Equilibrium Dynamics of the Toy Model of Dense Fluid: The Infinite Damping Limit†

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We investigate the equilibrium dynamics of our recently proposed toy model of dense fluid in the infinite damping limit. Contrary to naive expectation, the correlators involving the velocity-like variables do not quickly relax away. Instead, after a very fast transient relaxation, they exhibit rather slow relaxations due to the coupling to the density-like variable. Hence, the so-called "hopping" processes are not suppressed even in the large damping limit. These hopping processes can only be controlled by tuning the parameter δ^*, which is the ratio of the numbers of the components of the velocity-like and the density-like variables in the model. We analytically prove that there must exist an ergodic-to-nonergodic phase transition for δ^* such that $0 < \delta^* < 1$. The slow dynamics and the dynamic transition in the model are distinct from those in the idealized mode coupling theory.

I. Introduction

Although a complete understanding for the liquid−glass transition problem[1] seems to be still far from being achieved, there has been tremendous progress toward this lofty goal since the development of the mode coupling theory (MCT) of supercooled liquids.[2] As a first principle approach, the MCT has not only enjoyed considerable success in explaining the slowing down of the weakly supercooled liquids[3,4] but also had enormous impact on the area by stimulating further experiments, computer simulations, and other theoretical developments. The most successful application of the MCT was made for colloidal systems with repulsive interactions.[5,6] A recent further application of the MCT for the colloids with short-ranged attractions[7,8] has predicted the existence of the two types of glass phases and the reentrant behavior of glass−fluid−gel phases, which has been observed later experimentally.[9] It is therefore very important to understand the physical basis underlying this success of the MCT.

We have recently proposed a mean field type toy model of dense fluid[10] to gain further insights into the nature of the MCT. To motivate in more detail why we have constructed the model, we first describe some prominent features of the MCT. The current formulation of the MCT is based on an approximation scheme, which factorizes the four-body time correlation of densities into the bilinear product of the two-body density correlation functions. This so-called factorization approximation[11−13] is totally uncontrolled, making it very hard to improve the theory in a systematic way, which is a primary source rendering unclear the nature of the MCT. Via the factorization approximation, one can obtain a closed dynamic equation for the density correlation function in the idealized version of the MCT,[14,15] which predicts a sharp dynamic transition to a nonergodic state at a certain temperature. However, the MCT does not provide any information on the nature of this nonergodic state since the theory focuses primarily on the dynamic evolution of the density correlation function. The extended versions of the MCT[16−20] are found to round off the sharp transition via the so-called hopping processes, leaving some remnants of the transition, but it seems to be fair to state that the physical picture of the hopping process in the extended version of the MCT remains obscure.

In view of these features of the MCT, we deemed it worthwhile to construct a mean field type stochastic model of fluid, which contains reversible mode coupling but is exactly solvable without relying upon uncontrolled approximation. This was inspired by the works[21,22] in which random coupling models involving an infinte component order parameter have been shown to be exactly analyzed by mean field type concepts. We also hoped to obtain some information on the nature of the nonergodic state via the corresponding Fokker−Planck equation for the probability distribution function if the model exhibits a dynamic transition to a nonergodic state.

Our toy model consists of a stochastic dynamics for the density-like and the velocity-like variables in order to mimic the dynamics of fluid. The statics of the model is made rather trivial (i.e., Gaussian Hamiltonian); hence, there is no complex energy landscape. The kinetics is responsible for the glassy behavior in the model. While this feature is shared by the kinetically constrained models,[23] which have noninteracting Hamiltonian, and the systems of infinitely thin needles,[24−27] which has *no* static correlation, it is contrasted by the mean field spherical p-spin model[28,29] whose Hamiltonian is p-spins interacting with random coupling, rendering a well-known complex energy landscape. However, in contrast to various kinetically constrained models, our toy model has *reversible* nonlinear mode coupling so that the resulting dynamics closely imitates the MCT for fluids.[2] Reversible terms mean the terms not containing dissipative processes, which give rise to irreversible increase of entropy. In hydrodynamics, viscous terms are irreversible but the Euler convective term is reversible.

In earlier development, we worked with the model with the same number of components N for both the density-like variables a_i and the velocity-like variables b_i.[10a] In the limit N

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going to infinity, the model becomes exactly solvable and we were able to obtain the self-consistent closed equations for the four equilibrium correlation functions $C_a(t)$, $C_b(t)$, $C_{ab}(t)$, and $C_{ba}(t)$. The Laplace transformed form (denoted by the superfix L) of the density correlation function can be written in the form

$$C_a^L(z) = \frac{T}{\omega^2}\cdot\left\{z + \sum_{aa}^L(z) + \frac{\omega^2[1-\sum_{ab}^L(z)]^2}{z+\gamma+\sum_{bb}^L(z)}\right\}^{-1}$$

where the kernels $\sum^L(z)$ values are the Laplace transforms of the memory kernel $\sum(t)$ values. The kernel $\sum_{bb}^L(z)$ involves the biproduct of the density correlation function $C_a(t)$, whereas the so-called hopping kernels $\sum_{aa}^L(z)$ and $\sum_{ab}^L(z)$ involve not only $C_a(t)$ but also $C_b(t)$ as well as the cross correlators $C_{ab}(t)$ and $C_{ba}(t)$ (see eq 9 with $\delta^* = 1$ below). The structure of the theory in the model is found to be the same as that of the one-loop results of Schmitz, Dufty, and De (SDD)'s fluctuating hydrodynamic equations,[18] apart from the spatial degrees of freedom (our toy model is zero-dimensional in space). The above expression for the density correlation function, if the hopping kernels were absent, would give a closed equation for $C_a(t)$ alone, which would precisely correspond to the schematic version of the idealized MCT.[14,15] However, contrary to our expectation that the mean field type model may give rise to a dynamic transition, it is found that the so-called hopping processes still enter even in the mean field level in our model due to the presence of the velocity-like variables. This surprising feature of the model also indicates that the hopping processes here are not the barrier-crossing ones since there is no energy barrier in the Gaussian Hamiltonian of the model. Furthermore, it turns out that we find no parameter (temperature) region in which these hopping kernels become self-consistently small so that one can see a continuous slowing down as the "rounded transition" is approached. Although SDD has raised the possibility of this self-consistent suppression of the strength of the hopping kernels in their work,[18] they have not explicitly shown that that is indeed the case. It would be interesting to investigate whether extended versions of the toy model that include spatial degrees of freedom[42] can qualitatively change this conclusion.

At this point, we were fairly disappointed with the fact that neither the sharp dynamic transition of the idealzed MCT nor the glassy behavior were found in our mean field model with reversible mode coupling. Remember that our model is one of the types of models that are purported to describe transitions in real fluids. It was later realized that one way of controlling the strength of the hopping kernels is to assign different numbers of components M and N for the velocity-like and the density-like variables, respectively. To keep the mean field nature of the model, M and N are taken to the infinity while their ratio $\delta^* \equiv M/N$ is kept finite in the range $0 < \delta^* < 1$. This is a kind of way of blocking the channels through which the system can relax. We thus found that a toy model with $\delta^* < 1$ qualitatively changes the nature of the model. Indeed, reducing δ^* is found to lead to the suppression of the hopping process since the hopping kernels become proportional to δ^*, and the model can exhibit a sharp ergodic-to-nonergodic phase transition. We have also shown that the diffusion matrix of the reduced Fokker–Planck equation (see eqs 6 and 7 below) involving the density variables only, which is obtained by adiabatically eliminating the velocity-like variables in the large damping limit, become singular for $0 < \delta^* < 1$, and hence that the nonequilibrium stationary state can exist. The nonergodic phase should correspond to this nonequilibrium stationary state.

Another new key feature in the model with $\delta^* < 1$ is natural emergence of a new slow variable $\hat{a}_\alpha \equiv K_{i\alpha}a_i$, where $K_{i\alpha}$ is the linear coupling constant. Thus, we have two kinds of density correlation functions, $C_a(t) \equiv (1/N)\,\langle a_i(t)a_i(0)\rangle$ and $C_a^K(t) \equiv (1/M)\,\langle \hat{a}_\alpha(t)\hat{a}_\alpha(0)\rangle = (1/M)\,K_{i\alpha}K_{j\alpha}\,\langle a_i(t)a_j(0)\rangle$, where the summation is implied for repeated indices here and after. Note that the matrix $K_{i\alpha}K_{j\alpha}$ cannot be an identity matrix δ_{ij} for $\delta^* < 1$. Only when $\delta^* = 1$, this matrix becomes an identity matrix and then $C_a(t) = C_a^K(t)$. The mean field analysis of the model for $\delta^* < 1$ thus gives a self-consistently closed set of equations for five correlation functions, $C_a(t)$, $C_a^K(t)$, $C_b(t)$, $C_{ab}(t)$, and $C_{ba}(t)$, where the correlators involving the velocity-like variables are defined as $C_b(t) \equiv (1/M)\,\langle b_\alpha(t)b_\alpha(0)\rangle$, $C_{ab}(t) \equiv (1/M)\,\langle \hat{a}_\alpha(t)b_\alpha(0)\rangle$, and $C_{ba}(t) \equiv (1/M)\,\langle b_\alpha(t)\hat{a}_\alpha(0)\rangle$. The Laplace transformed density correlation functions can be written down as

$$C_a^{KL}(z) = \frac{T}{\omega^2}\cdot\left\{z + \sum_{aa}^L(z) + \frac{\omega^2[1-\sum_{ab}^L(z)]^2}{z+\gamma+\sum_{bb}^L(z)}\right\}^{-1}$$

and

$$C_a^L(z) - \delta^* C_a^{KL}(z) = \frac{T}{\omega^2}\cdot\frac{(1-\delta^*)}{z+\sum_{aa}^L(z)}$$

Because the hopping kernels $\sum_{aa}$ and $\sum_{ab}$ vanish as δ^* goes to zero (see eq 9 below), we would have obtained in the limit $\delta^* \to 0$ the self-consistent MC equation for $C_a^K(t)$ if the kernel $\sum_{bb}^L(z)$ were a function of bilinear product of $C_a^K(t)$ instead of $C_a(t)$. Here again, although for the model with $\delta^* < 1$ we can control the strength of the hopping kernels by tuning δ^* and thereby can obtain the glassy behavior and the kinetic transition, we simply failed to obtain the original form of the ideal MCT. Because of the distinction between these two density correlation functions for $\delta^* < 1$, the slow dynamics and the resulting dynamic transition are different from the ones in the original idealized MCT.

Another possible way of reducing the strength of the hopping kernels is to consider the case of the damping constant for the velocity-like variables going to infinity. Here, one might expect that the velocity-like variables quickly decay away in very short times and as a result the hopping strength would vanish. In this paper, we pursue this possibility and present the equilibrium dynamics of the model in the large overdamping limit. As shown in detail below, the above expectation turns out to be too naive: the velocity-like variable does not simply relax away within a short time. Instead, after a very fast transitent, it relaxes rather slowly due to the complex coupling to the density-like variables. As a result, the hopping does not vanish even in the infinite damping limit. This is shown through a nontrivial separation of transient dynamics and long time dynamics and through interesting scaling phenomenon in the late time dynamics. In addition to this, we present analytic arguments that there must be a dynamic ergodic–nonergodic phase transition at a finite coupling for a given value of δ^* such that $0 < \delta^* < 1$.

It seems to us that through all our efforts we come to an inevitable conclusion that the original MCT is not derived from our mean field toy model incorporating the reversible mode coupling. Our model provides a quite different dynamic scenario from the original MCT. This elusiveness of the original MCT casts some doubt on the possibility of obtaining the ideal MCT within mean field type models with reversible mode coupling. In fact, the mean field models, which give rise to the idealized MCT, are those with dissipative dynamics only. One class of these models are the p-spin models[30-32] and the spherical Amit-Roginsky model[33] with dissipative dynamics,[34] where the glassy dynamics and the dynamic transition are due to the complex energy landscape of the Hamiltonian with or without quenched disorder. Both models have additive thermal noises. Another class of models exhibiting a sharp transition very similar to the one of the idealized MCT is the kinetically constrained models[35,36] on Bethe lattices.[37,38] Because these models have noninteracting Hamiltonians, the sharp dynamic transition is due to the kinetic constraints. However, here again, the dynamics is of a dissipative nature. However, one should note that originally the ideal glass transition predicted by the MCT was applied for fluid, which has reversible mode coupling. It is well-known that the equilibrium dynamics of spherical p-spin model with $p = 3$ above the dynamic transition gives the schematic version of the ideal MCT. It has been suggested[39] from this correspondence that the original fomulation of the idealized MCT is a mean field description of the liquid−glass transition. The perplexing elusiveness of the ideal MCT within a mean field toy fluid model incorporating the reversible mode coupling seems to compel us to reassess the true nature of the above well-known correspondence.

When the infinite damping limit is taken *before* the mean field limit, the reversible mode coupling can be adiabatically eliminated to get a purely dissipative model still with the trivial Hamiltonian (see eqs 6 and 7 below). However, the diffusion matrix in the resulting Fokker−Planck equation becomes singular for $\delta^* < 1$, providing a room for nonergodicity. In this sense, our model has something more common to kinetically constrained models,[23] except that ours is derived from the model with reversible mode coupling of the type appeared in the original MCT.

II. Toy Model of Dense Fluid

A. Langevin Equation. Our toy model is defined as the following stochastic dynamic equations incorporating the reversible mode coupling between the N-component density-like variables $a_i(t)$ with $i = 1, 2, ..., N$ and the M-component velocity-like variables b_α with $\alpha = 1, 2, ..., M$.

$$\begin{aligned}
\partial_t a_i &= K_{i\alpha} b_\alpha + \omega J_{ij\alpha} a_j b_\alpha \\
\partial_t b_\alpha &= -\gamma b_\alpha - \omega^2 K_{j\alpha} a_j - \omega J_{ij\alpha}(\omega^2 a_i a_j - T\delta_{ij}) + f_\alpha \\
\langle f_\alpha(t) f_\beta(t')\rangle &= 2\gamma T \delta_{\alpha\beta}\delta(t-t') \\
\overline{J_{ij\alpha} J_{kl\beta}}^J &= \frac{g^2}{N^2}[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{\alpha\beta} + K_{i\beta}(K_{k\alpha}\delta_{jl} + K_{l\alpha}\delta_{jk}) + K_{j\beta}(K_{k\alpha}\delta_{il} + K_{l\alpha}\delta_{ik})]
\end{aligned} \tag{1}$$

Angular brackets and overbars indicate averages over thermal noise and quenched disorder, respectively. Throughout this work, we use Roman indices for the components of a and Greek for those of b. The model can be viewed as a set of oscillators with linear and random nonlinear mode couplings, $K_{i\alpha}$ and $J_{ij\alpha}$, respectively. Here, γ is the decay rate of the velocity-like variables b_α and ω gives a measure of the frequencies of oscillations of the density-like variables a_i. The thermal noises $f_\alpha(t)$ are independent Gaussian random variables with zero mean and variance $2\gamma T$, T being the temperature of the heat bath with which the system has a thermal contact.

The linear coupling $K_{i\alpha}$ satisfies the (one-sided) orthogonality for $M < N$

$$K_{i\alpha} K_{i\beta} = \delta_{\alpha\beta}, \quad K_{i\alpha} K_{j\alpha} \neq \delta_{ij} \tag{2}$$

For the special case of $M = N$, one can impose an additional condition $K_{i\alpha} = \delta_{i\alpha}$ and hence trivially $K_{i\alpha}K_{j\alpha} = \delta_{ij}$. We also note that $K_{i\alpha}$ governs linearized reversible dynamics of the model with the dynamical matrix Ω given by $\Omega_{ij} \equiv \omega^2 K_{i\alpha} K_{j\alpha}$. The mode coupling coefficients $J_{ij\alpha}$ are chosen to be the quenched Gaussian random variables with zero mean and variance specified in eq 1. It should be emphasized again that the present mean field model is distinct from other mean field models for glasses in that the model is constructed so as to closely mimic the MCT. In particular, the equation for a_i in eq 1 is analogous to the continuity equation of fluid, and the rhs of the equation for b_α in eq 1 is like the force acting on a fluid element.

B. Corresponding Fokker−Planck Equation. One can derive from the Langevin eq 1 the corresponding Fokker−Planck equation for the probability distribution function $P(\{a\},\{b\}, t)$ as follows:

$$\partial_t P(\{a\},\{b\}, t) \equiv \mathcal{L} P(\{a\},\{b\}, t) \tag{3}$$

where the Fokker−Planck operator $\mathcal{L}$ can be decomposed into $\mathcal{L} \equiv \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{MC}$ with

$$\begin{aligned}
\mathcal{L}_0 &\equiv \frac{\partial}{\partial b_\alpha}\gamma\left(T\frac{\partial}{\partial b_\alpha} + b_\alpha\right), \quad \mathcal{L}_1 \equiv K_{j\alpha}\left(-\frac{\partial}{\partial a_j} b_\alpha + \frac{\partial}{\partial b_\alpha}\omega^2 a_j\right), \\
\mathcal{L}_{MC} &\equiv J_{ij\alpha}\left[-\frac{\partial}{\partial a_i}\omega a_j b_\alpha + \frac{\partial}{\partial b_\alpha}\omega(\omega^2 a_i a_j - T\delta_{ij})\right]
\end{aligned} \tag{4}$$

It is readily shown, independent of the distribution of $J_{ij\alpha}$, that the equilibrium stationary distribution is given by

$$P_{EQ}(\{a\},\{b\}) = cst.\exp\left(-\sum_{j=1}^{N}\frac{\omega^2}{2T}a_j^2 - \sum_{\alpha=1}^{M}\frac{1}{2T}b_a^2\right) \tag{5}$$

where *cst.* is the normalization factor.

One can adiabatically eliminate the variables $\{b\}$ in the limit of large γ before taking the mean field limit and obtain the Fokker−Planck equation for the reduced probability distribution $\tilde{P}(\{a\}, t)$ containing only the $\{a\}$ variables:

$$\frac{\partial \tilde{P}(\{a\}, t)}{\partial t} = \frac{\partial}{\partial a_i}\left[Q_{ij}(\{a\})\left(\frac{\partial}{\partial a_j} + \frac{\omega^2}{T}a_j\right)\tilde{P}(\{a\}, t)\right] \tag{6}$$

where the diffusion matrix $Q_{ij}(\{a\})$ is given by

$$Q_{ij}(\{a\}) \equiv \frac{T}{\gamma}(K_{i\alpha} + \omega J_{ik\alpha} a_k)(K_{j\alpha} + \omega J_{jl\alpha} a_l) \tag{7}$$

We have pointed out that the diffusion matrix $Q_{ij}(\{a\})$ is singular for $M < N$, which implies that the Fokker−Planck eq 6 can have a nonequilibrium stationary solution.[10d] These nonequilibrium stationary solutions can precisely correspond to the nonergodic states observed in the present model. Also note that although the corresponding Langevin equation for eqs 6 and 7 becomes purely dissipative, it has a multiplicative thermal noise $\eta_i(t)$ with density-dependent variance $\langle \eta_i(t)\eta_j(t')\rangle = 2Q_{ij}(\{a\})\,\delta(t - t')$. This is a contrasting feature as compared to other dissipative mean field models such as the spherical p-spin model where the thermal noise is additive. It can be said that the reversible mode coupling appeared in the original toy model is reflected in the multiplicative structure of the thermal noise in the reduced dissipative toy model.[19]

III. Infinite Damping Limit and Scaling Properties

We have derived before via a standard generating functional method[10] the following closed set of equations for the five correlation functions, which is exact in the limit M, $N \rightarrow \infty$ with the ratio $\delta^* \equiv M/N$ kept finite with $0 \leq \delta^* \leq 1$:

$$\partial_t C_a(t) = \delta^* C_{ba}(t) - \sum_{aa}\otimes C_a(t) - \delta^* \sum_{ab}\otimes C_{ba}(t)$$
$$\partial_t C_{ab}(t) = C_b(t) - \sum_{aa}\otimes C_{ab}(t) - \sum_{ab}\otimes C_b(t)$$
$$\partial_t C_a^K(t) = C_{ba}(t) - \sum_{aa}\otimes C_a^K(t) - \sum_{ab}\otimes C_{ba}(t)$$
$$\gamma^{-1}\,\partial_t C_{ba}(t) = C_{ba}(t) - \frac{\omega^2}{\gamma}C_a^K(t) - \gamma^{-1}\sum_{ba}\otimes C_a^K(t) - \gamma^{-1}\sum_{bb}\otimes C_{ba}(t)$$
$$\gamma^{-1}\,\partial_t C_b(t) = -C_b(t) - \frac{\omega^2}{\gamma}C_{ab}(t) - \gamma^{-1}\sum_{ba}\otimes C_{ab}(t) - \gamma^{-1}\sum_{bb}\otimes C_b(t) \quad (8)$$

where the kernels Σ are given by

$$\sum_{aa}(t) \equiv \delta^* \frac{g^2\omega^4}{T}\,[C_a(t)C_b(t) + \delta^* C_{ab}(t)C_{ba}(t)],$$
$$\sum_{ab}(t) \equiv -2\delta^* \frac{g^2\omega^4}{T}\,C_a(t)C_{ba}(t),$$
$$\sum_{ba}(t) \equiv 2\delta^* \frac{g^2\omega^4}{T}\,C_a(t)C_{ab}(t), \quad \sum_{bb}(t) \equiv 2\frac{g^2\omega^6}{T}\,C_a^2(t) \quad (9)$$

and the symbols $\otimes$ have the following meaning: $A\otimes B(t) \equiv \int_0^t ds\, A(t-s)B(s)$. Note that the kernel $\Sigma_{bb}(t)$ does not involve the parameter δ^*. In deriving eqs 8 and 9, we have used the flucutuation−dissipation relations between the correlation and the response functions, which hold for the Gaussian Hamiltonian.[41] The initial conditions for the correlators are given by their equilibrium values

$$C_a(0) = C_a^K(0) = T/\omega^2, \quad C_b(0) = T, \quad C_{ab}(0) = C_{ba}(0) = 0 \quad (10)$$

For the case of $\delta^* = 1$ in which $C_a(t) = C_a^K(t)$, eq 8 has the same structure as the self-consistent one-loop results derived by SDD.[18] If there were no hopping kernels Σ_{aa} and Σ_{ab} (or equivalently Σ_{ba}), eq 8 would give the closed equation for $C_a(t) = C_a^K(t)$ only, which is a schematic version of the idealized MCT. We note that the hopping kernels arise from the nonlinear coupling between the density-like and the velocity-like variables $\omega J_{ij\alpha}a_j b_\alpha$ in the equation for the density-like variables a_i, whereas the kernel Σ_{bb} from the density nonlinearity $\omega^3 J_{ij\alpha}a_i a_j$ in the equation for the velocity-like variables b_α in eq 1.

As described in the Introduction, it is a unique feature of the model that the hopping strength can be adjusted by varying the parameter δ^* and at the same time a new kind of slow variable [whose correlator is $C_a^K(t)$] appears in the dynamics for $\delta^* < 1$. In this section, we analyze the closed set of eq 8 in the limit $\gamma \rightarrow \infty$. To this end, we find it instructive first to look at the linear dynamics ($g = 0$).

A. Linear Dynamics. Setting $g = 0$ in eq 8, all of the kernels drop out, and one has

$$\partial_t C_a(t) = \delta^* C_{ba}(t), \quad \partial_t C_{ab}(t) = C_b(t), \quad \partial_t C_a^K(t) = C_{ba}(t),$$
$$\gamma^{-1}\,\partial_t C_{ba}(t) = -C_{ba}(t) - \frac{\omega^2}{\gamma}C_a^K(t),$$
$$\gamma^{-1}\,\partial_t C_b(t) = -C_b(t) - \frac{\omega^2}{\gamma}C_{ab}(t) \quad (11)$$

It is easy to obtain the exact solution of the above linear equations, which in the limit of $\gamma \rightarrow \infty$ can be written as

$$C_b(t) = T\left(1 + \frac{\omega^2}{\gamma^2}\right)e^{-\gamma t} - \frac{\omega^2}{\gamma^2}Te^{-\omega^2 t/\gamma},$$
$$C_{ab}(t) = -C_{ba}(t) = -\frac{T}{\gamma}(e^{-\gamma t} - e^{-\omega^2 t/\gamma}),$$
$$C_a^K(t) = \frac{T}{\gamma^2}(1 - e^{-\gamma t}) + \frac{T}{\omega^2}e^{-\omega^2 t/\gamma},$$
$$C_a(t) = \frac{\delta^* T}{\gamma^2}(1 - e^{-\gamma t}) + \frac{T}{\omega^2}[(1-\delta^*) + \delta^* e^{-\omega^2 t/\gamma}] \quad (12)$$

The form of these solutions reveals a general structure of the dynamics, which holds in the nonlinear situation as well. In particular, the solution eq 12 shows that each correlator consists of two parts. The fast dynamics is associated with the time scale $\tau_s \equiv \gamma^{-1}$ whereas the slow dynamics is with the time scale $\tau_l \equiv \gamma/\omega^2$. Hence, as $\gamma \rightarrow \infty$, the fast dynamics forms an extremely narrow time layer in the short time region, in which the relaxation is very rapid. The fast parts of the density correlators C_a and C_a^K have extra factors γ^{-2} as compared to the slow ones and hence are negligible. The slow part of $C_a(t)$ has a nonvanishing component $(1 - \delta^*)T/\omega^2$ as $t \rightarrow \infty$. Both slow and fast parts of the correlators C_{ab} and C_{ba} have an extra factor γ^{-1}. The velocity correlator C_b has a "big" fast part, which has the amplitude $O(1)$, and a "small" slow part, which has the amplitude $O(\gamma^{-2})$. It turns out that this fast part of $C_b(t)$ makes an important contribution when we try to derive the equations for the slow dynamics in the nonlinear case. Note that slow part of $C_a(t) = T/\omega^2\,[(1-\delta^*) + \delta^* e^{-t/\tau_l}]$ has the amplitude T/ω^2 for $t \ll \tau_l$ and $(T/\omega^2)\,(1-\delta^*)$ for $t \gg \tau_l$.

B. Nonlinear Dynamics. In light of the form of the linear solution eq 12, defining the rescaled time $\hat{t} \equiv t/\tau_l$, we make the following ansatz for the correlators in the presence of nonlinear interaction ($g > 0$), which separates the fast and slow parts as

$$C_b(t) = T\left(1 + \frac{\omega^2}{\gamma^2}\right)e^{-\gamma t} + T\frac{\omega^2}{\gamma^2}\mathcal{C}_b(\hat{t}),$$
$$C_{ab}(t) = -\frac{T}{\gamma}e^{-\gamma t} + \frac{T}{\gamma}\,\mathcal{C}_{ab}(\hat{t}), \quad C_{ba}(t) = \frac{T}{\gamma}e^{-\gamma t} + \frac{T}{\gamma}\,\mathcal{C}_{ba}(\hat{t}),$$
$$C_a^K(t) = \frac{T}{\gamma^2}(1 - e^{-\gamma t}) + \frac{T}{\omega^2}\mathcal{C}_a^K(\hat{t}),$$
$$C_a(t) = \frac{\delta^* T}{\gamma^2}(1 - e^{-\gamma t}) + \frac{T}{\omega^2}\mathcal{C}_a(\hat{t}) \quad (13)$$

Substituting this ansatz to eq 8, we obtain the equations for the dimensionless correlators $C(\hat{t})$ values in the limit $\gamma \rightarrow \infty$

$$\partial_{\hat{t}}\mathcal{C}_a(\hat{t}) = \delta^*\mathcal{C}_{ba}(\hat{t}) - \lambda\delta^*\mathcal{C}_a(\hat{t}) - \Psi_{aa}\otimes\mathcal{C}_a(\hat{t}) - \delta^*\Psi_{ab}\otimes\mathcal{C}_{ba}(\hat{t})$$
$$\partial_{\hat{t}}\mathcal{C}_{ab}(\hat{t}) = \mathcal{C}_b(\hat{t}) - \lambda\delta^*\mathcal{C}_{ab}(\hat{t}) - \Psi_{ab}(\hat{t}) - \Psi_{aa}\otimes\mathcal{C}_{ab}(\hat{t}) - \Psi_{ab}\otimes\mathcal{C}_b(\hat{t})$$
$$\partial_{\hat{t}}\mathcal{C}_a^K(\hat{t}) = \mathcal{C}_{ba}(\hat{t}) - \lambda\delta^*\mathcal{C}_a^K(\hat{t}) - \Psi_{aa}\otimes\mathcal{C}_a^K(\hat{t}) - \Psi_{ab}\otimes\mathcal{C}_{ba}(\hat{t})$$
$$\mathcal{C}_{ba}(\hat{t}) = -\mathcal{C}_a^K(\hat{t}) - \Psi_{ba}\otimes\mathcal{C}_a^K(\hat{t}) - \Psi_{bb}\otimes\mathcal{C}_{ba}(\hat{t})$$
$$\mathcal{C}_b(\hat{t}) = -\mathcal{C}_{ab}(\hat{t}) - \Psi_{bb}(\hat{t}) - \Psi_{ba}\otimes\mathcal{C}_{ab}(\hat{t}) - \Psi_{bb}\otimes\mathcal{C}_b(\hat{t}) \quad (14)$$

where $\lambda \equiv g^2 T$ is the effective coupling constant in the model and the scaled memory kernels Ψ are given by

$$\Psi_{aa}(\hat{t}) \equiv \lambda\delta^*[\mathcal{C}_a(\hat{t})\mathcal{C}_b(\hat{t}) + \delta^*\mathcal{C}_{ab}(\hat{t})\mathcal{C}_{ba}(\hat{t})],$$
$$\Psi_{ab}(\hat{t}) \equiv -2\lambda\delta^*\mathcal{C}_a(\hat{t})\mathcal{C}_{ba}(\hat{t}),$$
$$\Psi_{ba}(\hat{t}) \equiv -2\lambda\delta^*\mathcal{C}_a(\hat{t})\mathcal{C}_{ab}(\hat{t}), \quad \Psi_{bb}(\hat{t}) \equiv 2\lambda\mathcal{C}_a^2(\hat{t}) \quad (15)$$

Because we are dealing with the slow part of the dynamics, we would lose the fast relaxation occurring in the thin layer of the early time region in the limit of infinite damping. Hence, the intial conditions for the correlators involving the velocity variable become *different* from those for the original dynamic equations. The initial conditions for the density correlators $\mathcal{C}_a$ and $\mathcal{C}_a^K$ are not modified by the nonlinear terms:

$$\mathcal{C}_a(0) = \mathcal{C}_a^K(0) = 1 \quad (16)$$

The initial conditions for the correlators involving the b variables are given by

$$\mathcal{C}_{ba}(0) = -\mathcal{C}_a^K(0) = -1, \quad \mathcal{C}_{ab}(0) = 1,$$
$$\mathcal{C}_b(0) = -\mathcal{C}_{ab}(0) - 2\lambda\mathcal{C}_a^2(0) = -(1 + 2\lambda) \quad (17)$$

where the first condition comes from the fourth member of eq 14 by setting $\hat{t} = 0$. The initial condition for $\mathcal{C}_{ab}(\hat{t})$ follows from the time-reversal symmetry $\mathcal{C}_{ab}(\hat{t}) = -\mathcal{C}_{ba}(\hat{t})$. The initial value of $\mathcal{C}_b(\hat{t})$ follows from the last member of eq 14. Note that $C_b(0)$ depends on the coupling strength λ.

Performing the Laplace transform $\mathcal{C}^L(\sigma) \equiv \int_0^\infty d\hat{t}\ e^{-\sigma\hat{t}}\ \mathcal{C}(\hat{t})$ on eq 14, one can express the correlators in terms of the memory kernels:

$$\mathcal{C}_a^{KL}(\sigma) = \left\{\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma) + \frac{[1 - \Psi_{ab}^L(\sigma)]^2}{[1 + \Psi_{bb}^L(\sigma)]}\right\}^{-1}$$

$$\mathcal{C}_a^L(\sigma) = \frac{(1-\delta^*)}{\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma)} + \delta^*\mathcal{C}_a^{KL}(\sigma)$$
$$= \frac{1}{\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma)} \cdot$$
$$\left\{1 - \delta^*\frac{[1 - \Psi_{ab}^L(\sigma)]^2}{[\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma)]\,[1 + \Psi_{bb}^L(\sigma)] + [1 - \Psi_{ab}^L(\sigma)]^2}\right\}$$

$$\mathcal{C}_{ab}^L(\sigma) = -\ \mathcal{C}_{ba}^L(\sigma) = \frac{[1 - \Psi_{ab}^L(\sigma)]}{[\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma)]\,[1 + \Psi_{bb}^L(\sigma)] + [1 - \Psi_{ab}^L(\sigma)]^2}$$

$$\mathcal{C}_b^L(\sigma) = -1 + \frac{[\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma)]}{[\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma)]\,[1 + \Psi_{bb}^L(\sigma)] + [1 - \Psi_{ab}^L(\sigma)]^2} \quad (18)$$

Expanding $\Psi_{aa}^L(\sigma)$ for vanishing σ as $\Psi_{aa}^L(\sigma) = \Psi_{aa}^L(0) + \phi_{aa}^L(0)\sigma$ with $\phi_{aa}^L(\sigma) \equiv d\Psi_{aa}^L(\sigma)/d\sigma$, one can express $[\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma)]$ in the limit $\sigma \to 0$ using eq 15 as

$$\sigma + \lambda\delta^* + \Psi_{aa}^L(\sigma) = \mu\sigma + \lambda\delta^* + \Psi_{aa}^L(0) = \mu\sigma + \lambda\delta^*\{1 + \int_0^\infty \mathrm{d}t\,[\mathcal{C}_a(t)\mathcal{C}_b(t) + \delta^*\mathcal{C}_{ab}(t)\mathcal{C}_{ba}(t)]\} \quad (19)$$

where $\mu \equiv 1 + \phi_{aa}^L(0) = 1 - \lambda\delta^* \int_0^\infty \mathrm{d}t\ t\,[\mathcal{C}_a(t)\mathcal{C}_b(t) + \delta^*\mathcal{C}_{ab}(t)\mathcal{C}_{ba}(t)]$. Here and after, we suppress hats on t. The above small σ expansion is based on the physical expectation that the correlators involving the b variables $\mathcal{C}_{ab}(t)$, $\mathcal{C}_{ba}(t)$, and $\mathcal{C}_b(t)$ always decay to zero at long times so that neither $\Psi_{aa}^L(\sigma)$ nor $\Psi_{ab}^L(\sigma)$ develop a $1/\sigma$ pole for vanishing σ.

We therefore see from eqs 18 and 19 that a necessary condition for the nonergodicity of the density correlators $\mathcal{C}_a$ and $\mathcal{C}_a^K$ is given by

$$\lambda\delta^* + \Psi_{aa}^L(0) \equiv \lambda\delta^*\{1 + \int_0^\infty \mathrm{d}t\,[\mathcal{C}_a(t)\mathcal{C}_b(t) + \delta^*\mathcal{C}_{ab}(t)\mathcal{C}_{ba}(t)]\} = 0 \quad (20)$$

When this nonergodicity condition is fulfilled, the Laplace-transformed density correlators develop $1/\sigma$ pole in the limit of vanishing σ: $\mathcal{C}_a^L(\sigma) = f_a/\sigma$ and $\mathcal{C}_a^{KL}(\sigma) = f_a^K/\sigma$ where f_a and f_a^K are called the nonergodicity parameters. This means that $\mathcal{C}_a(t)$ and $\mathcal{C}_a^K(t)$ can have nondecaying components in the long time limit: $\mathcal{C}_a(t \to \infty) \equiv f_a$ and $\mathcal{C}_a^K(t \to \infty) \equiv f_a^K$. The nonergodicity parameters f_a and f_a^K are then related to each other as

$$f_a^K = \frac{2\lambda f_a^2}{2\mu\lambda f_a^2 + [1 - \Psi_{ab}^L(0)]^2}, \quad f_a = \frac{(1-\delta^*)}{\mu} + \delta^* f_a^K \quad (21)$$

where $\Psi_{ab}^L(0) \equiv -2\lambda\delta^* \int_0^\infty \mathrm{d}t\ \mathcal{C}_a(t)\mathcal{C}_{ba}(t) = 2\lambda\delta^* \int_0^\infty \mathrm{d}t\ \mathcal{C}_a(t)\mathcal{C}_{ab}(t)$. The first expression in eq 21 is obtained from the first member of eq 18 via the fact that the dominant term of $\Psi_{bb}^L(\sigma)$ for small σ is given by $\Psi_{bb}^L(\sigma) = 2\lambda f_a^2/\sigma$. In contrast to the case of the idealized MCT, the analytic determination of the dynamic transition line is not possible here since the correlators $\mathcal{C}_{ab}(t)$ and $\mathcal{C}_b(t)$ are involved in eq 21 via μ and $\Psi_{ab}^L(0)$.

Also, when the nonergodicity condition eq 20 holds, we note from the last two members of eq 18 that

$$\mathcal{C}_{ab}^L(0) \equiv \int_0^\infty \mathrm{d}t\ \mathcal{C}_{ab}(t) = \frac{[1 - \Psi_{ab}^L(0)]}{2\mu\lambda f_a^2 + [1 - \Psi_{ab}^L(0)]^2},$$
$$\mathcal{C}_b^L(0) \equiv \int_0^\infty \mathrm{d}t\ \mathcal{C}_b(t) = -1 \quad (22)$$

These equations imply that $\mathcal{C}_{ab}(t)$ and $\mathcal{C}_b(t)$ should decay to zero in the long times even in the nonergodic phase, which provides a posterior consistency for the above physical assumption that $\mathcal{C}_{ab}(t)$ and $\mathcal{C}_b(t)$ are ergodic. This is also in accord with the numerical solutions of eq 14, which show that the correlators involving the b variables monotonically relax to zero for general values of λ and δ^*.

C. Presence of the Dynamic Transition. In this section, we analytically show that the model must possess a dynamic transition at a nonzero coupling λ for a given δ^* such that $0 < \delta^* < 1$. To this end, we first show that the model has a nonergodicity at $\lambda = 0$. We then proceed to prove that the system becomes ergodic in the limit of $\lambda \to \infty$ for nonzero δ^*.

1. $\lambda = 0$. In this case, the solution is obtained from eqs 12 and 13 as

$$\mathcal{C}_a^{(0)}(t) = (1-\delta^*) + \delta^* e^{-t},$$
$$\mathcal{C}_a^{K(0)}(t) = \mathcal{C}_{ab}^{(0)}(t) = -\mathcal{C}_b^{(0)}(t) = -\mathcal{C}_{ba}^{(0)}(t) = e^{-t} \quad (23)$$

Therefore, in the zeroth order, only the density correlator $\mathcal{C}_a(t)$ exhibits a nonergodic behavior. This nonergodicity is intimately connected to the singularity of the dynamic matrix $\Omega_{ij} \equiv \omega^2 K_{i\alpha}K_{j\alpha}$. The nonergodicity found here is reminiscent of the nonergodicity in noninteracting systems such as ideal gas.

2. *$\lambda \to \infty$ Limit.* We now show that in the limit $\lambda \to \infty$, the system is always ergodic as long as $\delta^* > 0$. We make the

following scaling ansatz of the correlators under the rescaling of time $t \to \bar{t} \equiv \lambda t$:

$$C_a(t) \equiv \bar{\mathcal{C}}_a(\bar{t}), \quad \mathcal{C}_a^K(t) \equiv \bar{\mathcal{C}}_a^K(\bar{t}), \quad \mathcal{C}_{ab}(t) \equiv \bar{\mathcal{C}}_{ab}(\bar{t}),$$
$$\mathcal{C}_{ba}(t) \equiv \bar{C}_{ba}(\bar{t}) \ \mathcal{C}_b(t) \equiv \lambda \bar{C}_b(\bar{t}) \tag{24}$$

Note that unlike other correlators, $\mathcal{C}_b(t)$ has one extra λ factor in front. This is natural since its initial value scales with λ (see eq 17), i.e., $\mathcal{C}_b(t = 0) \to -2\lambda$ as λ goes to infinity.

Using the ansatz eq 24, one can readily obtain the equations for $\bar{\mathcal{C}}_a$, $\bar{\mathcal{C}}_a^K$, and $\bar{\mathcal{C}}_b$ from eqs 14 and 15:

$$\partial_{\bar{t}} \bar{\mathcal{C}}_a(\bar{t}) = -\delta^* \bar{\mathcal{C}}_a(\bar{t}) - \delta^* \int_0^{\bar{t}} d\bar{s}\, [\bar{\mathcal{C}}_a \bar{\mathcal{C}}_b](\bar{t} - \bar{s}) \bar{\mathcal{C}}_a(\bar{s})$$
$$\partial_{\bar{t}} \bar{\mathcal{C}}_a^K(\bar{t}) = -\delta^* \bar{\mathcal{C}}_a^K(\bar{t}) - \delta^* \int_0^{\bar{t}} d\bar{s}\, [\bar{\mathcal{C}}_a \bar{\mathcal{C}}_b](\bar{t} - \bar{s}) \bar{\mathcal{C}}_a^K(\bar{s})$$
$$\bar{\mathcal{C}}_b(\bar{t}) = -2\bar{\mathcal{C}}_a^2(\bar{t}) - \int_0^{\bar{t}} d\bar{s}\, 2\bar{\mathcal{C}}_a^2(\bar{t} - \bar{s}) \bar{\mathcal{C}}_b(\bar{s}) \tag{25}$$

where we have not written the equations for $\bar{\mathcal{C}}_{ab}$ and $\bar{\mathcal{C}}_{\mathrm{ba}}$. Note in eq 25 that the cross-correlators $\bar{\mathcal{C}}_{ab}$ and $\bar{\mathcal{C}}_{\mathrm{ba}}$ are absent since the terms involving them drop out in the limit $\lambda \to \infty$. This renders the equations for $\bar{\mathcal{C}}_a$ and $\bar{\mathcal{C}}_{\mathrm{b}}$ closed. Furthermore, the second member of eq 25 suggests that $\bar{C}_a(\bar{t}) = \bar{\mathcal{C}}_a^K(\bar{t})$ for all times since $\bar{\mathcal{C}}_{\mathrm{a}}(0) = \bar{\mathcal{C}}_a^K(0)$.

Let us focus on the closed equations for $\bar{\mathcal{C}}_a$ and $\bar{\mathcal{C}}_b$. Their Laplace transforms (represented by superfixes L) are given, respectively, by

$$\bar{\mathcal{C}}_a^L(\bar{\sigma}) = \frac{1}{\bar{\sigma} + \delta^*(1 + [\bar{\mathcal{C}}_a(\bar{t})\bar{\mathcal{C}}_b(\bar{t})]^L(\bar{\sigma}))},$$
$$\bar{\mathcal{C}}_b^L(\bar{\sigma}) = \frac{-2[\bar{\mathcal{C}}_a^2(\bar{t})]^L(\bar{\sigma})}{1 + 2[\bar{\mathcal{C}}_a^2(\bar{t})]^L(\bar{\sigma})} \tag{26}$$

From the expression for $\bar{C}_b^L(\bar{\sigma})$ in eq 26, we obtain

$$1 + \bar{\mathcal{C}}_b^L(\bar{\sigma}) = \frac{1}{1 + 2[\bar{\mathcal{C}}_a^2(\bar{t})]^L(\bar{\sigma})} > 0 \text{ for all } \bar{\sigma} \tag{27}$$

Suppose that $\bar{\mathcal{C}}_a(\bar{t})$ is ergodic, i.e., it monotonically relaxes to zero from its initial value $\bar{\mathcal{C}}_a(0) = 1$. This assumption, together with eqs 26 and 27, implies since $\bar{\mathcal{C}}_b^L(\bar{\sigma}) < 0$

$$1 + [\bar{\mathcal{C}}_a(\bar{t})\bar{\mathcal{C}}_b(\bar{t})]^L(\bar{\sigma}) > 1 + \bar{\mathcal{C}}_b^L(\bar{\sigma}) > 0 \text{ for all } \bar{\sigma} \tag{28}$$

Equation 26 with eq 28 tells that $\bar{\mathcal{C}}_a(\bar{t})$ is indeed ergodic, which makes the assumption of $\bar{\mathcal{C}}_a(\bar{t})$ being ergodic self-consistent. That is, an ergodic solution exists, and in particular, we see that in the long time limit $\bar{\mathcal{C}}_a(\bar{t})$ exhibits an exponential relaxation of the form

$$\bar{\mathcal{C}}_a(\bar{t}) = \exp(-\bar{t}/\bar{t}_r) \text{ with } \bar{t}_r^{-1} \equiv \delta^*[1 + \int_0^\infty d\bar{t}\ \bar{\mathcal{C}}_a(\bar{t})\bar{\mathcal{C}}_b(\bar{t})] \tag{29}$$

We have tested the validity of the scaling ansatz eq 24 by comparing the numerical solutions of eqs 25 and 14 with a large value of λ, e.g., $\lambda = 10$. As shown in Figure 1a, the two numerical solutions (except $\bar{\mathcal{C}}_b$) become almost identical in the entire time region. As for $\bar{\mathcal{C}}_b$, the diagreement is seen only in the short time region, which is due to the finiteness of the value $\lambda = 10$. It is expected that this difference diminishes as λ becomes bigger. Shown in Figure 1b is the density correlator $\bar{\mathcal{C}}_a(\bar{t})$ for various values of δ^*. It continuously slows down as δ^* becomes smaller. The trivial nonergodicity occurs for $\delta^* = 0$, which is the only nonergodicity expected in the limit of $\lambda = \infty$.

In this subsection, we have demonstrated that for a fixed nonzero δ^* the model becomes always ergodic in the limit $\lambda = \infty$. Because the model is nonergodic at $\lambda = 0$ for $0 < \delta^* < 1$, we can imagine the following two scenarios for what would happen for finite λ: (i) An infinitesimal λ immediately destroys the nonergodic phase present in the linear dynamics. (ii) There exists a dynamic ergodicity−nonergodicity transition at some nonzero finite value λ_c. We demonstrate below that the model is in accord with scenario (ii).

3. *$\delta^* \to 0^+$ Limit.* In the above subsections, we considered the two extreme cases of $\lambda = 0$ and $\lambda = \infty$ for a given δ^*. Here, we consider the case of $\delta^* = 0^+$ for a fixed λ. This region corresponds to the most extreme nonergodic phase. In this case, Ψ_{bb} is the only nonvanishing kernel. Then setting $\delta^* = 0$ and substituting $\Psi_{bb}(t) \equiv 2\lambda C_a^2(t)$ in eq 14, we obtain

$$\partial_t \mathcal{C}_a(t) = 0, \quad \partial_t \mathcal{C}_{ab}(t) = \mathcal{C}_b(t), \quad \partial_t \mathcal{C}_a^K(t) = \mathcal{C}_{ba}(t),$$
$$\mathcal{C}_{ba}(t) = -\mathcal{C}_a^K(t) - 2\lambda \int_0^t ds\ \mathcal{C}_a^2(t - s)\mathcal{C}_{ba}(s),$$
$$\mathcal{C}_b(t) = -\mathcal{C}_{ab}(t) - \Psi_{bb}(t) - 2\lambda \int_0^t ds\ \mathcal{C}_a^2(t - s)\mathcal{C}_b(s) \tag{30}$$

We can readily obtain the solution of this set of equations:

$$\mathcal{C}_a(t) = \mathcal{C}_a(0) = 1, \quad \mathcal{C}_a^K(t) = \frac{2\lambda}{1 + 2\lambda} + \frac{1}{1 + 2\lambda} e^{-(1+2\lambda)t},$$
$$\mathcal{C}_{ab}(t) = -\mathcal{C}_{ba}(t) = e^{-(1+2\lambda)t},$$
$$\mathcal{C}_b(t) = -(1 + 2\lambda)e^{-(1+2\lambda)t} \tag{31}$$

We thus see that $\mathcal{C}_a$ does not decay at all, and $\bar{\mathcal{C}}_a^K$ has a nonvanishing component $2\lambda/(1 + 2\lambda)$ (for $\lambda > 0$). Note therefore that the system becomes nonergodic for arbitrary nonzero λ. This solution is consistent with those for the previous two cases $\lambda = 0$ and $\lambda = \infty$. That is, the $\lambda = 0$ limit of eq 31 is the same as the $\delta^* = 0$ limit of eq 23. Taking the large λ limit in eq 31 yields the solution for t less than $O(\lambda^{-1})$

$$\mathcal{C}_a(t) = \mathcal{C}_a^K(t) = 1, \quad \mathcal{C}_{ab}(t) = e^{-2\lambda t}, \quad \mathcal{C}_b(t) = -2\lambda e^{-2\lambda t}$$

This agrees with the solution of eq 25 with $\delta^* = 0$. Therefore, in the $\delta^*-\lambda$ plane, the $\lambda = 0$ line and the $\delta^* = 0$ line yield the nonergodic phases, whereas the other two lines $\delta^* = 1$ and $\lambda = \infty$ yield the ergodic phases (see Figure 5).

4. *Small δ^* and Small λ Behavior.* In this subsection, we show that in the small λ and δ^* region in the parameter space, the density correlators retain their nonergodic behavior. Because of the smallness of λ and δ^*, ignoring the terms proportional to $\lambda\delta^*$ or $\lambda\delta^{*2}$ in eq 14, we have

$$\partial_t \mathcal{C}_a(t) = \delta^* \mathcal{C}_{ba}(t), \quad \partial_t \mathcal{C}_{ab}(t) = \mathcal{C}_b(t), \quad \partial_t \mathcal{C}_a^K(t) = \mathcal{C}_{ba}(t),$$
$$\mathcal{C}_{ba}(t) = -\mathcal{C}_a^K(t) - 2\lambda \int_0^t ds\ \mathcal{C}_a^2(t - s)\mathcal{C}_{ba}(s),$$
$$\mathcal{C}_b(t) = -\mathcal{C}_{ab}(t) - \Psi_{bb}(t) - 2\lambda \int_0^t ds\ \mathcal{C}_a^2(t - s)\mathcal{C}_b(s) \tag{32}$$

Note that the only difference between eqs 30 and 32 is the equation for $\mathcal{C}_a(t)$. We note from the first and the third members of eq 32 that

$$\partial_t \mathcal{C}_a(t) - \delta^* \partial_t \mathcal{C}_a^K(t) = 0 \tag{33}$$

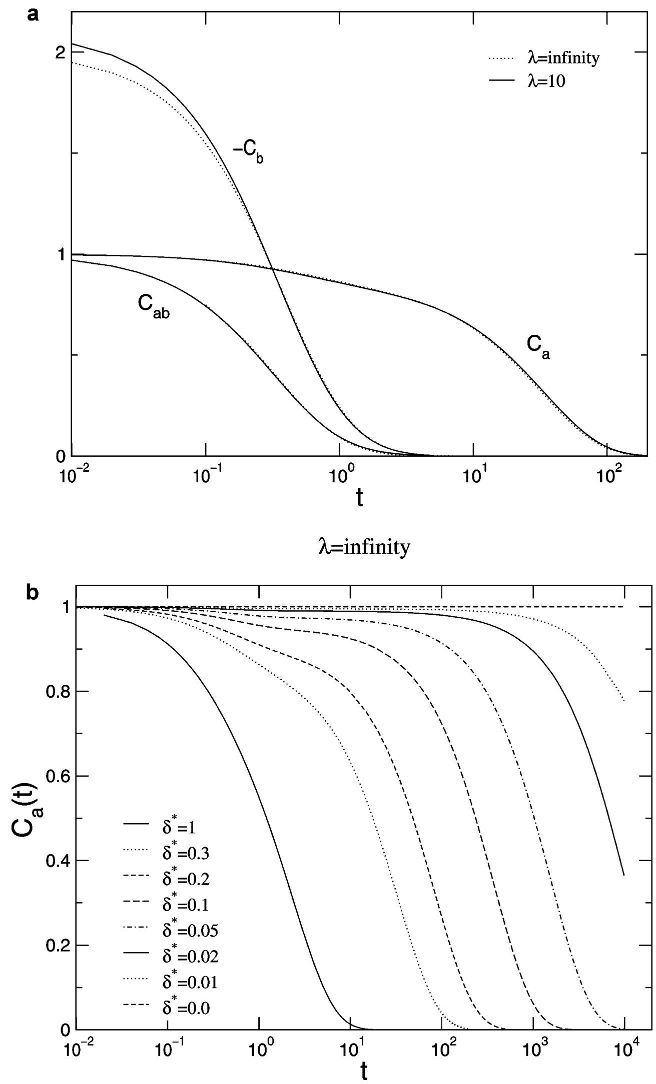

Figure 1. (a) Comparison of the numerical solutions of the full dynamic eq 14 with $\lambda = 10$ (solid lines) and eq 25, which corresponds to the $\lambda = \infty$ limit (dotted lines) for $\delta^* = 0.3$. (b) The density correlator $\mathcal{C}_a(t)$ at $\lambda = \infty$ for various values of δ^*. $\mathcal{C}_a(t)$ is (trivially) nonergodic only at $\delta^* = 0$.

This immediately gives the relationship between $\mathcal{C}_a$ and $\mathcal{C}_a^K$:

$$\mathcal{C}_a(t) = (1 - \delta^*) + \delta^* \mathcal{C}_a^K(t) \quad (34)$$

Using eqs 32 and 34, one can readily obtain the closed equation for $\mathcal{C}_a(t)$ as

$$\partial_t \mathcal{C}_a(t) = -\mathcal{C}_a(t) + (1 - \delta^*) - 2\lambda \int_0^t ds\ \mathcal{C}_a^2(t - s)\frac{d\mathcal{C}_a(s)}{ds} \quad (35)$$

Although this equation is similar to the schematic MCT equation for the density correlation function except the constant term $(1 - \delta^*)$ (and the inertial term), one has to remember that the validity of this equation is limited to the small region of the parameter space where the nonergodic solution is anticipated. The nonergodicity parameter $f_a \equiv \mathcal{C}_a(t \to \infty)$ satisfies the equation

$$f_a = (1 - \delta^*) - 2\lambda f_a^2(f_a - 1) \quad (36)$$

Note that the solution is always nonergodic, i.e., $f_a > 0$, as expected. Because we are dealing with the case of small λ and δ^*, the nonergodicity parameters of the density correlators are approximately given by using eqs 34 and 36

$$f_a = (1 - \delta^*) + 2\lambda\delta^*(1 - \delta^*)^2, \quad f_a^K = 2\lambda(1 - \delta^*)^2 \quad (37)$$

We also point out that the eq 32 satisfies the nonergodicity condition eq 19 since

$$\int_0^\infty \mathrm{d}t\, [\mathcal{C}_a(t)\mathcal{C}_b(t) + \delta^* \mathcal{C}_{ab}(t)\mathcal{C}_{ba}(t)] = \int_0^\infty \mathrm{d}t \left[\mathcal{C}_a(t)\frac{d\mathcal{C}_{ab}(t)}{\mathrm{d}t} + \mathcal{C}_{ab}(t)\frac{d\mathcal{C}_{\mathrm{a}}(t)}{\mathrm{d}t}\right] = \mathcal{C}_a(\infty)\mathcal{C}_{ab}(\infty) - \mathcal{C}_a(0)\mathcal{C}_{ab}(0) = -1 \quad (38)$$

where the ergodicity of $\mathcal{C}_{ab}(t)$, i.e., $\mathcal{C}_{ab}(t \to \infty) = 0$ was used. We further note that the second member of eq 22 is satisfied since

$$\int_0^\infty \mathrm{d}t\ \mathcal{C}_b(t) = \int_0^\infty \mathrm{d}t\ \partial_t \mathcal{C}_{ab}(t) = \mathcal{C}_{ab}(\infty) - \mathcal{C}_{ab}(0) = -1 \quad (39)$$

We compared the numerical solution of eq 32 with that of the full dynamic eq 14 for $\lambda = 0.05$ and $\delta^* = 0.1$, which is shown in Figure 2a,b. While the two solutions become almost identical for $\mathcal{C}_a^K$, $\mathcal{C}_{ab}$, and $\mathcal{C}_b$, there is a small but discernible difference in the value of the noergodicity parameter for $\mathcal{C}_a$. This result demonstrates that the apparent nonergodicity of $\mathcal{C}_a$ and $\mathcal{C}_a^K$ in the full dynamic eq 14 is the genuine one since we analytically know the existence of the nonergodicity in eqs 34 and 35.

IV. Numerical Solutions

Here, we obtain numerical solutions of the full dynamic eq 14 with initial conditions eqs 16 and 17. We have integrated eq 14 up to $t = 10^3$ with the integration time step $\mathrm{d}t = 10^{-3}$. Shown in Figures 3 and 4 are $\mathcal{C}_a$ and $\mathcal{C}_a^K$ for various values of λ for the two cases of $\delta^* = 0.1$ and $\delta^* = 0.3$. We see from these figures that the nonergodic behavior sets in at larger value of λ for smaller δ^*. This is natural since stronger nonergodicity is expected for smaller δ^*. For the same reason, the plateau region of $\mathcal{C}_a^K$ is more extended in the case of smaller δ^* for a given λ. The large decrease in the values of the plateau of $\mathcal{C}_a^K(t)$ with decreasing λ can be understood from the λ dependence of the plateau at $\delta^* = 0^+$, i.e., $f_a^K = 2\lambda/(1 + 2\lambda)$. We have also obtained a phase diagram shown in Figure 5. For a fixed value of δ^*, we monitor the long time (the longest simulation time is $t_{\max} = 10^3$) behavior of the density correlation functions $\mathcal{C}_a(t)$ and $\mathcal{C}_a^K(t)$ by varying the dimensionless coupling strength λ. We find an abrupt change in the long time behavior of these two density correlators for the very small change of λ around the critical value $\lambda_c(\delta^*)$. That is, the correlators become frozen up to the maximum time in the simulation time window for $\lambda = \lambda_c - \epsilon$, whereas they are observed to decay toward zero within the simulation time window for $\lambda = \lambda_c + \epsilon$ where ϵ is a small positive number; $\epsilon = 0.01-0.001$. For all probed points in Figure 5, we observe this abrupt change in the long time behavior of the density correlation functions. In this regard, this ergodic-to-nonergodic dynamic phase transition appears to be a first-order transition.

V. Summary and Concluding Remarks

In this work, we have presented the equilibrium dynamics of the toy model for dense fluid, in which the velocity-like variables undergo a large damping whose rate tends to infinity.

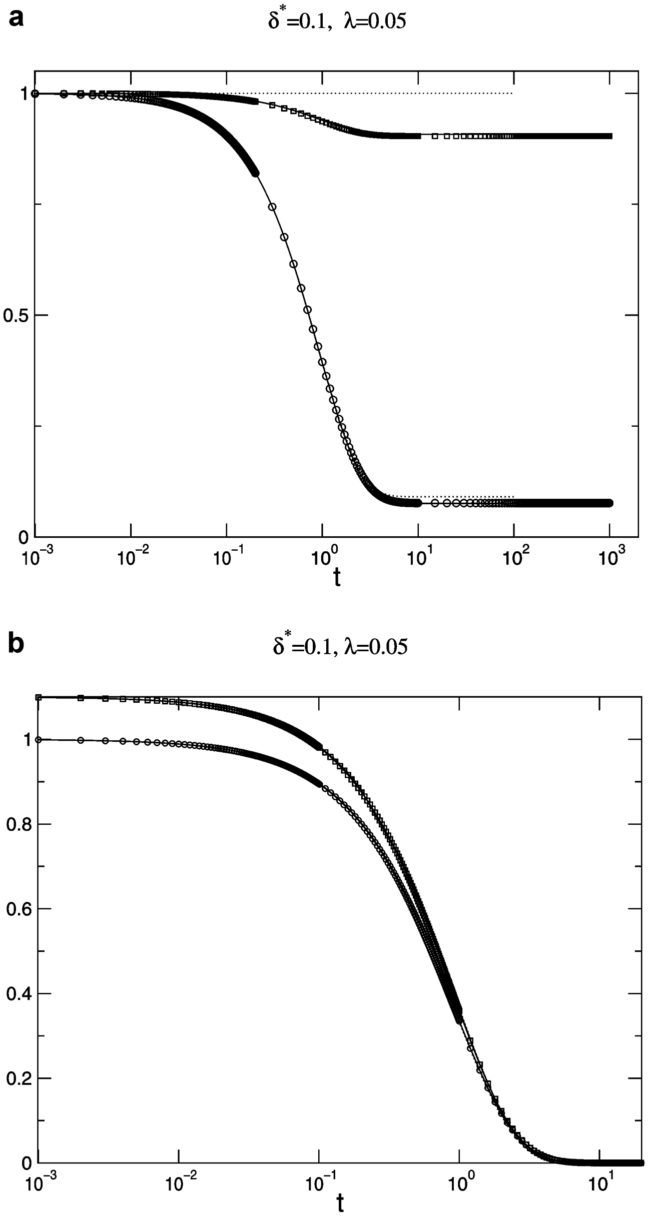

Figure 2. (a) Comparison of the numerical solutions of the full dynamic eq 14 [$\mathcal{C}_a$ (square) and $\mathcal{C}_a^K$ (circle)] and eq 32 (solid lines) for $\delta^* = 0.1$ and $\lambda = 0.05$. The dotted lines correspond to the correlators for $\delta^* = 0$, eq 31. (b) Comparison of the numerical solutions of the full dynamic eq 14 [$-\mathcal{C}_b$ (square) and $\mathcal{C}_{ab}$ (circle)] and eq 32 (solid lines) for $\delta^* = 0.1$ and $\lambda = 0.05$. The dotted lines correspond to the correlators at $\delta^* = 0$, eq 31. For the correlators involving b variables, the solutions of eqs 14 and 32 are almost identical with those for $\delta^* = 0$.

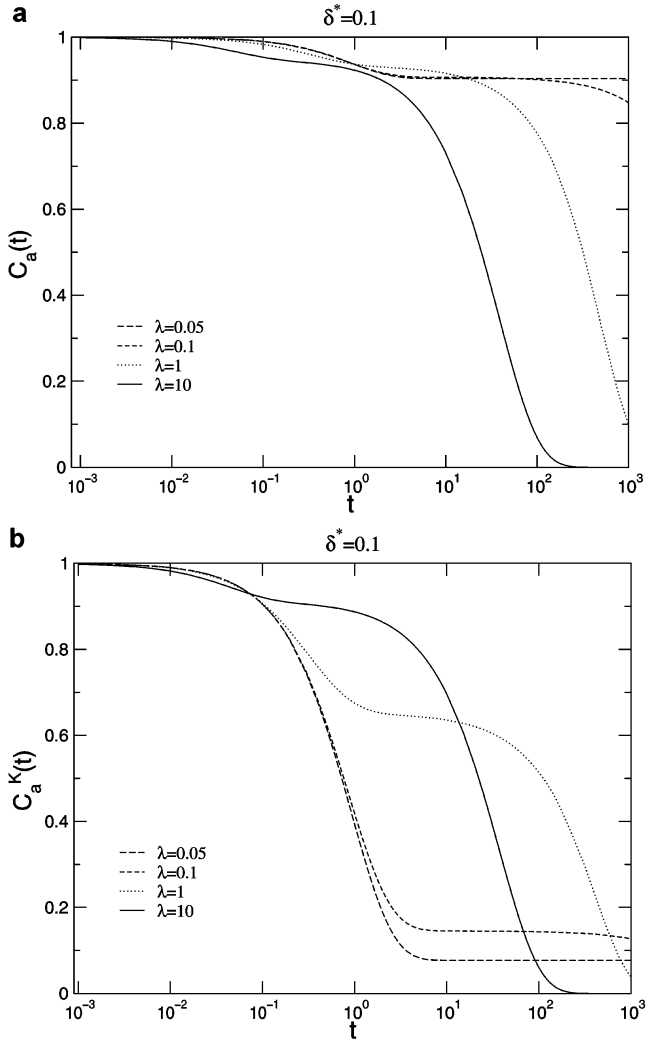

Figure 3. Numerical solutions of the full dynamic eq 14 for various values of λ at a fixed value of $\delta^* = 0.1$. The nonergodicity sets between $\lambda = 0.05$ and $\lambda = 0.1$.

We observe that through nonlinear couplings to the density-like variables, the correlators involving the velocity-like variables exhibit rather slow relaxations after rapid transient relaxations. As a result, we find that the strength of the hopping kernels does not get reduced even in the infinite damping limit of the velocity-like variables. Using a scaling analysis, we have derived a new closed set of equations for the correlators and have analytically proven that there should exist a sharp dynamic transition for δ^* in the range $0 < \delta^* < 1$. We also have obtained numerically an accurate phase diagram in the (δ^*, λ) plane. It should be emphasized that the slow dynamics and the corresponding dynamic transition to the nonergodic state in the present model are qualitatively different from those observed in the idealized MCT. In the idealized MCT, a sharp transition is smeared out into a crossover by adding coupling to the velocity variable (hopping), whereas in our model a sharp transition remains up to the critical value $\lambda_c(\delta^*)$ of the coupling.

It should be emphasized that in the present work (except section IIB), we took the large damping limit *after* taking the mean field limit M, $N \rightarrow \infty$ with the ratio $\delta^* \equiv M/N$ fixed. On the other hand, the Fokker−Planck equation for the reduced probability distribution, eq 6 with eq 7, is obtained when the large damping limit is taken first, in which the variables b_α have been adiabatically eliminated. To answer the question as to whether these opposite orders of taking limits yield the same dynamics or not, one should develop the method of analyzing the Fokker−Planck equation, eq 6 with eq 7 or the corresponding Langevin equation with multiplicative thermal noise[40] in the mean field limit. Although we have not obtained the resulting MCT type equation in this case, the Fokker−Planck equation, eqs 6 and 7 tell that it still has a standard nonlinear feedback mechanism and that the singular nature of the diffusion matrix $Q_{ij}(\{a\})$ can lead to a nonergodic transition. Because we aim at results that can be deduced without uncontrolled approximations, we did not rely on irreducible memory function method, which is useful to judge where to introduce uncontrollable factorization approximation.

In our model, we have been restricting ourselves to the simplest statics, i.e., the case of Gaussian Hamiltonian. This is because first we wanted to demonstrate that the kinetic effect

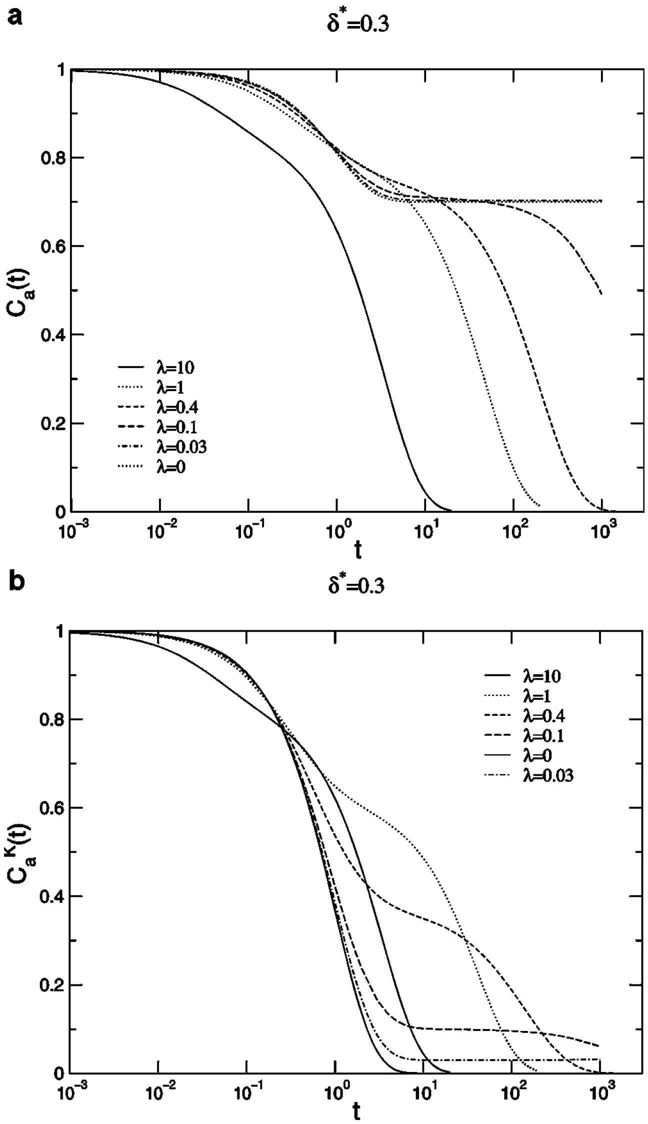

Figure 4. Numerical solutions of the full dynamic eq 14 for various values of λ at a fixed value of $\delta* = 0.3$. Here, the nonergodic behavior sets in at λ smaller than 0.1.

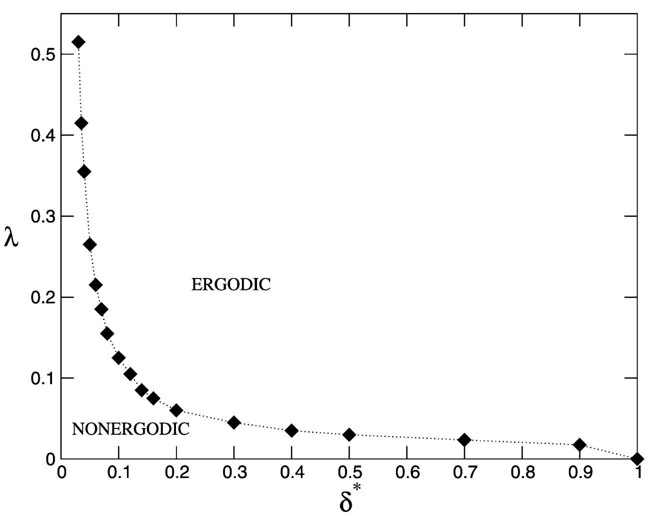

Figure 5. Dynamic transition line obtained from the numerical solutions of the full dynamic eq 14. The ergodic (nonergodic) phase is the region above (below) the transition line.

alone (brought by the reversible mode coupling nonlinearity) can cause a glassy dynamics. Another motivation is that for Gaussian Hamiltonian there exist the exact fluctuation−dissipation relationships between the correlation functions and the response functions, which we have used in our analysis. We thus so far have not explored the effects of the non-Gaussian statics in our model. In connection to this, there was an interesting recent suggestion[12] that the terms cutting off the sharp dynamic transition in MCT may be canceled by the terms from the non-Gaussian part of the Hamiltonian. It would be interesting to see if in our model the non-Gaussian statics can provide a mechanism with which to eliminate or suppress the hopping terms.

One other possible extension of the toy model is to include spatial degrees of freedom (SDF) as was recently attempted by the present authors.[42] It would be interesting to try to make the corresponding analyses of the toy model with SDF to make the statements such as those we made for our original toy model. If the model exhibits a sharp ergodic−nonergodic transition, one can investigate in a controlled way various singular behaviors near transition such as those found for the original MCT[2h], that is, the various power law property for the β-relaxation and the time−temperature superposition law for the α-relaxation, etc. These would be nontrivial questions since the driving mechanism for nonergodicity in our model is different from that in the original MCT.

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