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CLOSED FORM STIFFNESS MATRICES AND ERROR ESTIMATORS FOR PLANE HIERARCHIC TRIANGULAR ELEMENTS

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SUMMARY

Closed form expressions for the stiffness matrix and a simple error estimator and error indicator are derived for plane straight sided triangular finite elements in elasticity problems. The calculation of the error estimator is performed on an element by element basis, and is found to be very accurate and efficient. In general, the solutions for benchmark problems using the error indicators for selective refinement of the regions show accelerated convergence when compared to the convergence rate of solutions using uniform mesh refinement. Evaluation of the stiffness matrices and error estimators using explicit formulations is found to be several times faster than numerical integration.

INTRODUCTION

Closed form representations of element stiffness matrices may be preferred if they offer significant improvements in computational efficiency and accuracy when compared with other procedures. Efficient element formulation is particularly important in adaptive problems where many meshes may be generated and solved in the course of a specific problem solution. Triangular elements have a complete polynomial basis and therefore offer certain advantages over quadrilateral elements which employ incomplete polynomials. Zienkiewicz *et al.*,[1] Cook[2] and Dow and Byrd[3] have discussed the inherent difficulty presented by the incomplete polynomials used in quadrilateral elements. Triangular elements also play an important role in adaptive finite element analysis methods,[4–6] and Shephard[7,8] has observed that triangles may be better suited for use with general purpose mesh generation schemes than quadrilateral elements. In addition, the advantages to be gained from using the hierarchic formulation are well known and have been discussed in by Zienkiewicz *et al.*[9] and Szabo *et al.*[10]

The above observations suggest the need for a family of more efficient triangular elements employing the hierarchic formulation. Tinawi[11] and Subramaniam and Bose[12] have demonstrated the use of closed form integration in the formulation of the stiffness matrix for non-hierarchic triangular elements. The work presented here demonstrates that stiffness matrices and error estimators for hierarchic straight edged triangular elements can also be obtained in closed form.

The estimation of the error in finite element stress analysis is often accomplished by using the fact that a uniform h-version or p-version mesh will eventually converge to the exact solution at a

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uniform logarithmic rate as the number of degrees of freedom is increased. This procedure requires three solutions that lie in the asymptotic range. An estimate of the 'true' solution can then be found.[13,14] This method suffers from the fact that a uniform mesh reduction (or p-level increase) is assumed, but if this assumption is correct the convergence is usually very slow; however, if the assumption is not correct the estimate will naturally be poor.

The rates of convergence of the h- and the p-method have been established by Babuska and Szabo.[15] Babuska and Dorr[16] showed that faster convergence could be obtained using an h–p method that uses an *a priori* analysis to optimally refine the mesh. Early efforts at error estimation and adaptive mesh refinement were made using the strain energy of the element as an indicator. Szabo and Dunavant[17] have shown that the element strain energy is a poor error indicator for selecting elements whose p-level is to be increased in order to obtain rapid convergence. The error inherent in the finite element solution has two parts;[18] the first part is proportional to the error in the second derivatives of the displacement within the element and is attributed to the fact that the solution satisfies the continuity equation only in an integral sense and not at every point in the domain. The second, and usually the more dominant part of the error for low p-level elements, is caused by discontinuities in the first derivatives of the displacement at the boundaries of the element, since only displacement continuity is enforced across inter-element boundaries. A number of error indicators have been proposed that include the contributions of both these sources of error.

$$\|e\|^2 = C_1 \int_\Omega r^2 \, d\Omega + C_2 \int_I J^2 \, dI \tag{1}$$

Babuska and Rhienboldt[19] suggested the use of norms based on the residual forces over the domain. Kelly[20,21] and Kelly and Isles[22] proposed the partitioning of a set of fictitious residual 'error forces' between the elements followed by a complementary analysis to solve for the error, and have shown that an upper bound for the error in strain energy could be obtained. The difficulty with the methods that include the residual has been the fact that they are very complex, and as a result, the calculation of the indicator involves a very large numerical computation effort.

Zienkiewicz and Zhu[23] proposed a simple and computationally efficient error estimator in the energy norm by obtaining a smoothed solution that is a better approximation of the stress distribution than the stress distribution obtained from the finite element solution. The difference between these solutions is used to obtain an estimate of the error. Zienkiewicz and Zhu[24] have also employed this method successfully for the analysis of the plate bending problem and a similar approach has been used by Jirousek and Venkatesh.[25] Byrd[26] has implemented this method in the development of several similar indicators for which the improved solution is obtained by employing various local smoothing assumptions. The closed form error estimator and error indicator presented here are based on the method that employs local nodal stress averaging as used by Byrd.

THE ELEMENT FORMULATION

The area or natural co-ordinates L_1, L_2 and L_3 of any point inside a triangle are related by[27]

$$L_1 + L_2 + L_3 = 1 \tag{2}$$

In the assumed displacement formulation of the stiffness matrices, the displacement (u, v) at any point within the element is given in terms of the nodal displacement vectors $\delta\mathbf{u}$ and $\delta\mathbf{v}$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \begin{bmatrix} \delta\mathbf{u} \\ \delta\mathbf{v} \end{bmatrix} \tag{3}$$

where $\mathbf{N}$ is the shape function in terms of the area co-ordinates. The three node CST element, for example, has a shape function

$$\mathbf{N} = [L_1 \quad L_2 \quad L_3] \tag{4}$$

where each term in $\mathbf{N}$ corresponds to the displacement of a triangle vertex. The hierarchic LST element belonging to the hierarchic family of triangular elements described by Babuska *et al.*[28] has a shape function given by

$$\mathbf{N} = [L_1 \quad L_2 \quad L_3 \quad 2L_1L_2 \quad 2L_2L_3 \quad 2L_3L_1] \tag{5}$$

Using (2) to eliminate L_3, the shape functions may be written as functions of L_1 and L_2 alone in the form

$$\mathbf{N} = \mathbf{S}^{\mathrm{T}}\mathbf{Q} \tag{6}$$

where $\mathbf{Q}$ is a square matrix of order r that contains constant coefficient terms. Peano[29] has shown that the r entries of the vector $\mathbf{S}$ are of the form $L_1^m L_2^n$ and together form a linearly independent set that spans the space of polynomials of order p, where

$$r = (p+2)(p+1)/2 \tag{7}$$

The vector $\mathbf{S}$ for the LST element is

$$\mathbf{S} = [1 \quad L_1 \quad L_2 \quad L_1^2 \quad L_1L_2 \quad L_2^2] \tag{8}$$

The expression for the strain $\boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}^{\mathrm{T}} \tag{9}$$

The global co-ordinates of any point in the element are given in terms of the element area co-ordinates by

$$[x \quad y] = [L_1 \quad L_2 \quad L_3] \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \tag{10}$$

We use (2) to represent L_3 in terms of L_1 and L_2 and, upon differentiation, we get the inverse of the Jacobian

$$\boldsymbol{\Gamma} = \begin{bmatrix} \dfrac{\partial L_1}{\partial x} & \dfrac{\partial L_2}{\partial x} \\ \dfrac{\partial L_1}{\partial y} & \dfrac{\partial L_2}{\partial y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} y_{23} & y_{31} \\ x_{32} & x_{13} \end{bmatrix} \tag{11}$$

where $x_{ij} = x_i - x_j$, and J is the determinant of the Jacobian, which is twice the area of the triangle, and is given by

$$J = x_{13}y_{23} - x_{32}y_{31}$$

The derivatives of the displacements in global and local co-ordinates are related by

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{\Gamma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma} \end{bmatrix} \mathbf{g} \tag{12}$$

where

$$\mathbf{g}^{\mathrm{T}} = \left[\frac{\partial u}{\partial L_1} \quad \frac{\partial u}{\partial L_2} \quad \frac{\partial v}{\partial L_1} \quad \frac{\partial v}{\partial L_2} \right]$$

We differentiate the expressions for u and v in (3) to get

$$\mathbf{g} = \mathbf{R}\begin{bmatrix} \delta\mathbf{u} \\ \delta\mathbf{v} \end{bmatrix} \tag{13}$$

where $\mathbf{R}$ is a matrix of derivatives of the shape functions,

$$\mathbf{R} = \begin{bmatrix} \mathbf{N}_{,L_1} & \mathbf{0} \\ \mathbf{N}_{,L_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{,L_1} \\ \mathbf{0} & \mathbf{N}_{,L_2} \end{bmatrix} \tag{14}$$

Thus, the strain can be written as

$$\boldsymbol{\varepsilon} = \mathbf{PR}\begin{bmatrix} \delta\mathbf{u} \\ \delta\mathbf{v} \end{bmatrix} \tag{15}$$

where

$$\mathbf{P} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix} \tag{16}$$

The strain energy U is given by

$$U = \frac{1}{2}\int\int \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\varepsilon}\, t\, \mathrm{d}y\,\mathrm{d}x \tag{17}$$

where $\mathbf{D}$ is the elasticity matrix. Substituting for $\boldsymbol{\varepsilon}$ using (15) and changing the variables of integration to natural co-ordinates, we get

$$U = \frac{1}{2}\int_0^1 \int_0^{1-L_2} \mathbf{g}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{D}\mathbf{P}\mathbf{g}\, tJ\, \mathrm{d}L_1\,\mathrm{d}L_2 \tag{18}$$

where t is the thickness of the element. Substituting for $\mathbf{g}$ using (13) and setting $\mathbf{G} = \mathbf{P}^{\mathrm{T}}\mathbf{D}\mathbf{P}$, we get

$$U = \frac{1}{2}[\delta\mathbf{u}^{\mathrm{T}} \quad \delta\mathbf{v}^{\mathrm{T}}] \quad [\mathbf{K}] \quad \begin{bmatrix} \delta\mathbf{u} \\ \delta\mathbf{v} \end{bmatrix} \tag{19}$$

where $[\mathbf{K}]$ is the element stiffness matrix, and for an element of uniform thickness,

$$[\mathbf{K}] = tJ\int_0^1 \int_0^{1-L_2} \mathbf{R}^{\mathrm{T}}\mathbf{G}\mathbf{R}\, \mathrm{d}L_1\,\mathrm{d}L_2 \tag{20}$$

Note that $\mathbf{G}$ is a square matrix of order 4, which depends only on the triangle geometry and the material constants, and is independent of the order of the element, and $\mathbf{R}$ is a matrix of polynomials in L_1 and L_2. Substituting for $\mathbf{R}$ using (14) and integrating using the formula

$$\int_0^1 \int_0^{1-L_2} L_1^p L_2^q\, \mathrm{d}L_1\,\mathrm{d}L_2 = \frac{p!\,q!}{(p+q+2)!} \tag{21}$$

we obtain the stiffness matrix in the form

$$[\mathbf{K}] = tJ\begin{bmatrix} \mathbf{XX} & \mathbf{XY} \\ \mathbf{XY}^{\mathrm{T}} & \mathbf{YY} \end{bmatrix} \tag{22}$$

where

$$\begin{aligned} \mathbf{XX} &= \mathbf{G}_{11}\mathscr{A} + \mathbf{G}_{12}(\mathscr{B} + \mathscr{B}^{\mathrm{T}}) + \mathbf{G}_{22}\mathscr{C} \\ \mathbf{XY} &= \mathbf{G}_{13}\mathscr{A} + \mathbf{G}_{14}(\mathscr{B} + \mathscr{B}^{\mathrm{T}}) + \mathbf{G}_{24}\mathscr{C} \\ \mathbf{YY} &= \mathbf{G}_{33}\mathscr{A} + \mathbf{G}_{34}(\mathscr{B} + \mathscr{B}^{\mathrm{T}}) + \mathbf{G}_{44}\mathscr{C} \end{aligned} \tag{23}$$

where $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are matrices of constants

$$\begin{aligned} \mathscr{A} &= \mathbf{Q}^{\mathrm{T}}\mathscr{A}^*\mathbf{Q} \\ \mathscr{B} &= \mathbf{Q}^{\mathrm{T}}\mathscr{B}^*\mathbf{Q} \\ \mathscr{C} &= \mathbf{Q}^{\mathrm{T}}\mathscr{C}^*\mathbf{Q} \end{aligned} \tag{24}$$

and

$$\begin{aligned} \mathscr{A}^* &= \int_0^{1-L_2}\int_0^{1-L_2} \mathbf{S}^{\mathrm{T}}_{,L_1}\mathbf{S}_{,L_1}\,\mathrm{d}L_1\,\mathrm{d}L_2 \\ \mathscr{B}^* &= \int_0^{1-L_2}\int_0^{1-L_2} \mathbf{S}^{\mathrm{T}}_{,L_1}\mathbf{S}_{,L_2}\,\mathrm{d}L_1\,\mathrm{d}L_2 \\ \mathscr{C}^* &= \int_0^{1-L_1}\int_0^{1-L_2} \mathbf{S}^{\mathrm{T}}_{,L_2}\mathbf{S}_{,L_2}\,\mathrm{d}L_1\,\mathrm{d}L_2 \end{aligned} \tag{25}$$

The matrices $\mathscr{A}^*$, $\mathscr{B}^*$ and $\mathscr{C}^*$ are calculated using (21) since the matrix $\mathbf{S}$ and its derivatives contain only polynomials in L_1 and L_2.

$$\begin{aligned} \mathscr{A}^*_{m,n} &= \frac{(i_m - j_m)(i_n - j_n)(i_m + i_n - j_m - j_n - 2)!\,(j_m + j_n)!}{(i_n - j_n)!} \\ \mathscr{B}^*_{m,n} &= \frac{(i_m - j_m)j_n(i_m + i_n - j_m - j_n - 1)!\,(j_m + j_n - 1)!}{(i_n - j_n)!} \\ \mathscr{C}^*_{m,n} &= \frac{j_m j_n(i_m + i_n - j_m - j_n)!\,(j_m + j_n - 2)!}{(i_n - j_n)!} \end{aligned} \tag{26}$$

here

$$m = \frac{i_m(i_m - 1)}{2} + j_m + 1$$

$$n = \frac{i_n(i_n - 1)}{2} + j_n + 1$$

Thus, using (22)–(26), we can obtain the stiffness matrix in closed form.

THE ERROR ESTIMATOR

Since the finite element procedure is based on minimizing the total potential energy, an error estimate in the energy norm is the most natural one to use. The error in the energy norm for an element is

$$\| e_i \| = \sqrt{\int_{\Omega} e_{\sigma}^{T} C e_{\sigma} \, d\Omega} \tag{27}$$

where e_σ is the error in the calculated stress and $\mathbf{C}$ is the material compliance matrix. If the stress distribution from a finite element method solution is examined, it will be seen that a node shared by two elements will usually have two different stress values. This is because, although displacement continuity is guaranteed across inter-element boundaries, strain continuity is not enforced. It is then reasonable to assume that in problems involving a single material, the actual stress at the node would lie somewhere between the values calculated from each element; so an improved estimate of the stress distribution can be obtained by finding a smoothed stress solution $\boldsymbol{\sigma}^*$. An approximation for the error in the stress is obtained from the difference between this improved solution and the finite element solution by

$$\mathbf{e}_{\sigma} = \boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}} \tag{28}$$

where $\hat{\boldsymbol{\sigma}}$ is the stress distribution obtained directly from the finite element solution. Zienkiewicz and Zhu[23] have shown that the smoothed solution $\boldsymbol{\sigma}^*$ is in fact a more accurate representation of the stress distribution than $\hat{\boldsymbol{\sigma}}$. The mathematical analysis of this estimator is provided in Reference 30. The solution, however, requires the inversion of a large, problem dependent matrix whose order is the same as the number of degrees in the problem and accomplished by using an efficient iterative scheme. Byrd[26] has shown for benchmark problems that indicators that use averaged nodal stresses (NA1), so that smoothing is done on a local basis, are comparable in accuracy to the globally smoothed Zienkiewicz–Zhu error indicator.

Byrd observed that an improved solution that is smoothed on a local basis can also be obtained by averaging the stresses at each node, using nodal stresses calculated for each element associated with the node. The stress are interpolated over the element rather than globally using the element shape function N,

$$\boldsymbol{\sigma}^* = \begin{bmatrix} \mathrm{N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{N} \end{bmatrix} \bar{\boldsymbol{\sigma}}^* \tag{29}$$

where the averaged nodal stress is given by

$$\bar{\boldsymbol{\sigma}}^* = \begin{bmatrix} \bar{\boldsymbol{\sigma}}_x^* \\ \bar{\boldsymbol{\sigma}}_y^* \\ \bar{\boldsymbol{\sigma}}_{xy}^* \end{bmatrix} \tag{30}$$

and is calculated by simple averaging of the stress in each element associated with a given node. For example, in the case of the CST element, this is equivalent to interpolating a linear stress distribution over the element using the averaged stresses. In case of higher order elements, it is not necessary to use the same shape functions that were used to calculate the stiffness matrix for interpolating the stresses. Care must be taken, however, if the mid-side and internal shape functions chosen do not correspond to nodal displacements, as is the case in hierarchic elements.

The appropriate function of the derivatives of the smoothed stresses will then have to be calculated at the internal and mid-side nodes.

The element error estimator is thus obtained from (27),

$$\|e_i\|^2 = \int_\Omega (\boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}})^{\mathrm{T}} \mathbf{C}(\boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}})\,\mathrm{d}\Omega \tag{31}$$

and expanding,

$$\|e_i\|^2 = \int_\Omega \boldsymbol{\sigma}^{*\mathrm{T}} \mathbf{C}\boldsymbol{\sigma}^*\,\mathrm{d}\Omega - 2\int_\Omega \boldsymbol{\sigma}^{*\mathrm{T}} \mathbf{C}\hat{\boldsymbol{\sigma}}\,\mathrm{d}\Omega + \int_\Omega \hat{\boldsymbol{\sigma}}^{\mathrm{T}} \mathbf{C}\hat{\boldsymbol{\sigma}}\,\mathrm{d}\Omega \tag{32}$$

The first term and last terms are twice the strain energy as calculated from the smoothed stress solution and from the finite element method solution, and the middle term corresponds to the coupling between the two stress distributions. All three terms may be obtained in closed form for plane triangular elements. The finite element method solution stress distribution is obtained from

$$\hat{\boldsymbol{\sigma}} = \mathbf{D}\boldsymbol{\varepsilon} \tag{33}$$

The first term can be written in the form

$$\bar{\boldsymbol{\sigma}}^{*\mathrm{T}} \begin{bmatrix} C_{11}\mathbf{W} & C_{12}\mathbf{W} & C_{13}\mathbf{W} \\ C_{21}\mathbf{W} & C_{22}\mathbf{W} & C_{23}\mathbf{W} \\ C_{31}\mathbf{W} & C_{32}\mathbf{W} & C_{33}\mathbf{W} \end{bmatrix} \bar{\boldsymbol{\sigma}}^* \tag{34}$$

and the second term can be written as

$$\bar{\boldsymbol{\sigma}}^{*\mathrm{T}} \begin{bmatrix} \Gamma_{11}\mathbf{X} + \Gamma_{12}\mathbf{Y} & \mathbf{0} \\ \mathbf{0} & \Gamma_{21}\mathbf{X} + \Gamma_{22}\mathbf{Y} \\ \Gamma_{21}\mathbf{X} + \Gamma_{22}\mathbf{Y} & \Gamma_{11}\mathbf{X} + \Gamma_{12}\mathbf{Y} \end{bmatrix} \begin{bmatrix} \delta\mathbf{u} \\ \delta\mathbf{v} \end{bmatrix} \tag{35}$$

where

$$\begin{aligned} \mathbf{W} &= \int_0^{1-L_1} \int_0^{1-L_2} \mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}L_1\,\mathrm{d}L_2 \\ \mathbf{X} &= \int_0^{1-L_1} \int_0^{1-L_2} \mathbf{N}^{\mathrm{T}}\mathbf{N}_{,L_1}\,\mathrm{d}L_1\,\mathrm{d}L_2 \\ \mathbf{Y} &= \int_0^{1-L_1} \int_0^{1-L_2} \mathbf{N}^{\mathrm{T}}\mathbf{N}_{,L_2}\,\mathrm{d}L_1\,\mathrm{d}L_2 \end{aligned} \tag{36}$$

The matrices **W**, **X** and **Y** are easily calculated using (21) since the shape matrix **N** and its derivatives contain only polynomials in L_1 and L_2. The matrices **W**, **X** and **Y** are calculated in advance and stored in the program code. The last term in (32) is obtained from the stiffness matrix and nodal displacements and is twice the energy of the finite element solution,

$$\|\hat{u}_i\| = [\delta\mathbf{u}^{\mathrm{T}} \quad \delta\mathbf{v}^{\mathrm{T}}] \quad [\mathbf{K}] \begin{bmatrix} \delta\mathbf{u} \\ \delta\mathbf{v} \end{bmatrix} \tag{37}$$

Thus the error in the energy norm for each element can be calculated readily in closed form using expressions (34), (35) and (37). The energy norm for the entire finite element model is obtained by

$$\|\hat{u}\|^2 = \sum \|\hat{u}_i\|^2 \tag{38}$$

and error in energy norm from

$$\|e\|^2 = \sum \|e_i\|^2 \tag{39}$$

The corrected energy norm is

$$\|u\|^2 = \|\hat{u}\|^2 + \|e\|^2 \tag{40}$$

and the error estimator is calculated from

$$\eta = \frac{\|e\|}{\|u\|} \tag{41}$$

The calculation of an element refinement ratio also follows Zienkiewicz and Zhu.[23] If a maximum permissible error fraction $\bar{\eta}$ is specified for the solution, a mean permissible error for an element can be calculated from

$$\bar{e}_m = \bar{\eta}\sqrt{\frac{\|u\|^2}{m}} \tag{42}$$

where m is the number of elements. The refinement ratio ξ_i is obtained for each element,

$$\xi_i = \frac{\|e_i\|}{\bar{e}_m} \tag{43}$$

All elements for which $\xi_i > 1$ must be subdivided to the extent specified by the magnitude of ξ_i and the analysis repeated. If the actual error $\|e\|_{\mathrm{E}}$ for the solution for benchmark problems can be calculated, an estimate of the effectivity of the indicator θ can be obtained comparing the predicted error in the solution to the actual error achieved,

$$\theta = \frac{\|e\|}{\|e\|_{\mathrm{E}}} \tag{44}$$

RESULTS

The closed form stiffness matrix and error estimator expressions discussed above have been employed to solve several evaluation problems and the results are given below. Figure 1 compares Byrd's results[26] found from a series of uniform mesh models for an end loaded cantilever beam with length to breadth ratio of 10·0 with the error estimates calculated using the closed form expressions for the hierarchic LST element. Byrd has suggested that the error is always underestimated, and the calculated estimates bear this out. Zienkiewicz and Zhu[23] suggested multiplying by empirical factors of 1·3 for CST elements and 1·4 for LST elements, so as to obtain a more accurate estimate of the error. These empirical factors were not found to improve the error estimates consistently for the locally smoothed method.

The problem of a square plate with a square hole is shown in Figure 2. A singularity occurs at the re-entrant corner. Figure 3 shows a series of meshes obtained by uniform element size reduction along with the adaptive mesh obtained using the error indicators of the coarsest uniform mesh. The numbers within each element is ξ_i the refinement ratio. The LST model solution error is shown in Figure 4 as a function of the number of degrees of freedom for the meshes considered. This indicates that the solution converges faster when the mesh refinement is done adaptively using the error indicator rather than using uniform mesh refinement. In the case shown, the adaptive remeshing (including the empirical multiplying factor of 1·4 for the error

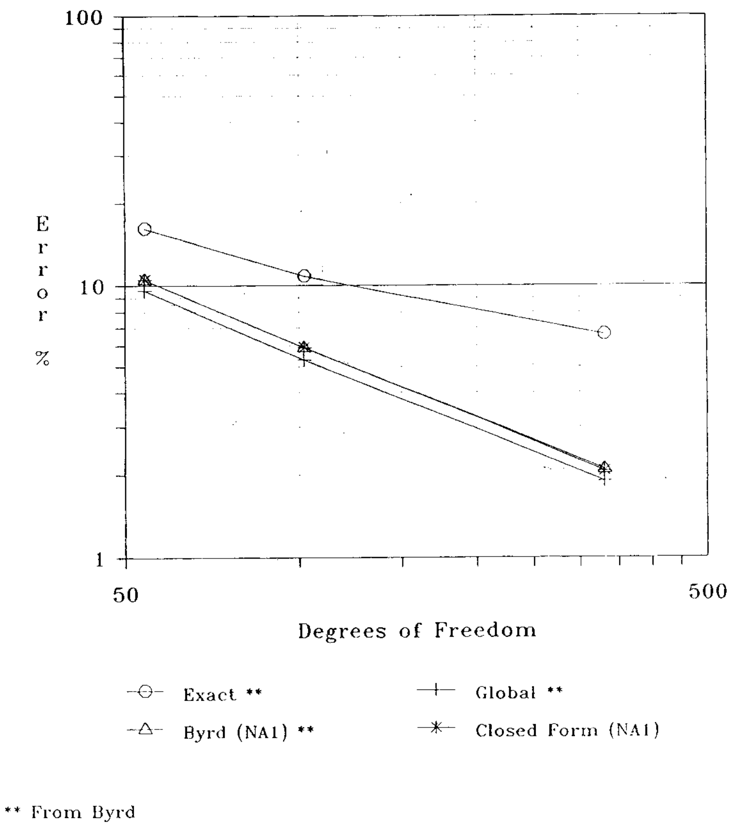


Figure 1. Comparison of the exact and estimated errors for the LST element model of an end loaded cantilever beam

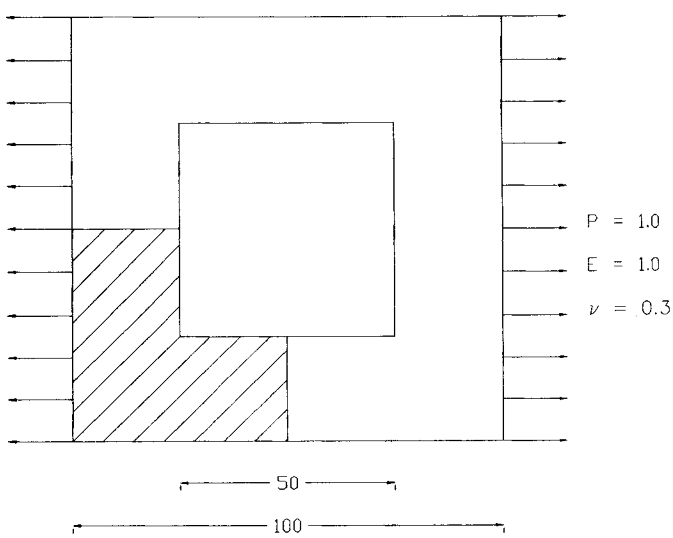


Figure 2. Square plate with a square hole under tension. Owing to symmetry a quarter section is used for the FEM model

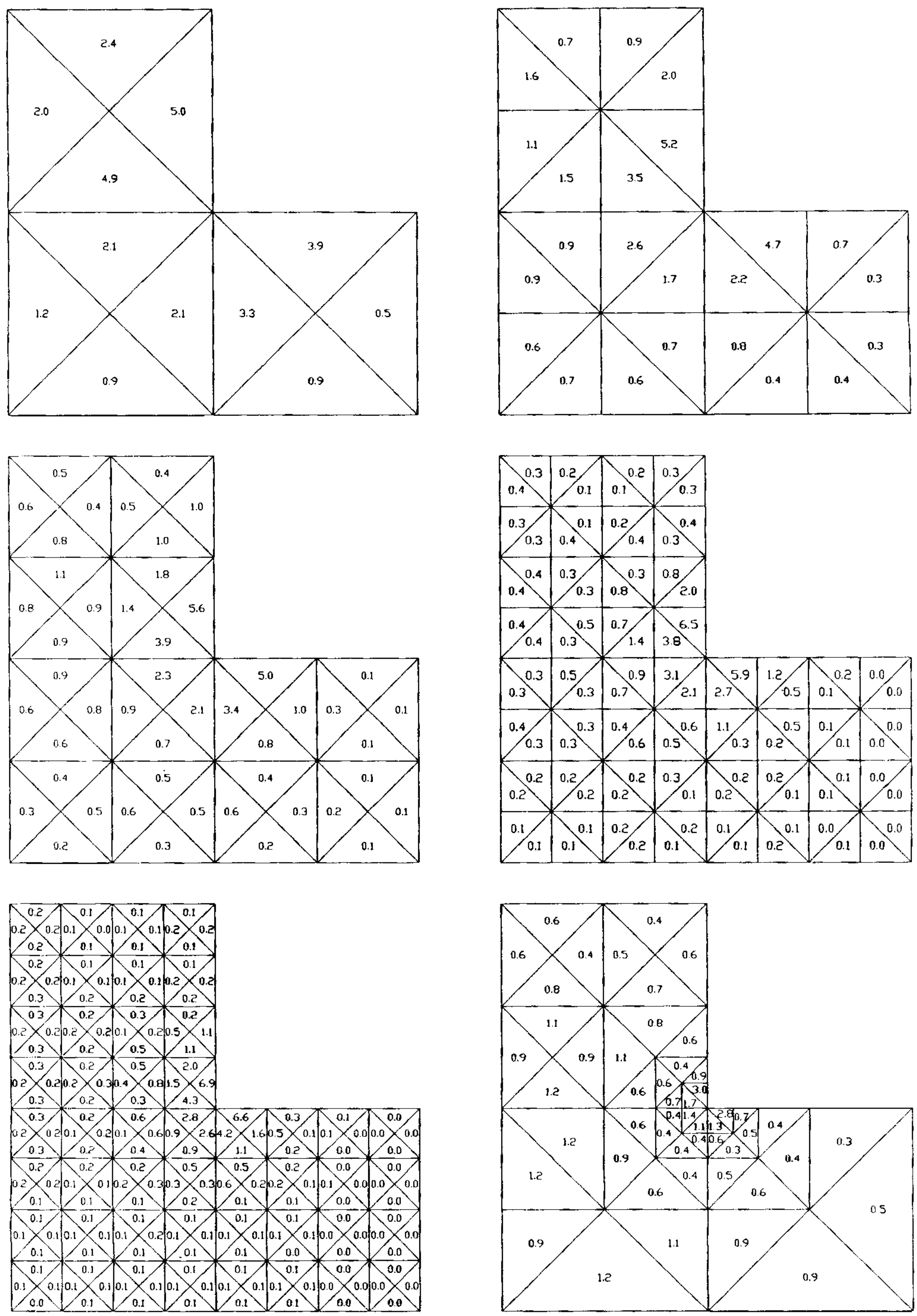


Figure 3. Uniform meshes and the adaptive mesh used to achieve 5 per cent error. The numbers in each element are the element refinement ratios

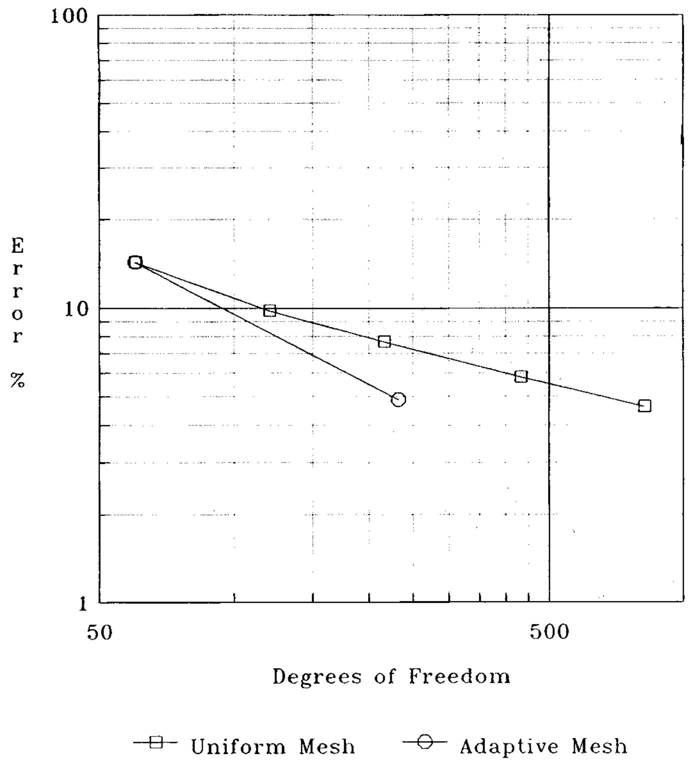


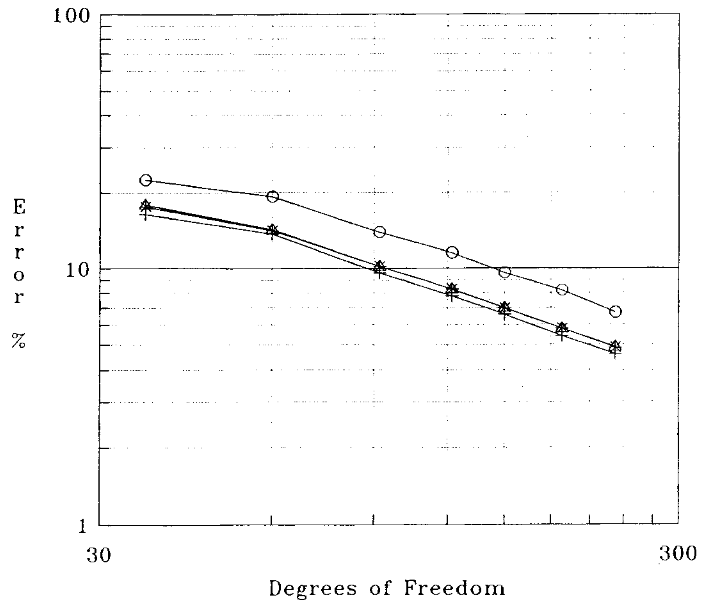
Figure 4. Comparison of the estimated errors for the uniform and adaptive meshes using LST elements to model the square plate problem

indicators) achieved the arbitrarily chosen acceptable error level (5 per cent) in two steps. A comparison of the error estimators for LST elements presented in Byrd[26] is shown for the square plate under tension in Figure 5. (The meshes used are similar to the last mesh of Figure 3. See Reference 26.) The results we obtain agree exactly with those of Byrd. It can be seen that the error is always underestimated.

The top loaded cylinder problem as presented by Zienkiewicz and Zhu was solved using the same meshes presented in Reference 23. In Figure 6(a) is shown the results obtained using CST elements; Figure 6(b) shows the results for LST elements. η is the exact error, η_G and θ_G are the error estimator and effectivity from Reference 23, and η_L and θ_L are the estimator and effectivity for the locally smoothed method presented in this work. The estimated error index is shown for each element.

The problem of a thick walled cylinder under internal pressure was solved using the meshes presented in Reference 23 as adaptive meshes for this problem, Figure 7. The errors estimators are observed to be more accurate when no singularity is present.

In comparing the results of this work and that of Zienkiewicz and Zhu[23] for regions with curved sides (Figures 6 and 7) it must be kept in mind that this work is restricted to elements with straight sides whereas in Reference 23 curved side elements were employed. With straight edged elements, increased accuracy is achieved by h-refinement rather than by p-refinement since increasing the number of elements allows the geometry to be matched more closely.



** From Byrd[24]

Figure 5. Comparison of the exact and estimated errors for the LST element model of the square plate problem

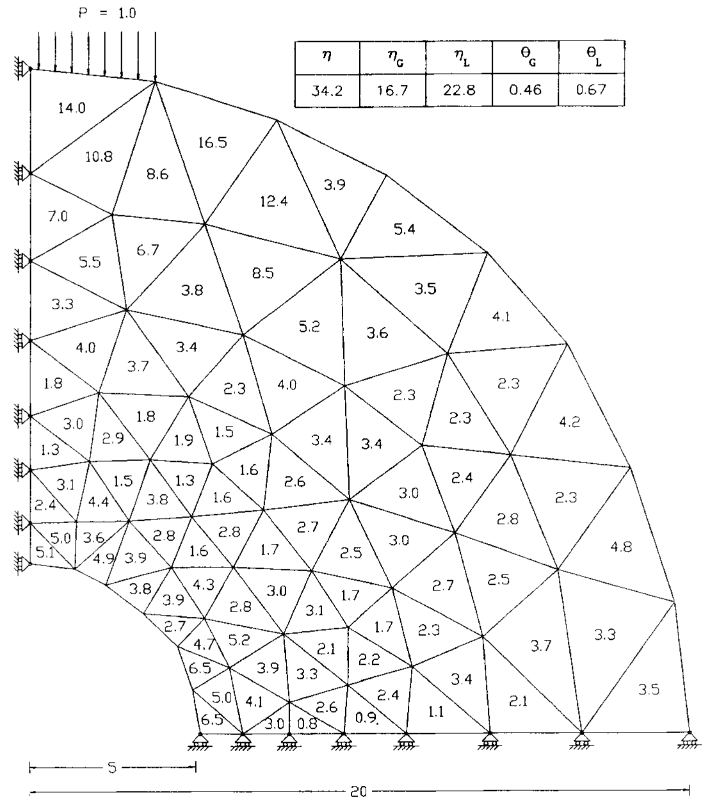


Figure 6(a). Error estimator, effectivity and element refinement ratios for the CST element model of a top loaded cylinder

η	η_G	η_L	θ_G	θ_L
18.3	8.7	9.7	0.47	0.53

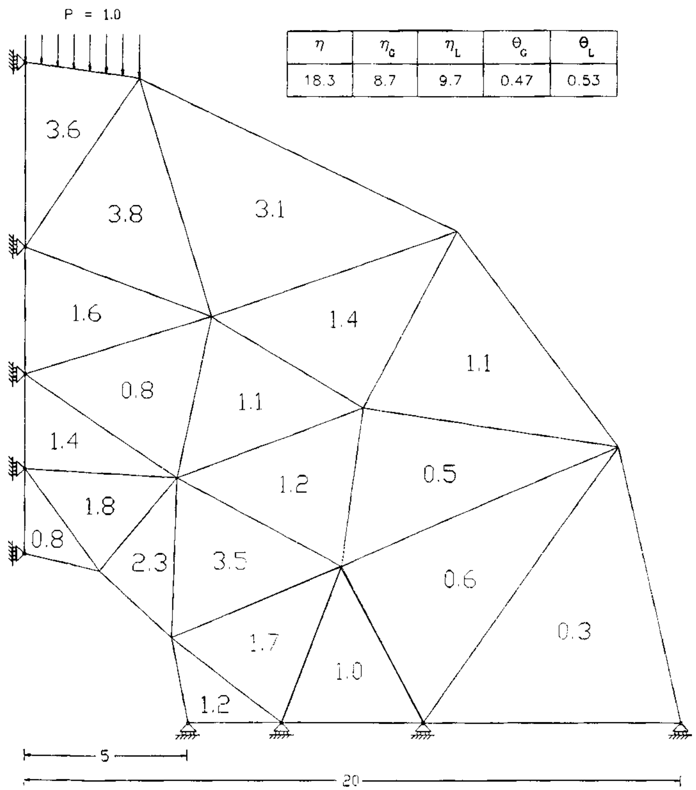


Figure 6(b). Error estimator, effectivity and element refinement ratios for the LST element model of a top loaded cylinder

η	η_G	η_L	θ_G	θ_L
24.8	15.7	20.5	0.62	0.83

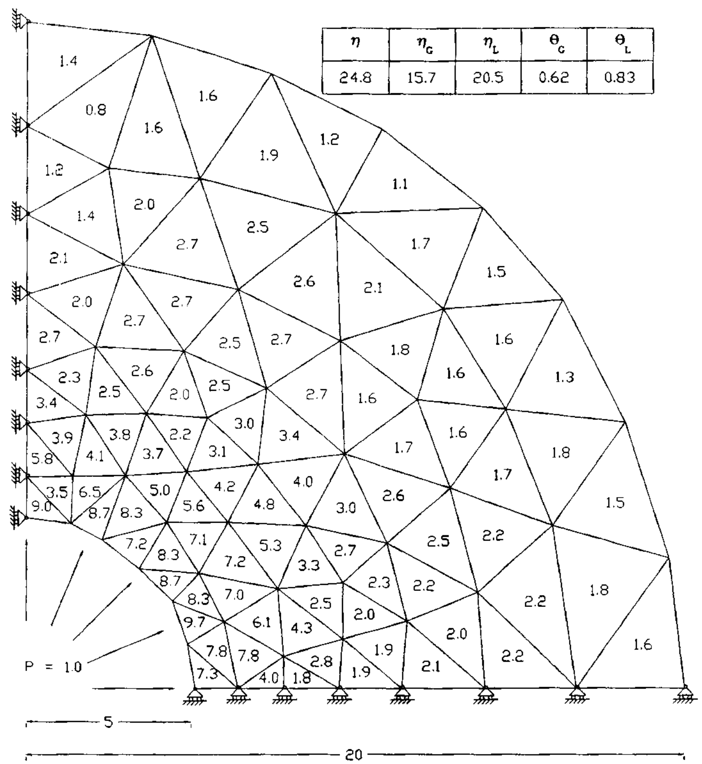


Figure 7(a). Error estimator, effectivity and element refinement ratios for the CST element model of a thick walled cylinder under internal pressure

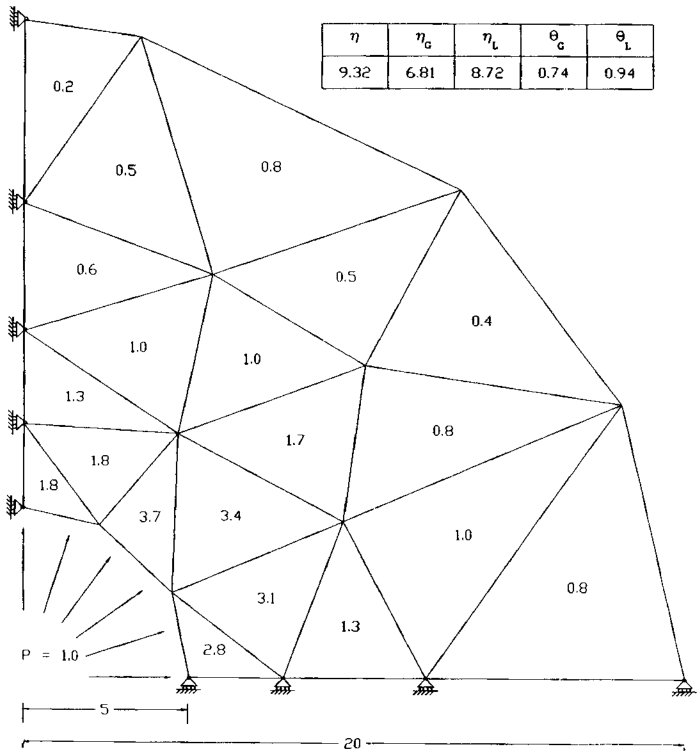


Figure 7(b). Error estimator, effectivity and element refinement ratios for the LST element model of a thick walled cylinder under internal pressure

In the examples presented here the time required to calculate the stiffness matrices for global matrix assembly using the closed form expressions was so small that it was found to be more efficient to recalculate them again for stress recovery instead of storing the matrices in a temporary file and reading them again. When compared to Gaussian numerical integration it was found that the stiffness matrix calculation using closed form integration was significantly faster, by a factor of 2 for CST elements and 6 for LST elements.

CONCLUSION

A procedure to obtain closed form expressions for triangular elements of arbitrary p-level has been presented. Efficient and accurate routines to calculate the stiffness matrices and error estimators for families of hierarchic elements can therefore be coded directly into a computer program. The calculated error estimators compare well with published results. The error is estimated from the difference of two stress values which approach each other as the accuracy of the solution improves. Triangular elements are attractive from the point of view that they have a complete polynomial basis, they are well suited for adaptive mesh generation, and the availability of closed form stiffness matrices and error estimators makes their calculation efficient and accurate. When using straight sided elements for instances of geometries with curved boundaries, many elements of moderate polynomial order must be used; if a coarse mesh is to be employed, curved sided elements of high polynomial order must be developed separately by numerical integration, for use at the curved boundaries.

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