3.10 Earth Rotation Variations

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Nomenclature

$\boldsymbol{\Omega}_0$	Mean angular velocity vector
Ω_0	Mean Earth angular velocity = 2π radian per sidereal day; $=7.292115\times10^{-5}$ rad s^{-1} = 1 cpsd
a	Mean radius of the Earth = 6371 km
a_e	Equatorial radius of the Earth = 6378 km
θ	Colatitude
λ	Longitude
T_d	Period of the relative orbital motion of the Earth and the Sun in solar days: $T_d = 365.2422$ days
H_d	Dynamical ellipticity of the Earth (astronomy community): $H_d = \frac{C-\bar{A}}{C} \approx \frac{C-A}{C}$, $H_d = \frac{e}{(1+e)}$
e	Dynamical ellipticity of the Earth (geodesy community): $e = \frac{C-\bar{A}}{A} \approx \frac{C-A}{C}$, $e = \frac{H_d}{1-H_d}$
e'	Triaxiality coefficient of the Earth: $e' = \frac{B-A}{A+B} = \frac{B-A}{2A}$
A, B	Equatorial moments of inertia: $A \approx B$
$\bar{A}$	Mean equatorial moment of inertia: $\bar{A} = \frac{A+B}{2}$
C	Polar moment of inertia: $C > B > A$
ϵ	Obliquity
ϵ_0	Mean obliquity at J2000
$\psi_A(t)$	Precession in longitude of the figure axis: more precisely of the CIP
$\dot{\psi}_A$	Precession rate in longitude of the figure axis: more precisely of the CIP
$\Delta\psi$	Nutation in longitude of the figure axis: more precisely of the CIP
$\epsilon_A(t)$	Mean obliquity of date
$\dot{\epsilon}_A$	Precession rate in obliquity of the figure axis: more precisely of the CIP
$\Delta\epsilon$	Nutation in obliquity of the figure axis: more precisely of the CIP
X, Y	Coordinates of the CIP in the GCRF: $X+iY=\Delta\psi\sin\epsilon+i\Delta\epsilon$
$\tilde{\eta}$	Nutation of the CIP: $\tilde{\eta} = \Delta\psi\sin\epsilon + i\Delta\epsilon = X + iY$
$\tilde{\eta}_n(t)$	Circular nutation component: $\tilde{\eta}_n(t) = -i\tilde{\eta}(\omega_n)e^{i\Xi_n(t)}$
ω_n	Frequency of nutation in the CRF: $\Xi_n(t) = \omega_n t + \mathcal{X}_n$
ν	Frequency of nutation in the CRF in cpsd: $\omega_n = \nu\Omega_0$
$\mathcal{X}_n$	Phase of component of frequency ω_n of $\tilde{\eta}$
$\tilde{\eta}(\omega_n)$	Amplitude of component of frequency ω_n of $\tilde{\eta}$
$\tilde{\eta}_{Rig}(\omega_n)$	Rigid Earth nutation amplitude of component of frequency ω_n
$\Xi(t)$ or $\Xi_n(t)$	Nutation argument: $\Xi(t) = n_1 l + n_2 l' + n_3 F + n_4 D + n_5 \Omega + \sum_{i=1}^{9} n_i \lambda_i$

ℓ	Mean anomaly of the Moon
ℓ'	Mean anomaly of the Sun
F	One of the fundamental nutation arguments: $F = L - \Omega$
L	Mean longitude of the Moon
D	Mean elongation of the Moon from the Sun
Ω	Mean longitude of the Moon's ascending node
λ_i	Mean longitude of the planet i
A, A', A''	Amplitudes of the nutation in longitude: $\Delta\psi(t) = \sum_{n}[(A_n + A_n^{'}t) \sin \Xi_n(t) + A_n^{''} \cos \Xi_n(t)]$
B, B', B''	Amplitudes of the nutation in obliquity: $\Delta\varepsilon(t) = \sum_{\omega}[(B_n + B_n^{'}t) \cos \Xi_n(t) + B_n^{''} \sin \Xi_n(t)]$
X_s, X_c	Sine and cosine components of X: $X = Y_s \sin \Xi + X_c \cos \Xi$
Y_s, Y_c	Sine and cosine components of Y: $Y = Y_c \cos \Xi + Y_s \sin \Xi$
$\eta^{p,\mathrm{ip}}$	Amplitude of the prograde in-phase component: $\eta^{p,\mathrm{ip}} = \frac{1}{2}\left(\frac{\omega_n}{\lvert\omega_n\rvert} X_s - Y_c\right)$
$\eta^{p,\mathrm{op}}$	Amplitude of the prograde out-of-phase component: $\eta^{p,\mathrm{op}} = \frac{1}{2}\left(X_c + \frac{\omega_n}{\lvert\omega_n\rvert} Y_s\right)$
$\eta^{r,\mathrm{ip}}$	Amplitude of the retrograde in-phase component: $\eta^{r,\mathrm{ip}} = -\frac{1}{2}\left(\frac{\omega_n}{\lvert\omega_n\rvert} X_s + Y_c\right)$
$\eta^{r,\mathrm{op}}$	Amplitude of the retrograde out-of-phase component: $\eta^{r,\mathrm{op}} = \frac{1}{2}\left(X_c - \frac{\omega_n}{\lvert\omega_n\rvert} Y_s\right)$
$\mathbf{\Omega}$	Earth's angular velocity vector: $\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$
$\mathbf{m}$	Wobble: $\mathbf{\Omega} = \mathbf{\Omega}_0 + \mathbf{\Omega}_0\mathbf{m}$
m_z	Wobble axial component: $m_z = \frac{\Omega_z}{\Omega_0} - 1 = \frac{-\Delta\mathrm{LOD}}{\mathrm{LOD}}$
m_x, m_y	Wobble equatorial components: $m_x = \frac{\Omega_x}{\Omega_0}$ and $m_y = \frac{\Omega_y}{\Omega_0}$
$\tilde{m}$	Complex sum of the two previously mentioned components: $\tilde{m} = m_x + \mathrm{i}m_y$ $\tilde{m} = \tilde{m}(\omega_w)e^{i}(\omega_w t + \mathcal{X}_w)$ $\Omega_0\tilde{m} = i\dot{\tilde{\eta}}e^{-\mathrm{i}\Omega_0(t-t_0)}$
$\omega_w = \omega$	Frequency of the equatorial wobble: $\omega_w = \omega_n - \Omega_0$
σ	Frequency of the equatorial wobble in cpsd: $\sigma = \frac{\omega_w}{\Omega_0} = \nu - 1$
$\Theta_w(t) = \omega t + \mathcal{X}_w$	Argument of the equatorial wobble
$\mathcal{X}_w$	Phase of the equatorial wobble
$\tilde{m}(\omega_w)$	Amplitude of the component of frequency ω_w of $\tilde{m}$: $\tilde{m}(\omega_w) = -\frac{\omega_n}{\Omega_0}\tilde{\eta}(\omega_n) = \frac{\omega_n}{\Omega_0}\tilde{p}(\omega_p)$
$\mathbf{H}$	Angular momentum vector
$\mathbf{\Gamma}$	Torque vector
(H_x, H_y, H_z)	Components of $\mathbf{H}$ in space
$(\Gamma_x, \Gamma_y, \Gamma_z)$	Components of $\mathbf{\Gamma}$ in space
X_H, Y_H	Normalized components of $\mathbf{H}$ in space: $X_H = \frac{H_X}{C\Omega_0}$, $Y_H = \frac{H_Y}{C\Omega_0}$
$\Delta\psi_H$	Nutation in longitude of the angular momentum axis
$\Delta\varepsilon_H$	Nutation in obliquity of the angular momentum axis
$\psi_{A,H}(t)$	Precession in longitude of the angular momentum axis
$\dot{\psi}_{A,H}$	Precession rate in longitude of the angular momentum axis
(H_x, H_y, H_z)	Components of $\mathbf{H}$ in TRF
$(\Gamma_x, \Gamma_y, \Gamma_z)$	Components of $\mathbf{\Gamma}$ in TRF
$\tilde{\Gamma}$	Complex sum of the first two components previously mentioned: $\tilde{\Gamma} = \Gamma_x + \mathrm{i}\Gamma_y$
$\mathbf{r}$	Position vector of a point within the Earth relative to the geocenter: $\mathbf{r} = (x, y, z)$
$\mathbf{r_B}$	Position vector of a celestial body B: $\mathbf{r}_B = (x_B, y_B, z_B)$ in TRF; relative to the geocenter: $\mathbf{r}_B = (X_B, X_B, X_B)$ in CRF
r_B	Distance from the geocenter to B
$\tilde{r}_B$	Complex sum of the first two components of $\mathbf{r}_B$ in terrestrial frame: $\tilde{r}_B = x_B + \mathrm{i}y_B$
$W_B(\mathbf{r})$	Gravitational potential of a celestial body B
G	Constant of gravitation
M_B	Mass of a celestial body B
M_E	Mass of the Earth
θ_B	Colatitude of a celestial body B
λ_B	Longitude of a celestial body B
$\rho(\mathbf{r})$	Density of matter at $\mathbf{r}$
ϕ_1, ϕ_2	Time-dependent coefficients of xz and yz in degree 2 tesseral tide generating potential (TGP) of degree 2 and order 1 = tesseral potential
$\tilde{\phi}$	Complex sum of the previously mentioned two components: $\tilde{\phi} = \phi_1 + \mathrm{i}\phi_2$
$\tilde{\Gamma}$	Torque as a function of $\tilde{\phi}$ and of the 21-coefficient of the TGP: $\tilde{\Gamma} = -iAe\Omega_0^2\tilde{\phi}$, $\tilde{\Gamma} = \left(\frac{15}{8\pi}\right)^{1/2} iAeF_{21}$
$F_{nm}(r_B, \theta_B, \lambda_B)$	Coefficient of the TGP
$F_{nm}^{*}(r_B, \theta_B, \lambda_B)$	Coefficient of the TGP
H_{ω}^{nm}	Spectral component amplitude in terms of 'height' of the TGP
$\Theta_{\omega}(t)$	Argument of the tidal component: $\Theta_{\omega}(t) = \omega t + \mathcal{X}_{\omega}$
ω	Tidal frequency
χ_{ω}	Tidal phase
τ	Doodson argument called lunar time: $\tau = 180° + \mathrm{GMST} - s + \lambda$
s	Doodson argument, mean tropic longitude of the Moon

h	Doodson argument, mean tropic longitude of the Sun
p	Doodson argument, mean tropic longitude of the Moon's perigee
N'	Doodson argument, mean tropic longitude of the Moon's node
p_s	Doodson argument, mean tropic longitude of the Sun's perigee
GMST	Greenwich hour angle of the mean equinox: $\mathrm{GMST} = \mathrm{GMST}_0 + \Omega_0 t + \cdots = 280.^{\circ}4606 + \Omega_0 t + \cdots$
GST	Greenwich hour angle of the true equinox
C_{nm}, S_{nm}	Geopotential coefficients: $C_{n0} = -J_2$
K	Hamiltonian for the computation of rigid Earth nutation
c	Semimajor axis of the orbit of B around the Earth (B could be the apparent Sun); also distance from geocenter of B on a circular motion in the orbital plane (for the simple case considered in the chapter)
p	Mean angular velocity of B around the Earth in uniform motion along a circle of radius c centered at the geocenter: $=2\pi$ radians per revolution $p^2 = GM_B/c^3$
n_S	Mean angular velocity of the apparent Sun $= 2\pi$ radians per year
F	Normalization factor of $\boldsymbol{\Gamma}$ in the CRF: $F = (3GM_B/2c^3)\ Ae \sin\varepsilon$
$\mathbf{V}$	Vector of which the equatorial part executes an elliptical motion
V_1, V_2	Components of $\mathbf{V}$
A_s	Amplitude of the sine component of (V_1, V_2): $V_1 = A_s \sin(\omega t + \alpha)$
B_c	Amplitude of the cosine component of (V_1, V_2): $V_2 = B_c \cos(\omega t + \alpha)$
ω	Frequency of the motion
α	Phase at time $t = 0$
V^+, V^-	Amplitudes of the circular motions: $V^+ = (A_s - B_c)/2$, $V^- = (A_s - B_c)/2$
V^p	Amplitudes of the circular prograde motions: $V^p = \frac{1}{2}\left(\frac{\omega}{\lvert\omega\rvert} A_s - B_c\right)$
V^r	Amplitudes of the circular retrograde motions: $V^r = -\frac{1}{2}\left(\frac{\omega}{\lvert\omega\rvert} A_s - B_c\right)$
$T_n(\omega_n)$	Nutation transfer function: $T_n(\omega_n) \equiv \frac{\tilde{\eta}(\omega_n)}{\tilde{\eta}\mathrm{Rig}(\omega_n)}$
$T_w(\omega_w)$	Wobble transfer function: $T_w(\omega_w) \equiv \dfrac{\tilde{m}(\omega_n)}{\tilde{m}\mathrm{Rig}(\omega_w)}$ $T_n(\omega_n) = T_w(\omega_w)$
c_{ij}	Components of the incremental inertia tensor
$\tilde{c}$	Complex sum of two of the previously mentioned components: $\tilde{c} = c_{13} + \mathrm{i}c_{23}$
k	Love number for mass redistribution potential
k_f	Fluid Love number: $k_f = 3(C - A)G/(\Omega_0^2 a^5)$
$\boldsymbol{\Omega}_f$	Incremental core angular velocity vector
m_{fx}, m_{fy}	Incremental core wobble equatorial components
$\tilde{m}_f$	Complex sum of the two previously mentioned components: $\tilde{m}_f = m_{fx} + \mathrm{i}m_{fy}$
$\mathbf{H}_f$	Angular momentum vector of the core
$\tilde{c}_f$	Complex sum of two of the components of the core incremental inertia tensor: $\tilde{c}_f = c_{f13} + ic_{f23}$
k	Compliance for the Earth incremental moment of inertia; concerns the centrifugal and external potential: $k = \frac{k}{k_f} e$
ξ	Compliance for the Earth incremental moment of inertia concerns the core centrifugal potential
γ	Compliance for the core incremental moment of inertia concerns the centrifugal and external potential
β	Compliance for the core incremental moment of inertia concerns the core centrifugal potential
σ_E	CW frequency for a rigid Earth also called Euler frequency: $\sigma_E = e\Omega_0$
$\sigma_{CW}^{\mathrm{elastic}}$	CW frequency for an elastic Earth: $\sigma_{CW}^{\mathrm{elastic}} = \Omega_0(e - k)$
σ_{CW}	CW frequency for Earth with a fluid core: $\sigma_{CW} = \Omega_0 \frac{A}{A_m}(e - k)$
m_{sx}, m_{sy}	Incremental inner core wobble equatorial components
$\tilde{m}_s$	Complex sum of the two previously mentioned components: $\tilde{m}_f = m_{fx} + \mathrm{i}m_{fy}$
$\tilde{\Theta}_s$	Tilt of the solid inner core: $\dot{\tilde{\Theta}}_s = \tilde{m}_s$
K^{CMB}	Coupling constant at the core–mantle boundary
K^{ICB}	Coupling constant at the inner core boundary
$(x_p - y_p)$	Coordinates of the CIP in the TRF
$\tilde{p}$	Complex sum of the previously mentioned components: $\tilde{p} = x_p - iy_p$ $\tilde{p} = \tilde{p}(\omega_p)e^{i(\omega_p t + \mathcal{X}_p)}$ $\tilde{p} = -(X + iY)e^{-i\Omega_0(t - t_0)}$ $\tilde{p} = -\tilde{\eta}e^{-i\Omega_0(t - t_0)}$
ω_p	Frequency of polar motion in TRF: $\omega_p = \omega_n - \Omega_0$

χ_p	Phase of the component of frequency ω_p of $\tilde{p}$
$\tilde{p}(\omega_p)$	Amplitude of the component of frequency ω_p of $\tilde{p}$: $\tilde{p}(\omega_p) = -\tilde{\eta}(\omega_n)$
$\rho_0(r)$	Mean density at radius r
$\mu_0(r)$	Rigidity modulus at radius r: $\mu_0(r) = 0$ in the fluid part
$\kappa_0(r)$	Incompressibility modulus at radius r
$\lambda_0(r)$, $\mu_0(r)$	Lamé parameters: $\lambda_0(r) = \kappa_0(r) - \frac{2}{3}\mu_0(r)$
$\phi_0(r)$	Gravity potential at radius r
$\psi_0(r)$	Centrifugal potential
$\Phi_0(r)$	Initial hydrostatic potential: $\Phi_0(r) = \phi_0(r) + \psi_0(r)$
$\tilde{g}$	Gradient of the gravity potential: $\tilde{g} = \frac{dW_0}{dr}$
g	Gravity: $g = 2\pi G\rho_0 r + \frac{r}{2}\frac{d^2W_0}{dr}$
σ	Frequency expressed in cpsd
$D^l_{mn}(\theta, \lambda)$	Generalized spherical harmonics (GSH)
l	GSH degree: l in $[,+\infty]$
m	GSH order: m in $[-l, +l]$
n	GSH third integer: n in $[-l, +l]$
$\hat{e}_-, \hat{e}_0, \hat{e}_+$	Basis vector used in GSH approach
Φ^{ext}	External gravitational potential
Φ^E_1	Mass redistribution potential: spherical harmonics components $\Phi^{Em}_{1l}(r)$
$\Phi^{Em}_{1l}(r)$	Related to derivative of the mass redistribution potential: $g^{Em}_{1l}(r) = \frac{d\Phi^{Em}_{1l}(r)}{dr} + 4\pi G\rho_0 U^m_l(r)$
s	Displacement vector
$S^{m-}_l, S^{m0}_l, S^{m+}_l$	Components of s
$U^m_l(r)$	Radial component of s: $U^m_l(r) = S^{m0}_l(r)$
$V^m_l(r)$	Spheroidal tangential component of s: $V^m_l(r) = S^{m+}_l(r) + S^{m-}_l(r)$
$W^m_l(r)$	Toroidal tangential component of s: $W^m_l(r) = S^{m+}_l(r) - S^{m-}_l(r)$
X^m_l	Dilatation: $X^m_l = \frac{dU^m_l}{dr} + \frac{1}{r}\left(L^l_0 V^m_l + 2U^m_l\right)$
$\mathbf{T}(\mathbf{r}, t)$	Incremental stress tensor: components $T^{mj}_{li}(r)$
$P^m_l(r)$	Pressure: $P^m_l(r) = T^{m0}_{l0}(r)$
$Q^m_l(r)$	Spheroidal tangential stress: $Q^m_l(r) = T^{m+}_{l0}(r) - T^{m-}_{l0}(r)$
$R^m_l(r)$	Toroidal tangential stress: $R^m_l(r) = T^{m+}_{l0}(r) - T^{m-}_{l0}(r)$
$\eta^{Um}_{1l}, \eta^{Um}_{2l}, \eta^{Um}_{3l}$, $\eta^{Vm}_{1l}, \eta^{Vm}_{2l}, \eta^{Vm}_{3l}$, $\eta^{Wm}_{1l}, \eta^{Wm}_{2l}, \eta^{Wm}_{3l}$	Intermediate notation used in the scalar equations in the frequency domain and using GSH

Abbreviations

μas	Microarcsecond
CIO	Celestial Intermediate Origin
CIP	Celestial intermediate pole
CMB	Core–mantle boundary
cpsd	Cycle per sidereal day
CRF	Celestial reference frame
CW	Chandler wobble
ECRF	Ecliptic celestial reference frame
FCN	Free core nutation
FICN	Free inner core nutation
FOC	Fluid outer core
GAST	Greenwich Apparent Sidereal Time
GCRF	Geocentric Celestial Reference Frame
GMST	Greenwich Mean Sidereal Time
GSH	Generalized spherical harmonics
GST	Greenwich Sidereal Time
ICB	Inner core boundary
ICRF	International Celestial Reference Frame
ICW	Inner core wobble
ITRF	International Terrestrial Reference Frame
J2000	Event (epoch) at the geocenter at date 2000 January 1.5
LOD	Length of day
mas	Milliarcsecond ($=4.8481 \times 10^{-9}$ rad)
NDFW	Nearly diurnal free wobble (FCN as seen in the TRF)
PFCN	Prograde free core nutation (FICN = PFCN)
SIC	Solid inner core
TGP	Tide-generating potential
TRF	Terrestrial reference frame
VLBI	Very long baseline interferometry

3.10.1 Introduction–Concepts–Overview

3.10.1.1 Manifestations of Variations in Earth Rotation and Orientation

Variations in the orientation in space of an Earth-fixed reference frame (Earth orientation variations) are driven by variations in Earth rotation, that is, in the angular velocity vector of Earth rotation. They are manifested as variations in the direction of Earth-related axes in space (precession and nutation; see Figure 1) as well as those relative to a terrestrial reference frame (TRF) (wobble and polar motion) and also as variations in the angular speed of rotation (or spin rate, for brevity) that translate into variations in the length of day (LOD).

The axes of interest are the figure axis, rotation axis, and the angular momentum axis. The term 'figure axis' is used herein for the axis of maximum moment of inertia of the 'static' Earth, that is, of the Earth with time-dependent deformations caused by tidal and other forces disregarded. (The axis of maximum moment of inertia of the dynamically deforming Earth has indeed been called the figure axis in some of the literature, but it is of little interest.) The rotation axis is simply the direction of the angular velocity vector (also called the rotation vector). All the axes may be thought of as unit vectors in the directions of the respective axes. Nutations of all three axes are distinct, yet closely related.

With every axis is associated a pole, which is the intersection of the axis with a geocentric sphere of unit radius (in arbitrary units). The motion of the axis is faithfully reflected in the motion of the associated pole and also in the motion of the plane perpendicular to the axis (the equator of that axis), so the nutation, for instance, may and often is referred to as the nutation of the associated pole or equator.

Precession of any axis is the secular (smoothly varying) part of the motion of the axis in space, that is, relative to the directions of 'fixed' stars. (Closest to the ideal of fixed stars are the quasars, the most distant objects in space.) It is the 'mean' motion of the axis in space. Nutation is the oscillatory part of the motion of the axis, with a large multiplicity of periods; it causes the instantaneous position of the axis to move around its mean position along an oscillatory path (see Figures 1–3).

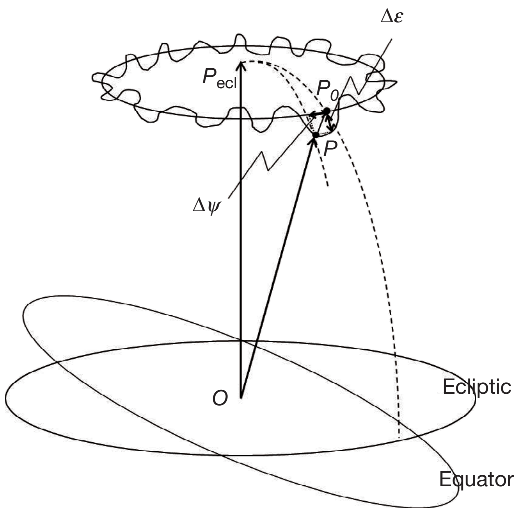

Figure 1 Precession and nutation of an Earth-related axis. Precession carries the pole of the axis at a uniform rate in an anticlockwise sense along a circle on the surface of the celestial sphere, centered on the normal to the ecliptic plane. Nutation consists of small deviations from this uniform motion, both over the precessional path (nutation in longitude, $\Delta\psi$) and perpendicular to it, on the celestial sphere (nutation in obliquity, $\Delta\varepsilon$). The resulting path of the pole appears wiggly, as in the figure.

The term 'wobble' is used herein in a very broad sense, for any periodic or quasiperiodic motion of the instantaneous rotation axis with respect to the figure of the Earth, irrespective of the frequency or the physical origin of the motion. The essence of wobble is the temporally varying offset (angular separation) of the direction of the rotation axis from that of the figure axis. Wobble is characterized by the variation of the equatorial components of the rotation axis in a terrestrial frame of reference. The Earth is made up broadly of three regions, the mantle and the two core regions within it, the fluid outer core (FOC), and the solid inner core (SIC). The term 'solid Earth,' as generally used, includes all three regions while excluding the fluid layers at the surface, namely, the oceans and the atmosphere. The three regions of the solid Earth have different rotation vectors, though their motions are mutually coupled. Therefore, each region has its own wobble. The term 'wobble,' when used without specific mention of the region, is intended to be that of the mantle.

The term 'polar motion' has been in use to refer generally to any motion of the Earth's pole of rotation relative to the Earth and, in particular, to the wobble as well as for secular polar motion. The latter is a steady drift of the mean pole of rotation (with the periodic wobbles averaged out) relative to the Earth; it may be viewed as the present-day trend of the polar wander, which has been taking place over geologic timescales. In recent usage, polar motion has taken on a technical meaning, to be explained later on (see Sections 3.10.2.5.1 and 3.10.12.4), as the motion of a conceptual pole called the celestial intermediate pole (CIP) with frequencies outside a specified range. A different aspect of the variations in Earth rotation is the variation in its spin rate or, its equivalent, the LOD variation. The mean rate of Earth rotation is one cycle per sidereal day (cpsd), equivalent to an angular speed $\Omega_0 = 2\pi$ radians per sidereal day $= 7.292115 \times 10^{-5}$ rad s^{-1}. (The sidereal day is the period of rotation of the Earth relative to the most distant

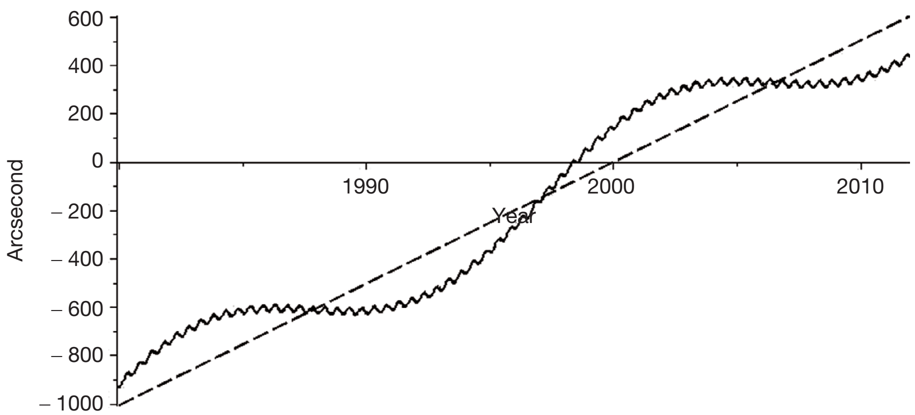

Figure 2 Precession and nutation of the axis in longitude. The dashed straight line represents the precessional motion, and the deviation of the curve from the straight line represents nutation. To make this deviation clearly visible, the angle of nutation has been scaled up by a factor of ten relative to the precession.

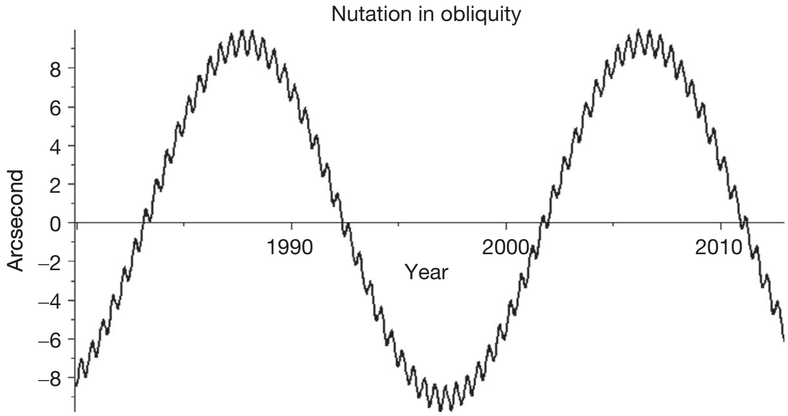

Figure 3 Nutation of the axis in obliquity.

'fixed' stars. It is $T_d/(T_d + 1)$ times the solar day of 24 h, where T_d, equal to 365.2422, is number of solar days in the period of the relative orbital motion of the Earth and the Sun. The solar day is the period of rotation of the Earth relative to direction toward the Sun.) The small deviations from the uniform rate of rotation give rise to LOD variations.

3.10.1.2 Free Rotational Modes

The variations in Earth rotation/orientation are made up of both free motions and forced motions of the various axes, apart from variations in the spin rate.

The free motions, which are not driven by any external torque, conserve the Earth's total angular momentum. The angular momentum is a product of the moment of inertia (which, in general, is a tensor) and the angular velocity, and its conservation requires that if one changes with time, the other must undergo compensating changes. This happens whenever the rotation (angular velocity) vector of the mantle or other region of the Earth is not along a principal axis, causing the inertia tensor (in a space-fixed frame) to vary as the rotation progresses; this in turn, causes a compensatory variation of the rotation vector. The free nutation/wobble normal modes are the manifestations of this phenomenon. The three-layer Earth has four such modes: the Chandler wobble (CW) with a period of about 430 days; the free core nutation (FCN) with a period of about −430 days in the celestial frame, along with its associated wobble (the nearly diurnal free wobble or NDFW), which has nearly diurnal frequency in the terrestrial frame as its name implies; the less prominent free inner core nutation (FICN, also called the prograde free core nutation or PFCN), which has a rather long period, apparently around 1000 days, in the celestial frame; and the inner core wobble (ICW) with a still longer period in the terrestrial frame. See Annex A for further explanations. They are excited by various geophysical processes. They have been found, observationally, to have variable amplitudes and are not amenable to theoretical prediction. Since the frequencies of the FCN and FICN/PFCN modes lie in the midst of the low-frequency band wherein the most important nutation components lie, the resonances associated with these modes play an important role in the nutational responses of the Earth. Though the CW frequency lies well outside this band when viewed in the celestial frame, its influence on nutation is not completely negligible unlike that of the ICW.

3.10.1.3 Forced Motions

The forced nutation, wobble, and precessional motions of the axes are excited almost wholly by the gravitational pulls of the Sun, Moon, and planets on the nonspherical (nearly ellipsoidal) Earth and are different aspects of the gyroscopic response of the rotating Earth to the gravitational torque. The deformations produced by the direct action of the spatially varying gravitational force (as distinct from the gravitational torque) on the solid Earth, as well as by the loading effect of ocean tides and the gravitational attraction of the solid Earth by the redistributed ocean mass, play an important role in rotation variations; so does the deforming action of the centrifugal perturbation caused by the rotation variations themselves. Torques exerted by atmospheric pressure tides (arising primarily from the varying thermal input from solar radiation) also make minor contributions to variations in the directions of the axes. The forced motions of the axes due to the gravitational torques on the solid Earth are accurately predictable, given a sufficiently accurate knowledge of the relevant Earth parameters and of the orbital motions of the various celestial bodies.

The nutational motions consist of numerous periodic components with a discrete spectrum of frequencies determined by the spectrum of the gravitational potential, at the Earth, of the solar system bodies. The frequencies of spectral components with nonnegligible amplitudes extend from large negative values to large positive values for the nutations of any of the axes mentioned in Section 3.10.1.2. (By 'large,' we mean several times the frequency of the diurnal Earth rotation.) Positive frequencies correspond to prograde motions, which are in the same sense as that of the diurnal Earth rotation, while negative frequencies are for retrograde motions (in the opposite sense).

Figure 4 shows the observations of LOD variations and polar motion.

The dominant causes of polar motion (other than the tidally excited wobbles) are the atmosphere and the ocean, at timescales of a few days to a few years in the terrestrial frame (see Chapter 3.09). These geophysical fluids are also primarily responsible for the LOD variations at timescales of a few days to a few years in the terrestrial frame, while the interaction

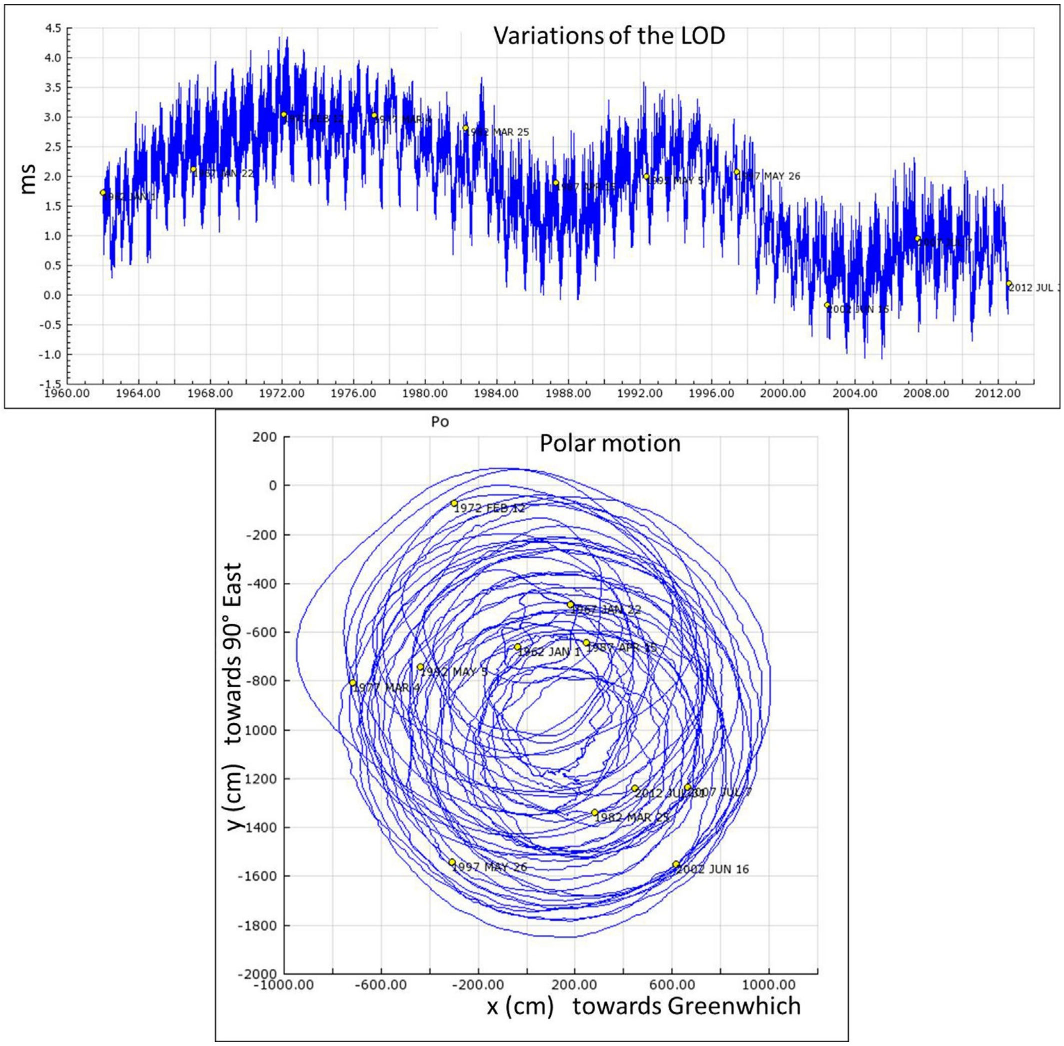

Figure 4 Observation of length-of-day variations and polar motion. Figure built from the IERS website.

of the magnetic field in the fluid core with the mantle is invoked to explain the variations at decadal timescales (see Chapter 3.09). There is a discrete spectrum in the LOD variations, arising primarily from the changes in the axial moment of inertia resulting from the deforming action of the tidal potential and to a very much smaller extent by tidal torques: the axial component of the tidal torque depends on the difference between the two principal moments of inertia in the equatorial plane (A and B), which is less than 1% of the difference between the polar (C) and equatorial moments of inertia.

3.10.1.4 Earth's Response to Gravitational Torques

The extent of the Earth's response to gravitational torques is determined by the values of a number of Earth parameters, which represent various aspects of the structure and properties of the Earth's interior: the matter distribution within the Earth, the existence and mutual couplings of the FOC and SIC regions, the couplings of these regions to the mantle, and the rheological and other properties of the Earth's interior. By far, the most important of the Earth's parameters is the dynamical ellipticity, which is the fractional difference between the moments of the Earth's inertia about the polar and equatorial axes; it is a reflection of the Earth's 'equatorial bulge.' The response to the external forcing is modified by the tidal and other excitations of the fluid layers at the surface, namely, the oceans and the atmosphere.

3.10.1.5 Low-Frequency Nutations

The spectrum of the nutational motion is discrete and is determined by the spectra of the orbital motions of the solar system bodies. The major spectral components of the nutation have frequencies within the low-frequency band (between −0.5 and 0.5 cpsd); many of these components have amplitudes that are larger than those of the spectral components in high-frequency bands (frequencies exceeding 0.5 cpsd) by five orders of magnitude. The low-frequency forced nutations arise from the action of external gravitational potentials on the axially symmetrical part of the matter distribution within the Earth. The distribution is, in fact, both axially symmetrical and ellipsoidal to a very high degree of approximation. The principal moments of inertia are then A, B, and C, with $C > B > A$ and $B \approx A$, and the dominant role in Earth rotation variations (other than those in the LOD) is played by the dynamical ellipticity defined either by $H_d = (C-A)/C$ or by $e = (C-A)/A \approx 1/304.5$, the former parameter being favored by astronomers and the latter by geophysicists. There exists a residual (nonellipsoidal) axially symmetrical part of the density distribution; it too gives rise to low-frequency nutations but with amplitudes much lower than those of the dominant nutations due to e. These are closely followed by the semidiurnal nutations (frequencies between 1.5 and 2.5 cpsd) arising from the triaxiality of the Earth, that is, the small fractional difference $(B-A)/A$ between the two equatorial moments of inertia; its value happens to be about $2e^2$, of the second order in the dynamical ellipticity. The high-frequency nutations, for which the nonaxially symmetrical part of the density distribution is responsible, have even smaller amplitudes as already stated.

In view of the preceding text, most theoretical formulations of Earth rotation variations are, justifiably, based on axially symmetrical ellipsoidal Earth models characterized by the dynamical ellipticity e and are therefore concerned with low-frequency nutations only.

3.10.1.6 Gravitational Perturbations and Displacement Fields

Different elements of matter in the Earth are subjected to differing gravitational accelerations by the Moon (or Sun or other body), because they are at different distances from the external body. Therefore, they undergo position-dependent displacements from their unperturbed positions (i.e., positions in the absence of the gravitational forces). In the case of a deformable Earth model, the displacement field $\mathbf{s}(\mathbf{r})$ over the whole Earth can be analyzed into a rotational part wherein the relative distances between different points remain undisturbed, and a deformational part. One approach to the theory of the gravitational perturbations of the Earth seeks to solve the equation of motion governing the field $\mathbf{s}(\mathbf{r})$, given a detailed model of the matter density and elastic constants as functions of position in the Earth's interior, to determine thereby both the deformations (the so-called solid Earth tides) and the rotational motions (nutations and precession) produced by the perturbing potential. Treatments employing the displacement field approach use only axially symmetrical ellipsoidal Earth models (see Dehant, 1987a,b; Smith, 1974; Wahr, 1981). Spherically symmetrical models are constructed from information provided by seismic data; the axially symmetrical ellipsoidal structure is then derived from such models using a theory (hydrostatic equilibrium) that determines the flattening of the constant density surfaces that is caused by the centrifugal effects of the Earth's mean rotation at a constant rate around the polar axis. Among the most widely used models is PREM of Dziewonski and Anderson (1981), modified to the ellipsoidal structure based on hydrostatic equilibrium theory. Further developments in the direction of incorporating nonhydrostatic equilibrium and triaxiality inside the Earth has only been undertaken very recently and with the objective of application to other planets or moons of our solar system than the Earth (Trinh et al., 2011).

From this spherically symmetrical Earth model, one has to go over to one that is ellipsoidal in structure when studying Earth rotation variations. The hypothesis of hydrostatic equilibrium is invoked to go over to an axially symmetrical ellipsoidal model: ellipticity of the Earth is an essential ingredient in the theory of rotation variations, as we have repeatedly seen.

3.10.1.7 Gravitational Torque on the Earth

A simpler approach focuses directly on the rotation variations by employing the equation of motion governing rotational motion, namely, the angular momentum balance equation (torque equation). The gravitational torque on the Earth is dependent solely on the matter distribution within the Earth and on the positions of the gravitating bodies relative to the TRF. Information on the motions of the solar system bodies is derived from the ephemerides. As for the matter distribution, one could rely on detailed models of the spherically symmetrical density profiles within the Earth constructed from seismic studies, which are converted to axially symmetrical ellipsoidal models constructed in the manner indicated in the last section. The torque on an idealized model with the previous symmetry involves only one Earth parameter, namely, the dynamical ellipticity, which represents the fractional difference between the moments of inertia about polar and equatorial axes. This parameter is sufficient to account for all but a very small part of the nutational motion. But when a fuller account is needed, one has to consider other features of the Earth's structure, which are represented by the higher-degree geopotential coefficients, on which other parts of the torque depend. These coefficients are estimated from space geodetic observations of the orbits of artificial Earth satellites.

3.10.1.8 Earth Response to the Torque

When considering the rotation variations of a wholly solid model Earth, its angular momentum can be expressed as a product of its inertia tensor and angular velocity vector. The torque equation in the rotating terrestrial frame then becomes an equation for the angular velocity components; the wobble and spin rate variations are related directly to these components and may be obtained by solving this equation. But when the existence of a fluid core region and the freedom it has to rotate with a different angular velocity from that of the mantle are taken into account, the torque equation for the whole Earth involves the angular velocities of both the mantle and the fluid core, and so, one needs to consider also the torque equation for the core region alone. Couplings of the core to the mantle enter into the latter equation and influence the rotation variations of the mantle that are observable. Examples of such couplings are the so-called inertial coupling due to the fluid pressure on the ellipsoidally shaped core–mantle boundary (CMB) and the electromagnetic (EM) coupling and possibly also viscous coupling at the CMB. The differential wobble motions of the mantle and the fluid core, on the one hand, and the presence of magnetic fields crossing over from the fluid core to the mantle side, on the other hand, cause induction of currents in the conducting layers at the bottom of the mantle; and the interaction of the magnetic field with the induced currents all over the conducting mantle layer is responsible for the EM coupling. Simultaneous solution of the equations for the whole Earth and for the core yields the wobbles of both the mantle and the core. Inclusion of the inner core into the theory brings in a pair of additional equations, one governing its angular velocity variations and the other, its orientation relative to the mantle.

3.10.1.9 Role of Deformations in Earth Rotation

The inertia tensor of each of the regions (or layers) is not just that of the unperturbed Earth: It includes the perturbations associated with deformations produced directly by the gravitational potentials of the external bodies as well as by the centrifugal potentials due to the variations in the angular velocity vectors of all three layers. The perturbations of each of the inertia tensors are made up, therefore, of parts that are proportional to the strength of the perturbing potential and to the wobbles of the three different regions. The proportionality constants are deformability parameters (also called compliances), which are intrinsic properties of the Earth model; they may be computed by integration of the equations governing the deformations of the Earth, given a detailed model of the matter density and elastic constants as functions of position in the Earth's interior. While the compliance parameters are real for an elastic Earth, the anelastic behavior of the mantle results in small complex increments to their elastic-Earth values. Loading of the crust by the ocean tides raised by the gravitational attraction of the external bodies causes further deformation, which again increments the inertia tensors and thereby influences the rotation variations. The increments due to ocean tidal effects, including the effect of ocean tidal currents, are nontrivial and have been calculated by different methods. One method that has been recently employed represents the deformational effect of ocean tides in terms of effective increments to the compliance parameters, which are not only complex but also frequency-dependent.

3.10.1.10 Interplay of Various Phenomena

In order to get results of high accuracy for Earth rotation variations, it is also necessary to take into account the mutual influences of a variety of phenomena. The tidal deformations of the Earth, including the contribution from the centrifugal perturbation caused by Earth rotation variations, as well as from ocean loading, affect the rotations of all three regions through the changes in their inertia tensors; the deformations also perturb the gravitational potential of the Earth itself, with a reciprocal effect on the deformations themselves; they affect the ocean tides through movements of the ocean floor and through the effect of the perturbation of the Earth's gravitational potential on the ocean surface; and the consequent modification of the ocean tidal loading and tidal currents influences both the deformations and the rotation variations. In brief, there is an interplay of the various phenomena, which is a factor in determining the effects of each one of them, and it is necessary to take account of this interplay while seeking to obtain accurate results for any of these effects. A chart summarizing all the interactions between the phenomena involved is presented in Figure 5.

3.10.1.11 Precision of Observations: Challenge to Theory

The precision of recent nutation estimates has surpassed 0.2 milliarcseconds (mas), and that of the estimates of individual spectral components is of the order of 10 microarcseconds (μas) for many of the more prominent components except for those with periods exceeding 1 year, for which the uncertainties are

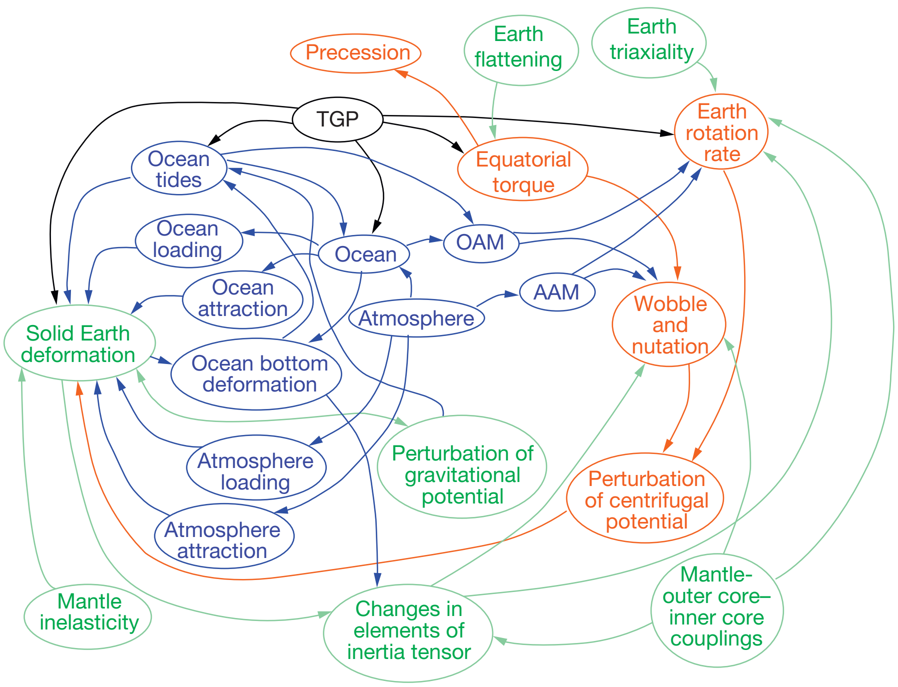

Figure 5 Chart representing the interactions and physical processes involved in Earth orientation variations and in Earth deformations.

higher ($1\ \text{mas} = 4.848 \times 10^{-9}$ rad, $1\ \mu\text{as} = 4.848 \times 10^{-12}$ rad). To match this level of precision, all effects that make contributions at the µas level to nutations have to be identified and taken into account. One such effect that had been ignored till very recently is the torque produced by the action of the gravitational potential of solar system bodies on the perturbations in the matter distribution within the Earth due to the action of the potential itself. Recent work has shown that the small contributions to nutation and precession arising from such second-order torque terms are not negligible in this context. General relativistic effects (geodesic precession and nutation) and effects of atmospheric pressure are also significant.

3.10.2 Earth Orientation/Rotation Variables: Reference Frames

As a preliminary observation, we note that the direction of any of the coordinate axes or other axes like the Earth's rotation axis or the angular momentum axis may be conveniently represented by its point of intersection with a geocentric unit sphere called the celestial sphere; the point is called the pole of the axis. Variations of the coordinates of the pole reflect the motions of the axis. The definition of the variables representing the motions of a chosen axis (or its pole) calls for the use of a suitable reference frame. For motions in space (precession and nutation), a celestial reference frame (CRF) has to be used, while wobble and polar motion are to be referred to a terrestrial frame. The celestial and terrestrial frames considered here are all geocentric: the coordinate axes originate from the geocenter.

There are two types of CRFs in use.

3.10.2.1 Ecliptic Celestial Reference Frame

It has the ecliptic as the principal plane (the plane containing the first two Cartesian coordinate axes) to which the third axis is perpendicular. The ecliptic plane may be thought of as the plane of the Earth's orbit around the Sun (or more correctly, as the plane of the orbit of the center of mass of the Earth–Moon system). The circle along which this plane intersects the celestial sphere provides a useful representation of the ecliptic plane and may also be referred to as the ecliptic, and the intersection point of the normal to the ecliptic with the celestial sphere is called the pole of the ecliptic. Similarly, the intersection of the equatorial plane (perpendicular to the axis of interest) with the celestial sphere will be called the equator, and the intersection of the axis itself with the celestial sphere is known as the pole of the equator. The intersections of the equator and the ecliptic are the equinoxes. The equinox at which the Sun crosses over from south of the equator to the north in its annual journey along the ecliptic (as perceived from the Earth) is the vernal (spring) equinox and the other is the autumn equinox. The first axis of the ecliptic reference frame is chosen to be along the nodal line (line of intersection) of the ecliptic and equatorial planes and is taken in the direction of the vernal equinox.

3.10.2.1.1 Precession

The angle between the equatorial and ecliptic planes (or between their polar axes) is the obliquity ε. A steady rotational motion of the equatorial plane or its polar axis about the first axis of the ecliptic frame constitutes the precession in obliquity, which is extremely small and is often ignored; the additional oscillatory (multiply periodic) rotation about the same axis is the nutation in obliquity, $\Delta\varepsilon$. The precession in longitude is a steady rotational motion of the Earth-related axis around the normal to the ecliptic plane, keeping an angle ε_0 to this normal, where $\varepsilon_0 = \varepsilon - \Delta\varepsilon$. Precession takes the pole of this axis over an arc of a circle on the celestial sphere, which is of radius $\sin \varepsilon_0$ and is centered on the normal. The angular size of this arc as seen from the center of the circle is denoted by $\psi_A(t)$; its linear dimension is evidently $\psi_A(t) \sin\varepsilon_0$. The nutation in longitude is an additional oscillatory rotation $\Delta\psi$ about the same normal (over the same arc). The direction of the precessional motion of the Earth's pole is defined to be the direction of increasing longitude; it is opposite to that of the Earth–Moon system around the normal to the ecliptic.

At the present rate of precession, which is very close to 50 arcseconds per year, the pole of the Earth's axis would take about 25 770 years to complete a full circuit around the pole of the ecliptic.

In defining an ecliptic-based frame, one has to contend with a complication: The ecliptic itself has a very slow rotation in space around a vector lying in the ecliptic. One may choose a reference frame based on the 'ecliptic of epoch,' that is, the ecliptic at a fixed epoch (instant of time). The standard choice of epoch is J2000, which is at 12 h Universal time on 1 January, 2000. Alternatively, one may choose a frame based on the ecliptic plane of the time of interest (ecliptic of date). The nutation and precession referred to the fixed ecliptic represent the actual motion of the nutating axis in space. The precession in this sense is only in longitude; it is often called the lunisolar precession because it is caused almost entirely by torques exerted by the Moon and Sun. But the classical definition of nutation and precession is with reference to the moving ecliptic; the precession in longitude thus defined includes a small part arising from the motion of the ecliptic and is called the general precession. There is also a small precession in obliquity arising from the ecliptic motion.

3.10.2.1.2 Nutation

The classical nutation variables $\Delta(t)$ and $\Delta\varepsilon(t)$ representing the oscillatory part of the motions in longitude and obliquity were defined, strictly speaking, relative to the moving ecliptic. The purely periodic components of both $\Delta\psi$ and $\Delta\varepsilon$ remain unaltered even if the fixed ecliptic is used, though the so-called Poisson terms do change (see Bretagnon et al., 1997). Poisson terms are products of t or powers of t multiplied by sine or cosine factors.

A representation of the nutational motion is provided in Figure 6.

$\Delta\psi$ goes up to about ± 19 arcseconds and $\Delta\varepsilon$ up to about ± 10 arcseconds.

3.10.2.2 Geocentric Celestial Reference Frame

This is a frame having the mean equatorial plane of J2000 as its principal plane and having no rotation in space. The qualification 'mean' in this context means that the nutational part of the displacement of the figure axis/equator is excluded while fixing the position of the equator at this epoch. The first axis

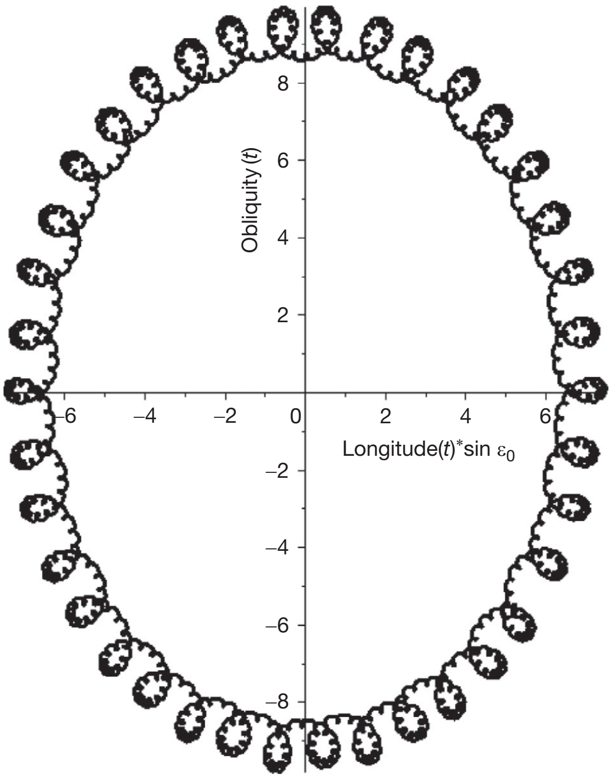

Figure 6 The trace, on the celestial sphere, of the nutational part alone (excluding precession) of the motion of the pole of an Earth axis. The position of the pole at J2000 is indicated. The main feature, the overall elliptical shape, is a reflection of the dominant 18.6-year nutation, and the loops over the ellipse represent primarily the semiannual nutation. The motion of the pole around the ellipse is in the clockwise sense.

(X-axis) of the frame is in the direction of the vernal equinox. This is the direction of increasing longitude referred to the ecliptic of J2000, and the Y-axis of the Geocentric Celestial Reference Frame (GCRF) is in the direction of increasing obliquity.

In the GCRF, the displacement of the pole of the axis of interest due to the precession and nutation is represented by the coordinates $X(t)$, $Y(t)$. For times that are not too far from J2000, one may take

$$\begin{aligned} X(t) &= [\psi_{\mathrm{A}}(t) + \Delta\psi(t)] \sin \varepsilon_0 \\ Y(t) &= \varepsilon_{\mathrm{A}}(t) + \Delta\varepsilon(t) \end{aligned} \qquad [1]$$

where $\varepsilon_{\mathrm{A}}(t)$ is the mean obliquity ε_0 at J2000 plus the precession in obliquity. But when the time difference from J2000 gets large, corrections have to be applied to these lowest-order expressions. The rate of precession in obliquity, $\dot{\varepsilon}_{\mathrm{A}}(t)$, is only about 0.25 milliarcseconds per year (mas year^{-1}), which is extremely small compared with $\dot{\psi}_{\mathrm{A}}(t)$, which is about 50 400 mas year^{-1}.

3.10.2.3 International Terrestrial Reference Frame

A terrestrial frame, by definition, is corotating with the Earth. The International Terrestrial Reference Frame (ITRF) is realized by a set of three mutually orthogonal 'Earth-fixed' axes, with its origin at the center of mass of the unperturbed Earth. The positions of the axes are determined by the coordinates inferred for an adequate number of observation stations on the *unperturbed* Earth from their positions on the real Earth as determined by appropriate observations (e.g., from a network of very long baseline interferometry (VLBI) stations); one has to remove, from the station position data thus obtained, the time-dependent displacements suffered by the station through the effects of solid Earth tides, tectonic plate motions, etc., through the use of suitable models. The principal plane of this reference frame is the plane of the true equator, perpendicular to the instantaneous direction of the figure axis, which is the third axis of the frame; its first axis (x-axis) lies on the Greenwich meridian, and the y-axis direction, orthogonal to the other two, is such as to make the frame a right-handed one. When we refer to a terrestrial frame, it is to be understood, by default, to be the ITRF unless otherwise specified.

3.10.2.4 Principal Axis Frame

This is a terrestrial frame that differs from the ITRF only in having the two equatorial coordinate axes along the two principal axes of inertia in the equatorial plane. Its x-axis lies about 15° west of the Greenwich meridian. In setting up the torque equations governing the rotational motion of the Earth, it is convenient and customary to make use of this reference frame.

The wobble variables are defined, whether in the ITRF or in the principal axis frame, as

$$m_x = \Omega_x/\Omega_0 \quad \text{and} \quad m_y = \Omega_y/\Omega_0 \qquad [2]$$

where Ω_x and Ω_y are the first two components of the angular velocity vector $\mathbf{\Omega}$ of the Earth (or more correctly, of the mantle) and Ω_0 is the mean angular velocity. Wobble is an inescapable accompaniment of nutation and vice versa. Wobble is intimately related to the nutation of the figure axis, as we shall see, and plays an essential role in various approaches to the theoretical treatment of nutation.

3.10.2.5 The CIP and CIO

The most direct and most precise information about the varying orientation of the TRF in space is obtained from VLBI. This technique uses observations of the time delays between the arrival times of radio signals from the most distant (and hence the most 'space-fixed') sources of radio waves in space at different members of an array of radio antennas spread over the globe to obtain precise estimates of the orientation of the terrestrial frame relative to a celestial frame defined by such sources. The observations are made at intervals of several days (typically five or seven) and have been carried out for over two decades; they make it possible to estimate with very high precision the variations in orientation, that is, precession, obliquity rate, nutations of the figure axis, and rotation rate variations, having periods exceeding a few days. Similarly, it is the low-frequency variations in direction of the rotation axis in the TRF that are susceptible to estimation with high precision. The conventions adopted by the International Astronomical Union (IAU) conform to this reality.

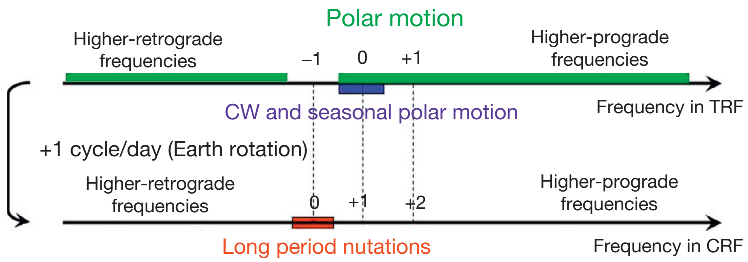

Figure 7 Ranges of polar motion frequencies (marked in green) and of nutation–precession frequencies (marked in red). Precession is induced by the zero-frequency component of the gravitational spectrum in the celestial frame.

3.10.2.5.1 Nutation and polar motion in the conventions: CIP

A pole called the CIP, intermediate between the poles of the terrestrial and CRFs, is introduced such that its nutation, that is, its motion relative to the space-fixed frame (GCRF), has only low-frequency spectral components (between −0.5 and 0.5 cpsd; see Figure 7). It is this motion of the CIP that is defined in the conventions as nutation. This low-frequency band in space maps into the 'retrograde diurnal' band in the terrestrial frame, with frequencies between −1.5 and −0.5 cpsd, because of the prograde rotation of this frame in space at the rate of 1 cpsd. Motions of the CIP in the terrestrial frame with frequencies outside this band are defined, according to the conventions, as polar motion. The low-frequency part of this band, which includes the CW and the tidally induced wobbles, dominates the polar motion. However, the amplitude of the polar motion is quite different from that of the wobble to which it is related, as will be shown in Section 3.10.7 (see also Capitaine, 1990, 2000; Capitaine et al., 2000, 2003).

3.10.2.5.2 LOD variations in the conventions: CIO

Variation in the spin rate around the z-axis, represented by m_z, and the consequent variation ΔLOD in the LOD are related to the z component Ω_z of $\mathbf{\Omega}$:

$$\begin{aligned} m_z &= (\Omega_z/\Omega_0) - 1 \\ \Delta\mathrm{LOD} &= -(\mathrm{LOD})m_z \end{aligned} \qquad [3]$$

In the scheme of analysis of VLBI data, variations in the angular velocity component in the direction of the CIP are estimated rather than those in the z component; these observations are performed at the mas level. Computation of the LOD variations from the spin rate variations estimated in this manner is in accord with the current conventions. The spin rate, in this context, is the rate of change of the angle of rotation around the CIP axis, measured from a point on the equator of the CIP, which is so chosen as to have no rotation about this axis. This point is called the Conventional International Origin (CIO; see Capitaine, 1986, 1990, 2000; Capitaine and Guinot, 1988; Capitaine et al., 1986, 2000, 2003; Guinot, 1979).

3.10.3 Equations of Rotational Motion

The most direct approach to the theoretical treatment of Earth rotation variations is based on the torque equations, which are statements of angular momentum conservation.

3.10.3.1 Equation of Motion in a Celestial Frame

In an inertial (space-fixed) reference frame, the torque equation is simply

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{\Gamma} \qquad [4]$$

So we have, trivially,

$$\mathbf{H} = \int \mathbf{\Gamma}\,\mathrm{d}t \qquad [5]$$

This result is independent of the structure of the body that is subjected to the torque.

The Earth's rotation axis being very close to the normal to the Earth's mean equator and hence to the Z-axis of the reference frame, the components H_X, H_Y are very small compared with H_Z, which is closely approximated by $C\Omega_0$; so $|\mathbf{H}| \approx C\Omega_0$. The X and Y components of the *unit vector* in the direction of $\mathbf{H}$, which characterize the precession and nutation *of the angular momentum axis*, are then

$$\begin{aligned} X_H &= \frac{H_X}{C\Omega_0} = \frac{\int \Gamma_X\,\mathrm{d}t}{C\Omega_0} \\ Y_H &= \frac{H_Y}{C\Omega_0} = \frac{\int \Gamma_Y\,\mathrm{d}t}{C\Omega_0} \end{aligned} \qquad [6]$$

where the subscript H identifies quantities pertaining to the angular momentum axis. The precession and nutation in longitude, $\psi_{A,H}$ and $\Delta\psi_H$, and the nutation $\Delta\epsilon_H$ in obliquity are related to X_H and Y_H through equations of the same form as eqn [1]:

$$X_H = \left(\psi_{A,H} + \Delta\psi_H\right)\sin\epsilon, \quad Y_H = \Delta\epsilon_H \qquad [7]$$

Explicit evaluation of the nutation quantities has to wait till the torque acting on the Earth and its time dependence are determined.

3.10.3.2 Equation of Motion in a Terrestrial Frame

Consider now the description of the Earth's rotational motion as seen from an Earth-fixed reference frame. The frame itself is rotating in space, and we denote its angular velocity vector by $\mathbf{\Omega}$. The torque equation governing the variation of the angular momentum vector $\mathbf{H}$ in this reference frame is

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} + \boldsymbol{\Omega} \times \mathbf{H} = \boldsymbol{\Gamma} \quad [8]$$

where the vectors **H** and **Γ** are now referred to the terrestrial frame. On writing the equation out in terms of the Cartesian components, we get

$$\begin{aligned} \dot{H}_x + \Omega_y H_z - \Omega_x H_y &= \Gamma_x \\ \dot{H}_y + \Omega_z H_x - \Omega_x H_z &= \Gamma_y \\ \dot{H}_z + \Omega_x H_y - \Omega_y H_x &= \Gamma_z \end{aligned} \quad [9]$$

where $\dot{H}_x = \mathrm{d}H_x/\mathrm{d}t$, and so on.

The aforementioned equations, unlike the corresponding ones in an inertial frame, involve not just **H** but **Ω** too. Therefore, it is necessary, in order to solve these equations, to make use of the relationship of the former to the latter. If the Earth were wholly solid, the relation would be $\mathbf{H} = [\mathbf{C}] \cdot \boldsymbol{\Omega}$, where [**C**] is the Earth's inertia tensor. This is no longer true if core regions are present. While the mantle rotates with angular velocity **Ω**, the other regions rotate, in general, with angular velocities differing from **Ω**; these differences would result in additional contributions to **H** from the core regions. Furthermore, the inertia tensors of the different regions will have to include contributions from tidal deformations of the respective regions when the deformability of the Earth is to be taken into account. Thus, it is necessary to use as inputs various aspects of the Earth's structure and properties before the torque equation in the terrestrial frame can be formulated, and additional equations that govern the variation of the angular velocities of the core regions will have to be considered simultaneously. We consider first the simplest case of a rigid Earth.

3.10.3.2.1 Euler equations for a rigid body

For a rigid body, equations in eqn [9] reduce to the Euler equations when coordinate axes parallel to the principal axes of the body are chosen. With such a choice,

$$H_x = A\Omega_x, \quad H_y = B\Omega_y, \quad H_z = C\Omega_z \quad [10]$$

where A, B, C stand for the principal moments of inertia in increasing order of magnitude, as usual. Substituting into eqn [9], we obtain

$$\begin{aligned} A\dot{\Omega}_x + (C - B)\Omega_y \Omega_x &= \Gamma_x \\ B\dot{\Omega}_y + (A - C)\Omega_z \Omega_x &= \Gamma_y \\ C\dot{\Omega}_z + (B - A)\Omega_x \Omega_y &= \Gamma_z \end{aligned} \quad [11]$$

These are the celebrated Euler equations.

For the rotation variations of the Earth, the fractional variations in the angular velocity, namely, m_x, m_y, m_z, are far smaller than unity. (For the real Earth, they are actually of the order of 10^{-8} or smaller for the forced wobbles and about 10^{-6} for the free CW.) So we may approximate the Euler equations to the first order in these small quantities:

$$\begin{aligned} A\Omega_0 \frac{\mathrm{d}m_x}{\mathrm{d}t} + \Omega_0^2 (C - B) m_y &= \Gamma_x \\ A\Omega_0 \frac{\mathrm{d}m_y}{\mathrm{d}t} + \Omega_0^2 (A - C) m_x &= \Gamma_y \\ C\Omega_0 \frac{\mathrm{d}m_z}{\mathrm{d}t} &= \Gamma_z \end{aligned} \quad [12]$$

3.10.3.2.2 Axially symmetrical ellipsoidal case: Wobble motion

We specialize now to the Earth considered as an axially symmetrical, ellipsoidal, rigid body with no core.

The $(B-A)$ term in the third of the Euler equations [12] then drops out, and Γ_z vanishes too, because of axial symmetry. It follows that H_z remains constant, equal to $C\Omega_0$, that is, $m_z = 0$. Furthermore, with $(C-B)$ now equal to $(C-A)$, the addition of i times the second of the aforementioned equations to the first one yields a single equation involving the complex quantities:

$$\tilde{m} = m_x + im_y, \quad \tilde{\Gamma} = \Gamma_x + i\Gamma_y \quad [13]$$

The combined (complex) equation is

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}t} - ie\Omega_0 \tilde{m} = \frac{\tilde{\Gamma}}{A\Omega_0} \quad [14]$$

where e is the dynamical ellipticity in the sense in which this term is commonly used in the geophysics literature:

$$e = (C - A)/A \quad [15]$$

Solution of the equation in the preceding text is a trivial matter:

$$\tilde{m}(t) = \mathrm{e}^{ie\Omega_0 t}\left[\tilde{m}(0) + \int \frac{\tilde{\Gamma}(t)}{A\Omega_0} \mathrm{e}^{-ie\Omega_0 t}\,\mathrm{d}t\right] \quad [16]$$

Free Wobble: The first term, $\tilde{m}(0)\mathrm{e}^{ie\Omega_0 t}$, does not involve the torque. It is a free motion of the angular velocity vector around the figure axis with frequency $e\Omega_0$ or e cpsd, as may be seen readily by decomposing $\tilde{m}$ into its real and imaginary parts: $m_x = \tilde{m}(0)\cos e\Omega_0 t$, $m_y = \tilde{m}(0)\sin e\Omega_0 t$, assuming $\tilde{m}(0)$ to be real. It represents, in fact, the Euler free wobble, with e cpsd as its eigenfrequency. In the case of more realistic nonrigid Earth models, the free mode becomes the CW, with a frequency differing from $e\Omega_0$.

Forced Wobble: The second term in eqn [16] is the forced wobble, which can be evaluated once the time dependence of $\tilde{\Gamma}$ is known.

Let us consider a single spectral component of the torque, with frequency $\sigma\Omega_0$, that is, σ cpsd:

$$\tilde{\Gamma}(t) = \tilde{\Gamma}(\sigma)\mathrm{e}^{i\sigma\Omega_0 t} \quad [17]$$

Then, eqn [14] simplifies to

$$\mathrm{i}(\sigma - e)\tilde{m}(\sigma) = \tilde{\Gamma}(\sigma)/\left(A\Omega_0^2\right) \quad [18]$$

so that

$$\tilde{m}(\sigma) = -i\frac{\tilde{\Gamma}(\sigma)}{A\Omega_0^2(\sigma - e)} \quad [19]$$

The factor $1/(\sigma - e)$ represents the *resonance* in the forced wobble that is associated with the Euler free wobble having the eigenfrequency e cpsd. In later sections where we consider both rigid and nonrigid Earth models, we identify the rigid case by adding a subscript Rig (standing for 'Rigid'). Thus, $\tilde{m}$, m_x, etc., of this section would be denoted by $\tilde{m}_{\mathrm{Rig}}$, m_{Rig_x}, etc.

3.10.4 The Tidal Potential and Torque

Before going on to the treatment of more realistic Earth models, let us examine the gravitational effects of a celestial body B of mass M_{B} on the Earth.

3.10.4.1 Potential, Acceleration, and Torque

If the position vector of B relative to the geocenter is $\mathbf{r}_B$, then its gravitational potential at $\mathbf{r}$ in the Earth is

$$\begin{aligned} W_B(\mathbf{r}) &= -\frac{GM_B}{|\mathbf{r}_B - \mathbf{r}|} \\ &= \frac{GM_B}{r_B}\left(1 - \frac{2\mathbf{r}\cdot\mathbf{r}_B}{r_B^2} + \frac{r^2}{r_B^2}\right)^{-1/2} \end{aligned} \quad [20]$$

Since $r = |\mathbf{r}| \le a_e \approx 6378$ km, and the nearest solar system body is the Moon at a geocentric distance of about 400 000 km, $r/r_B < 0.016$, and therefore, an expansion in powers of (r/r_B) is useful:

$$W_B(\mathbf{r}) = -\frac{GM_B}{r_B}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}_B}{r_B^2} - \frac{r^2}{2r_B^2} + \frac{3(\mathbf{r}\cdot\mathbf{r}_B)^2}{2r_B^4} + \cdots\right) \quad [21]$$

The acceleration of a mass element at $\mathbf{r}$ is then

$$-\Delta W_B(\mathbf{r}) = -\frac{GM_B}{r_B}\left(\frac{\mathbf{r}_B}{2r_B^2} + \frac{3(\mathbf{r}\cdot\mathbf{r}_B)\mathbf{r}_B - r_B^2\mathbf{r}}{r_B^4} + \cdots\right) \quad [22]$$

The first term represents a uniform acceleration toward the external body. In case the body is the Sun or the Moon, it is this centripetal acceleration which keeps the Earth and the other body going in an orbit around each other. The second term in the acceleration is position-dependent and causes different mass elements in the Earth to move relative to one another to the extent allowed by the elastic forces counteracting it. The result is a deformation of the solid Earth, which is manifested as the *solid Earth tide*, and also a redistribution of water mass in the oceans, the *ocean tide*. The nonuniform acceleration is referred to as the tidal acceleration. The term in the potential W_B that gave rise to this tidal acceleration is designated as the *tidal potential* of degree 2. Succeeding terms in the expansion of W_B, which are not explicitly shown, constitute tidal potentials of degrees >2. The torque on the whole Earth, which is responsible for the forced variations in Earth rotation, is

$$\begin{aligned} \mathbf{\Gamma} &= \int -\rho(\mathbf{r})\mathbf{r} \times \Delta W_B(\mathbf{r})\,\mathrm{d}^3 r \\ &= \frac{GM_B}{r_B}\int \rho(\mathbf{r})\left(\frac{\mathbf{r}\times\mathbf{r}_B}{r_B^2} + \frac{3(\mathbf{r}\cdot\mathbf{r}_B)(\mathbf{r}\times\mathbf{r}_B)}{r_B^4} + \cdots\right)\mathrm{d}^3 r \end{aligned} \quad [23]$$

where the integration is over the volume of the Earth. The first term in eqn [23] involves $\int \rho(\mathbf{r})\mathbf{r}\,\mathrm{d}^3 r$, which is simply the Earth's mass times the position vector of the center of mass; this vector is null because the center of mass is the origin (geocenter) itself. The torque is therefore given by the integral of the second term in the previous expression, apart from the much smaller higher-degree terms represented by the set of dots at the end.

3.10.4.2 Torque Components in Terrestrial Frame

We evaluate the components of the torque integral in a TRF by expressing $\mathbf{r}$ and $\mathbf{r}_B$ in terms of its components with respect to that frame. We choose the principal axis frame, which is defined as one in which the inertia tensor is diagonal. The x-axis of this frame is the principal axis of the least moment of inertia A; the longitude α of this axis in the ITRF is about $-14°0.93$.

Since the elements C_{ij} of the tensor are defined by

$$C_{ij} = \int \rho(\mathbf{r})\left(r^2\delta_{ij} - x_i x_j\right)\mathrm{d}^3\mathbf{r} \quad [24]$$

where δ_{ij} is the Kronecker delta function, we have, in the principal axis reference frame,

$$\begin{pmatrix} C_{11} \\ C_{22} \\ C_{33} \end{pmatrix} = \int \rho(\mathbf{r})\begin{pmatrix} (y^2+z^2) \\ (z^2+x^2) \\ (x^2+y^2) \end{pmatrix}\mathrm{d}^3 = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \quad (25)$$

$$\begin{aligned} \int \rho(\mathbf{r})xy\,\mathrm{d}^3 r &= \int \rho(\mathbf{r})yz\,\mathrm{d}^3 \\ &= \int \rho(\mathbf{r})zx\,\mathrm{d}^3 r = 0 \end{aligned} \quad [26]$$

If the time-dependent variations in the density function $\rho(\mathbf{r})$ and hence in the inertia tensor due to tidal deformations and other causes were also taken into account, the result of the integrations in eqn [25] would get modified to $A + c_{11}$, $B + c_{23}$, $C + c_{33}$ and those in eqn [26] to $-c_{12}$, $-c_{23}$, $-c_{31}$, respectively, where the c_{ij} are the contributions from the deformations. They would lead to second-order contributions to the torque, which are very small; they will be ignored in the following. But they are not entirely negligible at the levels of accuracy required at present, and we shall refer to them again in Sections 3.10.9.1 and 3.10.9.6 on numerical modeling.

Now, we can express the factor $(\mathbf{r}\cdot\mathbf{r}_B)(\mathbf{r}\times\mathbf{r}_B)$ in the torque integral eqn [23] in terms of the components of $\mathbf{r}$ and $\mathbf{r}_B$ and then evaluate the integral using eqns [25] and [26]. Thus, we obtain

$$\begin{pmatrix} \Gamma_x \\ \Gamma_y \\ \Gamma_z \end{pmatrix} = \frac{3GM_B}{r_B^5}\begin{pmatrix} (C-B)y_B z_B \\ (A-C)x_B z_B \\ (B-A)x_B y_B \end{pmatrix} \quad [27]$$

Replacing the coordinates of the celestial body in terms of the polar coordinates $(r_B, \theta_B, \lambda_B)$, where θ_B is its colatitude and λ_B is its longitude in the terrestrial frame that we are using, we rewrite the equatorial components of the torque as

$$\begin{aligned} \Gamma_x &= \frac{3GM_B}{r_B^3}\bar{A}(e - e') \times \cos\theta_B \sin\theta_B \sin\lambda_B \\ \Gamma_y &= \frac{3GM_B}{r_B^3}\bar{A}(e + e') \times \cos\theta_B \sin\theta_B \cos\lambda_B \end{aligned} \quad [28]$$

wherein $\bar{A} = (A+B)/2$, and

$$e = (C - \bar{A})/\bar{A}, \quad e' = (B - A)/(2\bar{A}) \quad [29]$$

The parameter e', proportional to $(B - A)$, is a manifestation of the triaxiality of the Earth, and e as defined here is the dynamical ellipticity of the triaxial Earth; the latter dominates the torque because e' is only about $e/300$.

The complex combination $\tilde{\Gamma} = \Gamma_x + i\Gamma_y$ is

$$\tilde{\Gamma} = -i\frac{3GM_B}{r_B^3}\bar{A}\cos\theta_B\sin\theta_B\left(e e^{i\lambda_B} + e' \mathrm{e}^{-i\lambda_B}\right) \quad [30]$$

Now, as seen from the rotating Earth, all celestial objects move westward in sky, that is, in a retrograde sense (opposite to the eastward rotation of the Earth itself, as a consequence of this rotation). Their angular velocities $\mathrm{d}\lambda_B/\mathrm{d}t$ are close to -1 cpsd. In fact, the temporal variation of the torque due to the orbital motion of the Sun or the Moon has spectral components with frequencies on both sides of -1 cpsd even if the orbits are taken to be circular, as will be demonstrated in the next couple of sections. The fact that the orbits are elliptical and that even these are perturbed by the influence of third bodies (e.g., perturbation of the lunar orbit around the Earth by the attraction of the Sun and the planets on the Moon) leads to a rich spectrum of frequencies bunched very close to and centered at -1 cpsd. This band of frequencies is referred to as the *retrograde diurnal band*. An examination of eqn [28] shows now that the main part proportional to $(C-\bar{A})$ of the equatorial projection (Γ_x, Γ_y) of the torque vector rotates in the retrograde sense as a consequence of the negative rate of change of λ_B and that this motion has the same retrograde diurnal spectrum; the triaxiality part of the vector, on the other hand, rotates in the prograde sense and has a prograde diurnal spectrum.

Another point of interest is that the products $y_B z_B$ and $x_B z_B$ present in Γ_x and Γ_y in eqn [27] come from the terms

$$-\frac{3GM_B}{r_B^5}(x_B z_B xz + y_B z_B yz) = -\Omega_0^2(\phi_1 xz + \phi_2 yz) \qquad [31]$$

of the potential eqn [21]. The dimensionless quantities ϕ_1, ϕ_2 and their complex combination $\tilde{\phi} = \phi_1 + i\phi_2$ were introduced by Sasao et al. (1980) and have been used extensively in further developments based on their formalism. A potential that is a linear combination of xz and yz is called a *tesseral* potential of degree 2. Thus, the equatorial components of the gravitational torque due to the degree 2 terms in the potential are produced solely by the tesseral part of these terms, and the only Earth parameters on which this torque depends are the differences among the three principal moments of inertia, which, in turn, depend on the dynamical ellipticity e and the triaxiality (fractional difference between the two equatorial moments of inertia). Since higher-degree terms in the expansion of the full potential are of much smaller magnitude as noted earlier, the degree 2 tesseral potential, which has its spectrum in the retrograde diurnal band, is responsible for the dominant part of the wobbles and hence of the nutations too, as we shall see. The residual small part of the wobble and nutation arise from the action of potentials of degree >2 on moments of the Earth's matter distribution that are of degree >2 (the moments of inertia being of degree 2).

3.10.4.3 Expansion of the Potential: Tidal Spectrum

The terms up to degree 2 in (r/r_B) in the expansion of the potential $W_B(\mathbf{r})$ of an external body B were shown in eqn [21]. A further expansion of each of the tidal terms (of degree $n \geq 2$) can be made into parts of different orders m from 0 to n, which are distinguished by the nature of their dependence on the direction of the position $\mathbf{r}_B$ of the body B and that of the Earth point $\mathbf{r}$ relative to the terrestrial reference frame (ITRF). This dependence is through the surface spherical harmonic functions Y_n^m of the colatitude θ and longitude λ of $\mathbf{r}$ in the ITRF, as well as the Y_n^m of θ_B and λ_B of $\mathbf{r}_B$ in the same frame. (Note that λ and λ_B in this frame are equal to α plus the corresponding quantities relative to the principal axis frame of the last section.)

The spherical harmonics are defined by

$$Y_n^m(\theta, \lambda) = N_{nm} P_n^m(\cos\theta) e^{im\lambda} \qquad [32]$$

and

$$Y_n^{-m} = (-1)^m Y_n^{m*} \qquad [33]$$

with * standing for complex conjugation and with

$$N_{nm} = (-1)^m \left(\frac{2n+1}{4\pi}\right)^{1/2} \left(\frac{(n-m)!}{(n+m)!}\right)^{1/2} \qquad [34]$$

for $n = 0, 1, 2, \ldots, m = 0, 1, \ldots, n$. P_n^m are the associated Legendre functions; they are derived from the Legendre polynomials P_n, which are defined through the expansion

$$\left(1 - 2cw + w^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(c) w^n \qquad [35]$$

$$P_n^m(c) = \left(1 - c^2\right)^{m/2} \frac{d^m P_n(c)}{dc^m} \qquad [36]$$

$P_n(c)$ is a polynomial of degree n in c and has unit value at $c = 1$. The spherical harmonics Y_n^m for all n and $m = -n, -n+1, \ldots, n-1, n$, form an orthonormal set:

$$\int Y_q^{p*}(\theta, \lambda) Y_n^m(\theta, \lambda) \sin\theta \, d\theta \, d\lambda = \delta_{qn}\delta_{pm} \qquad [37]$$

The integration here is over the unit sphere and δ_{qn} is the Kronecker delta function: δ_{qn} is 1 if $q = n$ and is 0 otherwise. The expansion of the potential eqn [20] in terms of the Y_m^m is, after dropping terms of degrees $n = 1, 2$, which do not produce any tidal deformations or torques,

$$W_B(\mathbf{r}) = -\sum_{n=2}^{\infty} r^n \sum_{m=0}^{n} \left[F_{nm}^*(r_B, , \theta_B, , \lambda_B) Y_n^m(\theta, \lambda) + F_{nm}(r_B, \theta_B, \lambda_B) Y_n^{m*}(\theta, \lambda)\right] \qquad [38]$$

with

$$F_{nm}(r_B, \theta_B, \lambda_B) = \frac{GM_B}{r_B^{n+1}} \frac{(2-\delta_{m_0})2\pi}{2n+1} \times Y_n^m(\theta_B, \lambda_B) \qquad [39]$$

and the complex conjugate of it provides the expression for $F_{nm}^*(r_B, \theta_B, \lambda_B)$. It may be observed that the part of W_B belonging to any specific degree n and order m contains the phase factors $e^{im(\lambda-\lambda_B)}$ and $e^{im(\lambda-\lambda_B)}$. Each of the phases is 0 on the meridian $\lambda = \lambda_B$, and as the celestial bodies move westward in the sky (i.e., in the retrograde sense) at a rate close to once per day, as seen from the terrestrial frame, the previously mentioned meridian moves with it; other constant phase meridians move in tandem. So, for any $m \neq 0$, this term constitutes a retrograde tidal wave sweeping over the volume of the Earth, with m cycles over the globe (because of the factor m multiplying λ).

Representation of the time dependence of the potential through its spectral expansion is of great value. The most widely used form is that of the expansion by Cartwright and Taylor (1971), though a number of tidal tables of much higher accuracy are now available (Hartmann and Soffel, 1994; Hartmann and Wenzel, 1995a,b; Hartmann et al., 1999; Roosbeek, 1999; Roosbeek and Dehant, 1998):

$$F^{*}_{nm}(r_{\mathrm{B}},\theta_{\mathrm{B}},\lambda_{\mathrm{B}}) = \left(g_e/a_e^n\right)\sum_{\omega} H_{\omega}^{nm}\zeta_{nm}\mathrm{e}^{i\Theta_{\omega}(t)} \qquad [40]$$

wherein the amplitude of the spectral component is given in units of length as the equivalent 'height' H_{ω}^{nm}. The factor ζ_{nm} is defined to be unity when $n-m$ is even and $-i$ when $n-m$ is odd. With these values for ζ_{nm}, the dependence of the tidal component on the longitude λ and time appears through the factor $\cos(\lambda+\Theta_{\omega}(t))$ for $(n-m)$ even and $\sin(\lambda+\Theta_{\omega}(t))$ for $(n-m)$ odd, when the real part is taken in eqn [38].

$\Theta_{\omega}(t)$ is the argument of the tidal component; the frequency ω is nonnegative and is equal to $\mathrm{d}\Theta_{\omega}/\mathrm{d}t$:

$$\Theta_{\omega}(t) = \omega t + \mathcal{X}_{\omega} \qquad [41]$$

where χ_{ω} is a constant. $\Theta_{\omega}(t)$ does have a dependence on n and m, which is not indicated explicitly. Now, the mean motion of B in the sky (i.e., the mean value of $\mathrm{d}\lambda_{\mathrm{B}}/\mathrm{d}t$) relative to the terrestrial frame is close to -1 cpsd, and $\mathrm{e}^{i\Theta_{\omega}(t)}$ for given n and m originates from $\mathrm{e}^{-im\lambda_{\mathrm{B}}}$ (modulated by other factors depending on θ_{B} and r_{B}), as is evident from the expression for F^{*}_{nm}. Hence, the frequency spectrum of $\mathrm{e}^{i\Theta\omega(t)}$ must lie in a band around m cpsd. Since the angle of a spectral component of the wave of order m is now $(\lambda+\Theta_{\omega}(t))$, it constitutes a retrograde wave with frequency $-\omega$ lying within the retrograde band around $-m$ cpsd.

The roles of the various solar system bodies in the argument $\Theta_{\omega}(t)$ are displayed through its expansion in terms of Doodson fundamental arguments of the tides:

$$\Theta_{\omega} = n_1(\tau-\lambda) + n_2 s + n_3 h + n_4 p + n_5 N' + n_6 p_{\mathrm{s}} + \cdots \qquad [42]$$

where $n_1=m$ and the set of dots stand for terms that are multiples of the mean tropic longitudes of the planets from Mercury to Saturn. Of the fundamental arguments that are explicitly shown, s, h, p, $-N$, p_{s} are the mean tropic longitudes of the Moon, Sun, Moon's perigee, Moon's node, and the Sun's perigee, respectively. The periods relating to these arguments range from 27.32 days for s to about 209 years for p_{s}. The Doodson argument τ is $180^{\circ}+\mathrm{GMST}-s+\lambda$ when expressed in degrees. GMST is the Greenwich Mean Sidereal Time, which is the time measured by the angle traversed along the mean equator of date by the Greenwich meridian at the mean sidereal rotation rate Ω_0:

$$\mathrm{GMST} = \mathrm{GMST}_0 + \Omega_0 t + \cdots \qquad [43]$$

$$\mathrm{GMST}_0 = 280^{\circ}.4606\cdots \qquad [44]$$

where t is measured from J2000. The period associated with $\tau-\lambda+s$ is that of the mean sidereal rotation, namely, 1 sidereal day. It is the term $n_1\tau=m\tau$ in Θ_{ω} that causes the frequencies in potentials of order m to lie within a band centered at m cpsd.

3.10.4.4 Degree 2 Potential and Torque: Spectra

The tidal torque on an ellipsoidal Earth arises from the degree 2 part of the potential W_{B}, as has been noted earlier. The degree 2 potential, in turn, is made up of zonal, tesseral, and sectorial parts, which are of orders m=0, 1, and 2, respectively. The associated Legendre functions appearing in the three orders are

$$\begin{aligned} P_2(\cos\theta) &= \frac{3\cos^2\theta - 1}{2} \\ P_2^1(\cos\theta) &= 3\sin\theta\cos\theta \\ P_2^2(\cos\theta) &= 3\sin^2\theta \end{aligned} \qquad [45]$$

These are factors in the spherical harmonics $Y_2^m(\theta,\lambda)$ and appear with the coefficients $F^{*}_{2m}(\theta_{\mathrm{B}},\lambda_{\mathrm{B}})$ in the tidal potential. Their time dependence enters through $\mathrm{e}^{i\Theta_{\omega}(t)}$ with $n_1=m=0$, 1, 2, respectively, as we see from eqns [40] and [42]. Therefore, the spectra of the zonal, tesseral, and sectorial potentials lie in the low-frequency, prograde diurnal, and prograde semidiurnal bands, respectively, as will be clear from the discussion at the end of the last section.

3.10.4.4.1 Equatorial components of torque

We can now carry over $\tilde{\Gamma}$ of eqn [30], which was with respect to the principal axis frame, to the ITRF. In doing so, we have to make the replacement $\lambda_{\mathrm{B}}\to(\lambda_{\mathrm{B}}-\alpha)$, as the x-axis of the principal axis frame is at longitude α in the ITRF as already noted. Secondly, the transformation of the vector components between the two frames makes $\tilde{\Gamma}$ in the ITRF $\mathrm{e}^{i\alpha}$ times $\tilde{\Gamma}$ of the other frame though Γ_z remains unchanged. In consequence of these two changes, eqn [30] goes over into

$$\begin{aligned} \tilde{\Gamma} &= -i\frac{GM_{\mathrm{B}}}{r_{\mathrm{B}}^3}P_2^1(\cos\theta_{\mathrm{B}})\bar{A}\left(e\mathrm{e}^{i\lambda_{\mathrm{B}}} + e'\mathrm{e}^{2i\alpha}\mathrm{e}^{-i\lambda_{\mathrm{B}}}\right) \\ &= i\frac{GM_{\mathrm{B}}}{r_{\mathrm{B}}^3}\left(\frac{24\pi}{5}\right)^{1/2}\bar{A}\times\left(eY_2^1(\theta_{\mathrm{B}},\lambda_{\mathrm{B}}) + e'\mathrm{e}^{2i\alpha}Y_2^{1*}(\theta_{\mathrm{B}},\lambda_{\mathrm{B}})\right) \end{aligned} \qquad [46]$$

where the definitions [32] and [34] have been used in the second step. One sees immediately by comparison with eqn [40] that the ellipticity term in the last expression is F^{*}_{21} apart from constant factors and that the triaxiality term is proportional to F_{21}. Then, eqn [40] taken together with the ensuing discussion shows that the spectra of these two terms lie in retrograde diurnal and prograde diurnal bands, respectively.

We have observed earlier that it is the tesseral part which produces the equatorial torque on an ellipsoidal Earth; this torque drives the wobbles. The form employed in eqn [31] for the tesseral potential may be related readily to F_{21}. We note that

$$-\Omega_0^2(\phi_1 xz + \phi_2 yz) = -\frac{\Omega_0^2 r^2}{3}\mathrm{Re}\left(\tilde{\phi}^{*}(t)P_2^1(\cos\theta)\mathrm{e}^{i\lambda}\right) \qquad [47]$$

where $\tilde{\phi}=\phi_1+i\phi_2$. Comparison with the $n=2$, $m=1$ term in eqn [38] after writing $P_{21}\mathrm{e}^{i\lambda}$ as $-(24\pi/5)^{1/2}Y_{21}$ enables us to identify the dimensionless quantity $\tilde{\phi}^{*}$ as

$$\tilde{\phi}^{*}(t) = -\left(\frac{15}{8\pi}\right)^{1/2}\frac{F^{*}_{21}}{\Omega_0^2} \qquad [48]$$

We turn our attention now to the equatorial torque $\tilde{\Gamma}$. One can express it in terms of F_{21} and F^{*}_{21} by using the definition of Y_2^1 from eqn [32]. One finds, after some elementary algebra, that

$$\tilde{\Gamma} = \left(\frac{15}{8\pi}\right)^{1/2} i\bar{A}\left(eF_{21} + e'\mathrm{e}^{2i\alpha}F^{*}_{21}\right) \qquad [49]$$

An alternative expression for $\tilde{\Gamma}$ in terms of $\tilde{\phi}$ may be obtained by using eqn [48]:

$$\tilde{\Gamma} = -iA\Omega_0^2\left(e\tilde{\phi} + e'e^{2i\alpha}\tilde{\phi}^*\right) \qquad [50]$$

This form, especially for the ellipticity part, is widely used in works following the approach of Sasao et al. (1980).

We write down now the spectral expansion of $\tilde{\phi}$ by making use of the relation [48] and the expansion of F_{21}^* from eqn [40]:

$$\tilde{\phi} = -i\left(\frac{15}{8\pi}\right)^{1/2}\frac{g_e}{a_e^2\Omega_0^2}\sum_{\omega} H_{\omega}^{21}e^{-i\Theta_\omega(t)} \qquad [51]$$

The expansion of the torque follows immediately. An important point to note is that the ellipticity part of the torque involves $\tilde{\phi}$ with the time dependence $e^{-\Theta_\omega(t)} = e^{-i(\omega t+\chi_\omega)}$ for the spectral terms, so that the spectral frequencies are negative (since ω is positive); when combined with the fact that $m=1$ for the tesseral potential, it follows that the spectrum is in the retrograde diurnal band.

In the work of Sasao et al. (1980), and in many other related works, it is customary to use the notation $\sigma\Omega_0$ for the (negative) spectral frequencies involved and $\tilde{\phi}(\sigma)$ for the amplitude of the spectral component. With this notation, a typical component of $\tilde{\phi}$ is given by

$$\tilde{\phi}_\sigma(t) = -i\tilde{\phi}(\sigma)e^{i(\sigma\Omega_0 t+\chi_\sigma)} \qquad [52]$$

with

$$\sigma = -\omega/\Omega_0, \quad \chi_\sigma = -\chi_\omega \qquad [53]$$

with the amplitude $\tilde{\phi}(\sigma)$ related to the height H_ω^{21} of Cartwright and Taylor (1971) by

$$\tilde{\phi}(\sigma) = \left(\frac{15}{8\pi}\right)^{1/2}\frac{g_e}{a_e^2\Omega_0^2}H_\omega^{21} \qquad [54]$$

Finally, we obtain an explicit expression for the wobble amplitude $\tilde{m}_R(\sigma)$ of a rigid axially symmetrical Earth by introducing the ellipticity part of the torque eqn [50] into the solution [19] of the torque equation:

$$\tilde{m}_R(\sigma) = -\frac{e\tilde{\phi}(\sigma)}{\sigma - e} \qquad [55]$$

Tables of the components of the tidal potential list the values of H_ω for the various components identified by the respective sets of values of the multipliers $n_1, n_2, \ldots, n_6$ of the Doodson arguments. The amplitude $\tilde{\phi}(\sigma)$ of any tide of interest may be computed using eqn [54] after picking out the H_ω of that tide from a tide table. The rigid Earth wobble amplitude may then be computed from eqn [55], given the value of e.

3.10.4.4.2 Axial component of the torque

The z component of the torque, given in eqn [27], may be rewritten with the replacement $\lambda_B \to (\lambda_B - \alpha)$, as

$$\begin{aligned}\Gamma_z &= -i\frac{3GM_B}{4r_B^3}(B-A)\sin^2\theta_B\left(e^{2i(\lambda_B-\alpha)} - e^{-2i(\lambda_B-\alpha)}\right) \\ &= -i\frac{GM_B}{r_B^3}(B-A)\left(\frac{6\pi}{5}\right)^{1/2}\left[e^{-2i\alpha}Y_2^2 - e^{2i\alpha}Y_2^{2^*}\right]\end{aligned} \qquad [56]$$

This translates into

$$\Gamma_z = -i\left(\frac{15}{8\pi}\right)^{1/2}\bar{A}e'\left(e^{-2i\alpha}F_{22} - e^{2i\alpha}F_{22}^*\right) \qquad [57]$$

The spectral expansion of this torque may be obtained as a special case of eqn [40]. Since $m=\pm 2$ in the present case, the two terms within parentheses have their spectra in the retrograde and prograde semidiurnal bands, respectively. The amplitudes are very small because e'/e is only about 1/300 as already noted.

3.10.4.5 Terms of General Degree and Order in the Torque

The torque generated by any higher-degree term in the potential may be obtained by taking $(-\mathbf{r}\times\nabla)$ of the (nm) part of eqn [38], then multiplying by $\rho(\mathbf{r})$, and integrating over the volume of the Earth. On using the fact that

$$\left[(\mathbf{r}\times\nabla)_x + i(\mathbf{r}\times\nabla)_y\right]Y_n^m = i[(n-m)(n+m+1)]^{1/2}Y_n^{m+1} \qquad [58]$$

for any m, positive or negative, and that $Y_n^{m*} = (-1)^m Y_n^{-m}$, it turns out that the equatorial part of the torque is given by

$$\begin{aligned}\tilde{\Gamma}^{nm} &= iM_E a_e^n\frac{2n+1}{16\pi N_{nm}}\left\{(C_{n,m-1} - iS_{n,m-1})\frac{2F_{n,m}}{(2-\delta m, 1)}\right. \\ &\left.\quad -(n+m)(n-m+1)(C_{n,m+1} + iS_{n,m+1})F_{n,m}^*\right\}\end{aligned} \qquad [59]$$

where C_{nm} and S_{nm} are geopotential coefficients defined by

$$\begin{aligned}&M_E a_e^n(C_{n,m} + iS_{nm}) \\ &= (2-\delta_{m,0})\frac{(n-m)!}{(n+m)!}\int \rho(\mathbf{r})r^n P_n^m(\cos\theta)e^{im\lambda}\,d^3r\end{aligned} \qquad [60]$$

Here, M_E is the mass and a_e the equatorial radius of the Earth. It may be seen from the spectral representation [40] of F_{nm}^* and its complex conjugate that the spectra of the two terms of eqn [59] lie, in general, in the two bands centered at $-m$ cpsd and m cpsd, respectively. (In the special case $m=0$, it turns out that both parts of the torque involve $(C_{n1}+iS_{n1})$ and that the spectra of the two parts together make up the band of width 1 cpsd centered at $m=0$.)

The coefficients C_{n0} of order 0, also denoted by $-J_n$, represent axially symmetrical structures. They appear only through the first term of eqn [59] with $m=1$. The torque resulting from the existence of C_{n0} may be seen to reduce to

$$i\frac{GM_B M_E}{r_B}\left(\frac{a_e}{r_B}\right)^n C_{n0}P_{n1}(\cos\theta_B)e^{i\lambda_B} \qquad [61]$$

(Note that all S_{n0} vanish.)

The axially symmetrical ellipsoidal Earth is a special case with $n=2$, for which

$$C_{20} = -J_2 = -\left(\bar{A}/M_E a_e^2\right)e \qquad [62]$$

It is of some interest to note that the triaxiality enters through

$$\begin{aligned}C_{22} &= \left(\bar{A}/2M_E a_e^2\right)e'\cos 2\alpha \\ S_{22} &= \left(\bar{A}/2M_E a_e^2\right)e'\sin 2\alpha\end{aligned} \qquad [63]$$

In this context, α stands for the longitude of the direction, in the ITRF, of the equatorial axis of the least moment of inertia A. (This angle was introduced in Section 3.10.4.2. It is to be kept in mind that α is used elsewhere to denote the right ascension.)

The z component of the general torque may also be obtained by making use of the identity $(\mathbf{r}\times\nabla)_z Y_n^m = imY_n^m$. One finds that

$$\Gamma_z^{nm} = -iM_\mathrm{E}a_\mathrm{e}^n \frac{2n+1}{4\pi}\frac{m}{N_{nm}(2-\delta_{m0})} \times \left\{(C_{nm} - iS_{nm})F_{nm} - (C_{n,m} + iS_{n,m})F^*_{n,m}\right\} \quad [64]$$

3.10.5 Torque in Celestial Frame: Nutation–Precession in a Simple Model

3.10.5.1 Torque Components in Celestial Frame

We considered in the last section the torque and rotation variations referred to the TRF. One might ask: What if we employed the components (X, Y, Z) of $\mathbf{r}$ and $(X_\mathrm{B}, Y_\mathrm{B}, Z_\mathrm{B})$ of $\mathbf{r}_\mathrm{B}$ in a space-fixed equatorial reference frame (say, the GCRF) and wished to compute $(\Gamma_X, \Gamma_Y, \Gamma_Z)$ in that frame on the same lines as in the foregoing section? The problem that one encounters in attempting such a course is that neither the Earth's z-axis (axis of maximum moment of inertia) nor the Earth's rotation vector stays aligned with the Z-axis of the celestial frame, as a result of precession–nutation and wobble. However, we may ignore the small time-dependent offsets among these axes for the limited purpose of a lowest-order calculation of the torque, and then, if we also ignore the small deviation of the Earth from axial symmetry, the axial symmetry persists about the Z-axis too. Considering these approximations, one may evaluate the integral [23] in the space-fixed frame by expressing the integrand in terms of coordinates (X, Y, Z) and $(X_\mathrm{B}, Y_\mathrm{B}, Z_\mathrm{B})$ and noting that the results of eqns [25] and [26] remain valid with these replacements. The components of $\mathbf{\Gamma}$ in the space-fixed frame are then found to be

$$\Gamma_X = \frac{3GM_\mathrm{B}}{r_\mathrm{B}^5}(C-A)Y_\mathrm{B}Z_\mathrm{B}$$
$$\Gamma_Y = -\frac{3GM_\mathrm{B}}{r_\mathrm{B}^5}(C-A)Z_\mathrm{B}X_\mathrm{B} \quad [65]$$

with $\Gamma_Z = 0$.

The values of r_B, X_B, Y_B, Z_B as functions of time may be taken, for any of the celestial bodies, from the appropriate ephemerides such as VSOP (Variations Séculaires des Orbites Planétaires) for the Sun and the planets (Bretagnon and Francou, 1988) and ELP (Éphéméride Lunaire Parisienne) for the Moon (Chapront-Touzé and Chapront, 1988). The time dependence of the torque components is then known. The nutation of the angular momentum axis may then be obtained by direct integration in view of eqns [6] and [7]. An interesting fact that is of importance is that though the orbital motions of both the Sun and the Moon, as seen from a nonrotating geocentric frame, are in the prograde sense relative to Earth's rotation, the torque exerted by each of the bodies contains both prograde and retrograde components relative to such a frame. A treatment based on a simplified model of the orbital motion of the Sun will bring out clearly why this happens, besides providing a concrete example of the calculation of nutation and precession.

3.10.5.2 Simple Model: Sun in a Circular Orbit

We consider an axially symmetrical ellipsoidal Earth subjected to the tidal potential of the Sun, which is supposed, for illustrative purposes, to be in uniform motion with angular velocity p along a circle of radius c centered at the geocenter and lying in the plane of the ecliptic. We take the space-fixed reference frame to be equatorial celestial frame (GCRF), with its X-axis pointing in the direction of the vernal equinox. (This is the direction of increasing ecliptic longitude, that is, of positive $\Delta\psi$. The motion in obliquity of the figure axis is in the direction of the Y-axis.) The angle between the principal plane and the ecliptic is of course the obliquity ε, which we shall treat as a constant. Then, the coordinates of the Sun in our reference system are

$$X_\mathrm{B} = c\cos pt, \quad Y_\mathrm{B} = c\cos\varepsilon\sin pt, \quad Z_\mathrm{B} = c\sin\varepsilon\sin pt \quad (66)$$

Note that $(X_\mathrm{B}, Y_\mathrm{B})$ describes the projection of the Sun's circular orbit from the ecliptic plane on to the equatorial plane; the projected orbit is, not surprisingly, an ellipse.

3.10.5.3 The Torque in the Simple Model

The time dependence of the torque eqn [65] can now be displayed in explicit form:

$$\Gamma_X = F\cos\epsilon(1 - \cos 2pt)$$
$$\Gamma_Y = -F\sin 2pt, \quad \Gamma_Z = 0 \quad [67]$$

where

$$F = \left(3GM_\mathrm{B}/2c^3\right)Ae\sin\epsilon \quad [68]$$

We know from Kepler's laws that $GM_\mathrm{B} = n_\mathrm{B}^2c^3$, where n_B is the 'mean motion,' that is, the mean orbital angular velocity of the body, which is what we denoted by p earlier. (In the case of an elliptical orbit, c has to be the semimajor axis of the orbit in Kepler's relation.)

Thus, the overall magnitude of the solar torque on the Earth is determined by p and the obliquity ε, besides the Earth parameter $(C-A)$. One may get an idea of the strength of the torque from the value of F. For the rotation variations induced by the torque, the nondimensional quantity $F/(\Omega_0^2A)$ is more relevant:

$$\frac{F}{\Omega_0^2A} = \frac{3}{2}\frac{p^2}{\Omega_0^2}e\sin\epsilon \quad [69]$$

We know that $p/\Omega_0 = 1/366.2422$, since the period of the relative orbital motion of the Sun and the Earth is 1 year of 365.2422 solar days or 366.2422 sidereal days. We also have, from the IERS Conventions 2000 (McCarthy, 2003), $\sin\varepsilon = 0.397777$ and $\cos\varepsilon = 0.917482$. Therefore, $F/(\Omega_0^2A) = 4.449\times10^{-6}e$; it is a dimensionless measure of the amplitude of the torque. On using the value $e = 0.0032845$ from a recent estimate, one finds it to be 1.461×10^{-8}.

Returning to eqns [67] and [69], we take note of three important features that are evident: the presence of a constant (time independent) term in the X component of the torque, the semiannual frequency of the periodic part of both the equatorial components of the torque, and the fact that the periodic part of the variation of (Γ_x, Γ_y) is an elliptical motion in the equatorial plane.

3.10.5.4 Nutation and Precession in the Model

With the components of $\mathbf{\Gamma}$ given in the celestial frame by eqn [67], explicit computation of the motion of the angular momentum axis in space may be done trivially using the eqn [5] according to which $X_H = \varepsilon \Gamma_X$ dt and $Y_H = \varepsilon \Gamma_Y$ dt. We find thus that

$$X_H = \frac{3p^2}{2\Omega_0} H_\mathrm{d} \sin\varepsilon \cos\varepsilon [t - (1/2p)\sin 2pt]$$
$$Y_H = \frac{3p^2}{2\Omega_0} H_\mathrm{d} \sin\varepsilon [(1/2p)\cos 2pt] \qquad [70]$$

where $H_\mathrm{d} = (C-A)/C = e/(1+e)$. The periodic parts of X_H and Y_H evidently describe an *elliptical* nutation.

Now, we resolve the two-vector with components X_H, Y_H into two parts as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = D\left[(1+\cos\varepsilon)\begin{pmatrix} -\sin 2pt \\ \cos 2pt \end{pmatrix} + (1-\cos\varepsilon)\begin{pmatrix} \sin 2pt \\ \cos 2pt \end{pmatrix}\right] \qquad [71]$$

where $D = (3p/8\Omega_0)\ H_\mathrm{d} \sin\varepsilon$. The first vector on the right-hand side rotates in the prograde sense (the same as the rotation of the Earth, from the positive direction of the X-axis toward that of the Y-axis), and the second one's rotation is retrograde. Thus, the aforementioned equation constitutes a resolution of the elliptical motion of the angular momentum axis into prograde and retrograde circular motions. The factor $(1-\cos\varepsilon)$ in the amplitude of the retrograde component makes it clear that its presence (despite the motion of the Sun being in the prograde sense in the ecliptic plane) is because of the nonzero obliquity of the ecliptic relative to the equator. It is another matter that D becomes zero and therefore the torque as a whole vanishes if $\varepsilon = 0$.

The complex combination of X_H and Y_H is

$$X_H + iY_H = \mathrm{i}(3p/8\Omega_0) H_\mathrm{d} \sin\varepsilon \times \left[(1+\cos\varepsilon)\mathrm{e}^{2ipt} + (1-\cos\varepsilon)\mathrm{e}^{-2ipt}\right] \qquad [72]$$

The prograde part is characterized by a positive frequency ($2p$) and the retrograde part by a negative frequency ($-2p$). This is a very general property, as will be seen in the next section.

It is worth emphasizing here that the results [70] and [72] (and also eqn [73]) are for the angular momentum axis and not for the figure axis. These results are independent of whether the Earth is rigid or nonrigid and of whether or not it has a core, since there was no need to input any information about the structure and properties of the Earth (other than that of axial symmetry) in order to obtain the solution for the angular momentum vector. In contrast, the motions of the figure axis and the rotation axis are strongly influenced by the Earth's properties. As a consequence, studies on those motions are the ones that serve to provide glimpses into the properties of the Earth's interior.

We return now to eqn [66] and rewrite their periodic parts in terms of the classical nutation variables $\Delta\psi_H$ and $\Delta\varepsilon_H$ using eqn [1]:

$$\Delta\psi_H = -(3p/4\Omega_0) H_\mathrm{d} \cos\epsilon \sin 2pt$$
$$\Delta\epsilon_H = (3p/4\Omega_0) H_\mathrm{d} \sin\epsilon \cos 2pt \qquad [73]$$

The coefficients of $\sin 2pt$ and $\cos 2pt$ in these expressions are known as the *coefficients of nutation* in longitude and obliquity.

For this semiannual nutation of angular frequency $2p = 2n_\mathrm{s} = 4\pi$ rad year^{-1}, one finds the coefficients in longitude and latitude to be -1269 and 550 mas, respectively (this corresponds to a quasi-circular motion at the Earth's surface of about 16 m as seen from space). All these numbers are very close to those obtained from accurate treatments of the problem. The largest of the nutation terms is of lunar origin. It has a period of about 18.6 years, which arises from the precession of the lunar orbit around the ecliptic. The coefficients in longitude and latitude of this nutation are about $-17\,000$ mas and 9200 mas, respectively. A representation of the nutational motion is provided in Figures 1, 2, 3, and 6.

As for the term in X_H, which is linear in t, it represents precession in longitude. The precession rate is

$$\dot{\psi}_{\mathrm{A,H}} = \frac{1}{\sin\epsilon}\frac{\mathrm{d}X_{H\,\mathrm{sec}}}{\mathrm{d}t} = \left(3p^2/2\Omega_0\right) H_\mathrm{d} \cos\epsilon \qquad [74]$$

On using the values given earlier for the various parameters, one finds that the precession rate due to solar gravitational torque as given by eqn [74] is 7.7291×10^{-5} rad year^{-1} or, equivalently, 15 943 mas year^{-1}. Lunar attraction too gives rise to precessional motion, a little more than double that due to the Sun; the greater role of the Moon despite its mass being very much smaller than that of the Sun is a consequence of its being very much closer to the Earth. The total precession rate is about 50 000 mas year^{-1}.

3.10.6 Elliptical Motions: Prograde and Retrograde Circular Components

We have observed in the previous example that the periodic part of the motion of the vector (X_H, Y_H) describes an elliptical path. Periodic motions of the figure axis or the rotation axis in space are also elliptical. For our purposes, the resolution of the elliptical nutations of the axes and of torque vectors responsible for the nutations into pairs of counterrotating circular components is important, because the Earth's nutational responses to the two circular components are far from similar. In the following, we consider specifically the elliptical motions in the celestial frame; but it may be kept in mind that the resolution into circular components can be carried out equally well for motions, such as the wobble, in the terrestrial frame (and indeed, for any elliptical motion in general).

3.10.6.1 The Spectrum and the Fundamental Arguments of Nutation

The example of the last section is a gross oversimplification of the orbital motion of the Sun as seen from the GCRF. The actual orbit is elliptical and not circular, and even the elliptical orbit is perturbed by the Moon's pull on the Earth and other influences; consequently, the orbital motion has a spectrum of frequencies, which will be reflected in the spectrum of the solar torque on the Earth. The contributions from the orbital motions of the Moon and the Sun (perturbed by the planetary influences) make up all but a small part of the spectrum; the remainder consists of the relatively minor but nonignorable contributions from the planets. An argument $\Xi(t)$ of the sine and cosine terms in a typical spectral component of the torque bears therefore the imprint of more than one of these bodies and is expressed as a linear combination, with integer coefficients, of a number of *fundamental arguments*, each of which relates to the motion of one or the other of the solar system bodies. The fundamental arguments relating to the lunar and solar motions are the Delaunay arguments. They are

- $\ell =$ mean anomaly of the Moon,
- $\ell' =$ mean anomaly of the Sun,
- $F = L - \Omega$,
- $D =$ mean elongation of the Moon from the Sun,
- $\Omega =$ mean longitude of the ascending node of the Moon,

where L is the mean longitude of the Moon.

Additional fundamental arguments relating to the planets are $\lambda_1, \lambda_2, \ldots, \lambda_8$, which are the mean longitudes with respect to the fixed equinox and ecliptic of J2000, in a solar system barycentric reference frame, of Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune, respectively. Each of these planetary arguments is taken to be linear in time. One more argument that appears in planetary terms in nutation is the precession angle p_a, in which a t^2 term is also retained.

The argument of a general term is then

$$\Xi(t) = n_1 l + n_2 l' + n_3 F + n_4 D + n_5 \Omega + \sum_{i=1}^{9} n_i \lambda_i \qquad [75]$$

where $\lambda_9 = p_a$. The frequency of this spectral term is $\mathrm{d}\Xi/\mathrm{d}t$.

A listing of the fundamental arguments as functions of time may be found in the IERS Conventions 2000 (McCarthy, 2003). The periods corresponding to the frequencies of the lunar and solar arguments l, l', F, D, and Ω are 27.5545, 365.2596, 27.2122, 29.5306, and -6798.384 solar days, respectively. The last of these periods, which is about 18.6 years, is the period of the precession of the orbital plane of the Moon around the normal to the ecliptic. The period corresponding to L is 27.3216 solar days; it is the period of revolution of the Moon in its orbit. All these arguments are very nearly but not exactly linear in t, meaning that the periods are very slowly varying with time. The periods used to identify them are the values pertaining to J2000. The periods of the planets, which relate to the arguments λ_i ($i = 1, 2, \ldots, 8$), range from about 88 days for Mercury to about 60 000 days for Neptune, and the period relating to p_a is about 25 770 years. It is clear then that the frequencies of the fundamental arguments are much smaller than once per day; all the $\Xi(t)$ of interest to the nutations of an axially symmetrical ellipsoidal Earth have the same property. Thus, they are all in the low-frequency band, that is, between -0.5 and 0.5 cpsd.

Bretagnon et al. (1997) observed that ℓ' and Ω are so close to $2l - 2D - \lambda_3$ and $\lambda_3 + D - F$, respectively, that observations over periods of the order of thousands of years would be needed to distinguish them. Therefore, in developing their nutation series, they dropped ℓ and Ω from the fundamental set, replacing them by the aforementioned combinations plus small corrections that are linear in t; they also dropped the argument p_a, which has a very long period. Furthermore, the highly accurate nutation series that they have developed in terms of the truncated set of 11 fundamental arguments are not for the classical $\Delta\psi$ and $\Delta\varepsilon$, which are referred to the moving ecliptic, but are for the periodic and Poisson terms in two of the Euler angles (which they denote by ψ and ω) of the transformation from the fixed ecliptic of J2000 to the terrestrial equatorial frame of date. An unfortunate consequence of these differences is that direct comparison of their SMART97 nutation series with other series is not possible. One has to keep in mind, in addition, differences in signs between the different sets of variables: for example, $\omega_0 = -\varepsilon_0$ and $\Delta\omega$ corresponds to $-\Delta\varepsilon$ with the effects of ecliptic motion ignored.

3.10.6.2 Resolution of Elliptical Motions into Pairs of Circular Motions

With the coordinate frame chosen to be the GCRF in which the X- and Y-axes are in the directions of increasing longitude and latitude, respectively, the components Γ_1 and Γ_2 of the torque acting on an ellipsoidal Earth have the forms

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} P_s \sin \Xi(t) \\ Q_c \cos \Xi(t) \end{pmatrix} \qquad [76]$$

with

$$\Xi(t) = (\omega_n t + \chi_\omega) \qquad [77]$$

where ω_n is the nutation frequency and χ_ω is the initial phase. The curve traced by the tip of this two-dimensional vector is circular if $Q_c = \pm P_s$, and elliptical otherwise. It should be kept in mind that Γ_1, Γ_2, P_s, Q_s (and other similar quantities introduced below) refer to a particular spectral component, though we have avoided adding an index ω to keep the notation simpler.

The Earth's response to the gravitational tidal forcing is not strictly in phase with the forcing, for a variety of reasons: Anelasticity of the mantle causes the tidal deformation of the solid Earth to lag behind; the ocean tide raised by the potential and hence the deformation produced by ocean loading are very much out of phase with the forcing and so on; and these effects affect the deformable Earth's rotational response to the gravitational torque. Consequently, the Cartesian components $X(t)$, $Y(t)$ of the nutation, for example, will not be in phase with the torque; $X(t)$ and $Y(t)$ will each have both cosine and sine terms:

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} X_s \sin \Xi + X_c \cos \Xi \\ Y_c \cos \Xi + Y_s \sin \Xi \end{pmatrix} \qquad [78]$$

X_s and Y_c belong to the part of the nutation that is 'in phase' with the torque, while X_c and Y_s pertain to the 'out-of-phase' part.

In most of the existing literature, the expansions into cosine and sine terms of various frequencies are written for $\Delta\psi$ and $\Delta\varepsilon$ rather than for X and Y, and the coefficients in those expansions are called the *coefficients of nutation*; thus, $X_s/\sin\varepsilon$ and Y_c are the in-phase coefficients of nutation, and $X_c/\sin\varepsilon$ and Y_s are the out-of-phase ones.

We can now resolve the two-vector with components (X, Y) into parts represented by the following two-vectors, each of which executes a prograde or retrograde circular motion:

$$\begin{pmatrix} \sin \Xi \\ -\cos \Xi \end{pmatrix}, \quad \begin{pmatrix} -\sin \Xi \\ -\cos \Xi \end{pmatrix}, \quad \begin{pmatrix} \cos \Xi \\ \sin \Xi \end{pmatrix}, \quad \begin{pmatrix} \cos \Xi \\ -\sin \Xi \end{pmatrix} \qquad [79]$$

A little reflection shows that if the $\Xi(t)$ is increasing with time, that is, if the frequency ω_n is positive, the first and third of the previously mentioned two-vectors rotate in the sense from the positive direction of the first axis of the equatorial reference frame toward that of the second axis. This is the same sense in which the Earth rotates and is therefore said to be *prograde*. The second and fourth two-vectors rotate in the opposite or *retrograde* sense. The roles are reversed if ω_n is negative.

Expanding in terms of these prograde and retrograde vectors, we now have

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \eta^{+,\,\mathrm{ip}}\begin{pmatrix} \sin\Xi \\ -\cos\Xi \end{pmatrix} + \eta^{-,\,\mathrm{ip}}\begin{pmatrix} -\sin\Xi \\ -\cos\Xi \end{pmatrix} + \eta^{+,\,\mathrm{op}}\begin{pmatrix} \cos\Xi \\ \sin\Xi \end{pmatrix} + \eta^{-,\,\mathrm{op}}\begin{pmatrix} \cos\Xi \\ -\sin\Xi \end{pmatrix} \quad (80)$$

wherein the parts which come from the in-phase and out-of-phase parts are identified by the superscripts 'ip' and 'op,' respectively. The relations connecting the *amplitudes* $\eta^{\pm,\mathrm{ip}}$ of the in-phase parts of the nutation and $\eta^{\pm,\mathrm{op}}$ of the out-of-phase parts to the *coefficients of nutation* present in eqn [78] may be immediately written down. We have

$$\begin{aligned} \eta^{+,\,\mathrm{ip}} &= \frac{1}{2}(X_s - Y_c), \quad \eta^{-,\,\mathrm{ip}} = -\frac{1}{2}(X_s + Y_c) \\ \eta^{+,\,\mathrm{op}} &= \frac{1}{2}(X_c + Y_s), \quad \eta^{-,\,\mathrm{op}} = \frac{1}{2}(X_c - Y_s) \end{aligned} \quad [81]$$

Recall now that if $\omega_n > 0$, the prograde components of the motion are those with the coefficients $\eta^{+,\mathrm{ip}}$ and $\eta^{+,\mathrm{op}}$ and the retrograde ones have the coefficients $\eta^{-,\mathrm{ip}}$ and $\eta^{+,\mathrm{op}}$, while the connections are reversed if $\omega_n < 0$. So, if we identify the prograde and retrograde amplitudes by superscripts p and r, respectively, in the place of $\pm$, we see that for $\omega_n = \pm|\omega_n|$, $\eta^{p,\mathrm{ip}} = \eta^{\pm,\mathrm{ip}}$ and $\eta^{p,\mathrm{op}} = \eta^{\pm,\mathrm{op}}$, while $\eta^{r,\mathrm{ip}} = \eta^{\mp,\mathrm{ip}}$ and $\eta^{r,\mathrm{op}} = \eta^{\mp,\mathrm{op}}$. These relations may be written, with the use of eqn [81], as the following expressions for the prograde and retrograde amplitudes in terms of the coefficients of nutation:

$$\begin{aligned} \eta^{p,\,\mathrm{ip}} &= \frac{1}{2}\left(\frac{\omega_n}{|\omega_n|}X_s - Y_c\right) \\ \eta^{r,\,\mathrm{ip}} &= -\frac{1}{2}\left(\frac{\omega_n}{|\omega_n|}X_s - Y_c\right) \\ \eta^{p,\,\mathrm{op}} &= \frac{1}{2}\left(X_c + \frac{\omega_n}{|\omega_n|}Y_s\right) \\ \eta^{r,\,\mathrm{op}} &= \frac{1}{2}\left(X_c - \frac{\omega_n}{|\omega_n|}Y_s\right) \end{aligned} \quad [82]$$

Conversely,

$$\begin{aligned} X_s &= \frac{\omega_n}{|\omega_n|}\left(\eta^{p,\,\mathrm{ip}} - \eta^{r,\,\mathrm{ip}}\right), \quad Y_c = -\left(\eta^{p,\,\mathrm{ip}} + \eta^{r,\,\mathrm{ip}}\right) \\ X_c &= \left(\eta^{p,\,\mathrm{op}} + \eta^{r,\,\mathrm{op}}\right), \quad Y_s = \frac{\omega_n}{|\omega_n|}\left(\eta^{p,\,\mathrm{op}} - \eta^{r,\,\mathrm{op}}\right) \end{aligned} \quad [83]$$

In the literature, the relations in the preceding text are practically always written in terms of $\Delta\psi$ and $\Delta\varepsilon$ rather than in terms of X and Y. The transition from the relations obtained earlier is accomplished by the replacements

$$\begin{aligned} X_s &\to A\sin\varepsilon, \quad Y_c \to B \\ X_c &\to A''\sin\varepsilon, \quad Y_s \to B'' \end{aligned} \quad [84]$$

In these relations, ε is treated as a constant equal to ε_0 (i.e., neglecting the small increments to ε_0 from nutation in obliquity).

Complex combinations like $\tilde{\Gamma} = \Gamma_1 + i\Gamma_2$ and $X + iY$ are of considerable interest in the context of nutations. (We have already employed such combinations to advantage in Section 3.10.3.2.2.) It may be verified, by applying the developments presented earlier, that $X + iY$ may be written as

$$X + iY = \tilde{\eta}^p e^{i(\omega_n/|\omega_n|)\Xi} + \tilde{\eta}^r e^{-i(\omega_n/|\omega_n|)\Xi} \quad [85]$$

with

$$\tilde{\eta}^p = \eta^{p,\,\mathrm{ip}} + i\eta^{p,\,\mathrm{op}}, \quad \tilde{\eta}^r = \eta^{r,\,\mathrm{ip}} + i\eta^{r,\,\mathrm{op}} \quad [86]$$

$\tilde{\eta}^p$ and $\tilde{\eta}^r$ are the complex amplitudes of the prograde and retrograde components of the nutation with the in-phase coefficients X_s, Y_c and out-of-phase coefficients X_c, Y_s as in eqn [78].

It is to be noted that whichever term of eqn [85] has a positive value for the frequency (ω_n in the first term and $-\omega_n$ in the second one) that multiplies $i\ t$ in the phase factor represents a prograde circular motion and the other, with a negative value of the frequency, represents retrograde motion. This is a fact worth keeping in mind for future reference.

3.10.6.3 Nutation Series

Up to this point, we have been focusing on one spectral component of the nutational motion. To represent the total motion, we have to add to the quantities appearing in the expressions on the right-hand side of eqn [78] an index n labeling the spectral terms of the various frequencies $\omega_n = \mathrm{d}\Xi_n/\mathrm{d}t$ present in the nutation and then sum over all the terms. The resulting series provide the spectral representation of $X(t)$ and $Y(t)$. The series for $\Delta\psi$ is $1/\sin\varepsilon$ times that of $X(t)$, while the series for $\Delta\varepsilon$ is the same as that of $Y(t)$.

However, the series that are thus obtained are not quite complete. A reason is that the spectral components of the torque are not strictly harmonic as would appear from the expressions in eqn [76]; the amplitudes have a slow variation with time, which may be taken to be linear. Moreover, the fundamental arguments of nutation (and hence the argument $\Xi(t)$, which appears in the torque terms) are not strictly linear in time, which is to say that the nutation frequencies also are varying, though very slowly. The combined effect of the slow variations of the amplitudes and the arguments is that the amplitudes of the spectral terms in the nutation series are not constants. The time dependence of the amplitudes is adequately taken into account in the classical nutation series for $\Delta\psi$ and $\Delta\varepsilon$ through a linear dependence on time. With this enhancement, the series referred to in the last paragraph take the following form:

$$\begin{aligned} \Delta\psi(t) &= \sum_n \left[\left(A_n + A'_n t\right)\sin\Xi_n(t) + A''_n\cos\Xi_n(t)\right] \\ \Delta\varepsilon(t) &= \sum_n \left[\left(B_n + B'_n t\right)\cos\Xi_n(t) + B''_n\sin\Xi_n(t)\right] \end{aligned} \quad [87]$$

Nutation series have the same general form as in the preceding text irrespective of whether they are for the figure axis, rotation axis, or the angular momentum axis, but the coefficients of nutation differ from one to the other.

Nutations have periods extending from thousands of years down to fractions of a day, as we have seen earlier. The largest of them are the 18.6-year, 9.3-year, annual, semiannual, and 13.66-day nutations. The in-phase coefficients A_n and B_n of the nutations of the figure axis are shown in Tables 1 and 2 for these five periods. The values are from recent computations

Table 1 Coefficients in longitude and obliquity of prominent nutations (in mas)

Period[a]	Rigid Earth		Nonrigid Earth	
	Longitude	Obliquity	Longitude	Obliquity
−18.6 years	−17 281	9228	−17 208	9205
−9.3 years	209	−90	207	−90
1 year	126	0	148	7
0.5 years	−1277	553	−1317	573
13.66 days	−222	95	−228	98

[a]Some of the periods appear with negative signs. They have been historically assigned to those nutations in which the retrograde component dominates over the prograde one (compare **Table 2**).

Table 2 Amplitudes of prograde and retrograde circular nutations (in mas)

Period[a]	Rigid Earth		Nonrigid Earth	
	Retrograde	Prograde	Retrograde	Prograde
18.6 years	−8051	−1177	−8025	−1180
9.3 years	87	4	86	4
1 year	−25	25	−33	26
0.5 year	−23	−531	−25	−548
13.66 days	−3	−92	−4	−94

[a]The periods shown are for the prograde nutations. The retrograde nutations have the corresponding negative values for their periods.

and are given both for a nonrigid Earth model and for a hypothetical rigid Earth. The entries in the table bring out the fact that the nutations of the figure axis are significantly affected by nonrigidity of the Earth, especially at some frequencies; in particular, the amplitude of the retrograde annual nutation of the nonrigid Earth is over 30% higher than that of the rigid Earth, while that of the prograde annual nutation is hardly affected by nonrigidity. The reason for the differing levels of sensitivity of different spectral components is that certain normal modes and associated resonances (especially the FCN) exist and affect the Earth's response to the torque in a frequency-dependent fashion in the case of the nonrigid Earth but not in the rigid case, as will be seen from the theory presented in later sections.

3.10.7 Kinematic Relations between the Nutation of the Figure Axis and the Wobble

Consider the motion of the Earth's figure axis relative to the equatorial CRF of J2000 (see Section 3.10.2.2). Recall that the coordinates of the pole of this axis in this frame at an instant t are $X(t)$, $Y(t)$. The Z coordinate of the pole is of no interest here.

The motion of the pole in the interval $(t, t+\mathrm{d}t)$ changes its coordinates by $(\mathrm{d}X, \mathrm{d}Y, 0)$. We transform this infinitesimal vector to the TRF now, in order to establish its relation to the components of the wobble. We seek to obtain the transformed components only to the first order in small quantities. For this limited purpose, we may take the transformation matrix to the zeroth order, meaning that we ignore the first-order offsets between the Z-axis of the celestial frame, on the one hand, and the figure and rotation axes of the Earth, on the other, and also ignore the variations in the spin rate. In this approximation, the equators of the celestial and terrestrial frames coincide, and the transformation from the former to the latter is simply a uniform rotation with angular velocity Ω_0 about the Z-axis of the celestial frame. So the components of the infinitesimal displacement vector of the figure axis pole in the terrestrial frame are

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}X\cos\Omega_0(t-t_0) + \mathrm{d}Y\sin\Omega_0(t-t_0) \\ \mathrm{d}y &= -\mathrm{d}X\sin\Omega_0(t-t_0) + \mathrm{d}Y\cos\Omega_0(t-t_0) \end{aligned} \qquad [88]$$

where t_0 is the instant of coincidence of the first axes of the two frames. The errors in these expressions are of the second order in the neglected small quantities.

The displacement $(\mathrm{d}x, \mathrm{d}y)$ of the figure axis pole in the terrestrial frame is, however, determined directly by the angular velocity vector. The Earth's rotation with angular velocity $\boldsymbol{\Omega}$ has the effect that any mantle-fixed vector $\mathbf{V}$ gets displaced by an amount $(\boldsymbol{\Omega}\times\mathbf{V})\mathrm{d}t$ in an infinitesimal time interval $\mathrm{d}t$. In particular, the displacement of the z-axis during the interval $(t, t+\mathrm{d}t)$ is $(\boldsymbol{\Omega}\times\hat{\mathbf{Z}})\,\mathrm{d}t$, with components $(\Omega_y, -\Omega_x, 0)\,\mathrm{d}t=\Omega_0(m_y, -m_x, 0)\mathrm{d}t$ relative to the terrestrial frame of the instant t. Since the z-axis is the figure axis, we have thus

$$\mathrm{d}x = \Omega_0 m_y\,\mathrm{d}t, \quad \mathrm{d}y = -\Omega_0 m_x\,\mathrm{d}t \qquad [89]$$

We obtain the relations that we seek on equating the expressions [88] and [89]:

$$\begin{aligned} \Omega_0 m_x &= \dot{X}\sin\Omega_0(t-t_0) - \dot{Y}\cos\Omega_0(t-t_0) \\ \Omega_0 m_y &= \dot{X}\cos\Omega_0(t-t_0) - \dot{Y}\sin\Omega_0(t-t_0) \end{aligned} \qquad [90]$$

The familiar complex combination of this pair of equations now yields

$$\begin{aligned} \Omega_0\tilde{m} &= i\big(\dot{X}+i\dot{Y}\big)\mathrm{e}^{-i\Omega_0(t-t_0)} \\ &= i\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}t}\mathrm{e}^{-i\Omega_0(t-t_0)} \end{aligned} \qquad [91]$$

where

$$\tilde{\eta} = X + iY \qquad [92]$$

It is important to note that no dynamical considerations have been invoked while deriving the result [91]. Therefore, it is valid for rigid and nonrigid Earth models alike. Since it is of purely kinematic origin, it is commonly referred to as the *kinematic relation*.

3.10.7.1 Kinematic Relations in the Frequency Domain

For a circular nutation of some frequency ω_n (using the subscript n here for nutation),

$$\begin{aligned} X &= \tilde{\eta}(\omega_\mathrm{n})\cos(\omega_\mathrm{n}t+\chi_\mathrm{n}) \\ Y &= \tilde{\eta}(\omega_\mathrm{n})\sin(\omega_\mathrm{n}t+\chi_\mathrm{n}) \\ X+iY &= \tilde{\eta} = \tilde{\eta}(\omega_\mathrm{n})\mathrm{e}^{i(\omega_\mathrm{n}t+\chi_\mathrm{n})} \end{aligned} \qquad [93]$$

where χ_n is a phase that depends on the frequency ω_n. On introducing this expression into the first of the equalities in eqn [91], it becomes obvious that the form of $\tilde{m}$ is given by

$$\tilde{m} = \tilde{m}(\omega_w) e^{i(\omega_w t + \chi_w)} \quad [94]$$

with

$$\tilde{m}(\omega_w) = -\frac{\omega_n}{\Omega_0} \tilde{\eta}(\omega_n), \quad \omega_w = \omega_n - \Omega_0 \quad [95]$$

(The subscript w labels quantities pertaining to the wobble.) These equations constitute the kinematic relations in the spectral domain; they correspond to the kinematic relation [91] in the time domain.

An additional relation, which is not usually written down explicitly, is obtained on considering the zero-frequency part of $\dot{X}$, which is $\dot{\psi}_A \sin\varepsilon$ where $\dot{\psi}_A$ is the rate of precession in longitude. It is evident from eqn [91] that the corresponding spectral component of $\tilde{m}$ has frequency $(-\Omega_0)$. It follows then that

$$\Omega_0 \tilde{m}(-\Omega_0) = i\dot{\psi}_A \sin\varepsilon - \dot{\varepsilon}_A \quad [96]$$

With ε approximated by ε_0, the previous equation expresses the precession rate $\dot{\psi}_A$ in terms of the wobble amplitude at the frequency $-\Omega_0$ (-1 cpsd) as $-i(\Omega_0/\sin\varepsilon_0)\tilde{m}(-\Omega_0)$.

Another relation that is of importance concerns polar motion. We recall from Section 3.10.2.5.1 that the motion of the CIP in the celestial frame involves only frequencies in the low-frequency bands. This means that for all other frequencies, the CIP coincides with the pole of the celestial frame. But these are the frequencies, after transforming to the terrestrial frame, that are contained in polar motion. It follows that polar motion is effectively the motion of the pole of the celestial frame relative to the terrestrial frame. But the reverse of this motion, that is, the motion of the pole of the terrestrial frame (which is the pole of the figure axis) in the celestial frame, is the nutation $\tilde{\eta}$ of the figure axis. Therefore, the spectral component of frequency ω_p of the polar motion as defined by the IAU conventions may be obtained simply by transforming the negative of the corresponding spectral component of the nutation of the figure axis from eqn [93] to the terrestrial frame:

$$\begin{gathered} \tilde{p}_{\omega_p}(t) = -\tilde{\eta}(\omega_n) e^{i(\omega_n t + \chi_n)} e^{-i\Omega_0 t} \\ \omega_p = \omega_w = \omega_n - \Omega_0 \end{gathered} \quad [97]$$

On using the kinematic relation [95] between $\tilde{\eta}$ and $\tilde{m}$, we finally obtain the relation between the amplitudes of the polar motion and the corresponding wobble:

$$\tilde{p}(\omega_p) = -\tilde{\eta}(\omega_n) = \frac{\Omega_0}{\omega_n} \tilde{m}(\omega_w) \quad [98]$$

Clearly, a polar motion of the CIP, in the frequency domain in which it is defined, is not identical to the wobble of the same frequency.

3.10.7.2 Implications of the Kinematic Relations

Now, we examine briefly the implications of the kinematic relations.

First and foremost is that these relations provide a formal proof of the existence of a circular nutation associated with every circular wobble and *vice versa*; the frequency of the nutation is algebraically greater than that of the associated wobble by 1 cpsd. This important general relation is simply a consequence of the Earth's diurnal rotation in space with angular velocity Ω_0. This relationship was already seen in a qualitative way earlier: it was noted in Section 3.10.4.2 that the spectrum of the torque in the terrestrial frame is in the retrograde diurnal band, and it was shown explicitly in Section 3.10.5.2 that the spectrum in the celestial frame is in the low-frequency band.

The amplitude relation in eqn [95] shows that the amplitude $\tilde{\eta}(\omega_n)$ of the nutation contains a factor $1/\omega_n$, which simulates a normal mode at $\omega_n = 0$. Unlike a proper normal mode, it has no connection to any property of the rotating Earth. It has been referred to in the literature as the 'tilt-over mode,' following Smith (1974) and Wahr (1981). The resonance factor $1/\omega_n$ produces a great enhancement of the nutation amplitude compared with the associated wobble amplitude when the frequency is very small, their ratio $(-\Omega_0/\omega_n)$ being, apart from a minus sign, the period of the nutation in sidereal days. Thus, the amplitude of the retrograde 18.6-year nutation (period $= -6798$ solar days $= -6816$ sidereal days) is 6816 times as large as that of the corresponding retrograde diurnal wobble. There is no such resonant kinematic factor in other frequency bands, and this is a major reason for the overriding importance of the low-frequency band in nutations. As far as other normal modes are concerned, each wobble normal mode and the resonance in the wobble amplitude that is associated with it have their counterparts in the nutation amplitude, of course with a shift in frequencies by 1 cpsd.

As an immediate application of the kinematic relations, we use the amplitude relation in eqn [95] to arrive at the amplitude $\tilde{\eta}(\omega_n)$ of the nutation of the *rigid Earth* that is produced by a torque $\tilde{\Gamma}(\sigma)$ having frequency σ cpsd in the terrestrial frame. Introducing the wobble amplitude eqn [19] of the rigid Earth into this relation, we obtain

$$\tilde{\eta}_R(\omega_n) = i\frac{\tilde{\Gamma}(\omega_w)}{A(\omega_w - \Omega_0 e)(\Omega_0 + \omega_w)} \quad [99]$$

As for the precession rate, one finds from eqn [19] at the particular frequency $\omega_w = -\Omega_0$, together with eqn [96], that

$$\dot{\psi}_A = \frac{\tilde{\Gamma}(-\Omega_0)}{A\Omega_0^2(1+e)} \quad [100]$$

3.10.7.3 Transfer Function

Consider two Earth models, one rigid and the other nonrigid, both of which are forced by the same torque $\tilde{\Gamma}(\omega_w)$. (In the simplest case when both are axially symmetrical and ellipsoidal, this means that both must have the same value for the dynamical ellipticity e.) Let the wobbles of the rigid and nonrigid Earth models be $\tilde{m}_R(\omega_w)$ and $\tilde{m}(\omega_w)$, respectively. The former may be taken from $\tilde{m}$ of eqn [19] of Section 3.10.3.2.2, which is for the rigid Earth. Both $\tilde{m}_R$ and $\tilde{m}$ are proportional to $\tilde{\Gamma}(\omega_w)$, and therefore, the ratio $\tilde{m}(\omega_w)/\tilde{m}_R(\omega_w)$ is independent of the torque. Moreover, as was observed in the last section, the kinematic relation [91] and its frequency domain version [95] are valid for any type of Earth model, rigid or nonrigid. Therefore, it follows from the latter equation that

$$T_{\rm n}(\omega_{\rm n}) \equiv \frac{\tilde{\eta}(\omega_{\rm n})}{\tilde{\eta}_{\rm R}(\omega_{\rm n})} = \frac{\tilde{m}(\omega_{\rm w})}{\tilde{m}_{\rm R}(\omega_{\rm w})} \equiv T_{\rm w}(\omega_{\rm w}) \qquad [101]$$

These ratios, which are independent of the forcing torque as noted earlier, are the *transfer functions* for nutation and wobble; the two are equal. It is useful to write these relations also with the alternative notation for frequencies:

$$T_{\rm n}(\nu) \equiv \frac{\tilde{\eta}(\nu)}{\tilde{\eta}_{\rm R}(\nu)} = \frac{\tilde{m}(\sigma)}{\tilde{m}_{\rm R}(\sigma)} \equiv T_{\rm w}(\sigma) \qquad [102]$$

where ν and σ are the frequencies of the nutation and the wobble, respectively, expressed in cpsd:

$$\nu = \omega_{\rm n}/\Omega_0, \quad \sigma = \omega_{\rm w}/\Omega_0, \quad \nu = \sigma + 1 \qquad [103]$$

the last equality being one of the kinematic relations.

The transfer functions are of great utility for the following reason. If one solves for wobble amplitude $\tilde{m}(\sigma)$ of the Earth model of interest by solving the relevant equations of motion with inputs from the Earth's structure pertaining to that model, one can compute the transfer function $T_{\rm w}(\sigma)$ (which is independent of the amplitude of the forcing), by dividing the amplitude by $\tilde{m}_{\rm R}(\sigma)$ as given by eqn [19] or [55]. Since $T_{\rm n}(\nu) = T_{\rm w}(\sigma)$, we can immediately obtain the nutation amplitudes of the chosen Earth model as

$$\tilde{\eta}(\nu) = T_{\rm w}(\sigma)\tilde{\eta}_{\rm R}(\nu) = -T_{\rm w}(\sigma)\tilde{m}_{\rm R}(\sigma)/(1+\sigma) \qquad [104]$$

where the last step makes use of the kinematic relations. Of course, $\tilde{m}_{\rm R}(\sigma)$ may be readily calculated using the tide tables, as explained in eqn [55].

Alternatively, $\tilde{n}_{\rm R}(\nu)$ in the last equation may be taken directly from tables of rigid Earth nutation amplitudes. Nutations of the rigid Earth have been studied extensively, and highly accurate results are available for the rigid Earth nutation amplitudes. The important point is that the computation of the torques exerted by the solar system bodies, starting with their time-dependent positions as given by the ephemerides, needs to be done only once, at the stage of working out the rigid Earth amplitudes. The use of the transfer functions enables one to avoid repeating this exercise for each nonrigid Earth model.

3.10.7.4 Relations Connecting the Nutations of Different Axes: Oppolzer Terms

The nutation of the rotation axis can be related to that of the figure axis with the aid of the kinematic relation [91], as we shall now show. This relation, like the kinematic relation, holds good whether the Earth is rigid or nonrigid.

By definition, the temporal variations of the vectors from the pole of the Z-axis of the celestial frame to the poles of the figure axis and the rotation axis, as seen from the celestial frame, constitute the nutations of these two axes. We denote the complex combinations of the (equatorial) components of the pole of the figure axis in the celestial frame by $(X+iY)$ and that of the rotation axis by $(X_{\rm R}+iY_{\rm R})$. (Caution: In this section, R does not stand for 'Rigid' as elsewhere, but for 'rotation.') Now, it is evident that the difference $(X_{\rm R}+iY_{\rm R})-(X+iY)$ gives the complex representation of the components (in the celestial frame) of the vector from the figure axis pole to the rotation pole. It is the variation of this vector, as seen in the *terrestrial* frame, that constitutes the wobble, represented by m; its components in the celestial frame will then be represented by $\tilde{m}e^{i\Omega_0(t-t_0)}$, under the same approximations as in the second paragraph of Section 3.10.7. Consequently, we have

$$(X_R + iY_R) - (X + iY) = \tilde{m}e^{i\Omega_0(t-t_0)} \qquad [105]$$

We can now use the kinematic relation [91] to eliminate $\tilde{m}$ from the previous equation. The result is the relation that we seek:

$$X_R + iY_R = (X + iY) + \frac{i}{\Omega_0}\left(\dot{X} + i\dot{Y}\right) \qquad [106]$$

For a spectral component of the nutation with frequency $\omega_{\rm n}$, one writes

$$\begin{aligned} X + iY &= \tilde{\eta}(\omega_{\rm n})e^{i(\omega_{\rm n}t+\chi_{\rm n})} \\ X_{\rm R} + iY_{\rm R} &= \tilde{\eta}_{\rm R}(\omega_{\rm n})e^{i(\omega_{\rm n}t+\chi_{\rm n})} \end{aligned} \qquad [107]$$

and the foregoing relation becomes

$$\tilde{\eta}_{\rm R}(\omega_{\rm n}) = \left(1 - \frac{\omega_{\rm n}}{\Omega_0}\right)\tilde{\eta}(\omega_{\rm n}) \qquad [108]$$

We see that the fractional difference between the nutation amplitudes of the rotation axis and the figure axis is the frequency of the nutation expressed in cpsd.

It is of interest to note that the precession of the figure axis in longitude, which is represented by $\dot{\psi}_{\rm A}t$, leads to the precession part $\dot{\psi}_{\rm A}(t + i/\Omega_0)$ in $(X_{\rm R}+iY_{\rm R})$. Thus, one finds that the rotation pole has, expectedly, the same rate of precession in longitude as the pole of the figure axis, but the rotation pole is offset by a constant amount $\left(\dot{\psi}_{\rm A}/\Omega_0\right)$ in obliquity.

One may ask whether the nutation of the angular momentum axis is also related in a simple manner to the nutations of the other two axes. The answer is in the negative, in general. The reason is that the relation of the angular momentum vector to the rotation vector depends very much on the structure of the Earth, and so the relation between the nutations of the angular momentum and rotation axes differs from one Earth model to another. The particular case of a rigid Earth is of considerable interest, however, because classical treatments of the rotation variations of the rigid Earth have had, as their primary output, the nutations of the angular momentum axis; the nutations of the figure and rotation axes were then inferred. The relations used for this last step will now be derived, taking the Earth to be a rigid axially symmetrical ellipsoid. The components of the angular momentum vector in a principal axis frame are then $H_x = A\Omega_0 m_x$, $H_y = A\Omega_0 m_y$, $H_z = C\Omega_0(1+m_z)$, and the equatorial components of the unit vector along $\mathbf{H}$ are $(A/C)\ m_x$, $(A/C)m_y$ to the lowest order in m_x, m_y, m_z. These are the components of the vector offset of the pole of the angular momentum axis from that of the figure axis (the z-axis); they are simply (A/C) times the corresponding components (m_x, m_y) for the pole of the rotation axis. The scalar factor (A/C) is of course independent of whether the offset vectors of the angular momentum axis and the rotation axis are viewed from the terrestrial or the celestial frame. Therefore, we have

$$(X_H + iY_H) - (X + iY) = (A/C)[(X_{\rm R} + iY_{\rm R}) - (X + iY)] \qquad [109]$$

This means that when considering the offset of the angular momentum axis, a relation of the form [105] holds with $\tilde{m}$ replaced by $(A/C)\tilde{m} = \tilde{m}/(1+e)$. Consequently, we have, instead of eqn [106], the relation

$$X_H + iY_H = (X + iY) + \frac{i}{(1+e)\Omega_0}\left(\dot{X} + i\dot{Y}\right) \qquad [110]$$

The corresponding relation for the amplitudes of the respective spectral components of frequency ω_n is

$$\tilde{\eta}_H(\omega_n) = \left(1 - \frac{\omega_n}{(1+e)\Omega_0}\right)\tilde{\eta}(\omega_n) \qquad [111]$$

Therefore,

$$\tilde{\eta}(\omega_n) - \tilde{\eta}_H(\omega_n) = \frac{\omega_n}{(1+e)\Omega_0 - \omega_n}\tilde{\eta}_H \qquad [112]$$

This is called the *Oppolzer term for the figure axis*. It enables one to compute the amplitude of the nutation of the figure axis, given that of the angular momentum axis. The *Oppolzer term for the rotation axis* can be obtained by combining the previous equation with eqn [108]. The result is that

$$\tilde{\eta}_R(\omega_n) - \tilde{\eta}_H(\omega_n) = -\frac{e\omega_n}{(1+e)\Omega_0 - \omega_n}\tilde{\eta}_H \qquad [113]$$

It must be kept in mind that the expressions for the Oppolzer terms [112] and [113] are valid only for the rigid Earth model considered, unlike the kinematic relation and the relation [108] between $\tilde{\eta}_R$ and $\tilde{\eta}$, which are independent of the Earth model.

Traditionally, nutation tables for the rigid Earth have been presented as tables of the coefficients in the spectral expansions of $\Delta\psi_H(t)$ and $\Delta\varepsilon_H(t)$ in terms of real simple harmonic (cosine and sine) functions of time, together with the corresponding coefficients for the Oppolzer terms. In the case of nonrigid Earth models, tables are presented, as a rule, for the nutations of the figure axis.

3.10.8 Rigid Earth Nutation

By 'rigid Earth,' we mean a rigid Earth model having the same density function $\rho(\mathbf{r})$ as for the actual Earth. It has therefore the same principal moments of inertia, and other moments of higher degree of the matter distribution, as the actual Earth. These parameters are determined from the spatial structure of the Earth's own external gravitational field, inferred from its influence on the orbits of low-flying Earth satellites observed by satellite laser ranging. The determination of the rotation variations should then, in principle, be a simple matter, if the time dependence of the gravitational potentials of the Moon, Sun, and the planets at the Earth is known. This dependence is determined by the orbital motions of the Moon around the Earth and of the Earth and other solar system bodies around the Sun. The orbits deviate from simple Kepler's ellipses for a variety of reasons: (1) perturbation of the lunar orbit around the Earth by the attraction of the Moon by the Sun and of the Earth's orbit around the Sun by the Moon's gravitation (the so-called three-body effects), (2) additional perturbations of these orbits by the planets (resulting in the 'indirect planetary effect' on Earth rotation) and also the so-called planetary-tilt effect on the lunar orbit, and (3) the 'J_2-tilt effect' due to the perturbation of the Moon's orbit by the noncentral part of the Earth's gravitational potential that arises from the Earth's ellipticity. All these orbit perturbations are reflected in the lunar and solar gravitational potentials at the Earth and hence in their torques on the Earth. Additional torques arise from the direct action of the planets on the ellipticity of the Earth, referred to as the 'direct planetary effect.' The ELP2000 ephemerides provide the time-dependent position of the Moon in analytic form, and VS0P2000 presents similar information for the Sun and the planets. These ephemerides are recent versions of the ELP and VSOP series of ephemerides beginning with Chapront-Touzé and Chapront (1983) and Bretagnon and Francou (1988), respectively. The ephemerides of the Sun and the Moon include the perturbations mentioned in the preceding text. Treatments of the rotation variations of the rigid Earth fall into two broad classes: those based on the Hamiltonian formulation of mechanics and those making use of the torque equation governing rotational motion.

The Hamiltonian approach begins generally with a treatment of the 'main problem' where the influences of the Moon and Sun on the Earth's rotation are considered, taking into account the Sun's perturbation of the lunar orbit and the Moon's perturbation of the Earth–Sun orbital motion and, in recent treatments, also the planetary perturbations of both these orbits. The direct planetary torques on the Earth are treated separately, as are other second-order effects like the 'crossed-nutation coupling,' a term that refers to the change in the torque on the Earth due to the changes in the orientation of the Earth itself (due to precession–nutation) and the consequent increments to these motions.

The torque approach makes direct use of the ephemerides, which already take account of all the orbit perturbations. In either approach, the Earth parameter, which plays the dominant role in the gravitational torque on the Earth, is the dynamical ellipticity, but triaxiality and the higher-degree geopotential coefficients that play lesser roles are also considered. In addition, second-order increments to the gravitational torque exerted by the Moon and Sun due to changes in Earth's orientation caused by nutation and precession are also taken into account in recent rigid Earth nutation theories.

3.10.8.1 Hamiltonian Approach

In the Hamiltonian approach, the starting point is the Hamiltonian K of the rotating Earth. (The standard notation for the Hamiltonian is H; it has been changed here because of the use of H for another quantity in the present context.) K is the sum of the Earth's rotational kinetic energy and the potential energy of its gravitational interaction with the Moon and Sun. (The relatively small interactions with the planets are dealt with separately.) Recent computations of the Earth's precession and nutation by Souchay et al. (1999) are the culmination of developments starting from the work of Kinoshita (1977) based on the Hamiltonian approach. The presentation in the succeeding text is a very brief overview of that work.

The Hamiltonian of a system is a function of a set of coordinate variables q_i and corresponding momentum variables p_i, which, taken together, describe the state of the system at any given time; (q_i, p_i) for any given i is said to constitute a canonically conjugate pair. Given the expression $K(q, p)$ for the Hamiltonian, the equations of motion are given by

$$\dot{q}_i = \frac{\partial K}{\partial p_i}, \quad \dot{p}_i = \frac{\partial K}{\partial q_i} \qquad [114]$$

for all i.

The Hamiltonian for the rotational motion of the Earth is

$$K = \sum_i \frac{L_x^2}{2A} + \frac{L_y^2}{2B} + \frac{L_z^2}{2C} + M_B W_E(r_B, \theta_B, \lambda_B) \qquad [115]$$

wherein the kinetic energy involves the components of the angular momentum vector in the terrestrial frame (which are denoted here by L_x, L_y, L_z in a departure from earlier notation) and the potential energy is expressed through the potential of the Earth at the position of the body B. (To be specific, we take the body to be the Moon; the case of the Sun is entirely similar). Since θ_B and λ_B characterize the direction of the Moon in the TRF, they depend on the orientation of this frame in space, that is, in relation to an inertial reference frame. This orientation may be represented by the transformation that carries the latter frame to the former.

It is customary, in Hamiltonian treatments of the rotation variations of the rigid Earth, to place the nutation and precession of its angular momentum axis in the foreground and to infer from them subsequently the motions of the figure and rotation axes. Consequently, the inertial plane (which is the principal plane normal to the third axis of the inertial frame), the 'angular momentum plane' (or Andoyer plane) normal to the angular momentum vector, and the principal plane of the terrestrial frame (the equatorial plane) are all made use of in the transformation from the inertial frame to the terrestrial one (see Figure 8 for the definition of the angles used). The transformation consists of (1) a rotation through an angle h about the third axis of the inertial frame to bring its first axis into a new position along the nodal line (intersection) of the inertial and angular momentum planes,

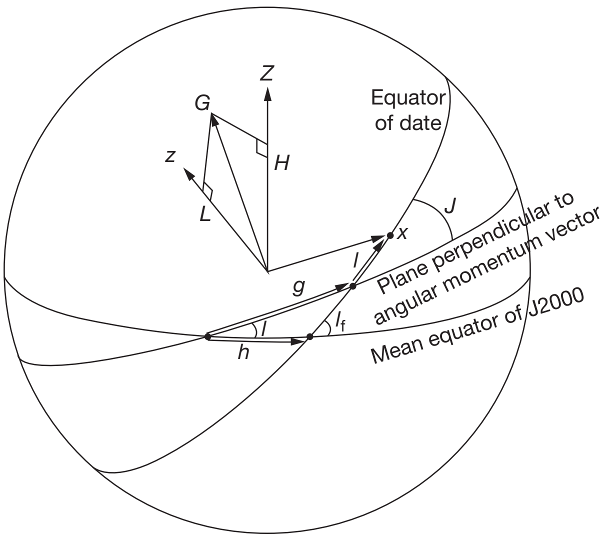

Figure 8 Angles used in the transformations between the celestial and terrestrial frames in the Hamiltonian approach for rigid Earth nutation computation.

(2) a rotation through an angle J about this new axis to carry the third axis of the inertial frame into the direction of the angular momentum vector, (3) a rotation about this vector through an angle g to take the first axis from the earlier mentioned nodal line to the node of the angular momentum plane and the equatorial plane, (4) a rotation through angle I about this node to carry the third axis over from the angular momentum axis to the third axis of the terrestrial frame, and (5) a rotation through an angle l about the last-mentioned axis to bring the first axis into alignment with the first axis (at 0 longitude) of the terrestrial frame. The rotation angles h, g, l are taken as the coordinate variables q_i ($i = 1, 2, 3$), describing Earth orientation, and the angular momentum components H, G, L along the axes about which these rotations are made are taken as the conjugate momentum variables p_i for the purpose of the Hamiltonian formulation. (H and L are evidently the components of the angular momentum vector along the Z-axis of the inertial frame and the z-axis of the terrestrial frame, respectively, and G is the magnitude of the angular momentum vector.) These three pairs of canonically conjugate variables are called the Andoyer variables (Andoyer, 1923). The two other quantities I and J are not independent of these: $\cos I = H/G$, $\cos J = L/G$.

The nutation and precession of the angular momentum axis in longitude consist of the motions of this axis around the third axis of the inertial plane. This is precisely what variations Δh in the angle h describe. The nutation–precession in obliquity is represented by the variations ΔI in the inclination of the angular momentum axis to the inertial axis; this quantity may be expressed in terms of variations in the Andoyer variables H and G: $\Delta I = [(H/G)\Delta G - \Delta H]/(G \sin I)$. They may be determined, therefore, by solving the Hamiltonian equations for the Andoyer variables.

To set up the Hamiltonian equations, it is necessary to express the total energy given by eqn [115] as a function of the Andoyer variables representing Earth rotation, on the one hand, and the position coordinates of the Moon with reference to the inertial frame, on the other. The orbital motion of the Moon makes its coordinates time-dependent. The ephemerides (ELP2000 in the case of the Moon) express the time dependence of each of the position coordinates as a spectral expansion in which the argument of each term is an integer linear combination of the Delaunay fundamental arguments and additional arguments referring to the motions of the planets (see Section 3.10.6.1); the planetary arguments enter because of the gravitational perturbations they produce on the lunar orbit. Expansions of the same kind are also provided for the functions of the coordinates, which appear in the spherical harmonic expansion of the potential in the celestial frame. With the time dependence of the potential from the orbital motions of the solar system bodies thus made explicit, the Hamiltonian equations $\dot{h} = \partial K/\partial H$, $\dot{H} = -\partial K/\partial h$ and so on yield the three pairs of time derivatives as functions of the six Andoyer variables and of the time variable appearing through the spectral expansions.

Kinoshita (1977) applied a method due to Hori (1966) to facilitate the solution of these equations by means of a transformation of the Andoyer variables and their functions, which preserves the form of Hamilton's equations while expressing them in terms of the new (transformed) variables. The transformation, which is carried out perturbatively, is designed to cause the new Hamiltonian to have only a secular variation with time, while all periodicities enter through the 'determining function' through which the transformation is accomplished. A detailed treatment of the transformation and the subsequent solution of the equations are beyond our scope. The interested reader may refer to Kinoshita (1977). For the inclusion of planetary influences, see Souchay and Kinoshita (1996, 1997). Various second-order effects are treated in later papers of Souchay, Kinoshita, and collaborators (Folgueira et al., 1998a,b; Kinoshita and Souchay, 1990; Souchay and Kinoshita, 1996, 1997), and the complete results are presented by Souchay et al. (1999) as the REN-2000 nutation series. This is the rigid Earth series used by Mathews et al. (2002) for convolution with their transfer function in constructing the MHB2000 series on which the IAU2000A nutation series is based.

3.10.8.2 Torque Approach

This approach is based on the solution of the torque equation either in the celestial frame or in the terrestrial frame, introduced in Sections 3.10.3.1 and 3.10.3.2, respectively.

3.10.8.2.1 Axially symmetrical Earth: simplified treatment

The Earth's structure parameters (the principal moments of inertia and geopotential coefficients of higher orders) are constant in the terrestrial frame, and therefore, the torque on the Earth is readily determined in this frame, as noted earlier. Transformation of the torque to the celestial frame is easy only if the Earth is taken to be axially symmetrical and the effects of the Earth's nutation, precession, and wobble on the torque are ignored. We have seen in Section 3.10.4.5 that the structure parameters that appear in the torque on an axially symmetrical Earth are the J_n (equal to $-C_{n0}$) and that the torque proportional to J_n are given by eqn [61] in the terrestrial frame. Under the approximations stated earlier, the corresponding torque in the celestial frame may be obtained simply by replacing θ_B and λ_B by $(\pi/2-\delta_B)$ and α_B, respectively, where δ_B is the declination and α_B the right ascension of the body B in the equatorial CRF. The reasoning is the same as in the special case $n=2$ considered in Section 3.10.5.1 and will not be repeated here. The relations between the Cartesian coordinates used there and the polar coordinates δ and α are $X_B=r_B \cos\delta_B \cos\alpha_B$, $Y_B=r_B\cos\delta_B\sin\alpha_B$, $Z_B=r_B\sin\delta_B$.

The time dependence of the torque may now be made explicit by using the ELP or VSOP ephemerides, as the case may be. The nutation of the angular momentum axis is then trivially obtained by integration of the torque (see Section 3.10.3.1). This approach has been used by Roosbeek and Dehant (1998); they identified the corrections that need to be applied to the nutation series obtained directly from the integration. Their final results constitute the RDAN (Roosbeek-Dehant Analytical Nutation) nutation series. Elegant analytic derivations have been provided by Williams (1994) for the contributions to nutation and precession from various small effects.

3.10.8.2.2 Rigorous treatment of the general case

The transformation of the torque equations taking account of the full range of geopotential coefficients in the terrestrial frame (ITRF) to equivalent equations in the fixed ecliptic celestial reference frame (ECRF) of J2000 can be carried out rigorously, as was done by Bretagnon et al. (1997, 1998). (We denote this celestial frame by ECRF0, with 0 serving as a reminder that it is based on a fixed ecliptic.) The transformation is effected through a sequence of three rotations. The angles of the rotations, which these authors represent by ψ, ω, and φ, are the Euler angles of the transformation. The first two rotations bring one over from the ecliptic plane to the equatorial plane while taking the first axis of ECRF0 over into the nodal line of the two planes; the rotation through φ is about the third axis of the ITRF and takes the first axis from the nodal line to the first axis of the ITRF. If one desires that the final frame be the principal axis frame rather than the ITRF, one needs only to replace φ by $\tilde{\varphi}=\varphi+\alpha$, where α is the longitude (in the ITRF) of the equatorial principal axis of the least moment of inertia A.

The variations of φ (or of $\tilde{\varphi}$) are closely related to LOD variations. The variations of ψ and $(-\omega)$ as a result of the variable rotation of the Earth represent the nutation–precession in longitude and obliquity, respectively. The precession here is the so-called lunisolar precession reflecting the actual motions of the equator (or equivalently, of the figure axis) and differs from the general precession, which is defined relative to the moving ecliptic and therefore includes an ecliptic motion term (see the last paragraph of Section 3.10.2.1.1).

The torque equations in the principal axis frame are Euler equations for the angular velocity components. The angular velocity components Ω_x, Ω_y, Ω_z in the principal axis frame, which Bretagnon et al. (1997, 1998) denote by (p, q, r), are related to $\dot{\psi}$, $\dot{\omega}$, $\dot{\varphi}$ through the (exact) kinematic relations

$$\begin{aligned}\Omega_x &= \dot{\psi}\sin\omega\sin\tilde{\varphi}+\dot{\omega}\cos\tilde{\varphi}\\ \Omega_y &= \dot{\psi}\sin\omega\cos\tilde{\varphi}-\dot{\omega}\sin\tilde{\varphi}\\ \Omega_z &= \dot{\psi}\cos\omega+\dot{\varphi}\end{aligned} \quad [116]$$

On introducing these expressions for the angular velocity components into the Euler equation [11], one obtains a set of coupled second-order differential equations for the Euler angles, which involve small nonlinear terms arising from the small deviations of ω and $\dot{\varphi}$ from their constant mean values. The torque components that appear on the right-hand side of the equations are in the principal axis frame. Their expressions may be obtained from the torque components referred to the ITRF, which are given in eqns [59] and [64]. For the complex combination of the equatorial components, all one needs to do is to multiply the former expression by $e^{i\alpha}$, where α is the longitude of the x-axis of the principal axis frame in the ITRF (see eqn [63]). The axial component of the torque is the same in both frames. Of course, the expressions in the preceding text have to be summed over n and m to get the torque due to the full potential of the external body.

The time dependence of the torque produced by the Moon, Sun, or planet is obtainable by using the analytic expressions given in the relevant ephemerides for the position coordinates of that body. The coupled differential equations may then be solved to any desired accuracy (given the Earth's structure parameters to sufficient accuracy) by an iterative process, which is needed because of the nonlinearities. Corrections for relativistic effects (Brumberg et al., 1992; Fukushima, 1991; Klioner, 1998; Soffel and Klioner, 1998) have to be applied to the solutions thus obtained. The final nutation series constructed in this manner by Bretagnon et al. (1997, 1998) is named SMART97.

3.10.9 Axially Symmetrical Ellipsoidal Nonrigid Earth: Torque Equations and Solutions

In the foregoing sections, we considered the wobble and nutation in some simplified models, then established a number of basic results of general applicability, and briefly dealt with treatments of the nutations of the rigid Earth. We turn now to a study of more realistic models having one of more of the following features: deformability, existence of a fluid core and SIC, etc. The torque approach will be employed for these studies.

We observe at the outset that the basic equation of motion governing the rotation variations as seen from the surface of the Earth in relation to a terrestrial frame continues to be the torque equation [8] for the whole Earth; the angular velocity $\boldsymbol{\Omega}$ appearing in the equation is that of the mantle. An additional equation is needed to take account of the rotation variations of the fluid core, which are coupled to, but not the same as, those of the mantle. Two more equations are needed when an SIC is present, as will be seen shortly.

3.10.9.1 Deformable Wholly Solid Earth

If the Earth were wholly solid, the only difference from the rigid case would be in the inertia tensor. The deformation from the direct action of the tidal potential, as well as indirect effects like the perturbations of the centrifugal resulting from Earth rotation variations, has the consequence that the inertia tensor $[\mathbf{C}]$ is no longer diagonal. So the angular momentum components become

$$\begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} A + c_{11} & c_{12} & c_{13} \\ c_{21} & A + c_{22} & c_{+} \\ c_{31} & c_{32} & C + c_{33} \end{pmatrix} \begin{pmatrix} m_x\Omega_0 \\ m_y\Omega_0 \\ (1+m_z)\Omega_0 \end{pmatrix} = \Omega_0 \begin{pmatrix} Am_x + c_{13} \\ Am_y + c_{23} \\ C(1+m_z) + c_{33} \end{pmatrix} \qquad [117]$$

with the neglect of second-order terms, which involve the product of any of the C_{ij} with m_x or m_y or m_z. The components of $\boldsymbol{\Omega}\times\mathbf{H}$ may be obtained using the previously mentioned. Its components, leaving out the second-order terms, are $(Cm_y-(Am_y+c_{23}),\ -Cm_x+(Am_x+c_{13}),\ 0)$. When these are substituted into the torque equation $\mathrm{d}\mathbf{H}/\mathrm{d}t+\boldsymbol{\Omega}\times\mathbf{H}=\boldsymbol{\Gamma}$, one obtains the following equations:

$$\begin{aligned} \Omega_0(A\dot{m}_x + \Omega_0\dot{c}_{13}) + (C-A)\Omega_0^2 m_y - \Omega_0^2 c_{23} &= \Gamma_x \\ \Omega_0\left(A\dot{m}_y + \Omega_0\dot{c}_{23}\right) - (C-A)\Omega_0^2 m_y - \Omega_0^2 c_{13} &= \Gamma_y \\ C\Omega_0\dot{m}_z + \dot{c}_{33} &= \Gamma_z \end{aligned} \qquad [118]$$

Now, we take the complex combination of the first two equations and denote $c_{13}+ic_{23}$ by $\tilde{c}_3$. Then, on using for the torque the expression [50] with $e'=0$ (for axial symmetry), we obtain the following equation:

$$A\frac{\mathrm{d}\tilde{m}}{\mathrm{d}t} - ie\Omega_0 A\tilde{m} + \frac{\mathrm{d}\tilde{c}_3}{\mathrm{d}t} + i\Omega_0\tilde{c}_3 = -ieA\Omega_0\tilde{\phi} \qquad [119]$$

For a spectral component with frequency $\sigma\Omega_0$, we have then

$$A(\sigma-e)\tilde{m}(\sigma) + (\sigma+1)\tilde{c}_3(\sigma) = -eA\tilde{\phi}(\sigma) \qquad [120]$$

In order to solve this equation for $\tilde{m}(\sigma)$, we need to know how $\tilde{c}_3$ is related to the wobble $\tilde{m}$ and to the external forcing $\tilde{\phi}$. One can show from deformation theory that the contribution to $\tilde{c}_3$ is $-A\kappa\tilde{\phi}$. Since the value of κ could be frequency-dependent, as we shall see later, we take this relation to be in the frequency domain. The parameter κ is a measure of the deformability of the Earth under forcing by $\tilde{\phi}$ and is referred to as a compliance; it is closely related to the Love number k of the unperturbed Earth:

$$\kappa = \frac{\Omega_0^2 a^5}{3GA}k = \frac{ek}{k_\mathrm{f}}, \qquad k_\mathrm{f} = \frac{3GAe}{\Omega_0^2 a^5} \qquad (121)$$

where A is the mean radius of the Earth and G is the constant of universal gravitation, $G=6.67259\times10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$. (The Love number k is defined by the statement that the incremental gravitational produced at the surface of the Earth by the redistribution of matter within the Earth as a result of the action of the lunisolar potential is k times the lunisolar potential at the surface.) The so-called fluid Love number k_f plays the same role for a hypothetical wholly fluid Earth. Other Love numbers h and l relate the tide-induced vertical and horizontal displacements at the Earth's surface to the lunisolar potential. (In these cases, the displacement at the surface is h/g times the lunisolar potential.) There are also load Love numbers $k',\ h',\ l'$, which pertain to the potential perturbation and the displacements caused by loading on the Earth's surface. The contribution from the incremental centrifugal potential (associated with the wobble $\tilde{m}$) to $\tilde{c}_3$ is $Ak\tilde{m}$, for the simple reason that the difference between the centrifugal potentials associated with the perturbed and unperturbed angular velocities $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}_0$ is identical to the tesseral potential given by eqn [31], except for the replacement of $\boldsymbol{\phi}_1,\ \boldsymbol{\phi}_2$ by $-m_x,\ -m_y$. Thus,

$$\tilde{c}_3 = Ak\left(\tilde{m} - \tilde{\phi}\right) \qquad [122]$$

On substituting this expression for $\tilde{c}_3$ in eqn [120], the equation becomes

$$[\sigma - e + (\sigma+1)k]\tilde{m}(\sigma) = -[e-(\sigma+1)k]\tilde{\phi}(\sigma). \qquad (123)$$

One sees, by setting $\tilde{\phi}(\sigma)=0$, that the frequency $\sigma_\mathrm{E}=e$ of the Euler free wobble mode of the rigid Earth is now replaced by $\sigma_1=(e-k)/(1+k)$. It turns out that k is close to $e/3\approx0.001$, and so $\sigma_1\approx(2/3)\sigma_\mathrm{E}$, meaning that the period of the free wobble of the deformable Earth with no fluid regions is approximately 50% longer than that of the rigid Earth. As for the wobble response to forcing by $\tilde{\phi}(\sigma)$, the change from the rigid case consists in a replacement of e by the frequency-dependent effective value $e-(\sigma+1)k$. It becomes clear, on recalling that $(1+\sigma)$ lies within the low-frequency band, that the modification becomes larger as a moves away from -1 but is never very large, as $|\sigma+1|$ is at most of the order of 0.1 for spectral components of any significance.

3.10.9.2 Two-Layer Earth, with Mantle and Fluid Core

3.10.9.2.1 Equations of motion

When the rotation of the mantle is not about its symmetry axis, the fluid flow within the core consists of a rotation of the fluid as a whole about yet another axis, plus a residual flow needed to make the flow at the ellipsoidal CMB normal to the boundary. The angular velocity of the rotational flow is also not along the symmetry axis; we denote it by $\mathbf{\Omega}_\mathrm{f}$. The differential angular velocity between the core and the mantle is then

$$\omega_\mathrm{f} = \mathbf{\Omega}_\mathrm{f} - \mathbf{\Omega} \qquad [124]$$

with components $\Omega_0(m_{\mathrm{f}x}, m_{\mathrm{f}y}, m_{\mathrm{f}z})$. (Superscripts or subscripts f are used to identify quantities pertaining to the fluid core. $m_{\mathrm{f}x}$ and $m_{\mathrm{f}y}$ represent the differential wobble between the core and the mantle.) The contribution $[\mathbf{C}]_\mathrm{f}\cdot\omega_\mathrm{f}$ to the angular momentum of the Earth from this differential rotation has to be added now to $[\mathbf{C}]\cdot\mathbf{\Omega}$ to get the total $\mathbf{H}$. Consequently, the $[\mathbf{C}]_\mathrm{f}\cdot(\mathrm{d}\omega_\mathrm{f}/\mathrm{d}t)$ and $\mathbf{\Omega}\times([\mathbf{C}]_\mathrm{f}\cdot\omega_\mathrm{f})$ will appear on the left-hand side of the torque equation in addition to the terms in $[\mathbf{C}]$ that we had earlier. The components of these can be readily evaluated, noting that the components of $[\mathbf{C}]_\mathrm{f}\cdot\omega_\mathrm{f}$ are $(A_\mathrm{f}m_{\mathrm{f}x}, A_\mathrm{f}m_{\mathrm{f}y}, C_\mathrm{f}m_{\mathrm{f}z})$ to the first order in small quantities. On using these, we find the new equation of motion to be

$$A\Omega_0\frac{\mathrm{d}\tilde{m}}{\mathrm{d}t} - ieA\Omega_0^2\tilde{m} + A_\mathrm{f}\Omega_0\frac{\mathrm{d}\tilde{m}_\mathrm{f}}{\mathrm{d}t} + iA_\mathrm{f}\Omega_0^2\tilde{m}_\mathrm{f} + \Omega_0\frac{\mathrm{d}\tilde{c}_3}{\mathrm{d}t} + i\Omega_0^2\tilde{c}_3 = \tilde{\Gamma} \qquad [125]$$

This is no longer an equation for $\tilde{m}$ alone: it involves $\tilde{m}_\mathrm{f}$ also.

A second equation for $\tilde{m}$ and $\tilde{m}_\mathrm{f}$ is provided by the torque equation for the fluid core alone. Unlike the equation for the whole Earth, wherein the internal interactions between its different parts were of no relevance, the equation for the core alone must include torques exerted by all bodies outside itself, namely, the mantle as well as the celestial bodies. The obvious interaction mechanism comes from the impact of the flowing fluid core on the CMB (which is nonspherical) and the resulting inertial reaction from the mantle on the core, when the axis of the global rotation of the core is not aligned with the symmetry axis. This reaction is reflected in the perturbations of the various dynamical quantities within the fluid, like the pressure, gravitational potential, and density. The combined effect of all this is an inertial torque on the core that is at the first order proportional to the differential wobble ω_f and the dynamical ellipticity e_f of the core. (e_f is defined, analogously to e, as $(C_\mathrm{f}-A_\mathrm{f})/A_\mathrm{f}$.) Mathematically, this result comes from a proof that the total torque $\mathbf{\Gamma}_\mathrm{f}$ on the fluid core can be reduced to $(\mathbf{\Omega}+\omega_\mathrm{f})\times H_\mathrm{f}$ to the first order in the surface flattening of the CMB. The torque arises from forces derived from the pressure P and the gravitational potential (of the Earth itself as well as of celestial bodies) and is represented by the integral

$$\mathbf{\Gamma}_\mathrm{f} = -\int r\times\left(\nabla P + \rho\nabla\phi_\mathrm{g}\right)\mathrm{d}V \qquad [126]$$

where the integration is over the instantaneous volume of the core. The proof is not very simple and will not be presented here (see, for instance, the annex in Mathews et al., 1991). Given this result, the torque equation for the core, which is of exactly the same form as eqn [8] except for the replacement of $\mathbf{H}$ by $\mathbf{H}_\mathrm{f}$ and $\mathbf{\Gamma}$ by $\mathbf{\Gamma}_\mathrm{f}$, reduces to

$$\frac{\mathrm{d}\mathbf{H}_\mathrm{f}}{\mathrm{d}t} - \omega_\mathrm{f}\times\mathbf{H}_\mathrm{f} = 0 \qquad [127]$$

This is the equation in reduced form for the angular momentum of the core. An equivalent form was derived by Poincaré (1910) by taking the fluid core to be homogeneous and incompressible assuming for the flow velocity in the fluid a form that is linear in x, y, z and satisfies, from the outset, the condition of being normal to the core boundary. A derivation without making such simplifying assumptions was given in elegant form by Sasao et al. (1980), using ideas based on a much more complicated derivation by Molodensky (1961).

To obtain the equation of motion for $\tilde{m}_\mathrm{f}$, we need to express the preceding equation in component form. The components of $\mathbf{H}_\mathrm{f}$ may be obtained in the same manner as those of $\mathbf{H}$:

$$\mathbf{H}_\mathrm{f} = \Omega_0\begin{pmatrix} A_\mathrm{f}(m_x + m_{\mathrm{f}x}) + c_{13}^\mathrm{f} \\ A_\mathrm{f}(m_y + m_{\mathrm{f}y}) + c_{23}^\mathrm{f} \\ C_\mathrm{f}(1 + m_z + m_{\mathrm{f}z}) + c_{33}^\mathrm{f} \end{pmatrix} \qquad [128]$$

Using these to express eqn [127] in terms of components, and then taking the complex combination of the first two components, we obtain

$$A_\mathrm{f}\Omega_0\dot{\tilde{m}} + A_\mathrm{f}\Omega_0\dot{\tilde{m}}_\mathrm{f} + i(1+e_\mathrm{f})A_\mathrm{f}\Omega_0^2\tilde{m}_\mathrm{f} + \Omega_0\dot{\tilde{c}}_{3\mathrm{f}} = 0 \qquad [129]$$

The pair of coupled eqns [125] and [129] govern the temporal variations of $\tilde{m}$ and $\tilde{m}_\mathrm{f}$. On going over to the frequency domain, the dynamical variables to be solved for are $\tilde{m}(\sigma)$ and $\tilde{m}_\mathrm{f}(\sigma)$; and c_3 and $c_{3\mathrm{f}}$ go over into $c_3(\sigma)$ and $c_{3\mathrm{f}}(\sigma)$. The last two quantities have to be written, as before, in terms of the forcing and the dynamical variables. We now have, in the frequency domain,

$$\begin{aligned}\tilde{c}_3 &= A\left[k\left(\tilde{m}-\tilde{\phi}\right)+\xi\tilde{m}_1\right]\\ \tilde{c}_{3\mathrm{f}} &= A_\mathrm{f}\left[\gamma\left(\tilde{m}-\tilde{\phi}\right)+\beta\tilde{m}_\mathrm{f}\right]\end{aligned} \qquad [130]$$

The terms proportional to $\tilde{m}_\mathrm{f}$ represent the deformations due to the centrifugal effect of the differential wobble of the core. There are now four compliances in all, representing the deformabilities of the Earth and of the core under the centrifugal forcing due to wobbles of the whole Earth or due to the differential wobbles of the core. All of them are of order 10^{-3} or smaller, though the precise values depend on the Earth model used:

$$\begin{aligned} &k\approx 1.0\times10^{-3}, \quad \xi\approx 2\times10^{-4}\\ &\gamma\approx 2\times10^{-3}, \quad \beta\approx 6\times10^{-4}\end{aligned} \qquad [131]$$

On substituting the previous expressions into the spectral components of eqns [125] and [129], we obtain the following coupled equations:

$$\begin{aligned}&[\sigma - e + (\sigma+1)k]\tilde{m}(\sigma) + (\sigma+1)(A_\mathrm{f}/A+\xi)\tilde{m}_\mathrm{f}(\sigma)\\ &\quad = -[e-(\sigma+1)k]\tilde{\phi}(\sigma)\end{aligned} \qquad [132]$$

$$\sigma(1+\gamma)\tilde{m}(\sigma) + (\sigma+1+\beta\sigma+e_\mathrm{f})\tilde{m}_\mathrm{f}(\sigma) = \sigma\gamma\tilde{\phi}(\sigma) \qquad [133]$$

An important point to note is that for $\sigma = -1$, which corresponds to zero frequency relative to the CRF, the first of the preceding equations reduces to $(\sigma-e)\tilde{m}(\sigma) = -e\tilde{\phi}(\sigma)$, which is precisely the same as for the rigid Earth. This is an example of a general property, which Poincaré named *gyrostatic rigidity*: at zero frequency in an inertial frame, the nutation of a nonrigid Earth and the associated wobble in the terrestrial frame are the same as for a rigid Earth with the same value of e.

3.10.9.2.2 Wobble normal modes

We are now in a position to determine the wobble normal modes and eigenfrequencies of the two-layer Earth by solving the homogeneous equations obtained by setting $\tilde{\phi}(\sigma) = 0$. To the lowest order in the ellipticities and compliance parameter, we find the eigenfrequencies to be

$$\sigma_1 = \sigma_{\mathrm{CW}} = \frac{A}{A_{\mathrm{m}}}(e - k)$$
$$\sigma_2 = \sigma_{\mathrm{NDFW}} = -\left(1 + \frac{A}{A_{\mathrm{m}}}(e_{\mathrm{f}} - \beta)\right) \qquad [134]$$

where A_{m} is the moment of inertia of the mantle: $A_{\mathrm{m}} = A - A_{\mathrm{f}}$. The labels CW and NDFW stand for 'CW' and 'NDFW,' respectively. The CW frequency of the two-layer Earth differs from that of the wholly solid Earth just by a factor (A/A_{m}), which is about 9/8. As for the NDFW, its frequency is $-1-1/430$ cpsd, justifying the characterization as nearly diurnal. (In nutation, this mode appears as the FCN, or more precisely, the retrograde free core nutation (RFCN), with the retrograde frequency $-(A/A_{\mathrm{m}})(e_{\mathrm{f}}-\beta) \approx -1/430$ cpsd.) This mode has a very important role to play in the forced wobbles, because of its location in the middle of the retrograde diurnal band in which the forcing frequencies lie: Its presence results in resonant enhancement of several of the prominent wobbles and the corresponding nutations. In particular, the amplitude of the retrograde annual nutation with frequency $\approx -1/366$ cpsd is enhanced by over 30% relative to that of the same nutation of the rigid Earth, because its frequency σ in the terrestrial frame is $\approx$–1–1/366 cpsd, very close to the FCN eigenfrequency.

3.10.9.2.3 Solution of the wobble equations: resonances and nutation amplitudes

Solution of the pair of eqns [132] and [133] with the forcing potential $\tilde{\phi}(\sigma)$ present yields the forced wobble $\tilde{m}(\sigma)$ of the mantle and the differential wobble $\tilde{m}_{\mathrm{f}}(\sigma)$ of the fluid core. To the first order in the compliances and ellipticities, one finds that

$$\tilde{m}(\sigma)\frac{(\sigma+1)[eA + \gamma\sigma A_{\mathrm{f}} - (1+\sigma)kA]}{A_{\mathrm{m}}(\sigma-\sigma_1)(\sigma-\sigma_2)}\tilde{\phi}(\sigma) \qquad [135]$$

The two factors appearing in the denominator give rise, not surprisingly, to resonances associated with the eigenfrequencies σ_1 and σ_2. The expression for the amplitude $\tilde{m}_{\mathrm{f}}(\sigma)$ of the differential wobble of the fluid core may be similarly written down.

The amplitude of the nutation associated with the wobble [135] may be obtained by applying the kinematic relation $\tilde{\eta}(v) = -\tilde{m}(\sigma)/v$, with $v = 1+\sigma$. Given the values of the Earth parameters, numerical evaluation may be done using $\tilde{\phi}(\sigma)$ obtained from a tide table.

Alternatively, one may take the transfer function $T_{\mathrm{w}}(\sigma) = T_{\mathrm{n}}(v)$ obtained by dividing the preceding expression by the rigid Earth wobble amplitude and $\tilde{m}_{\mathrm{R}}(\sigma) = e\tilde{\phi}(\sigma)/(e-\sigma)$ by multiplying it by rigid Earth nutation amplitude for the frequency $v = \sigma + 1$, for example, from REN2000 of Souchay et al. (1999), to obtain the nutation amplitude for the nonrigid Earth. In this case, one does not need to use the amplitude of the tidal potential. This is the approach adopted by Mathews et al. (2002).

3.10.9.3 Coupling of the Core and the Mantle at the CMB

It was mentioned in the beginning of **Section 3.10.9.2.1** that the pressure and gravitational forces acting on the fluid core result in an effective inertial coupling between the core and the mantle, which is proportional to the dynamical ellipticity of the core. No other couplings were included in the equations of that section. The efforts to refine nutation theory in order to achieve close agreement between theoretical predictions and the high-precision observational data available now have led to the investigation of other coupling mechanisms that might be significant, especially EM and viscous couplings at the CMB.

The magnetic fields generated in the fluid core by the geodynamo mechanism extend into the mantle across the CMB. As differential motions between the solid and fluid sides of the CMB take place as a result of the differential wobble between the mantle and the fluid core, the field lines remain 'frozen' in the fluid because of the very high conductivity of the fluid and move with it, and the continuing portion of the lines on the mantle side tends to sweep through the mantle material. But if there is a highly conducting layer at the bottom of the mantle, the attempted motion of the field lines induces currents, which, on the one hand, interact with the magnetic field and produce a Lorentz force on the matter and produce a magnetic field that increments the main magnetic field $\mathbf{B}_0$, which existed in the absence of the wobbles. The Lorentz forces all through the conducting layer result in a torque on the mantle and an equal reaction by the mantle on the fluid core. The strength of this torque that couples the two regions depends on the conductivity of the mantle layer, on the frequency of the wobble causing the relative motion, and on the strength and configuration of the main field over the surface of the CMB. If the field is sufficiently weak, the torque is 45° out of phase with the wobble and has a strength proportional to the mean squared of the radial component B_{r} of $\mathbf{B}_0$ over the boundary surface; otherwise, the proportionality is only approximate, and the phase difference too depends on the overall strength and configuration of the field. In any case, the equatorial torque on the fluid core can be expressed as

$$\tilde{\Gamma}^{\mathrm{CMB}} = -i\Omega_0^2 A_{\mathrm{f}} K^{\mathrm{CMB}} \tilde{m}_{\mathrm{f}} \qquad [136]$$

where K^{CMB} is a complex coupling constant depending on the physical quantities mentioned earlier. The viscosity of the fluid core is widely believed to be negligible; in case it is not, both EM and viscous effects can be encompassed under an expression of the same form as in the preceding text, with K^{CMB} becoming dependent also on the coefficient of viscosity. The mathematical derivation of the expression for K^{CMB} in terms of the relevant physical parameters will not be presented here. The interested reader may refer to Buffett (1996a, 1996b, 2010), Buffett et al. (2002), Mathews and Guo (2005), Koot et al. (2008, 2010), and Koot and Dumberry (2011).

A simple reflection will show that the effect of these couplings can be incorporated into the theory of **Section 3.10.9.2** simply by adding K^{CMB} to the coefficient $\tilde{m}_{\mathrm{f}}$ in eqn [133].

This replacement does not affect σ_{CW} of the CW. But σ_{NDFW} does get modified; the change consists in the replacement of $(e_{\mathrm{f}}-\beta)$ in eqn [134] by $(e_{\mathrm{f}}-\beta+K^{\mathrm{CMB}})$, making it complex. Consequently, the NDFW resonance acquires a width.

3.10.9.4 Anelasticity and Ocean Tide Effects

It has been known, from seismological data on the oscillational normal modes of the Earth and on the propagation of seismic waves, that the response of the material of the Earth's mantle to stresses is not strictly elastic. A small part of the response is not instantaneous; it decays over a finite though short period of time. This part of the response is said to be 'anelastic.' When transformed to the frequency domain, the anelastic response manifests itself through a frequency dependence of the rheological parameters like the Young's modulus and rigidity modulus; in addition, these parameters become complex. Wahr and Bergen (1986) showed that anelasticity effects are in the range of tenths of mas on the amplitudes (both in phase and out of phase) of a number of spectral components of nutation. The anelasticity model that they employed would produce increments $\Delta^{\mathrm{AE}}k$, $\Delta^{\mathrm{AE}}\gamma$, etc., proportional to

$$\left\{1-\left(\frac{\omega_{\mathrm{m}}}{\omega}\right)^{\alpha}\right\}\cot\frac{\alpha\pi}{2}+i\left(\frac{\omega_{\mathrm{m}}}{\omega}\right)^{\alpha} \qquad [137]$$

to the various compliances. Here, ω is the frequency of the tidal potential responsible for the nutation of interest and ω_{m} is a reference frequency in the band of frequencies of the seismic waves used for estimating the elastic rheological parameters; it is taken to be equivalent to a period of 1, 200, or 300 s in different models (Dziewonski and Anderson, 1981; Widmer et al., 1991). The power law index α is typically considered to be around 0.15 (between 0.1 and 0.2). Evaluation of the proportionality constant has to be done by integration of the deformation equations for the Earth model.

The computation of the effects of loading of the Earth's crust by ocean tides, as well as those of ocean tidal currents, was first done by Wahr and Sasao (1981) using the theoretical formalism of Sasao and Wahr (1981). They included the effect of ocean tides in eqns [125] and [129] by adding the ocean tidal contributions $\tilde{c}_3^{\mathrm{o}}$ and $\tilde{c}_{3\mathrm{f}}^{\mathrm{o}}$ to $\tilde{c}_3$ and $\tilde{c}_{3\mathrm{f}}$, respectively, where the added quantities reflect the changes in the inertia tensors of the whole Earth and of the fluid core due to the deformation produced by the tidal redistribution of ocean mass, together with the equivalent of the angular momentum content of the tidal currents in the oceans. Using estimates of these quantities obtained with the help of ocean tidal models, they found increments of up to about 1 mas to the amplitudes of the leading nutations. Mathews et al. (2002) expressed $\tilde{c}_3^{\mathrm{o}}$ and $\tilde{c}_{3\mathrm{f}}^{\mathrm{o}}$ in terms of incremental compliances $\Delta^{\mathrm{OT}}k$ and $\Delta^{\mathrm{OT}}\gamma$. These were computed using empirical formulas (separately for the mass redistribution part and for the tidal current part) representing their frequency dependences; the parameters in the formula were estimated with the help of available data, in particular, the ocean tide angular momentum data provided by Chao et al. (1996) for four prominent tidal components; the reference in the preceding text may be consulted for details. The results that Mathews et al. (2002) obtained for the increments to the nutations from ocean tidal (starting from the results of Chao et al., 1996) as well as anelasticity effects are close to those of earlier authors, for example, Wahr and Sasao (1981) and Wahr and Bergen (1986).

3.10.9.5 Inclusion of SIC

In the presence of the SIC, the fluid layer is more appropriately referred to as the FOC; and all the quantities labeled by the subscript f now refer to the FOC. Quantities pertaining to the SIC will be identified by the subscript s. The SIC has a moment of inertia A_{s} that is only 1/1400 of that of the whole Earth, and so it would seem that it could not influence the rotation of the Earth as a whole to any significant extent. (Note that the mantle moment of inertia is $A_{\mathrm{m}}=A-A_{\mathrm{f}}-A_{\mathrm{s}}$ now.) Recent investigations, beginning with Mathews et al. (1991), De Vries and Wahr (1991), Dehant et al. (1993), and Legros et al. (1993), have revealed however that a new NDFW appears as a result of the role of the inner core in the dynamics and that the associated resonance has an appreciable effect on the amplitudes of a few nutations.

The inner core is floating within the fluid core, and its symmetry axis is free to go out of alignment with that of the mantle. Consequently, two new equations enter the picture. One is the torque equation governing the time variations of the differential wobble $\tilde{m}_{\mathrm{s}}$ of the SIC, that is, of the complex combination of the equatorial components of $(\mathbf{\Omega_s}-\Omega)/\Omega_0$; the other is a kinematic equation relating $\tilde{m}_{\mathrm{s}}$ to $\tilde{n}_{\mathrm{s}}$, the complex version of equatorial components of the offset $(\mathbf{i_s}-\Omega_0/\Omega_0)$ of the inner core symmetry axis $\mathbf{i}_{\mathrm{s}}$ from that of the mantle. Thus, one has a set of four equations, which couple the variables $\tilde{m}$, $\tilde{m}_{\mathrm{f}}$, $\tilde{m}_{\mathrm{s}}$, $\tilde{n}_{\mathrm{s}}$ to one another.

The torque equation involves gravitational and other torques acting on the inner core. The gravitational torque is made up of the torque exerted by the celestial body and a torque arising from the misalignment between the ellipsoidal inner core and the rest of the Earth (which is also of ellipsoidal symmetry except for a slight realignment of the inner core boundary (ICB) because of the tilt of the inner core). Both parts are proportional to the ellipticity e_{s} of the SIC; the internal gravitational part is proportional also to the density contrast between the SIC and the fluid core just outside the ICB and involves a gravitational coupling constant α_{g} between the SIC and the rest of the Earth. An additional torque arises from EM (and possibly viscous) couplings at the ICB. It may be expressed as $\Gamma^{\mathrm{ICB}}=-i\Omega_0^2A_{\mathrm{s}}K^{\mathrm{ICB}}\tilde{m}_{\mathrm{s}}$; the complex ICB coupling constant K^{ICB} appears in the torque equations for both regions coupled by it, namely, the FOC and the SIC.

The expression for $\mathbf{H}$ now gets an additional contribution from $\tilde{m}_{\mathrm{s}}$, and the moments of inertia of all the regions bear the imprint of the misalignment $\tilde{n}_{\mathrm{s}}$. Furthermore, $\tilde{c}_3$, $\tilde{c}_{3\mathrm{f}}$, and the new $\tilde{c}_{3\mathrm{s}}$ all involve terms proportional to $\tilde{m}_{\mathrm{s}}$ besides those in $\tilde{m}$ and $\tilde{m}_{\mathrm{f}}$, and a number of new compliances enter the picture.

3.10.9.5.1 Solution of the equations for the three-layer Earth

Solution of the four coupled linear equations with the tidal potential $\tilde{\phi}$ set equal to zero yields two new normal modes in addition to the earlier two. One has a frequency depending on e_{s} and the ratio of the density contrast mentioned earlier to the mean density of the inner core; it is very close to -1 cpsd.

The corresponding nutation, with a very small prograde frequency, is referred to as the PFCN or FICN. The resonance associated with this mode has a nontrivial effect on forced nutations at nearby frequencies. The frequency of the other wobble normal mode, called the inner core wobble (ICW) mode, is almost an order of magnitude smaller than that of the CW, and its effect is quite ignorable.

The forced wobbles $\tilde{m}$, $\tilde{m}_f$, $\tilde{m}_s$ as well as the offset $\tilde{n}_s$ of the inner core axis for any excitation frequency σ are found by solving the set of four equations with $\tilde{\phi}(\sigma)$ taken to be nonzero. Division of $\tilde{m}(\sigma)$ by the rigid Earth amplitude $\tilde{m}_R(\sigma)$ from eqn [55] gives the transfer function $T_w(\sigma) = T_n(\nu)$, and multiplication of this quantity by the rigid Earth amplitude $\tilde{\eta}_R(\nu)$ of the nutation of frequency $\nu = \sigma + 1$ finally gives the nutation amplitude $\tilde{\eta}(\nu)$ for the nonrigid Earth.

The magnitude to which the resonance associated with each normal mode affects the amplitudes of forced nutation for a given Earth model may be displayed through the resonance expansion of the transfer function for that model. For the three-layered Earth, which has four eigenmodes, the expansion takes the form

$$T_w(\sigma) = R + R'(1+\sigma) + \frac{R_{CW}}{\sigma - \sigma_{CW}} + \frac{R_{FCN}}{\sigma - \sigma_{FCN}} + \frac{R_{FICN}}{\sigma - \sigma_{FICN}} + \frac{R_{ICW}}{\sigma - \sigma_{ICW}} \qquad [138]$$

The form of the expansion in the preceding text presumes that the dynamical flattening used in the construction of the rigid Earth nutation series accurately represents the flattening of the real Earth. It was found by Mathews et al. (2002), on requiring the best fit of the results of nonrigid Earth nutation theory to observational data, that a slightly different value of the flattening parameter is called for. It is necessary then to modify the transfer function accordingly (see Mathews et al., 2002, for more detail).

3.10.9.6 Confronting Theory with Observations

The main aim of the nutation theory is to provide theoretical models from which the nutation amplitudes/coefficients can be computed accurately, making it possible to match the observationally estimated series by the computed nutation series at about the same level as the uncertainties in the observational estimates (a couple of tenths of mas) when observational inputs are used for the inherently unpredictable contributions to variations in orientation related to excitations of geophysical origin (atmospheric and nontidal ocean effects, the free wobbles and nutations, etc.). Such a theoretical model, combined with an accurate model for LOD variations, can then be used to make accurate predictions for the Earth's orientation in space at future epochs.

For highly accurate computations to be possible, one needs sufficiently accurate values of the relevant Earth parameters. The other essential input, namely, accurate values of the amplitudes of the components of the tidal potential or, alternatively, the values of the rigid Earth nutation amplitudes, is provided by existing theories. Given all these, the precession and low-frequency nutation due to the action of the degree 2 tesseral gravitational potential of the solar system bodies at the first order in the potential may be calculated to high accuracy using the theory outlined in Section 3.10.9 as applied to a three-layer Earth, taking into account the effects dealt with in Section 3.10.9.3 and subsequent sections. However, to get a complete accounting of the nutations, one needs to add some other contributions:

(i) Contributions to nutations and precession from torques of the second order in the perturbing potential, which arise from the action of each of the three parts (zonal, tesseral, and sectorial) of the potential on the increment to the density function $\rho(\mathbf{r})$ that is produced by the tidal deformation produced by the other two parts. These have been evaluated in part by Folgueira et al. (1998a, 1998b), Souchay and Folgueira (2000), and Mathews et al. (2002). A complete calculation done since by Lambert and Mathews (2005) shows that mutual cancelations limit the total effect to about 30 μas on the 18.6-year nutation (about a third of the earlier result) and that the effect on precession is practically nil.
(ii) Geodesic precession and nutation, arising from the difference between kinematic and dynamical quantities in general relativity. This relativistic contribution is about -19.2 mas year^{-1} to the precession and about -30 and $+30$ μas, respectively, to the prograde and retrograde annual nutations; the effect on other nutations is much smaller.
(iii) The contribution from atmospheric pressure tides generated by solar thermal effects that is of the order of 0.1 mas at the prograde annual period and possibly nonnegligible at a couple of other periods.
(iv) Contributions of the order of 0.1 mas or smaller to a couple of low-frequency nutations from the degree 3 tidal potential acting on J_3 and about 25 μas year^{-1} to precession from the degree 4 potential acting on J_4 (see, for instance, Souchay et al., 1999; Williams, 1994).

Other variations in Earth rotation with spectra outside the low-frequency band in the terrestrial frame are produced by potentials of degrees >2 and of orders $m \neq 1$; they are grouped under polar motions of the CIP and are not considered as nutations. For a treatment of these, see Mathews and Bretagnon (2003).

Contributions from the effects (i)–(iv) in the preceding text form part of the observed nutation and precession. These contributions have to be subtracted out from the observational estimates of the precession rate and the coefficients/amplitudes of low-frequency nutations, before comparing to the results of computations based on the theory of Section 3.10.9.

The values of the Earth parameters appearing in the theory are obtained by computations based on an Earth model. Only very rough estimates can be made of the EM coupling constants K^{CMB} and K^{ICB}, assuming some model for the structure and strengths of the core magnetic fields crossing the CMB and ICB. If, on using these parameter values, computational results on the precession rate and nutation amplitudes turn out to have unacceptably large residuals relative to their observational estimates, one is forced to conclude that the value(s) of one or more of the Earth parameters do not reflect the properties of the real Earth accurately enough, implying that the detailed Earth model used for computation of the parameter values needs to be suitably modified or refined. One has then the option of adjusting the values of those parameters to which the offending residuals are most sensitive. Such a course was adopted by Gwinn et al. (1986) and Herring et al. (1986), and it led to the inference that the ellipticity e_f of the core has to be about 5% higher than its Earth model value. This deduction was based on the observation that the retrograde annual nutation, in which a large residual of about 2 mas was found, is strongly influenced by the FCN resonance as the FCN eigenfrequency is quite close to the retrograde annual and that a rather small adjustment of e_f would change the eigenfrequency enough to eliminate the residual. The adjustment needed to e_f has since been modeled as a result of mantle convection, which causes the flattening of the CMB to deviate from the hydrostatic equilibrium value (Defraigne et al., 1996), an effect that is not taken into account in Earth models constructed from radial seismic data taken together with the assumption of hydrostatic equilibrium. It had been noted even earlier by Wahr (1981), from observational data on the precession rate, that the ellipticity e of the whole Earth has to be higher than the hydrostatic equilibrium value by over 1%.

In recent versions of torque equations, which take account of the inner core as well as the couplings at the CMB and the ICB between the fluid core and the adjoining solid regions, there are parameters like K^{CMB} and K^{ICB} for which no inputs are available from Earth models though the estimates of the mean squared magnetic field at the CMB inferred from satellite measurements of magnetic fields outside the Earth make an initial estimate for K^{CMB} possible. Confronting nutation theory with observational data provides a means of obtaining estimates with relatively small uncertainties for such parameters.

In constructing the nutation model MHB2000 on which the current IAU model IAU2000A is based, Mathews et al. (2002) employed a least squares procedure to obtain the best fit of the results of their theory to estimates of nutation and precession (see Herring et al., 2002) from a long VLBI data set, which was then up-to-date. Before the fitting, the small contributions listed under items (i)–(iv) in the preceding text were subtracted from the observational estimates, for reasons explained earlier. The least squares process involves repeated adjustments and optimization of the values of a few Earth parameters, namely, the ellipticities e, e_f, compliances κ, γ, the imaginary part of K^{CMB}, and the real and imaginary parts of K^{ICB}, while retaining values based on the PREM Earth model for all the other parameters. (The real part of K^{CMB} appears in combination with e_f in the equations of the theory, and the two could not be independently estimated. Instead, a theoretical relation between the real and imaginary part of K^{CMB} has been used, and e_f only has been estimated.) The theoretical values for the fit were obtained from the solution of the equations for the three-layer Earth as outlined in **Section 3.10.9.5.1**, including the various effects considered in the sections preceding it. The rigid Earth nutation amplitudes used to multiply the transfer function were taken by these authors from the REN2000 series of Souchay et al. (1999). In carrying out the numerical evaluation of the nutation amplitudes and the precession rate, an a priori set of values was employed for the Earth parameters occurring in the equations. The values of the selected Earth parameters were optimized by a fitting process, which seeks a least squares minimization of the residuals between the observational estimates and the theoretically computed values. The process was iterative in view of the nonlinear dependence of the nutation amplitudes on the parameters being optimized. Within each step of the iterative process, another iterative loop was necessary. The reason was that the strong frequency dependence of the ocean tide increments $\triangle^{\mathrm{OT}}\kappa$, $\triangle^{\mathrm{OT}}\gamma$, etc., to the compliance parameters that appear in the set of wobble equations was not known a priori; it had to be determined through an ocean response model involving free parameters as well as frequency-dependent body tide and load Love numbers, which depend on the wobbles, which, in turn, have to be found by solving the wobble equations, and these equations involve the incremental compliances just mentioned, which we are trying to determine. The ocean responses known observationally for a very small number of tidal frequencies served as the input in this inner loop. In this process, the mutual influences of nutation–wobble, solid Earth deformation, and ocean tidal responses all come into full play in determining each type of response.

The final results are a set of optimized values for the Earth parameters chosen for adjustment by the process of arriving at the 'best fit' between the numbers from theory and observational data, on the one hand, and the MHB2000 nutation series on the other. The series is obtained from the nutation series that is output at the end of the iteration process (i.e., from the equations containing the optimized values for the adjusted parameters) by adding the small contributions, which had been subtracted out earlier.

The best-fit values of e and e_f are higher than their hydrostatic equilibrium values; these are interpreted as effects of mantle convection, as mentioned earlier. The excess (nonhydrostatic) part of e_f corresponds to an extra difference of a little under 400 m between the equatorial and polar radii of the CMB. The estimates obtained for K^{CMB} and K^{ICB} call for higher root-mean-squared (rms) values for the magnetic fields at the CMB and ICB than what are suggested by geodynamo models in general and also, in the case of the CMB, higher than the extrapolated values from satellite observations of magnetic fields along the satellite's orbit. No independent observations are available for the fields at the ICB.

If the observed complex coupling constant K^{CMB} is assumed to result from a purely EM coupling at the CMB, then it can be explained by an rms strength of the magnetic field at the CMB of the order of 0.7 mT, if the electrical conductivity of the outer core and of the lowermost mantle is fixed to $5\times10^{5}\ \mathrm{S\ m^{-1}}$ (Buffett et al., 2002; Koot et al., 2010). For lower values of the conductivities, the rms strength of the field must be higher. From observations of the surface magnetic field, the large-scale components of the field (spherical harmonic degrees smaller than 14) at the CMB are known to have an rms strength of the order of 0.3 mT. Therefore, nutation observations suggest that a large part of the energy of the magnetic field at the CMB is contained in the small-scale components (harmonic degrees larger than 14), which are unobservable at the Earth's surface.

The possibility that the coupling constants might include the effect of viscous drag at the boundaries of the fluid core has been considered by Mathews and Guo (2005), Deleplace and Cardin (2006), Koot et al. (2010), and Koot and Dumberry (2011). These studies of viscomagnetic coupling at the CMB show that the viscous torque becomes significant only for values of the outer core viscosity of the order of $10^{-2}\ \mathrm{m^2\ s^{-1}}$.

However, these values are in disagreement with inferences obtained from other sources (such as laboratory experiments on iron alloys at high pressure and temperature and theoretical 'ab initio' computations; see Alfè et al., 2000; Rutter et al., 2002), even if the viscosity is assumed to be an effective eddy viscosity due to turbulent motions (Buffett and Christensen, 2007). For realistic values of the outer core viscosity (i.e., lower than 10^{-4} m^2 s^{-1}), the viscous coupling does not contribute significantly to the observed dissipation.

Recently, another mechanism has been proposed to explain the observed coupling constant at the CMB. This mechanism is a variant of the EM coupling that requires the existence of a stratification of the light elements at the top of the core and of a topography at the CMB (Buffett, 2010). If the stratification at the top of the core is strong (with a buoyancy frequency $N=0.09$ s^{-1}), this mechanism is able to explain the observed dissipation with an rms strength of the magnetic field at the CMB of 0.5 mT, with a lowermost mantle conductivity of 10^3 S m^{-1} and a CMB topography with a height of 120 m and horizontal length scale of 10^5 m.

In the nutation model adopted by the IAU in 2000, the only topographic feature that is considered in the nutation computation is the flattening of the CMB and the hydrostatic flattening of ICB. In reality, however, the CMB shape is bumped. Wu and Wahr (1997) had computed the effects of the topography on the nutation and had shown that some of the topography coefficients may induce large contributions (at a few tenths of mas level). This is due to the fact that some of the nutation frequencies may be close to inertial waves frequencies and are thus in resonance (amplified). However, these computations do not consider the presence of an inner core, which may alter the resonance frequencies.

3.10.10 Nutation–Precession from the Displacement Field

3.10.10.1 Displacement Field Approach

Perturbation by the tidal potential produces displacements of elements of matter from their equilibrium positions in the otherwise uniformly rotating Earth. The field of these displacements over the whole interior of the Earth may be analyzed into deformational and rotational parts; the first part is manifested as Earth tides and the second as rotation variations. The formulation and solution of the equations that govern the displacement field enable one to determine the precessional and nutational motions from the rotational part of the displacement field.

The approach of computing Earth orientation variations from the equations governing the field of displacements produced within the Earth by the tidal potential has been developed by Smith (1974), Wahr (1981), Dehant (1987a, 1987b), Dehant (1990), and Rogister (2001). This approach relies on detailed seismic models of the matter density and rheological parameters as functions of position within the Earth's interior; it can yield only accuracies commensurate with those of available Earth models.

3.10.10.1.1 Reference system used

The reference frame used here has its origin at the geocenter (center of mass of the Earth) and is tied to the hydrostatic equilibrium Earth; it rotates with the uniform velocity $\mathbf{\Omega}_0=\Omega_0\hat{e}_z$. It is thus different from the reference frame used in the torque approach.

3.10.10.1.2 Earth model parameters and unknowns used

Consider first a spherical Earth model. In this case, the physical parameters depend only on the radius r. The following are the relevant quantities:

- $\rho_0(r)$: the mean density on a sphere of radius r
- $\mu_0(r)$: the mean rigidity modulus on a sphere of radius r (it is 0 in the fluid part)
- $\kappa_0(r)$: the mean incompressibility modulus on a sphere of radius r
- $\lambda_0(r)=\kappa_0(r)-2/3(\mu_0(r))$
- $\phi_0(r)$: mean initial self-gravity potential on a sphere of radius r

$\lambda_0(r)$ and $\mu_0(r)$ are called Lamé parameters; $\phi_0(r)$ is determined by the density function

$$\phi_0(r) = -G\int_{V_{\text{in}}}\frac{\rho_0(r')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V_{\text{in}} \qquad [139]$$

where V_{in} is the volume included in the sphere of radius r. As the Earth in these theories is considered as being in equilibrium under rotation at a constant rate, it has an equatorial bulge that can be computed from the expression of the effect of the rotation at a long timescale (hydrostatic equilibrium). In such equilibrium, the equal rheological properties are constant on any self-gravity equipotential surface. For such an ellipsoidal rotating Earth of flattening $\epsilon(r)$, the preceding parameters depend not only on the radius but also on the colatitude θ, and one must add the centrifugal potential to the self-gravitational potential $\Phi_0(r)=\phi_0(r)+\psi_0(r)$. The centrifugal force related to the rotation $\mathbf{\Omega}_0$ is the negative gradient of the centrifugal potential, which is $-\frac{1}{2}\left[\mathbf{\Omega}_0^2 r^2-(\mathbf{\Omega}_0\cdot\mathrm{r})^2\right]$ It may be expressed as $-\nabla[\psi_0(r)+\psi_1(r)P_2(\cos(\theta))]$ where $\psi_0(r)=\frac{-\Omega^2 r^2}{3}$ at a distance r from the center of the Earth.

Let us take the density as an example. It can be written at the first order in the flattening:

$$\rho(r,\theta)=\rho_0(r)+\rho_2(r)P_2(\cos(\theta)) \qquad [140]$$

where

$$\rho_2(r)=\frac{2}{3}\varepsilon(r_0)r_0\frac{\mathrm{d}\rho_0}{\mathrm{d}r} \qquad [141]$$

and P_2 is the Legendre polynomial of degree 2. One can write down similar expressions for the increments to the Lamé parameters $\lambda_0(r), \mu_0(r)$ and to $\Phi_0(r)$ in terms of $\mu_2(r), \lambda_2(r)$ and $\Phi_2(r)$.

The parameters in the preceding text are given as input to the deformation equations that will be integrated to obtain the nutations. The unknowns of our system are (1) the components of the vector field $s(\mathbf{r}, t)$ representing the displacements of the points $\mathbf{r}$ in the Earth caused by the gravitational action of the external body, (2) the components of the stress tensor $\mathbf{T}(\mathbf{r}, t)$ (or more precisely in this context, the components of $\mathbf{T}(\mathbf{r}, t)\bullet\hat{e}_r$, the stress components on a surface element normal to the radius vector), and (3) the mass redistribution potential Φ_1^{E} induced by the deformational response to the external potential.

3.10.10.1.3 Basic equations

3.10.10.1.3.1 Poisson equation

The Poisson equation relates the mass redistribution potential to the displacement field, more specifically to the dilatation and the density gradient:

$$\nabla^2 \Phi_1^{\mathrm{E}} = -4\pi G \nabla \cdot [(\rho_0 + \rho_2 P_2)\mathrm{s}] \quad [142]$$

3.10.10.1.3.2 Stress–strain relationship

The stress–strain relationship expresses the incremental stress tensor in terms of the deformation:

$$\mathbf{T} = (\lambda_0 + \lambda_2 P_2)(\nabla \cdot \mathrm{s})\mathbf{I} + (\mu_0 + \mu_2 P_2)\left(\nabla \cdot \mathrm{s} + (\nabla \cdot \mathrm{s})^{\mathrm{T}}\right) \quad [143]$$

where **I** is the unit matrix and where superscript T indicates that we must take the transpose of the matrix.

3.10.10.1.3.3 Motion equation

This is the equation relating the forces acting on a body and its deformations. It involves the internal stress, the self-gravity, the rotation of the reference frame, the external gravitational potential $\boldsymbol{\Phi}^{\mathrm{ext}}$, and the mass redistribution potential:

$$\nabla \cdot \mathbf{T} = (\rho_0 + \rho_2 P_2)(\eta_1 + \eta_2 + \eta_3) \quad [144]$$

where the η_i are defined by

$$\eta_1 = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \nabla \Phi_1^{\mathrm{E}} + \nabla \Phi^{\mathrm{ext}} - (\nabla \cdot s)\nabla \Phi_0 + \nabla(s \cdot \nabla \Phi_0) \quad [145]$$

$$\eta_2 = 2\boldsymbol{\Omega} \wedge \frac{\mathrm{d}s}{\mathrm{d}t} \quad [146]$$

$$\eta_3 = -(\nabla \cdot s)\nabla(\Phi_2 P_2) + \nabla(\mathrm{s} \cdot \nabla(\Phi_2 P_2)) \quad [147]$$

3.10.10.1.3.4 Boundary conditions

The quantities appearing in the previous set of equations have to obey boundary conditions appropriate to the boundaries involved. With the use of the generic symbol $\hat{\mathbf{n}}$ for the unit normal to any boundary surface, the boundary conditions are the following:

1. Continuity of $\hat{\mathbf{n}} \cdot \mathbf{s}, \hat{\mathbf{n}} \cdot \mathbf{T}, \Phi_1$ and $\hat{\mathbf{n}} \cdot (\nabla \Phi_1 + 4\pi G \rho s)$ across every boundary, with $\hat{\mathbf{n}} \cdot \mathbf{T}$ required to vanish at the outer (free) surface, assuming that there is no surface loading.
2. Continuity of s across any welded boundary between two solid regions.
3. Conditions at fluid–solid boundaries: Inviscid fluids cannot support shear stresses. Consequently, the part of $\hat{\mathbf{n}} \cdot \mathbf{T}$ that is perpendicular to $\hat{\mathbf{n}}$ vanishes on the fluid side of the boundary, so it must vanish on the solid side too, in view of the previously stated continuity condition on $\hat{\mathbf{n}} \cdot \mathbf{T}$. Similarly, if the core fluid has no viscosity, it can 'glide' along the solid boundary, making it possible for the components of s that are tangential to the boundary to be discontinuous across the boundary.
4. Conditions at $r = 0$: Except for nonzero stress at the center of the Earth, all the variables ($\boldsymbol{\Phi}_1$, $\nabla \boldsymbol{\Phi}_1$, and s) vanish at the center.

3.10.10.1.4 Generalized spherical harmonics (GSH) expansions: radial functions

Solution of the equations of the problem is facilitated by reducing the equations to those for individual spectral components of the variables governed by the equations. For a spectral component of frequency σ cpsd, the time dependence appears through the factor $e^{i\sigma\Omega_0 t}$, and therefore, the time derivative $\partial/\partial t$ appearing in eqns [145] and [146] may be replaced by $i\sigma\Omega_0$. The fundamental eqns [142]–[144] then become partial differential equations in the spatial variables only.

As a second step, one projects these equations on to a basis ($\hat{e}_-$, $\hat{e}_0$, $\hat{e}_+$) related to the classical basis vectors ($\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\lambda$) for a reference frame appropriate for spherical coordinates. The unit vectors ($\hat{e}_-$, $\hat{e}_0$, $\hat{e}_+$) constitute the canonical basis defined by

$$\begin{aligned} \hat{e}_- &= \frac{1}{\sqrt{2}}(\hat{e}_\theta - \mathrm{i}\hat{e}_\lambda) \\ \hat{e}_0 &= \hat{e}_r \\ \hat{e}_+ &= -\frac{1}{\sqrt{2}}(\hat{e}_\theta + \mathrm{i}\hat{e}_\lambda) \end{aligned} \quad [148]$$

This basis is used in conjunction with the GSH D^l_{mn}, which are functions of the colatitude θ and the longitude ϕ and which depend on three integer numbers: l in the interval [0, $+\infty$], called the degree, m in [$-l$, $+l$], called the order, and n, determined by the tensor quantity considered in [$-l$, $+l$] (n is in [-1, $+1$] for a vector and in [-2, $+2$] for a tensor, etc.) (see also **Annex B** where canonical decompositions of vector, scalar, and tensor fields are illustrated). Then, the coefficients of any particular GSH in the expansions of the fields s, Φ_1^{E}, etc., and hence in the various terms of the earlier mentioned partial differential equations, are functions of r only; when the expansions are used, these equations reduce to coupled ordinary differential equations for the radial functions. For the displacement field s, for instance, one writes the expansion

$$\begin{aligned} \mathrm{s}(r,\theta,\lambda) &= s^- \hat{e}_- + s^0 \hat{e}_0 + s^+ \hat{e}_+ = \begin{pmatrix} s_- \\ s_0 \\ s_+ \end{pmatrix} \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \begin{pmatrix} S_l^{m_-}(r) D^l_{m_-}(\theta,\lambda) \\ S_l^{m_0}(r) D^l_{m_0}(\theta,\lambda) \\ S_l^{m_+}(r) D^l_{m_+}(\theta,\lambda) \end{pmatrix} \end{aligned} \quad [149]$$

We may reexpress eqn [149] in a form which makes explicit the fact that the vector field s is made up of three geometrically distinct parts. For this purpose, define $U_l^m(r), V_l^m(r), W_l^m(r)$ by

$$U_\ell^m(r) = S_\ell^{m0}(r) \quad [150]$$

$$V_\ell^m(r) = S_\ell^{m+}(r) + S_\ell^{m-}(r) \quad [151]$$

$$W_\ell^m(r) = S_\ell^{m+}(r) - S_\ell^{m-}(r) \quad [152]$$

Similarly, the decomposition of the radial tension is given by

$$\mathbf{T} \cdot \hat{e}_r = \sum_{l=0}^{\infty} \sum_{m=-l}^{\infty} \begin{pmatrix} T_{l0}^{m-} D^l_{m-} \\ T_{l0}^{m0} D^l_{m0} \\ T_{l0}^{m+} D^l_{m+} \end{pmatrix} \quad [153]$$

In terms of these quantities, the expansion of s goes over into

$$s(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}U_l^m D_{m0}^l(\theta,\phi)\hat{e}_0 +\frac{1}{2}V_l^m\left[D_{m-}^l(\theta,\phi)\hat{e}_+ + D_{m+}^l(\theta,\phi)\hat{e}_-\right] +\frac{1}{2}W_l^m\left[D_{m-}^l(\theta,\phi)\hat{e}_+ - D_{m+}^l(\theta,\phi)\hat{e}_-\right] \quad [154]$$

We observe that the first of the three terms in the preceding text is a radial vector field, since $\hat{e}_0$ is simply the unit vector $\hat{\mathbf{r}}$ in the radial direction. The second and third terms are transverse to the radial direction; the former (like the radial part) is deformational and is called the spheroidal part of $s(r, \boldsymbol{\theta}, \boldsymbol{\phi},)$, while the latter is purely rotational and is called the toroidal part. (The two deformational parts may more precisely be designated as the radial and transverse spheroidal fields.) Any other vector field may also be similarly decomposed to reflect the geometric characters of the three different parts. For instance, starting with the GSH expansion

$$\mathbf{T}\cdot\hat{e}_r=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\begin{pmatrix}T_{l0}^{m-}D_{m-}^l\\ T_{l0}^{m0}D_{m0}^l\\ T_{l0}^{m+}D_{m+}^l\end{pmatrix} \quad [155]$$

of the vector field $\mathrm{T}\bullet\hat{e}_r$ representing the radial component of the stress tensor, one may reduce it to the same form as eqn [154], with (U_l^m, V_l^m, W_l^m) replaced by (P_l^m, Q_l^m, R_l^m) where

$$P_l^m(r)=T_{l0}^{m0}(r) \quad [156]$$

$$Q_l^m(r)=T_{l0}^{m+}(r)+T_{l0}^{m-}(r) \quad [157]$$

$$R_l^m(r)=T_{l0}^{m+}(r)-T_{l0}^{m-}-(r) \quad (158)$$

Besides the two vector fields in the preceding text, the scalar field $\Phi_1^{\mathrm{E}}(r,\theta,\lambda)$ too appears in the differential equations of our problem. The GSH expansion of this field

$$\Phi_1^{\mathrm{E}}(r,\theta,\lambda)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Phi_{1l}^{\mathrm{Em}}(r)D_{m0}^l(\theta,\lambda) \quad [159]$$

introduces a new radial function $\Phi_{1l}^{\mathrm{Em}}(r)$. It is convenient to introduce another radial function

$$g_{1l}^{\mathrm{Em}}(r)=\frac{\mathrm{d}\Phi_{1l}^{\mathrm{Em}}(r)}{\mathrm{d}r}+4\pi G\rho_0 U_l^m(r) \quad (160)$$

From now on, we shall omit to write the dependence on r, for simplifying the writing.

The system of equations for our problem may now be expressed as a set of equations for the eight radial functions $U_l^m(r),\ldots, R_l^m(r), \Phi_{1l}^{\mathrm{Em}}(r), g_{1l}^{\mathrm{Em}}(r)$.

3.10.10.1.5 Equations for radial functions

3.10.10.1.5.1 Poisson equation

The projection of the Poisson equation on the GSH basis provides one with an equation of the second order in d/dr. The introduction of the new variable g_1^{E} enables us to reduce this second-order equation to a pair of first-order equations:

$$\frac{\mathrm{d}\Phi_{1l}^{\mathrm{Em}}}{\mathrm{d}r}=g_{1l}^{\mathrm{Em}}-4\pi G\rho_0 U_l^m \quad [161]$$

$$\begin{aligned}\frac{\mathrm{d}g_{1l}^{\mathrm{Em}}}{\mathrm{d}r}=&\frac{l(l+1)}{r^2}\Phi_{1l}^{\mathrm{Em}}-\frac{2}{r}g_{1l}^{\mathrm{Em}}-\frac{L_0^l}{r}4\pi G\rho_0 V_l^m\\ &-4\pi G\sum_{l'=l-2}^{|l+2|}\begin{vmatrix}l&2&l'\\1&0&1\\m&0&m\end{vmatrix}\frac{L_0^l}{r}\rho_2\begin{Bmatrix}V_{l'}^m\\W_{l'}^m\end{Bmatrix}\\ &+4\pi G\sum_{l'=l-2}^{|l+2|}\begin{vmatrix}l&2&l'\\0&0&0\\m&0&m\end{vmatrix}\left[\frac{\mathrm{d}\rho_2}{\mathrm{d}r}U_{l'}^m+\frac{2}{r}\rho_2 U_{l'}^m\right.\\ &\left.+\rho_2\left(\frac{1}{\beta_0}P_{l'}^m-\frac{\lambda_0}{\beta_0 r}\left(L_0^{l'}V_{l'}^m+2U_{l'}^m\right)\right)\right]\end{aligned} \quad [162]$$

where $\beta_0=\lambda_0+2\mu_0$ and $\beta_2=\lambda_2+2\mu_2$ and L_0^l and $L_0^{l'}$ are defined by

$$L_0^l=\sqrt{\frac{l(l+1)}{2}} \quad [163]$$

$$L_0^{l'}=\sqrt{\frac{l'(l'+1)}{2}} \quad [164]$$

The last factor in the second term of eqn [162] consists of two elements written as a column enclosed by curly brackets. The column is meant to be equal to its top element if l and l' have the same parity and to the bottom element if l and l' are of opposite parity. The same shorthand notation may be found in eqns [166] and [167] and in a number of equations further in the succeeding text.

3.10.10.1.5.2 Stress–strain relationship

The projection of the stress–strain relationship on the GSH basis provides three differential equations of the first order in d/dr:

$$\begin{aligned}\frac{\mathrm{d}U_l^m}{\mathrm{d}r}=&\frac{1}{\beta_0}P_l^m-\frac{\lambda_0}{\beta_0 r}\left(L_0^l V_l^m+2U_l^m\right)-\sum_{l'=l-2}^{|l+2|}\begin{vmatrix}l&2&l'\\0&0&0\\m&0&m\end{vmatrix}\\ &\times\left[\frac{\beta_2}{\beta_0}P_{l'}^m-2\frac{\lambda_0\mu_2-\lambda_2\mu_0}{\beta_0^2 r}\left(L_0^{l'}V_{l'}^m+2U_{l'}^m\right)\right]\end{aligned} \quad [165]$$

$$\begin{aligned}\frac{\mathrm{d}V_l^m}{\mathrm{d}r}=&\frac{1}{\mu_0}Q_l^m-\frac{1}{r}\left(V_l^m+2L_0^l U_l^m\right)\\ &-\frac{\mu_2}{\mu_0}\sum_{l'=l-2}^{|l+2|}\begin{vmatrix}l&2&l'\\1&0&1\\m&0&m\end{vmatrix}\begin{Bmatrix}Q_{l'}^m\\R_{l'}^m\end{Bmatrix}\end{aligned} \quad [166]$$

$$\frac{\mathrm{d}W_l^m}{\mathrm{d}r}=\frac{1}{\mu_0}R_l^m+\frac{1}{r}W_l^m-\frac{\mu_2}{\mu_0}\sum_{l'=l-2}^{|l+2|}\begin{vmatrix}l&2&l'\\1&0&1\\m&0&m\end{vmatrix}\begin{Bmatrix}R_{l'}^m\\Q_{l'}^m\end{Bmatrix} \quad [167]$$

The last factor in each of the last two equations consists of two elements written as a column enclosed by curly brackets. The column is meant to be equal to its top element if l and l' have the same parity and to the bottom element if l and l' are of opposite parity. The same shorthand notation may be found in a number of equations further in the succeeding text.

3.10.10.1.5.3 Motion equation

Finally, the projection of the motion equation on the GSH basis provides three differential equations of the first order in d/dr:

$$\frac{dP_l^m}{dr} = -\frac{2}{r}P_l^m - \frac{L_0^l}{r}Q_l^m + \frac{2}{r}\left[\lambda_0 X_l^m + \frac{\mu_0}{r}\left(L_0^l V_l^m + 2U_l^m\right)\right] + \rho_0\left(\eta_{1l}^{Um} + \eta_{2l}^{Um} + \eta_{3l}^{Um}\right) + \sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 0 & 0 & 0 \\ m & 0 & m \end{vmatrix} \left[\rho_2\left(\eta_{1l'}^{Um} + \eta_{2l'}^{Um} + \eta_{3l'}^{Um}\right) + \frac{2}{r}\left(\lambda_2 X_{l'}^m + \frac{\mu_2}{r}\left(L_0^{l'} V_{l'}^m + 2U_{l'}^m\right)\right)\right] \quad [168]$$

$$\frac{dQ_l^m}{dr} = -\frac{3}{r}Q_l^m - \frac{2\left(L_2^l\right)^2}{r^2}\mu_0 V_l^m + \frac{2L_0^l}{r}\left[\lambda_0 X_l^m + \frac{\mu_0}{r}\left(L_0^l V_l^m + 2U_l^m\right)\right] + \rho_0\left(\eta_{1l}^{Vm} + \eta_{2l}^{Vm} + \eta_{3l}^{Vm}\right) + \sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 1 & 0 & 1 \\ m & 0 & m \end{vmatrix}\rho_2\begin{Bmatrix} \eta_{1l'}^{Vm} + \eta_{2l'}^{Vm} \\ \eta_{1l'}^{Wm} + \eta_{2l'}^{Wm} \end{Bmatrix} + \frac{2L_0^l}{r}\sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 0 & 0 & 0 \\ m & 0 & m \end{vmatrix} \times \left[\lambda_2 X_{l'}^m + \frac{\mu_2}{r}\left(L_0^{l'} V_{l'}^m + 2U_{l'}^m\right)\right] + \frac{2L_2^l}{r^2}\mu_2\sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 2 & 0 & 2 \\ m & 0 & m \end{vmatrix} L_2^{l'}\begin{Bmatrix} V_{l'}^m \\ W_{l'}^m \end{Bmatrix} \quad [169]$$

$$\frac{dR_l^m}{dr} = -\frac{3}{r}R_l^m + \frac{2\left(L_2^l\right)^2}{r^2}\mu_0 W_l^m + \rho_0\left(\eta_{1l}^{Wm} + \eta_{2l}^{Wm} + \eta_{3l}^{Wm}\right) + \sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 1 & 0 & 1 \\ m & 0 & m \end{vmatrix}\rho_2\begin{Bmatrix} \eta_{1l'}^{Wm} + \eta_{2l'}^{Wm} \\ \eta_{1l'}^{Vm} + \eta_{2l'}^{Vm} \end{Bmatrix} + \frac{2L_2^l}{r^2}\mu_2\sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 2 & 0 & 2 \\ m & 0 & m \end{vmatrix} L_2^{l'}\begin{Bmatrix} W_{l'}^m \\ V_{l'}^m \end{Bmatrix} \quad [170]$$

where L_0^l, $L_0^{l'}$ and L_2^l were defined previously by eqns [163] and [164], and $L_2^{l'}$ are defined by

$$L_2^l = \sqrt{\frac{(l-1)(l+2)}{2}} \quad [171]$$

$$L_2^{l'} = \sqrt{\frac{(l'-1)(l'+2)}{2}} \quad [172]$$

Note that X_l^m appearing in eqns [168] and [169] are defined by

$$X_l^m = \frac{dU_l^m}{dr} + \frac{1}{r}\left(L_0^l V_l^m + 2U_l^m\right) \quad [173]$$

The radial, spheroidal, and toroidal scalars $\eta_{il'}^{Um}$, $\eta_{il'}^{Vm}$, and $\eta_{il'}^{Wm}$ are defined by

$$\eta_{1l}^{Um} = -\Omega^2\sigma^2 U_l^m + g_{1l}^{Em} - \frac{2(g+\tilde{g})}{r}U_l^m - \frac{L_0^l}{r}\tilde{g}V_l^m \quad [174]$$

$$\eta_{1l}^{Vm} = -\Omega^2\sigma^2 V_l^m - \frac{L_0^l}{r}\left(\Phi_{1l}^{Em} + \tilde{g}U_l^m\right) \quad [175]$$

$$\eta_{1l}^{Wm} = -\Omega^2\sigma^2 W_l^m \quad [176]$$

$$\eta_{2l}^{Um} = 2\sigma\Omega^2\sum_{l'=|l-1|}^{|l+1|}\begin{vmatrix} l & 2 & l' \\ 0 & 1 & -1 \\ m & 0 & m \end{vmatrix}\begin{Bmatrix} V_{l'}^m \\ -W_{l'}^m \end{Bmatrix} \quad [177]$$

$$\eta_{2l}^{Vm} = -2\sigma\Omega^2\sum_{l'=|l-1|}^{|l+1|}\left[\begin{vmatrix} l & 2 & l' \\ -1 & -1 & 0 \\ m & 0 & m \end{vmatrix}\begin{Bmatrix} 2U_{l'}^m \\ 0 \end{Bmatrix} + \begin{vmatrix} l & 2 & l' \\ -1 & 0 & -1 \\ m & 0 & m \end{vmatrix}\begin{Bmatrix} V_{l'}^m \\ -W_{l'}^m \end{Bmatrix}\right] \quad [178]$$

$$\eta_{2l}^{Wm} = 2\sigma\Omega^2\sum_{l'=|l-1|}^{|l+1|}\left[\begin{vmatrix} l & 2 & l' \\ -1 & -1 & 0 \\ m & 0 & m \end{vmatrix}\begin{Bmatrix} 0 \\ 2U_{l'}^m \end{Bmatrix} + \begin{vmatrix} l & 2 & l' \\ -1 & 0 & -1 \\ m & 0 & m \end{vmatrix}\begin{Bmatrix} -W_{l'}^m \\ V_{l'}^m \end{Bmatrix}\right] \quad [179]$$

$$\eta_{3l}^{Um} = \sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 0 & 0 & 0 \\ m & 0 & m \end{vmatrix} \times \left[\frac{d\Phi_2}{dr}\left(\frac{dU_{l'}^m}{dr} - X_{l'}^m\right) + \frac{d^2\Phi_2}{dr^2}U_{l'}^m\right] + \sqrt{3}\sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 0 & 1 & -1 \\ m & 0 & m \end{vmatrix} \times \left(\frac{1}{r}\frac{d\Phi_2}{dr} - \frac{\Phi_2}{r^2} - \frac{\Phi_2}{r}\frac{d}{dr}\right)\begin{Bmatrix} V_{l'}^m \\ -W_{l'}^m \end{Bmatrix} \quad [180]$$

$$\eta_{3l}^{Vm} = -\frac{2L_0^l}{r}\frac{d\Phi_2}{dr}\sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 0 & 0 & 0 \\ m & 0 & m \end{vmatrix}U_{l'}^m - \sqrt{3}\frac{\Phi_2}{r}\sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 0 & 1 & -1 \\ m & 0 & m \end{vmatrix}\begin{Bmatrix} V_{l'}^m \\ -W_{l'}^m \end{Bmatrix} + 2\sqrt{3}\frac{\Phi_2}{r}\sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 1 & 1 & 0 \\ m & 0 & m \end{vmatrix}\begin{Bmatrix} X_{l'}^m \\ 0 \end{Bmatrix} \quad [181]$$

$$\eta_{3l}^{Wm} = 2\sqrt{3}\frac{\Phi_2}{r}\sum_{l'=l-2}^{|l+2|}\begin{vmatrix} l & 2 & l' \\ 1 & 1 & 0 \\ m & 0 & m \end{vmatrix}\begin{Bmatrix} 0 \\ X_{l'}^m \end{Bmatrix} \quad [182]$$

where the superscripts U, V, and W indicate that the quantities $\eta_{il}^{U,V,W,m}$ appear in the equation for $dU_l^m/dr, dV_l^m/dr$, and dW_l^m/dr, respectively. $\tilde{g}$ and g appearing in these equations are defined by $\tilde{g} = \frac{dW_0}{dr}$ and $g = 2\pi G\rho_0 r + \frac{r}{2}\frac{d^2W_0}{dr}$.

It is to be noted that σ, the frequency of excitation of the deformation, enters the equations of motion through the expressions in eqns [174]–[179]. It is important to note that the radial differential equations obtained previously do not couple field components of different orders m. Furthermore, the radial functions that are coefficients of the spheroidal (including radial) parts of degree l in the GSH expansions are coupled with coefficients of toroidal parts of degrees $l-1$, $l+1$, $l-3$, $l+3$... and with coefficients of spheroidal parts of degrees $l-2$, $l+2$, $l-4$, $l+4$....

3.10.10.1.6 Solutions of the radial equations

Let us now arrange the set of radial functions contained in the spheroidal vector fields as a column vector $\bar{\sigma}_l^m$ and those in the toroidal fields as a column vector $\bar{\tau}_l^m$. In view of the nature of the couplings of the different parts of the field as just explained, the differential equations for the $\bar{\sigma}_l^m$ and $\bar{\tau}_l^m$ have the structure of the following matrix-differential equation, wherein ε is a generic symbol standing for the matrix blocks made up of the first-order parts of the matrix elements that arise from the ellipticity of the Earth's structure and A_0 is for the zeroth-order parts, which remain nonvanishing when ellipticity is neglected. (The dimensions of the matrix blocks A_0 and the values of the elements of the blocks depend on their positions in the matrix (eqn [183]) and similarly for the blocks ε. All the blocks are rectangular.)

$$\begin{pmatrix} \cdot & & & & & & & & \\ & \cdot & & & & & & & \\ \varepsilon & \varepsilon & A_0+\varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & \varepsilon & A_0+\varepsilon & \varepsilon & \varepsilon & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & \varepsilon & A_0+\varepsilon & \varepsilon & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & \varepsilon & \varepsilon & A_0+\varepsilon & \varepsilon & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & \varepsilon & \varepsilon & A_0+\varepsilon & \varepsilon & \varepsilon \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \end{pmatrix} \times \begin{pmatrix} \cdot \\ \cdot \\ \bar{\sigma}_{l-2}^m \\ \bar{\tau}_{l-1}^m \\ \bar{\sigma}_l^m \\ \bar{\tau}_{l+1}^m \\ \bar{\sigma}_{l+2}^m \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f(\Phi^{\text{ext}}) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad [183]$$

where $f(\Phi^{\text{ext}})$ represents a normalized function of the forcing potential Φ^{ext}, which is taken to be of degree l and order m, and $\bar{\sigma}_l^m$ and $\bar{\tau}_l^m$ stand for the following column vectors:

$$\bar{\sigma}_l^m = \begin{pmatrix} U_l^m \\ P_l^m \\ \Phi_{1l}^{\text{Em}} \\ g_{1l}^{\text{Em}} \\ V_l^m \\ Q_l^m \end{pmatrix} \qquad [184]$$

and

$$\bar{\tau}_l^m = \begin{pmatrix} W_l^m \\ R_l^m \end{pmatrix} \qquad [185]$$

The system of equations to be solved is infinite since l may take any integer value $\geq |m|$.

In order to solve the system, it is necessary to truncate at the first order in the small quantities, considering that $\bar{\sigma}_{l+2}^m$ and $\bar{\sigma}_{l-2}^m$ are small as well. With the degree l and order m fixed, one considers the following supertruncated system:

$$\begin{pmatrix} A_0+\varepsilon & \varepsilon & \varepsilon \\ \varepsilon & A_0+\varepsilon & \varepsilon \\ \varepsilon & \varepsilon & A_0+\varepsilon \end{pmatrix} \begin{pmatrix} \bar{\tau}_{l-1}^m \\ \bar{\sigma}_l^m \\ \bar{\tau}_{l+1}^m \end{pmatrix} = \begin{pmatrix} 0 \\ f(\Phi^{\text{ext}}) \\ 0 \end{pmatrix} \qquad [186]$$

consisting in only the ten ordinary differential equations in d/dr, involving the following ten variables: $U_l^m, P_l^m, \Phi_{1l}^{\text{Em}}, g_{1l}^{\text{Em}}, V_l^m, Q_l^m, W_{l-1}^m, R_{l-1}^m, W_{l+1}^m$, and R_{l+1}^m.

For the actual solution of these equations, one starts then from five independent solutions that do not diverge at $r=0$. One integrates from the center up to the surface of the Earth using as inputs the rheological properties λ_0 and μ_0 as functions of r, as well as the density function ρ_0, and using the boundary conditions at the ICB and at the CMB (such as continuity of the potential, continuity of the radial displacement, and continuity of the stress). One reaches the surface with five independent solutions, to which one applies the five surface boundary conditions. This ensures the uniqueness of the solution and is done for one given frequency.

If one wants to search for the Earth's normal modes, one has to set $\Phi^{\text{ext}}=0$ in the truncated equations, integrate similarly inside the Earth (ignoring the right-hand side of eqn [186]), and impose the boundary conditions on the solutions of the resulting system of equations. At the surface, the boundary conditions to be satisfied correspond to algebraic relations that have a solution only when the determinant of the matrix of the system equals zero, which is a relation involving the frequency. The remaining steps are then to search by successive iterations for the frequency that allows the system to be solved (that corresponds to a determinant of the matrix equal to zero).

3.10.11 Atmospheric Tides and Nontidal Effects from Surficial Fluids

The effects of fluid masses at the Earth's surface on nutation must be considered as they produce observable effects. They consist of ocean tidal effects, atmospheric effects, and nontidal ocean effects. The first of these has been dealt with in Section 3.10.9.4.

Atmospheric effects on nutation are mainly on the prograde annual component corresponding to the one solar day period in a terrestrial frame; the diurnal cycle in the temperature of the atmosphere is responsible for the effect. Modulation of the diurnal frequency by the annual and semiannual seasonal variations results in effects at additional frequencies, which translate into semiannual and terannual prograde frequencies and the retrograde annual and zero frequencies in the celestial frame. These effects are not easy to compute, and in any case, the amplitudes of the atmospheric signals keep changing with time. Further consideration of the atmospheric effects on nutation should be made in the time domain. Nevertheless, a new estimation of the atmospheric contributions to Earth's nutations based on three reanalyses of atmospheric global circulation models (GCM) has been recently used by Koot and de Viron (2011) in order to estimate the complex amplitudes of the periodic terms in the atmospheric forcing. Unlike previous estimations based on operational GCMs, the results obtained from the three reanalysis GCMs are in good agreement, and it was possible to show that the atmospheric contributions to nutations are quite small. The implications of the atmospheric contribution to the estimation of Earth's deep interior properties from nutation observations have been studied as well. Koot and de Viron (2011) showed that this contribution is too small to affect significantly the estimation of these properties.

Atmospheric effects on the ocean also induce changes in the nutation. These too are difficult to compute. In practice, the annual prograde nutation contribution from both direct atmospheric effect and induced atmospheric ocean effect on nutation is fitted on the observation. de Viron et al. (2005) have shown that the effects computed from general circulation models using the angular momentum approach fit the 'observed' value surprisingly well.

3.10.12 New Conventions for Earth Rotation Variations

3.10.12.1 CIP and Relation of its Motions to the Nutation and Wobble

The CIP is a conceptual pole (see Capitaine, 2000; Capitaine et al., 2000), the motion of which corresponds to the precessional motion, and nutational motions with periods greater than 2 days, of the ITRF pole in space. Its position is represented by two coordinates (X, Y) in the International Celestial Reference Frame (ICRF), which include a constant offset from the pole of the ICRF besides the lunisolar precessions and nutations in longitude and obliquity. Reduction of observational data requires the use of the transformation between the ICRF and the ITRF, which involves a succession of elementary rotations (represented by matrices), which may be chosen in different ways. One of the alternative expressions is in terms of the rotation, which carries the pole of the ICRF into the intermediate pole (CIP) and the ICRF into an intermediate reference frame, a rotation of this frame around the CIP, representing the axial rotation of the Earth, and a further rotation of the resulting reference frame to bring it into coincidence with the ITRF and the CIP to the pole of the ITRF.

Precession and nutation, as stated previously, are defined now by the motion of the CIP relative to the ICRF, and wobble is the motion of the CIP relative to the ITRF. The rotation around the CIP will be dealt with in more detail in the next section.

3.10.12.2 Nonrotating Origin

Time based on Earth rotation, as classically defined, is the Greenwich Apparent Sidereal Time (GAST, or simply Greenwich Sidereal Time (GST)). GST is the arc length measured along the equator of date from the equinox of date to the Greenwich meridian, which marks the axis Ox that serves as the origin of longitude. If the axis of rotation (the pole of the equator) is kept in a fixed direction in space (and if the ecliptic remained stationary), GST thus defined would give the angle of Earth rotation exactly. However, the equatorial plane is moving in space because of nutation and precession, and this gives rise to small motions of the equinox along the equator that mimic small rotations around the pole of the equator, though they are not caused by axial rotation around the pole of the equator. GST is therefore no longer an exact measure of the angle of axial rotation. In order to obtain a rigorous measure of time based on Earth rotation, it is necessary, therefore, to define an origin on the equator, which has no motion along the instantaneous equator or, differently stated, which has no rotation around the pole of the equator. Such an origin is referred to as the nonrotating origin (NRO) (Guinot, 1979).

According to the conventions adopted by the IAU in 2000 and the IUGG in 2003, the axial rotation of the Earth is to be measured around the CIP, so the equator of the CIP is the relevant equator for the NRO, which is to replace the equinox in the definition of time. To distinguish the newly defined NRO from other origins, which had earlier been referred to by the same name, the nomenclature CIO (Celestial Intermediate Origin) and TIO have been adopted for the NROs on the CIP equator, by IAU2000 resolutions (see Capitaine, 1986, 1990; Capitaine et al., 1986).

If $\hat{n}$ is the unit vector in the direction of the CIP and if Ω is the angular velocity of the Earth, the defining condition on the NRO, namely, that it should have no rotation around the CIP, is given by (see Capitaine, 1986, 1990; Capitaine et al., 1986)

$$\hat{n}\cdot\Omega = 0 \qquad [187]$$

An equivalent way of stating the condition is in terms of the velocity of the CIO – that the velocity, say $\dot{x}$, should be parallel to the instantaneous direction of the CIP:

$$\dot{x} = k\hat{n} \qquad [188]$$

On taking the scalar product with $\hat{n}$, one finds that $k = x\cdot\hat{n}$, the last step being a consequence of the fact that $x\cdot\hat{n}$ and its time derivative vanish since x is on the equator of $\hat{n}$. Hence, the defining condition takes the form introduced by Kaplan (2005):

$$\dot{x} = -(x\cdot\hat{n})\hat{n} \qquad [189]$$

Considering that $\dot{x} = \Omega\wedge x$, and that consequently $\dot{x}\cdot\Omega = 0$, one immediately sees that eqns [188] and [189] are perfectly equivalent.

3.10.12.3 Definition of Universal Time

Until the implementation of the IAU2000 resolutions, UT1 was defined from GMST through the relation

$$\mathrm{GMST}(T_u, \mathrm{UT1}) = \mathrm{GMST}_{0h\mathrm{UT1}}(T_u) + r\mathrm{UT1}$$

which relates the increase of GMST during day T_u to the 'time of day' (UT1), with $\mathrm{GMST}_{0h\mathrm{UT1}}(T_u)$ given by a formula that is not strictly linear in T_u (see IERS Conventions, 1996; Capitaine et al., 1986, 2000, 2003; McCarthy, 1996).

Universal time UT1 was intended to be a measure of the true accumulated rotation of the Earth after a specified initial epoch. The stellar angle is defined to be just that. GMST does not represent the true magnitude of the accumulated rotation. The new definition of UT1 is in terms of $\theta: \theta = \theta_0 + k(\mathrm{UT1} - \mathrm{UT1}_0)$, where k is a scale factor that accounts for the fact that the rate of change of θ differs slightly from that of the old UT1. In this case, the new day of UT1 remains close to the mean solar day. The value of θ_0 is chosen to ensure continuity of the new UT1 with the old one at the epoch of switchover from the old to the new (2003 January 1.0, Julian Day 2452640.5).

3.10.12.4 Transformation between ICRF and ITRF

Reduction of observational data requires the use of the transformation between the ICRF and the ITRF, which involves a succession of elementary rotations (represented by matrices) that may be chosen in different ways as stated previously. They may, for instance, be rotations through the classical Euler angles. An alternative expression is in terms of the rotation that carries the pole of the ICRF into an intermediate pole, a rotation around this pole, and the rotation that brings this pole into coincidence with the pole of the ITRF.

Classically, the matrix transformation that allows us to pass from the TRF to the CRF or vice versa was related to precession, nutation, LOD, and polar motion by

$$[\mathrm{TRF}] = WR_3(\mathrm{GST})PN[\mathrm{CRF}] \qquad [190]$$

or

$$[\mathrm{CRF}] = PN'R_3(-\mathrm{GST})W'[\mathrm{TRF}] \qquad [191]$$

where W contains the polar motion contribution, R_3 indicates a rotation around the third (z) axis, and PN contains the precession–nutation contributions, respectively (see Mueller, 1981).

GST is the Greenwich Sidereal Time of date (also called the Greenwich Apparent Sidereal Time, GAST), that is, the angle on the true equator of date between the true equinox of date and the Greenwich meridian, which is the origin of longitude in the terrestrial frame (direction of Ox). The transformation [190] is the classical one involving an intermediate frame associated with the true equator and equinox of date.

In the NRO approach, one has

$$[\mathrm{TRF}] = MR_3(\theta)M'[\mathrm{CRF}] \qquad [192]$$

where M and M' are related to the CIP coordinates in space and in the terrestrial frame, respectively. The transformation [192] is with reference to an intermediate frame based on the concept of the CIP and NRO (adopted by the IAU in 2000 and the IUGG in 2003). As explained earlier, the CIP position is represented by two coordinates (X, Y) in the ICRF. One has

$$[\mathrm{TRF}] = R_1\left(y_p\right)R_2\left(x_p\right)R_3(-s')R_3(-\theta) \times R_3(s)\begin{pmatrix} 1 & -XY/2 & X \\ -XY/2 & 1 & Y \\ -X & -Y & 1 \end{pmatrix}[\mathrm{CRF}] \qquad [193]$$

where R_1, R_2, and R_3 refer to rotations about the x-, y- and z-axes of the frame that one is in at the relevant stage of the transformation. Thus, in the decomposition of the transformation between the CRF and TRF into polar motion, axial rotation about the CIP, and nutation–precession, θ appears in combination with s and s' defined by Capitaine et al. (1986).

A Annex 1

A.1 Rotational Normal Mode Definitions

The existence of more than one rotational normal mode for the Earth is due to the presence of the core regions that are not rigidly attached to one another or to the mantle region and the consequent possibility of differential rotations between different regions.

In each of the rotational modes, the rotation axes of the three regions, as well as the symmetry axis of the SIC, are offset from the symmetry axis of the mantle by different angles. (The magnitude of the angular offset of the rotation axis of each region from the mantle symmetry axis is the amplitude of the wobble of that region.) All four axes rotate with a common frequency (the frequency of the normal mode) around the mantle symmetry axis. The rotational normal modes not only contribute directly to Earth rotation variations but also give rise to resonances in the forced nutations and wobbles when the frequency of forcing is close to one or the other of the normal mode eigenfrequencies. The four rotational modes are the following:

- The CW is a low-frequency mode with a period of about 432 days in a TRF. If observed, it is characterized by an offset of the direction of the axis of rotation of the mantle from the figure axis of the ellipsoidal Earth, accompanied by a motion of the rotation axis around the figure axis in the prograde sense, that is, in the same sense as that of the diurnal rotation of the Earth. In this mode, the rotation axes of the three regions are nearly coincident. This is the only rotational eigenmode that could exist if the Earth were wholly solid. (The period would of course be different in that case.) The counterpart of the CW for a rigid Earth is the Euler wobble with a period close to 300 days, which is determined solely by the Earth's dynamical ellipticity $(C-A)/A$, that is, the fractional difference between the moments of inertia about the polar and equatorial axes. In fact, the period, when expressed in cpsd, is just $(1/e)$.
- The FCN is a free mode related to the existence of a flattened fluid core inside the Earth. If observed, it is associated with an offset of the rotation axis of the fluid core from the mantle symmetry axis (i.e., the amplitude of the wobble of the FOC), far larger than that of the rotation axis of the mantle itself. The frequency of this mode is nearly diurnal as seen in a terrestrial frame; for this reason, the motion as seen from the terrestrial frame is referred to as the NDFW. The motion of the rotation axes is accompanied by a circular motion of the mantle symmetry axis in space, which is the FCN; it is retrograde, that is, in the opposite sense to that of the diurnal Earth rotation and has a period of about 430 days. Ellipticity of the core CMB is essential for the existence of the FCN mode. It causes the equatorial bulge of the fluid mass to impinge on the mantle at the CMB in the course of the differential rotation of the core and the mantle about noncoincident axes; the pressure of the fluid on the boundary and the inertial reaction on the fluid core itself sustain the differential rotation between the core and the mantle in the FCN mode.
- The FICN is a mode related to the existence of a flattened inner core inside the fluid core. If excited, the amplitude of the wobble of the SIC dominates over the wobble amplitudes of the other regions; the figure axis of the SIC too is offset by almost the same amount as the rotation axis of the SIC. The FICN has, like the FCN, a nearly diurnal retrograde period in a TRF. The motion of the axes in space is prograde in this mode, unlike in the case of the FCN. For this reason, the FICN is also referred to as the PFCN; when it is necessary to emphasize the distinction between this mode and the FCN, the latter is referred to as the retrograde free core nutation (RFCN). The period of the FICN in space is about 1000 days. As in the case of the FCN, the fluid pressure acting on the ellipsoidal boundary (the ICB in this case) plays an important role here; but an even larger role is played by the gravitational coupling of the SIC to the mantle due to the offset between the symmetry axes of the two ellipsoidal regions.

- The ICW is a mode also related to the existence of a flattened inner core inside the fluid core. If excited, the offset of the symmetry axis of the SIC from that of the mantle exceeds the offsets of the three rotation axes by several orders of magnitude. It has a long period (longer than the CW) in a terrestrial frame and is prograde. The ellipsoidal structure of the SIC and of the rest of the Earth and the density contrast between the SIC and the fluid at the ICB are essential elements in the generation of this normal mode.

B Annex 2

B.1 Generalized Spherical Harmonics

The GSH function D^l_{mn} is a function of θ and φ depending on three integer numbers l in $[0, +\infty]$, called degree; m in $[-l, +l]$, called order; and n in $[-l, +l]$. Its definition is the following:

$$D^l_{mn}(\theta,\varphi) = (-1)^{m+n} P^{mn}_l(\mu) e^{im\varphi} \qquad [194]$$

where $\mu = \cos\theta$ and P^{mn}_l is the generalized Legendre function:

$$P^{mn}_l(\mu) = \frac{(-1)^{l-n}}{2^l(l-n)!}\sqrt{\frac{(l-n)!(l+m)!}{(l+n)!(l-m)!}}\sqrt{\frac{1}{(1-\mu)^{m-n}(1+\mu)^{m+n}}}$$
$$\frac{d^{l-m}}{d\mu^{l-m}}\left((1-\mu)^{l-n}(1+\mu)^{l+n}\right) \qquad [195]$$

The product of two GSH is expressed using the so-called J-square coefficients related to the Wigner symbols as seen in the text (see Edmonds, 1960):

$$D^{l_1}_{m_1 n_1} D^{l_2}_{m_2 n_2} = \sum_{l=|l_2-l_1|}^{l=|l_2+l_1|} \begin{vmatrix} l & l_1 & l_2 \\ n & n_1 & n_2 \\ m & m_1 & m_2 \end{vmatrix} D^l_{mn} \qquad [196]$$

where $n = n_1 + n_2$ and $m = m_1 + m_2$. The GSH form a basis on which the components of a tensor, of a vector, or of a scalar can be decomposed. The canonical component (α_1, α_2) of a tensor $\mathbf{T}$ of order 2 can be written as

$$\mathbf{T}^{\alpha 1,\alpha 2}(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} T^{\alpha 1,\alpha 2}(r) D^l_{mn}(\theta,\varphi) \qquad [197]$$

where $n = \sum_{i=1}^{2}\alpha_i$. The canonical components of a vector

$$\mathrm{v}(r,\theta,\varphi) = v^-(r,\theta,\varphi)\hat{e}_- + v^0(r,\theta,\varphi)\hat{e}_0 + v^+(r,\theta,\varphi)\hat{e}+ = \begin{bmatrix} v^-(r,\theta,\varphi) \\ v^0(r,\theta,\varphi) \\ v^+(r,\theta,\varphi) \end{bmatrix} \qquad [198]$$

can be decomposed as

$$\mathrm{v}(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \begin{bmatrix} v^{m-}_l(r) D^l_{m-}(\theta,\varphi) \\ v^{m0}_l(r) D^l_{m0}(\theta,\varphi) \\ v^m_l+(r) D^l_{m+}(\theta,\varphi) \end{bmatrix} \qquad [199]$$

For a scalar function $f(r, \theta, \varphi)$, the decomposition in GSH is

$$f(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f^m_l(r) D^l_{m0}(\theta,\varphi) \qquad [200]$$

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