Math. Nachr. **282,** No. 1, 99 – 103 (2009) / **DOI** 10.1002/mana.200710724

A note on Schatten-class membership of Hankel operators with anti-holomorphic symbols on generalized Fock-spaces

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Received 29 October 2007, revised 19 March 2008, accepted 29 March 2008
Published online 15 December 2008

Key words Hankel operator, Schatten-class, Fock space
MSC (2000) 47B35, 32A36

In this paper we investigate Hankel operators with anti-holomorphic L^2-symbols on generalized Fock spaces A^2_m in one complex dimension. The investigation of the mentioned operators was started in [4] and [3]. Here, we show that a Hankel operator $H_{\overline{f}}$ with anti-holomorphic L^2-symbol is in the Schatten-class $\mathcal{S}_p$ if and only if the symbol is a polynomial $\overline{f} = \sum_{k=0}^{N} b_k \overline{z}^k$ with degree N satisfying $2N < m$ and $p > \frac{2m}{m-2N}$. The result has been proved independently before in the recent work [2], which also considers the case of several complex variables. However, in addition to providing a different proof for the result the present work shows that the methodology developed in [4] and [3] can be adopted in order to work to characterize Schatten-class membership.

1 Introduction

Let A^2_m denote the generalized Fock space

$$A^2_m = \left\{ g \text{ entire} : \|g\|^2_m := \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^m} \, d\lambda(z) < \infty \right\}$$

and denote by L^2_m the corresponding L^2-space. Here, $d\lambda$ denotes the Lebesgue measure in $\mathbb{C} \cong \mathbb{R}^2$. It is well-known that A^2_m is the closure of the polynomials with respect to the L^2-norm (see [1]). The expressions

$$c^2_{n,m} = \int_{\mathbb{C}} |z^n|^2 e^{-|z|^m} \, d\lambda(z), \quad n, m \in \mathbb{N},$$

are the so-called moments. For sake of notational simplicity, we will denote the moments by c^2_n, if the dependence on m is obvious. Remember that the Hankel operator $H_{\overline{f}} : A^2_m \to {A^2_m}^{\perp}$ with symbol $\overline{f} = \sum_{k=0}^{\infty} b_k \overline{z}^k \in L^2_m$ is given by

$$H_{\overline{f}} = (I - P)\overline{f} : A^2_m \to {A^2_m}^{\perp},$$

where $P : L^2_m \xrightarrow{\perp} A^2_m$ denotes the Bergman projection. In [3] we have proved the following result (for the special case of monomials see also [4].):

Theorem 1.1 *Let $H_{\overline{f}}$ be a Hankel operator with symbol $\overline{f} = \sum_{k=0}^{\infty} b_k \overline{z}^k \in L^2_m$. This operator can only be bounded if the symbol is a polynomial $\overline{f} = \sum_{k=0}^{N} b_k \overline{z}^k$. Moreover, the Hankel operator with symbol $\overline{f} = \sum_{k=0}^{N} b_k \overline{z}^k$ is bounded if and only if $2N \leq m$.*

In addition, for operators with monomial symbols Schatten-class membership ($\mathcal{S}_p$) of the Hankel operators has been characterized.

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Theorem 1.2 *Let $p > 0$. For $2k < m$ the Hankel-operators*

$$H_{\overline{z}^k} = (I - P)\overline{z}^k : A^2_m \longrightarrow {A^2_m}^{\perp}$$

are in $\mathcal{S}_p$ if and only if $p > \frac{2m}{m-2k}$. For $2k \geq m$ the Hankel-operators are not in $\mathcal{S}_p$.

The aim of this note is to characterize Schatten-class membership of Hankel operators with general anti-holomorphic symbols. We will prove the following proposition:

Theorem 1.3 *Let $H_{\overline{f}}$ be a Hankel operator with polynomial-symbol $\overline{f} = \sum_{k=0}^N b_k \overline{z}^k$. Then the operator belongs to $\mathcal{S}_p$ if and only if $m > 2N$ and $p > \frac{2m}{m-2N}$.*

In order to prove the above theorem we first derive the following proposition.

Proposition 1.4 *Let $\overline{f} = \sum_{j=0}^N a_j \overline{z}^j$. Furthermore, let P_{1k} be the projection onto the closure of the linear span (denoted by $\langle\langle\cdot\rangle\rangle$ in the following) of the set $\left\{\overline{z}^{k-p} z^{Nl-p} \mid l \in \mathbb{N}; p \leq k \text{ and } p \leq Nl\right\}$ and let P_0 be the projection onto $\left\langle\left\langle\{z^{Nl}, l \in \mathbb{N}\}\right\rangle\right\rangle$. Let*

$$\widetilde{H}_{k,\overline{f}} = P_{1k} H_{\overline{f}} P_0, \quad k \in \mathbb{N}.$$

Then we have

$$\widetilde{H}^*_{k,\overline{f}} \widetilde{H}_{k,\overline{f}}\left(z^{lN}/c_{lN}\right) = |a_k|^2 \left(\frac{c^2_{lN+k}}{c^2_{lN}} - \frac{c^2_{lN}}{c^2_{lN-k}}\right) z^{lN}/c_{lN}$$

for $l \geq 1$ and

$$\widetilde{H}^*_{k,\overline{f}} \widetilde{H}_{k,\overline{f}}\left(z^0/c_0\right) = |a_k|^2 \frac{c^2_k}{c^2_0}.$$

Finally, $\widetilde{H}^*_{k,\overline{f}} \widetilde{H}_{k,\overline{f}}(z^p/c_p) = 0$ for $p \neq lN$ for all $l \in \mathbb{N}$.

The above proposition is more general than it looks at first sight. It is known from Theorem 1.1 that a Hankel operator with anti-holomorphic symbol can only be bounded if the symbol is a polynomial. Therefore, the proposition characterizes all Schatten-class Hankel operators with general anti-holomorphic L^2-symbols. It has to be mentioned, that the above result has been proved independently earlier by H. Bommier and H. Youssfi ([2]). In contrast to [2] the present work uses methods developed in [4] and [3]. The reasoning applied here also works in the case of several complex dimensions. However, for sake of simplicity we only formulate and prove the result in case of one complex dimension as in [4] and [3].

2 Proof of the result

In this section we prove Theorem 1.3. The idea of the proof is the following: Since the operators $H^*_{\overline{f}} H_{\overline{f}}$ are not diagonal operators with respect to the standard basis $\{z^n/c_n, n \in \mathbb{N}\}$, Schatten-class membership cannot be easily described directly. One approach is to show that Schatten-class membership ($\mathcal{S}_p$) of $H_{\overline{f}}$ implies the Schatten-class membership of the operators $H_{\overline{z}^k}$. From this the result can easily be induced. Our approach will use a slightly weaker result, but the mentioned result is obtained in the remark after the proof. It is obvious from Theorem 1.2, that for $p > \frac{2m}{m-2N}$ the Hankel operator with symbol $\overline{f} = \sum_{k=0}^N b_k \overline{z}^k$ is in $\mathcal{S}_p$. Therefore one direction in the proof of Theorem 1.3 is obvious. The other direction is discussed in the proof below. First, we prove Proposition 1.4.

Proof of Proposition 1.4. Let $\overline{f} = \sum_{j=0}^N a_j \overline{z}^j$ and remember that P_{1k} is the projection onto $\left\langle\left\langle\left\{\overline{z}^{k-p} z^{Nl-p} \mid l \in \mathbb{N}; p \leq k \text{ and } p \leq Nl\right\}\right\rangle\right\rangle$ and that P_0 is the projection onto $\left\langle\left\langle\left\{z^{Nl}, l \in \mathbb{N}\right\}\right\rangle\right\rangle$. First, note that

$$\widetilde{H}^*_{k,\overline{f}} \widetilde{H}_{k,\overline{f}} = P_0 H^*_{\overline{f}} P_{1k} H_{\overline{f}} P_0.$$

For $l \geq 1$ large enough we have

$$H_{\overline{f}}\big(z^{lN}\big) = \sum_{j=0}^{N} a_j z^{lN} \overline{z}^j - \sum_{j=0}^{N} a_j \frac{c_{lN}^2}{c_{lN-j}^2} z^{lN-j}$$

and therefore

$$P_{1k} H_{\overline{f}}\big(z^{lN}\big) = a_k z^{lN} \overline{z}^k - a_k \frac{c_{lN}^2}{c_{lN-k}^2} z^{lN-k}.$$

Since $H_{\overline{f}}^* = \big[(I-P)M_{\overline{f}}P\big]^* = PM_f(I-P)$ and $P_{1k}H_{\overline{f}}\big(z^{lN}\big) \in {A_m^2}^{\perp}$, it follows that

$$\begin{aligned} H_{\overline{f}}^* P_{1k} H_{\overline{f}}\big(z^{lN}\big) &= P\left(\sum_{j=1}^{N} a_k \overline{a_j} z^{lN+j} \overline{z}^k - \sum_{j=1}^{N} a_k \overline{a_j} \frac{c_{lN}^2}{c_{lN-k}^2} z^{lN-k+j}\right) \\ &= \sum_{j=1}^{N} a_k \overline{a_j} z^{lN-k+j} \left(\frac{c_{lN+j}^2}{c_{lN+j-k}^2} - \frac{c_{lN}^2}{c_{lN-k}^2}\right). \end{aligned}$$

Finally,

$$\widetilde{H}_{k,\overline{f}}^* \widetilde{H}_{k,\overline{f}}\big(z^{lN}/c_{lN}\big) = |a_k|^2 \left(\frac{c_{lN+k}^2}{c_{lN}^2} - \frac{c_{lN}^2}{c_{lN-k}^2}\right) z^{lN}/c_{lN}.$$

For $l = 0$, we have $H_{\overline{f}}\big(z^0\big) = \sum_{j=0}^{N} a_j \overline{z}^j$ and therefore $P_{1k}H_{\overline{f}}\big(z^{lN}\big) = a_k \overline{z}^k$. Hence

$$H_{\overline{f}}^* P_{1k} H_{\overline{f}}\big(z^0\big) = \sum_{j=k}^{N} a_k \overline{a_j} z^{lN-k+j} \frac{c_{lN+j}^2}{c_{lN+j-k}^2}$$

and

$$\widetilde{H}_{k,\overline{f}}^* \widetilde{H}_{k,\overline{f}}\big(z^{lN}/c_{lN}\big) = |a_k|^2 \frac{c_k^2}{c_{lN}^2} z^0/c_0.$$

Obviously $\widetilde{H}_{k,\overline{f}}^* \widetilde{H}_{k,\overline{f}}(z^p/c_p) = 0$ for $p \neq lN$ for all $l \in \mathbb{N}$. This finishes the proof. □

Next, we prove Theorem 1.3.

P r o o f o f T h e o r e m 1.3. If the operator $H_{\overline{f}}$ is in $\mathcal{S}_p$ then the operators

$$\widetilde{H}_{k,\overline{f}} = P_{1k} H_{\overline{f}} P_0, \quad k \in \mathbb{N},$$

are in the Schatten-class as well. We know from Proposition 1.4 that the operator $\widetilde{H}_{k,\overline{f}}^* \widetilde{H}_{k,\overline{f}}$ is diagonal with respect to the basis $\{z^n/c_n \mid n \in \mathbb{N}\}$. The strictly positive eigenvalues are $|a_k|^2 \left(\frac{c_{lN+k}^2}{c_{lN}^2} - \frac{c_{lN}^2}{c_{lN-k}^2}\right)$. $\Big($In case $l = 0$ the eigenvalue is $|a_k|^2 \left(\frac{c_{lN+k}^2}{c_{lN}^2}\right).\Big)$ As in [3] it can be shown now that the operator $\widetilde{H}_{k,\overline{f}}$ is in $\mathcal{S}_p$ if and only if $p > \frac{2m}{m-2k}$ (see the proof of Theorem 5 and Proposition 3 in [3] for details). For sake of completeness we shortly sketch the proof. First, note that

$$c_{l,m}^2 = \int_{\mathbb{C}} \big|z^l\big|^2 e^{-|z|^m}\, d\lambda(z) = \frac{2\pi}{m} \Gamma\left(\frac{2l+2}{m}\right).$$

Using Stirling's formula

$$\Gamma(x+1) = x^x e^{-x} \sqrt{2\pi x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + O\big(1/x^3\big)\right)$$

one can show that

$$\frac{c_{l+k}^2}{c_l^2} - \frac{c_l^2}{c_{l-k}^2} \approx \mathcal{C}(k,m) l^{\frac{2k}{m}-1}$$

for some constant $\mathcal{C}(k,m)$. Therefore, since

$$\sum_{l=1}^{\infty} \left(|a_k|^2 \left(\frac{c_{lN+k}^2}{c_{lN}^2} - \frac{c_{lN}^2}{c_{lN-k}^2} \right) \right)^{p/2}$$

converges if and only if (given $a_k \neq 0$)

$$\sum_{l=1}^{\infty} \frac{1}{l^{p/2(1-\frac{2k}{m})}} < \infty$$

it is obvious that $\widetilde{H}_{k,\overline{f}}$ is in $\mathcal{S}_p$ if and only if $p > \frac{2m}{m-2k}$. This finishes the proof. □

Remark 2.1 Let $\overline{f} = \sum_{j=1}^{N} a_j \overline{z}^j$ and $a_k \neq 0$. It is known from [3, p. 412] that for $l \geq k$

$$H_{\overline{z}^k}^* H_{\overline{z}^k} \left(\frac{z^l}{c_l} \right) = \left(\frac{c_{l+k}^2}{c_l^2} - \frac{c_l^2}{c_{l-k}^2} \right) \frac{z^l}{c_l}$$

and otherwise

$$H_{\overline{z}^k}^* H_{\overline{z}^k} \left(\frac{z^l}{c_l} \right) = \frac{c_{l+k}^2}{c_l^2} \frac{z^l}{c_l}.$$

Therefore $H_{\overline{z}^k}$ is in $\mathcal{S}_p$ if and only if

$$\sum_{l=k}^{\infty} \left(\frac{c_{l+k}^2}{c_l^2} - \frac{c_l^2}{c_{l-k}^2} \right)^{\frac{p}{2}} < \infty.$$

As above, this is equivalent to

$$\sum_{l=k}^{\infty} \frac{1}{l^{p/2(1-\frac{2k}{m})}} < \infty.$$

Since the limiting behavior of the sequence

$$\left(\frac{c_{lN+k}^2}{c_{lN}^2} - \frac{c_{lN}^2}{c_{lN-k}^2} \right)_{l \in \mathbb{N}}$$

is the same as the one of

$$\left(\frac{c_{l+k}^2}{c_l^2} - \frac{c_l^2}{c_{l-k}^2} \right)_{l \in \mathbb{N}},$$

the Hankel-operator $H_{\overline{z}^k}$ is in $\mathcal{S}_p$ if and only if the Hankel operator $\widetilde{H}_{k,\overline{f}}$ is. This can be seen as follows. Since

$$\frac{c_{l+k}^2}{c_l^2} - \frac{c_l^2}{c_{l-k}^2} \approx \mathcal{C}(k,m) l^{\frac{2k}{m}-1} \quad \text{and} \quad \frac{c_{lN+k}^2}{c_{lN}^2} - \frac{c_{lN}^2}{c_{lN-k}^2} \approx \mathcal{C}(k,m) (lN)^{\frac{2k}{m}-1}$$

it follows from $\widetilde{H}_{k,\overline{f}} \in \mathcal{S}_p$ that

$$\sum_{l=k}^{\infty} \frac{1}{(lN)^{p/2(1-\frac{2k}{m})}} < \infty,$$

which implies

$$\sum_{l=k}^{\infty} \frac{1}{l^{p/2(1-\frac{2k}{m})}} < \infty.$$

Therefore, the operator $H_{\overline{z}^k}$ is in $\mathcal{S}_p$ and consequently Schatten-class membership of $H_{\overline{f}}$ implies Schatten-class membership of $H_{\overline{z}^k}$.

Acknowledgements The author would like to thank H. Youssfi for many interesting conversations and the University of Marseille (Université de Provence) for a kind invitation. In addition, helpful comments from two anonymous referees are gratefully acknowledged.

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