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A SOLUTION OF TWO-DIMENSIONAL TOPOLOGICAL QUANTUM GRAVITY

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We give a formulation of two-dimensional topological gravity without matter in terms of a supersymmetric conformally invariant field theory and derive a path integral expression for the physical amplitudes. A careful analysis of the contact terms of the physical operators reveals the presence of a non-commutative algebra, isomorphic to the Virasoro algebra. We show that this algebra completely determines all the amplitudes at arbitrary genus, which coincide with those of the one-matrix model at the $k = 1$ critical point.

1. Introduction

The recently obtained exact solutions of matrix models of two-dimensional gravity [1] have lead to considerable progress in the understanding of low-dimensional non-critical string theory [2–7]. The matrix model approach proved to be especially powerful for doing non-perturbative calculations and has provided a number of surprising results. In particular it has revealed an unexpected and yet mysterious relation between two-dimensional gravity and the KdV-hierarchy [4, 8]. There is increasing evidence that the amplitudes of all minimal models coupled to gravity can be described very directly in terms of the generalized KdV equations [8]. Not only does one find the correct scaling dimensions of the operators, also part of the fusion rules are reproduced [9].

In spite of all this progress, a vast number of questions are still unanswered. In particular there is still a big gap in the understanding of the relation between the matrix model formulation and the conventional approach to two-dimensional

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gravity, based on conformal field theory coupled to Liouville theory or light-cone gravity [10]. Clearly, the continuum formulation is by far not enough developed to be able to produce definite numbers for the physical amplitudes, but even qualitative predictions of the matrix models, such as the existence of an infinite number of physical operators, have not yet been understood from the continuum point of view.

More successful, in this respect, has been the topological approach to two-dimensional gravity [11–15]. Soon after the breakthrough in the matrix models Witten conjectured that the one-matrix model is equivalent to two-dimensional topological gravity and showed that the amplitudes at genus zero and one are the same in both theories [15]. At genus zero these results were also obtained from a field theoretical viewpoint by Distler [16]. More recently, this relationship has been extended to n-matrix models, which were shown to have the same properties as topological gravity coupled to certain topological matter systems [17]. Motivated by these developments, we study in this paper the topological approach to two-dimensional gravity in more detail.

If all the conjectured identifications are correct, they imply that there is a very interesting connection between the topological and the conventional formulation of two-dimensional gravity [15–17]. This relation suggests that the latter theory is in fact also in essence topological. Intuitively this can indeed be understood, since gravity coupled to minimal models only has a finite number of degrees of freedom. Physically one therefore expects that its correlation functions are essentially independent of the distance between operators [18], and this absence of a notion of distance is the basic characteristic of a topological theory.

As the first step in our study of two-dimensional topological gravity we develop a field theoretical formulation, suitable for doing explicit computations. We find that topological gravity is conveniently described in terms of a supersymmetric conformally invariant field theory, with similar properties as fermionic string theory. It consists of a ghost sector, which was previously studied in refs. [11, 16], and a "Liouville sector".

Our aim is to use this field theoretical formulation to study the physical amplitudes of the theory, and to make contact with the results of the one-matrix models. The first relation we will derive is the so-called "puncture equation" [17], which is closely related to the "string equation" of the one-matrix model [6]. It expresses the fact that one of the operators in the theory, the "puncture operator", only interacts with the other operators through contact terms. This is not surprising in a topological field theory, because the only topological statement one can make about the positions of two operators is whether they coincide or not. It seems therefore a natural idea to try to see whether all interactions in topological gravity are contact interactions.

While pursuing this line of thought, we found in collaboration with R. Dijkgraaf that from the KdV equations one can derive an infinite set of recursion relations

[19], generalizing the puncture equation. These relations indeed seemed to imply that all operators have essentially only contact interactions. In this paper we will use our formulation of topological gravity to show that this is indeed true, and furthermore, to give an independent derivation of the same recursion relations. We further show that these relations, which are given in eq. (7.13), uniquely fix all the physical amplitudes. Their derivation from the KdV and string equation, which, combined with our results, establishes the equivalence between topological gravity and the one-matrix model, will be discussed in a separate publication [19].

A crucial ingredient in the derivation of the recursion relations is the fact that the contact terms between the operators are not symmetric and form a non-commutative algebra. This algebra turns out to be isomorphic to the Virasoro algebra, although its geometrical interpretation is very different. The recursion relations then follow from a simple consistency requirement of the theory.

The organization of this paper is as follows. In sect. 2 we give the lagrangian based on the formulation of two-dimensional gravity as ISO(2) gauge theory [12, 14], and construct the BRST-invariant operators. The gauge fixing is discussed in sect. 3. In sect. 4 we derive a path integral representation of these amplitudes. Here we will make contact with the abstract cohomological definition of the amplitudes given in ref. [15]. In sect. 5 we discuss the derivation of two special recursion relations, the "puncture equation" and the "dilaton equation". The structure of the contact terms are studied in sect. 6. Finally in sect. 7 we complete the derivation of the recursion relations.

2. Two-dimensional topological gravity

In this section our aim is to obtain a convenient field theoretical formulation of two-dimensional topological gravity. As described in ref. [15], such a theory is characterized by the fact that the only physical operators correspond to stable cohomology classes of the moduli space of (punctured) Riemann surfaces.

2.1. THE LAGRANGIAN OF TOPOLOGICAL GRAVITY

To explain the general method [20–23] for obtaining the topological field theory associated with some moduli space, let us take as an example the moduli space of flat gauge fields $A = A^a_\alpha(x)\tau^a \, \mathrm{d}x^\alpha$,

$$\mathscr{M}_{\text{gauge}} = \{A \mid \mathrm{d}A + A \wedge A = 0\}/\mathscr{G},$$

on a Riemann surface Σ. As the first step, one introduces fermionic fields $\psi = \psi^a_\alpha(x)\tau^a \, \mathrm{d}x^\alpha$, which are needed to represent forms on the moduli space $\mathscr{M}_{\text{gauge}}$. The fields ψ are the superpartners of A with respect to a supersymmetry charge Q_{s}, satisfying $\{Q_{\mathrm{s}}, A\} = \psi$ and $Q_{\mathrm{s}}^2 = 0$. Next one constructs a supersymmetric lagrangian S, for which the space of classical solutions coincides with $\mathcal{M}_{\text{gauge}}$. In our example, the simplest lagrangian satisfying these requirements is

$$S = \int \operatorname{tr}(\pi(\mathrm{d}A + A \wedge A)) + \int \operatorname{tr}(\chi \mathrm{D}\psi), \tag{2.1}$$

with $\mathrm{D} = \mathrm{d} + [A, \cdot\,]$. The field π and its anti-commuting partner χ are Lagrange multipliers, and do not contribute to the physical degrees of freedom of the theory. The supersymmetry variations are

$$\delta_s A = \psi, \qquad \delta_s \chi = \pi. \tag{2.2}$$

The action S is invariant under local gauge transformations, and in addition has the local fermionic symmetry $\delta\psi = \mathrm{D}\hat{\alpha}$. Both symmetries will have to be fixed via an appropriate BRST procedure. The equation of motion for ψ, $\mathrm{D}\psi = 0$, implies that the physical degrees of freedom of ψ represent elements of the tangent space to $\mathcal{M}_{\text{gauge}}$. Consequently, we may view the (BRST-improved) supersymmetry charge Q as an exterior derivative on $\mathcal{M}_{\text{gauge}}$. These observations make it intuitively clear that there will be a one-to-one correspondence between the physical, i.e. Q-closed modulo Q-exact, observables of the above theory and cohomology classes on the moduli space $\mathcal{M}_{\text{gauge}}$.

We now want to apply this idea to construct the lagrangian for topological gravity. In order to exploit the similarity with the gauge theory situation described above, we will use the vielbein formalism. So we introduce a zweibein $e^{\pm} = e^{\pm}_{\alpha}(x)\mathrm{d}x^{\alpha}$ and an SO(2) spin connection $\omega = \omega_{\alpha}(x)\mathrm{d}x^{\alpha}$, which for the moment is an independent field, but eventually will be determined in terms of $e^{\pm}$ by the torsion constraints,

$$\begin{aligned} \mathrm{D}e^{+} &= \mathrm{d}e^{+} - \omega \wedge e^{+} = 0, \\ \mathrm{D}e^{-} &= \mathrm{d}e^{-} + \omega \wedge e^{-} = 0. \end{aligned} \tag{2.3}$$

The moduli space $\mathcal{M}$ of Riemann surfaces can be represented as the space of all zweibeins, modulo diffeomorphisms, local Lorentz and Weyl transformations. Our aim in this section is to obtain an action, similar to (2.1), which is invariant under the first two local symmetries, and which constraints the Weyl degree of freedom by its field equations. A convenient way to eliminate the Weyl mode is to restrict the curvature $R(x) = \mathrm{d}\omega(e(x))$ to a particular value $R_0(x)$, and represent $\mathcal{M}$ as

$$\mathcal{M} = \{(e^{+}, e^{-}); \mathrm{d}\omega(e) = R_0\}/\mathrm{Diff} \times \mathrm{lL}.$$

The usual choice is to take the curvature two-form R_0 to be a constant times $e^{+} \wedge e^{-}$. For surfaces with $g > 1$ this constant curvature condition implies that the zweibein $e^{\pm}$ together with the spin connection ω describe a flat $SL(2,\mathbb{R})$ connection

$$A=(\omega,e^{+},e^{-}).$$

A possible starting point, therefore, would be to take the action (2.1) with gauge group $SL(2,\mathbb{R})$ as the action for two-dimensional topological gravity [12,14]. Although this is indeed a correct approach, we will in fact make a different choice, which will prove to be much more convenient for doing explicit computations.

Since our aim is to construct a topological theory, there is no need for the curvature R_0 to be a smooth function. We can therefore choose to fix the Weyl degree of freedom by imposing R_0 to vanish everywhere, except for possible δ-function singularities. We will see that these curvature singularities, which are necessary in order that the integral $I=(1/2\pi)\int R_0$ equals the Euler number, can in a natural way be associated with operator insertions. This procedure, as it turns out, has the big advantage that we will be able to use the powerful techniques of conformal field theory.

As the starting point for the study of two-dimensional topological gravity we take the following lagrangian:

$$S=\int(\pi_0\,\mathrm{d}\omega+\pi_+\,\mathrm{D}e^{+}+\pi_-\,\mathrm{D}e^{-})+\int(\chi_0\,\mathrm{d}\psi^{0}+\chi_+\,\mathrm{D}\psi^{+}+\chi_-\,\mathrm{D}\psi^{-}),\tag{2.4}$$

which has as field equations the zero-curvature condition, the torsion constraints and the corresponding equations for the superpartners. The explicit form of the super torsion constraints is

$$\begin{aligned}\mathrm{D}\psi^{+}&=\mathrm{d}\psi^{+}-\omega\wedge\psi^{+}+e^{+}\wedge\psi^{0}=0,\\ \mathrm{D}\psi^{-}&=\mathrm{d}\psi^{-}+\omega\wedge\psi^{-}-e^{-}\wedge\psi^{0}=0.\end{aligned}\tag{2.5}$$

The action (2.4) is in fact identical to the topological gauge theory action (2.1) with gauge group equal to the two-dimensional Poincaré group ISO(2). However, the underlying local symmetry group of the theory is not ISO(2) gauge transformations, but diffeomorphisms $\times$ local Lorentz rotations, which act on the spin-connection and zweibein as*

$$\begin{aligned}\delta\omega&=\mathrm{d}\alpha+\xi\cdot\mathrm{d}\omega,\\ \delta e^{\pm}&=\pm\alpha e^{\pm}+\mathrm{D}(\xi\cdot e^{\pm})+\xi\cdot\mathrm{D}e^{\pm}.\end{aligned}\tag{2.6}$$

These are equivalent to ISO(2) transformations only when the zweibein is non-invertible and when we impose the field equations. The transformation properties

* Here, for later convenience, we have shifted the local Lorentz parameter α with an amount $\xi\cdot\omega$ with respect to the conventional definition.

of the fermionic fields ψ^0 and $\psi^{\pm}$ follow by applying the global supersymmetry $\delta_s \omega = \psi^0$ and $\delta_s e^{\pm} = \psi^{\pm}$ to eq. (2.6). We have

$$\delta\psi^0 = \xi \cdot d\psi^0 ,$$

$$\delta\psi^{\pm} = \pm\alpha\psi^{\pm} + D(\xi\cdot\psi^{\pm}) + \xi\cdot D\psi^{\pm} . \tag{2.7}$$

The presence of the global supersymmetry implies that in addition to the diffeomorphisms and local Lorentz rotations there are also corresponding local fermionic symmetries. These act on the fermionic fields as

$$\hat{\delta}\psi^0 = d\hat{\alpha} + \hat{\xi}\cdot d\omega ,$$

$$\hat{\delta}\psi^{\pm} = \pm\hat{\alpha}e^{\pm} + D(\hat{\xi}\cdot e^{\pm}) + \hat{\xi}\cdot De^{\pm} , \tag{2.8}$$

and leave the spin connection and zweibein invariant.

It is possible to think of the lagrangian (2.4) as resulting from gauge fixing a lagrangian which is identically zero [21]. The fields ψ and χ are then interpreted as ghost and antighost fields. We have chosen not to follow such a presentation, because it requires some "inspiration" to pick the right gauge-fixed lagrangian. However, in order to motivate the structure of the BRST transformations, which we are about to discuss, it is useful to keep this interpretation of the lagrangian (2.4) in mind.

2.2. THE BRST COHOMOLOGY

In order to do computations, the local gauge symmetries described above will have to be gauge fixed. This will give rise to ghost fields for each of these local symmetries. The physical operators of the theory are then obtained by considering the BRST cohomology. Because the BRST transformation rules of the fields $\omega, e^{\pm}$, their superpartners $\psi_0, \psi^{\pm}$ and the ghost fields do not depend on the specific gauge choice, we can study the BRST cohomology of our theory before discussing the details of the gauge fixing.

Following refs. [21,22] we write the total BRST variation δ_{brst} of the fields as the sum of the supersymmetry variation δ_s plus a second part, which is obtained by replacing the gauge parameters in eqs. (2.6)–(2.8) by the corresponding ghost fields. In the following we will work on-shell, so that the transformation rules will have the form of ISO(2) transformations. The anti-commuting ghosts associated with local Lorentz rotations and diffeomorphisms are denoted by c_0 and $c = c^{\alpha}(x)$, and their commuting superpartners by γ_0 and $\gamma = \gamma^{\alpha}(x)$.

Let us first consider the local Lorentz sector of the theory consisting of the (super-) spin connection ω, ψ_0 and the ghosts c_0, γ_0. In this sector the BRST variations are given by

$$\delta_{\text{brst}}\omega = \psi_0 + \mathrm{d}c_0\,, \qquad \delta_{\text{brst}}c_0 = \gamma_0\,,$$
$$\delta_{\text{brst}} = \mathrm{d}\gamma_0\,, \qquad \delta_{\text{brst}}\gamma_0 = 0\,. \tag{2.9}$$

We see that the other fields do not appear and furthermore that these BRST variations are nilpotent. We can therefore study the BRST cohomology in the local Lorentz sector without bothering about the other fields [15].

It is easily seen from eq. (2.9) that the only BRST-invariant operators are given by BRST variations, which suggests that the BRST cohomology is trivial. However, the relevant cohomology in this situation is a so-called equivariant cohomology: one should only consider operators which are independent of the Lorentz ghost c_0. Physically this means that one defines the cohomology for gauge-invariant operators [20,22]. In this way one gets as non-trivial BRST-invariant operators the powers of the commuting ghost field γ_0,

$$\sigma_n^{(0)} = \gamma_0^n\,, \tag{2.10}$$

which are the operators first introduced in ref. [15]. The superscript (0) indicates that these fields are zero-forms on the Riemann surface. With each of the operators $\sigma_n^{(0)}$ one can associate operators $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$, through the relations [20]

$$\mathrm{d}\sigma_n^{(0)} = \delta_{\text{brst}}\sigma_n^{(1)}\,, \qquad \mathrm{d}\sigma_n^{(1)} = \delta_{\text{brst}}\sigma_n^{(2)}\,. \tag{2.11}$$

One finds

$$\sigma_n^{(1)} = n\psi_0\gamma_0^{n-1}\,,$$
$$\sigma_n^{(2)} = n\,\mathrm{d}\omega\,\gamma_0^{n-1} + \tfrac{1}{2}n(n-1)\psi_0 \wedge \psi_0\gamma_0^{n-2}\,. \tag{2.12}$$

One can think of the $\sigma_n^{(i)}$ as representing three different components of one single operator σ_n: we can formally write $\sigma_n = (\gamma_0 + \psi_0 + \mathrm{d}\omega)^n$ and the $\sigma_n^{(i)}$, $i = 0, 1, 2$, are then the projections on the zero-, one- and two-form component, respectively [21]. We will see later on that in physical amplitudes γ_0, ψ_0 and $\mathrm{d}\omega$ indeed represent three components of a curvature two-form on the moduli space of punctured Riemann surfaces. Finally notice that, although the one- and two-form components are not BRST invariant, eq. (2.11) shows that their line and surface integrals are.

For completeness we give here the BRST variations of the other fields in the theory,

$$\delta e^{\pm} = \psi^{\pm} \pm c_0 e^{\pm} + \mathrm{D}(c\cdot e^{\pm})\,,$$
$$\delta\psi^{\pm} = \pm c_0\psi^{\pm} + \mathrm{D}(c\cdot\psi^{\pm}) \pm \gamma_0 e^{\pm} + \mathrm{D}(\gamma\cdot e^{\pm})\,,$$
$$\delta c = \gamma + c\cdot\partial c\,,$$
$$\delta\gamma = c\cdot\partial\gamma - \gamma\cdot\partial c\,. \tag{2.13}$$

We believe that there are no new non-trivial BRST-invariant operators in this sector of the theory, although we don't have a proof of this fact.

3. Topological gravity as a free conformal field theory

We will now discuss a specific gauge-fixing procedure which preserves the global supersymmetry, and in addition leads to a formulation of topological gravity in terms of a free conformal field theory. Our treatment is related to that of refs. [11, 16, 24].

We first concentrate on the bosonic fields. To fix the invariance under diffeomorphisms we choose a conformal gauge. A zweibein (e^+, e^-) defined on a genus-g surface specifies a complex structure on it: locally on the surface one can choose complex coordinates $(z, \bar{z})$ such that the zweibein acquires the form

$$e^+ = e^{\phi_+} \mathrm{d}z, \qquad e^- = e^{\phi_-} \mathrm{d}\bar{z}, \tag{3.1}$$

with $\phi_+ = (\phi_-)^*$. In the conformal gauge this fact is exploited to factor out the diffeomorphisms by parametrizing the space of inequivalent zweibeins by means of the Weyl modes $\phi_\pm$ and the moduli of the Riemann surface. The associated jacobian is represented by the usual coordinate ghosts (b, c) and $(\bar{b}, \bar{c})$ of bosonic string theory. The local Lorentz transformations are conveniently fixed by imposing that

$$\phi_+ = \phi_- . \tag{3.2}$$

The corresponding local Lorentz ghosts (b_0, c_0) are non-dynamical, and can by their equation of motion be expressed in terms of the other fields. In particular, the BRST transformation of (3.2) gives that*

$$c_0 = \tfrac{1}{2}(\partial c + c\,\partial\phi - \bar{\partial}\bar{c} - \bar{c}\,\bar{\partial}\phi), \tag{3.3}$$

where $\phi = \phi_+ + \phi_-$. Finally, inserting these gauge choices into the action (2.4) and eliminating the spin connection via the torsion constraints, we obtain the following action for the bosonic sector:

$$S_{\mathrm{B}} = \int \pi\,\bar{\partial}\partial\phi + \int (b\,\bar{\partial}c + \bar{b}\,\partial\bar{c}) . \tag{3.4}$$

The field equation of the field π, which was previously denoted by π_0, is the zero-curvature equation $R = \bar{\partial}\partial\phi = 0$.

The gauge fixing of the local fermionic symmetry group (2.8) proceeds completely parallel to the bosonic case. To eliminate the super diffeomorphisms we impose as gauge-fixing condition the supersymmetry variation of the conformal

* Here we use that later on we will impose the Q_s-variation of (3.2) as a second gauge condition.

gauge (3.1),

$$\psi^{+} = e^{\phi_{+}}\psi_{+}\,\mathrm{d}z\,,\qquad \psi^{-} = e^{\phi_{-}}\psi_{-}\,\mathrm{d}\bar{z}\,, \tag{3.5}$$

and the super local Lorentz rotations are fixed via the condition

$$\psi_{+} = \psi_{-}\,. \tag{3.6}$$

The ghost sector associated with the super-diffeomorphisms consists of two conjugate pairs of commuting fields (β,γ) and $(\bar{\beta},\bar{\gamma})$, which are the superpartners of the coordinate ghosts (b,c) and $(\bar{b},\bar{c})$ and have the same conformal spin $(2,-1)$. The super local Lorentz ghosts are again non-dynamical. After eliminating ψ^0 via the constraints, the final result for the fermionic action is (here $\psi = \psi_{-} + \psi_{+}$)

$$S_{\mathrm{F}} = \int \chi\,\bar{\partial}\partial\psi + \int (\beta\,\bar{\partial}\gamma + \bar{\beta}\,\partial\bar{\gamma})\,. \tag{3.7}$$

The combined action $S = S_{\mathrm{B}} + S_{\mathrm{F}}$ is the complete gauge-fixed action for two-dimensional topological gravity. It describes a set of decoupled free conformally invariant theories. The fields π, ϕ and χ, ψ represent the "Liouville sector" of the theory. In principle it is possible to formulate topological gravity without this Liouville sector (see refs. [11, 16]); however, one is then forced to use non-covariant operators of the type considered in ref. [25].

Like any string theory in the conformal gauge, the gauge fixed action $S = S_{\mathrm{B}} + S_{\mathrm{F}}$ of topological string theory has a large group of residual symmetries, and correspondingly, a large number of conserved charges. The fact that it describes a free theory means that, at least locally, the degrees of freedom are split into a left-moving and a right-moving sector. Consequently, the conserved charges exist for left and right movers separately. The most important symmetries are:

(i) Global supersymmetry transformations, with associated supercharge

$$Q_{\mathrm{s}} = \oint (\partial\pi\,\psi + b\gamma)\,. \tag{3.8}$$

(ii) Conformal transformations, generated by the stress–energy tensor $T = T_{\mathrm{L}} + T_{\mathrm{gh}}$, with

$$\begin{aligned} T_{\mathrm{L}} &= \partial\pi\,\partial\phi + \partial^2\pi + \partial\chi\,\partial\psi\,,\\ T_{\mathrm{gh}} &= c\,\partial b + 2\,\partial c\, b + \gamma\,\partial\beta + 2\,\partial\gamma\,\beta\,. \end{aligned} \tag{3.9}$$

The conformal transformations commute with the supersymmetry transformations.

(iii) Spin--1 super conformal transformations, generated by $G = G_{\mathrm{L}} + G_{\mathrm{gh}}$, with

$$G_{\mathrm{L}} = \partial\chi\,\partial\phi + \partial^2\chi\,, \qquad G_{\mathrm{gh}} = c\,\partial\beta + 2\,\partial c\,\beta\,. \tag{3.10}$$

The spin-2 conformal field G is the super partner of the stress tensor: $\{Q_{\mathrm{s}}, G\} = T$.
(iv) BRST transformations, generated by the total BRST charge

$$\begin{aligned} Q &= Q_{\mathrm{s}} + Q_{\mathrm{v}}\,, \\ Q_{\mathrm{v}} &= \oint\!\left(c\left(T_{\mathrm{L}} + T_{\beta\gamma} + \tfrac{1}{2}T_{bc}\right) + \gamma G_{\mathrm{L}}\right). \end{aligned} \tag{3.11}$$

When we assign ghost numbers to all fields, by giving the Liouville sector (ϕ, π, ψ, χ) charge $(0, 0, 1, -1)$, and the ghost fields (c, b, γ, β) charge $(1, -1, 2, -2)$, all terms in the BRST generator Q will have ghost number 1. The BRST variation of the anti-ghosts b and β are given by T and $b + G$ respectively.

The symmetry structure we have just described is characteristic of topological gravity, also when it is coupled to topological matter. We notice that the structure of topological string theory is similar to that of superstring theory in the treatment of Friedan et al. [26], but there are of course some important differences.

Now let us discuss the gauge-fixed form of the physical operators. Since in our gauge choice the commuting local Lorentz ghost $\gamma_0 = \delta_{\mathrm{s}} c_0$ is non-dynamical, it is determined by its field equation. This field equation is given by the supersymmetry variation of eq. (3.3),

$$\gamma_0 = \tfrac{1}{2}\left(\partial\gamma + \gamma\,\partial\phi + c\,\partial\psi - \bar\partial\bar\gamma - \bar\gamma\,\bar\partial\phi - \bar c\,\bar\partial\psi\right). \tag{3.12}$$

It is easy to verify that the expression on the right-hand side is indeed BRST invariant. The physical operators in the gauge-fixed theory are therefore still given by eqs. (2.10) and (2.12) except that γ_0 now denotes the right-hand side of (3.12) and ω and ψ_0 are equal to the solution of the torsion constraints

$$(\omega, \bar\omega) = \tfrac{1}{2}(\partial\phi, -\bar\partial\phi)\,, \qquad (\psi_0, \bar\psi_0) = \tfrac{1}{2}(\partial\psi, -\bar\partial\psi)\,. \tag{3.13}$$

The physical operator $\sigma_n^{(i)}$ has ghost number $2n - i$.

Finally, an observation that will become important later on is that, if we allow ourselves to use the equations of motion for ψ, we can write γ_0 as a total BRST derivative. Namely, the field equation of ψ, $\bar\partial\partial\psi = 0$ implies that, at least locally on the Riemann surface, we can write

$$\psi(z, \bar z) = \psi(z) + \bar\psi(\bar z)\,. \tag{3.14}$$

Using the fact that the chiral fields satisfy $\psi = \{Q_{\mathrm{s}}, \phi\}$ and $\bar\psi = \{\bar Q_{\mathrm{s}}, \phi\}$ as an intermediate step, we can write γ_0 as a (double) BRST variation of the conformal factor ϕ,

$$\begin{aligned}\gamma_0 &= \tfrac{1}{2}\{Q, \psi - \bar{\psi}\}\\ &= \tfrac{1}{2}\{Q, \{Q_s - \bar{Q}_s, \phi\}\},\end{aligned} \tag{3.15}$$

where Q is the total (i.e. left plus right) BRST charge. Naively this would mean that γ_0, and thus all the physical operators σ_n, are BRST trivial. However, there are two reasons why we have to be careful in using the equations of motion for ψ. The first is that on a general Riemann surface there is an obstruction to make the left–right decomposition (3.14) globally: the chiral parts of ψ separately will not be single-valued around the cycles. Secondly, since we will be integrating over the positions of operators, it can happen that two operators meet at the same point. In this case there will in general be δ-function type "contact terms" between the two operators, which violate the equation of motion [27]. We will see that, because of these subtleties, the physical operators σ_n have non-trivial amplitudes, in spite of the fact that they are on-shell BRST trivial. The interpretation of eq. (3.15) will become more clear in sect. 4.

4. Physical amplitudes

In this section we will describe the definition of the physical amplitudes of the theory, using the functional integral approach. One of our aims will be to make contact with the topological definition of the amplitudes given by Witten [15].

In ref. [15] the amplitudes of topological gravity are defined as integrals over the moduli space $\mathcal{M}_{g,s}$ of genus-g surfaces with s punctures,

$$\langle \sigma_{n_1}\sigma_{n_2}\cdots\sigma_{n_s}\rangle = \int_{\mathcal{M}_{g,s}} \alpha_{(1)}^{n_1}\wedge\alpha_{(2)}^{n_2}\cdots\wedge\alpha_{(s)}^{n_s}, \tag{4.1}$$

where $\alpha_{(i)}^{n_i}$ denotes the n_i-fold wedge product of certain curvature two-forms $\alpha_{(i)}$, representing the first Chern class of the line bundle $\mathcal{L}_{(i)}$ over $\mathcal{M}_{g,s}$ described by the cotangent bundle to Σ at x_i. The integrand in (4.1) describes a volume form on $\mathcal{M}_{g,s}$ provided that $\Sigma_i(n_i - 1) = 3g - 3$. The amplitudes (4.1) are related to topological invariants introduced by Mumford [28]. One way to represent the $\alpha_{(i)}$, which was used in ref. [15], is as the divisor of a meromorphic section $s(x_i)$ of $\mathcal{L}_{(i)}$ over $\mathcal{M}_{g,s}$. This representation amounts to taking

$$\alpha_{(i)} = \partial\bar{\partial}\log|s(x_i)|^2, \tag{4.2}$$

where ∂ and $\bar{\partial}$ denote derivatives on $\mathcal{M}_{g,s}$. The right-hand side has support only where the meromorphic section $s(x_i)$ is 0 or ∞. In fact, because one has the freedom to change $\alpha_{(i)}$ by adding exact forms, it is also possible to relax the condition that $s(x_i)$ is meromorphic.

Using the topological definition described above, Witten derived various properties of the amplitudes and calculated them for genus $g \leqslant 1$. In this section we will show that the amplitudes (4.1) indeed represent the physical amplitudes of topological string theory as formulated in sect. 3.

4.1. THE (π, ϕ) FUNCTIONAL INTEGRAL

We begin with a few preliminary remarks concerning the functional integral formulation of the (π, ϕ) system. In local complex coordinates, this system is described by the free scalar action $S = \int \pi \bar{\partial} \partial \phi$. However, this form of the action is somewhat misleading, since it does not reflect the fact that ϕ transforms inhomogeneously under coordinate transformations. The stress tensor $T_{\pi\phi} = \partial \pi \partial \phi + \partial^2 \pi$ indeed has a modified form, and the operator product

$$T(z)\phi(w) \sim \frac{1}{(z-w)^2} + \frac{1}{z-w} \partial\phi(w) \tag{4.3}$$

shows that ϕ is not a scalar, but transforms as $\delta\phi = \xi \partial \phi + \partial \xi$. Therefore, if we want to define the (π, ϕ) functional integral we need to have a covariant description of the action, which is valid in all coordinate systems. To this end, let us introduce a background metric $\hat{g}_{ab}$ and (temporarily) redefine ϕ as the conformal factor with respect to this metric: $g_{ab} = \mathrm{e}^{\phi} \hat{g}_{ab}$. The new ϕ transforms as a scalar. The covariantized form of the action is

$$S_{\mathrm{L}}^{(\mathrm{cov})} = \int \sqrt{\hat{g}} \left(\hat{g}^{ab} \partial_a \pi \partial_b \phi + \tfrac{1}{2} \pi \hat{R} \right). \tag{4.4}$$

We see that the redefinition of ϕ has resulted in a coupling of π to the background curvature. A consequence of this term is that in order to obtain non-vanishing amplitudes, we have to insert operators

$$V_{q_i}(x_i) = \mathrm{e}^{q_i \pi(x_i)} \tag{4.5}$$

to cancel the background charge. These operators create δ-function curvature singularities $\sqrt{g} R(x) = \Sigma q_i \delta(x - x_i)$ of the *dynamical* metric $g_{ab} = \mathrm{e}^{\phi} \hat{g}_{ab}$. The integrated curvature must be equal to the Euler number

$$\sum_i q_i = 2g - 2. \tag{4.6}$$

Note that the operators V_{q_i} have conformal dimension 0.

The (π, ϕ) functional integral is now easily performed by the standard rules of gaussian integration. In this way every correlation function can be expressed solely in terms of scalar Green functions. The correlation functions of the original Liouville field ϕ are obtained from these expressions via the substitution $\phi \to \phi + \log\sqrt{\hat{g}}$. After this substitution, the dependence of the correlation functions on the background metric $\hat{g}_{ab}$ is reduced to an overall constant independent of the positions of the fields [29]. In particular one finds that ϕ satisfies the equation of motion

$$\left\langle \partial\bar{\partial}\phi(z) \prod_i e^{q_i \pi(x_i)} \right\rangle = \sum_i q_i \delta(z - x_i), \tag{4.7}$$

and this equation determines the correlation function $\langle \phi(z) \Pi_i e^{q_i \pi(x_i)} \rangle$ up to an overall additive constant independent of z. For example, if we make the particular choice that all $q_i = \pm 1$ and locate the x_i such that the right-hand side of (4.7) describes the divisor of some meromorphic one-form $s(z)$ on Σ, we obtain that

$$\left\langle \phi(z) \prod_i e^{\pm \pi(x_i)} \right\rangle = \log|s(z)|^2 + \text{const}. \tag{4.8}$$

The overall constant can be calculated via the functional integral. It has some residual dependence on the background metric $\hat{g}_{ab}$, but this dependence will cancel in physical amplitudes. Namely, as we will show later on, the expectation value $\langle \phi(x) \rangle$ will take over the role of $\log|s(x)|^2$ in the geometrical description (4.1) of the physical amplitudes, and one can show that a change of the metric will only multiply $s(x)$ by a single-valued function on $\mathcal{M}_g$ and therefore changes $\alpha_{(i)} = \partial\bar{\partial} \log|s(x_i)|^2$ only by an exact form.

To illustrate this last point, let us discuss as an example the one-point function of ϕ on the torus. In this case we do not need curvature insertions. For the background metric we can take the usual flat metric $ds^2 = (\mathrm{Im}\,\tau)^{-1}|dx + \tau dy|^2$, where the prefactor $(\mathrm{Im}\,\tau)^{-1}$ is needed for modular invariance. For the one-point function of Liouville field ϕ one finds in this case

$$\langle \phi \rangle = -\log \mathrm{Im}\,\tau. \tag{4.9}$$

However, an equally good choice for the metric is $ds^2 = |\eta(\tau)|^4 |dx + \tau dy|^2$, where $\eta(\tau)$ is the Dedekind η-function. In this case one finds

$$\langle \phi \rangle_{g=1} = \log|\eta(\tau)|^4. \tag{4.10}$$

Hence we have two different answers for $\langle \phi \rangle_{g=1}$. The difference, however, is a single-valued function on $\mathcal{M}_{g=1}$ and both results will therefore lead to the same answer for the physical amplitude.

4.2. DESCRIPTION OF THE AMPLITUDES

As in ordinary string theory, the amplitudes of topological strings can be written as integrals over the moduli space $\mathcal{M}_g$ of Riemann surfaces. In addition, however, global supersymmetry implies that we also have to include an integration over anti-commuting superpartners $\hat{m}_i$ of the moduli m_i. In a similar way as that the ordinary moduli arise because of the global obstruction to gauge away all traceless components of the zweibein (e^+, e^-), the origin of these fermionic moduli is that the fermionic gauge symmetries (2.8) are not sufficient to gauge away all traceless components of the fermionic fields (ψ^+, ψ^-). As a consequence, there remain $6g-6$ anti-commuting integrations.

When one varies the action with respect to the moduli, one obtains the stress tensor T folded into a Beltrami differential. Similarly one finds that the dependence of the action on the anti-commuting moduli $\hat{m}_i$ is given by

$$\frac{\partial S}{\partial \hat{m}_i} = G_i, \qquad G_i = \int_\Sigma d^2x\, \mu_i(x,\bar{x}) G(x), \tag{4.11}$$

where G is the supercurrent defined in (3.10) and μ_i is the Beltrami differential corresponding to m^i. From (4.11) we see that the integral over the anti-commuting moduli can be accounted for by inserting the product $\prod_{i=1}^{3g-3} G_i \overline{G}_i$ into the functional integral. Thus we arrive at the following definition of the physical amplitudes of topological string theory:

$$\langle \sigma_{n_1} \sigma_{n_2} \dots \sigma_{n_s} \rangle = \int_{\mathcal{M}_g} \int e^{-S} \prod_{i=1}^{3g-3} G_i \overline{G}_i \prod_j e^{q_j \tilde{\pi}(x_j)} \prod_{k=1}^{s} \int_\Sigma \sigma_{n_k}^{(2)}. \tag{4.12}$$

Here $\int e^{-S}$ is short-hand notation for the functional integral over all fields in the theory, with the restriction that the integral over the fields b and β runs over the space orthogonal to the Beltrami differentials μ_i. The operators $e^{q_i \tilde{\pi}}$, with $\tilde{\pi} = \pi + c\,\partial\chi + \bar{c}\,\bar{\partial}\chi$, are the BRST-invariant versions of the curvature creating operators (4.5). Finally, the $\sigma_{n_k}^{(2)}$ are the two-form components of the physical operators given in eq. (2.12). We will now make a few comments about eq. (4.12).

First let us check the ghost counting. The physical operators $\sigma_{n_k}^{(2)}$ have ghost number $2n-2$, whereas the supercurrents G_i have ghost number -1. Taking into account that the b and β zero modes are projected out, the sum of all ghost charges has to be zero. Thus we have the selection rule

$$\sum_k (n_k - 1) = 3g - 3. \tag{4.13}$$

This is the only selection rule in the theory: all amplitudes for which (4.13) is satisfied are non-vanishing. The counting (4.13) coincides with that given in ref. [15].

At first sight, the physical amplitudes (4.12) seem to depend on the positions x_i of the curvature insertions. However, we know that this can not be true, since we are describing a topological theory: how we distribute the curvature over Σ is nothing other than a gauge choice needed to fix the Weyl factor of the metric. Indeed, we can observe that the variation of $\mathrm{e}^{q\bar{\pi}}$ with respect to its position is given by a total BRST variation, $\mathrm{d}\,\mathrm{e}^{q\bar{\pi}} = q\{Q, \mathrm{d}\chi\,\mathrm{e}^{q\bar{\pi}}\}$, which – at least formally – shows that it decouples from physical amplitudes. Therefore, we have the freedom of choosing the curvature insertions wherever we want. Let us mention two natural choices.

One possibility is to make use of string field theory [30]. In string field theory one associates to a Riemann surfaces a unique set of interaction points x_i, and one can put a curvature singularity at each of these x_i. This procedure essentially amounts to choosing the curvature to coincide with the divisor of some preferred meromorphic one-differential. An alternative procedure, however, is to put the curvature singularities at the locations of the operator insertions σ_{n_i}. As we will see later on, this procedure has a number of advantages, but for the moment we will leave the positions of the $\mathrm{e}^{q_i\bar{\pi}}$ arbitrary.

We are now in a position to compare our field theoretical description (4.12) of the physical amplitudes with the topological definition of ref. [15], described at the beginning of this section. To make the two definitions coincide, what we need to argue is that the physical operators σ_{n_i} indeed represent the n_i-fold wedge product of the first Chern classes $\alpha_{(i)}$ on $\mathscr{M}_{g,s}$. Since σ_{n_i} is given by the n_i-fold product of $\gamma_0 + \psi_0 + \mathrm{d}\omega$, it is sufficient if we can show that

$$\alpha_{(i)} = (\gamma_0 + \psi_0 + \mathrm{d}\omega)(x_i)\,. \tag{4.14}$$

With this identification we mean that γ_0 represents the component of $\alpha_{(i)}$ given by a two-form on $\mathscr{M}_g$ and a function on Σ, ψ_0 the component which is a one-form on both $\mathscr{M}_g$ and Σ, and $\mathrm{d}\omega$ the two-form component on the surface. To establish (4.14) we first note that the formulas (3.15) and $\mathrm{d}\omega = \partial\bar{\partial}\phi$ have a direct correspondence with the representation of the two-form $\alpha_{(i)}$ as the double derivative $\alpha_{(i)} = \partial\bar{\partial}\log|s(x)|^2$ on $\mathscr{M}_{g,s}$. Indeed, eq. (4.8) shows that the expectation value $\langle\phi(x)\rangle$ can be identified with $\log|s(x)|^2$. The freedom of choice for the meromorphic one-form $s(x)$, used to represent $\alpha_{(i)}$, corresponds in the field theory to the freedom of choosing the positions of the curvature insertions and the background metric $\hat{g}_{ab}$. The second ingredient, which completes the identification (4.14), is the observation that the total BRST operator Q, as well as the global supersymmetry charge Q_s, act as exterior derivatives on moduli space. To see this one uses the fact that all operators in the functional integral (4.12) are BRST and Q_s invariant, except for the supercurrents G_i, which obey

$$\{Q, G_i\} = \{Q_s, G_i\} = T_i\,, \tag{4.15}$$

where $T_i = \int_\Sigma d^2x\, \mu_i(x) T(x)$. Since

$$\frac{\partial S}{\partial m_i} = T_i\,, \tag{4.16}$$

this implies that total BRST and Q_s variations give rise to amplitudes which are derivatives on moduli space $\mathcal{M}_g$. Thus in particular, the formula $\gamma_0 = \frac{1}{2}\{Q,\{Q_s - \overline{Q}_s, \phi\}\}$ shows that γ_0 indeed represents the two-form component of $\alpha_{(i)}$ on $\mathcal{M}_g$.

In the rest of this paper we will study the physical amplitudes using their field theoretical description (4.12). In principle one can use the correspondence described above to interpret our derivations and results in terms of the mathematical description of the amplitudes.

5. The puncture and the dilaton operator

The physical amplitudes (4.12) are written as integrals over the moduli space $\mathcal{M}_g$ and over the positions x_i of the operators. Strictly speaking, however, the region of integration is the moduli space $\mathcal{M}_{g,s}$ of punctured surfaces, which must be compactified by adding surfaces with nodes and with "coincident" punctures. The reason why this is important is that, as we will show in the subsequent sections, the main contribution to the amplitude come from contact interactions, i.e. when one operator coincides with another operator or a node. As will be explained below, in these situations one is not allowed to use the positions of the operators as integration parameters, and as a consequence, also the other components of σ_n, i.e. other than the two-form component $\sigma_n^{(2)}$, will become important. In order to be able to explain what the correct prescription in these situations is, we begin this section with a discussion of the puncture operator, which will play a prominent role in the rest of this paper.

5.1. PUNCTURES AND PICTURES

From ordinary string theory, we know that when we insert a vertex operator at a point x on a Riemann surface, we introduce two new moduli, namely the coordinates of the point x. These coordinates are "created" via the restriction of the diffeomorphism group to the subgroup leaving x fixed. In bosonic string theory, the operator implementing this restriction is $P(x) = c\bar{c}(x)$. In our case, the gauge group includes besides the diffeomorphism also the fermionic spin -1 symmetry generated by the superpartner G of the stress tensor. Supersymmetry requires that when we restrict diffeomorphisms to be 1 at x, the same restriction has to be imposed on the super-diffeomorphisms. The quantum operator imposing both these restrictions is the "puncture operator"

$$P(x) = c\bar{c}\delta(\gamma)\delta(\bar{\gamma})(x)\,. \tag{5.1}$$

Here $\delta(\gamma)(x)$ is the operator forcing the commuting ghost γ to be zero at x.

Hence the puncture operator creates two zero modes of b and two zero modes of β. Each of these zero modes is associated with a modulus of the puncture with the opposite statistics. Therefore, in addition to the ordinary coordinates of x, the puncture created by $P(x)$ also has two anti-commuting coordinates, over which we also have to integrate to obtain the physical amplitudes. Apart from the spin of γ and β, this situation is identical to that in fermionic string theory.

As is well known in that context, the existence of anti-commuting moduli leads to the phenomenon of "picture changing" [26]. In this language, (5.1) is the puncture operator in the -1 picture and the transformation to the 0 picture is the result of absorbing the β zero modes and performing the integral over the corresponding fermionic moduli. A β zero mode is projected out via a δ-function $\delta(\phi_x\beta)$ and the integral over a fermionic modulus leads to the insertion of a super translation generator $\{Q, \phi_x\beta\} = \phi_x(b+G)$. Applying this procedure to $P(x)$ we find the puncture operator in the 0 picture,

$$P^{(0)}(x) = 1. \tag{5.2}$$

Hence we can identify $P^{(0)}$ with $\sigma_0^{(0)}$.

The higher physical operators σ_n with $n>0$ also have their -1 picture versions. From eq. (5.2) we see that the puncture operator $P(x)$ in eq. (5.1) in fact behaves as an inverse picture changing operator. We can now use this fact to write the physical operators in the -1 picture as a product of the zero-form component and the puncture operator $P(x)$, which gives

$$\sigma_n^{(0)} \cdot P = 2^{-n}(\partial\gamma - \bar{\partial}\bar{\gamma})^n c\bar{c}\delta(\gamma)\delta(\bar{\gamma}). \tag{5.3}$$

These -1 picture operators are in fact the basic form of the physical operators σ_n. Physically, they first create a puncture on the surface and then insert the operator at the position of the puncture. In most geometric situations the super-coordinates of the puncture will be super-moduli of the punctured Riemann surface, and by performing the integration over these coordinates one converts the operators σ_n into the two-form versions $\sigma_n^{(2)}$. There are however special circumstances in which one does not integrate over the position of the vertex operators, but keeps it at a fixed position, and in this case one has to use the operators σ_n in the -1 picture. Such situations precisely arise when one wants to study contact interactions.

5.2. THE PUNCTURE AND THE DILATON EQUATION

We will now prove two special recursion relations for the general amplitudes $\langle\sigma_{n_1}\sigma_{n_2}\cdots\sigma_{n_s}\rangle$ on a general Riemann surface. Both recursion relations express certain amplitudes of $s+1$ σ-operators into ones with one operator less. The first, the "puncture equation", reads as follows:

$$\left\langle P\prod_{i=1}^{s}\sigma_{n_i}\right\rangle = \sum_{j=1}^{s}\left\langle \sigma_{n_j-1}\prod_{i\neq j}\sigma_{n_i}\right\rangle. \tag{5.4}$$

An equation of this form was first written down in the context of the one-matrix models in ref. [6]. In the context of topological gravity it has been described in ref. [17], where a mathematical derivation is given due to P. Deligne. The second equation we will prove is simpler and determines the coupling of the "dilaton" operator σ_1. It reads

$$\left\langle \sigma_1 \prod_{i=1}^{s} \sigma_{n_i} \right\rangle = (2g-2+s) \left\langle \prod_{i=1}^{s} \sigma_{n_i} \right\rangle . \tag{5.5}$$

Notice that the number $(2g-2+s)$ is the Euler number χ of the s-punctured surface. Hence eq. (5.5) states that σ_1 exactly couples to the Euler number, justifying the name dilaton. The above two equations hold for arbitrary genus g and number of punctures s, with the only restriction that $s \geqslant 3$ at $g=0$, and $s \geqslant 1$ at $g=1$.

We will first derive the puncture equation. The idea is to eliminate P from the correlator by integrating over its position. As discussed in subsect. 5.1, the two-form component $P^{(2)}$ of the puncture operator is identically zero. Naively this would mean that any correlator with a puncture operator in it vanishes. However, it is possible that there are contact terms between P and the other operators which invalidate this naive conclusion. We will now show that this is indeed what happens.

So let us investigate the contribution from the integral over the position x of P coming from the infinitesimal neighbourhood near one of the other operators $\sigma_{n_i}(x_i)$. For convenience we will choose a coordinate system centered at x_i, and we will use an operator formalism. Let us define the state

$$|\sigma_n\rangle = \sigma_n^{(0)}(0)|P\rangle , \tag{5.6}$$

where $|P\rangle = P(0)|0\rangle$ is the vacuum state in the -1 picture. We have to use the -1 picture, because in the following the position of σ_n is not integrated over, and is kept fixed. The contact term between P and σ_n is contained in the infinitesimal integral

$$\int_{D_\epsilon} P|\sigma_n\rangle = \int_{|x|<\epsilon} d^2x P(x)|\sigma_n\rangle . \tag{5.7}$$

In the search for contact terms it is crucial to choose a particular compactification of the moduli space, by adding in a suitable way the surfaces where two points "coincide". The compactification appropriate for string theory is the so-called stable compactification. The way in which it describes the coincidence of two points is depicted in fig. 1. The two points remain at a finite distance from each other, but are separated from the rest of the surface via the formation of a node, which can be described by the so-called "plumbing fixture", see for example ref.

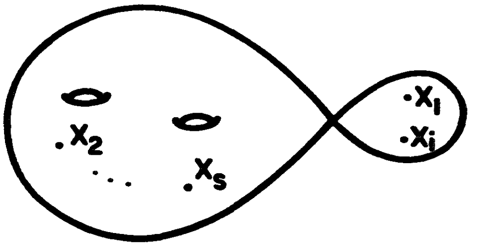

Fig. 1. The coincidence of two points x_1 and x_i is in the stable compactification of $\mathcal{M}_{g,s}$ represented by the situation in which both points are located on a sphere, separated from the rest of the surface by a node.

[31]. In this region of moduli space one no longer uses the positions of the two operators as moduli, but the width q of the neck between the two components of the surface, and the position of the node. For our problem this means in particular that instead of the two-form component of the P (which vanishes), we must use the -1 picture puncture operator $P(x)$ given in eq. (5.1).

A correct definition of the contact term (5.7), i.e. which correctly represents the geometric situation of fig. 1, is

$$\int_{D_\epsilon} P|\sigma_n\rangle = \int_{|q|<\epsilon} \mathrm{d}^2q|q|^{-2}G_0\overline{G}_0 q^{L_0}\bar{q}^{\overline{L}_0}P(1)|\sigma_n\rangle . \tag{5.8}$$

Here q is the modulus describing the degeneration of the node and the operator $q^{L_0}\bar{q}^{\overline{L}_0}$ is the evolution operator describing the propagation through the neck. The insertion of $G_0\overline{G_0}$ is the result of performing the integral over the anti-commuting superpartners of q*. In eq. (5.8) and in the following equations it is implicitly understood that the zero modes of the anti-ghosts corresponding to these moduli are projected out via the insertion of the $b_0\bar{b}_0$ and $\delta(\beta_0)\delta(\overline{\beta_0})$. Taking this into account, we may replace $P(1)$ in eq. (5.8) by 1. To determine (5.8) we make use of the fact that the physical operators satisfy the recursion relation [cf. eq. (3.15)]

$$|\sigma_n\rangle = \tfrac{1}{2}Q(Q_s-\overline{Q}_s)\phi(0)|\sigma_{n-1}\rangle , \tag{5.9}$$

and "partially integrate" the BRST derivatives. The only commutators which can contribute for $\epsilon \to 0$ are those with G_0 and $\overline{G}_0$. Using that $\{Q,G_0\} = \{Q_s,G_0\} = L_0$, and similar for $\overline{G}_0$, we can replace the right-hand side of eq. (5.8) by

$$\int_{|q|<\epsilon} \mathrm{d}^2q\,\partial_q\bar{\partial}_q\big(q^{L_0}\bar{q}^{\overline{L}_0}\phi(0)|\sigma_{n-1}\rangle\big) . \tag{5.10}$$

* Notice that the q-integral in eq. (5.8) can formally be performed yielding $\int_{D_\epsilon}P|\sigma_n\rangle = \Delta P(1)|\sigma_n\rangle$, where $\Delta = G_0\overline{G}_0/(L_0+\overline{L}_0)\delta(L_0-\overline{L}_0)$ is the topological string propagator.

This manipulation is known as the "canceled propagator argument" in string theory, where it is used to derive Ward identities, see for example ref. [32].

Clearly, the integral (5.10) is non-vanishing only if there is a $\log|q|^2$ singularity at $q=0$. From (4.3) it follows that $[L_0, \phi(0)] = \phi_0 + 1$, with $\phi_0 = \phi_0 \partial\phi$, and this in turn implies that

$$L_0 \phi(0)|\sigma_{n-1}\rangle = |\sigma_{n-1}\rangle, \tag{5.11}$$

since $\phi_0|\sigma_{n-1}\rangle = L_0|\sigma_{n-1}\rangle = 0$. We deduce from (5.11) that there is indeed a $\log|q|^2$ singularity given by

$$q^{L_0}\bar{q}^{\bar{L}_0}\phi(0)|\sigma_{n-1}\rangle = \log|q|^2|\sigma_{n-1}\rangle + \text{reg}. \tag{5.12}$$

Finally, inserting this into (5.8) and (5.10) gives the following very simple result for the contact term:

$$\int_{\mathrm{D}_\epsilon} P|\sigma_n\rangle = |\sigma_{n-1}\rangle. \tag{5.13}$$

To obtain the total amplitude we have to sum over the contact terms at all $\sigma_{n_i}(x_i)$, which gives the result (5.4). Note that since the neighbourhood D_ϵ is arbitrarily small we can symbolically write the contact term (5.13) as a δ-function contribution in $P(x)$,

$$P(x)|\sigma_n\rangle = \delta(x)|\sigma_{n-1}\rangle. \tag{5.14}$$

The only situation in which the puncture equation does not apply is for the three-punctured sphere. In that case the moduli space is a point, and the definition of the amplitude $\langle PPP\rangle$ does not involve an integral over the positions of the P's. It is customary to set $\langle PPP\rangle = 1$.

The proof of the dilaton equation (5.5) is rather similar. Again, we wish to perform the integral over the position of σ_1. The contribution away from the position of the other operators is given by the integral of the two-form component $\sigma_1^{(2)} = \mathrm{d}\omega$. This integral gives us the term on the right-hand side proportional $2g-2$, i.e. the total amount of curvature on the surface. The other term proportional to the number of punctures s again comes from the contact terms. A similar analysis as just described for the puncture operator now gives that

$$\int_{\mathrm{D}_\epsilon} \sigma_1|\sigma_n\rangle = |\sigma_n\rangle, \tag{5.15}$$

which directly leads to eq. (5.5).

The dilaton equation has also one exception, namely it can not be applied for the one-point function $\langle\sigma_1\rangle$ on the torus. Again the reason is that the position of the operator is not a modulus, and is not integrated over. Instead the amplitude has the form

$$\langle \sigma_1 \rangle_{g=1} = \int_{\mathscr{F}} d^2 q |q|^{-2} \operatorname{tr}\left[G_0 \overline{G}_0 q^{L_0} \bar{q}^{\bar{L}_0} \sigma_1 \right], \tag{5.16}$$

where $\mathscr{F}$ is the image of the standard fundamental domain under the map $q = e^{2\pi i \tau}$. We can again apply the "canceled propagator argument" to evaluate this amplitude. We write $\sigma_1 = \frac{1}{2}\{Q, \{Q_s - \overline{Q}_s, \phi\}\}$ and perform the partial integrations to obtain

$$\langle \sigma_1 \rangle_{g=1} = \int_{\mathscr{F}} d^2 q \, \partial_q \bar{\partial}_q \operatorname{tr}\left[q^{L_0} \bar{q}^{\bar{L}_0} \phi \right]. \tag{5.17}$$

The trace $\operatorname{tr}[q^{L_0} \bar{q}^{\bar{L}_0} \phi \rangle$ is just the one-point function of ϕ on the torus, given in eq. (4.10). Thus we find that the "tadpole" amplitude of σ_1 is equal to [15, 17]

$$\langle \sigma_1 \rangle_{g=1} = \int_{\mathscr{F}} d^2 q \, \partial_q \bar{\partial}_q \log |\eta(q)|^4 = \tfrac{1}{12}, \tag{5.18}$$

where we used that $\eta(q) \sim q^{1/24}$ for $q \to 0$.

6. The contact term algebra

In the light of the previous discussion of the puncture and dilaton equation it is a natural question to ask whether one can derive similar recursion relations for the higher operators σ_n. Again the idea would be that one tries to eliminate one of the σ_n from the correlator by integrating over its position on the surface Σ. Unfortunately, however, in the general case one expects that some complications arise, due to the fact that, besides the contact terms, there will also be a non-trivial contribution from the integral over the rest of the surface. In particular we know that on a genus-g surface we also have to insert operators $e^{q_i \bar{\pi}}$, with $\sum_i q_i = 2g - 2$, and it is clear that when we integrate $\sigma_n^{(2)}$ over one of these curvature singularities we find a non-zero result.

In order to get a handle on these curvature contributions, the idea which we will now pursue is to try and put them on the same footing as the contact terms. This can be achieved by choosing the locations of the curvature singularities to be at the positions of the physical operators σ_n themselves. To determine the amount of curvature we should associate with each operator σ_n, we consider the selection rule

$$\sum_i (n_i - 1) = 3g - 3. \tag{6.1}$$

This rule and the fact that the total curvature must be $2g - 2$ tells us that at σ_n we must locate $\frac{2}{3}(n-1)$ units of curvature. Thus we are led to consider the new operators

$$\tilde{\sigma}_n^{(0)} = \sigma_n^{(0)} \cdot \mathrm{e}^{\frac{2}{3}(n-1)\tilde{\pi}} . \tag{6.2}$$

The new two-form components $\tilde{\sigma}_n^{(2)}$ again follow by applying the "descent equation" (2.11), and the new -1 picture operators are still equal to $\tilde{\sigma}_n^{(0)} \cdot P$. Physically, the new and old operators should be equivalent, possibly up to a renormalization. However, as we will show, the important practical advantage of using the new operators is that in integrated correlation functions the only contributions come from contact interactions, either between the operators or with possible nodes of the surface. In this section we will determine the contact terms between the operators $\tilde{\sigma}_n$.

When a physical operator $\tilde{\sigma}_m$ is integrated over the Riemann surface Σ, it will in general pick up contact terms at the positions of the other operators. To be more precise, a contact term is the contribution to the integral of $\tilde{\sigma}_m$ which comes from the infinitesimal neighbourhood of another operator. This contribution can be expressed in terms of the physical operators themselves. Conservation of ghost number and of curvature imply that the contact term of $\tilde{\sigma}_m$ at $\tilde{\sigma}_n$ must be of the form

$$\int_{\mathrm{D}_\epsilon} \tilde{\sigma}_m |\tilde{\sigma}_n\rangle = A_m^n |\tilde{\sigma}_{n+m-1}\rangle , \tag{6.3}$$

where $|\tilde{\sigma}_n\rangle = \sigma_n^{(0)}(0)|P\rangle$, and D_ϵ again denotes an infinitesimal neighbourhood of the position of $\tilde{\sigma}_n$ and A_m^n are certain constants. At the end of this section we will describe how the constants A_m^n can be calculated via the method of sect. 5. Before doing this, however, we will discuss some of the general properties of these contact terms.

First let us re-examine the dilaton equation with these new operators. The dilaton operator $\tilde{\sigma}_1$ is the same as it was before and it therefore still couples to the Euler number. Physically, however, there is a difference, since in the new situation it gets its contributions solely from contact terms at the other operators $\tilde{\sigma}_n$. To express this fact, we now use eq. (6.1) to rewrite the dilaton equation (5.5) as

$$\left\langle \tilde{\sigma}_1 \prod_{i=1}^{s} \tilde{\sigma}_{n_i} \right\rangle = \sum_j \tfrac{1}{3}(2n_j + 1) \left\langle \tilde{\sigma}_{n_j} \prod_{i \neq j} \tilde{\sigma}_{n_i} \right\rangle . \tag{6.4}$$

All the terms on the right-hand side indeed come from contact terms

$$\int_{\mathrm{D}_\epsilon} \tilde{\sigma}_1 |\tilde{\sigma}_n\rangle = \tfrac{1}{3}(2n+1)|\tilde{\sigma}_n\rangle . \tag{6.5}$$

The factor $\frac{1}{3}(2n+1)$ is equal to the curvature $\frac{2}{3}(n-1)$ at $\tilde{\sigma}_n$ corrected by 1 due to the original contact term (5.15).

Eqs. (6.3) and (6.5) already exhibit a somewhat surprising feature. Namely, an important conclusion which immediately follows from these equations is that the operation of evaluating contact terms is apparently *non-commutative*. From eq. (6.5) we see that $\tilde{\sigma}_1$ acts as a "number operator" on the states $|\tilde{\sigma}_n\rangle$, while the other operators raise and lower its eigenvalues. In physical terms, the operation of taking the contact term of $\tilde{\sigma}_m$ at $\tilde{\sigma}_n$ adds an amount $\frac{2}{3}(m-1)$ to the curvature and therefore does not commute with taking the contact term with $\tilde{\sigma}_1$, which measures the curvature. Indeed, one finds that the commutator between the two operations is proportional to the curvature at $\tilde{\sigma}_m$.

One might worry about the fact that this non-commutativity of the contact terms seems to imply that the integrations over the positions of the operators also do not commute, which would clearly be unacceptable. However, a closer examination shows that this is not true. In fact, we can use the requirement that the integrals commute to obtain a consistency condition on the coefficients A_n^m. The argument is as follows. Consider the integrals of two operators $\tilde{\sigma}_n$ and $\tilde{\sigma}_m$ in a small neighbourhood of a third operator $\tilde{\sigma}_k$. We demand that

$$\int_{D_\epsilon}\tilde{\sigma}_n\int_{D_\epsilon}\tilde{\sigma}_m|\tilde{\sigma}_k\rangle=\int_{D_\epsilon}\tilde{\sigma}_m\int_{D_\epsilon}\tilde{\sigma}_n|\tilde{\sigma}_k\rangle\,. \tag{6.6}$$

Since we take the neighbourhood D_ϵ small, we can evaluate both sides of this equation by collecting the contact contributions to the integrals using eq. (6.3), while also taking into account the contact term between the first two operators $\tilde{\sigma}_n$ and $\tilde{\sigma}_m$. One obtains a condition on the coefficients A_n^m which can be written in the form of a commutator algebra,

$$A_n^{k+m-1}A_m^k-A_m^{k+n-1}A_n^k=C_{mn}A_{m+n-1}^k\,, \tag{6.7}$$

where $C_{mn}=A_n^m-A_m^n$ is the anti-symmetric part of the contact term between $\tilde{\sigma}_n$ and $\tilde{\sigma}_m$,

$$\int_{D_\epsilon}\tilde{\sigma}_n|\tilde{\sigma}_m\rangle-\int_{D_\epsilon}\tilde{\sigma}_m|\tilde{\sigma}_n\rangle=C_{mn}|\tilde{\sigma}_{n+m-1}\rangle\,. \tag{6.8}$$

Hence we see that the non-commutativity of the contact terms is indeed consistent with the requirement that the integrations over the operators commute, but implies that the coefficients A_m^n are not symmetric in n and m.

We will first determine the "structure constants" C_{mn}. Using eq. (6.5) and the consistency relation (6.7) we can compute C_{mn} for the special case $n=1$, and find that $C_{m1}=\frac{2}{3}(m-1)$. This is precisely equal to the curvature at $\tilde{\sigma}_m$. Thus we see that the physical origin of the asymmetry of the contact terms is again the fact that the operators $\tilde{\sigma}_m$ create curvature. This curvature is measured by the naive integral of $\tilde{\sigma}_1^{(2)} = \mathrm{d}\omega$ over $|\tilde{\sigma}_m\rangle$, but not in that of $\tilde{\sigma}_m^{(2)}$ over $|\tilde{\sigma}_1\rangle$, since in the -1 picture $\tilde{\sigma}_1$ does not contain $\mathrm{d}\omega$. It is reasonable to assume that the same physical effect causes the asymmetry of the general contact terms, which means that we can calculate the coefficients C_{mn} by concentrating on the contributions of the curvature. We compute the left-hand side of eq. (6.8) by naively integrating the two-form component of the first operator over the -1 picture version of the other operator whose position is kept fixed. Hence, the first term is proportional to the curvature at $\tilde{\sigma}_m$ as measured by $\tilde{\sigma}_n^{(2)}$, which contains a term $n\,\mathrm{d}\omega\,\gamma_0^{n-1}\cdot \mathrm{e}^{\frac{2}{3}(n-1)\tilde{\pi}}$, while in the second term the roles of $\tilde{\sigma}_n$ and $\tilde{\sigma}_m$ are reversed. In this way we find that the structure constants C_{mn} are given by

$$\begin{aligned} C_{mn} &= n\tfrac{2}{3}(m-1) - m\tfrac{2}{3}(n-1) \\ &= \tfrac{2}{3}(m-n)\,. \end{aligned} \tag{6.9}$$

Thus we obtain the surprising result that the contact term algebra (6.7), (6.8) is isomorphic to (half of) the Virasoro algebra!

We should note that the just described derivation of the "commutator" (6.8), although it leads to the correct result, is actually somewhat heuristic. Namely, as discussed before, in string theory we must use the moduli space of stable surfaces in which two operators are never allowed to meet at the same point. Therefore, their two-form components will never be able to measure the curvature located at the other operator. However, a justification of the previous calculation is that one can in principle choose to distribute the curvature over a small but finite region around the punctures, in which case the two-form components *are* able to measure it. We will verify eq. (6.9) by an independent calculation.

Let us now turn to the computation of the complete contact coefficients A_m^n. The result we will obtain is

$$A_m^n = \tfrac{1}{3}(2n+1)\,. \tag{6.10}$$

These coefficients indeed satisfy the consistency condition (6.6), with structure constants given by eq. (6.9). Notice that this result in particular implies that the puncture equation has been changed to

$$\left\langle \tilde{P}\prod_{i=1}^{s}\tilde{\sigma}_{n_i}\right\rangle = \sum_j \tfrac{1}{3}(2n_j+1)\left\langle \tilde{\sigma}_{n_j-1}\prod_{i\neq j}\tilde{\sigma}_{n_i}\right\rangle. \tag{6.11}$$

Comparing this with the old eq. (5.4) we conclude that the new operators $\tilde{\sigma}_n$ are renormalized with respect to the old σ_n according to

$$\tilde{\sigma}_n = 3^{-n}(2n+1)!!\sigma_n\,. \tag{6.12}$$

This renormalization has its origin in the fact that the definition of the $\tilde{\sigma}_n$ requires normal ordering.

In the analysis of contact terms an important role is played by the (bare) puncture operator P. As we explained in sect. 4, when two operators approach each other we no longer use their positions as moduli, and correspondingly we must work in the -1 picture. In this picture the operators are represented as

$$\tilde{\sigma}_m = \tilde{\sigma}_m^{(0)} \cdot P\,. \tag{6.13}$$

One can argue that, due to this form of the operators, the contact terms of $\tilde{\sigma}_m$ are determined once we know the contact terms of the bare puncture operator P at the $|\tilde{\sigma}_n\rangle$, via the following formula:

$$\int_{\mathrm{D}_\epsilon} \tilde{\sigma}_m |\tilde{\sigma}_n\rangle = \tilde{\sigma}_m^{(0)}(0) \cdot \int_{\mathrm{D}_\epsilon} P|\tilde{\sigma}_n\rangle\,. \tag{6.14}$$

Since the neighbourhood D_ϵ is arbitrarily small, this equation expresses the fact that the puncture operator $P(x)$ behaves near $\tilde{\sigma}_n$ as a δ-function. Our derivation of the puncture equation was already based on this fact. It is therefore reasonable to assume that the component $\tilde{\sigma}_m^{(0)}$ does not give rise to additional contributions to the contact term (6.14).

A direct consequence of (6.14) is that the coefficients A_m^n are independent of m. Conservation of curvature and ghost number tells us that the contact term of the puncture operator is of the form

$$\int_{\mathrm{D}_\epsilon} P|\tilde{\sigma}_n\rangle = A_n|\mathrm{e}^{\frac{2}{3}\pi}\tilde{\sigma}_{n-1}\rangle\,. \tag{6.15}$$

Substituting this into eq. (6.14) gives

$$\begin{aligned}\int_{\mathrm{D}_\epsilon} \tilde{\sigma}_m|\tilde{\sigma}_n\rangle &= A_n \tilde{\sigma}_m^{(0)}(0)|\mathrm{e}^{\frac{2}{3}\pi}\tilde{\sigma}_{n-1}\rangle \\ &= A_n|\tilde{\sigma}_{n+m-1}\rangle\,,\end{aligned} \tag{6.16}$$

and so indeed the coefficients are independent of m, namely $A_m^n = A_n$.

We will now calculate the coefficients A_n by explicitly evaluating the contact term of P at the $\tilde{\sigma}_n$. We will use the same method as in subsect. 4.2. We describe the coincidence of P and $\tilde{\sigma}_n$ via the geometric situation of fig. 1, and correspondingly represent the contact term as

$$\int_{D_\epsilon} P|\tilde{\sigma}_n\rangle = \int_{|q|<\epsilon} d^2q|q|^{-2}G_0\overline{G}_0 q^{L_0}\bar{q}^{\overline{L}_0}P(1)|\sigma_n\rangle. \tag{6.17}$$

Next we use that the $\tilde{\sigma}_n$ satisfy the following recursion relation:

$$|\tilde{\sigma}_n\rangle = \tfrac{1}{2}Q(Q_s - \overline{Q}_s)\phi(0)|e^{\frac{2}{3}\pi}\tilde{\sigma}_{n-1}\rangle, \tag{6.18}$$

and repeat the "canceled propagator argument" to rewrite eq. (6.17) as the integral of a double q-derivative. The key equation [replacing eq. (5.11)] which determines the coefficient of the $\log|q|^2$-term is

$$\begin{aligned} L_0\phi(0)|e^{\frac{2}{3}\pi}\tilde{\sigma}_{n-1}\rangle &= (\phi_0 + 1)|e^{\frac{2}{3}\pi}\tilde{\sigma}_{n-1}\rangle \\ &= \tfrac{1}{3}(2n+1)|e^{\frac{2}{3}\pi}\tilde{\sigma}_{n-1}\rangle, \end{aligned} \tag{6.19}$$

where we have used $[L_0, \phi(0)] = \phi_0 + 1$. This equation shows that the normal ordered product $:\phi\, e^{\frac{2}{3}(n-1)\pi}:$ has logarithmic conformal dimension $\frac{1}{3}(2n+1)$. Using the above two equations we can repeat the steps leading from (5.8) to (5.13) and obtain for the contact terms of P at $\tilde{\sigma}_n$,

$$\int_{D_\epsilon} P|\tilde{\sigma}_n\rangle = \tfrac{1}{3}(2n+1)|e^{\frac{2}{3}\pi}\tilde{\sigma}_{n-1}\rangle. \tag{6.20}$$

Combining this with eq. (6.16) leads to the final result (6.10) for the coefficients A^n_m.

As a final remark, we notice that in this section we have not assumed that the only contributions of the $\tilde{\sigma}_n$ integrals come from contact terms. However, as we will show in sect. 7, the fact that the contact terms form a non-commutative algebra in fact *implies* that there are no other contributions, except at surfaces with nodes.

7. Recursion relations for the amplitudes

In this section we will analyze the complete amplitudes of two-dimensional topological gravity in more detail. We will show that they satisfy an infinite number of recursion relations, one corresponding to each physical operator σ_n. The first two relations are the puncture and the dilaton equation discussed in sect. 5. To derive this infinite set of relations we will make use only of the contact term algebra derived in sect. 6. We will not explicitly use the field theoretical formulation, but our arguments will depend in an essential way on the fact that such a formulation exists.

A genus-g amplitude can schematically be written as a multiple integral of the form

$$\langle \sigma_{n_1} \dots \sigma_{n_s} \rangle_g = \int_{\mathcal{M}_g} \int_{\Sigma} \mathrm{d}x_s \dots \int_{\Sigma} \mathrm{d}x_1 \langle \sigma_{n_1}(x_1) \dots \sigma_{n_s}(x_s) \rangle . \tag{7.1}$$

Here the $\sigma_n(x_i)$ are the physical operators given in eq. (6.2), i.e. with curvature singularities at their positions. In eq. (7.1) we have purposely written the integrals in a particular order. Namely, we imagine that we first integrate over the position x_1 of the operator σ_{n_1}, then over the position x_2 of σ_{n_2}, etc. At first sight this seems a rather unnecessary specification since the final result for the total amplitude should of course be independent of the order of integration. However, because of the non-commutative nature of the "contact term algebra" described in sect. 6, the requirement that one can interchange the order of integration becomes rather non-trivial, and as we will see, extremely restrictive.

Let us consider the integral over the position x_1 of the first operator in eq. (7.1). In order to separate the contact interactions of σ_{n_1} we decompose this integral in the following way:

$$\int_{\Sigma} \mathrm{d}x_1 = \sum_{i=2}^{s} \int_{\mathrm{D}_i} \mathrm{d}x_1 + \int_{\Delta} \mathrm{d}x_1 + \int_{\Sigma^{\#}} \mathrm{d}x_1 . \tag{7.2}$$

Here D_i is a small neighbourhood around the position x_i of the operator σ_{n_i}, $i = 2, \dots, s$. The second term is only present when the surface Σ is developing one or more nodes; Δ stands in this case for the infinitesimal neighbourhood(s) of the node(s). Finally, $\Sigma^{\#}$ is what remains of the surface after removing the infinitesimal regions D_i and Δ. We will use the stable compactification of the moduli space, for which none of the neighbourhoods $\mathrm{D}_1, \dots, \mathrm{D}_n$ and Δ are allowed to intersect. The situation is shown in fig. 2.

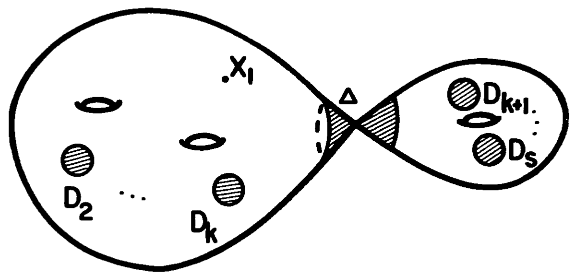

Fig. 2. The integral over the position x_1 is divided into the integration over the small regions D_i around the other punctures, that over the neighbourhood Δ of a (possible) node, plus the integral over the rest of the surface.

The amplitude (7.1) can thus be decomposed into a sum of contributions,

$$\left\langle \sigma_{n_1} \prod_i \sigma_{n_i} \right\rangle = \sum_{i=2}^{s} \left\langle \sigma_{n_1} \prod_i \sigma_{n_i} \right\rangle_{\mathrm{D}_j} + \left\langle \sigma_{n_1} \prod_i \sigma_{n_i} \right\rangle_{\Delta} + \left\langle \sigma_{n_1} \prod_i \sigma_{n_i} \right\rangle_{\Sigma^{\#}}, \quad (7.3)$$

where the subscripts D_j, Δ and $\Sigma^{\#}$ indicate the region of integration of the first operator σ_{n_1}. The corresponding terms in eq. (7.3) will be called the contact, factorization and surface term respectively. We will show below that the surface term actually vanishes. To this end let us first discuss the other two terms in eq. (7.3), representing the contact interactions of the operator σ_{n_1}, in more detail.

The contact interaction between σ_{n_1} and the operator σ_{n_j} is contained in the integral over the neighbourhood D_j. Using our results (6.3) and (6.10) of sect. 6, we find for this contact term

$$\left\langle \sigma_{n_1} \prod_i \sigma_{n_i} \right\rangle_{\mathrm{D}_j} = \tfrac{1}{3}(2n_j + 1) \left\langle \sigma_{n_1 + n_j - 1} \prod_{i \neq j} \sigma_{n_i} \right\rangle. \quad (7.4)$$

The factorization terms, which arise from the integral over the neighbourhood Δ of a node, can also be represented in terms of physical correlation functions. This can be understood as follows. In the stable compactification of moduli space, the situation in which the operator σ_{n_1} approaches a node is described via the formation of a second node, as shown in fig. 3. In the field theory one represents this situation by sandwiching σ_{n_1} between two topological string propagators, which are written as integrals over the two moduli q_1 and q_2, describing the formation of the two nodes. The factorization terms should represent the result of these two integrations. Both integrations can be performed via the canceled propagator argument, by twice applying the recursion relation (5.9) to σ_{n_1}. The corresponding δ-function contribution to the integral over q_1 and q_2 can be represented as insertions of physical operators at each side of the nodes.

Applying the above reasoning to a node associated with the pinching of a handle, we expect that the factorization term in the amplitude (7.3) will have the

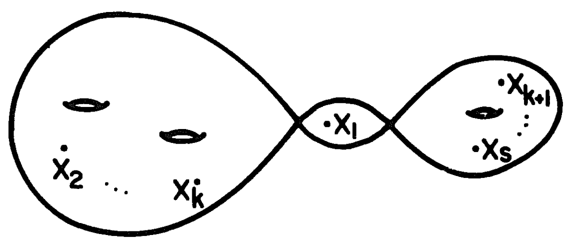

Fig. 3. In the stable compactification the situation in which the point x_1 approaches the position of a node is described via the formation of a second node.

following form*:

$$\left\langle \sigma_{n_1} \prod_i \sigma_{n_i} \right\rangle_{\Delta, g} = \sum_{k=2}^{n_1} B^k_{n_1} \left\langle \sigma_{k-2} \sigma_{n_1 - k} \prod_i \sigma_{n_i} \right\rangle_{g-1} . \tag{7.5}$$

Here we took into account the conservation of ghost number. Note that in evaluating these factorization terms the dimension of the moduli space decreases with 2.

For the second type of node, associated with cycles which divide the surface Σ into two parts, the contribution will be similar as (7.5), except that the right-hand side is given in terms of products of lower-genus amplitudes,

$$\left\langle \sigma_{n_1} \prod_i \sigma_{n_i} \right\rangle_{\Delta', g} = \sum_{k=2}^{n_1} B'^k_{n_1} \sum_{\substack{S = X \cup Y \\ g = g_1 + g_2}} \left\langle \sigma_{k-2} \prod_{i \in X} \sigma_{n_i} \right\rangle_{g_1} \left\langle \sigma_{n-k} \prod_{j \in Y} \sigma_{n_j} \right\rangle_{g_2} . \tag{7.6}$$

Here we used the notation $\Sigma_{S = X \cup Y}$ to denote the sum over all divisions of the set of labels $S = \{i_1, \dots, i_s\}$ in two subsets X and Y.

The coefficients B^k_n and B'^k_n are in principle computable by studying the factorization process very carefully. Because such a factorization analysis is local, both coefficients are expected to be equal, possibly up to a combinatorical factor arising from considerations involving the modular group. Here we will not attempt to perform such an analysis, and instead we will determine B^k_n and B'^k_n by requiring that the factorization terms are compatible with the contact terms.

Substituting eqs. (7.4)–(7.6) into eq. (7.3) gives the complete definition of the decomposition rule of the amplitude (7.1) with respect to integration of the first operator σ_{n_1}. There is a similar decomposition rule for all the operator σ_{n_i}. Furthermore, it is possible to apply these decompositions repeatedly, where by definition in the surface term $\langle \sigma_{n_1} \Pi_i \sigma_{n_i} \rangle_{\Sigma^\#}$ in eq. (7.3) there are no contact terms between σ_{n_1} and the other operators.

We are now in a position to formulate more precisely the consistency requirement following from the fact that the physical amplitudes must be symmetric in all operators,

$$\left\langle \sigma_n \sigma_m \prod_{n_i} \sigma_{n_i} \right\rangle = \left\langle \sigma_m \sigma_n \prod_{n_i} \sigma_{n_i} \right\rangle . \tag{7.7}$$

This equation symbolizes the condition that, when we first perform the integral over the position of σ_n, applying the decomposition (7.3)–(7.6), and then over the position of σ_m, the result should be the same as when we reverse role of σ_n and σ_m, i.e. first integrate out σ_m and then σ_n. The resulting expressions on both sides, however, will not be manifestly the same, and we therefore obtain a non-trivial constraint relating the contact, factorization and surface terms.

* Here one could in principle allow that the factorization coefficients B^k_n depend on the genus g. However, via the reasoning described below, one can show that this is not the case.

The above consistency requirement (7.7) is the global version of the local consistency condition (6.6) on the contact terms. In that case it implied that the contact terms form a (non-commutative) algebra. We will now show that the global requirement (6.6) implies that this contact term algebra is in fact represented on the space of amplitudes, via the evaluation of contact and factorization terms.

Let us assemble the various contributions to the difference between both sides of eq. (7.7). Physically it is clear that the order of the integrals over the positions of σ_n and σ_m only matters when the operators come near each other, thus the only non-zero contributions can come from the various contact iteractions. As seen from eq. (6.8), the contact term between σ_n and σ_m indeed depends on the integration order, and this leads to a contribution $\frac{2}{3}(m-n)\langle\sigma_{n+m-1}\prod_i\sigma_{n_i}\rangle$ to the commutator. In addition, there is a non-trivial commutator of the contact interactions of σ_n and σ_m at the other operators and nodes. The requirement that all contributions should cancel gives a relation of the following form:

$$\tfrac{2}{3}(m-n)\left\langle\sigma_{n+m-1}\prod_i\sigma_{n_i}\right\rangle=\sum_i\mathscr{R}_{\mathrm{D}_i}+\mathscr{R}_{\Delta}\,, \tag{7.8}$$

where $\mathscr{R}_{\mathrm{D}_i}$ denotes the commutator of the contact terms of σ_n and σ_m at σ_{n_i}, and $\mathscr{R}_\Delta$ that at the nodes. Thus we find that the amplitude $\langle\sigma_{n+m-1}\prod_i\sigma_{n_i}\rangle$ can be expressed solely in terms of contact interactions! The results of sect. 6 give that

$$\mathscr{R}_{\mathrm{D}_i}=\tfrac{2}{3}(m-n)\left\langle\sigma_{n+m-1}\prod_i\sigma_{n_i}\right\rangle_{\mathrm{D}_i}, \tag{7.9}$$

and for the factorization terms $\mathscr{R}_\Delta$ one finds expressions precisely of the form (7.5) and (7.6), with $\sigma_{n_i}=\frac{2}{3}(m-n)\sigma_{n+m-1}$, except that the coefficients $\frac{2}{3}(m-n)B^k_{m+n-1}$ are replaced by $\frac{1}{3}(2m-2k+1)B^k_m-\frac{1}{3}(2n+2k+1)B^k_n$, and the same for the B'^k_n.

How do we interpret the result (7.8)? Comparing it with the decomposition formula (7.3), we find that the only consistent interpretation is that the separate terms in eq. (7.8) have to be identified with the corresponding terms in eq. (7.3). This observation has some important consequences. First of all, we see that the surface term in eq. (7.3) is indeed identically zero! In addition, the identification of the factorization terms

$$\mathscr{R}_\Delta=\tfrac{2}{3}(m-n)\left\langle\sigma_{n+m-1}\prod_i\sigma_{n_i}\right\rangle_\Delta \tag{7.10}$$

leads to a set of relations between the factorization coefficients B^k_n. Let us take

$m \geqslant n \geqslant 2$; then for $2 \leqslant k \leqslant n$ we find the relation

$$\tfrac{1}{3}(2m-2k+1)B_m^k - \tfrac{1}{3}(2n-2k+1)B_n^k = \tfrac{2}{3}(m-n)B_{m+n-1}^k, \qquad (7.11)$$

and for $n \leqslant k \leqslant m$ we simply get $B_m^k = B_{m+n-1}^k$. The same equations hold for the $B_n'^k$. It is easy to see that these relations are satisfied only when the coefficients B_n^k are independent of n and k, and similar for $B_n'^k$. To fix them completely one has to use the consistency requirement (7.7) for the case where one of the operators is the puncture operator P. We write the solution as $B_n^k = b/a$ and $B_n'^k = 1/a$, where the constants a and b are given by

$$a = 18\langle PPP\rangle, \qquad b = 24\langle\sigma_1\rangle. \qquad (7.12)$$

We have now determined all the terms in the decomposition (7.3). Putting all ingredients together we can write it as a set of recursion relations for the physical amplitudes. Expressed in terms of the complete amplitudes: $\langle\dots\rangle = \sum_g \lambda^{2g-2}\langle\dots\rangle_g$ these relations take the form

$$\Big\langle\sigma_{n+1}\prod_{i\in S}\sigma_{n_i}\Big\rangle = \sum_j \tfrac{1}{3}(2n_j+1)\Big\langle\sigma_{n+n_j}\prod_{i\neq j}\sigma_{n_i}\Big\rangle + \frac{1}{a}\sum_{k=1}^{n}\Big\{b\Big\langle\sigma_{k-1}\sigma_{n-k}\prod_{i\in S}\sigma_{n_i}\Big\rangle$$

$$+ \sum_{S=X\cup Y}\Big\langle\sigma_{k-1}\prod_{i\in X}\sigma_{n_i}\Big\rangle\Big\langle\sigma_{n-k}\prod_{j\in Y}\sigma_{n_j}\Big\rangle\Big\}. \qquad (7.13)$$

Here we used the fact that the genus expansion of each amplitude only contains one term. The string coupling constant is contained in the constant $a = 18\lambda^{-2}$. These equations are recursive in the sense that the dimension of the moduli space, which is equal to $\sum_i n_i$, is reduced by 1 for the contact terms and by 2 for the factorization terms. Therefore by repeatedly applying these relations one can successively eliminate all operators σ_n and reduce any amplitude to an expression in terms of the basic building blocks $\langle PPP\rangle$ and $\langle\sigma_1\rangle$. The values (7.12) for the coupling constants a and b are such that the final expressions one obtains for the amplitudes in terms of $\langle PPP\rangle$ and $\langle\sigma_1\rangle$ are unique, i.e. independent of the order in which one has applied the recursive relation (7.13).

The recursion relations (7.13) completely determine all amplitudes of two-dimensional topological gravity. As mentioned in sect. 1, we have found an independent derivation of the same relations starting from the results of refs. [2–6] for the one-matrix model [19]. We may therefore conclude that topological gravity and the $k=1$ one-matrix model indeed lead to the same amplitudes at arbitrary genus, as was conjectured by Witten [15].

One of the main results of this paper is that the contact terms in topological gravity give rise to a non-commutative algebra, isomorphic to the Virasoro algebra.

Moreover we have shown that this algebra is represented non-linearly on the amplitudes through the recursion relations (7.13). An interesting feature is that the consistency of the relations requires that they contain amplitudes of different genus. To determine some of the unknown coefficients we have relied on certain consistency relations. Of course it would be interesting to obtain the factorization coefficients by a more direct calculation, using the mathematical or our field theoretical description of the theory.

We have found that relations of the same type as (7.13) are also present in the multicritical points of the one-matrix model. The interpretation of these relations, as well as their generalization to the n-matrix model, will be discussed in a forthcoming paper [19].

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