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Properties of minimal mathematical expectations☆

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Abstract

The aim of this paper is to discuss how to compute a class of minimal mathematical expectations by using backward stochastic differential equations. We also prove that the minimal mathematical expectation operator still preserves some properties of the mathematical expectation operator.

Keywords: Backward stochastic differential equation (BSDE); Minimal mathematical expectation; Minimal conditional mathematical expectation

1. Notation

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $\{W_t\}_{t\geq 0}$ denote d-dimensional standard Brownian motion on this probability space. For simplicity, we assume $d = 1$. We also assume $\{\mathcal{F}_t\}$ is the natural filtration generated by Brownian motion, i.e.,

$$\mathcal{F}_t := \sigma(W_s : s \leq t).$$

Let $T > 0$ be a given finite horizon. We assume $\mathcal{F}_T = \mathcal{F}$. Set

$L^p(\Omega, \mathcal{F}, P) := \{\xi : \xi$ is a $\mathcal{F}_T$-measurable random variable with $E|\xi|^p < \infty\}$, $p > 1$.

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$\mathcal{L}(\Omega, \mathcal{F}, P) := \{\xi :$ there exists $p > 1$ such that $\xi \in L^p(\Omega, \mathcal{F}, P)\}$.
$L^2(0, T; \mathcal{F}, R^n) := \{V : V_t$ is $\mathcal{F}_t$-adapted n-dimensional process with $E \int_0^T |V_t|^2 \, dt < \infty\}$.
$\mathcal{Q}$ is the set of all probability measures Q equivalent to P such that $\frac{dQ}{dP} \in L^2(\Omega, \mathcal{F}, P)$.

For any $Q \in \mathcal{Q}$, let $M_t := E[\frac{dQ}{dP}|\mathcal{F}_t]$, then $\{M_t\}$ is a square integrable martingale. According to the martingale representation theorem, there exists $z \in L^2(0, T; \mathcal{F}, R^d)$ such that $M_t = 1 + \int_0^t z_s \, dW_s$. Let $\theta_t := \frac{z_t}{M_t}$, then $M_t = 1 + \int_0^t \theta_s M_s \, dW_s$. Upon solving this stochastic differential equation, one can obtain

$$M_t = \exp\left\{-\frac{1}{2}\int_0^t |\theta_s|^2 \, ds + \int_0^t \theta_s^\top \, dW_s\right\}, \qquad 0 \le t \le T.$$

In particular, let $t = T$, we have

$$\frac{dQ}{dP} = \exp\left\{-\frac{1}{2}\int_0^T |\theta_s|^2 \, ds + \int_0^T \theta_s^\top \, dW_s\right\}, \tag{1}$$

where $\theta^\top$ denotes θ's transpose.

Eq. (1) means that for each probability measure $Q \in \mathcal{Q}$, there exists a stochastic process $\{\theta_t\}$ such that $\frac{dQ}{dP}$ is generated by $\{\theta_t\}$ via (1). We denote such a Q by Q^θ and call Q^θ the probability measure generated by $\{\theta_t\}$. We will discuss minimal mathematical expectation with respect to a subset $\mathcal{P}$ of $\mathcal{Q}$, which is denoted by

$$\mathcal{P} := \left\{Q^\theta \in \mathcal{Q} : Q^\theta \text{ is generated by } \{\theta_t\} \text{ and } \sup_{0\le t\le T} |\theta_t| \le \mu\right\}, \tag{2}$$

where $\mu > 0$ is a given constant.

Definition 1. Let $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, we call $\mathcal{E}[\xi]$ and $\mathcal{E}[\xi|\mathcal{F}_t]$ defined respectively by

$$\mathcal{E}[\xi] := \inf_{Q\in\mathcal{P}} E_Q\xi; \qquad \mathcal{E}[\xi|\mathcal{F}_t] := ess \inf_{Q\in\mathcal{P}} E_Q[\xi|\mathcal{F}_t], \qquad 0 \le t \le T$$

the minimal mathematical expectation and the minimal conditional mathematical expectation of the random variable ξ on $\mathcal{P}$.

We can define the maximal (conditional) mathematical expectation of the random variable ξ on $\mathcal{P}$ by an analogous method.

Since $\forall Q \in \mathcal{P}$ and $p > 1$, by the Cauchy–Schwarz inequality, we have

$$E_Q|\xi| \le (E|\xi|^p)^{\frac{1}{p}} e^{\frac{1}{2}(q-1)\mu^2 T}, \qquad \forall Q \in \mathcal{P} \tag{3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, thus the minimal (maximal) mathematical expectation in the definition above is well-defined.

Minimal (maximal) mathematical expectations arise from the pricing of contingent claim incomplete markets (see for example [1]). The applications of minimal (maximal) mathematical expectations in economics can be found in [2]. A recent paper by Chen and Davison [3] deals with the relation between minimal (maximal) mathematical expectations and Choquet expectation. However, because of the non-linearity of minimal (maximal) mathematical expectation, it is difficult to discuss the properties of minimal (maximal) mathematical expectations. In this paper, we will study the properties of minimal mathematical expectation by using g-expectation introduced in [4].

2. Main results

Suppose μ is the constant used in the definition of $\mathcal{P}$ in (2). For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, let (y_t, z_t) be the solution of the following backward stochastic differential equation (BSDE):

$$y_t = \xi - \int_t^T \mu|z_s|\mathrm{d}s - \int_s^T z_s \,\mathrm{d}W_s, \qquad 0 \le t \le T.$$

Peng [4] introduced the notions of g-expectation $\mathcal{E}_{-\mu}[\xi]$ and conditional g-expectation $\mathcal{E}_{-\mu}[\xi \mid \mathcal{F}_t]$ of ξ by

$$\mathcal{E}_{-\mu}[\xi] = y_0, \qquad \mathcal{E}_{-\mu}[\xi|\mathcal{F}_t] := y_t, \qquad 0 \le t \le T.$$

The following lemma shows the relation between minimal mathematical expectation and g-expectation:

Lemma 1. *Suppose $\xi \in L^2(\Omega, \mathcal{F}, P)$, μ is the constant defined in (2). Then*

$$\mathcal{E}[\xi] = \mathcal{E}_{-\mu}[\xi]$$

and

$$\mathcal{E}[\xi|\mathcal{F}_t] = \mathcal{E}_{-\mu}[\xi|\mathcal{F}_t].$$

Furthermore, if $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, for any $n \ge 1$, let $\xi_n := \xi I_{[|\xi| \le n]}$. Then

$$\mathcal{E}[\xi] = \lim_{n\to\infty} \mathcal{E}_{-\mu}[\xi_n], \qquad \mathcal{E}[\xi|\mathcal{F}_t] = \lim_{n\to\infty} \mathcal{E}_{-\mu}[\xi_n|\mathcal{F}_t] \qquad \text{in } \mathcal{L}(\Omega, \mathcal{F}, P). \tag{4}$$

Proof. The first claim is from Proposition 3.1 in [5]. Now let us prove (4).

Since

$$\inf_{Q\in\mathcal{P}} E_Q\xi_n = \inf_{Q\in\mathcal{P}}[E_Q(\xi_n - \xi) + E_Q\xi] \ge \inf_{Q\in\mathcal{P}} E_Q(\xi_n - \xi) + \inf_{Q\in\mathcal{P}} E_Q\xi.$$

Thus,

$$\mathcal{E}_{-\mu}[\xi_n] - \mathcal{E}[\xi] \ge \inf_{Q\in\mathcal{P}} E_Q(\xi_n - \xi).$$

Similarly,

$$\mathcal{E}[\xi] - \mathcal{E}_{-\mu}[\xi_n] \ge \inf_{Q\in\mathcal{P}} E_Q(\xi - \xi_n),$$

which means $\mathcal{E}_{-\mu}[\xi_n] - \mathcal{E}[\xi] \le \sup_{Q\in\mathcal{P}} E_Q(\xi_n - \xi)$.

Hence,

$$\inf_{Q\in\mathcal{P}} E_Q(\xi_n - \xi) \le \mathcal{E}_{-\mu}[\xi_n] - \mathcal{E}[\xi] \le \sup_{Q\in\mathcal{P}} E_Q(\xi_n - \xi).$$

Applying inequality (3), $\sup_{Q\in\mathcal{P}} E_Q|\xi_n - \xi| \to 0$ as $n \to \infty$. The second equality in (4) can be proven in the same way.

The proof of (4) is complete.

Applying Lemma 1 and Theorem 37.3 in [4], we obtain that the following properties in [4] are still true for $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$.

Lemma 2. *Assume $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$. Then,*

(i) $\mathcal{E}[\xi] = \mathcal{E}[\mathcal{E}[\xi|\mathcal{F}_t]]$, $0 \le t \le T$;
(ii) *If* $\zeta \ge 0$ *is a* $\mathcal{F}_t$*-measurable r.v., then* $\mathcal{E}[\zeta\xi|\mathcal{F}_t] = \zeta\mathcal{E}[\xi|\mathcal{F}_t]$, $0 \le t \le T$.

Remark 1. If μ in $\mathcal{P}$ is zero, then the minimal expectation $\mathcal{E}[\xi]$ is the traditional mathematical expectation $E[\xi]$. Lemma 2 becomes $E[\xi] = E[E[\xi|\mathcal{F}_t]]$ and $E[\zeta\xi|\mathcal{F}_t] = \zeta E[\xi|\mathcal{F}_t]$ whenever ζ is measurable to $\mathcal{F}_t$.

Applying Lemma 1, we can compute the minimal mathematical expectation and minimal conditional mathematical expectation of ξ by a BSDE.

Example 1. Suppose a is a constant, compute $\mathcal{E}[aW_T|\mathcal{F}_t]$ and $\mathcal{E}[aW_T]$.

Let $y_t = \mathcal{E}[aW_T|\mathcal{F}_t]$, then there exists $\{z_t\}$ such that (y_t, z_t) is the solution of the BSDE:

$$y_t = aW_T - \mu\int_t^T |z_s|\mathrm{d}s - \int_t^T z_s\,\mathrm{d}W_s, \qquad 0 \le t \le T.$$

Solving the BSDE, $y_t = aW_t - |a|\mu(T-t)$; $z_t = a$, $0 \le t \le T$. That is,

$$\mathcal{E}[aW_T|\mathcal{F}_t] = aW_t - |a|\mu(T-t); \qquad \mathcal{E}[aW_T] = -|a|\mu T.$$

Example 2. Compute $\mathcal{E}[\mathrm{e}^{2\mu W_T}|\mathcal{F}_t]$ and $\mathcal{E}[\mathrm{e}^{2\mu W_T}]$.

Applying the Itô formula to $\mathrm{e}^{2\mu x}$, we obtain that $(y_t, z_t) = (\mathrm{e}^{2\mu W_t}, 2\mu\mathrm{e}^{2\mu W_t})$ is the solution of the BSDE:

$$y_t = \mathrm{e}^{2\mu W_T} - \mu\int_t^T |z_s|\mathrm{d}s - \int_t^T z_s\,\mathrm{d}W_s, \qquad 0 \le t \le T.$$

Thus, $\mathcal{E}[\mathrm{e}^{2\mu W_T}|\mathcal{F}_t] = \mathrm{e}^{2\mu W_t}$ and $\mathcal{E}[\mathrm{e}^{2\mu W_T}] = 1$.

Applying Example 1, we can obtain Lemma 3:

Lemma 3. *The operator of the minimal mathematical expectation* $\mathcal{E}[\cdot]$ *is linear if and only if* $\mathcal{P} = \{P\}$.

Proof. Obviously, if $\mathcal{P} = \{P\}$, $\mathcal{E}[\cdot]$ is linear. Conversely, assume $\mathcal{E}[\cdot]$ is linear, i.e., for any ξ, $\eta \in \mathcal{L}(\Omega, \mathcal{F}, P)$, $\mathcal{E}[\xi+\eta] = \mathcal{E}[\xi] + \mathcal{E}[\eta]$. For $a > 0$, let us choose $\xi := aW_T$ and $\eta := -aW_T$, by Example 1, $\mathcal{E}[\xi] = \mathcal{E}[\eta] = -a\mu T$. Since $\mathcal{E}[\cdot]$ is linear, thus $\mathcal{E}[aW_T - aW_T] = \mathcal{E}[aW_T] + \mathcal{E}[-aW_T]$, that is, $a\mu T = 0$, thus $\mu = 0$. From the definition of $\mathcal{P}$ in (2), the proof is complete.

For P-Brownian motion $\{W_t\}$, we know that it has the following properties w.r.t. $E[\cdot]$. For any $0 \le t \le r \le s \le T$ and function h (which satisfies some conditions), we have

(i) $E[h(W_s - W_r)|\mathcal{F}_r] = E[h(W_s - W_r)]$;
(ii) $E[h(W_s - W_r)] = E[h(W_{s-r})]$ and $E[h(W_s - W_r)]$ is the function of $s - r$.

A natural question is whether the minimal mathematical expectation operator still preserves the above properties. We will try to discuss it.

Example 3 shows that $\mathcal{E}[h(W_T - W_r)]$ does not depend on $T - r$.

Example 3. Let $h(x) = \mathrm{e}^{2\mu x}$, by Lemmas 1 and 2(ii):

$$\mathcal{E}[h(W_T - W_r)|\mathcal{F}_r] = \mathrm{e}^{-2\mu W_r}\mathcal{E}[\mathrm{e}^{2\mu W_T}|\mathcal{F}_r] = 1.$$

Hence, applying Lemma 2(i),

$$\mathcal{E}[h(W_T - W_r)] = \mathcal{E}[\mathcal{E}[h(W_T - W_r)|\mathcal{F}_r]] = 1$$

which does not depend on $T - r$. However, we have the following theorem:

Theorem 1. *Suppose $\{W_t\}$ is a P-standard Brownian motion, if $h : R^d \longrightarrow R$ is a Borel measurable function such that for any $T \geq s > r \geq 0$, $h(W_s - W_r) \in \mathcal{L}(\Omega, \mathcal{F}, P)$. Then*

$$\mathcal{E}[h(W_s - W_r)|\mathcal{F}_r] = \mathcal{E}[h(W_s - W_r)].$$

Proof. It is direct from Lemma 1 and Proposition 4.2 in [5].

Example 4 shows that for some $Q \in \mathcal{P}$, $E_Q[h(W_T - W_r)] \neq E_Q[h(W_{T-r})]$.

Example 4. Let Q be the probability measure generated by $\theta_t := \frac{\mu}{1+t}$. For simplicity, we assume $\mu = 1$. Then $Q \in \mathcal{P}$, but $E_Q[W_T - W_r] \neq E_Q[W_{T-r}]$ and $E_Q[W_T - W_r]$ is not a function of $T - r$.

Indeed, let $y_t^1 := E_Q[W_T - W_r|\mathcal{F}_t]$ and $y_t^2 := E_Q[W_{T-r}|\mathcal{F}_t]$, then (y_t^i) $(i = 1, 2)$ are the solutions of the following BSDEs corresponding to the terminal value $\xi^1 = W_T - W_r$ and $\xi^2 = W_{T-r}$.

$$y_t^i = \xi^i - \int_t^T \frac{1}{1+s} z_s^i \,\mathrm{d}s - \int_t^T z_s^i \,\mathrm{d}W_s, \qquad 0 \leq t \leq T, i = 1, 2.$$

Thus,

$$y_t^1 = W_{t\vee r} - W_r + \ln\left(\frac{1+t\vee r}{1+T}\right), \qquad z_t^1 = I_{[r,T]}(t);$$

$$y_t^2 = W_{t\wedge T-r} + \ln\left(\frac{1+t\wedge T-r}{1+T-r}\right); \qquad z_t^2 = I_{[0,T-r]}(t), \qquad 0 \leq t \leq T.$$

That is,

$$E_Q[W_T - W_r] = y_0^1 = \ln\left(\frac{1+r}{1+T}\right) \neq \ln\left(\frac{1}{1+T-r}\right) = y_0^2 = E_Q[W_{T-r}].$$

Thus $E_Q[W_T - W_r]$ obviously is not a function of $T - r$. But we have the following theorem.

Theorem 2. *If $h \in C^{2+\alpha}(R^d)$ and $|h(x)| \leq c(1 + |x|)$ such that for any $0 \leq r \leq s \leq T$ and $h(W_s - W_r) \in \mathcal{L}(\Omega, \mathcal{F}, P)$, where $\alpha > 0, c > 0$. Then, $\forall T \geq r \geq 0$,*

(a) $\mathcal{E}[h(W_T - W_r)] = \mathcal{E}[h(W_{T-r})]$,
(b) $\mathcal{E}[h(W_T - W_r)]$ *is a function of* $T - r$.

Proof. Without loss of generality, we assume $h(W_s - W_r) \in L^2(\Omega, \mathcal{F}, P)$, otherwise, for any $n, j \geq 1$, we can choose $h_j^{(n)}(x) := h(x)\mathrm{e}^{-jk_n(x)}$, where $k_n(x)$ is defined by:

$$k_n(x) := \begin{cases} |x-n|^3, & |x| > n; \\ 0, & |x| \leq n. \end{cases} \tag{5}$$

It is easy to check that for each n, $h_j^{(n)}(x) \to h(x)I_{[|x|\leq n]}(x)$ as $j \to \infty$ and $h_j^{(n)} \in C^{2+\alpha}(R^d)$, $h_j^{(n)}(W_s - W_r) \in L^2(\Omega, \mathcal{F}, P)$.

Let us choose a sequence of continuous functions $\Psi_n(x)$, $(n = 1, 2, \ldots)$ as in the proof of Theorem IV-3.2 in [6] and let

$$\Phi_n(x) := \int_0^{|x|} \mathrm{d}y \int_0^y \Psi_n(u)\,\mathrm{d}u.$$

Let $\overline{W}_t := W_{t+r} - W_r$, $0 \le t \le T - r$. For each Φ_n, let $(\overline{y}^n, \overline{z}^n)$ and (y^n, z^n) be the solutions of the following equations respectively:

$$\overline{y}_t^n = h(\overline{W}_{T-r}) - \mu \int_t^{T-r} \Phi_n(\overline{z}_s^n)\,\mathrm{d}s - \int_t^{T-r} \overline{z}_s^n\,\mathrm{d}\overline{W}_s, \qquad 0 \le t \le T - r. \tag{6}$$

$$y_t^n = h(W_{T-r}) - \mu \int_t^{T-r} \Phi_n(z_s^n)\,\mathrm{d}s - \int_t^{T-r} z_s^n\,\mathrm{d}W_s, \qquad 0 \le t \le T - r. \tag{7}$$

According to Lemma 3.1 in [7], for each $n \ge 1$, $T - r$ and $h(x)$, there exists a unique solution $V^n \in C^{1+2}([0, T - r] \times R^d)$ solving the partial differential equation

$$\begin{cases} \dfrac{\partial V^n}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i=1}^{d} \dfrac{\partial^2 V^n}{\partial x_i^2} - \mu \Phi_n\left(\dfrac{\partial V^n}{\partial x}\right) = 0 \\ V^n(T - r, x) = h(x), \qquad x \in R^d, n = 1, 2, \ldots, \end{cases} \tag{8}$$

where

$$\frac{\partial V^n}{\partial x} := \left(\frac{\partial V^n}{\partial x_1}, \frac{\partial V^n}{\partial x_2}, \ldots, \frac{\partial V^n}{\partial x_d}\right).$$

By the generalized Feynman–Kac formulation (Proposition 4.3 [5]), $\overline{y}_t^n = V^n(t, \overline{W}_t)$ and $y_t^n = V^n(t, W_t)$, $n = 1, 2, \ldots$, particularly, $\overline{y}_0^n = y_0^n = V^n(0, 0)$.

On the other hand, in Eqs. (6) and (7), note that $n \to \infty$, $\Phi_n(x) \uparrow |x|$ (see Theorem IV-3.2 [6]). Applying Lemma 2.1 in [5], $\overline{y}_0^n \to \overline{y}_0$, $y_0^n \to y_0$, where $\overline{y}_0$ and y_0 are the values of the solutions $\{\overline{y}_t\}$ and $\{y_t\}$ of the BSDEs (9) and (10) at time $t = 0$:

$$\overline{y}_t = h(\overline{W}_{T-r}) - \mu \int_t^{T-r} |\overline{z}_s|\mathrm{d}s - \int_t^{T-r} \overline{z}_s\,\mathrm{d}\overline{W}_s, \qquad 0 \le t \le T - r, \tag{9}$$

$$y_t = h(W_{T-r}) - \mu \int_t^{T-r} |z_s|\mathrm{d}s - \int_t^{T-r} z_s\,\mathrm{d}W_s, \qquad 0 \le t \le T - r. \tag{10}$$

From $\overline{y}_0^n = y_0^n$, $\overline{y}_0 = y_0$. Applying Lemma 1, $y_0 = \mathcal{E}[h(W_{T-r})]$, $\overline{y}_0 = \mathcal{E}[h(W_T - W_r)]$, the proof of (a) is complete.

(b) Let $\Phi_n(x)$ be the function defined in the proof of (a). By Theorem 3.1 in [7], for each $n \ge 1$, there exists a unique solution $V^n \in C^{1+2}([0, T] \times R^d)$ satisfying the following partial differential equation:

$$\begin{cases} \dfrac{\partial V^n}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i=1}^{d} \dfrac{\partial^2 V^n}{\partial x_i^2} - \mu \Phi_n\left(\dfrac{\partial V^n}{\partial x}\right) = 0 \\ V^n(0, x) = h(x), \qquad x \in R^d, n = 1, 2 \ldots. \end{cases}$$

Applying the Itô formula to $V^n(T - t, W_t - W_r)$ and Theorem 2.1 in [5], we have

$$V^n(T - t, W_t - W_r) = y_t^n,$$

where (y_t^n) is the solution of equation

$$y_t^n = h(W_T - W_r) - \mu \int_t^T \Phi_n(z_s^n)\, ds - \int_t^T z_s^n\, dW_s, \qquad r \le t \le T.$$

In particular, $y_r^n = V^n(T - r, 0)$ obviously is not a function of $T - r$.

Applying Theorem 1 and noting that $\Phi_n(x) \uparrow |x|$, as $n \to \infty$, we obtain

$$y_r^n \to \mathcal{E}[h(W_T - W_r)|\mathcal{F}_r] = \mathcal{E}[h(W_T - W_r)].$$

Since y_r^n is a function of $T - r$, the proof of (b) is complete.

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