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A Sampling Theorem and Wintner's Results on Fourier Coefficients

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1. INTRODUCTION

In a recent paper J. R. Higgins [6] surveyed the Shannon sampling theorem and its variants. The present paper arises from the reconsideration, aimed towards a sampling theorem, of the book of Wintner [9]. The idea of this book was to obtain precise formulae for the Fourier coefficients of a function from a countable set of its values on the unit circle. The very early workers in the field had obtained formal results of this kind and Wintner was attempting to give rigorous proofs. He succeeded in this for the Fourier cosine coefficients but concluded that the Fourier sine coefficients were more elusive.

The main theorem of the present paper is the following: Let $\omega_{kn} = e^{2\pi i/kn}$, denote the Möbius function by μ, and let H^1 denote the Hardy space.

THEOREM. *If $f(z) \in H^1(U)$, where U: $|z| < 1$ and $f'(z) \in \mathrm{Lip}_1$ on $|z| = 1$, then the Taylor coefficients of $f(z)$ are given by*

$$c_n = \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m).$$

In Section 2 we derive this theorem in two ways. The first uses a modification to Wintner's approach and to do this we summarize [9] where appropriate. The second stems from [7], in which the present authors obtained the above theorem for a function analytic inside the unit disk. The formula which holds on any circle of radius < 1 is extended to the boundary using a theorem of Hardy and Littlewood.

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2. The Sampling Theorem

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function of period 1, and define

$$s_n(x) = \frac{1}{n} \sum_{m=1}^{n} f\left(x + \frac{m}{n}\right). \tag{1}$$

Then $s_n(x)$ has period 1 and if f is Riemann-integrable,

$$\lim_{n \to \infty} s_n(x) = \int_x^{x+1} f(t)\, dt = \int_0^1 f(t)\, dt. \tag{2}$$

Concerning the convergence of the s_n's to the integral, we have (cf. [9, p. 4]):

Proposition 1. *If a function $f(x)$ of period 1 satisfies $f' \in \mathrm{Lip}_1$, then*

$$\left| \int_0^1 f(t)\, dt - \frac{1}{n} \sum_{m=1}^{n} f\left(x + \frac{m}{n}\right) \right| \leqslant C/n^2$$

uniformly in x, where C is the Lipschitz constant.

Recall: $f \in \mathrm{Lip}_\alpha$ on $[a, b]$, with $0 < \alpha \leqslant 1$, if there is a constant $C > 0$ such that

$$|f(x) - f(y)| \leqslant C|x - y|^\alpha$$

for all x, y in $[a, b]$.

Of central importance in the sequel is the Möbius function, μ, which has the property that

$$\sum_{d \mid m} \mu(d) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m > 1. \end{cases}$$

A main result of Wintner [9, p. 6] is the following connection, via the Möbius function, between number theory and Fourier series.

Theorem 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function of period 1 such that $f' \in \mathrm{Lip}_1$. Normalize f by taking*

$$a_0 = \int_x^{x+1} f(t)\, dt = \int_0^1 f(t)\, dt = 0,$$

and let $s_n(x)$ *be given as in* (1). *Then each of the series*

$$g_n(x) = \sum_{k=1}^{\infty} \mu(k)\, s_{nk}(x), \qquad n = 1, 2, 3, \ldots \tag{3}$$

is absolutely-uniformly convergent and

$$f(x) = \sum_{n=1}^{\infty} g_n(x)$$

is absolutely-uniformly convergent, where

$$g_n(x) = a_n \cos 2\pi nx + b_n \sin 2\pi nx.$$

Instrumental in the proof of the convergence of (3) is the error estimate given by Proposition 1.

Furthermore, taking $x = 0$, the theorem, together with (1), shows that the Fourier cosine coefficients of f are given by the Bruns formula

$$a_n = \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f\left(\frac{m}{kn}\right), \qquad n = 1, 2, 3, \ldots, \tag{4}$$

which was apparently first published in [2].

In order to consider functions in $|z| \leqslant 1$ we introduce a change of notation. Letting $\omega_{kn} = e^{2\pi i/kn}$, (4) becomes

$$a_n = \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m), \tag{5}$$

whenever $f(z)$ is real-valued of period 2π on $|z| = 1$, such that $f'(z) \in \mathrm{Lip}_1$.

In [7] the present authors obtained precisely the representation of (5) for the *Taylor coefficients* of an analytic function in $|z| < 1$. The approach in [7] was entirely different from that of Wintner, and relied heavily on an obscure result due to J. L. Walsh [8]. The connection between the two approaches lies in the following.

THEOREM 2. *If* $f(z) \in H^1(U)$, *where* $U\colon |z| < 1$, *and* $f'(z) \in \mathrm{Lip}_1$ *on* $|z| = 1$, *then the Taylor coefficients of* $f(z)$ *are given by*

$$c_n = \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m).$$

Proof. The hypotheses imply $f(z)$ has the Fourier series expansion at points on $|z|=1$,

$$f(e^{i\theta})=\sum_{n=0}^{\infty} c_n e^{in\theta},$$

where $c_{-n}=0$ for $n=1, 2, 3, \ldots$, and moreover, the c_n's are also the Taylor coefficients (cf. [4]). Then

$$\begin{aligned} f(e^{i\theta}) &= \sum_{n=0}^{\infty} (a_n+ib_n)\, e^{in\theta} \\ &= \sum_{n=0}^{\infty} (a_n \cos n\theta - b_n \sin n\theta) \\ &\quad + i \sum_{n=0}^{\infty} (b_n \cos n\theta + a_n \sin n\theta) \\ &= u(e^{i\theta}) + iv(e^{i\theta}). \end{aligned}$$

By (5) applied to the functions u and v, respectively, as $u', v' \in \mathrm{Lip}_1$,

$$a_n=\sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} u(\omega_{kn}^m), \qquad b_n=\sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} v(\omega_{kn}^m),$$

for $n=1, 2, 3, \ldots$. It follows that

$$c_n=a_n+ib_n=\sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m),$$

as desired.

Thus, under the hypotheses of the theorem, this result can be viewed as an extension to $|z|=1$ of the formula given by the authors in [7] for the Taylor coefficients of any analytic $f(z)$ in U. In fact we now give a second proof of Theorem 2 using this idea.

A Second Proof. Since f' satisfies a Lipschitz condition on $|z|=1$, it follows by the theorem of Hardy and Littlewood [5, p. 413] that there exists a constant C such that

$$|f'(re^{i\theta})-f'(re^{i\theta'})| \leqslant C|re^{i\theta}-re^{i\theta'}|, \qquad 0<r\leqslant 1.$$

Proposition 1 can then be applied to each circle of radius r and

$$\left|\frac{1}{kn} \sum_{m=1}^{kn} f(r\omega_{kn}^m)\right| \leqslant \frac{C}{k^2n^2}, \qquad 0<r\leqslant 1 \text{ (again taking } a_0=0). \tag{6}$$

The present authors' version of the theorem for analytic functions [7] gives

$$c_n = \frac{1}{r^n} \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(r\omega_{kn}^m), \qquad 0 < r < 1.$$

Let $\varepsilon > 0$ be given. Then by (6) there exist r_0 fixed and N sufficiently large such that for $r \geqslant r_0$,

$$\begin{aligned} &\left| c_n - \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m) \right| \\ &\quad = \left| \frac{1}{r^n} \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(r\omega_{kn}^m) - \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m) \right| \\ &\quad < \frac{\varepsilon}{2} + \left| \frac{1}{r^n} \sum_{k=1}^{N} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(r\omega_{kn}^m) - \sum_{k=1}^{N} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m) \right|. \end{aligned}$$

Since the second term involves a finite sum it can be made less than $\varepsilon/2$ for r sufficiently close to 1. Hence we can conclude that

$$\left| c_n - \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m) \right| < \varepsilon$$

and the proof is complete.

Remarks. (i) Suppose $f' \in \mathrm{Lip}_1$ on $|z| = 1$ and its Fourier coefficients satisfy $c_n = 0$, $n < -k$, $k > 0$. Then $g(z) = z^k f(z)$ satisfies the hypotheses of Theorem 2. By applying this theorem to g one can obtain formulae for c_n, $n \geqslant -k$, in terms of $f(\omega_{kn}^m)$, $1 \leqslant k < \infty$, $1 \leqslant m \leqslant kn$.

(ii) Setting $x = 1/4n$ in $g_n(x) = a_n \cos 2\pi nx + b_n \sin 2\pi nx$ we obtain a formula for b_n on another set of points. Thus a real-valued harmonic function satisfying the hypotheses of Theorem 2 can be recaptured from its values on two separate countable sets of points. This seems to parallel the results [1, 3] concerning the Shannon sampling theorem.

Regarding the error due to truncating the infinite series for the c_n's, we have:

PROPOSITION 2. *Let $f(z) \in H^1(U)$, $f'(z) \in \mathrm{Lip}_1$ on $|z| = 1$ with Lipschitz constant C. Then the truncation error satisfies*

$$\left| c_n - \sum_{k=1}^{N} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m) \right| \leqslant \frac{C}{n^2} \sum_{k=N+1}^{\infty} \frac{1}{k^2}.$$

Proof.

$$\begin{aligned}
&\left| c_n - \sum_{k=1}^{N} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m) \right| \\
&\quad = \left| \sum_{k=N+1}^{\infty} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(\omega_{kn}^m) \right| \\
&\quad \leqslant \frac{C}{n^2} \sum_{k=N+1}^{\infty} \frac{1}{k^2},
\end{aligned}$$

applying Theorem 2 and Proposition 1, respectively.

Remark. If $f(z)$ is simply analytic in $|z|<1$, then on any circle $|z|=r<1$, $f'(z)$ satisfies a Lipschitz condition

$$|f'(re^{i\theta}) - f'(re^{i\theta'})| \leqslant C_r |re^{i\theta} - re^{i\theta'}|.$$

Hence, computing the Taylor coefficients of $f(z)$ via $|z|=r$, the truncation error likewise satisfies

$$\begin{aligned}
&\left| c_n - \frac{1}{r^n} \sum_{k=1}^{N} \frac{\mu(k)}{kn} \sum_{m=1}^{kn} f(r\omega_{kn}^m) \right| \\
&\quad \leqslant \frac{C_r}{r^n n^2} \sum_{k=N+1}^{\infty} \frac{1}{k^2}.
\end{aligned}$$

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