K)$ the set of all integers $v$ for which there exists a PBD $B(K, 1; v)$. A PBD $B(\{k\}, 1; v)$ is essentially a *balanced incomplete block design* (BIBD) with parameters $v$, $k$, and $\lambda = 1$.

Let $K$ and $M$ be sets of positive integers. A *group divisible design* (GDD), denoted by $\mathrm{GD}(K, 1, M; v)$, is a triple $(X, \mathbb{G}, \mathbb{B})$ where

1. $X$ is a $v$-set (of *points*),
2. $\mathbb{G}$ is a collection of non-empty subsets of $X$ (called *groups*) with sizes in $M$ and which partition $X$,
3. $\mathbb{B}$ is a collection of subsets of $X$ (called *blocks*) each with size at least two in $K$,
4. no block meets a group in more than one point, and
5. every pairset $\{x, y\}$ of points not contained in a group is contained in exactly one block.

The *group-type* (or *type*) of a GDD $(X, \mathbb{G}, \mathbb{B})$ is the multiset $\{|G| : G \in \mathbb{G}\}$ and we shall use the "exponential" notation for its description, for example, a group-type $1^i 2^j 3^k \ldots$ denotes $i$ occurrences of groups of size 1, $j$ occurrences of groups of size 2, and so on. A *weighting* of a GDD $(X, \mathbb{G}, \mathbb{B})$ is any mapping $w : X \to Z^+ \cup \{0\}$. A $\mathrm{GD}(\{k\}, 1, \{m\}; v)$ will be denoted simply by $\mathrm{GD}(k, 1, m; v)$. We shall say that a GDD $(X, \mathbb{G}, \mathbb{B})$ is a $K$–GDD if $|B| \in K$ for every $B \in \mathbb{B}$.

A *transversal design* (TD), denoted by $\mathrm{TD}(k, m)$, is a GDD with $km$ points, $k$ groups of size $m$ and $m^2$ blocks of size $k$ where each block meets each group in precisely one point; that is, each block is a transversal of the collection of groups. It is well-known that a $\mathrm{TD}(k, m)$ is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order $m$.

A set of blocks (in a PBD or GDD) that partitions the point set is called a *parallel class*. If the blocks of a PBD or GDD can be partitioned into parallel classes, then the design is said to be *resolvable*. It is well-known that the existence of a $\mathrm{TD}(k + 1, m)$ is equivalent to the existence of a resolvable $\mathrm{TD}(k, m)$.

We shall make use of PMDs with "holes" (briefly HPMDs). For a brief description, we denote by $DK_{n_1, n_2, \ldots, n_h}$ the complete multipartite directed graph with vertex set $X = \cup_{1 \le i \le h} X_i$, where $X_i (1 \le i \le h)$ are disjoint sets with $|X_i| = n_i$, $v = \sum_{1 \le i \le h} n_i$, and where two vertices $x$ and $y$ from different sets $X_i$ and $X_j$ are joined by exactly one arc from $x$ to $y$ and one arc from $y$ to $x$. If $DK_{n_1, n_2, \ldots, n_h}$ can be decomposed into $k$-circuits such that for any $r$, $1 \le r \le k - 1$, and any two vertices $x$ and $y$ from different sets $X_i$ and $X_j$ there is exactly one circuit along which the directed distance from $x$ to $y$ is $r$, then we call $(X, \mathbb{B})$ a *holey perfect Mendelsohn design* (briefly denoted by $(v, k, 1)$-HPMD) where $\mathbb{B}$ is the collection of all $k$-circuits. The set $X_i$, $1 \le i \le h$, is called a *hole* and the vector $(n_1, n_2, \ldots, n_h)$ is called the *type* of the HPMD. A $(v, k, 1)$-HPMD of type $(1, 1, \ldots, 1, n)$ is equivalent to a $k$-$\mathrm{IPMD}(v, n)$. For convenience we shall denote the HPMD by the triple $(X, \mathbb{G}, \mathbb{B})$ where $\mathbb{G}$ represents the collection of holes.

In constructions of GDDs and PBDs, the "weighting" technique and Wilson's Fundamental Construction (see [12]) are quite often used, where we start with a "master" GDD and small input designs to obtain a new GDD. Similar techniques will be applied in our constructions of HPMDs, where we either start with an HPMD and use GDDs as input designs or start with a GDD and use some HPMDs as input designs. For more details of this technique, the reader is referred to [3, 12, 15].

The technique of filling in holes will be very useful in our constructions. The following lemma is fairly obvious.

**Lemma 2.1.** *If there exist both a k-IPMD$(v, n)$ and a k-IPMD$(n, m)$, then there exists a k-IPMD$(v, m)$.*

The following construction is a variation of the block design analogue of [11, Theorem 9] and represents a slight modification of [13, Theorem 2.4].

**Lemma 2.2** *Suppose that $(X, \mathbb{B})$ is a $(v, K, 1)$-PBD admitting t disjoint parallel classes $B_{r1}, B_{r2}, \ldots, B_{rk_r}$, $1 \leq r \leq t$. For every r, $1 \leq r \leq t$, suppose that there is an integer $n_r$ such that a k-IPMD $(|B_{rs}| + n_r, n_r)$ exists for every s, $1 \leq s \leq k_r$. Let $\mathbb{B}^*$ be the collection of blocks not belonging to any parallel class used. Suppose that there is an $(m, k, 1)$-PMD for every block in $\mathbb{B}^*$ of size $m \in K$. Then there exists a k-IPMD$(v + n, n)$ where $n = n_1 + n_2 + \ldots n_t$.*

As an application of Lemma 2.2, we have the following useful lemma.

**Lemma 2.3.** *Suppose that there exists a resolvable TD$(t, m)$. For $i = 1, 2, \ldots, m$, suppose that there exists an integer $n_i \geq 0$ such that there is a k-IPMD$(t + n_i, n_i)$. Further suppose that a k-IPMD$(m + u, u)$ exists. Then there exists a k-IPMD$(tm + n, n)$ where $n = u + \sum_{1 \leq i \leq m} n_i$.*

*Proof.* A resolvable TD$(t, m)$ admits $m$ parallel classes of blocks of size $t$ and one parallel class (of groups) of size $m$. We can then apply Lemma 2.2 to obtain the desired result. □

**Lemma 2.4.** *If there exist a TD$(5, m)$ and a 4-IPMD$(m + 2, 3)$, then there exists a 4-IPMD$(5m + 2, 7)$.*

*Proof.* Let $(X, \mathbb{G}, \mathbb{B})$ be a TD$(5, m)$ where $\mathbb{G} = \{G_1, G_2, G_3, G_4, G_5\}$ and $|G_i| = m$. We shall adjoin two infinite points to this TD and form a 4-IPMD$(5m + 2, 7)$ based on the set $X \cup \{\infty_1, \infty_2\}$ as follows: First, we select a block of size 5 from $\mathbb{B}$ say $B = \{b_1, b_2, b_3, b_4, b_5\}$ where $B \cap G_i = \{b_i\}$. Then we perform the following:

(1) form a (5,4,1)-PMD on the blocks of $\mathbb{B} \backslash \{\{B\}\}$;
(2) for $i = 1, 2, \ldots, 5$ we form a 4-IPMD$(m + 2, 3)$ based on the set $G_i \cup \{\infty_1, \infty_2\}$ with a hole of size 3 on the set $\{\infty_1, \infty_2, b_i\}$;
(3) we discard the block $B$ to obtain 4-IPMD$(5m + 2, 7)$ based on the set $X \cup \{\infty_1, \infty_2\}$ with a hole of size 7 based on the set $\{b_1, b_2, b_3, b_4, b_5, \infty_1, \infty_2\}$. □

The following lemma is contained in [15, Theorem 5.3].

**Lemma 2.5.** *If there exist a $(v, k, 1)$-HPMD of type $(n_1, n_2, \ldots, n_h)$ and a k-IPMD$(n_i + m, m)$ for $2 \leq i \leq h$, then there exists a k-IPMD$(v + m, n_1 + m)$. Moreover, if there exists a k-IPMD$(n_1 + m, m)$, then there exists a k-IPMD$(v + m, m)$.*

In most cases, our direct method of construction will be a variation of the method using difference sets in the construction of BIBDs (see, for example, [5]). Instead of listing all the blocks of a design, it will be sufficient to give the group $G$ (usually the cyclic group $Z_n$) acting on a set of base blocks to be developed. Below we shall give an illustration of this type of construction. Further examples are provided in the Appendix.

**Lemma 2.6.** *There exists a 4-IPMD$(v, 3)$ for $v \in \{10, 11, 14, 15, 22\}$.*

*Proof.* For $v = 10$, we let $X = Z_7 \cup \{\infty_1, \infty_2, \infty_3\}$, $Y = \{\infty_1, \infty_2, \infty_3\}$, and $\mathbb{B}$ be the collection of blocks obtained by developing the following modulo 7:

$$(\infty_1, 0, 2, 1) \quad (\infty_2, 0, 4, 2) \quad (\infty_3, 0, 1, 4).$$

It is readily checked that $(X, Y, \mathbb{B})$ is a 4-IPMD(10,3).

For $v = 11$, let $X = Z_8 \cup \{\infty_1, \infty_2, \infty_3\}$, $Y = \{\infty_1, \infty_2, \infty_3\}$. Let $\mathbb{B}_1 = \{(i, 2 + i, 4 + i, 6 + i) : i = 0, 1\}$, Let $\mathbb{B}_2$ be the collection of blocks obtained by developing the following modulo 8:

$$(\infty_1, 0, 1, 6) \quad (\infty_2, 0, 3, 7) \quad (\infty_3, 0, 6, 5).$$

If we let $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2$, then it is easy to check that $(X, Y, \mathbb{B})$ is a 4-IPMD(11,3).

For $v = 14$, let $X = Z_{11} \cup \{\infty_1, \infty_2, \infty_3\}$, $Y = \{\infty_1, \infty_2, \infty_3\}$, and $\mathbb{B}$ the collection of blocks obtained by developing the following modulo 11:

$$(0, 2, 7, 8) \quad (\infty_1, 0, 4, 1) \quad (\infty_2, 0, 6, 2) \quad (\infty_3, 0, 9, 8).$$

Then $(X, Y, \mathbb{B})$ is a 4-IPMD(14,3).

For $v = 15$, let $X = Z_{12} \cup \{\infty_1, \infty_2, \infty_3\}$, $Y = \{\infty_1, \infty_2, \infty_3\}$. Let $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2$, where $\mathbb{B}_1 = \{(i, 3 + i, 6 + i, 9 + i) : i = 0, 1, 2\}$ and $\mathbb{B}_2$ is obtained by developing the following modulo 12:

$$(0, 2, 1, 7) \quad (\infty_1, 0, 4, 2) \quad (\infty_2, 0, 1, 9) \quad (\infty_3, 0, 7, 4).$$

Then $(X, Y, \mathbb{B})$ is a 4-PMD(15,3).

Finally, for $v = 22$, let $X = Z_{19} \cup \{\infty_1, \infty_2, \infty_3\}$, $Y = \{\infty_1, \infty_2, \infty_3\}$, and $\mathbb{B}$ the collection of blocks obtained by developing the following modulo 19:

$$(0, 3, 8, 7) \quad (0, 4, 12, 10) \quad (0, 7, 1, 17) \quad (\infty_1, 0, 6, 17)$$
$$(\infty_2, 0, 1, 16) \quad (\infty_3, 0, 10, 5).$$

Then $(X, Y, \mathbb{B})$ is a 4-IPMD(22,3). □

**Lemma 2.7.** *If $q = 2s + 1$ is a prime power where $s \geq 3$, $s \not\equiv 0 (mod\ 4)$, then there exists a 4-IPMD$(q + s, s)$.*

*Proof.* When $s$ is odd, the result has already been established by Zhang [13, Lemma 3.1]. If $s \equiv 2 (\text{mod } 4)$, we let $X = \text{GF}(q) \cup \{\infty_i : 1 \leq i \leq s\}$, $Y = \{\infty_i : 1 \leq i \leq s\}$. Let $x$

be a primitive element of GF($q$) and $\mathbb{B}$ be the collection of blocks obtained by developing the following base blocks under the additive group of GF($q$):

$$\{(\infty_{2i+1}, 0, x^{4i}, x^{4i}(1 + x^2)), (\infty_{2i+2}, 0, x^{4i+1}, x^{4i+1}(1 + x^2)) : i = 0, 1, \ldots, (s/2) - 1\}$$

where $s \equiv 2 (\mathrm{mod}\ 4)$. Then $(X, Y, \mathbb{B})$ is a 4-IPMD$(q + s, s)$. □

The following are useful input designs in the construction of HPMDs. Our next lemma comes from Zhang [14, Lemma 3.16].

**Lemma 2.8.** *There exists a* $(16, 4, 1)$*-HPMD of type* $4^4$.

**Lemma 2.9.** *There exist a* $(20, 4, 1)$*-HPMD of type* $4^5$ *and a* $(24, 4, 1)$*-HPMD of type* $4^6$.

*Proof.* It is well-known (see [7]) that there exist a $(21, 5, 1)$-BIBD and a $(25, 5, 1)$-BIBD. We delete one point from each of the BIBD to obtain $\{5\}$-GDDs of types $4^5$ and $4^6$, respectively. Replacing each block in the above GDDs with a $(5, 4, 1)$-PMD, we obtain the desired HPMDs of types $4^5$ and $4^6$. □

**Lemma 2.10.** *There exist a* $(22, 4, 1)$*-HPMD of type* $4^4 6^1$ *and a* $(26, 4, 1)$*-HPMD of type* $4^5 6^1$.

*Proof.* For a $(22, 4, 1)$-HPMD of type $4^4 6^1$, we take $X = Z_{16} \cup \{\infty_1, \infty_2, \ldots, \infty_6\}$, $\mathbb{G} = \{\{i, i + 4, i + 8, i + 12\} : i = 0, 1, 2, 3\} \cup \{\{\infty_i : i = 1, 2, \ldots, 6\}\}$. Let $\mathbb{B}$ be the collection of blocks obtained by developing the following modulo 16:

$$(\infty_1, 0, 2, 9) \quad (\infty_2, 0, 5, 14) \quad (\infty_3, 0, 15, 10) \quad (\infty_4, 0, 6, 3)$$
$$(\infty_5, 0, 1, 11) \quad (\infty_6, 0, 3, 1).$$

Then $(X, \mathbb{G}, \mathbb{B})$ is readily checked to be a (22,4,1)-HPMD of type $4^4 6^1$.

For a (26,4,1)-HPMD of type $4^5 6^1$, we let $X = Z_{20} \cup \{\infty_1, \infty_2, \ldots, \infty_6\}$, $\mathbb{G} = \{\{i, i + 5, i + 10, i + 15\} : i = 0, 1, 2, 3, 4\} \cup \{\{\infty_i : i = 1, 2, \ldots, 6\}$. Let $\mathbb{B}$ be the collection of blocks obtained by developing the following modulo 20:

$$(0, 1, 3, 14) \quad (\infty_1, 0, 7, 16) \quad (\infty_2, 0, 17, 9) \quad (\infty_3, 1, 0, 13)$$
$$(\infty_4, 0, 8, 6) \quad (\infty_5, 0, 16, 19) \quad (\infty_6, 0, 4, 18).$$

It is readily checked that $(X, \mathbb{G}, \mathbb{B})$ is a $(26, 4, 1)$-HPMD of type $4^5 6^1$. □

Applying the weighting-technique (see, for example, [12, 15]) with Lemmas 2.8–2.10, we obtain the main tool for our recursive constructions.

**Lemma 2.11.** *Suppose that there exists a* TD$(6, t)$, $0 \le u, a, b \le t$ *and* $0 \le a + b \le t$. *Then there exists a* $(16t + 4u + 4a + 6b, 4, 1)$*-HPMD of type* $(4t)^4 (4u)^1 (4a + 6b)^1$.

*Proof.* In all but the last two groups of a TD$(6, t)$, we give the points weight 4. In the next to last group, we give $u$ points weight 4 and the remaining points weight 0. In the last group, we give $a$ points weight 4, give $b$ points weight 6 and the remaining points weight 0. Using the HPMDs of types $4^4$, $4^5$, $4^6$, $4^4 6^1$, and $4^5 6^1$, we readily obtain the desired HPMD of type $(4t)^4 (4u)^1 (4a + 6b)^1$. □

Before proceeding, we wish to remark that it is evident that some of our lemmas in this section will require transversal designs. For our purposes, the necessary transversal designs can be obtained from [5, 6, 7]. In particular, we shall make use of the following well-known results.

**Lemma 2.12.** *For every prime power q, there exists a* $TD(q+1, q)$.

**Lemma 2.13.** *If* $m \geq 5$ *is an odd integer, then there exists a* $TD(6, m)$.

For all practical purposes, we shall view the existence of a $(v, 4, 1)$-PMD as being equivalent to either a 4-IPMD$(v, 0)$ or a 4-IPMD$(v, 1)$. As an application of Lemma 2.11, we can now state the main construction.

**Lemma 2.14.** *Let* $t \geq 5$ *be an odd integer. Suppose that* $0 \leq u, a, b \leq t$ *and* $0 \leq a + b \leq t$. *Then the following hold:*

(1) *There exists a* 4-*IPMD*$(16t + 4u + 4a + 6b, 4a + 6b)$, *provided that* $u \neq 1, 2, 3$.
(2) *There exists a* 4-*IPMD*$(16t + 4u + 4a + 6b + 1, 4a + 6b + 1)$.

*Proof.* If $t \geq 5$ is an odd integer, then there exists a TD$(6, t)$ from Lemma 2.13. We can then apply Lemma 2.11 to obtain a $(16t + 4u + 4a + 6b, 4, 1)$-HPMD of type $(4t)^4(4u)^1(4a + 6b)^1$. Since there is a $(v, 4, 1)$-PMD for $v = 4t$ where $t \geq 5$ and $v = 4u$, $u \neq 1, 2, 3$, the proof of (1) follows from an application of Lemma 2.5 with $m = 0$. For the proof of (2), we can adjoin one point to the HPMD and use the fact that there exist a 4-IPMD$(4t + 1, 1)$ and a 4-IPMD$(4u + 1, 1)$ to obtain the desired result by applying Lemma 2.5 with $m = 1$. □

The following lemma will also be useful in taking care of a few special cases.

**Lemma 2.15.** *If there exist a* $TD(5, t)$ *and a* 4-*IPMD*$(t + m, m)$, *then there exists a* 4-*IPMD*$(5t + m, t + m)$.

*Proof.* Since a $(5, 4, 1)$-PMD exists, the existence of a TD$(5, t)$ implies the existence of a $(5t, 4, 1)$-HPMD of type $t^5$. We can then apply Lemma 2.5 to obtain the desired result. □

## 3. CONSTRUCTION OF 4-IPMDs

For convenience, we shall briefly write IPMD$(v, n)$ for a 4-IPMD$(v, n)$ in what follows. Direct constructions have provided us with IPMD$(v, 7)$ for most of the values of $v$ where $22 \leq v \leq 86$ and $v \equiv 2$ or $3 \pmod 4$. These constructions are given in the Appendix and the results can be summarized as follows:

**Lemma 3.1.** *If* $v \equiv 2$ *or* $3 \pmod 4$, $22 \leq v \leq 86$, *and* $v \notin \{47, 59, 62, 67, 70, 79, 82, 83\}$, *then there exists an* $IPMD(v, 7)$.

*Proof.* The constructions are given in the Appendix. Note that for $v \in \{71, 74, 75, 78, 86\}$ the existence of the IPMD$(v, 7)$ follows from the existence of an IPMD$(23, 7)$ and an IPMD$(27, 7)$ by applying Lemma 2.1. □

**Lemma 3.2.** *If $v \in \{47, 62, 67\}$, then there exists an IPMD$(v, 7)$.*

*Proof.* In each case, we apply Lemma 2.4. For $v = 47, 62, 67$, we choose $m = 9, 12, 13$, respectively, and the necessary ingredients come from Lemma 2.6. □

We require the following special construction.

**Lemma 3.3** *There exists an IPMD*(59, 7).

*Proof.* Applying the weighting technique with a (13, 4, 1)-PMD and a resolvable TD(4, 4), we can obtain a (52, 4, 1)-HPMD of type $4^{13}$ based on the set $X = Z_{13} \times Z_4$ and which has sub-HPMDs of type $1^{13}$ on $Z_{13} \times \{i\}$, $i \in Z_4$. For each $i \in Z_4$, replace the sub-HPMD on $Z_{13} \times \{i\}$ by a 4-HPMD(19, 6) based on $(Z_{13} \times \{i\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$. The 4-IPMD(19, 6) exists from Lemma 2.7. This gives rise to a (58, 4, 1)-HPMD of type $4^{13}6^1$. We can then apply Lemma 2.5 with $m = 1$ to adjoin another infinite point to this HPMD to obtain the desired IPMD(59, 7). □

**Lemma 3.4.** *If $v \in \{70, 79\}$, then there exists an IPMD$(v, 7)$.*

*Proof.* For $v = 70$, we apply Lemma 2.7 with $s = 23$ to obtain an IPMD(70, 23) and the result follows from the existence of an IPMD(23, 7). For $v = 79$, we apply Lemma 2.7 with $s = 26$ to get an IPMD(79, 26) and the result follows from the existence of an IPMD(26, 7). □

**Lemma 3.5.** *If $v \in \{82, 83, 87, 90, 91, 94, 95, 98, 99, 114\}$, then there exists an IPMD$(v, 7)$.*

*Proof.* In what follows, we shall make use of the results in Lemma 3.1 and apply Lemma 2.1 for the end result. For $v \in \{82, 87, 98\}$ we shall first apply Lemma 2.15 with $t \in \{15, 16, 19\}$. We then use the fact that an IPMD(22, 7) and an IPMD(23, 7) exist from Lemma 3.1; and an IPMD(22, 3) exists from Lemma 2.6. For the remaining cases we shall apply Lemma 2.3. For $v = 83$, start with a resolvable TD(7, 8) to which we adjoin 27 infinite points by using the existence of an IPMD(10, 3) and an IPMD(11, 3). This gives an IPMD(83, 27) and the result follows. For $v \in \{90, 94\}$, we start with a resolvable TD(7, 9). For $v = 90$, we adjoin 27 infinite points to this TD using the existence of an IPMD(10, 3) and a (9, 4, 1)-PMD. This gives an IPMD(90, 27) and the result follows. For $v = 94$, we adjoin 31 points to the TD using an IPMD(10, 3) and an IPMD(13, 4) from Zhang [13]. This gives an IPMD(94, 31) and the desired result. For $v = 91$, we apply Lemma 2.3 by starting with a resolvable TD(8, 8) and then add 27 points by using an IPMD(11, 3). This gives an IPMD(91, 27) and the result follows. For $v \in \{95, 99\}$, we start with a resolvable TD(8, 9) and adjoin 23 and 27 infinite points, respectively, using the existence of an IPMD(11, 3) and a (9, 4, 1)-PMD. We thus obtain an IPMD(95, 23) and an IPMD(99, 27). For $v = 114$, we start with a resolvable TD(8, 11) and add 26 infinite points, using an IPMD(9, 1), an IPMD(11, 3), and an IPMD(16, 5) which comes from Lemma 2.7. We then obtain an IPMD(114, 26) and the desired result follows. This completes the proof of the lemma. □

**Lemma 3.6.** *If $v \equiv 2 (mod\ 4)$ and $v \geq 102$, then there exists an IPMD$(v, 7)$.*

*Proof.* We shall apply Lemma 2.14 (1). For convenience, we let $v = 16t + 4u + 4a + 6b$ and $n = 4a + 6b$, where $t \geq 5$ is an odd integer and $0 \leq u, a, b \leq t$, $0 \leq a + b \leq t$ and $u \neq 1, 2, 3$. If $t = 5$, we can choose $u = 0, 4, 5$ and suitable choices of $a$ and $b$ will produce values of $n \in \{22, 26, 30\}$. Consequently, if $102 \leq v \leq 130$ and $v \neq 114$, then there exists an IPMD$(v, n)$ for some $n \in \{22, 26, 30\}$. But an IPMD(114, 7) exists from Lemma 3.5, and so an IPMD$(v, 7)$ exists for all $v \equiv 2 (\text{mod } 4)$ where $102 \leq v \leq 130$. For $t \geq 7$, we can make suitable choices of $a$ and $b$ to produce values of $n \in \{22, 26, 30, 34, 38, 42\}$. We can also choose $u = 0$ or $4 \leq u \leq t$. Hence, if $16t + 22 \leq v \leq 20t + 42$, then there exists an IPMD$(v, n)$ for some $n \in \{22, 26, 30, 34, 38, 42\}$, which implies the existence of an IPMD$(v, 7)$. For any two consecutive odd integers $t_1, t_2 \geq 7$, the intervals obtained for $v$ will overlap. Consequently, there exists an IPMD$(v, 7)$ for all $v \geq 134$ where $v \equiv 2 (\text{mod } 4)$, and the lemma is proved. □

**Lemma 3.7.** *If $v \equiv 3 (\text{mod } 4)$ and $v \geq 103$, then an IPMD$(v, 7)$ exists.*

*Proof.* The proof is similar to that of Lemma 3.6, except we apply Lemma 2.14 (2). Let $v = 16t + 4u + 4a + 6b + 1$ and $n = 4a + 6b + 1$, where $t \geq 5$ is an odd integer and $0 \leq u, a, b \leq t$, $0 \leq a + b \leq t$. If $t = 5$, we can choose $0 \leq u \leq 5$ and suitable choices of $a$ and $b$ can be made to obtain values of $n \in \{23, 27, 31\}$. Consequently, for all values of $v \equiv 3 (\text{mod } 4)$ where $103 \leq v \leq 131$, there exists an IPMD$(v, n)$ for some $n \in \{23, 27, 31\}$ and hence an IPMD$(v, 7)$ exists. For $t \geq 7$, suitable choices of $a$ and $b$ can be made to obtain values of $n \in \{23, 27, 31, 35, 39, 43\}$. We can choose $0 \leq u \leq t$ and so for $16t + 23 \leq v \leq 20t + 43$, there exists an IPMD$(v, n)$ for some $n \in \{23, 27, 31, 35, 39, 43\}$; and hence an IPMD$(v, 7)$ exists. Evidently, we can obtain an IPMD$(v, 7)$ for all $v \geq 135$ where $v \equiv 3 (\text{mod } 4)$ and the proof of the lemma is complete. □

Combining the results of Lemmas 3.1–3.7, we have proved the main result:

**Theorem 3.8.** *There exists an IPMD$(v, 7)$ if and only if $v \equiv 2$ or $3 (\text{mod } 4)$ and $v \geq 22$.*

## 4. AN APPLICATION TO PMCDs

The result of Theorem 3.8 can be applied to the problem of finding minimal coverings for $(v, 4, 1)$-PMCDs. We first make the following observation.

**Lemma 4.1.** *If $v \equiv 0$ or $1 (\text{mod } 4)$ and $v \notin \{4, 8, 12\}$, then $c(v, 4) = C(v, 4) = v(v - 1)/4$.*

*Proof.* If $v \equiv 0$ or $1 (\text{mod } 4)$ and $v \notin \{4, 8, 12\}$, then there exists a $(v, 4, 1)$-PMD which provides an exact minimal covering. □

Next, we have the following useful result.

**Lemma 4.2.** $c(7, 4) = C(7, 4) = 11$.

*Proof.* Let $X = Z_5 \cup \{\infty_1, \infty_2\}$ and $\mathbb{C}$ be the following collection of blocks:

$$(\infty_1, \infty_2, 2, 3) \quad (\infty_1, \infty_2, 3, 4) \quad (\infty_1, 0, 4, \infty_2) \quad (\infty_1, 1, 0, \infty_2)$$
$$(\infty_1, 2, \infty_2, 0) \quad (\infty_1, 3, \infty_2, 1) \quad (\infty_1, 4, 1, 2) \quad (\infty_2, 4, 2, 1)$$
$$(0, 1, 3, 2) \quad (0, 3, 1, 4) \quad (0, 2, 4, 3).$$

Then it is readily checked that $(X, \mathbb{C})$ is a (7,4,1)-PMCD with $C(7,4)$ blocks. □

As a consequence of Theorem 3.8, we can prove the following:

**Theorem 4.3.** *For all integers $v \geq 20$, $c(v,4) = C(v,4) = \lceil v(v-1)/4 \rceil$.*

*Proof.* If $v \equiv 0$ or 1(mod 4), then the result follows from Lemma 4.1. If $v \equiv 2$ or 3(mod 4) and $v \geq 22$, then there exists an IPMD$(v, 7)$ from Theorem 3.8. If we fill in the hole of this IPMD with a copy of the (7,4,1)-PMCD given in the proof of Lemma 4.2, then it is easy to verify that the resulting design is a $(v, 4, 1)$-PMCD with $C(v,4)$ blocks. This completes the proof of the theorem. □

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**Note added in proof:** It is now known that a (12,4,1)-PMD exists. Consequently, the possible exception $v = 12$ can be removed from Theorem 1.1, and this result also improves Lemma 4.1.

## 6. APPENDIX: INCOMPLETE PMDs WITH BLOCK SIZE 4

An IPMD$(v, 7)$ for $v \in \{22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 66\}$
points: $Z_{v-7} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$
blocks: develop the following base blocks modulo $v - 7$:

$v = 22$: $(\infty_1, 0, 4, 3)$ $(\infty_2, 0, 2, 11)$ $(\infty_3, 0, 8, 5)$ $(\infty_4, 0, 1, 6)$
$(\infty_5, 0, 10, 8)$ $(\infty_6, 0, 3, 14)$ $(\infty_7, 0, 6, 13)$

$v = 26$: $(\infty_1, 0, 2, 18)$ $(\infty_2, 0, 4, 17)$ $(\infty_3, 0, 1, 15)$ $(\infty_4, 0, 5, 14)$
$(\infty_5, 0, 6, 13)$ $(\infty_6, 0, 11, 7)$ $(\infty_7, 0, 3, 11)$ $(0, 17, 16, 7)$

$v = 30$: $(\infty_1, 0, 2, 1)$ $(\infty_2, 0, 9, 19)$ $(\infty_3, 0, 1, 6)$ $(\infty_4, 0, 12, 8)$
$(\infty_5, 0, 6, 14)$ $(\infty_6, 0, 16, 10)$ $(\infty_7, 0, 14, 11)$ $(0, 18, 16, 20)$
$(0, 11, 3, 16)$

$v = 34$: $(\infty_1, 0, 3, 26)$ $(\infty_2, 0, 1, 23)$ $(\infty_3, 0, 15, 5)$ $(\infty_4, 0, 11, 10)$
$(\infty_5, 0, 2, 16)$ $(\infty_6, 0, 18, 15)$ $(\infty_7, 0, 6, 13)$ $(0, 20, 18, 23)$
$(0, 12, 20, 6)$ $(0, 9, 19, 11)$

$v = 38$: $(\infty_1, 0, 7, 30)$ $(\infty_2, 0, 15, 2)$ $(\infty_3, 0, 1, 3)$ $(\infty_4, 0, 13, 7)$
$(\infty_5, 0, 21, 19)$ $(\infty_6, 0, 22, 18)$ $(\infty_7, 0, 19, 14)$ $(0, 24, 23, 28)$
$(0, 16, 5, 22)$ $(0, 10, 21, 25)$ $(0, 8, 20, 17)$

$v = 42$: $(\infty_1, 0, 9, 33)$ $(\infty_2, 0, 10, 31)$ $(\infty_3, 0, 1, 5)$ $(\infty_4, 0, 12, 8)$
$(\infty_5, 0, 26, 25)$ $(\infty_6, 0, 28, 23)$ $(\infty_7, 0, 17, 15)$ $(0, 5, 16, 6)$
$(0, 7, 21, 13)$ $(0, 2, 22, 19)$ $(0, 3, 11, 29)$ $(0, 13, 28, 16)$

$v = 46$: $(\infty_1, 0, 13, 1)$ $(\infty_2, 0, 19, 3)$ $(\infty_3, 0, 6, 30)$ $(\infty_4, 0, 25, 12)$
$(\infty_5, 0, 8, 25)$ $(\infty_6, 0, 28, 24)$ $(\infty_7, 0, 30, 22)$ $(0, 36, 7, 5)$
$(0, 14, 4, 24)$ $(0, 7, 19, 18)$ $(0, 1, 6, 17)$ $(0, 32, 2, 6)$
$(0, 3, 5, 21)$

$v = 50$: $(\infty_1, 0, 6, 42)$ $(\infty_2, 0, 10, 4)$ $(\infty_3, 0, 18, 37)$ $(\infty_4, 0, 3, 35)$
$(\infty_5, 0, 24, 10)$ $(\infty_6, 0, 33, 31)$ $(\infty_7, 0, 2, 23)$ $(0, 5, 21, 8)$
$(0, 17, 16, 36)$ $(0, 8, 17, 21)$ $(0, 23, 15, 28)$ $(0, 11, 2, 29)$
$(0, 28, 11, 42)$ $(0, 38, 7, 4)$

$v = 54$: $(\infty_1, 0, 9, 46)$ $(\infty_2, 0, 6, 40)$ $(\infty_3, 0, 22, 8)$ $(\infty_4, 0, 14, 35)$
$(\infty_5, 0, 19, 14)$ $(\infty_6, 0, 29, 18)$ $(\infty_7, 0, 7, 27)$ $(0, 5, 21, 9)$
$(0, 8, 23, 17)$ $(0, 17, 13, 36)$ $(0, 45, 16, 1)$ $(0, 2, 6, 19)$
$(0, 27, 11, 37)$ $(0, 40, 5, 8)$ $(0, 24, 2, 46)$

$v = 58$: $(\infty_1, 0, 11, 48)$ $(\infty_2, 0, 4, 44)$ $(\infty_3, 0, 5, 38)$ $(\infty_4, 0, 1, 35)$
$(\infty_5, 0, 23, 19)$ $(\infty_6, 0, 38, 30)$ $(\infty_7, 0, 2, 41)$ $(0, 18, 15, 42)$
$(0, 46, 25, 9)$ $(0, 16, 1, 20)$ $(0, 25, 2, 34)$ $(0, 10, 23, 22)$
$(0, 3, 11, 25)$ $(0, 6, 18, 12)$ $(0, 7, 5, 27)$ $(0, 44, 8, 10)$

$v = 66$: $(\infty_1, 0, 12, 54)$ $(\infty_2, 0, 11, 7)$ $(\infty_3, 0, 50, 47)$ $(\infty_4, 0, 9, 50)$
$(\infty_5, 0, 33, 17)$ $(\infty_6, 0, 18, 37)$ $(\infty_7, 0, 10, 24)$ $(0, 44, 1, 14)$
$(0, 21, 2, 24)$ $(0, 7, 15, 32)$ $(0, 58, 4, 27)$ $(0, 6, 10, 25)$
$(0, 20, 8, 34)$ $(0, 53, 23, 21)$ $(0, 48, 13, 5)$ $(0, 30, 20, 56)$
$(0, 46, 18, 57)$ $(0, 37, 6, 58)$

An IPMD$(v, 23)$ for $v \in \{74, 78\}$
points: $Z_{v-23} \cup \{\infty_i : 1 \le i \le 23\}$

blocks: develop the following base blocks modulo $v - 23$:

$v = 74$: $(\infty_1, 0, 23, 1)$ $(\infty_2, 0, 13, 2)$ $(\infty_3, 0, 19, 3)$ $(\infty_4, 0, 7, 4)$
$(\infty_5, 0, 20, 5)$ $(\infty_6, 0, 2, 6)$ $(\infty_7, 0, 46, 44)$ $(\infty_8, 0, 9, 8)$
$(\infty_9, 0, 15, 42)$ $(\infty_{10}, 0, 14, 10)$ $(\infty_{11}, 0, 25, 12)$ $(\infty_{12}, 0, 5, 13)$
$(\infty_{13}, 0, 32, 14)$ $(\infty_{14}, 0, 12, 36)$ $(\infty_{15}, 0, 30, 16)$ $(\infty_{16}, 0, 6, 17)$
$(\infty_{17}, 0, 16, 33)$ $(\infty_{18}, 0, 45, 20)$ $(\infty_{19}, 0, 28, 21)$ $(\infty_{20}, 0, 41, 29)$
$(\infty_{21}, 0, 31, 23)$ $(\infty_{22}, 0, 3, 24)$ $(\infty_{23}, 0, 34, 25)$ $(0, 1, 19, 41)$

$v = 78$: $(\infty_1, 0, 9, 21)$ $(\infty_2, 0, 17, 5)$ $(\infty_3, 0, 51, 42)$ $(\infty_4, 0, 23, 16)$
$(\infty_5, 0, 32, 26)$ $(\infty_6, 0, 35, 32)$ $(\infty_7, 0, 39, 45)$ $(\infty_8, 0, 26, 53)$
$(\infty_9, 0, 38, 43)$ $(\infty_{10}, 0, 18, 8)$ $(\infty_{11}, 0, 21, 46)$ $(\infty_{12}, 0, 28, 27)$
$(\infty_{13}, 0, 20, 49)$ $(\infty_{14}, 0, 3, 1)$ $(\infty_{15}, 0, 15, 52)$ $(\infty_{16}, 0, 7, 48)$
$(\infty_{17}, 0, 2, 44)$ $(\infty_{18}, 0, 10, 41)$ $(\infty_{19}, 0, 34, 15)$ $(\infty_{20}, 0, 30, 19)$
$(\infty_{21}, 0, 11, 33)$ $(\infty_{22}, 0, 8, 24)$ $(\infty_{23}, 0, 33, 25)$ $(0, 1, 20, 5)$
$(0, 4, 17, 41)$

An IPMD(86, 27)
points: $Z_{59} \cup \{\infty_i : 1 \leq i \leq 27\}$
blocks: develop the base blocks modulo 59:

$(\infty_1, 0, 6, 58)$ $(\infty_2, 0, 19, 2)$ $(\infty_3, 0, 4, 3)$ $(\infty_4, 0, 56, 54)$
$(\infty_5, 0, 28, 6)$ $(\infty_6, 0, 21, 7)$ $(\infty_7, 0, 12, 8)$ $(\infty_8, 0, 14, 9)$
$(\infty_9, 0, 10, 49)$ $(\infty_{10}, 0, 24, 11)$ $(\infty_{11}, 0, 35, 12)$ $(\infty_{12}, 0, 17, 46)$
$(\infty_{13}, 0, 26, 14)$ $(\infty_{14}, 0, 34, 15)$ $(\infty_{15}, 0, 16, 43)$ $(\infty_{16}, 0, 32, 17)$
$(\infty_{17}, 0, 5, 18)$ $(\infty_{18}, 0, 8, 19)$ $(\infty_{19}, 0, 1, 39)$ $(\infty_{20}, 0, 2, 21)$
$(\infty_{21}, 0, 30, 22)$ $(\infty_{22}, 0, 33, 23)$ $(\infty_{23}, 0, 15, 35)$ $(\infty_{24}, 0, 31, 25)$
$(\infty_{25}, 0, 42, 33)$ $(\infty_{26}, 0, 9, 32)$ $(\infty_{27}, 0, 3, 28)$ $(0, 7, 29, 11)$

An IPMD($v$, 7) for $v \in \{23, 27, 31, 35, 39, 43, 51\}$
points: $Z_{v-7} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$
blocks: $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2$ where $\mathbb{B}_1 = \{(i, t + i, 2t + i, 3t + i) : 0 \leq i \leq t - 1$ and $t =$

$(v - 7)/4\}$ and $\mathbb{B}_2$ is obtained by developing the following modulo $v - 7$:

$$v = 23: (\infty_1, 0, 5, 1) \quad (\infty_2, 0, 6, 3) \quad (\infty_3, 0, 7, 5) \quad (\infty_4, 0, 8, 7)$$
$$(\infty_5, 0, 1, 10) \quad (\infty_6, 0, 2, 12) \quad (\infty_7, 0, 8, 14)$$
$$v = 27: (\infty_1, 0, 4, 19) \quad (\infty_2, 0, 8, 2) \quad (\infty_3, 0, 11, 4) \quad (\infty_4, 0, 3, 15)$$
$$(\infty_5, 0, 7, 6) \quad (\infty_6, 0, 10, 8) \quad (\infty_7, 0, 2, 11) \quad (0, 16, 13, 19)$$
$$v = 31: (\infty_1, 0, 4, 2) \quad (\infty_2, 0, 1, 21) \quad (\infty_3, 0, 2, 20) \quad (\infty_4, 0, 13, 6)$$
$$(\infty_5, 0, 9, 8) \quad (\infty_6, 0, 3, 10) \quad (\infty_7, 0, 16, 11) \quad (0, 11, 19, 10)$$
$$(0, 12, 9, 19)$$

$$v = 35: (\infty_1, 0, 11, 1) \quad (\infty_2, 0, 1, 4) \quad (\infty_3, 0, 14, 5) \quad (\infty_4, 0, 2, 6)$$
$$(\infty_5, 0, 22, 18) \quad (\infty_6, 0, 17, 16) \quad (\infty_7, 0, 15, 13) \quad (0, 20, 17, 23)$$
$$(0, 12, 20, 5) \quad (0, 9, 2, 18)$$
$$v = 39: (\infty_1, 0, 4, 31) \quad (\infty_2, 0, 1, 30) \quad (\infty_3, 0, 14, 29) \quad (\infty_4, 0, 17, 6)$$
$$(\infty_5, 0, 18, 9) \quad (\infty_6, 0, 13, 12) \quad (\infty_7, 0, 25, 19) \quad (0, 24, 22, 29)$$
$$(0, 16, 4, 23) \quad (0, 10, 21, 27) \quad (0, 2, 14, 10)$$
$$v = 43: (\infty_1, 0, 1, 27) \quad (\infty_2, 0, 5, 32) \quad (\infty_3, 0, 17, 2) \quad (\infty_4, 0, 29, 28)$$
$$(\infty_5, 0, 8, 20) \quad (\infty_6, 0, 14, 30) \quad (\infty_7, 0, 20, 14) \quad (0, 31, 1, 34)$$
$$(0, 28, 17, 4) \quad (0, 13, 11, 26) \quad (0, 3, 7, 18) \quad (0, 19, 5, 29)$$

$$v = 51: (\infty_1, 0, 7, 3) \quad (\infty_2, 0, 17, 40) \quad (\infty_3, 0, 3, 28) \quad (\infty_4, 0, 19, 7)$$
$$(\infty_5, 0, 35, 30) \quad (\infty_6, 0, 27, 17) \quad (\infty_7, 0, 26, 18) \quad (0, 5, 20, 26)$$
$$(0, 21, 19, 32) \quad (0, 38, 10, 7) \quad (0, 10, 12, 11) \quad (0, 4, 5, 13)$$
$$(0, 22, 2, 30) \quad (0, 20, 6, 35)$$
$$v = 55: (\infty_1, 0, 23, 4) \quad (\infty_2, 0, 11, 5) \quad (\infty_3, 0, 5, 41) \quad (\infty_4, 0, 41, 40)$$
$$(\infty_5, 0, 14, 10) \quad (\infty_6, 0, 9, 30) \quad (\infty_7, 0, 2, 26) \quad (0, 18, 3, 31)$$
$$(0, 30, 20, 45) \quad (0, 4, 19, 16) \quad (0, 19, 6, 28) \quad (0, 7, 17, 9)$$
$$(0, 27, 16, 2) \quad (0, 1, 14, 22) \quad (0, 6, 1, 17)$$
$$v = 63: (\infty_1, 0, 12, 5) \quad (\infty_2, 0, 11, 45) \quad (\infty_3, 0, 19, 18) \quad (\infty_4, 0, 29, 9)$$
$$(\infty_5, 0, 37, 24) \quad (\infty_6, 0, 42, 40) \quad (\infty_7, 0, 20, 41) \quad (0, 9, 27, 12)$$
$$(0, 16, 17, 39) \quad (0, 45, 12, 8) \quad (0, 32, 14, 53) \quad (0, 33, 8, 43)$$
$$(0, 10, 1, 16) \quad (0, 6, 13, 10) \quad (0, 26, 20, 28)$$

An IPMD$(v, 23)$ for $v \in \{71, 75\}$

points: $Z_{v-23} \cup \{\infty_i : 1 \leq i \leq 23\}$
blocks: $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2$ where $\mathbb{B}_1 = \{(i, t+i, 2t+i, 3t+i) : 0 \leq i \leq t-1$ and $t = (v-23)/4\}$ and $\mathbb{B}_2$ is obtained by developing the following modulo $v - 23$:

$$
\begin{aligned}
v = 71:\ & (\infty_1,0,2,47)\quad (\infty_2,0,22,2)\quad (\infty_3,0,19,3)\quad (\infty_4,0,4,44)\\
& (\infty_5,0,26,5)\quad (\infty_6,0,1,6)\quad (\infty_7,0,18,7)\quad (\infty_8,0,23,8)\\
& (\infty_9,0,41,39)\quad (\infty_{10},0,3,10)\quad (\infty_{11},0,42,37)\quad (\infty_{12},0,24,12)\\
& (\infty_{13},0,30,13)\quad (\infty_{14},0,9,34)\quad (\infty_{15},0,16,15)\quad (\infty_{16},0,29,16)\\
& (\infty_{17},0,10,31)\quad (\infty_{18},0,34,30)\quad (\infty_{19},0,38,29)\quad (\infty_{20},0,8,28)\\
& (\infty_{21},0,13,27)\quad (\infty_{22},0,11,26)\quad (\infty_{23},0,6,23)\\
v = 75:\ & (\infty_1,0,2,1)\quad (\infty_2,0,33,6)\quad (\infty_3,0,47,21)\quad (\infty_4,0,8,17)\\
& (\infty_5,0,11,25)\quad (\infty_6,0,15,2)\quad (\infty_7,0,24,12)\quad (\infty_8,0,27,3)\\
& (\infty_9,0,37,23)\quad (\infty_{10},0,32,24)\quad (\infty_{11},0,29,18)\quad (\infty_{12},0,6,13)\\
& (\infty_{13},0,22,45)\quad (\infty_{14},0,20,41)\quad (\infty_{15},0,18,37)\quad (\infty_{16},0,45,42)\\
& (\infty_{17},0,1,36)\quad (\infty_{18},0,3,8)\quad (\infty_{19},0,30,9)\quad (\infty_{20},0,4,47)\\
& (\infty_{21},0,42,38)\quad (\infty_{22},0,16,33)\quad (\infty_{23},0,34,32)\quad (0,12,22,16)
\end{aligned}
$$

## REFERENCES


[1] F. E. Bennett, *Conjugate orthogonal Latin squares and Mendelsohn designs,* Ars Combinatoria **19** (1985), 51–62.

[2] F. E. Bennett and Chen Maorong, *Incomplete perfect Mendelsohn designs,* Ars Combinatoria **31** (1991), 211–216.

[3] F. E. Bennett, K. T. Phelps, C. A. Rodger, and L. Zhu, *Constructions of perfect Mendelsohn designs,* Discrete Math. **103** (1992), 139–151.

[4] F. E. Bennett, Zhang Xuebin, and L. Zhu, *Perfect Mendelsohn designs with block size four,* Ars Combinatoria **29** (1990), 65–72.

[5] T. Beth, D. Jungnickel, and H. Lenz, *Design Theory,* Bibliographisches Institut, Zurich, 1985 (D. Jungnickel, *Design Theory: An update,* Ars Combinatoria **28** (1989), 129–199).

[6] A. E. Brouwer, *The number of mutually orthogonal Latin squares—a table up to order 10000,* Research Report ZW 123/79, Mathematisch Centrum, Amsterdam, 1979.

[7] H. Hanani, *Balanced incomplete block designs and related designs,* Discrete Math. **11** (1975), 255—369.

[8] N. S. Mendelsohn, "A natural generalization of Steiner triple systems," in *Computers in Number Theory,* A. O. L. Atkin and B. J. Birch, (Editors), Academic Press, New York, 1971, pp. 323–338.

[9] N. S. Mendelsohn, "Combinatorial designs as models of universal algebras," in *Recent Progress in Combinatorics,* Academic Press, New York and London, 1969, pp. 123–132.

[10] N. S. Mendelsohn, *Perfect cyclic designs,* Discrete Math. **20** (1977), 63–68.

[11] M. J. Pelling and D. G. Rogers, *Stein quasigroups I: Combinatorial aspects,* Bull. Austral. Math. Soc. **18** (1978), 221–236.

[12] R. M. Wilson, *Constructions and uses of pairwise balanced designs,* Mathematical Centre Tracts **55** (1974), 18–41.

[13] Zhang Xuebin, *On the existence of* $(v, 4, 1)$*-PMD,* Ars Combinatoria **29** (1990), 3–12.

[14] Zhang Xuebin, *Constructions of resolvable Mendelsohn designs,* Ars Combinatoria, to appear.

[15] L. Zhu, *Perfect Mendelsohn designs,* J. Comb. Math. and Comb. Comp. **5** (1989), 43–54.

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