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THE GLOBAL ELEMENT METHOD APPLIED TO FLUID FLOW PROBLEMS

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Abstract—The Global Element Method is used to solve the steady parallel fully developed flow in ducts of various cross sections, including some singular cases. Also considered are pulsatile flow and a coupled problem combining heat and fluid flow.

1. INTRODUCTION

Laminar flow of viscous fluids through uniform straight ducts of general cross section was treated by O'Brien [1] using a finite difference method. Non uniform grids were used to deal with curved boundaries.

However, Duck [2] pointed out that near sharp corners in the cross section, singularities in the flow could cause significant errors, and presented a method for treating the flow in ducts of "pinched" cross section, using conformal mapping onto a semicircle.

In the present paper the Global element Method (GEM) [3–5] is applied to two singular cross sections treated by Duck [2], the bifurcating Cassinian and the pinched circle.

Analytical results on the flow between non-coaxial circular cylinders have been known for some time (Heyda [6]; Redberger and Charles [7]), while MacDonald [8] has extended these to give exact formulae which relate the rate of mass flow to the axial pressure gradient. The wall shear stress and the drag of the fluid on unit length of each cylinder are also treated. In the present paper these exact results are employed as a further check on the accuracy of the Global Element Method.

A coupled version of the GEM program is employed to treat the fluid flow and heat transfer in a duct of cuspidal cross section simulating a blocked pressurised water reactor channel (Hassan and Barrow [9]).

The purpose of the paper is to illustrate that the Global Element Method is of wide application in fluid flow problems, including singular cases, and that new geometries and modifications to the equations or boundary conditions may be incorporated with great ease. Such problems are of wide interest and cannot always be treated by analytic or semi-analytic methods; where numerical methods are applied, the singularities in the plane inhibit the achievement of high accuracy. We treat three examples here to show that the Global Element Method provides a flexible and easy-to-use tool capable of high accuracy for a wide range of flow problems.

2. THE GLOBAL ELEMENT METHOD (GEM)

The Global Element Method (Delves and Hall [3]) is a variational (Ritz–Galerkin) technique which combines the advantages of global methods (very rapid convergence for simple regions) with the ability to handle complicated geometries. The region is divided into a (small) number of subregions, on each of which a rapidly convergent expansion is constructed. Continuity across the boundaries between regions is treated, together with the other boundary conditions, by incorporating suitable terms in the functional. Details of the functional are given in Delves and Hall [3], while Delves and Phillips [4] give more information on the numerical techniques employed.

The GEM2 program as presently implemented solves second order elliptic partial differential equations of the form:

$$-\nabla A(x)\nabla u + B(x)u + \mathbf{C}(x)\cdot\nabla u = g,$$

presenting the solution in the form of a Chebyshev expansion in each subregion (Delves and Phillips [4]; McKerrell, Phillips and Delves [10]).

Four sided and three sided subregions can be employed, and are mapped onto the unit square or a sector of a circle respectively by a "blending function map" (Gordon and Hall [5]). In the latter case the mapping may be chosen to incorporate an anticipated singularity at the vertex. (See also Kermode, Delves and McKerrell [11].) A wide variety of applications of the method (with particular emphasis on rates of convergence with increasing number of terms in the expansion) is given in Ref. [12], while the modification of part of the algorithm to a parallel processing environment (the ICL DAP) is described in Ref. [13].

3. GEM AND FLUID FLOW IN A DUCT

For incompressible flow in a duct of a Newtonian fluid of viscosity μ the momentum equation reduces to:

$$\rho\frac{\partial w}{\partial t} = -\frac{\mathrm{d}p}{\mathrm{d}z} + \mu\nabla^2 w, \tag{3.1}$$

where w is the single velocity component in the direction (z) of the duct, and p and ρ are the pressure and density respectively.

For steady flow $\partial w/\partial t = 0$ so we have:

$$\nabla^2 w = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z}. \tag{3.2}$$

For oscillatory flow of angular frequency ω we can write

$$p = re[P\exp(i\omega t)],\ w = re(W\exp(i\omega t)],$$

then the equation for oscillatory flow is:

$$\nabla^2 W - \frac{i\rho\omega}{\mu}W = \frac{1}{\mu}\frac{\mathrm{d}P}{\mathrm{d}z}. \tag{3.3}$$

We follow Duck [2] and solve the normalised problem:

$$\nabla^2 W - i\beta^2 W = 1 \tag{3.4}$$

3.1. Cassinian and pinched circular cross section

The Cassinian cross section is given in the complex plane by:

$$|z - z_1|\,|z + z_1| = 1,$$

where we take z_1 to be real and positive. The form of the curve is determined by whether z_1 is less than, equal to, or greater than unity, and the three cases corresponding to the results below ($z_1 = 0.5, 1.0, 1.5$) are illustrated in Figs 1(a), (b) and (c). The pinched circular cross section (for example half of a branching circular pipe) is illustrated in Fig. 1(d) for the case $\alpha = 0.75$.

In each case Figs 1(a)–(d) we used a single three sided element for the part of the cross section in the first quadrant. The appropriate symmetry is imposed by taking Neumann boundary conditions on the axes.

In Table 1 we give the points where the lines of constant velocity (real or imaginary component) cross the axes, in the case $\beta = 1$. The values of velocity are chosen to correspond with the curves plotted in Duck [2] Figs 2, 3, 4 and 7. Agreement is excellent in the case of Fig. 2, except that his curve labelled $w_i = 0.02$ should apparently correspond

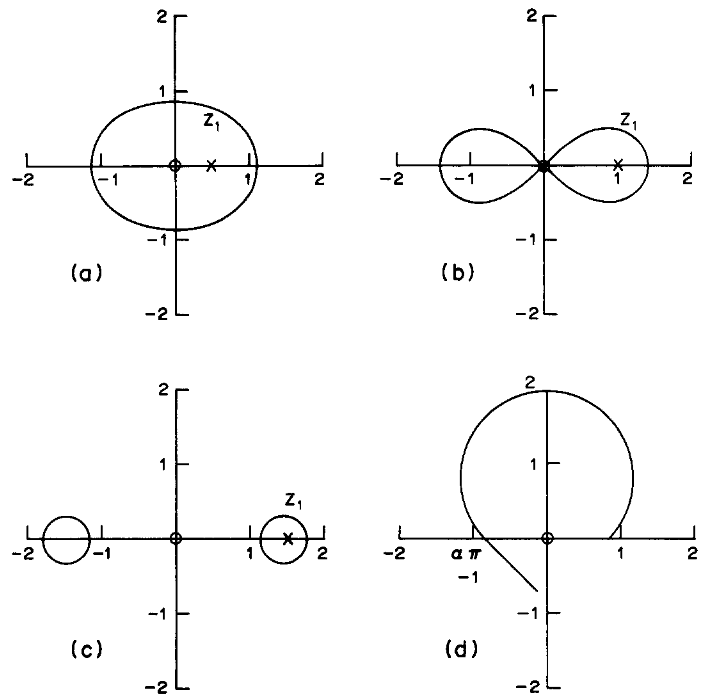

Fig. 1. The Cassinian for (a) $z_1 = 0.5$; (b) $z_1 = 1.0$; (c) $z_1 = 1.5$; pinched circle for (d) $\alpha = 0.75$.

Table 1. Points where curves of constant w_r and w_i cross the axes

Fig. 1.	Fig. in Ref. [2]	w_r	w_i	Axis	
(a)	Fig. 2			x	y
		−0.17		0.58	0.445
			0.02	0.745	0.577
			0.025	0.64	0.49
		−0.12		0.78	0.58
			0.01	0.93	0.72
		−0.05		0.99	0.77
			0.003	1.03	0.82
(b)	Fig. 3			x	x
			0.001	0.25	1.31
		−0.04		0.35	1.24
			0.0025	0.435	1.18
		−0.06		0.52	1.09
			0.0035	0.58	1.045
		−0.07		0.64	0.98
(c)	Fig. 4			x	x
			0.0001	1.15	1.77
		−0.015		1.22	1.70
			0.0004	1.27	1.05
		−0.023		1.31	1.61
			0.0005	1.32	1.60
		−0.028		1.40	1.53
(d)	Fig. 7			x	y
		−0.4		0.13	0.58
			0.13	0.31	0.80
		−0.35		0.37	1.05
			0.1	0.50	1.20
		−0.25		0.58	1.45
			0.06	0.68	1.55
		−0.1		0.76	1.80

Cassinian with $\beta = 1$, (a) $z_1 = 0.5$, (b) $z_1 = 1.0$, (c) $z_1 = 1.5$. Pinched circle with $\beta = 1$, (d) $\alpha = 0.75$.

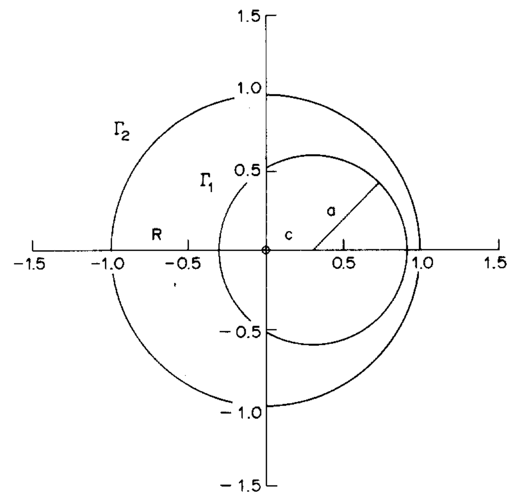

Fig. 2. Cross section for non-coaxial circular cylinders: $a = 0.6$, $c = 0.3$.

to $w_i = 0.025$. Agreement in the other cases is apparently less good, but comparison is limited by the fact that the results in Duck [2] are presented graphically.

3.2. Non-coaxial circular cylinders

We consider fully developed laminar incompressible flow between two cylinders as illustrated in Fig. 2. The outer cylinder has radius 1, the inner cylinder radius a, and c is the distance between the centres. (We have $c > 0$, $a + c \leqslant 1$.)

The velocity component $w(x, y)$ parallel to the axes of the cylinders satisfies the governing equation (Poisson), which we take, following Macdonald [8] in the form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -1$$

with boundary conditions:

$$w = 0 \text{ on } \Gamma_1 : (x - c)^2 + y^2 = a^2,$$
$$w = 0 \text{ on } \Gamma_2 : x^2 + y^2 = 1.$$

An exact result for $w(x, y)$ was first given by Heyda [6], while Redberger and Charles [7] gave numerical results for a number of quantitites including the mass flow per unit time:

$$Q = \iint_R w(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

MacDonald [8] gave exact formulae for Q, and for the wall shear stress τ^i, τ^o on the inner and outer cylinder respectively.

The global element method was used to calculate $w(x, y)$, Q, τ^i, and τ^o using a single element (curvilinear quadrilateral) for the half cross section $y \geqslant 0$ with Neumann boundary conditions on the x axis.

Agreement with the exact results of MacDonald [8] was excellent for a variety of values of a and c. For $N = 10$ (we use $N \times N$ expansion functions) all errors were less than 0.5%, while average or integrated quantities (such as Q) were in agreement to at least 6 significant figures.

Pulsatile flow was also considered, using eqn (3.4). In this case no exact results were available, but results for Q_1 and Q_2 (the real and imaginary parts of Q) were consistent with those obtained by MacDonald using a Fourier series method [14].

Table 2. Real and imaginary parts of Q for non-coaxial circular cylinders.

			Q_1		Q_2	
a	c	β	GEM	asymptotic formula	GEM	asymptotic formula
0.5	0.5	16.348	0.001200	0.001259	−0.007417	−0.007394
0.5	0.5	32.696	0.001599	0.001678	−0.002024	−0.002023
0.5	0.5	65.392	0.000019	0.000022	−0.000527	−0.000528

Comparison of GEM results with those from the asymptotic formula for the fully eccentric case.

For the fully eccentric case ($a + c = 1$) MacDonald [14] has derived the following asymptotic formula valid for large $\beta(1 - a)$; i.e. for high frequency (with the inner cylinder not of very small radius).

$$Q = \frac{\pi}{i\beta^2}\left\{1 - a^2 - \frac{\sqrt{2}}{\beta}(1 + a)(1 - i) - \frac{2\sqrt{(2a)}}{\sqrt{\pi}\beta^{3/2}(1 - a)^{1/2}}\left[-\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right]\right.$$
$$\left. - \frac{1}{2^{5/2}(\pi a(1 - a))^{1/2}\beta^{5/2}}[3(1 - a)e^{-i\pi/8} + e^{3i\pi/8}(1 + a^2)/(1 - a)]\right\}$$

Results were obtained from the GEM program for values of β up to 65 and were found to be consistent with the asymptotic formula. These results are shown in Table 2.

3.3. A coupled problem in a region of cuspidal section

Fully developed fluid flow and heat transfer for a 4-cusp duct of cross section as in Fig. 3 has been studied by Hassan and Barrow [9]. This cross section simulates the situation which might ensue in a loss of coolant accident in a pressursied water reactor when adjacent fuel rods balloon and come into contact. Particular attention is directed to the turbulent case, but in the present paper only laminar flow is treated. Hassan and Barrow [9] show that results for uniform temperature on the boundary are similar to those for more realistic boundary conditions; a typical real problem is given by:

$$\nabla^2 W = -1 \quad \text{in} \quad R, W = 0 \quad \text{on} \quad \partial R,$$
$$\nabla^2 T = 100\, W \quad \text{in} \quad R, T = 0 \quad \text{on} \quad \partial R.$$

The maximum W_0 of W and the minimum T_0 of T occur at the origin.

The problem was treated using the coupled version of GEM [15]. Results were obtained using (a) a single four sided element, and also using a single three sided element for (b)

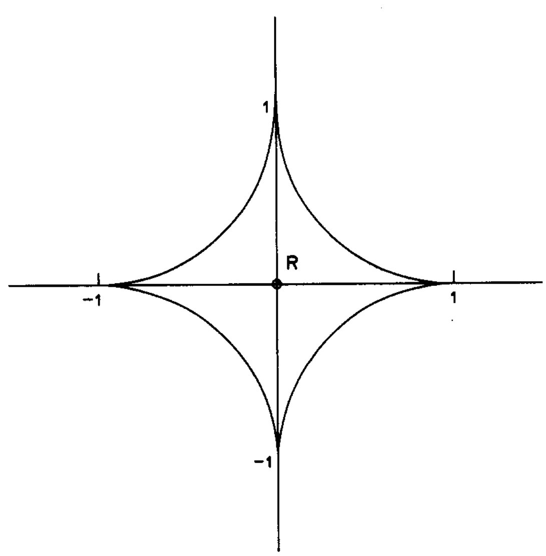

Fig. 3. 4-cusp cross section.

the first quadrant, or (c) the first octant. The singular geometry (the cusp) caused no problem to the GEM program as the blending function map (Gordon and Hall [5]) maps the singularity away.

Of course the above calculation does not necessitate a program for coupled problems as the first equation can be solved for W and Poisson's equation with the now known W used to find T. This was rather faster but gave identical results:

$$W_0 = 0.05357 \pm 0.00001, \quad T_0 = -0.21442 \pm 0.00003.$$

These results are consistent with the calculations of Hassan and Barrow [16] ($W_0 = 0.048 \pm 0.005$, $T_0 = -0.20 \pm 0.02$).

4. CONCLUSION

The flexibility of the Global Element Method has been demonstrated by applying it to a variety of fluid flow problems leading to elliptic partial differential equations to be solved in a 2-D region, including some singular regions, coped with adequately by transformation using the blending function map [5] into a square or semicircular region. Pulsatile flow has been treated in a standard manner using complex velocities, and coupled problems have also been included.

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