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On extensions of some Flugede–Putnam type theorems involving (p,k)-quasihyponormal, spectral, and dominant operators

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A Hilbert space operator S is called (p,k)-quasihyponormal if $S^{*k}((S^*S)^p - (SS^*)^p)S^k \geq 0$ for an integer $k \geq 1$ and $0 < p \leq 1$. In the present note, we consider (p,k)-quasihyponormal operator $S \in B(\mathcal{H})$ such that $SX = XT$ for some $X \in B(\mathcal{K},\mathcal{H})$ and prove the Fuglede–Putnam type theorems when the adjoint of $T \in B(\mathcal{K})$ is either (p,k)-quasihyponormal or dominant or a spectral operator.

1 Introduction

Let $\mathcal{H}$, $\mathcal{K}$ be complex Hilbert spaces and let $B(\mathcal{H})$, $B(\mathcal{K})$ be the Banach algebras of operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. We use the notations $[\operatorname{ran} S]$ and $(\ker S)^\perp$ to denote, respectively, the closure of the range $\operatorname{ran} S$ and the orthogonal complement of the kernel $\ker S$ of $S \in B(\mathcal{H})$. S is called (p,k)-quasihyponormal if $S^{*k}((S^*S)^p - (SS^*)^p)S^k \geq 0$ for an integer $k \geq 1$ and $0 < p \leq 1$. S is called p-quasihyponormal if S is $(p,1)$-quasihyponormal. S is called p-hyponormal if $(S^*S)^p - (SS^*)^p \geq 0$ and log-hyponormal operator if S is invertible and $\log(S^*S) - \log(SS^*) \geq 0$. S is called dominant if, for $z \in \mathbb{C}$, there exists $M_z \geq 0$ such that $(S-z)^*(S-z) \geq M_z^2(S-z)(S-z)^*$.

The celebrated Fuglede–Putnam theorem ([7], [17]) says that if $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$ are normal operators and if $SX = XT$ for some $X \in B(\mathcal{K},\mathcal{H})$, then $S^*X = XT^*$, $[\operatorname{ran} X]$ reduces S, $(\ker X)^\perp$ reduces T and $S|_{[\operatorname{ran} X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators. Among the various extensions of this theorem for non-normal operators, the most recent one is the following (cf. [3], [10]–[12]).

Proposition 1.1 ([4], [24]) *Let $S \in B(\mathcal{H})$ and $T^* \in B(\mathcal{K})$ be either log-hyponormal or p-hyponormal operators. If $SX = XT$ for some $X \in B(\mathcal{K},\mathcal{H})$, then $S^*X = XT^*$, $[\operatorname{ran} X]$ reduces S, $(\ker X)^\perp$ reduces T and $S|_{[\operatorname{ran} X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

Our basic objective for the exposition is to seek some extensions of the above theorem involving the class of (p,k)-quasihyponormal operators due to Kim [13] along with the well-known classes of dominant and spectral operators. Note that a p-hyponormal operator due to Aluthge [1] is (p,k)-quasihyponormal. For $S \in B(\mathcal{H}), T \in B(\mathcal{K})$ and $X \in B(\mathcal{K},\mathcal{H})$, we say that the FP theorem holds for the triplet (S,X,T) if $SX = XT$ implies $S^*X = XT^*$, $[\operatorname{ran} X]$ reduces S, $(\ker X)^\perp$ reduces T and $S|_{[\operatorname{ran} X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators. Also we say that the FP theorem holds for (S,T) if it holds for (S,X,T) with all $X \in B(\mathcal{K},\mathcal{H})$. Before we initiate our study, it is worth noting that the mere fact thats S is (p,k)-quasihyponormal and T^* is either (p,k)-quasihyponormal or dominant or spectral does not guarantee the validity of the Fuglede–Putnam type theorems. To see this, just consider the operators $S = X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $T = 0$. This will be the motivation

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for us to explore suitable conditions on S or T or X in order to extend some well-known Fuglede–Putnam type theorems.

Proposition 1.2 ([13], [21]) *Let $S \in B(\mathcal{H})$ be (p,k)-quasihyponormal.*

(i) *If the range of S^k is not dense, then*

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad on \quad \mathcal{H} = \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k}$$

where S_1 is p-hyponormal, $S_3^k = 0$ and $\sigma(S) = \sigma(S_1) \cup \{0\}$.

(ii) *If $\mathcal{M} \subset \mathcal{H}$ is an invariant subspace of S, then the restriction $S|_{\mathcal{M}}$ is (p,k)-quasihyponormal.*

Proposition 1.3 (Takahashi [20]) *Let $S \in B(\mathcal{H})$ and let $T \in B(\mathcal{K})$. Then the following assertions are equivalent.*

(i) *If $SX = XT$ where $X \in B(\mathcal{K}, \mathcal{H})$, then $S^*X = XT^*$.*

(ii) *If $SX = XT$ where $X \in B(\mathcal{K}, \mathcal{H})$, then $[\operatorname{ran} X]$ reduces S, $(\ker X)^\perp$ reduces T, and $S|_{[\operatorname{ran}\, X]}, T|_{(\ker X)^\perp}$ are normal.*

Remark 1.4 In (ii), $X_1 : (\ker X)^\perp \ni x \to Xx \in [\operatorname{ran} X]$ is a quasiaffinity (i.e., X_1 is injective and has dense range) such that $S|_{[\operatorname{ran}\, X]} X_1 = X_1 T|_{(\ker X)^\perp}$. Hence $S|_{[\operatorname{ran}\, X]}, T|_{(\ker X)^\perp}$ are unitarily equivalent normal operators by a corollary of the Fuglede–Putnam theorem (see Theorem 1.6.4 of [18] and its proof).

Example 1.5 If T is p-hyponormal and $q < p$, then T is q-hyponormal by the Löwner–Heinz inequality. We consider converse implication. A simple way to make a p-hyponormal operator T which is not q-hyponormal for any $q > p$ (hence it is not subnormal) is to consider a matrix valued unilateral shift. Let $\mathcal{H} = \oplus_0^\infty \mathbb{C}^2 \ni x = \begin{pmatrix} x_0 & x_1 & x_2 & \dots \end{pmatrix}$ and

$$T \begin{pmatrix} x_0 & x_1 & x_2 & \dots \end{pmatrix} = \begin{pmatrix} 0 & A_0 x_0 & A_1 x_1 & A_2 x_2 & \dots \end{pmatrix}.$$

Since

$$T^*T = A_0^* A_0 \oplus A_1^* A_1 \oplus \dots$$

and

$$TT^* = 0 \oplus A_0 A_0^* \oplus A_1 A_1^* \oplus \dots,$$

we have that T is p-hyponormal ($(TT^*)^p \le (T^*T)^p$) if and only if

$$\left(A_j A_j^*\right)^p \le \left(A_j^* A_j\right)^p \quad \text{for} \quad j = 0, 1, 2, \dots$$

We show 2 examples. At first we define $T = T_{B,A}$ by

$$A = \begin{pmatrix} \left(\dfrac{3}{2}\right)^{1/2p} & 0 \\ 0 & \left(\dfrac{3}{4}\right)^{1/2p} \end{pmatrix} = A_1 = A_2 = \dots,$$

$$B = \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{pmatrix} = A_0.$$

Then

$$A^{2p} - B^{2p} = \begin{pmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{4} \end{pmatrix} \ge 0,$$

but

$$A^{2q} - B^{2q} = \begin{pmatrix} \left(\frac{3}{2}\right)^{q/p} - \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \left(\frac{3}{4}\right)^{q/p} - \frac{1}{2} \end{pmatrix} \not\geq 0$$

for any $q > p$ because

$$\det\left(A^{2q} - B^{2q}\right) = \left(\frac{9}{8}\right)^{q/p} \left(1 - \frac{1}{2}\left(\left(\frac{4}{3}\right)^{q/p} + \left(\frac{2}{3}\right)^{q/p}\right)\right) < 0$$

by strict convexity of the function $t^{q/p}$.

Secondly, we define $T = T_{B,A}$ by

$$A = 7^{1/2p} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + 3^{1/2p} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = A_1 = A_2 = \ldots,$$

$$B = \begin{pmatrix} 4^{1/2p} & 0 \\ 0 & 1 \end{pmatrix} = A_0.$$

Then

$$A^{2p} - B^{2p} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \geq 0,$$

but

$$A^{2q} - B^{2q} = 7^{q/p} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + 3^{q/p} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 4^{q/p} & 0 \\ 0 & 1 \end{pmatrix} \not\geq 0$$

for all $q > p$. Because, let $t = q/p$ and

$$\det\left(A^{2q} - B^{2q}\right) = 21^t + 4^t - \frac{1}{2}\left(28^t + 12^t + 7^t + 3^t\right) = f(t).$$

Then $f(1) = 0$ but $f'(1) = \frac{7}{2}\log 7 + \frac{27}{2}\log 3 - 32\log 2 < 0$.

These examples shows that if $0 < p < q$, then a class of q-hyponormal operators strictly includes a class of p-hyponormal operators.

Next, we show 2 examples of (p,k)-quasihyponormal operator T which is not $(q,k-1)$-quasihyponormal for any $q > p$. (It is trivial that if T is $(p,k-1)$-quasihyponormal, then T is (p,k)-quasihyponormal. However there is a $(p,1)$-quasihyponormal operator T which is not $(q,1)$-quasihyponormal for any $0 < q < p$ by [22], [23]. Hence, if $0 < q < p$, then p-hyponormality means q-hyponormality, but $(p,1)$-hyponormality does not mean $(q,1)$-hyponormality. It seems that implication relations for (p,k)-quasihyponrmal operators are complicated.)

At first, we consider above unilateral shift T on $\mathcal{H} = \bigoplus_0^\infty \mathbb{C}^2$. Since T is (p,k)-quasihyponormal if and only if

$$(A_j A_j^*)^p \leq (A_j^* A_j)^p \quad \text{for} \quad j = k, k+1, k+2, \ldots,$$

we have (p,k)-quasihyponormal operator T by defining $A_0, A_1, \ldots, A_{k-1}$ are any, $A_k = B$ and $A = A_{k+1} = A_{k+2} = \ldots$. Also, T is not (p,k)-quasihyponormal for any $q > p$.

Secondly, Proposition 1.2 implies that if T is (p,k)-quasihyponormal, then T has a matrix decomposition $\left(\begin{smallmatrix} T_1 & T_2 \\ 0 & T_3 \end{smallmatrix}\right)$ where T_1 is p-hyponormal and $T_3^k = 0$. Conversely, let T have such matrix decomposition. It seems an interesting problem that to get a good condition of T_2 which means (p,k)-quasinormality of T. Obviously, if $T_2 = 0$, then T is (p,k)-quasihyponormal. Let

$$T = \begin{pmatrix} T_{A,B} & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \bigoplus_0^\infty \mathbb{C}^2 \oplus \bigoplus_0^\infty \mathbb{C}^2$$

where

$$\begin{aligned} T_2 x &= \begin{pmatrix} Cx_0 & 0 & 0 & 0 & \dots \end{pmatrix}, \\ T_3 x &= \begin{pmatrix} 0 & Cx_0 & 0 & 0 & \dots \end{pmatrix} \end{aligned}$$

for $x = \begin{pmatrix} x_0 & x_1 & x_2 & \dots \end{pmatrix} \in \bigoplus_0^\infty \mathbb{C}^2$ with $0 \le C$. Then T is $(p,1)$-quasihypnormal if and only if

$$B\bigl(A^{2p} - B^{2p}\bigr)B \ge 0, \quad C\bigl(B^{2p} - 2^{p+1}C^{2p}\bigr)C \ge 0,$$

and T is $(p,2)$-quasihyponormal if and only if

$$CB\bigl(A^{2p} - B^{2p}\bigr)BC \ge 0.$$

Hence, if $C = \gamma B$ with $\gamma > 2^{-\frac{p+1}{2p}}$, then T is $(p,2)$-quasihyponormal but not $(p,1)$-quasihyponormal. Also, T is not $(q,1)$-quasihyponormal for any $q > p$. By similar way, we can get (p,k)-quasihyponormal operator T which is not $(q,k-1)$-quasihyponormal for any $q > p$.

2 Fuglede–Putnam Theorem

First of all, we would like to list some known results as propositions and to prove a couple of lemmas essential for our work.

Proposition 2.1 ([25], [19]) *Let $S \in B(\mathcal{H})$ be dominant and let $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace of S.*

(i) *Then the restriction $S|_{\mathcal{M}}$ is dominant.*

(ii) *If the restriction $S|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces S.*

Lemma 2.2 *Let $S \in B(\mathcal{H})$ be (p,k)-quasihyponormal. If $S|_{[\operatorname{ran} S^k]}$ is normal, then $\bigl[\operatorname{ran} S^k\bigr]$ reduces S.*

Proof. Decompose S into

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \bigl[\operatorname{ran} S^k\bigr] \oplus \ker S^{*k}$$

as in Proposition 1.2. Let P be the orthogonal projection of $\mathcal{H}$ onto $\bigl[\operatorname{ran} S^k\bigr]$. Then

$$\begin{pmatrix} (S_1^* S_1)^p & 0 \\ 0 & 0 \end{pmatrix} \ge P(S^*S)^p P \ge P(SS^*)^p P \ge P(SPS^*)^p P \ge \begin{pmatrix} (S_1 S_1^*)^p & 0 \\ 0 & 0 \end{pmatrix}$$

by Hansen's inequlity [8] and Löwner–Heinz's inequality [9], [15]. Since $S_1 = S|_{[\operatorname{ran} S^k]}$ is normal, the above inequality means that we can write

$$(SS^*)^p = \begin{pmatrix} (S_1^* S_1)^p & A \\ A^* & B \end{pmatrix}.$$

Let $(SS^*)^{p/2} = \left(\begin{smallmatrix} X & Y \\ Y^* & Z \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} = P(SS^*)^{p/2} P \ge P(SPS^*)^{p/2} P = \begin{pmatrix} (S_1^* S_1)^{p/2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $X \geq (S_1^* S_1)^{p/2}$. On the otherhand,

$$(SS^*)^p = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}^2 = \begin{pmatrix} X^2 + YY^* & XY + YZ \\ Y^*X + ZY^* & Y^*Y + Z^2 \end{pmatrix},$$

we have

$$(S_1^* S_1)^p = X^2 + YY^* \geq X^2.$$

Hence $Y = 0$ and

$$(SS^*)^{p/2} = \begin{pmatrix} (S_1^* S_1)^{p/2} & 0 \\ 0 & Z \end{pmatrix}.$$

Hence

$$SS^* = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \begin{pmatrix} S_1^* & 0 \\ S_2^* & S_3^* \end{pmatrix} = \begin{pmatrix} S_1 S_1^* + S_2 S_2^* & S_2 S_3^* \\ S_3 S_2^* & S_3 S_3^* \end{pmatrix} = \begin{pmatrix} S_1^* S_1 & 0 \\ 0 & Z^{2/p} \end{pmatrix}.$$

Thus $S_2 S_2^* = 0$ and $S_2 = 0$. □

Lemma 2.3 ([14, Lemma 10]) *Let $S \in B(\mathcal{H})$ be (p, k)-quasihyponormal and $\mathcal{M}$ an invariant subspace of S. If $S|_{\mathcal{M}}$ is an injective normal operator, then $\mathcal{M}$ reduces S.*

Remark 2.4 In Lemma 2.3, if the injectivity of $S|_{\mathcal{M}}$ is abolished, $\mathcal{M}$ may not reduce S. Take any nilpotent operator S with $S^{k-1} \neq 0 = S^k$. Then $S|_{[\mathrm{ran}\ S^{k-1}]} = 0$ is normal. If $\left[\mathrm{ran}\ S^{k-1}\right]$ reduces S, then $S^* S^{k-1} \mathcal{H} \subset \left[\mathrm{ran}\ S^{k-1}\right]$. Hence $S^{*k-1} S^{k-1} \mathcal{H} \subset \left[\mathrm{ran}\ S^{k-1}\right]$ and

$$\ker S^{k-1} = \ker S^{*k-1} S^{k-1} \supset \ker S^{*k-1}.$$

Since $S^{*k} = S^{*k-1} S^* = 0$, we have $S^{k-1} S^* = 0$. Hence $S^{k-1} S^{*k-1} = 0$, and hence $S^{k-1} = 0$. This is a contradiction.

First we consider the situation such that S and T^* are (p, k)-quasihyponormal operators.

Theorem 2.5 *Let $S \in B(\mathcal{H})$ and $T^* \in B(\mathcal{K})$ be (p, k)-quasihyponormal operators. If either S or T^* is injective, then the FP-theorem holds for (S, T).*

Proof. Decompose S, T, and X into

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = [\mathrm{ran}\ X] \oplus \ker X^*,$$

$$T = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = (\ker X)^\perp \oplus \ker X,$$

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : (\ker X)^\perp \oplus \ker X \longrightarrow [\mathrm{ran}\ X] \oplus \ker X^*.$$

Then the equation $SX = XT$ gives

$$S_1 X_1 = X_1 T_1, \tag{2.1}$$

where S_1 and T_1^* are (p, k)-quasihyponormal operators by Proposition 1.2 and X_1 is a quasiaffinity. Assume first that S is injective. Obviously, then S_1 is injective. From (2.1), it is easy to see that T_1 is injective or equivalently, $\mathrm{ran}\ T_1^*$ is dense. Incidentally, T_1^* comes out to be a p-hyponormal operator. In particular, $\ker T_1^* \subset \ker T_1$ and so $\ker T_1^* = 0$. Using (2.1), it is not difficult to show that S_1^* is injective, thereby S_1 is p-hyponormal. Next, assume that T^* is injective. Then T_1^*, and hence by (2.1), S_1^* is injective. Clearly S_1 is an injective p-hyponormal operator and therefore T_1 is injective, again by (2.1). As a consequence of this, T_1^* is p-hyponormal. Thus we have shown that if either S or T^* is injective, then S_1 and T_1^* are both p-hyponormal operators. Applying Proposition 1.1 to (2.1), we derive $S_1^* X_1 = X_1 T_1^*$ and S_1, T_1 are normal operators. Since S_1 and T_1 are injective, Lemma 2.3 tells us that $S_2 = T_2 = 0$, so it completes the proof. □

Theorem 2.6 *Let $S \in B(\mathcal{H})$ be p-hyponormal and $T^* \in B(\mathcal{K})$ (p,k)-quasihyponormal. If $X \in B(\mathcal{K},\mathcal{H})$ is injective, then the FP-theorem holds for (S,X,T).*

P r o o f. If T is injective, then T^* is p-hyponormal and so the result follows from Proposition 1.1. Assume that $\ker T \neq \{0\}$. Decompose X and T as

$$X = \begin{pmatrix} X_1 & X_2 \end{pmatrix} : \left[\operatorname{ran} T^{*k}\right] \oplus \ker T^k \longrightarrow \mathcal{H},$$

and

$$T = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = \left[\operatorname{ran} T^{*k}\right] \oplus \ker T^k.$$

Then the condition $SX = XT$ is equivalent to

$$SX_1 = X_1T_1 + X_2T_2, \tag{2.2}$$

and

$$SX_2 = X_2T_3. \tag{2.3}$$

Since $T_3^k = 0$, we deduce $S^kX_2 = 0$ from (2.3). Therefore the assumption that S is p-hyponormal implies $X_2T_3 = SX_2 = 0$. The injectivity of X_2 gives $T_3 = 0$. Hence $\ker T = \ker T^k$, thereby T^* is p-quasihyponormal. In particular, T_1^* is p-hyponormal by Lemma 1 of [23]. Now combining the equation $SX_2 = 0$ with (2.2), we obtain $S(SX_1) = (SX_1)T_1$. Since S and T_1^* are p-hyponormal, the FP theorem holds for (S, SX_1, T_1) by Proposition 1.1. Next we assert that T is normal. First we verify the relation $\ker SX_1 = \ker T_1$. If $T_1x = 0$, then the equation $S(SX_1 - X_1T_1) = 0$ gives $S_2X_1x = 0$ and hence $SX_1x = 0$ or $x \in \ker SX_1$. On the other hand if $SX_1x = 0$, then by (2.2), we have

$$0 = X_1T_1x + X_2T_2x = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} T_1x \\ T_2x \end{pmatrix}.$$

Because X is injective, $T_1x = 0 = T_2x$. This finishes the verification of $\ker SX_1 = \ker T_1$. Therefore $\ker T_1$ reduces T_1 and $T|_{\ker T_1}$ and hence T_1 is normal. Now Lemma 2.2 tells us that $T_2 = 0$. Thus $T = T_1 \oplus 0$ is normal, so it completes the proof. □

Theorem 2.7 *Let $S \in B(\mathcal{H})$ and $T^* \in B(\mathcal{K})$ be (p,k)-quasihyponormal operators with reducing kernels. Then the FP theorem holds for (S,T).*

P r o o f. Decompose S and T^* into normal and pure parts as

$$S = S_1 \oplus S_2 \quad \text{on} \quad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

and

$$T^* = T_1^* \oplus T_2^* \quad \text{on} \quad \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2,$$

where S_1, T_1^* are normal parts and S_2, T_2 pure parts. Notice that S_2, T_2^* are injective due to the underlying kernel conditions on S and T^*. Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} : \mathcal{K}_1 \oplus \mathcal{K}_2 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Then $SX = XT$ implies

$$\begin{pmatrix} S_1X_1 & S_1X_2 \\ S_2X_3 & S_2X_4 \end{pmatrix} = \begin{pmatrix} X_1T_1 & X_2T_2 \\ X_3T_1 & X_4T_2 \end{pmatrix}.$$

Since S_2 is an injective (p,k)-quasihyponormal operator and T_1 normal, an application of Theorem 2.5 to the relation $S_2X_3 = X_3T_1$ yields $S_2^*X_3 = X_3T_1^*$, $[\operatorname{ran} X_3]$ reduces S_2 and $S_2|_{[\operatorname{ran} X_3]}$ is normal. Consequently, $X_3 = 0$ as S_2 is pure. In the same vein, one can derive $X_2 = 0$ from $S_1X_2 = X_2T_2$ and $X_4 = 0$ from $S_2X_4 = X_4T_2$. Since $S_1X_1 = X_1T_1$ and S_1, T_1 are normal, the theorem follows from Proposition 1.1. □

Theorem 2.8 *Let $S \in B(\mathcal{H})$ and $T^* \in B(\mathcal{K})$ be (p,k)-quasihyponormal operators. Suppose* $\ker S \subset \ker S^{*k}$. *If $X \in B(\mathcal{K},\mathcal{H})$ satisfies* $\ker S^{*k} \subset \ker X^*$*, then the FP theorem holds for (S,X,T).*

P r o o f. Decompose S, T, and X into

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k},$$
$$T = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = (\ker X)^\perp \oplus \ker X,$$
$$X = \begin{pmatrix} X_1 & 0 \\ X_2 & 0 \end{pmatrix} : (\ker X)^\perp \oplus \ker X \longrightarrow \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k}.$$

First we show $X_2 = 0$. Let $x \in (\ker X)^\perp$. Then $X_2 x \in \ker S^{*k} \subset \ker X^*$. Hence

$$X^* X_2 x = \begin{pmatrix} X_1^* & X_2^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ X_2 x \end{pmatrix} = \begin{pmatrix} X_2^* X_2 x \\ 0 \end{pmatrix} = 0.$$

This implies $X_2 = 0$. Since $SX = XT$, we have $S_1 X_1 = X_1 T_1$. Furthermore, X_1 is injective. By our hypothesis,

$$\ker S_1 \subset \ker S \cap \left[\operatorname{ran} S^k\right] \subset \ker S^{*k} \cap \left[\operatorname{ran} S^k\right] = \{0\}.$$

This enable us to apply Theorem 2.5 to conclude that $S_1^* X_1 = X_1 T_1^*$, $[\operatorname{ran} X_1]$ reduces S_1, $(\ker X_1)^\perp$ reduces T_1, and $S_1|_{[\operatorname{ran} X_1]}, T_1|_{(\ker X_1)^\perp} = T_1$ are unitarily equivalent normal operators. Hence $T_2 = 0$ by Lemma 2.3. Next, decompose S, X into

$$S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{13} \end{pmatrix} \quad \text{on} \quad \mathcal{H} = [\operatorname{ran} X] \oplus \ker X^*,$$
$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} : (\ker X)^\perp \oplus \ker X \longrightarrow [\operatorname{ran} X] \oplus \ker X^*.$$

Then $S_{11} = S|_{[\operatorname{ran} X]} = S_1|_{[\operatorname{ran} X_1]}$ is an injective normal operator. Hence $S_{12} = 0$ by Lemma 2.3. Since $SX = XT$, we have $S_{11} X_{11} = X_{11} T_1$. Then $S_{11}^* X_{11} = X_{11} T_1^*$ by Proposition 1.1 and $S^* X = X T^*$. □

Remark 2.9 If S is assumed to be p-quasihypormal, then the condition "$\ker S \subset \ker S^{*k}$" being used only to show the injectivity is not needed. To see this, note that $S^k = \begin{pmatrix} S_1^k & S_1 S_2^{k-1} \\ 0 & 0 \end{pmatrix}$. Since $\ker S_1$ reduces S_1, it is easy to see from the decomposition S^k that $\ker S_1 \subset \ker S^{*k}$. This inclusion will imply the injecitivity of S_1.

According to Theorem 3 of [24], the FP theorem holds if S is either p-hyponormal or log-hyponormal and T^* dominant. Here consider a situation in which S is (p,k)-quasihyponormal.

Theorem 2.10 *Let $S \in B(\mathcal{H})$ be (p,k)-quasihyponormal and let $T^* \in B(\mathcal{K})$ be dominant. If either S or T^* is injective, then the FP theorem holds for (S,T).*

P r o o f. Decompose S, T, X into

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = [\operatorname{ran} X] \oplus \ker X^*,$$
$$T = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = (\ker X)^\perp \oplus \ker X,$$
$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : (\ker X)^\perp \oplus \ker X \longrightarrow [\operatorname{ran} X] \oplus \ker X^*.$$

Then the implication of the hypothesis $SX = XT$ is

$$S_1 X_1 = X_1 T_1. \tag{2.4}$$

Note that S_1 is (p,k)-quasihyponormal by Proposition 1.2, T_1^* is dominant by Proposition 2.1 and X_1 is a quasiaffinity. First consider the case when S is injective. Then S_1 is injective. Therefore, by the virtue of (2.4), T_1 is also injective. Since T_1^* is dominant, it turns out to be injective. Once again we invoke (2.4) to infer that S_1^* is injective and therefore S_1 is p-hyponormal. Applying Theorem 3 of [24] to (2.4), we find $S_1^*X_1 = X_1T_1^*$, where S_1 and T_1 are normal operators. Since S_1 is injective, Lemma 2.3 shows that $S_2 = 0$. Also $T_2 = 0$ by Proposition 2.1. Now the desired conclusion is immediate. Next we consider the case in which T^* is injective. Clearly T_1^* and therefore S_1^* is injective from (2.4). Ultimately, S_1 turns out to be p-hyponormal. Conclude as before that $S_1^*X_1 = X_1T_1^*$, where S_1 and T_1 are injective normal operators and hence $S_2 = T_2 = 0$. This establishes the proof. □

Corollary 2.11 *Let $S \in B(\mathcal{H})$ be a (p,k)-quasihyponormal operator with reducing kernel and let $T^* \in B(\mathcal{K})$ be a dominant operator. Then the FP theorem holds for (S,T).*

Proof. By our hypothesis, $S = S_1 \oplus 0$ with respect to $\mathcal{H} = (\ker S)^\perp \oplus \ker S$ and $T = T_1 \oplus 0$ with respect to $\mathcal{K} = (\ker T)^\perp \oplus \ker T$. Decompose X into

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} : (\ker T)^\perp \oplus \ker T \longrightarrow (\ker S)^\perp \oplus \ker S.$$

Then

$$S_1X_1 = X_1T_1, \tag{2.5}$$

$$S_1X_2 = 0, \tag{2.6}$$

and

$$X_3T_1 = 0. \tag{2.7}$$

Since both S_1 and T_1^* are injective, (2.6) and (2.7) will imply $X_2 = X_3 = 0$. Also an application of Theorem 2.10 to (2.5) gives $S_1^*X_1 = X_1T_1^*$. Combining all these facts, we arrive at the desired conclusion. □

Corollary 2.12 *Let $S \in B(\mathcal{H})$ be dominant and let $T^* \in B(\mathcal{K})$ be (p,k)-quasihyponormal. If either S is injective or $\ker T^*$ reduces T^*, then the FP theorem holds for (S,T).*

Proof. Since $SX = XT$, we have $T^*X^* = X^*S^*$. Hence $T^{**}X^* = X^*S^{**}$ by Theorem 2.10. Thus $S^*X = XT^*$. The rest follows from Proposition 1.3. □

Theorem 2.13 *Let $S \in B(\mathcal{H})$ be a (p,k)-quasihyponormal operator and let $N \in B(\mathcal{K})$ be a normal operator. Let $X \in B(\mathcal{K},\mathcal{H})$ have a dense range and $SX = XN$. Then S and N are unitarily equivalent normal operators and $S^*X = XN^*$.*

Proof. Decompose S, N into

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k},$$

and

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = [\operatorname{ran} N] \oplus \ker N^*.$$

Since $S^kX = XN^k$ and X has a dense range, we have

$$\left[X\left(\left[\operatorname{ran} N^k\right]\right)\right] = \left[\operatorname{ran} S^k\right].$$

We remark $\left[\operatorname{ran} N^k\right] = [\operatorname{ran} N]$ as N is normal. Define

$$X_1 : [\operatorname{ran} N] \ni x \longrightarrow Xx \in \left[\operatorname{ran} S^k\right].$$

Then X_1 has a dense range and

$$X_1 N_1 x = X N x = S X x = S_1 X_1 x$$

for $x \in [\operatorname{ran} N]$. Hence

$$S_1 X_1 = X_1 N_1 .$$

Since S_1 is p-hyponormal, there exists a quasiaffinity Y and a hyponormal operator $\tilde{S}_1$ such that

$$\tilde{S}_1 Y = Y S_1$$

(see [11, p. 310]. $\tilde{S}_1$ is the Aluthge transform of S_1 if $1/2 \le p$ and the second Aluthge transform of S_1 if $0 < p < 1/2$.) Hence

$$\tilde{S}_1 Y X_1 = Y S_1 X_1 = Y X_1 N_1$$

and YX_1 has a dense range. Then $\tilde{S}_1$ is normal by Proposition 1.1. This implies that $S_1 = S|_{[\operatorname{ran} S^k]}$ is normal by [16]. Consequently, $\left[\operatorname{ran} S^k\right]$ reduces S and $S_2 = 0$ by Lemma 2.3. Since $X[\operatorname{ran} N] \subset \left[\operatorname{ran} S^k\right]$, we can write

$$X = \begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix} : [\operatorname{ran} N] \oplus \ker N^* \longrightarrow \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k} .$$

Then $SX = XN$ implies

$$\begin{pmatrix} S_1 X_1 & S_1 X_2 \\ 0 & S_3 X_3 \end{pmatrix} = \begin{pmatrix} X_1 N_1 & 0 \\ 0 & 0 \end{pmatrix} .$$

Since X has a dense range, X_3 has a dense range. Hence $S_3 X_3 = 0$ implies $S_3 = 0$. Thus $S = \left(\begin{smallmatrix} S_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is normal and the rest follows from Proposition 1.1. □

Finally we consider the Fuglede–Putnam type theorems in case S is a (p, k)-quasihyponormal operator and T, a spectral operator.

Theorem 2.14 *Let $S \in B(\mathcal{H})$ be a (p, k)-quasihyponormal operator and let $T \in B(\mathcal{K})$ be a spectral operator with $\ker T^* \subset \ker T$. If $SX = XT$ for a quasiaffinity $X \in B(\mathcal{K}, \mathcal{H})$, then $S^* X = X T^*$, S is normal and T a scalar operator similar to S.*

Proof. The kernel condition on T makes it obvious that $\ker T^* = \ker T^{*k}$ or $[\operatorname{ran} T] = \left[\operatorname{ran} T^k\right]$. Now the relation $SX = XT$ combined with the assumption on X forces $S^k X = X T^k$ and therefore $\left[\operatorname{ran} S^k\right] = \left[X\left[\operatorname{ran} T^k\right]\right] = [X[\operatorname{ran} T]]$. This leads to the decomposition of X into

$$X = \begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix} : [\operatorname{ran} T] \oplus \ker T^* \longrightarrow \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k} ,$$

where X_1 a quasiaffinity. Decompose S and T as

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k} ,$$

and

$$T = T_1 \oplus 0 \quad \text{on} \quad \mathcal{K} = [\operatorname{ran} T] \oplus \ker T^* .$$

Note that S_1 is p-hyponormal and T_1 a spectral operator. Because it is known that $T|_{\mathcal{M}}$ is spectral if $E(\tau)\mathcal{M} \subset \mathcal{M}$ for all Borel set $\tau \subset \mathbb{C}$ where $E(\quad)$ denotes the resolution of the identity of T (Theorem 12.3 of [5]). Moreover, $SX = TX$ if and only if

$$S_1 X_1 = X_1 T_1 , \tag{2.8}$$

$$S_3X_3 = 0, \tag{2.9}$$

and

$$S_1X_2 + S_2X_3 = 0. \tag{2.10}$$

Applying Theorem 11 of [10] to (2.8), one can see that $S_1^*X_1 = X_1T_1^*$, S_1 is a normal operator and T_1 is a scalar operator similar to S_1. Evidently, $S_2 = 0$ in view of Lemma 2.2. Since X_3 has a dense range, (2.9) gives $S_3 = 0$. Thus the proof is over. □

Another theorem in the same spirit is:

Theorem 2.15 *Let $S \in B(\mathcal{H})$ be a (p,k)-quasihyponormal operator for which $\ker S = \ker S^2$ and let $T \in B(\mathcal{K})$ be a spectral operator. If $SX = XT$ for a quasiaffinity $X \in B(\mathcal{K},\mathcal{H})$, then $S^*X = XT^*$, S is normal and T a scalar operator similar to S.*

P r o o f. Consider decompositions of S and T as

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k},$$

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = \left[\operatorname{ran} T^k\right] \oplus \ker T^{*k}.$$

Since $S^kX = XT^k$, $X\left[\operatorname{ran} T^k\right] = \left[\operatorname{ran} S^k\right]$. This enables us to decompose X as

$$X = \begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix} : \left[\operatorname{ran} T^k\right] \oplus \ker T^{*k} \longrightarrow \left[\operatorname{ran} S^k\right] \oplus \ker S^{*k}.$$

Clearly, $X_1T_1^k = S_1^kX_1$. Observe that X_1 is a quasiaffinity and T_1 is a spectral operator. Since S_1 is p-hyponormal, S_1^k is p/k-hyponormal by Theorem 2 of [2]. Therefore the equation $X_1T_1^k = S_1^kX_1$ will imply that S_1^k is a normal operator and T_1^k is a scalar operator similar to S_1^k. By Corollary 2 of [2], we conclude that S_1 is normal and hence $S_2 = 0$ by Lemma 2.2. Since $S_3^k = 0$, $S^{*k}x = 0$ implies $S^kx = \begin{pmatrix} S_1^k & 0 \end{pmatrix} \left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) = 0$; thus $\ker S^{*k} \subset \ker S^k$. Therefore, by using the kernel condition on S, we arrive at the inclusion $\ker S^* \subset \ker S$. This, in turn, will imply $\ker S^* = \ker S^{*k}$. In particular, $\ker S_3^* = \ker S_3^{*k}$. As a consequence of this, we find $S_3 = 0$, forcing S to be normal. Now we finish the argument by invoking Corollary 4 of [6]. □

Corollary 2.16 *Let $S \in B(\mathcal{H})$ be p-quasihyponormal and let $T \in B(\mathcal{K})$ be a spectral operator. If $SX = XT$ for a quasiaffinity $X \in B(\mathcal{K},\mathcal{H})$, then $S^*X = XT^*$, S is normal and T a scalar operator similar to S.*

P r o o f. It is clear that $\|(S^*S)^{p/2}Sx\| \geq \|(SS^*)^{p/2}Sx\|$ for all $x \in \mathcal{H}$; in particular $\ker S = \ker S^2$. □

Corollary 2.17 *Let $S \in B(\mathcal{H})$ be (p,k)-quasihyponormal and let $T \in B(\mathcal{K})$ be a spectral operator with $\ker T^* = \ker T^{*2}$. If $SX = XT$ for some quasiaffinity $X \in B(\mathcal{K},\mathcal{H})$, then S is normal and T is a scalar operator similar to S.*

P r o o f. Since $SX = XT$, $X^*S^{*2} = T^{*2}X^*$. We remark $\ker S^* = \ker S^{*2}$. Because if $S^{*2}x = 0$, then $T^{*2}X^*x = 0$. Hence $T^*X^*x = 0$ by the assumption. Then $X^*S^*x = 0$ and $S^*x = 0$. Thus $\ker S^* = \ker S^{*k}$ or $[\operatorname{ran} S] = \left[\operatorname{ran} S^k\right]$. This implies S is p-quasihyponormal. Thus the result follows from Corollary 2.16. □

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