# *The Binomial k-Clique*

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**Abstract:** **A finite collection C of $k$-sets, where $k \geq 2$, is called a $k$-clique if every two $k$-sets (called lines) in C have a nonempty intersection and a $k$-clique is a called a maximal $k$-clique if $|C| < \infty$ and C is maximal with respect to this property. That is, every two lines in C have a nonempty intersection and there does not exist $A$ such that $|A| \leq k$, $A \notin C$ and $A \cap X \neq \emptyset$ for all $X \in C$. An elementary example of a maximal $k$-clique is furnished by the family of all the $k$-subsets of a $(2k-1)$-set. This $k$-clique will be called the binomial $k$-clique. This paper is intended to give some combinatorial characterizations of the binomial $k$-clique as a maximal $k$-clique. The techniques developed are then used to provide a large number of examples of mutually nonisomorphic maximal $k$-cliques for a fixed value of $k$.** 
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## 1. INTRODUCTION AND PRELIMINARIES

Let $k$ be a natural number, $k \geq 2$. A $k$-uniform hypergraph $\mathbf{C}$ is a finite family of $k$-sets (called its lines or lines, whose number is finite). $\mathbf{C}$ is called a $k$-clique if every pair of $k$-sets in $\mathbf{C}$ have a nonempty intersection. A $k$-clique is called a maximal $k$-clique (or a maximal clique) if there does not exist a set $A$ of size $\leq k$ such that $A \notin \mathbf{C}$ and $A \cap X \neq \emptyset$ for all $X \in \mathbf{C}$. A nonempty set $M$ of size $k$ is said to be a spanning set or a covering set of a $k$-clique $\mathbf{C}$ if $X \cap M \neq \emptyset$ for all $X \in \mathbf{C}$ and for a maximal $k$-clique $\mathbf{C}$ every covering set is a line in $\mathbf{C}$ and conversely every line in $\mathbf{C}$ is a covering set.

The maximal $k$-clique problem has attracted much attention in the last 30 years. A large number of constructions of maximal $k$-cliques are known in the literature (see the references) where the main focus was on finding maximal $k$-cliques (for a given $k$) with the smallest number of lines. In some sense, the problem we discuss here is of the opposite kind. We wish to study maximal $k$-cliques that have a large number of lines. Such maximal $k$-cliques have been considered in Drake [4], Tuza [9, 10] and Erdös-Lovász [6]). In particular, the object, which is of central focus in this paper, is a combinatorial characterization of the following well-known construction.

**Definition 1.1.** *Let $k \geq 2$ and $S$ be a set of order $2k-1$. Let $\mathbf{C}$ consist of all the $k$-subsets of $S$. This $k$-uniform hypergraph is called* **the binomial $k$-clique**.

It is easily seen that $\mathbf{C}$ is a $k$-clique as two disjoint sets in $\mathbf{C}$ would require $|S| \geq 2k$. Further $\mathbf{C}$ is a maximal $k$-clique because if $|E| \leq k$ and $E \notin \mathbf{C}$ then $E$ is not a $k$-subset of $S$ and hence $|E \cap S| \leq k-1$ so that $\exists$ some $X \in \mathbf{C}$ such that $X \cap E = (X \cap S) \cap E = X \cap (S \cap E) = \emptyset$. Note that for the binomial $k$-clique $\mathbf{C}$, we have $|\mathbf{C}| = \binom{2k-1}{k}$. Let $v$ denote the number of points in a maximal $k$-clique $\mathbf{C}$ and let $b$ denote the number of lines. By two-way counting of the set $\{(x, X) : x \in X \in \mathbf{C}\}$, we get $kb \leq v(b - 2k + 2)$. Among all the maximal $k$-cliques, $\mathbf{C}$, the binomial $k$-clique seems to have the largest ratio $\frac{b}{v}$. In fact, for binomial $k$-clique we have

$$\frac{b}{v} = \binom{2k-1}{k-1}\frac{1}{2k-1} = (k+1)C_k,$$

where $C_k$ represents the Catalan Number. We also note that a construction due to Drake [4] starts with a maximal $k$-clique, and recursively constructs maximal $(k+1)$-clique with the number of lines greater than the one given by the binomial $k$-clique.

This paper is intended to give a combinatorial characterization of the binomial clique. This is done in Sections 2 and 3. We then go on to construct a large number of maximal $k$-cliques that have the same number lines as those of the binomial $k$-clique but have very large number of points. This theme was initiated in a seminal paper of Erdös and Lovász [6]. In particular, we show that for a fixed $k$, we can construct a large number of nonisomorphic maximal $k$-cliques with the number of such nonisomorphic maximal $k$-cliques lower bounded by an exponential function in $k$ (and hence goes to infinity with $k$). The maximal $k$-clique theory has a close association with extremal hypergraphs. Let $\mathbf{C}$ be a finite family of $k$-element sets and $\nu(\mathbf{G})$ denote the maximum number of pairwise disjoint sets in $\mathbf{C}$. If $\nu(\mathbf{C}) = \mathbf{1}$, then any two lines in $\mathbf{C}$ have a non-empty intersection. The collection of $k$-sets $\mathbf{C}$ is said to be $\nu$-critical if replacing any member of $\mathbf{C}$ by a proper nonempty subset, the number of pairwise disjoint subsets becomes larger. When $\mathbf{C}$ is 1-critical, each line in $\mathbf{C}$ intersects with each other line and replacing a line by one of its proper subset, there exists at least one line in $\mathbf{C}$ disjoint with it. It is clear that all maximal $k$-cliques are 1-critical uniform $k$-hypergraphs but a 1-critical $k$-uniform hypergraph need not be a maximal $k$-clique. The following result is due to Tuza [10].

**Theorem 1.2.** *Let $\mathbf{C}$ be a maximal $k$-clique. Then $|\mathbf{C}| \leq (1 - e^{-1} + \epsilon_k)k^k$ where $0 < \epsilon_k < e^{-1}$ and tends to $0$ for large $k$.*

*It now makes sense to define $M(k)$ $(< \infty)$ to be the largest size $|\mathbf{C}|$ of a maximal $k$-clique $\mathbf{C}$.*

## 2. CHARACTERIZATIONS OF THE BINOMIAL CLIQUE

The set-up is as follows. $\mathbf{C}$ denotes a (maximal) $k$-clique. The members of $\mathbf{C}$ are denoted by upper case letters $A, B, X, Y,$ etc. The point set of $\mathbf{C}$ will be denoted by $V$ and the members of $V$ are denoted by lower case letters $a, x, y,$ etc. The term *flag* is used to denote a point-line pair $(x, X)$ where $x$ is contained in $X$. By $\dot\cup$, we mean disjoint union. Given a $k$-clique $\mathbf{C}$, we say that a *set $M$ is a blocking set of* $\mathbf{C}$ if $M \cap X \neq \emptyset\ \forall X \in \mathbf{C}$. Notice that if a $k$-clique has blocking set of size $< k$ then, we can add some more points not in the $k$-clique to make it a blocking set of size $k$. Also observe that $\mathbf{C}$ is a maximal $k$-clique iff there is no set $M$ of $k$ points blocking $\mathbf{C}$ such that $M \notin \mathbf{C}$. We also note that a blocking set has been called a covering or a spanning set in Section 1.

**Lemma 2.1.** *Let $\mathbf{C}$ be a maximal $k$-clique.*

*(a) Let $a \in A$ where $A$ is a line of $\mathbf{C}$. Then there exists a line $B$ such that $A \cap B = \{a\}$.*
*(b) Let $A \cap B = \{a\}$, where $A, B \in \mathbf{C}$, then $B$ is the unique line intersecting $A$at $a$ if and only if $A - \{a\} \cup \{b\} \in \mathbf{C}$ for all $b \in B, b \neq a$.*

*Proof.* Suppose such a $B$ does not exist. Then $A - \{a\}$ is a blocking set of $\mathbf{C}$, a contradiction as $\mathbf{C}$ is a maximal $k$-clique, proving (a). Suppose $B$ is the unique line such that $A \cap B = \{a\}$. Then the only line that is not blocked by $A - \{a\}$ is $B$. This forces that $A - \{a\} \cup \{b\} \in \mathbf{C}$ for all $b \in B$. Conversely suppose $A - \{a\} \cup \{b\} \in \mathbf{C}$ for all $b \in B$ with $b \neq a$. Let $B = \{a = b_1, b_2, \ldots, b_k\}$. Let $A_j = A - \{a\} \dot\cup \{b_j\}$ for $j = 2, 3, \ldots, k$. Suppose a line $Y$ meets $A$ in $a$ alone. Then $Y$ must intersect $A_j$ in $b_j$, $j = 2, 3, \ldots, k$. So $Y$ contains each of the point $a, b_2, b_3, \ldots, b_k$ and hence $Y = B$. □

An easy consequence of Lemma 2.1 is the following.

**Corollary 2.2.** *Let $\mathbf{C}$ be a maximal $k$-clique. Let $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ such that $X, Y \in \mathbf{C}$. Suppose $X$ is the unique line intersecting $Y$ at $x_1 = y_1$. Then there exist lines $Y_2, Y_3, \ldots, Y_k$ in $\mathbf{C}$ such that*

*(a) $X \cap Y_i = \{x_i\}$ for $i = 2, 3, \ldots, k$ and*
*(b) $X$ is the unique line that intersects $Y_i$ at $x_i$.*

*Proof.* $Y \cap X = \{x_1\}$ and $X$ is a unique such line. By Lemma 2.1(b) we have $Y - \{x_1\} \cup \{x_i\} \in \mathbf{C}$. Let $Y_i = Y - \{x_1\} \cup \{x_i\}$. Then $X \cap Y_i = \{x_i\}$ for $i = 2, 3, \ldots, k$. This proves (a). Note that $Y_i - \{x_i\} \cup \{x_j\} = Y_j \in \mathbf{C}$ for all $x_i \in X$. Hence by Lemma 2.1(b) again, $X$ is unique line that intersects $Y_i$ at $x_i$. □

**Definition 2.3.** *Let $\mathbf{C}$ be a maximal $k$-clique. Let $X \in \mathbf{C}$ and $x \in X$. A flag $(x, X)$ is said to be extremal if there exists a unique $Y$ such that $X \cap Y = \{x\}$. A line $X \in \mathbf{C}$ is said to be extremal if for all $x \in X$ the flag $(x, X)$ is extremal. A point $p$ is said to be extremal if each line containing $p$ is extremal.*

**Theorem 2.4.** *Let $k \geq 3$. If $\mathbf{C}$ is a maximal $k$-clique with an extremal point, then $\mathbf{C}$ is the binomial clique and conversely.*

*Proof.* Our set up and notations are as follows. $p$ is an extremal point in $\mathbf{C}$ (which we fix). $X \cap Y = \{p\}$, $X' = X - \{p\} = \{x_1, x_2, \ldots, x_{k-1}\}$, $Y' = Y - \{p\} = \{y_1, y_2, \ldots, y_{k-1}\}$. Converse follows easily and the proof of Theorem 2.4 is given through the following steps.

*Step 1*

(i) $\forall\, y \in Y'$, $X' \dot{\cup} \{y\} \in \mathbf{C}$ and

(ii) $\forall\, x \in X'$, $Y' \dot{\cup} \{x\} \in \mathbf{C}$. □

*Proof.* Follows from Lemma 2.1(b).

*Step 2*

(i) $\forall\, y \in Y', \forall\, x \in X'$, $Y - \{y\} \dot{\cup} \{x\} \in \mathbf{C}$.

(ii) $\forall\, x \in X', \forall\, y \in Y'$, $X - \{x\} \dot{\cup} \{y\} \in \mathbf{C}$. □

*Proof.* Let $x \in X'$. By Step 1, $U = \{x\} \dot{\cup} Y' \in \mathbf{C}$. Further $p \in X$ and hence $X$ is an extremal line and the unique line intersecting $X$ in $x$ must be $U$. By applying Step 1 again, $X - \{x\} \dot{\cup} \{y\} \in \mathbf{C}$ and (ii) holds by symmetry.

*Step 3* Let $r \geq 1$. Then $\forall\, \{x_1, x_2, \ldots, x_r\} \subseteq X'$ and $\forall\, \{y_1, y_2, \ldots, y_r\} \subseteq Y'$, we have

(i) $\{x_1, x_2, \ldots, x_r, y_r, \ldots, y_{k-1}\} \in \mathbf{C}$.

(ii) $\{y_1, y_2, \ldots, y_r, x_r, \ldots, x_{k-1}\} \in \mathbf{C}$.

(iii) $X - \{x_1, x_2, \ldots, x_r\} \dot{\cup} \{y_1, y_2, \ldots, y_r\} \in \mathbf{C}$.

(iv) $Y - \{y_1, y_2, \ldots, y_r\} \dot{\cup} \{x_1, x_2, \ldots, x_r\} \in \mathbf{C}$. □

*Proof.* Let $r = 1$ . Then (i) and (ii) follow from Step 1, and (iii) and (iv) follow from Step 2. So the assertion holds for $r = 1$. This is the basis of induction. Let $r \geq 2$ and assume that the result is true for $r - 1$. By induction hypothesis

$$\begin{aligned} L &= X - \{x_1, x_2, \ldots, x_{r-1}\} \dot{\cup} \{y_1, y_2, \ldots, y_{r-1}\} \\ &= \{p, y_1, \ldots, y_{r-1}, x_r, x_{r+1}, \ldots, x_{k-1}\} \in \mathbf{C}. \end{aligned}$$

Similarly,

$$\begin{aligned} M &= Y - \{y_1, y_2, \ldots, y_{r-1}\} \dot{\cup} \{x_1, x_2, \ldots, x_{r-1}\} \\ &= \{p, x_1, \ldots, x_{r-1}, y_r, y_{r+1}, \ldots, y_{k-1}\} \in \mathbf{C}. \end{aligned}$$

Here, $L$ and $M$ intersect at the unique point $p$. Hence, Step 1 is applicable to get $M - \{p\} \dot{\cup} \{x_r\} \in \mathbf{C}$. Let this line be denoted by $T$. Thus, $T = \{x_1, \ldots, x_{r-1}, x_r, y_r, y_{r+1}, \ldots, y_{k-1}\} \in \mathbf{C}$ which proves (i) and (ii) follows by symmetry.

We can thus assume that (i) and (ii) both hold for $r$. Further, $T$ and $L$ intersect at the unique point $x_r$. Here, $p$ is extremal and $p \in L$. So, $L$ also is extremal. Hence the flag $(x_r, L)$ is extremal. We see that $T$ must be the unique line that intersects $L$ in $x_r$. Applying Step 1, we get $L - \{x_r\} \dot{\cup} \{y_r\} \in \mathbf{C}$ (here $y_r \in T$ but $y_r \notin L$). So

$$\{p, y_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_{k-1}\} = X - \{x_1, x_2, \ldots, x_r\} \dot{\cup} \{y_1, y_2, \ldots, y_r\} \in \mathbf{C}.$$

This proves (iii) and (iv) follows by symmetry. We have thus completed proof of Step 3. □

*Proof of Theorem 2.4.* Let $S = X \cup Y = \{p, x_1, x_2, \ldots, x_{k-1}, y_1, y_2, \ldots, y_{k-1}\}$. Let $A$ be any $k$-subset of $S$. Let $p \in A$. Let $A \neq X, Y$. Note that $S = Y \dot{\cup} X'$. Let $|A \cap (Y - p)| = r$ where $r \geq 0$. Then, $|A \cap Y| = r + 1$ and therefore $|A \cap X'| = k - (r + 1) = (k - 1) - r$. So, by relabeling of points, we can assume that $A = X - \{x_1, x_2, \ldots, x_r\} \dot{\cup} \{y_1, y_2, \ldots, y_r\}$ and then $A \in \mathbf{C}$ by Step 3 (iii). On the other hand if $p \notin A$ then $|A \cap X'| = r$, for some $r \geq 1$ and then $|A \cap Y'| = k - r$. Thus by relabeling of points again, we can assume that $A = \{x_1, x_2, \ldots, x_r, y_r, \ldots, y_{k-1}\}$ and by Step 3 (i), $A \in \mathbf{C}$. So all the subsets of $S$ are in $\mathbf{C}$. It is then easily seen that $\mathbf{C}$ is the binomial $k$-clique. □

**Theorem 2.5.** *Let* $\mathbf{C}$ *be a maximal* $k$*-clique. Then the following are equivalent.*

- *(a)* $\mathbf{C}$ *is a binomial* $k$*-clique.*
- *(b) Given any flag* $(a, A)$, $\exists$ *at least* $k - 1$ *lines* $X$ *in* $\mathbf{C}$ *such that* $X \cap A = A - \{a\}$.
- *(c) Given any flag* $(a, A)$ $\exists$ *exactly* $k - 1$ *lines* $X$ *in* $\mathbf{C}$ *such that* $X \cap A = A - \{a\}$.

*Proof.* (a) implies, (b) follows by looking at the construction of the binomial $k$-clique. Let (b) hold. Each line $X$ with $X \cap A = A - \{a\} = A'$ has exactly one point $x$ such that $A' \dot{\cup} \{x\} = X$ and if $X \neq Y$ with $Y \cap A = A'$ and $A' \dot{\cup} \{y\} = Y$, we must have $x \neq y$. Let the set of lines that intersect $A$ in $A'$ be $T = \{X_j : j = 1, 2, \ldots, t\}$ where $X_j = A' \dot{\cup} \{x_j\}$. Then by assumption $t \geq k - 1$. Let $B$ be some line such that $B \cap A = \{a\}$. Then $A' \cap B = \emptyset$ and therefore $B$ must contain each $x_j$ and $|B - \{a\}| \leq k - 1$ implies $t \leq k - 1$. So, $t = k - 1$ and $B = \{a, x_1, x_2, \ldots, x_{k-1}\}$. If (c) holds then, continuing the above argument we see that the line $B$ such that $B \cap A = \{a\}$ is unique and hence each flag $(a, A)$ is extremal. So by Theorem 2.4, $\mathbf{C}$ is the binomial $k$-clique and hence (a) holds. □

**Corollary 2.6.** *Let* $\mathbf{C}$ *be a maximal* $k$*-clique containing a point* $p$ *with the following property: for any line* $X$ *containing the point* $p$ *and for each* $x \in X$ *there exist at least (and in fact exactly)* $k - 1$ *lines* $Y$ *such that* $X \cap Y = X - \{x\}$. *Then* $\mathbf{C}$ *is the binomial* $k$*-clique.*

**Example 2.7.** *(Erdös-Lovász [6]) Let* $\mathbf{C}$ *denote the maximal* 3*-clique on the point set* $V = \{1, 2, 3, 4, 5, 6, 7\}$. *The lines of* $\mathbf{C}$ *are from two sets* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *where* $\mathbf{C}_1$ *is given by*

$$\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$

*and* $\mathbf{C}_2$ *is given by*

$$\{\{1, 2, 5\}, \{3, 4, 5\}, \{1, 3, 6\}, \{2, 4, 6\}\{1, 4, 7\}, \{2, 3, 7\}\}.$$

*This clique is obtained from the binomial* 3*-clique on the point set* $\{1, 2, 3, 4, 5\}$ *by replacement of* 5 *with* 6 *in the lines* $\{1, 3, 5\}, \{2, 4, 5\}$ *and by the replacement of* 5 *with* 7 *in the lines* $\{1, 4, 5\}, \{2, 3, 5\}$. *The same idea is explored in Section 4 for all values of* $k$. *It is easy to verify that* $\mathbf{C}$ *is a maximal* 3*-clique. The flag* $(1, \{1, 2, 5\})$ *is not extremal since both* $\{1, 3, 6\}$ *and* $\{1, 4, 7\}$ *intersect* $\{1, 2, 5\}$ *in* 1 *and hence the line* $\{1, 2, 5\}$ *is not extremal. We thus see (by symmetry of the construction) that no line in* $\mathbf{C}_2$ *is extremal. On the other hand, each line in* $\mathbf{C}_1$ *is extremal. Since each point* $p$ *in* $V$ *is on some line* *of* $\mathbf{C}_2$*, no point of* $\mathbf{C}$ *is extremal. This shows that the hypothesis of Theorem 2.4 cannot be weakened, in general.*

## 3. FURTHER CHARACTERIZATIONS OF BINOMIAL $k$-CLIQUES

*As in the previous sections,* $\mathbf{C}$ *denotes a maximal* $k$*-clique. We also fix a natural number* $t$ *where* $1 \leq t \leq k-1$.

**Definition 3.1.** *Let* $X \in \mathbf{C}$*. Let* $\{p_1, p_2, \ldots, p_t\} \subset X$*, then* $(\{p_1, p_2, \ldots, p_t\}, X)$ *is said to be a* $t$*-flag if the* $t$ *points* $p_1, p_2, \ldots, p_t$ *are all distinct. Note that this generalizes the earlier definition of a flag.*

**Definition 3.2.** *Let* $X \in \mathbf{C}$ *and let* $\{p_1, p_2, \ldots, p_t\}$ *be a* $t$*-subset of* $X$*. The* $t$*-flag* $(\{p_1, p_2, \ldots, p_t\}, X)$ *is said to be* $t$*-extremal if there exists a unique set* $A = A(\{p_1, p_2, \ldots, p_t\}, X)$ *such that the following conditions hold.*

- *(a)* $|A| = k-1$.
- *(b)* $A \cap X = \emptyset$.
- *(c) For each subset* $B$ *of* $A$ *with* $|B| = k-t$ *there exists a line*

$$\{p_1, p_2, \ldots, p_t\} \dot{\cup} B \in \mathbf{C}$$

  *and these are the only lines that intersect* $X$ *in* $\{p_1, p_2, \ldots, p_t\}$.
- *(d) Exchange Property: Let* $Y \cap X = \{p_1, p_2, \ldots, p_t\}$ *and let*

$$Y = \{p_1, p_2, \ldots, p_t\} \dot{\cup} \{y_1, y_2, \ldots, y_{k-t}\}$$

  *where* $y_1, y_2, \ldots, y_{k-t} \in A$*. Let* $A - Y = \{z_1, z_2, \ldots, z_{t-1}\}$*. Then the* $t$*-flag* $(\{p_1, p_2, \ldots, p_t\}, Y)$ *satisfies the properties (a),(b), and (c) mentioned above and*

$$A' = A(\{p_1, p_2, \ldots, p_t\}, Y) = X - \{p_1, p_2, \ldots, p_t\} \dot{\cup} \{z_1, z_2, \ldots, z_{t-1}\}.$$

*Properties (a) through (d) will be collectively referred to as* $t$*-extremal properties.*

**Theorem 3.3.** *Let* $\mathbf{C}$ *be a maximal* $k$*-clique. Let* $2 \leq t \leq \frac{k}{2}$*.* $\mathbf{C}$ *is a binomial* $k$*-clique if and only if there exists a point* $p$ *in* $\mathbf{C}$ *such that, for every line* $X$ *containing* $p$*, there is a* $t$*-subset, say* $B$ *of* $X$*, which is* $t$*-extremal and there exists a* $t$*-subset of* $X - B$*, which is* $t$*-extremal.*

*Proof.* It is clear that if $\mathbf{C}$ is a binomial $k$-clique the conditions specified in the Theorem 3.3 are satisfied. Now suppose the conditions specified in the statement of the Theorem are satisfied. Let $X$ be a line containing $p$. Let $X = \{p = p_1, p_2, \ldots, p_t, x_1, x_2, \ldots, x_{k-t}\}$ and let the $t$-flag $(\{p_1, p_2, \ldots, p_t\}, X)$ be $t$-extremal. Let $Y = \{p_1, p_2, \ldots, p_t, y_1, y_2, \ldots, y_{k-t}\}$ and $X \cap Y = \{p_1, p_2, \ldots, p_t\}$. Let

$$\begin{aligned} A &= A(\{p_1, p_2, \ldots, p_t\}, X) \\ &= Y - \{p_1, p_2, \ldots, p_t\} \dot{\cup} \{z_1, z_2, \ldots, z_{t-1}\} \\ &= \{z_1, z_2, \ldots, z_{t-1}, y_1, y_2, \ldots, y_{k-t}\}. \end{aligned}$$

Thus the $2k-1$ points

$$p_1, p_2, \ldots, p_t, x_1, x_2, \ldots, x_{k-t}, y_1, y_2, \ldots, y_{k-t}, z_1, z_2, \ldots, z_{t-1}$$

are all distinct. The proof will be given in the following steps.

*Step 1* Let $x = x_i$ for $i = 1, 2, \ldots, k-t$ and $z = z_j$ for $j = 1, 2, \ldots, t-1$. Then $X - \{x\}\dot{\cup}\{z\} \in \mathbf{C}$. □

*Proof.* Clearly $X$ is a line intersecting $Y$ at $\{p_1, p_2, \cdots, p_t\}$. We have

$$\begin{aligned} A &= A(\{p_1, p_2, \ldots, p_t\}, X) \\ &= Y - \{p_1, p_2, \ldots, p_t\}\dot{\cup}\{z_1, z_2, \ldots, z_{t-1}\} \\ &= \{z_1, z_2, \ldots, z_{t-1}, y_1, y_2, \ldots, y_{k-t}\}. \end{aligned}$$

Then, the $t$-extremal property (d) implies that

$$\begin{aligned} A' &= A(\{p_1, p_2, \ldots, p_t\}, Y) \\ &= X - \{p_1, p_2, \ldots, p_t\}\dot{\cup}\{z_1, z_2, \ldots, z_{t-1}\} \\ &= \{z_1, z_2, \ldots, z_{t-1}, x_1, x_2, \ldots, x_{k-t}\}. \end{aligned}$$

If $W = \{x_1, x_2, \ldots, x_{k-t}\} - \{x_i\}\dot{\cup}\{z_j\}$ then $W$ is a $(k-t)$ subset of $A'$ and by the $t$-extremal property (c) applied to the $t$-flag $(\{p_1, p_2, \ldots, p_t\}, Y)$, $W\dot{\cup}\{p_1, p_2, \ldots, p_t\} \in \mathbf{C}$. So, $X - \{x\}\dot{\cup}\{z\} \in \mathbf{C}$. □

*Step 2* Let $x = x_i$ and $y = y_j$ for $i = 1, 2, \ldots, k-t$ and $j = 1, 2, \ldots, k-t$ then $X - \{x\}\dot{\cup}\{y\} \in \mathbf{C}$.

*Proof.* Since $\{y_1, y_2, \ldots, y_{k-t}\} - \{y_j\}\dot{\cup}\{z_1\}$ is a $(k-t)$-subset of

$$A = A(\{p_1, p_2, \ldots, p_t\}, X) = \{z_1, z_2, \ldots, z_{t-1}, y_1, y_2, \ldots, y_{k-t}\}$$

using Step 1 for $Y$, we get

$$Y' = \{p_1, p_2, \ldots, p_t, y_1, y_2, \ldots, y_{k-t}\} - \{y_j\}\dot{\cup}\{z_1\} = Y - \{y_j\}\dot{\cup}\{z_1\} \in \mathbf{C}.$$

So,

$$\{p_1, p_2, \ldots, p_t, z_1, y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{k-t}\} \in \mathbf{C}.$$

Further, $Y' \cap X = \{p_1, p_2, \ldots, p_t\}$. Also

$$\begin{aligned} A &= A(\{p_1, p_2, \ldots, p_t\}, X) \\ &= \{z_1, z_2, \ldots, z_{t-1}, y_1, y_2, \ldots, y_{k-t}\} \\ &= Y' - \{p_1, p_2, \ldots, p_t\}\dot{\cup}\{y_j, z_2, \ldots, z_{t-1}\}. \end{aligned}$$

By the $t$-extremal property (d), we get

$$\begin{aligned} A' &= A(\{p_1, p_2, \ldots, p_t\}, Y') \\ &= X - \{p_1, p_2, \ldots, p_t\}\dot{\cup}\{y_j, z_2, \ldots, z_{t-1}\} \\ &= \{y_j, z_2, \ldots, z_{t-1}, x_1, x_2, \ldots, x_{k-t}\}. \end{aligned}$$

Here, $\{x_1, x_2, \dots x_{k-t}\} - \{x_i\}\dot{\cup}\{y_j\} = \{y_j, x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_{k-t}\}$ is a $(k-t)$ subset of $A'$. By the $t$-extremal property (c),

$$\{y_j, x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_{k-t}\}\dot{\cup}\{p_1, p_2, \dots, p_t\} \in \mathbf{C}.$$

So $X - \{x_i\}\dot{\cup}\{y_j\} \in \mathbf{C}$. That is, $X - \{x\}\dot{\cup}\{y\} \in \mathbf{C}$. □

*By steps 1 and 2, for $1 \leq i \leq k-t$, there exists at least $k-1$ lines $Y$ such that $X \cap Y = X - \{x_i\}$.*

*Step 3* There exists at least $k-1$ lines $Y$ such that $X \cap Y = X - \{p_i\}$ for $i = 1, 2, \cdots, t$.

*Proof.* Consider $X = \{p_1, p_2, \dots, p_t, x_1, x_2, \dots, x_{k-t}\}$. By assumption there exists a $t$-subset of $X - \{p_1, p_2, \dots, p_t\}$ say $\{x_1, x_2, \dots, x_t\}$, which is $t$-extremal. Using Steps 1 and 2 and by changing the role of $\{p_1, p_2, \dots, p_t\}$ and $\{x_1, x_2, \dots, x_t\}$, we obtain at least $k-1$ lines $Y$ such that $X \cap Y = X - \{p_i\}$. □

*Proof of Theorem 3.3.* By steps 1 and 2, there exist at least $k-1$ lines that intersect $X$ at $X - \{x_i\}$ for $i = 1, 2, \dots, k-t$. Further by Step 3, there exist at least $k-1$ lines that intersect $X$ at $X - \{p_j\}$ for $j = 1, 2, \dots, t$. Since $X$ is an arbitrary line of $\mathbf{C}$ containing $p$, Corollary 2.6, shows that $\mathbf{C}$ is a binomial $k$-clique. □

## 4. GENERALIZED BINOMIAL $k$-CLIQUES

We begin with the following general construction of a maximal $k$-clique. This construction first appeared in the paper by Erdös and Lovász [6].

**Construction 4.1.** *Let $k \geq 3$ be a fixed integer. Let $S$ and $T$ be two disjoint sets of sizes $s$ and $2k-2$ respectively, where $s \geq 1$. Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$. The (maximal) $k$-clique* $\mathbf{C}$ *we now construct has $V = S\dot{\cup}T$ as its point-set. The line-set of* $\mathbf{C}$ *has lines of two types.* $\mathbf{C_1}$ *denotes the collection of all the $k$-subsets of $T$. Let $\mathbf{F} = \{A : A \subset T and |A| = k-1\}$. Note that for each member $A$ of* $\mathbf{F}$, *there is a unique member $B$ of* $\mathbf{F}$, *which is disjoint from $A$. Let*

$$\mathbf{G} = \{\{A, B\} : A, B \in \mathbf{F} \text{ and } A \cap B = \emptyset\}.$$

*Note that $|\mathbf{F}| = \binom{2k-2}{k-1}$ and hence $|\mathbf{G}| = \frac{1}{2}\binom{2k-2}{k-1}$. Define a surjective function $g : \mathbf{G} \to S$. Finally, if $g(\{A, B\}) = \alpha_i$, then let $\overline{A} = A\dot{\cup}\alpha_i$ and $\overline{B} = B\dot{\cup}\alpha_i$. Define $\mathbf{C_2} = \{\overline{A} : A \in \mathbf{F}\}$. Let* $\mathbf{C}$ *denote the disjoint union* $\mathbf{C_1}\dot{\cup}\mathbf{C_2}$. *This configuration is called* **the generalized binomial $k$-clique.**

**Theorem 4.2.** *(Erdös-Lovász [6]) The generalized binomial $k$-clique is a maximal $k$-clique with the number of lines equal to $\binom{2k-1}{k}$.*

**Definition 4.3.** *Let* $\mathbf{C}$ *denote the generalized binomial clique as in Construction 4.1. Let $i \in \{1, 2, \dots, s\}$ and let*

$$a_i = |\{\{A, B\} \in G : g(\{A, B\}) = \alpha_i\}|.$$

*We also assume without loss of generality (by relabeling the elements of $S$ if necessary) that the $a_1 \geq a_2 \geq \cdots \geq a_s \geq 1$. Then the monotonically decreasing sequence $(a_1, a_2, \ldots, a_s)$ of positive integers is called the* **frequency sequence associated with the generalized binomial $k$-clique C.**

We recall that the replication number of a point refers to the number of lines containing that point. The assertion of the following Lemma 4.4 with $a_1 = a_2 = \cdots = 2$ is due to Erdös and Lovász [6].

**Lemma 4.4.** *Let $\mathbf{C}$ be a generalized binomial $k$-clique with associated frequency sequence $(a_1, a_2, \ldots, a_s)$. Then the following assertions hold.*

*(a)* $v = |V| = 2k - 2 + s$ *and* $b = \binom{2k-1}{k}$ *and thus the number of lines in a generalized binomial $k$-clique is the same as those in the binomial $k$-clique.*
*(b)* *Let $\alpha_i \in S$. Then $\alpha_i$ has replication number $2a_i$.*
*(c)* *We have the following partition of the integer $\frac{1}{2}\binom{2k-2}{k-1}$:*

$$\frac{1}{2}\binom{2k-2}{k-1} = a_1 + a_2 + \cdots + a_s. \quad (*)$$

*Further, $\mathbf{C}$ is the binomial $k$-clique iff $|S| = 1$ (which is the case if the partition given in the previous sentence is the trivial partition with $s = 1$).*
*(d)* *If $\mathbf{C}$ is not the binomial clique, then replication numbers in $\mathbf{C}$ are $\binom{2k-2}{k-1}$ (if the point is in $T$) and $2a_i$ where $i = 1, 2, \ldots, s$.*
*(e)* *Isomorphic generalized binomial $k$-cliques have the same associated frequency sequences and in particular, if two distinct generalized binomial $k$-cliques have distinct frequency sequences, then they are not isomorphic.*

All the assertions are routine verifications; the following two Theorems are then direct consequences of Lemma 4.4.

**Theorem 4.5.** *Let $k \geq 3$. Then there exists a maximal $k$-clique with the number of points $v$ any number between $2k - 1$ and $2k - 2 + \frac{1}{2}\binom{2k-2}{k-1}$ with $\binom{2k-1}{k}$ lines.*

**Theorem 4.6.** *Let $k \geq 3$. The number of mutually nonisomorphic $k$-cliques is at least as large as $p(n)$ where $p(n)$ denotes the number of partitions of $n$ and $n = \frac{1}{2}\binom{2k-2}{k-1}$.*

## 5. CONCLUDING REMARKS

Tuza [9] has essentially proved that the maximal $k$-clique with $2k - 2 + \frac{1}{2}\binom{2k-2}{k-1}$ constructed in Theorem 4.5 has the largest number of points (among all the maximal $k$-cliques). We are not aware of the proof of the following conjecture: The binomial clique gives the largest ratio $\frac{b}{v}$, the ratio between number of lines and the number of points.

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