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Systems & Control Letters 25 (1995) 31–40

SYSTEMS & CONTROL LETTERS

Remarks on linearization of discrete-time autonomous systems and nonlinear observer design ☆

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Received 16 November 1993; revised 3 March 1994

Abstract

This paper presents necessary and sufficient conditions under which a discrete-time autonomous system with outputs is locally state equivalent to an observable linear system or a system in the nonlinear observer form (Krener and Isidori, 1983). In particular, an open problem raised in Lee and Nam (1991), namely the observer linearization problem, is solved for a nonlinear system which may not be invertible (i.e., *the mapping f may not be a local diffeomorphism*). As a consequence, the nonlinear observer design problem is solved by means of exact linearization techniques.

Keywords: Discrete-time autonomous systems; Exact linearization; State equivalence; Diffeomorphism; Nonlinear observer; Observability

1. Introduction

In the last decade, the problems of exact linearization via a change of coordinates and exact feedback linearization via a state transformation and a regular static state feedback have been extensively studied for continuous-time nonlinear systems. Such study, directly or indirectly, has had an impact on a variety of nonlinear control design problems, including disturbance decoupling, noninteracting control, nonlinear adaptive control and asymptotic stabilization via smooth state feedback, etc. For additional details, the reader is referred to the books by Isidori [2] and by Nijmeijer and Van der Schaft [11].

In the discrete-time nonlinear context, the equivalence of discrete-time nonlinear control systems to linear ones via a state transformation and with or without a smooth state feedback, has also been characterized by many authors; see for instance, the following papers as well as references therein [1, 3, 6, 8, 10].

A dual problem to feedback linearization of nonlinear control systems is the so-called the observer linearization problem [2, 4]. The problem is initially studied by Krener and Isidori in [4], in which a necessary and sufficient condition is given for an autonomous system with the single output to be locally equivalent to a system in the nonlinear observer form via a nonlinear state transformation. The generalizations of [4] to the multi-output nonlinear systems are carried out by Krener and Respondek [5] and by Xia and Gao [12]. Recently, a discrete-time counterpart of such result has been parallely obtained by Lee and

☆ Research supported in part by grants from the AFOSR and NSF.

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SSDI 0167-6911(94)00054-Y

Nam [7], under the restrictive assumption that a *discrete-time autonomous system is invertible*, or equivalently, the map $f: \mathbb{R}^n \to \mathbb{R}^n$, is a local diffeomorphism. For discrete-time autonomous systems which are *not invertible*, the observer linearization problem remains open. One of the purposes of this paper is to present a complete solution to this unsolved problem.

In this paper, we consider a discrete-time autonomous system with outputs

$$\Sigma: \quad x(k+1) = f(x(k)), \quad y(k) = h(x(k)), \tag{1.1}$$

where $x \in \mathbb{R}^n, f: \mathbb{R}^n \to \mathbb{R}^n$ and $h: \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions, with $f(0) = 0$ and $h(0) = 0$. Necessary and sufficient conditions are derived for Σ to be locally equivalent to an observable linear system

$$\Sigma_L: \quad z(k+1) = Az(k), \quad y(k) = Cz(k) \tag{1.2}$$

or a system in the nonlinear observer form

$$\Sigma_0: \quad z(k+1) = Az(k) + \varphi(y(k)), \quad y(k) = Cz(k) \tag{1.3}$$

by means of a change of coordinates $z = T(x)$. The first main result (Theorem 1 or 2) proposed in this paper can be regarded as a discrete analogous one of Theorem 5.13 given by Nijmeijer and Van der Schaft [11]. The second main result (Theorem 3 or 4) is an extension of Propositions 1 and 2 given in [7], but a major difference is that our solution to the observer linearization problem is not based on the restrictive hypothesis that f is a local diffeomorphism. Therefore, all the results developed in this paper are applicable to *noninvertible autonomous systems* with outputs that cannot be dealt with by Lee and Nam's paper [7].

2. State equivalent to linear observable systems

In this section, we study the problem of when a discrete-time autonomous system with outputs (1.1) is locally equivalent to a linear observable system (1.2). In an attempt to make our idea clear, we first present a solution to the problem in the single output case.

Theorem 1. *A discrete-time autonomous system* (1.1) *with the single output is locally equivalent to a linear observable system* (1.2) *via a state transformation* $z = T(x)$ *if and only if*

(i) *the pair* $((\partial f/\partial x)_{x=0}, (\partial h/\partial x)_{x=0})$ *is observable,*

(ii) $h \circ f^n(x) = \sum_{i=1}^{n} \alpha_i h \circ f^{i-1}(x)$, *for all* x *in a neighborhood of* $x = 0$ *and for some real constants* α_i, $1 \leqslant i \leqslant n$, *where* $f^i(x) \triangleq f(f^{i-1}(x))$ $\forall i \geqslant 1$ *and* $f^0(x) \triangleq x$.

Proof. *Necessity*: Suppose there is a state transformation $z = T(x)$ such that system (1.1) is locally equivalent to a linear observable system (1.2). Then

$$Az = T \circ f \circ T^{-1}(z) \quad \text{and} \quad Cz = h \circ T^{-1}(z).$$

Hence,

$$A = (T)_*|_{x=0} \left(\frac{\partial f}{\partial x}\right)_{x=0} (T^{-1})_*|_{z=0}, \qquad C = \left(\frac{\partial h}{\partial x}\right)_{x=0} (T^{-1})_*|_{z=0}. \tag{2.1}$$

Since the pair (A, C) is observable, so is the pair $((\partial f/\partial x)_{x=0}, (\partial h/\partial x)_{x=0})$. On the other hand, note that

$$h \circ f(x) = (h \circ T^{-1}) \circ (T \circ f(x)) = C \circ (Az) = CAz = CAT(x).$$

By induction, it is easy to show that

$$h \circ f^i(x) = CA^i T(x) \quad \text{for } i = 1, 2, \ldots, n.$$

Thus, condition (ii) follows immediately from observability.

Sufficiency: If condition (i) holds, it is clear that

$$z = T(x) \triangleq [h(x), h \circ f(x), \ldots, h \circ f^{n-1}(x)]^{\mathrm{T}} \tag{2.2}$$

is a change of coordinates, since $((\partial T(x)/\partial x)_{x=0} = (T)_*|_{x=0}$ is nonsingular. Moreover, let $z_{n+1}(k) \triangleq h \circ f^n(x(k))$. Then it follows from (2.2) that

$$z_i(k+1) = h \circ f^i(x(k)) = z_{i+1}(k) \quad \text{for } i = 1, 2, \ldots, n.$$

By condition (ii), there exist real constants α_i, $1 \leqslant i \leqslant n$, such that

$$z_n(k+1) = z_{n+1}(k) = h \circ f^n(x(k)) = \sum_{i=1}^{n} \alpha_i h \circ f^{i-1}(x(k)) = \sum_{i=1}^{n} \alpha_i z_i(k).$$

Therefore, we conclude that a state transformation defined by (2.2) transforms system (1.1) into a linear system (1.2) with

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_n \end{bmatrix} \quad \text{and} \quad C = [1 \quad 0 \quad \ldots \quad 0]. \tag{2.3}$$

Clearly, the pair (A, C) is observable.

Remark 1. It is clear that condition (i) of Theorem 1 is equivalent to for all x in a neighborhood U of $x = 0 \in \mathbb{R}^n$,

$$(\tilde{\mathrm{i}}) \qquad \dim\left(\operatorname{span}\left\{\frac{\partial h(x)}{\partial x}, \frac{\partial(h \circ f(x))}{\partial x}, \ldots, \frac{\partial(h \circ f^{n-1}(x))}{\partial x}\right\}\right) = n. \tag{2.4}$$

As a matter of fact, conditions (i) says that the matrix

$$\begin{bmatrix} \left(\dfrac{\partial h}{\partial x}\right)_{x=0} \\ \left(\dfrac{\partial h}{\partial x}\right)_{x=0}\left(\dfrac{\partial f}{\partial x}\right)_{x=0} \\ \vdots \\ \left(\dfrac{\partial h}{\partial x}\right)_{x=0}\left(\dfrac{\partial f}{\partial x}\right)_{x=0}^{n-1} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial h}{\partial x}\right)_{x=0} \\ \left(\dfrac{\partial(h \circ f(x))}{\partial x}\right)_{x=0} \\ \vdots \\ \left(\dfrac{\partial(h \circ f^{n-1}(x))}{\partial x}\right)_{x=0} \end{bmatrix}$$

is nonsingular. This, in turn, implies (2.4) immediately. Obviously, condition $(\tilde{\mathrm{i}})$ is a discrete version of condition of Lemma 9.1 given in [2].

Theorem 1 can be easily extended to the multi-output systems of the form (1.1), as given in the following statement whose proof is similar to Theorem 1 and can be found in [9].

Theorem 2. *A discrete-time autonomous system with outputs* (1.1) *is locally equivalent to a linear observable system if and only if there exist observability indices* $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m \geqslant 1$ *with* $\sum_{i=1}^{m} \mu_i = n$ *such that for all* x *in a neighborhood of* $x = 0$,

(i) $\dim(\operatorname{span}\{\partial h_1(x)/\partial x, \ldots, \partial(h_1 \circ f^{\mu_1 - 1}(x))/\partial x; \ldots; \partial h_m(x)/\partial x, \ldots, \partial(h_m \circ f^{\mu_m - 1}(x))/\partial x\}) = n$,

(ii) $h_i \circ f^{\mu_i}(x) \in \operatorname{span}\{h_1(x), \ldots, h_1 \circ f^{\mu_1 - 1}(x); \ldots; h_m(x), \ldots, h_m \circ f^{\mu_m - 1}(x)\}$ *for* $1 \leqslant i \leqslant m$.

Remark 2. Theorem 1 or 2 becomes globally valid if conditions (i) and (ii) are satisfied $\forall x \in \mathbb{R}^n$ and both the mapping $z = T(x)$ defined by (2.2) and the inverse mapping $x = T^{-1}(z)$ are smooth functions in $\mathbb{R}^n$.

3. State equivalent to systems in the nonlinear observer form

In this section, we turn our attention to the so-called the observer linearization problem [2, 4] in the discrete-time nonlinear context. Our goal is to investigate the question of when a discrete-time autonomous system with outputs (1.1) is locally equivalent to a discrete-time system in the nonlinear observer form (1.3). The problem has been studied by Lee and Nam [7]. A necessary and sufficient condition for the observer linearization problem to be solvable was given in [7], under the restrictive hypothesis that *the autonomous system is invertible.* However, *the problem remains unsolved if the system is not invertible or the map f is not diffeomorphic.* Next, we show how this open problem can be solved and how the invertibility assumption of the system can be removed.

Suppose the pair (A, C) is observable. It is clear that after a linear change of coordinates, the pair (A, C) can always be put into the following form

$$A = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = [0 \ \dots \ 0 \ \ 1]. \tag{3.1}$$

Without loss of generality, in what follows, we will assume that A and C are in the form (3.1). The following result is a technical lemma which will be used to prove the main theorem in this section. The proof of the lemma is straightforward and therefore omitted for reasons of space. The reader is referred to [9] for a detailed discussion.

Lemma 1. *Let $X \in \mathbb{R}^l$ and $Y \in \mathbb{R}^P$ be vectors. Suppose $F: \mathbb{R}^l \times \mathbb{R}^P \to \mathbb{R}$ is a given smooth function. Then*
(i) *there exist smooth functions $G_1: \mathbb{R}^l \to \mathbb{R}$ and $G_2: \mathbb{R}^P \to \mathbb{R}$ such that*

$$F(X, Y) = G_1(X) + G_2(Y)$$

if and only if $\partial^2 F(X, Y)/\partial(X, Y)^2$ is a diagonal matrix. Or, equivalently,

$$\frac{\partial^2 F}{\partial X \partial Y} = \frac{\partial^2 F}{\partial Y \partial X} = 0.$$

As a consequence,
(ii) *there exist smooth functions $g_i: \mathbb{R} \to \mathbb{R}$, $1 \leqslant i \leqslant l + P$, such that*

$$F(Z) \triangleq F(x_1, \dots, x_l; x_{l+1}, \dots, x_{l+P}) = \sum_{i=1}^{l+P} g_i(x_i) \tag{3.2}$$

if and only if the Hessian matrix of $F(Z)$ is diagonal.

On the basis of this technical lemma, we are able to prove the following main result which solves the observer linearization problem for discrete-time autonomous systems with outputs (1.1), without making the restrictive assumption that f is a local diffeomorphism. This is an essential difference from the work [7] due to Lee and Nam.

Theorem 3. *A discrete-time autonomous system with the single output is locally equivalent to a system in the nonlinear observer form* (1.3) *via a state transformation* $z = T(x)$ *if and only if*

(i) *the pair* $((\partial f/\partial x)_{x=0}, (\partial h/\partial x)_{x=0})$ *is observable,*

(ii) *the Hessian matrix of the function* $h \circ f^n \circ \psi^{-1}(s)$ *is diagonal, where* $x = \psi^{-1}(s)$ *is the inverse map of*

$$s = \psi(x) = [h(x), h \circ f(x), \ldots, h \circ f^{n-1}(x)]^{\mathrm{T}}. \tag{3.3}$$

Remark 3. As shown in Remark 1, condition (i) is equivalent to the statement that the Jacobi matrix of $s = \psi(x)$ is nonsingular for all x in a neighborhood of $x = 0$. In other words, condition (i) guarantees that the mapping $s = \psi(x): \mathbb{R}^n \to \mathbb{R}^n$ is a local diffeomorphism. Therefore, the inverse map $\psi^{-1}(s)$ exists and is locally well-defined.

Remark 4. Clearly, condition (ii) of Theorem 3 can be easily checked. Given a discrete-time autonomous system with outputs (1.1), if condition (i) is satisfied, one can construct the map $s = \psi(x)$ and compute the inverse map ψ^{-1}. Once $\psi^{-1}(s)$ is available, it is straightforward to judge whether

$$\frac{\partial^2 (h \circ f^n \circ \psi^{-1}(s))}{\partial s^2}$$

is a diagonal matrix or not. However, we note that sometimes it is tough to compute ψ^{-1}.

Proof of Theorem 3. *Necessity*: Similar to the proof of Theorem 1, note that by assumption,

$$Az = T \circ f \circ T^{-1}(z) - \varphi(Cz) \quad \text{and} \quad Cz = h \circ T^{-1}(z). \tag{3.4}$$

Thus,

$$A = (T)_* \left(\frac{\partial f}{\partial x}\right)_{x=0} (T^{-1})_* + \left(\frac{\partial \varphi}{\partial y}\right)_{y=0} \left(\frac{\partial h}{\partial x}\right)_{x=0} (T^{-1})_* \quad \text{and} \quad C = \left(\frac{\partial h}{\partial x}\right)_{x=0} (T^{-1})_*.$$

Since the pair (A, C) is observable, so is the pair

$$\left((T)_* \left(\frac{\partial f}{\partial x}\right)_{x=0} (T^{-1})_*, \left(\frac{\partial h}{\partial x}\right)_{x=0} (T^{-1})_*\right).$$

This implies that condition (i) holds, as shown in the proof of Theorem 1.

In order to show condition (ii), we first recall that A and C are as defined in (3.1). With this in mind, we deduce from (3.4) that

$$\begin{aligned}
&h(x) = h \circ T^{-1}(z) = Cz = z_n,\\
&z_1(k+1) = \varphi_1(z_n(k)),\\
&z_2(k+1) = z_1(k) + \varphi_2(z_n(k)),\\
&\quad\vdots\\
&z_n(k+1) = z_{n-1}(k) + \varphi_n(z_n(k)),
\end{aligned}$$

where z_i and φ_i, $1 \leqslant i \leqslant n$, denote the ith component of the vectors $z = T(x)$ and φ, respectively. From the relationship above, it can be easily seen that

$$\begin{aligned}
&z_n(k) = h(x(k)),\\
&z_{n-1}(k) = z_n(k+1) - \varphi_n(z_n(k)) = h \circ f(x(k)) - \varphi_n(h(x(k)))),\\
&\quad\vdots\\
&z_1(k) = z_2(k+1) - \varphi_2(z_n(k)) = h \circ f^{n-1}(x(k)) - \sum_{i=2}^{n} \varphi_i(h \circ f^{i-2}(x(k))).
\end{aligned} \tag{3.5}$$

In particular,

$$z_1(k+1) = \varphi_1(z_n(k)) = \varphi_1(h(x(k))) = h \circ f^n(x(k)) - \sum_{i=2}^{n} \varphi_i(h \circ f^{i-1}(x(k))).$$

Hence,

$$h \circ f^n(x) = \sum_{i=1}^{n} \varphi_i(h \circ f^{i-1}(x)). \tag{3.6}$$

By Remarks 1 and 2, the inverse map of $s = \psi(x)$ defined by (3.3) exists and is locally well-defined. Thus, it follows from (3.6) and (3.3) that

$$h \circ f^n \circ \psi^{-1}(s) = \sum_{i=1}^{n} \varphi_i(s_i). \tag{3.7}$$

This, in view of Lemma 1, proves condition (ii) and completes the proof.

Sufficiency: Suppose conditions (i) and (ii) are satisfied. By Lemma 1, there exist smooth functions φ_i, $1 \leqslant i \leqslant n$, such that (3.7) holds. Thus, (3.6) follows immediately from (3.7) and (3.3). Using the smooth functions φ_i ($1 \leqslant i \leqslant n$) thus obtained, one can construct a map $z = T(x)$ as defined in (3.5). Then a straightforward calculation shows that

$$(T)_*|_{x=0} = \left(\frac{\partial T}{\partial x}\right)_{x=0} = \begin{bmatrix} * & \cdots & * & 1 \\ \vdots & & 1 & 0 \\ * & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \left(\frac{\partial \psi}{\partial x}\right)_{x=0}.$$

From condition (i) and Remark 1, we conclude that $z = T(x)$ defined by (3.5) is a local diffeomorphism and qualifies as a state transformation. Under this particular change of coordinates, system (1.1) is transformed into a system in the nonlinear observer form (1.3) with A and C being in the form (3.1).

From the proof of Theorem 3, we observe that in order to construct a state transformation $z = T(x)$ as defined in (3.5), which changes system (1.1) into system (1.3), one needs to solve the smooth functions φ_i ($1 \leqslant i \leqslant n$) from Eq. (3.6) or (3.7). This can be done easily if we assume that the mapping φ appeared in (1.3) is normalized such that $\varphi(0) = 0$. In this case, it follows immediately from (3.7) that

$$\varphi_i(s_i) = h \circ f^n \circ \psi^{-1}(0, \ldots, 0, s_i, 0, \ldots, 0), \tag{3.8}$$

where $\psi(x)$ is defined by (3.3).

We conclude this section with an extension of Theorem 3 to the multi-output case. The following canonical forms of A and C are assumed.

$$A = \mathrm{diag}(A_1, \ldots, A_m), \qquad A_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{\mu_i \times \mu_i},$$

$$C = \begin{bmatrix} c_1 & & 0 \\ & c_2 & \\ & & \ddots & \\ 0 & & & c_m \end{bmatrix}_{m \times n}, \qquad c_i = [0 \quad 0 \ldots 0 \quad 1]_{1 \times \mu_i}, \tag{3.9}$$

for $1 \leqslant i \leqslant m$.

First, we present an intermediate result.

Lemma 2. *A discrete-time autonomous system with outputs* (1.1) *is locally equivalent to a system in the nonlinear observer form* (1.3) *with* (3.9) *via a state transformation* $z = T(x)$ *if, and only if, there exist observability indices* $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m \geqslant 1$ *with* $\sum_{i=1}^m \mu_i = n$ *such that for all* x *in a neighborhood of* $x = 0$,

(i) $\dim(\operatorname{span}\{\partial h_1(x)/\partial x, \dots, \partial(h_1 \circ f^{\mu_1 - 1}(x))/\partial x; \dots; \partial h_m(x)/\partial x, \dots, \partial(h_m \circ f^{\mu_m - 1}(x))/\partial x\}) = n$,

(ii) *the mapping* $\varphi: \mathbb{R}^m \to \mathbb{R}^n$ *appeared in* (1.3), *which is in the form*

$$\varphi(y) = \begin{bmatrix} \varphi_1(y) \\ \vdots \\ \varphi_m(y) \end{bmatrix} \in \mathbb{R}^n, \qquad \varphi_i(y) = \begin{bmatrix} \varphi_i^1(y) \\ \vdots \\ \varphi_i^{\mu_i}(y) \end{bmatrix} \quad \text{for } 1 \leqslant i \leqslant m,$$

is such that

$$h_i \circ f^{\mu_i}(x) = \sum_{l=1}^{\mu_i} \varphi_i^l(h_1 \circ f^{l-1}(x), \dots, h_m \circ f^{l-1}(x)) \quad \text{for } 1 \leqslant i \leqslant m. \tag{3.10}$$

The proof of Lemma 2 is analogous to the proof of Theorem 3 and can be done easily by using Lemma 1 [9]. Combining Lemma 2 with Lemma 1, we have the following theorem that is a generalization of Theorem 3 for the multi-output system (1.1).

Theorem 4. *A discrete-time autonomous system with outputs* (1.1) *is locally equivalent to a system in the nonlinear observer form* (1.3) *with* (3.9) *via a state transformation* $z = T(x)$ *if and only if there exist observability indices* $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m \geqslant 1$ *with* $\sum_{i=1}^m \mu_i = n$ *such that for all* x *in a neighborhood of* $x = 0$,

(i) $\dim(\operatorname{span}\{\partial h_1(x)/\partial x, \dots, \partial(h_1 \circ f^{\mu_1 - 1}(x))/\partial x; \dots; \partial h_m(x)/\partial x, \dots, \partial(h_m \circ f^{\mu_m - 1}(x))/\partial x\}) = n$,

(ii) $\partial^2(h_i \circ f^{\mu_i} \circ \psi^{-1}(s_1, \dots, s_m))/\partial s_i^l \partial s_j^p = 0$ *for* $1 \leqslant i, j \leqslant m$, $1 \leqslant l \leqslant \mu_i$, $1 \leqslant p \leqslant \mu_j$, *and* $l \neq p$, *where* $S = \psi(x) = [s_1, \dots, s_m]^{\mathrm{T}} = [\psi_1(x), \dots, \psi_m(x)]^{\mathrm{T}}$ *is a mapping and*

$$s_i = [s_i^1, \dots, s_i^{\mu_i}]^{\mathrm{T}} = \psi_i(x) = [h_i(x), \dots, h_i \circ f^{\mu_i - 1}(x)]^{\mathrm{T}} \in \mathbb{R}^{\mu_i} \quad \text{for } 1 \leqslant i \leqslant m.$$

The proof of Theorem 4 is strongly reminiscent of the proof of Theorem 3 and therefore is omitted for reasons of space. The reader is referred to [9] for additional details.

4. Nonlinear observer design and examples

For a class of discrete-time autonomous systems with outputs in the form (1.1), which is locally state equivalent to a linear observable system of the form (1.2) or a system in the nonlinear observer form (1.3), the nonlinear observer design problem can be easily solved by means of standard linear observer design techniques. For instance, it is well-known that a Luenberger observer for a linear observable system (1.2) is constructed as

$$\hat{z}(k+1) = A\hat{z}(k) + L(y(k) - C\hat{z}(k)) = (A - LC)\hat{z}(k) + Ly(k),$$

where L is chosen in such a way that all the eigenvalues are located on the open unit disc. In this case, the estimate state error $e_k = \hat{z}_k - z_k$ satisfies

$$e(k+1) = (A - LC)e(k)$$

and tends to zero as $k \to \infty$. This, in turn, implies that $\hat{x} = T^{-1}(\hat{z})$ is an estimate state of the nonlinear system (1.1) and $\hat{x}$ eventually approaches to the state x of system (1.1). Therefore, we conclude that system

$$\hat{x}(k+1) = T^{-1} \circ (AT(\hat{x}(k)) + L(y(k) - CT(\hat{x}(k)))) \tag{4.1}$$

is a nonlinear observer for the discrete-time nonlinear system (1.1), which is equivalent to a linear observable system (1.2) by a state transformation $z = T(x)$.

Note that the above construction of a Luenberger observer is also applicable to the nonlinear system (1.1) which is state equivalent to a system in the nonlinear observer form (1.3). In fact, it is easy to show that the following system

$$\hat{x}(k+1) = T^{-1} \circ (AT(\hat{x}(k)) + \varphi(y(k)) + L(y(k) - CT(\hat{x}(k)))) \tag{4.2}$$

is a nonlinear observer for system (1.1) if system (1.1) is locally equivalent to system (1.3) via a change of coordinates $z = T(x)$, and $\hat{x}$ converges to the state x of system (1.1) when $k \to \infty$.

In summary, we conclude that once we find a local linearizing state transformation $z = T(x)$ which changes system (1.1) into a linear observable system (1.2) or a system in the nonlinear observer form (1.3), the nonlinear observer problem can be solved immediately. We now present an example to show how a nonlinear observer can be constructed for system (1.1).

Example 1. Consider a discrete-time autonomous system with two outputs.

$$\Sigma_2: \quad \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = f(x(k)) = \begin{bmatrix} x_3^2(k)(x_4(k) + x_2^2(k)) \\ x_1(k) \\ x_2(k) \\ x_4(k) - x_1^2(k) + x_2^2(k) \end{bmatrix},$$

$$y(k) = h(x(k)) = \begin{bmatrix} h_1(x(k)) \\ h_2(x(k)) \end{bmatrix} = \begin{bmatrix} x_3(k) \\ x_4(k) + x_2^2(k) \end{bmatrix}.$$

Since the system in question is not invertible, the observer linearization problem cannot be solved by using Lee and Nam's result [7]. However, the problem can be solved by Theorem 4 proposed in this paper, as illustrated in the following.

First, a direct computation shows that

$$\dim\left(\mathrm{span}\left\{\left(\frac{\partial h_1}{\partial x}\right)_{x=0}, \left(\frac{\partial (h_1 \circ f)}{\partial x}\right)_{x=0}, \left(\frac{\partial (h_1 \circ f^2)}{\partial x}\right)_{x=0}, \left(\frac{\partial h_2}{\partial x}\right)_{x=0}\right\}\right)$$
$$= \dim\left(\mathrm{span}\left\{\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}\right\}\right) = 4.$$

Hence, condition (i) of Theorem 4 is satisfied. Second, by construction we have

$$s_1 = \begin{bmatrix} s_1^1 \\ s_1^2 \\ s_1^3 \end{bmatrix} = \psi_1(x) = \begin{bmatrix} h_1(x) \\ h_1 \circ f(x) \\ h_1 \circ f^2(x) \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \qquad s_2 = \psi_2(x) = h_2(x) = x_4 + x_2^2.$$

Hence,

$$S = \psi(x) = [x_3, x_2, x_1, x_4 + x_2^2]^{\mathrm{T}} = [s_1^{\mathrm{T}}, s_2]^{\mathrm{T}}.$$

The inverse map of $S = \psi(x)$ is given by

$$x = \psi^{-1}(S) = [s_1^3, s_1^2, s_1^1, s_2 - (s_1^2)^2]^{\mathrm{T}}.$$

Then it can be shown that

$$h_1 \circ f^{\mu_1} \circ \psi^{-1}(S) = x_3 \circ f^3 \circ \psi^{-1}(s_1^1, s_1^2, s_1^3, s_2) = (s_1^1)^2 s_2,$$

$$h_2 \circ f^{\mu_2} \circ \psi^{-1}(S) = (x_4 + x_2^2) \circ f \circ \psi^{-1}(s_1^1, s_1^2, s_1^3, s_2) = s_2.$$

This, in turn, implies that condition (ii) of Theorem 4 also holds. By Theorem 4, there exists a change of coordinates

$$z = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} h_1 \circ f^3(x) - \sum_{l=2}^{3} \varphi_1^l(h_1 \circ f^{l-2}, h_2 \circ f^{l-2}(x)) \\ h_1 \circ f^2(x) - \varphi_1^3(h_1(x), h_2(x)) \\ h_1(x) \\ h_2(x) \end{bmatrix} \tag{4.3}$$

such that Σ_2 can be transformed into a system in the nonlinear observer form, where φ_1^2 and φ_1^3 are components of the vector function $\varphi(y) = [\varphi_1^1(y), \varphi_1^2(y), \varphi_1^3(y), \varphi_2^1(y)]^{\mathrm{T}}$. From (3.10), it follows that

$$\begin{aligned} &h_2 \circ f(x) = (x_4 + x_2^2) \circ f(x) = x_4 + x_2^2 = y_2 = \varphi_2^1(y), \\ &h_1 \circ f^3(x) = x_3^2(x_4 + x_2^2) = y_1^2 y_2 = \varphi_1^1(y) + \varphi_1^2(y \circ f(x)) + \varphi_1^3(y \circ f^2(x)). \end{aligned} \tag{4.4}$$

Since $\mu_1 = 3$ and $\mu_2 = 1$, we obtain from (4.4) that

$$\varphi(y) = [y_1^2 y_2, 0, 0, y_2]^{\mathrm{T}}. \tag{4.5}$$

Substituting (4.5) into (4.3) yields

$$z = T(x) = [x_1, x_2, x_3, x_4 + x_2^2]^{\mathrm{T}}.$$

This state transformation changes Σ_2 into a system in the nonlinear observer form with

$$\begin{aligned} &A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ &\varphi(y) = \varphi(Cz) = [z_3^2 z_4, 0, 0, z_4]^{\mathrm{T}}. \end{aligned} \tag{4.6}$$

Now a nonlinear observer for Σ_2 can be constructed by (4.2) with (4.6) and $x = T^{-1}(z) = [z_1, z_2, z_3, z_4 - z_2^2]^{\mathrm{T}}$.

5. Conclusions

In this paper, we have presented necessary and sufficient conditions for a discrete-time autonomous system with outputs to be equivalent to a linear observable system or a system in the nonlinear observer form. In particular, we have shown how the open problem raised in [7], namely the observer linearization problem of discrete-time systems for which *f is not diffeomorphic*, can be solved by the method developed in this paper. As a consequence, the nonlinear observe design problem is solved for a class of linearizable nonlinear systems.

References

[1] J.W. Grizzle, *Feedback linearization of discrete-time systems*, Lecture Notes in Control and Information Sciences (Springer, Berlin, 1986) 273–281.
[2] A. Isidori, *Nonlinear Control Systems* (Springer, New York, 2nd ed., 1989).
[3] B. Jakubczyk, Feedback linearization of discrete-time systems, *Systems Control Lett.* 9 (1987) 411–416.

[4] A.J. Krener and A. Isidori, Linearization by output injection and nonlinear observers, *Systems Control Lett.* **3** (1983) 47–52.
[5] A.J. Krener and W. Respondek, Nonlinear observers with linearizable error dynamics, *SIAM J. Control Optim.* **23** (1985) 197–216.
[6] H.G. Lee, A. Arapostathis and S.I. Marcus, On the linearization of discrete time systems, *Internat. J. Control* **45** (1987) 1803–1822.
[7] W. Lee and K. Nam, Observer design for autonomous discrete-time nonlinear systems, *Systems Control Lett.* **17** (1991) 49–58.
[8] H.G. Lee and S.I. Marcus, Approximate and local linearization of nonlinear discrete-time systems, *Internat. J. Control* **44** (1986) 1103–1124.
[9] W. Lin and C.I. Byrnes, *Feedback Design Methods for Discrete-Time Nonlinear Control Systems* (Birkhäuser, Boston, to be published).
[10] S. Monaco and D. Normand-Cyrot, The immersion under feedback of a multidimensional discrete-time nonlinear system into a linear system, *Internat. J. Control* **38** (1983) 245–261.
[11] H. Nijmeijer and A. Van der Schaft, *Nonlinear Dynamic Control Systems* (Springer, Berlin, 1990).
[12] X.H. Xia and W.B. Gao, Nonlinear observer design by observer error linearization, *SIAM J. Control Optim.* **27** (1989) 199–216.