Time-Dependent Diffusion Coefficients in Periodic Porous Materials[†]

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We derive an approximate solution for the Laplace transform of the time-dependent diffusion coefficient, $D(t)$, of a molecule diffusing in a periodic porous material. In our model, the material is represented by a simple cubic lattice of identical cubic cavities filled with a solvent and connected by small circular apertures in otherwise reflecting cavity walls, the thickness of which can be neglected. The solution describes the decrease of $D(t)$ from its initial value, $D(0) = D$, where D is the diffusion constant in the free solvent, to its asymptotic value, $D(\infty) = D_{eff}$, which is much smaller than D. A simple heuristic formula for the mean-squared displacement of the diffusing molecule is suggested. The theoretically predicted results are in good agreement with the data obtained from Brownian dynamics simulations.

1. Introduction

Recent advances in materials science have enabled the creation of a new class of porous materials formed by periodic arrays of identical cavities connected by narrow channels.[1−10] Such materials, having a controlled microgeometry, can be used as unique environments for different chemical and biological processes.[1,2,10,11] An important potential application of these materials is in the area of controlled drug release.[10−14] By changing the matrix microstructure, it is possible to tune the diffusivity of a drug and therefore the rate of its efflux from the encapsulating matrix.

In this paper, we study diffusion in periodic porous materials filled with a solvent. Because of the constraints imposed by a matrix structure, the diffusivity of molecules will be decreased as compared to that taking place in the free solvent. The decrease is quite pronounced at long times, whereas at very short times the molecules are only slightly influenced by the constraints so that the diffusivity remains close to its value in the absence of constraints.[15−17] Provided that the diameters of connecting channels are small relative to the cavity sizes, the diffusion process can be viewed as consisting of two steps: first, a fast equilibration within the cavity, and then slow diffusion which can be described in terms of a random walk between neighboring cavities.[18]

We derive approximate solutions for the Laplace transforms of the mean-squared displacement and the related time-dependent diffusion coefficient of a particle moving in a three-dimensional simple cubic lattice of cavities filled with a solvent and connected by small circular apertures located in the centers of the cavity walls, as shown in Figure 1. The radius of the apertures is assumed to be much smaller than the length of the cavity whose walls are assumed to be negligibly thin. The approximate solution is derived in two steps. First, the initial problem is replaced by the problem of diffusion in a space separated into cubic boxes with homogeneous (i.e., without apertures) walls which are uniformly permeable. The wall
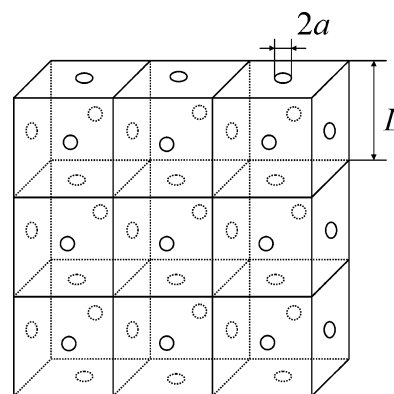
Figure 1. Fragment of a model periodic porous material formed by a simple cubic lattice of contacting cubic cavities. The cavity walls are assumed to be infinitely thin. The radius, a, of the cavity connecting apertures is assumed to be much smaller than the lattice period, L, $a \ll L$.

permeability is chosen so that the correct escape rate from the cavity is reproduced. This is done by using the method of boundary homogenization.[19−27] Second, the problem with uniformly permeable walls can be solved exactly by making use of a recently derived solution,[28] of a problem of diffusion in space with periodically spaced permeable partitions, first analyzed by Tanner.[29]

The approximate solutions are expressed in terms of fairly complicated Laplace transforms. An alternate heuristic formula for the mean-squared displacement is suggested. The analytical results for the mean-squared displacement are then compared to the results of Brownian dynamics simulations. There is good agreement between the two provided that the ratio of the aperture radius to the length of the cavity side is sufficiently small.

2. Theory

The aim of our analysis is that of finding the time-dependent behavior of the diffusion coefficient, $D(t)$. Because of the lattice symmetry, we calculate $D(t)$ by relating it to the mean-squared displacement along one of the principal axes of the cubic lattice of cavities, which we choose to be the x-direction, $\langle \Delta x^2(t) \rangle =$

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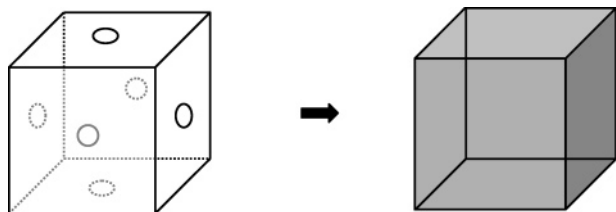
Figure 2. Schematic representation of the replacement of impermeable cavity walls with small apertures (left) by uniformly permeable walls with a properly chosen effective permeability, κ (right).

$\overline{\langle[x(t)-x(0)]^2\rangle}$. In this definition, the angular brackets indicate an average with respect to the particle position at time t, and the overbar indicates an average over the initial position of the particle. The latter will be assumed to be uniformly distributed in the cavity. We define $D(t)$ by the relation

$$D(t) = \frac{1}{2}\frac{d}{dt}\overline{\langle\Delta x^2(t)\rangle} \qquad (2.1)$$

The results of our analysis will be expressed as Laplace transforms of either of the functions appearing in this relation.

To escape from the cavity, a particle should reach one of the apertures. Being in the aperture, the particle with equal probability of $^1/_2$ either returns to the initial cavity or moves to the neighboring cavity. Keeping this scenario in mind, we replace the real impermeable cavity walls with small apertures by uniformly permeable walls, as shown in Figure 2. The wall permeability, κ, is chosen so that it leads to the correct escape rate from the cavity. To do this, we use the relation between κ and the effective trapping rate, κ_{abs}, of the wall with apertures replaced by perfectly absorbing circular patches.

$$\kappa = \kappa_{abs}/2 \qquad (2.2)$$

When finding κ_{abs}, the nonuniform boundary conditions on the wall (perfectly absorbing on the patches and reflecting on the rest of the wall) are replaced by a homogenized partially absorbing boundary condition which is uniform over the wall.

The idea underlying boundary homogenization is that the effect of local boundary conditions vanishes with distance from the surface. Sufficiently far from the surface, fields of flux and concentration become uniform in the lateral direction. This suggests that these fields can be reproduced by replacing the surface with nonuniform boundary conditions by a uniformly absorbing surface with a properly chosen value of κ_{abs}. Boundary homogenization is one of the approaches referred to as effective medium theories.[19-21] The original idea can be traced back to the work of Maxwell.[22] Homogenization of a reflecting boundary covered by small absorbing disks was first utilized by Shoup and Szabo.[23] This was done in their intuitively appealing derivation of the Berg–Purcell formula for the stationary flux of diffusing molecules to circular receptors on the cell surface.[24] Shoup and Szabo homogenized a surface randomly covered by nonoverlapping absorbing disks in the limiting case of low disk surface fraction. Their result was generalized by Zwanzig.[25] An extensive study of boundary homogenization for randomly distributed, nonoverlapping, perfectly or partially absorbing disks at arbitrary disk surface fraction is given in ref 26. A similar analysis for regular distributions of the absorbers on a surface is to be found in ref 27.

In the limiting case of a low surface fraction of absorbers, κ_{abs} is given by the product of the trap concentration and the Hill rate constant, k_{Hill}.[30] The latter determines the stationary flux of diffusing molecules to a perfectly absorbing disk on an

otherwise reflecting plane and is given by $k_{Hill} = 4Da$, where a is the disk radius and D is the diffusion constant of the molecules in free space. In our case, the trap concentration is $1/L^2$, where L is the length of the side of the cavity, and κ_{abs} is given by

$$\kappa_{abs} = \frac{k_{Hill}}{L^2} = \frac{4Da}{L^2} \qquad (2.3)$$

Upon substituting this into eq 2.2, we arrive at

$$\kappa = \frac{\kappa_{abs}}{2} = \frac{2Da}{L^2} \qquad (2.4)$$

This expression gives the effective wall permeability in terms of the geometric parameters a and L and the free-space diffusion constant, D.

When the real cavity walls with apertures have been replaced by uniformly permeable walls with the permeability, κ, given in eq 2.4, to find $D(t)$, we need to solve the problem of diffusion among periodically spaced permeable partitions separated by the distance L. A model of this type was suggested and analyzed approximately by Tanner,[29] and an exact solution in terms of a Laplace transform was recently derived by us in ref 28. Our solution describes the monotonic decrease of the diffusion coefficient, $D(t)$, from its initial value, $D(0) = D$, to its asymptotic value at long times, $D(\infty) = D_{eff}$, given by a formula first enunciated by Crick:[31]

$$D_{eff} = \frac{D\kappa L}{D + \kappa L} = \frac{2Da}{L + 2a} \qquad (2.5)$$

where the last term on the right-hand side is found by substituting for κ the expression given in eq 2.4. The expression in eq 2.5 gives D_{eff} as a function of D and the geometric parameters a and L. When $L \gg a$, this expression reduces to the result for D_{eff} given in ref 18.

The exact solution for the Laplace transform of the mean-squared displacement, as derived in ref 28, can be written in terms of dimensionless variables which correspond to $D = L = 1$. In this set of variables, the permeability, as calculated from eq 2.4, is $\kappa = 2a$, and the Laplace transform of the dimensionless mean-squared displacement is found to be

$$\mathcal{L}\{\overline{\langle\Delta x^2(\tau)\rangle}\} \equiv \int_0^\infty e^{-s\tau}\overline{\langle\Delta x^2(\tau)\rangle}\,d\tau = 2\{K(s) - L(s) + \Sigma_0(s)\,M_2(s) + 2[\Sigma_1(s) - \Sigma_3(s)]M_1(s) + [\Sigma_2(s) - 2\Sigma_4(s)]M_0(s)\} \quad (2.6)$$

where $\tau = Dt/L^2$ is the dimensionless time and the functions on the right-hand side of this formula are given in the Appendix. The corresponding Laplace transform of the diffusion coefficient is found from eq 2.1 which leads to the relation

$$\mathcal{L}\{D(\tau)\} = \frac{s}{2}\mathcal{L}\{\overline{\langle\Delta x^2(\tau)\rangle}\} \qquad (2.7)$$

This expression, in which $\mathcal{L}\{\overline{\langle\Delta x^2(\tau)\rangle}\}$ is given in eq 2.6 with κ given by eq 2.4, is the main result of this paper. Figure 3 illustrates the behavior of $D(\tau)$ predicted by the theory for $a = 0.01$ which corresponds to $\kappa = 0.02$. In this example, $D_{eff} = 0.02$, as found from eq 2.5, and can be seen from the inset in Figure 3.

In addition, we suggest a simple heuristic formula for $\overline{\langle\Delta x^2(t)\rangle}$ which is constructed to agree with known results at
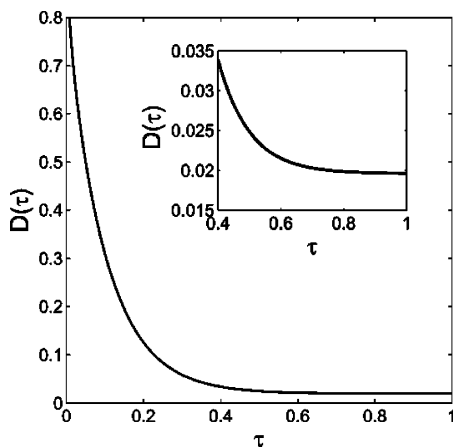
Figure 3. Behavior of $D(\tau)$ as a function of dimensionless time τ. The parameters used to generate the curves are $D = L = 1$ and $a = 0.01$, so that $\kappa = 0.02$. The values of $D(\tau)$ were found by numerically inverting the transform $\mathcal{L}\{D(\tau)\}$ given by eqs 2.6 and 2.7.

both $t = 0$ and $t = \infty$. The formula gives $\overline{\langle \Delta x^2(t) \rangle}$ in terms of the mean-squared displacement in an isolated cavity, $\langle \Delta x_{\text{iso}}^2(t) \rangle$, which can be found by using the eigenfunction expansion of Green's function. In dimensional variables, $\langle \Delta x_{\text{iso}}^2(t) \rangle$ has the form

$$\overline{\langle \Delta x_{\text{iso}}^2(t) \rangle} = \frac{L^2}{6}\left\{ 1 - \frac{48}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\exp\left[-\frac{\pi^2(2n+1)^2 Dt}{L^2} \right] \right\}$$

(2.8)

Our suggested heuristic formula is

$$\overline{\langle \Delta x^2(t) \rangle} \approx \frac{\overline{\langle \Delta x_{\text{iso}}^2(t) \rangle}[\langle \Delta x_{\text{iso}}^2(\infty) \rangle + 2D_{\text{eff}}t]}{\langle \Delta x_{\text{iso}}^2(\infty) \rangle}$$

$$= \overline{\langle \Delta x_{\text{iso}}^2(t) \rangle}\left[1 + \frac{24aDt}{(L+2a)L^2} \right]$$

(2.9)

where we have used the relation for D_{eff} in eq 2.5 and the asymptotic value $\langle \Delta x_{\text{iso}}^2(\infty) \rangle = L^2/6$. The heuristic formula reduces to $\overline{\langle \Delta x^2(t) \rangle} \approx \langle \Delta x_{\text{iso}}^2(t) \rangle$ as $t \rightarrow 0$ and $\overline{\langle \Delta x^2(t) \rangle} \approx \langle \Delta x_{\text{iso}}^2(\infty) \rangle + 2D_{\text{eff}}t$ in the limit $t \rightarrow \infty$.

A rationale for the formula in eq 2.9 is based on the fact that since the apertures are small the particle lifetime in the cavity, $L^3/(12Da)$, is much larger than the characteristic time of the intracavity equilibration, $L^2/(\pi^2 D)$. Therefore, at short times,

$$\overline{\langle \Delta x^2(t) \rangle}_{\text{short}} \approx \overline{\langle \Delta x_{\text{iso}}^2(t) \rangle}$$

(2.10)

The function $\langle \Delta x_{\text{iso}}^2(t) \rangle$ increases monotonically from zero to its equilibrium value $\langle \Delta x_{\text{iso}}^2(\infty) \rangle = L^2/6$. The slow diffusion at large times can be regarded as a result of a random walk between neighboring cavities.[18] Therefore, at long times, we can write

$$\overline{\langle \Delta x^2(t) \rangle}_{\text{long}} \approx \overline{\langle \Delta x_{\text{iso}}^2(\infty) \rangle} + 2D_{\text{eff}}t$$

(2.11)

To combine the long- and short-time behavior of $\overline{\langle \Delta x^2(t) \rangle}$, we take advantage of the fact that $\overline{\langle \Delta x^2(t) \rangle}_{\text{short}}$ and $\overline{\langle \Delta x^2(t) \rangle}_{\text{long}}$ approach the same asymptotic value, $\langle \Delta x_{\text{iso}}^2(\infty) \rangle$, as $t \rightarrow \infty$
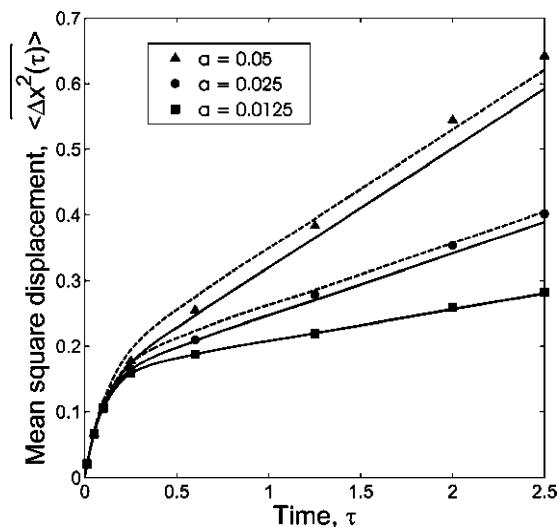
Figure 4. Comparison of the simulation results (symbols) with the dependence found by numerically inverting $\mathcal{L}\{\overline{\langle \Delta x^2(\tau) \rangle}\}$ given in eq 2.6 (solid curves) and that predicted by the heuristic approximation in eq 2.9 (dashed curves).

and 0, respectively. With this observation, we can write the following interpolation formula:

$$\overline{\langle \Delta x^2(t) \rangle} \approx \frac{\overline{\langle \Delta x^2(t) \rangle}_{\text{short}}\overline{\langle \Delta x^2(t) \rangle}_{\text{long}}}{\langle \Delta x_{\text{iso}}^2(\infty) \rangle}$$

(2.12)

This leads to the heuristic formula in eq 2.9 when the expressions in eqs 2.10 and 2.11 are substituted into this last equation. The form of the interpolation formula used in eq 2.12 was suggested by Visscher,[32] who analyzed the Kramers problem and interpolated the friction dependence of the rate constant between the low and intermediate-to-high friction regimes.

Brownian dynamics simulations based on runs of 10^4 particles initially uniformly distributed in a cavity were used to check the accuracy of our approximate theory. Three values of a were considered, $a = 0.0125$, 0.025, and 0.05, and we took $D = L = 1$. Figure 4 compares the simulation results with the dependence found by numerically inverting $\mathcal{L}\{\overline{\langle \Delta x^2(\tau) \rangle}\}$ given in eq 2.6 and with the dependence predicted by the heuristic formula in eq 2.9. We found excellent agreement between the numerical results and the two theoretical curves for the smallest aperture radius, $a = 0.0125$, for which the solid and dashed curves are nearly indistinguishable. An increase in the aperture radius, a, is expected to degrade the agreement between the theoretical and numerical results. This is because the fundamental premise behind the homogenization technique is based on the assumption that $L \gg a$. A point of some interest is that for the two larger radii, $a = 0.025$ and $a = 0.05$, the values of $\overline{\langle \Delta x^2(\tau) \rangle}$ found from the heuristic expression in eq 2.9 are in better agreement with the simulation results than those found by numerical inversion of the Laplace transform.

In summary, we have developed an approximate theory for evaluating the time-dependent diffusion coefficient of a particle that diffuses in a three-dimensional periodic porous material. The material is modeled by a simple cubic lattice of identical cavities filled with a solvent which allow contact between neighboring cavities through circular apertures centered on the sides of the cavity walls (cf. Figure 1). The results of the analysis

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are given in the form of the Laplace transform of the time-dependent diffusion coefficient, $D(t)$, which is initially equal to D, the diffusion constant in the free solvent, but later relaxes to $D_{eff} = D(\infty)$ which is much smaller than D due to the existence of constraining boundaries (Figure 3). The major step in our derivation is the replacement of the real cavity walls with small apertures by uniformly permeable walls with a properly chosen permeability (Figure 2). The theoretical results are found to be in good agreement with simulations, as seen in Figure 4.

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Appendix: Auxiliary Functions Needed to Evaluate eq 2.6

The required functions are the following:

$$P_1(s) = 2\kappa\sqrt{s}\cosh(\sqrt{s}) + (\kappa^2 + s)\sinh(\sqrt{s})$$

$$P_2(s) = \kappa\cosh(\sqrt{s/4}) + \sqrt{s}\sinh(\sqrt{s/4})$$

$$P_3(s) = \kappa\sinh(\sqrt{s/4}) + \sqrt{s}\cosh(\sqrt{s/4})$$

$$M_0(s) = \frac{1}{\sqrt{s}P_1(s)}\{\kappa[\cosh(\sqrt{s}) - 1] + \sqrt{s}\sinh(\sqrt{s})\}$$

$$M_1(s) = \frac{1}{sP_1(s)}\{\sqrt{s}[\cosh(\sqrt{s}) - 1] + \kappa[\sinh(\sqrt{s}) - \sqrt{s}]\} - \frac{1}{2}M_0(s)$$

$$M_2(s) = \frac{2}{s^{3/2}P_1(s)}\left\{\kappa\left[\cosh(\sqrt{s}) - 1 - \frac{s}{2}\right] + \sqrt{s}[\sinh(\sqrt{s}) - \sqrt{s}]\right\} - M_1(s) - \frac{M_0(s)}{4}$$

$$\hat{f}_r(s) = \frac{\kappa}{P_1(s)}[\sqrt{s}\cosh(\sqrt{s}) + \kappa\sinh(\sqrt{s})]$$

$$\hat{f}_{tr}(s) = \frac{\kappa\sqrt{s}}{P_1(s)}$$

$$\Sigma_0(s) = \frac{\kappa\sinh(\sqrt{s/4})}{\sqrt{s}P_2(s)[1 - \hat{f}_r(s) - \hat{f}_{tr}(s)]}$$

$$\Sigma_1(s) = \frac{1}{1 + \hat{f}_r(s) - \hat{f}_{tr}(s)}\Sigma_0(s)$$

$$\Sigma_2(s) = \frac{1 - \hat{f}_r(s) + \hat{f}_{tr}(s)}{1 - \hat{f}_r(s) - \hat{f}_{tr}(s)}\Sigma_1(s)$$

$$\Sigma_3(s) = \frac{\kappa[\sqrt{s/4}\cosh(\sqrt{s/4}) - \sinh(\sqrt{s/4})]}{sP_3(s)[1 + \hat{f}_r(s) - \hat{f}_{tr}(s)]}$$

$$\Sigma_4(s) = \frac{1}{1 - \hat{f}_r(s) - \hat{f}_{tr}(s)}\Sigma_3(s)$$

$$K(s) = \frac{\kappa}{s^{5/2}P_2(s)}\left[\sqrt{s}\cosh(\sqrt{s/4}) - \left(2 + \frac{s}{4}\right)\sinh(\sqrt{s/4})\right]$$

$$L(s) = \frac{2 + \kappa}{s^2 P_3(s)}[\sinh(\sqrt{s/4}) - (\sqrt{s/4})\cosh(\sqrt{s/4})]$$

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