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Hodographs and normals of rational curves and surfaces

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Abstract

Derivatives and normals of rational Bézier curves and surface patches are discussed. A non-uniformly scaled hodograph of a degree $m \times n$ tensor-product rational surface, which provides correct derivative direction but not magnitude, can be written as a degree $(2m-2) \times 2n$ or $2m \times (2n-2)$ vector function in polynomial Bézier form. Likewise, the scaled normal direction is degree $(3m-2) \times (3n-2)$. Efficient methods are developed for bounding these directions and the derivative magnitude.

Keywords: Rational curves; Rational surfaces; Hodographs; Normal vectors

1. Introduction

Derivatives and normal vectors of parametric curves and surfaces are important issues in computer graphics and geometric modeling. Although the derivative at any single point on a Bézier curve or surface can be calculated by subdivision, it is often useful to assess the possible range of derivatives or normal vectors for an entire curve segment of surface patch. For example, cusps and inflection points on a curve can be detected by this analysis. Bounds of derivative and normal directions also help in detecting intersections between two curves or surfaces (Sederberg and Meyers, 1988; Hohmeyer, 1992). In various other algorithms for curves and surfaces, efficiency can often be enhanced by *a priori* determination of a bound on derivative magnitude.

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The first derivative of a curve is sometimes called its *hodograph.* It is well-known that the hodograph of a degree n polynomial Bézier curve can be expressed as a degree $n-1$ polynomial Bézier curve. For a tensor-product surface, the hodograph is defined as the partial derivative of the surface with respect to each parameter s or t. If the surface is a polynomial Bézier patch of degree m in s and n in t, the hodographs are also tensor-product, and the degree of the s-hodograph is $m-1$ in s and n in t, while the degree of the t-hodograph is $m \times (n-1)$ (Boehm et al., 1984). Since the normal direction can be obtained as the cross product of the s- and t-hodographs, its degree is $(2m-1) \times (2n-1)$.

These simple results, however, do not apply to rational curves and surfaces. The hodograph of a degree n rational curve is a rational curve of degree $2n$. Since derivative direction is often more important than magnitude, Sederberg and Wang (1987) proposed a *scaled hodograph*, which expresses only the derivative direction, and is a degree $2n-2$ polynomial Bézier curve. Floater (1992) provided upper bounds of first and higher order derivative magnitudes for rational Bézier curves. However, derivatives and normals of rational surfaces have not been widely addressed.

We introduce a new approach for representing hodographs and normals. First, we apply it to rational curves and reconfirm the hodographs and the bounds of derivatives introduced by Sederberg and Wang (1987) and Floater (1992). We then apply it to rational surfaces to get scaled hodographs and normal direction in polynomial Bézier form, from which we devise efficient methods for bounding their direction and magnitude.

In this paper, *curve* generally means *rational Bézier curve*, and *surface* means *rational tensor-product Bézier patch.* Also, note that scaled hodographs provide exact directions, but not magnitudes.

2. Rational curves

In this section, we explain a new approach for bounding the derivative direction and magnitude of a curve. Although the final results are identical to those presented in (Sederberg and Wang, 1987) and (Floater, 1992), this new method extends directly to surfaces.

2.1. Direction between two homogeneous points

In projective geometry, points are usually specified in homogeneous form. A homogeneous point

$$\boldsymbol{P} = (X, Y, Z, W) \tag{1}$$

has the Cartesian coordinates

$$\tilde{\boldsymbol{P}} = (x, y, z) = \left(\frac{X}{W}, \frac{Y}{W}, \frac{Z}{W}\right). \tag{2}$$

In this paper, a single character in bold typeface signifies a homogeneous point, while one with a tilde denotes a Cartesian point or vector.

To aid in our discussion, we define:

$$\mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_2) \equiv (W_1X_2 - W_2X_1,\ W_1Y_2 - W_2Y_1,\ W_1Z_2 - W_2Z_1) \tag{3}$$

where $\boldsymbol{P}_i = (X_i, Y_i, Z_i, W_i)$. The "Dir" function indicates the direction of the Cartesian vector between two points, because

$$\mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_2) = W_1W_2(\tilde{\boldsymbol{P}}_2 - \tilde{\boldsymbol{P}}_1) \tag{4}$$

if $W_1W_2 \neq 0$. Furthermore, one can easily verify that "Dir" satisfies the following relations:

$$\begin{aligned}
&\mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_1) = \boldsymbol{0},\\
&\mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_2) = -\mathrm{Dir}(\boldsymbol{P}_2, \boldsymbol{P}_1),\\
&\mathrm{Dir}(k\boldsymbol{P}_1, \boldsymbol{P}_2) = \mathrm{Dir}(\boldsymbol{P}_1, k\boldsymbol{P}_2) = k\,\mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_2),\\
&\mathrm{Dir}(\boldsymbol{P}_1 + \boldsymbol{P}_2, \boldsymbol{P}_3) = \mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_3) + \mathrm{Dir}(\boldsymbol{P}_2, \boldsymbol{P}_3),\\
&\mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_2 + \boldsymbol{P}_3) = \mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_2) + \mathrm{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_3),
\end{aligned} \tag{5}$$

where k is a scalar value.

2.2. Hodograph of a rational curve

A rational Bézier curve $\boldsymbol{P}[t]$ is defined with homogeneous control points $\boldsymbol{P}_i = (X_i, Y_i, Z_i, W_i)$:

$$\boldsymbol{P}[t] = (X[t], Y[t], Z[t], W[t]) = \sum_{i=0}^{n} B_i^n[t]\boldsymbol{P}_i, \tag{6}$$

where

$$B_i^n[t] = \binom{n}{i}(1-t)^{n-i}t^i, \tag{7}$$

and n is the degree of the curve. In this paper, we assume that all $W_i > 0$. The derivative of the curve in homogeneous coordinates is

$$\boldsymbol{P}'[t] = (X'[t], Y'[t], Z'[t], W'[t]) = n\sum_{i=0}^{n-1} B_i^{n-1}[t](\boldsymbol{P}_{i+1} - \boldsymbol{P}_i) \tag{8}$$

and the derivative in Cartesian coordinates is

$$\begin{aligned}
\frac{d}{dt}\tilde{\boldsymbol{P}}[t] &= \frac{d}{dt}\left(\frac{X[t]}{W[t]}, \frac{Y[t]}{W[t]}, \frac{Z[t]}{W[t]}\right)\\
&= \frac{1}{(W[t])^2}(W[t]X'[t] - W'[t]X[t],\ W[t]Y'[t] - W'[t]Y[t],\\
&\qquad W[t]Z'[t] - W'[t]Z[t])\\
&= \frac{\mathrm{Dir}(\boldsymbol{P}[t], \boldsymbol{P}'[t])}{(W[t])^2}.
\end{aligned} \tag{9}$$

Since $(W[t])^2$ is a scalar function, $\mathrm{Dir}(\boldsymbol{P}[t],\boldsymbol{P}'[t])$ gives the derivative direction in Cartesian coordinates.

One might expect that the degree of $\mathrm{Dir}(\boldsymbol{P}[t],\boldsymbol{P}'[t])$ is $2n-1$. However, the degree is at most $2n-2$, which is shown as follows. Write

$$\boldsymbol{P}[t]=(1-t)\boldsymbol{F}_0[t]+t\boldsymbol{F}_1[t], \tag{10}$$

$$\boldsymbol{P}'[t]=n(\boldsymbol{F}_1[t]-\boldsymbol{F}_0[t]), \tag{11}$$

where

$$\boldsymbol{F}_0[t]=\sum_{i=0}^{n-1} B_i^{n-1}[t]\,\boldsymbol{P}_i, \tag{12}$$

$$\boldsymbol{F}_1[t]=\sum_{i=1}^{n} B_{i-1}^{n-1}[t]\,\boldsymbol{P}_i. \tag{13}$$

Therefore,

$$\begin{aligned}\mathrm{Dir}(\boldsymbol{P}[t],\boldsymbol{P}'[t]) &= \mathrm{Dir}((1-t)\boldsymbol{F}_0[t],n\boldsymbol{F}_1[t])-\mathrm{Dir}(t\boldsymbol{F}_1[t],n\boldsymbol{F}_0[t])\\ &= n\,\mathrm{Dir}(\boldsymbol{F}_0[t],\boldsymbol{F}_1[t])\\ &= n\sum_{k=0}^{2n-2}(1-t)^{2n-2-k}t^k \sum_{\substack{i+j=k+1\\ 0\leqslant i\leqslant n-1\\ 1\leqslant j\leqslant n}} \binom{n-1}{i}\binom{n-1}{j-1}\mathrm{Dir}(\boldsymbol{P}_i,\boldsymbol{P}_j).\end{aligned} \tag{14}$$

For each combination of i and j which satisfies $i+j=k+1$,

$$\begin{aligned}&\binom{n-1}{i}\binom{n-1}{j-1}\mathrm{Dir}(\boldsymbol{P}_i,\boldsymbol{P}_j)+\binom{n-1}{j}\binom{n-1}{i-1}\mathrm{Dir}(\boldsymbol{P}_j,\boldsymbol{P}_i)\\ &=\left[\binom{n-1}{i}\binom{n-1}{j-1}-\binom{n-1}{j}\binom{n-1}{i-1}\right]\mathrm{Dir}(\boldsymbol{P}_i,\boldsymbol{P}_j)\\ &=\binom{n}{i}\binom{n}{j}\frac{j-i}{n}\mathrm{Dir}(\boldsymbol{P}_i,\boldsymbol{P}_j)\\ &=\binom{n}{i}\binom{n}{k-i+1}\frac{k-2i+1}{n}\mathrm{Dir}(\boldsymbol{P}_i,\boldsymbol{P}_{k-i+1}).\end{aligned} \tag{15}$$

Having eliminated j, we get

$$\begin{aligned}&\mathrm{Dir}(\boldsymbol{P}[t],\boldsymbol{P}'[t])\\ &=\sum_{k=0}^{2n-2}(1-t)^{2n-2-k}t^k\sum_{i=\max(0,k-n+1)}^{\lfloor k/2\rfloor}(k-2i+1)\binom{n}{i}\binom{n}{k-i+1}\mathrm{Dir}(\boldsymbol{P}_i,\boldsymbol{P}_{k-i+1})\\ &=\sum_{k=0}^{2n-2}B_k^{2n-2}[t]\,\tilde{\boldsymbol{H}}_k\end{aligned} \tag{16}$$

and

$$\frac{d}{dt}\tilde{\boldsymbol{P}}[t] = \frac{1}{(W[t])^2}\sum_{k=0}^{2n-2} B_k^{2n-2}[t]\,\tilde{\boldsymbol{H}}_k \tag{17}$$

where

$$\tilde{\boldsymbol{H}}_k = \frac{1}{\binom{2n-2}{k}} \sum_{i=\max(0,k-n+1)}^{\lfloor k/2 \rfloor} (k-2i+1)\binom{n}{i}\binom{n}{k-i+1}\operatorname{Dir}(\boldsymbol{P}_i, \boldsymbol{P}_{k-i+1}). \tag{18}$$

Eq. (16) gives a scaled hodograph, and its control points are $\tilde{\boldsymbol{H}}_k$ $(0 \leqslant k \leqslant 2n-2)$. Eq. (17) gives the hodograph with the correct magnitude. These results are equivalent to the hodographs in (Sederberg and Wang, 1987). For a rational cubic Bézier curve, the scaled hodograph is degree four with control points:

$$\begin{aligned}
\tilde{\boldsymbol{H}}_0 &= 3\operatorname{Dir}(\boldsymbol{P}_0, \boldsymbol{P}_1),\\
\tilde{\boldsymbol{H}}_1 &= \tfrac{3}{2}\operatorname{Dir}(\boldsymbol{P}_0, \boldsymbol{P}_2),\\
\tilde{\boldsymbol{H}}_2 &= \tfrac{1}{2}\operatorname{Dir}(\boldsymbol{P}_0, \boldsymbol{P}_3) + \tfrac{3}{2}\operatorname{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_2),\\
\tilde{\boldsymbol{H}}_3 &= \tfrac{3}{2}\operatorname{Dir}(\boldsymbol{P}_1, \boldsymbol{P}_3),\\
\tilde{\boldsymbol{H}}_4 &= 3\operatorname{Dir}(\boldsymbol{P}_2, \boldsymbol{P}_3).
\end{aligned} \tag{19}$$

2.3. Bound of derivative direction

In Eq. (18), each coefficient of $\operatorname{Dir}(\boldsymbol{P}_i, \boldsymbol{P}_{k-i+1})$ is positive. This guarantees that the derivative direction is bounded by the convex hull of the vectors $\operatorname{Dir}(\boldsymbol{P}_a, \boldsymbol{P}_b)$ $(0 \leqslant a < b \leqslant n)$. For each $\operatorname{Dir}(\boldsymbol{P}_a, \boldsymbol{P}_b)$,

$$\operatorname{Dir}(\boldsymbol{P}_a, \boldsymbol{P}_b) = W_a W_b(\tilde{\boldsymbol{P}}_b - \tilde{\boldsymbol{P}}_a) = W_a W_b \sum_{i=a}^{b-1}(\tilde{\boldsymbol{P}}_{i+1} - \tilde{\boldsymbol{P}}_i), \tag{20}$$

which shows that $\operatorname{Dir}(\boldsymbol{P}_a, \boldsymbol{P}_b)$ is bounded by the convex hull of the $b-a$ vectors $\tilde{\boldsymbol{P}}_{i+1} - \tilde{\boldsymbol{P}}_i$ $(i = a, a+1, \ldots, b-1)$. As a result, $\operatorname{Dir}(\boldsymbol{P}[t], \boldsymbol{P}'[t])$ $(0 \leqslant t \leqslant 1)$ can be bounded by the convex hull of the n vectors $\tilde{\boldsymbol{P}}_{i+1} - \tilde{\boldsymbol{P}}_i$ $(i = 0, 1, \ldots, n-1)$. This result is commonly used for polynomial curves, but here we note that the bound is not violated if arbitrary positive weights are assigned to the control points of a polynomial Bézier curve.

2.4. Bound of derivative magnitude

We can also derive an upper bound of the derivative magnitude in the following way. Define

$$W_{\max} \equiv \max_{0 \leqslant i \leqslant n} W_i, \tag{21}$$

$$W_{\min} \equiv \min_{0 \leqslant i \leqslant n} W_i, \tag{22}$$

$$D_{\max} \equiv \max_{0 \leqslant i \leqslant n-1} \|\tilde{\boldsymbol{P}}_{i+1} - \tilde{\boldsymbol{P}}_i\|. \tag{23}$$

It is obvious that

$$W_{\min} \leqslant W[t] \leqslant W_{\max} \quad (0 \leqslant t \leqslant 1). \tag{24}$$

We also invoke the identity

$$\sum_{i=\max(0,k-n+1)}^{\lfloor k/2 \rfloor} (k-2i+1)^2 \binom{n}{i} \binom{n}{k-i+1} = n \binom{2n-2}{k} \tag{25}$$

which can be proven by applying Eq. (16) to the curve $P(t) = (t,0,0,1)$. In this case, $\mathrm{Dir}(P(t), P'(t)) = (1,0,0)$ and $P_i = (\frac{i}{n},0,0,1)$. From (16) it follows that $\tilde{H}_k = (1,0,0)$ for $k = 0,1,\ldots,2n-2$. Inserting this identity and $\mathrm{Dir}(P_i,P_j) = ((j-i)/n,0,0)$ in (18) one obtains (25).

$\mathrm{Dir}(\boldsymbol{P}_a, \boldsymbol{P}_b)$ $(0 \leqslant a < b \leqslant 1)$ can be bounded as follows:

$$\|\mathrm{Dir}(\boldsymbol{P}_a, \boldsymbol{P}_b)\| = W_a W_b \|\tilde{\boldsymbol{P}}_b - \tilde{\boldsymbol{P}}_a\| \leqslant W_a W_b \sum_{i=a}^{b-1} \|\tilde{\boldsymbol{P}}_{i+1} - \tilde{\boldsymbol{P}}_i\| \leqslant W_{\max}^2 (b-a) D_{\max}. \tag{26}$$

From Eqs. (18), (25), and (26),

$$\left\| \frac{d}{dt} \tilde{\boldsymbol{P}}[t] \right\| \leqslant \frac{n W_{\max}^2 D_{\max}}{W_{\min}^2} \sum_{k=0}^{2n-2} B_k^{2n-2}[t] = \frac{n W_{\max}^2 D_{\max}}{W_{\min}^2}. \tag{27}$$

This result is equivalent to (Floater, 1992).

3. Rational surfaces

3.1. Hodograph for each parameter direction

A tensor-product rational Bézier surface $\boldsymbol{P}[s,t]$ is defined with homogeneous control points $\boldsymbol{P}_{i,j} = (X_{i,j}, Y_{i,j}, Z_{i,j}, W_{i,j})$:

$$\begin{aligned} \boldsymbol{P}[s,t] &= (X[s,t], Y[s,t], Z[s,t], W[s,t]) \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m[s] B_j^n[t]\, \boldsymbol{P}_{i,j}. \end{aligned} \tag{28}$$

For such a surface, the derivative direction is defined independently for each parameter s,t. The s-derivative in Cartesian coordinates is

$$\begin{aligned}\frac{\partial}{\partial s}\tilde{\boldsymbol{P}}[s,t] &= \frac{\partial}{\partial s}\left(\frac{X[s,t]}{W[s,t]}, \frac{Y[s,t]}{W[s,t]}, \frac{Z[s,t]}{W[s,t]}\right) \\ &= \frac{1}{(W[s,t])^2}(W[s,t]X_s[s,t] - W_s[s,t]X[s,t], \\ &\qquad W[s,t]Y_s[s,t] - W_s[s,t]Y[s,t], \\ &\qquad W[s,t]Z_s[s,t] - W_s[s,t]Z[s,t]) \\ &= \frac{\mathrm{Dir}(\boldsymbol{P}[s,t], \boldsymbol{P}_s[s,t])}{(W[s,t])^2},\end{aligned} \tag{29}$$

where

$$\boldsymbol{P}_s[s,t] = (X_s[s,t], Y_s[s,t], Z_s[s,t], W_s[s,t]),$$

$$X_s[s,t] = \frac{\partial}{\partial s}X[s,t], \qquad Y_s[s,t] = \frac{\partial}{\partial s}Y[s,t], \qquad \ldots .$$

Thus, the s-derivative direction in Cartesian coordinates can be defined as $\mathrm{Dir}(\boldsymbol{P}[s,t], \boldsymbol{P}_s[s,t])$. In this section, we only focus on the s-derivative direction; the t-derivative direction can be derived in like manner.

The surface $\boldsymbol{P}[s,t]$ can be expressed,

$$\boldsymbol{P}[s,t] = \sum_{i=0}^{m} B_i^m[s]\,\boldsymbol{Q}_i[t], \tag{30}$$

where

$$\boldsymbol{Q}_i[t] = \sum_{j=0}^{n} B_j^n[t]\,\boldsymbol{P}_{i,j}. \tag{31}$$

Notice that Eq. (30) can be interpreted as a curve with the parameter s, with control points $\boldsymbol{Q}_i[t]$. This means that the s-derivative of $\boldsymbol{P}[s,t]$ can be viewed as simply the derivative of a curve. Thus, we can apply Eq. (16),

$$\begin{aligned}\mathrm{Dir}(\boldsymbol{P}[s,t], \boldsymbol{P}_s[s,t]) &= \sum_{k=0}^{2m-2}(1-s)^{2m-2-k}s^k \sum_{i=\max(0,k-m+1)}^{\lfloor k/2\rfloor}(k-2i+1) \\ &\qquad\times\binom{m}{i}\binom{m}{k-i+1}\mathrm{Dir}(\boldsymbol{Q}_i[t], \boldsymbol{Q}_{k-i+1}[t]).\end{aligned} \tag{32}$$

The next step is to compute $\mathrm{Dir}(\boldsymbol{Q}_a[t], \boldsymbol{Q}_b[t])$ $(0 \leqslant a < b \leqslant m)$. In an expression reminiscent of the curve case, we have

$$\begin{aligned}&\mathrm{Dir}(\boldsymbol{Q}_a[t], \boldsymbol{Q}_b[t]) \\ &\quad= \sum_{l=0}^{2n}(1-t)^{2n-l}t^l \sum_{j=\max(0,l-n)}^{\min(l,n)}\binom{n}{j}\binom{n}{l-j}\mathrm{Dir}(\boldsymbol{P}_{a,j}, \boldsymbol{P}_{b,l-j}).\end{aligned} \tag{33}$$

From Eqs. (32) and (33),

$$\mathrm{Dir}(\boldsymbol{P}[s,t],\boldsymbol{P}_s[s,t]) = \sum_{k=0}^{2m-2}\sum_{l=0}^{2n} B_k^{2m-2}[s]B_l^{2n}[t]\,\tilde{\boldsymbol{H}}_{k,l} \tag{34}$$

and

$$\frac{\partial}{\partial s}\tilde{\boldsymbol{P}}[s,t] = \frac{1}{(W[s,t])^2}\sum_{k=0}^{2m-2}\sum_{l=0}^{2n} B_k^{2m-2}[s]B_l^{2n}[t]\,\tilde{\boldsymbol{H}}_{k,l}, \tag{35}$$

where

$$\tilde{\boldsymbol{H}}_{k,l} = \sum_{i=\max(0,k-m+1)}^{\lfloor k/2\rfloor}\ \sum_{j=\max(0,l-n)}^{\min(l,n)} (k-2i+1)\binom{m}{i}\binom{m}{k-i+1}\binom{n}{j}\binom{n}{l-j} \times \mathrm{Dir}(\boldsymbol{P}_{i,j},\boldsymbol{P}_{k-i+1,l-j}) \Big/ \binom{2m-2}{k}\binom{2n}{l}. \tag{36}$$

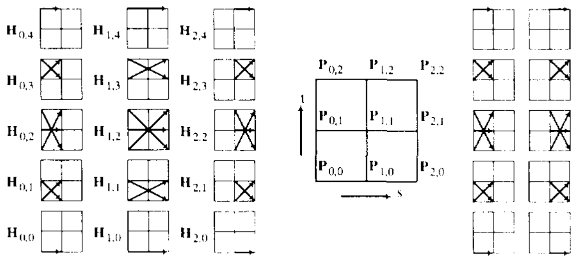
Eq. (34) gives a scaled hodograph for s, and its control points are $\tilde{\boldsymbol{H}}_{k,l}$. Eq. (35) is the hodograph for s with the correct magnitude.

The degree of the scaled hodograph is $2m-2$ in s and $2n$ in t. For a rational bilinear surface, the degree of the hodograph is 0 in s and 2 in t, and its control points are

$$\begin{aligned}
\tilde{\boldsymbol{H}}_{0,0} &= \mathrm{Dir}(\boldsymbol{P}_{0,0},\boldsymbol{P}_{1,0}),\\
\tilde{\boldsymbol{H}}_{0,1} &= \tfrac{1}{2}\mathrm{Dir}(\boldsymbol{P}_{0,0},\boldsymbol{P}_{1,1}) + \tfrac{1}{2}\mathrm{Dir}(\boldsymbol{P}_{0,1},\boldsymbol{P}_{1,0}),\\
\tilde{\boldsymbol{H}}_{0,2} &= \mathrm{Dir}(\boldsymbol{P}_{0,1},\boldsymbol{P}_{1,1}).
\end{aligned} \tag{37}$$

For a rational biquadratic surface, the hodograph is degree 2×4, and its control points are

$$\begin{aligned}
\tilde{\boldsymbol{H}}_{0,0} &= 2\,\mathrm{Dir}(\boldsymbol{P}_{0,0},\boldsymbol{P}_{1,0}),\\
\tilde{\boldsymbol{H}}_{0,1} &= \mathrm{Dir}(\boldsymbol{P}_{0,0},\boldsymbol{P}_{1,1}) + \mathrm{Dir}(\boldsymbol{P}_{0,1},\boldsymbol{P}_{1,0}),\\
\tilde{\boldsymbol{H}}_{0,2} &= \tfrac{1}{3}\mathrm{Dir}(\boldsymbol{P}_{0,0},\boldsymbol{P}_{1,2}) + \tfrac{4}{3}\mathrm{Dir}(\boldsymbol{P}_{0,1},\boldsymbol{P}_{1,1}) + \tfrac{1}{3}\mathrm{Dir}(\boldsymbol{P}_{0,2},\boldsymbol{P}_{1,0}),\\
\tilde{\boldsymbol{H}}_{0,3} &= \mathrm{Dir}(\boldsymbol{P}_{0,1},\boldsymbol{P}_{1,2}) + \mathrm{Dir}(\boldsymbol{P}_{0,2},\boldsymbol{P}_{1,1}),\\
\tilde{\boldsymbol{H}}_{0,4} &= 2\,\mathrm{Dir}(\boldsymbol{P}_{0,2},\boldsymbol{P}_{1,2}),\\
\tilde{\boldsymbol{H}}_{1,0} &= \mathrm{Dir}(\boldsymbol{P}_{0,0},\boldsymbol{P}_{2,0}),\\
\tilde{\boldsymbol{H}}_{1,1} &= \tfrac{1}{2}\mathrm{Dir}(\boldsymbol{P}_{0,0},\boldsymbol{P}_{2,1}) + \tfrac{1}{2}\mathrm{Dir}(\boldsymbol{P}_{0,1},\boldsymbol{P}_{2,0}),\\
\tilde{\boldsymbol{H}}_{1,2} &= \tfrac{1}{6}\mathrm{Dir}(\boldsymbol{P}_{0,0},\boldsymbol{P}_{2,2}) + \tfrac{2}{3}\mathrm{Dir}(\boldsymbol{P}_{0,1},\boldsymbol{P}_{2,1}) + \tfrac{1}{6}\mathrm{Dir}(\boldsymbol{P}_{0,2},\boldsymbol{P}_{2,0}),\\
\tilde{\boldsymbol{H}}_{1,3} &= \tfrac{1}{2}\mathrm{Dir}(\boldsymbol{P}_{0,1},\boldsymbol{P}_{2,2}) + \tfrac{1}{2}\mathrm{Dir}(\boldsymbol{P}_{0,2},\boldsymbol{P}_{2,1}),\\
\tilde{\boldsymbol{H}}_{1,4} &= \mathrm{Dir}(\boldsymbol{P}_{0,2},\boldsymbol{P}_{2,2}),\\
\tilde{\boldsymbol{H}}_{2,0} &= 2\,\mathrm{Dir}(\boldsymbol{P}_{1,0},\boldsymbol{P}_{2,0}),
\end{aligned}$$



(a) Control points of hodograph (b) Vectors for bounding

Fig. 1. s-Derivative of rational biquadratic surface.

$$\begin{aligned}
\tilde{\boldsymbol{H}}_{2,1} &= \mathrm{Dir}(\boldsymbol{P}_{1,0}, \boldsymbol{P}_{2,1}) + \mathrm{Dir}(\boldsymbol{P}_{1,1}, \boldsymbol{P}_{2,0}), \\
\tilde{\boldsymbol{H}}_{2,2} &= \tfrac{1}{3}\,\mathrm{Dir}(\boldsymbol{P}_{1,0}, \boldsymbol{P}_{2,2}) + \tfrac{4}{3}\,\mathrm{Dir}(\boldsymbol{P}_{1,1}, \boldsymbol{P}_{2,1}) + \tfrac{1}{3}\,\mathrm{Dir}(\boldsymbol{P}_{1,2}, \boldsymbol{P}_{2,0}), \\
\tilde{\boldsymbol{H}}_{2,3} &= \mathrm{Dir}(\boldsymbol{P}_{1,1}, \boldsymbol{P}_{2,2}) + \mathrm{Dir}(\boldsymbol{P}_{1,2}, \boldsymbol{P}_{2,1}), \\
\tilde{\boldsymbol{H}}_{2,4} &= 2\,\mathrm{Dir}(\boldsymbol{P}_{1,2}, \boldsymbol{P}_{2,2}).
\end{aligned} \tag{38}$$

Fig. 1(a) shows the structure of these control points.

3.2. Bound of derivative direction

Although $(2m-1)(2n+1)$ control points are required for the scaled hodograph, the derivative direction can be bounded by smaller number of vectors. For Eq. (32), the method in Section 2.3 can be applied, and thus, $\mathrm{Dir}(\boldsymbol{P}[s,t], \boldsymbol{P}_s[s,t])$ can be bounded by the convex hull of the m vectors $\mathrm{Dir}(\boldsymbol{Q}_i[t], \boldsymbol{Q}_{i+1}[t])$ $(i = 0, 1, \ldots, m-1)$. For Eq. (33), however, $\mathrm{Dir}(\boldsymbol{Q}_a[t], \boldsymbol{Q}_b[t])$ cannot be bounded in such a way. Therefore, $\mathrm{Dir}(\boldsymbol{P}[s,t], \boldsymbol{P}_s[s,t])$ can be bounded by the convex hull of the $m(2n+1)$ vectors

$$\sum_{j=\max(0,l-n)}^{\min(l,n)} \binom{n}{j}\binom{n}{l-j} \mathrm{Dir}(\boldsymbol{P}_{i,j}, \boldsymbol{P}_{i+1,l-j})$$
$$(i = 0, 1, \ldots, m-1;\ l = 0, 1, \ldots, 2n). \tag{39}$$

For example, the s-derivative direction of a biquadratic patch can be bounded by the following 10 vectors:

$$\begin{aligned}
&\mathrm{Dir}(\boldsymbol{P}_{0,0}, \boldsymbol{P}_{1,0}), \\
&2\,\mathrm{Dir}(\boldsymbol{P}_{0,0}, \boldsymbol{P}_{1,1}) + 2\,\mathrm{Dir}(\boldsymbol{P}_{0,1}, \boldsymbol{P}_{1,0}), \\
&\mathrm{Dir}(\boldsymbol{P}_{0,0}, \boldsymbol{P}_{1,2}) + 4\,\mathrm{Dir}(\boldsymbol{P}_{0,1}, \boldsymbol{P}_{1,1}) + \mathrm{Dir}(\boldsymbol{P}_{0,2}, \boldsymbol{P}_{1,0}), \\
&2\,\mathrm{Dir}(\boldsymbol{P}_{0,1}, \boldsymbol{P}_{1,2}) + 2\,\mathrm{Dir}(\boldsymbol{P}_{0,2}, \boldsymbol{P}_{1,1}), \\
&\mathrm{Dir}(\boldsymbol{P}_{0,2}, \boldsymbol{P}_{1,2}), \\
&\mathrm{Dir}(\boldsymbol{P}_{1,0}, \boldsymbol{P}_{2,0}), \\
&2\,\mathrm{Dir}(\boldsymbol{P}_{1,0}, \boldsymbol{P}_{2,1}) + 2\,\mathrm{Dir}(\boldsymbol{P}_{1,1}, \boldsymbol{P}_{2,0}),
\end{aligned}$$

$$\begin{aligned}&\mathrm{Dir}(\boldsymbol{P}_{1,0},\boldsymbol{P}_{2,2})+4\,\mathrm{Dir}(\boldsymbol{P}_{1,1},\boldsymbol{P}_{2,1})+\mathrm{Dir}(\boldsymbol{P}_{1,2},\boldsymbol{P}_{2,0}),\\&2\,\mathrm{Dir}(\boldsymbol{P}_{1,1},\boldsymbol{P}_{2,2})+2\,\mathrm{Dir}(\boldsymbol{P}_{1,2},\boldsymbol{P}_{2,1}),\\&\mathrm{Dir}(\boldsymbol{P}_{1,2},\boldsymbol{P}_{2,2}).\end{aligned}\tag{40}$$

Fig. 1(b) shows the structure of these vectors.

3.3. Bound of derivative magnitude

An upper bound of the derivative magnitude for a rational surface can be obtained in the following way. From Eqs. (29) and (32), by applying the method used in Section 2.4, the derivative magnitude can be bounded with the control points $\boldsymbol{Q}_i[t]$ in Eq. (31):

$$\begin{aligned}\left\|\frac{\partial}{\partial s}\tilde{\boldsymbol{P}}[s,t]\right\| &= \frac{m}{(W[s,t])^2}\left\|\sum_{k=0}^{2m-2}(1-t)^{2m-2-k}t^k\right.\\&\qquad\left.\times\sum_{\substack{i+j=k+1\\0\leqslant i\leqslant m-1\\1\leqslant j\leqslant m}}\binom{m-1}{i}\binom{m-1}{j-1}\mathrm{Dir}(\boldsymbol{Q}_i[t],\boldsymbol{Q}_j[t])\right\|\\&\leqslant\frac{mW_{\max}^2}{W_{\min}^2}\max_{0\leqslant i\leqslant m-1}\|\tilde{\boldsymbol{Q}}_{i+1}[t]-\tilde{\boldsymbol{Q}}_i[t]\|,\end{aligned}\tag{41}$$

where

$$W_{\max}\equiv\max_{\substack{0\leqslant i\leqslant m\\0\leqslant j\leqslant n}}W_{i,j},\tag{42}$$

$$W_{\min}\equiv\min_{\substack{0\leqslant i\leqslant m\\0\leqslant j\leqslant n}}W_{i,j}.\tag{43}$$

The distance between the control points $\boldsymbol{Q}_i[t]$ can be bounded as follows:

$$\begin{aligned}&\|\tilde{\boldsymbol{Q}}_{i+1}[t]-\tilde{\boldsymbol{Q}}_i[t]\|\\&\quad=\frac{\|\mathrm{Dir}(\boldsymbol{Q}_i[t],\boldsymbol{Q}_{i+1}[t])\|}{W_i[t]W_{i+1}[t]}\\&\quad\leqslant\frac{1}{W_{\min}^2}\left\|\sum_{l=0}^{2n}\frac{B_l^{2n}[t]}{\binom{2n}{l}}\sum_{j=\max(0,l-n)}^{\min(l,n)}\binom{n}{j}\binom{n}{l-j}\mathrm{Dir}(\boldsymbol{P}_{i,j},\boldsymbol{P}_{i+1,l-j})\right\|\\&\quad\leqslant\frac{S_{\max}}{W_{\min}^2},\end{aligned}\tag{44}$$

where $W_i[t]$ denotes the weight of $\boldsymbol{Q}_i[t]$, and

$$S_{\max}\equiv\max_{\substack{0\leqslant i\leqslant m-1\\0\leqslant l\leqslant 2n}}\left\|\frac{1}{\binom{2n}{l}}\sum_{j=\max(0,l-n)}^{\min(l,n)}\binom{n}{j}\binom{n}{l-j}\mathrm{Dir}(\boldsymbol{P}_{i,j},\boldsymbol{P}_{i+1,l-j})\right\|.\tag{45}$$

Therefore, we get the following upper bound:

$$\left\| \frac{\partial}{\partial s} \tilde{\boldsymbol{P}}[s,t] \right\| \leqslant \frac{m W_{\max}^2 S_{\max}}{W_{\min}^4}. \tag{46}$$

Notice that $S_{\max}$ is the maximum magnitude of the $m(2n+1)$ vectors

$$\frac{1}{\binom{2n}{l}} \sum_{j=\max(0,l-n)}^{\min(l,n)} \binom{n}{j} \binom{n}{l-j} \operatorname{Dir}(\boldsymbol{P}_{i,j}, \boldsymbol{P}_{i+1,l-j})$$
$$(i=0,1,\ldots,m-1;\ l=0,1,\ldots,2n), \tag{47}$$

each of which has the same direction as each vector in Eq. (39). This means that Eq. (47) forms a set of *bounding vectors* that can be used to bound both derivative direction and magnitude.

3.4. Normal direction

The normal direction can be obtained simply by taking the cross product of the s- and t-scaled hodographs, where the degree of the normal direction is $(4m-2)\times(4n-2)$. Hohmeyer (1992) pointed out that the degree can be reduced to $(3m-1)\times(3n-1)$. We show here that it can actually be reduced to $(3m-2)\times(3n-2)$.

First of all, we define the notation:

$$\operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3) \equiv \left(\begin{vmatrix} Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \\ W_1 & W_2 & W_3 \end{vmatrix}, \begin{vmatrix} Z_1 & Z_2 & Z_3 \\ X_1 & X_2 & X_3 \\ W_1 & W_2 & W_3 \end{vmatrix}, \begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ W_1 & W_2 & W_3 \end{vmatrix} \right), \tag{48}$$

which provides a normal direction of three homogeneous points. This notation "Nrm" satisfies the following relations:

$$\begin{aligned}
&\operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_1,\boldsymbol{P}_1) = \operatorname{Nrm}(\boldsymbol{P}_2,\boldsymbol{P}_1,\boldsymbol{P}_1) = \operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_1) = \operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_1,\boldsymbol{P}_2) = \boldsymbol{0},\\
&\operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3) = \operatorname{Nrm}(\boldsymbol{P}_2,\boldsymbol{P}_3,\boldsymbol{P}_1) = \operatorname{Nrm}(\boldsymbol{P}_3,\boldsymbol{P}_1,\boldsymbol{P}_2),\\
&\qquad = -\operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_3,\boldsymbol{P}_2) = -\operatorname{Nrm}(\boldsymbol{P}_2,\boldsymbol{P}_1,\boldsymbol{P}_3) = -\operatorname{Nrm}(\boldsymbol{P}_3,\boldsymbol{P}_2,\boldsymbol{P}_1),\\
&\operatorname{Nrm}(k\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3) = \operatorname{Nrm}(\boldsymbol{P}_1,k\boldsymbol{P}_2,\boldsymbol{P}_3) = \operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,k\boldsymbol{P}_3) = k\operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3),\\
&\operatorname{Nrm}(\boldsymbol{P}_1+\boldsymbol{P}_2,\boldsymbol{P}_3,\boldsymbol{P}_4) = \operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_3,\boldsymbol{P}_4) + \operatorname{Nrm}(\boldsymbol{P}_2,\boldsymbol{P}_3,\boldsymbol{P}_4),\\
&\operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2+\boldsymbol{P}_3,\boldsymbol{P}_4) = \operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_4) + \operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_3,\boldsymbol{P}_4),\\
&\operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3+\boldsymbol{P}_4) = \operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3) + \operatorname{Nrm}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_4),
\end{aligned} \tag{49}$$

where k is a scalar value.

The normal direction at $\boldsymbol{P}[s,t]$ is given by

$$\operatorname{Nrm}(\boldsymbol{P}[s,t], \boldsymbol{P}_s[s,t], \boldsymbol{P}_t[s,t]). \tag{50}$$

To calculate it efficiently, we define S_{ij} as follows:

$$
\begin{aligned}
S_{00} &\equiv \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} B_i^{m-1}[s]B_j^{n-1}[t]\, P_{i,j}, &\quad S_{01} &\equiv \sum_{i=0}^{m-1}\sum_{j=1}^{n} B_i^{m-1}[s]B_{j-1}^{n-1}[t]\, P_{i,j} \\
S_{10} &\equiv \sum_{i=1}^{m}\sum_{j=0}^{n-1} B_{i-1}^{m-1}[s]B_j^{n-1}[t]\, P_{i,j}, &\quad S_{11} &\equiv \sum_{i=1}^{m}\sum_{j=1}^{n} B_{i-1}^{m-1}[s]B_{j-1}^{n-1}[t]\, P_{i,j}.
\end{aligned}
\tag{51}
$$

Then P, P_s, and P_t are simply expressed with S_{ij}:

$$
\begin{aligned}
P[s,t] &= (1-s)(1-t)\, S_{00} + s(1-t)\, S_{10} + (1-s)t\, S_{01} + st\, S_{11}, \\
P_s[s,t] &= m\sum_{i=0}^{m-1}\sum_{j=0}^{n} B_i^{m-1}[s]B_j^n[t]\, (P_{i+1,j} - P_{i,j}) \\
&= m[(1-t)\, S_{10} + t\, S_{11} - (1-t)\, S_{00} - t\, S_{01}], \\
P_t[s,t] &= n\sum_{i=0}^{m}\sum_{j=0}^{n-1} B_i^{m}[s]B_j^{n-1}[t]\, (P_{i,j+1} - P_{i,j}) \\
&= n[(1-s)\, S_{01} + s\, S_{11} - (1-s)\, S_{00} - s\, S_{10}].
\end{aligned}
\tag{52}
$$

Substituting Eqs. (52) into Eq. (50) and applying Eq. (49), we get

$$
\begin{aligned}
&\mathrm{Nrm}(P[s,t], P_s[s,t], P_t[s,t]) \\
&\quad = mn[(1-s)(1-t)\, \mathrm{Nrm}(S_{00}, S_{10}, S_{01}) + s(1-t)\, \mathrm{Nrm}(S_{00}, S_{10}, S_{11}) \\
&\qquad + (1-s)t\, \mathrm{Nrm}(S_{00}, S_{11}, S_{01}) + st\, \mathrm{Nrm}(S_{10}, S_{11}, S_{01})].
\end{aligned}
\tag{53}
$$

Since the degree of S_{ij} is $(m-1)\times(n-1)$, the degree of the normal direction is $(3m-2)\times(3n-2)$. For example, the normal direction of a bilinear surface is degree 1×1:

$$
\begin{aligned}
\mathrm{Nrm}(P[s,t], P_s[s,t], P_t[s,t]) &= (1-s)(1-t)\, \mathrm{Nrm}(P_{0,0}, P_{1,0}, P_{0,1}) \\
&\quad + s(1-t)\, \mathrm{Nrm}(P_{0,0}, P_{1,0}, P_{1,1}) \\
&\quad + (1-s)t\, \mathrm{Nrm}(P_{0,0}, P_{1,1}, P_{0,1}) \\
&\quad + st\, \mathrm{Nrm}(P_{1,0}, P_{1,1}, P_{0,1}).
\end{aligned}
\tag{54}
$$

Table 1 summarizes the degree of scaled hodographs and normal direction for rational surfaces.

3.5. Bound of normal direction

There are two ways to create a bound of the normal direction. One is from the bounds of s- and t derivative directions, which is discussed in (Sederberg and Meyers, 1988) for polynomial surfaces: create a bounding cone for each derivative direction from which a bounding cone for the normals can be calculated. This idea can be applied

Table 1
Degree of scaled hodographs and normal directions

Patch	Bilinear	Biquadratic	Bicubic	$m \times n$
s-hodograph	0×2	2×4	4×6	$(2m-2) \times 2n$
t-hodograph	2×0	4×2	6×4	$2m \times (2n-2)$
normal	1×1	4×4	7×7	$(3m-2) \times (3n-2)$

Table 2
Number of vectors required for bounding

Patch	Bilinear	Biquadratic	Bicubic	$m \times n$
s-hodograph	3	10	21	$m(2n+1)$
t-hodograph	3	10	21	$(2m+1)n$
normal	4	25	64	$(3m-1)(3n-1)$

to a rational surface, where it is necessary to evaluate $m(2n+1)$ vectors for s-derivative and $(2m+1)n$ vectors for t-derivative. The other method is to calculate the normal direction by using the method in Section 3.4. Here, the set of control points can bound the direction.

For polynomial surfaces, Sederberg and Meyers (1988) claimed that the latter method generally gives a tighter bound, but requires much more computation. This is also true for rational surfaces. Table 2 shows the number of vectors required for the bounds. For both biquadratic and bicubic patches, the number of vectors required to bound normals is larger than total number of vectors for bounding s- and t-derivative directions. Furthermore, calculation of "Nrm" is more expensive than that of "Dir". Thus, we recommend that normal bounds be computed from derivative bounds as in (Sederberg and Meyers, 1988), except for bilinear surfaces.

4. Discussion

We have tried to find tighter or simpler bounds for derivative directions and normals of rational surfaces. The following hypotheses seem plausible, but have been proven false.

Hypothesis 1. *For a degree $m \times n$ rational surface $\boldsymbol{P}[s,t]$, the s-derivative direction is bounded by s-derivative directions of mn bilinear surfaces, each of which are defined by four control points:* $\boldsymbol{P}_{i,j}, \boldsymbol{P}_{i+1,j}, \boldsymbol{P}_{i,j+1}, \boldsymbol{P}_{i+1,j+1}$ $(i = 0,1,\dots,m-1;\ j = 0,1,\dots,n-1)$.

Hypothesis 2. *On a degree $m \times n$ rational surface $\boldsymbol{P}[s,t]$, the surface normal is bounded by the normals of* $4mn$ *triangles:*

$$\triangle \boldsymbol{P}_{i,j}\boldsymbol{P}_{i+1,j}\boldsymbol{P}_{i,j+1},\quad \triangle \boldsymbol{P}_{i,j}\boldsymbol{P}_{i+1,j}\boldsymbol{P}_{i+1,j+1},\quad \triangle \boldsymbol{P}_{i,j}\boldsymbol{P}_{i+1,j+1}\boldsymbol{P}_{i,j+1},\quad \triangle \boldsymbol{P}_{i+1,j}\boldsymbol{P}_{i+1,j+1}\boldsymbol{P}_{i,j+1}$$

$(i = 0,1,\dots,m-1;\ j = 0,1,\dots,n-1)$.

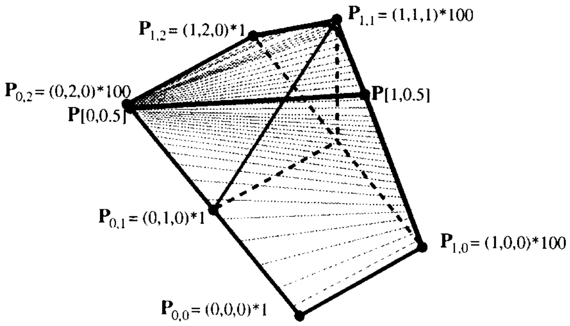


Fig. 2. A counterexample of the hypotheses.

Hypothesis 1 is true for any degree $m \times 1$ surface, and Hypothesis 2 is true for any bilinear surface (see Eq. (54)). However, they are not true in general. Fig. 2 shows a counterexample. For this degree 1×2 patch, the s-derivative direction and the surface normal at $P[1, 0.5]$ are $(1.00, -1.29, 0.66)$ and $(-0.81, -0.41, 0.42)$, respectively. These directions defy the hypothesized bounds. If Hypothesis 1 were true, the projection of the s-derivative onto xy-plane, i.e. $(1.00, -1.29, 0)$, should be inside of $(1, 1, 0)$ and $(1, -1, 0)$. If Hypothesis 2 were true, the normal direction should be bounded by $(0, -1, 1)$, $(0, 1, 1)$, $(-1, -1, 1)$, and $(-1, 1, 1)$.

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