Stochastic Resonance in a Generalized Quantum Kubo Oscillator

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We discuss stochastic resonance in a biased linear quantum system that is subject to multiplicative and additive noises. Starting from a microscopic system-reservoir Hamiltonian, we derive a *c*-number analogue of the generalized Langevin equation. The developed approach puts forth a quantum mechanical generalization of the "Kubo type" oscillator which is a linear system. Such a system is often used in the literature to study various phenomena in nonequilibrium systems via a particular interaction between system and the external noise. Our analytical results proposed here have the ability to reveal the role of external noise and vis-à-vis the mechanisms and detection of subtle underlying signatures of the stochastic resonance behavior in a linear system. In our development, we show that only when the external noise possesses a "finite correlation time" the quantum effect begins to appear. We observe that the quantum effect enhances the resonance in comparison to the classical one.

I. Introduction

Searching for methods capable of reliable modeling of the dynamics of classical and quantum systems in the presence of randomly fluctuating micro force-fields remains a major area of activity in the realm of chemical physics. Despite the progress in recent years, methodological challenges still persist. An interesting and nontrivial physical situation emerges when the quantum dynamics evolves far away from thermal equilibrium, and in this domain Einstein's and Onsager's relations no longer hold which can be tested experimentally.[1] A traditional approach to such a study involves a description in terms of nonlinear stochastic differential equations such as Langevin equations. Using this, one can investigate the effect of nonequilibrium fluctuations on the dynamics and the related transport processes. It is well documented in the literature that various interesting phenomena [such as noise-induced chaos,[2] stochastic resonance (SR),[3–7] and coherence resonance[8]] have emerged due to the interplay between noise and deterministic nonlinear dynamics. Consequently, investigation of the effect and nature of noise (white as well as colored) on the signal processes in nonlinear systems has revealed a plethora of intriguing phenomena. It is now found that different kinds of noise can be used to improve the signal processing, and thus noise can have constructive influence on complex systems.

The traditional physical ideas of stochastic resonance (SR) can be understood in a simple manner.[6] Conventional SR is a nonlinear effect that accounts for the optimum response of a dynamical system to an external force at a certain noise intensity. SR can be used to magnify weak signals embedded in a noisy ambient through the cooperation between signal and the noise. A lot of effort has been devoted to the precise detection and quantification of stochastic resonance. While one may contemplate several measures of SR, the signal-to-noise ratio (SNR)[9] and the spectral amplification factor (SAF)[10] are the dominant ones, with the theoretical analysis being carried in terms of the two-state approximation. In the case of the extended systems, a rather complete picture of the dependence of the output SNR on different parameters may be obtained from a knowledge of the nonequilibrium potential. In the SR case, noise modifies both the effective stiffness and the damping factor of the system. The SR phenomenon can appear in many different systems in the presence of a suitable amount of noise. The simple picture of SR described above does not provide a satisfactory description of various forms of SR, and a plethora of highly sophisticated models have emerged with the progress of time.

A recent article by Albert and Barabási[11] provides a clear review on complex networks (for example, the cell, a network of chemical reactions, and the internet, a network of physical links) which can be viewed as entities providing noise-induced phenomena in nonlinear systems. It is now a well-accepted fact that the cells are optimized to function in the presence of stochastic fluctuations. Living systems are open systems in which the interaction among the different component parts is nonlinear in character and the corresponding interactions with the environment are noisy causing modification of the signals, thereby making it difficult to detect and analyze them. Another important and ongoing consideration lies in understanding the synchronization on oscillator networks driven by noise.[12]

SR, a noise-induced phenomenon, developed by Benzi and co-workers,[3] has been put forth as a possible explanation for the periodic recurrence of Earth's climate between ice ages and periods of relative warmth with a period of about 10^5 years. Since then it has been considered as an important phenomenon throughout natural physical sciences and in many areas of engineering including laboratory experiments and model systems. In the last two decades, stochastic resonance and related phenomena have received substantial attention. Fauve and Heslot[13] were the first to experimentally verify the phenomenon of stochastic resonance in an AC driven Schmitt trigger. McNamara et al.[14] observed the stochastic resonance in a ring laser. This is the first observation of the amplification of the

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signal-to-noise ratio through the addition of injected noise in an optical device. Noise-induced dynamics of a laser system with optical feedback was studied by Masoller,[15] and it was shown that with increasing noise the dynamics of attractor jumping exhibits a resonant behavior as a result of the interplay of noise and the delayed feedback. Starting with the "Kubo type" oscillator based on the theory of multiplicative noise, Chowdhuri et al.[16] have posited how the fluctuation and decoherence, and their relationship, can be realized in classical chaos. In this regard, they have investigated the effect of noise on an extended chaotic system and thereby analyzed the interplay between the deterministic noise and a real random process. The observation of SR in chaotic systems has implications in the area of noisy information processing. Here we would like to mention that SR exhibits a deterministic behavior in random systems in the context of chaos, a random type of behavior in deterministic systems. Instances of this display the fact that determinism and randomness are complementary to each other, rather than being contradictory.

Recently, Sergeev and Olszewski[17] have derived an analytical solution for the Kubo-Anderson oscillator with a fluctuating frequency ω for an arbitrary distribution function which has been applied to explain various dynamical problems of solid state NMR, when the potential barrier for the mobility of magnetic nuclei is a stochastic function of time. SR has also been studied on pattern formation,[18] spatial order of spiral waves,[19] Ising model,[20] and so on. In vitro experiments[21] showed that hair cells exhibit SR, a counterintuitive nonlinear phenomenon where the detection of weak signals is enhanced by the presence of external noise. Recently, Dikshtein et al.[22] observed stochastic resonance for motion of flexible macromolecules in solution. Here it is pertinent to mention the fact that a change in the nature of noise that drives a system can lead to optimization of the system performance.[23] Gammaitoni et al.[24] have introduced an open-loop control scheme to monitor the enhancement or suppression of the spectral response for stochastic resonators. In each of the above mentioned examples, the particular phenomenon of interest has been characterized by the noise, with the noise playing a nondetrimental role. SR has also been invoked or observed in the context of entropic mechanisms:[25] quantum effects,[26] quantum computing;[27] pattern formation,[28] and device development,[29] and many others. Applications incorporating aspects of SR may yet prove revolutionary in such fields.

The SR has also been rationalized using the ideas of linear response theory (LRT). However, the validity of LRT crucially depends on the value of the frequency.[30] LRT provides a general description of SR, and its application to SR goes far beyond the case of thermal equilibrium. The application of LRT to SR reveals that the bistability or even static potential is not a prerequisite for the phenomenon to mark its appearance; rather, it is expected to appear in diverse systems and contexts.[31]

The above discussion suggests that the noise has a beneficial role in a variety of areas of natural sciences, and SR is an important and worth studying phenomenon, for nonlinear and stochastic dynamical systems which promise a broad spectrum of applications. SR is a paradigm of the constructive and cooperative effects of fluctuations on *nonlinear systems*. However, Sinha[32] reported SR in a completely noise-free situation considering thermal noise as a low-dimensional chaos. Cabrera et al.[33] have demonstrated that the stochastic resonance like behavior in systems with different nonlinearities can be observed even in the absence of periodic forces. For instance, the linear process behaves nonlinearly and exhibits noise-induced transitions and stochastic resonance in the influence of non-Markovian dichotomous noise.[34]

Undoubtedly, the paradigm of SR is basically an interplay between the underlying noises and nonlinearity of the system. Gitterman reported that the SR can occur for an arbitrary dichotomous noise and for a colored noise with short autocorrelation time even in a linear system.[35] Our goal, presently, is to extend the subject by studying the quantum SR phenomenon[36] in linear systems but also to involve many degrees of freedom through the construction of a quantum mechanical forced damped oscillator with fluctuating frequency described as a generalization of Kubo type oscillator as the underlying linear dynamical system. Here we aim to demonstrate how this opens up a plethora of possibilities for SR. Our findings may prove to be fruitful for various practical applications.

In our preceding work,[37] we have developed and applied the formalism to study the phenomenon of stochastic resonance of the Kubo type oscillator in the presence of nonequilibrium environment in the classical regime implementing the underlying phenomenological Langevin equation. There we culminated our discussion by stating that we would soon develop the corresponding quantum analogue. Essentially, the present article is an outcome of such a development. To meet this end, in this article we present the quantum realization of a Kubo type oscillator starting from microscopic Hamiltonian description and study the stochastic resonance therefrom. The underlying physics of microscopic realization of the Kubo type oscillator is also applicable in the context of the classical one.

The organization of the paper is as follows. In Section II, we put forth the general aspects pertaining to the development of quantum stochastic dynamics. The realization of the damped forced quantum oscillator will be presented in Section III. In Section IV, we discuss stochastic resonance in a quantum system. Our concluding remarks are given in the last section.

II. Quantum Stochastic Dynamics: General Aspects

In this section, we present the formal development of quantum stochastic dynamics by invoking the paradigmatic model and derive the quantum Langevin equation from a microscopic picture. Here we consider a particle of mass m to be connected to a heat bath comprised of a set of harmonic oscillators of unit mass with frequency set $\{\omega_j\}$. The total system-reservoir Hamiltonian can be written as[38]

$$\hat{H} = \frac{\hat{p}^2}{2m} + V_0(\hat{x}, t) + \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2} + \frac{1}{2}\left(\omega_j\hat{q}_j - \frac{c_j}{\omega_j}f(\hat{x})\right)^2\right] \quad (1)$$

where $\hat{x}$ and $\hat{p}$ are the coordinate and momentum operators, respectively, of the Brownian particle and $\{\hat{q}_j, \hat{p}_j\}$ are the set of coordinate and momentum operators of the heat bath oscillators. $V_0(\hat{x}, t)$ is the potential that acts as an external force field on the system. The system is coupled to the heat bath oscillators nonlinearly through the coupling function $f(\hat{x})$, and c_j's are the coupling constants. The coordinate and momentum operators follow the usual commutation relation $[\hat{x}, \hat{p}] = i\hbar$ and $[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}$. Eliminating the reservoir degrees of freedom in the usual way,[39,40] we obtain the operator quantum Langevin equation for the system variable as

$$m\ddot{\hat{x}} = -V_0'(\hat{x}, t) - f(\hat{x}(t))\int_0^t dt'\gamma(t - t')f'(\hat{x}(t'))\hat{p}(t') + f'(\hat{x}(t))\hat{\eta}(t) \quad (2)$$

where the noise operator $\hat{\eta}(t)$ [a stationary Gaussian operator noise] and the memory kernel $\gamma(t)$ are given by

$$\hat{\eta}(t) = \sum_{j=1}^{N}\left[\left\{\frac{\omega_j^2}{c_j}\hat{q}_j(0) - f(\hat{x}(0))\right\}\frac{c_j^2}{\omega_j^2}\cos(\omega_j t) + \frac{c_j}{\omega_j}\hat{p}_j(0)\sin(\omega_j t)\right] \quad (3)$$

and

$$\gamma(t) = \sum_{j=1}^{N}\frac{c_j^2}{\omega_j^2}\cos(\omega_j t) \quad (4)$$

The noise properties of $\hat{\eta}(t)$ can be derived by using suitable canonical thermal distribution of bath coordinates and momenta operators at $t = 0$ to obtain

$$\langle\hat{\eta}(t)\rangle_{QS} = 0 \quad (5)$$

and

$$\frac{1}{2}\langle\hat{\eta}(t)\hat{\eta}(t') + \hat{\eta}(t')\hat{\eta}(t)\rangle_{QS} = \frac{1}{2}\sum_{j=1}^{N}\frac{c_j^2}{\omega_j^2}\hbar\omega_j\coth\left(\frac{\hbar\omega_j}{2k_BT}\right)\cos(\omega_j(t - t')) \quad (6)$$

Here $\langle\cdots\rangle_{QS}$ implies quantum statistical average on bath degrees of freedom and is defined as

$$\langle\hat{O}\rangle_{QS} = \frac{\mathrm{Tr}[\hat{O}\exp(-\hat{H}_B/k_BT)]}{\mathrm{Tr}[\exp(-\hat{H}_B/k_BT)]} \quad (7)$$

for any bath operator $\hat{O}(\hat{q}_j, \hat{p}_j)$ and

$$H_B = \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2} + \frac{1}{2}\left(\omega_j\hat{q}_j - \frac{c_j}{\omega_j}f(\hat{x})\right)^2\right] \quad (8)$$

at $t = 0$. By trace, we mean the usual quantum statistical average. The true quantum dynamics of the Brownian particle is determined by the fact that the noise term $\hat{\eta}(t)$ is an operator in the whole Hilbert space of the system and environment. Being an operator, the commutator of $\hat{\eta}(t)$ does not vanish. This property of nonvanishing commutators is essential to be consistent with the Heisenberg uncertainty relation throughout the whole reduced dynamical evolution. For a detailed discussion in this direction, we refer to the recent paper of Hänggi and Ingold.[41]

Equation 6 is the fluctuation-dissipation relation (FDR) expressed in terms of noise operators appropriately ordered in the quantum mechanical sense, and consequently eq 2 can be viewed as the generalized quantum Langevin equation (GQLE) in an operator form. This quantum Langevin modeling actually stems out from an earlier work by Magalinskiĭ[42] employing eq 1 in the absence of the potential renormalization term. Several workers[43,44] have further exploited this idea of the quantum Langevin equation to study a varied class of quantum systems, which may be considered as a simple system coupled to heat baths. In this context, the comprehensive work of Hänggi and Ingold[41] is worth mentioning. It is pertinent to mention here that the GQLE (eq 2) includes both the state-dependent dissipation and the multiplicative noise. Although the quantum mechanical system-reservoir linear coupling model for microscopic description of additive noise and linear dissipation, which are related by FDR, is well-known over many decades in several fields,[39,45–47] the nature of nonlinear coupling and its consequences have been explored with renewed interest only recently. For example, it has been observed that the quantum dissipation can reduce the appearance of the metastable state and barrier drift in a double-well potential.[48] Tanimura and co-workers have extensively used nonlinear coupling in modeling the elastic and inelastic relaxation mechanisms and their interplay in vibrational and Raman spectroscopies.[49–51] Starting from a system-bath Hamiltonian in a molecular coordinate representation, Ishizaki and Tanimura[52] examined the applicability of a stochastic multilevel model for vibrational dephasing and energy relaxation in multidimensional infrared spectroscopy, where they considered an intramolecular anharmonic mode nonlinearly coupled to a colored noise bath at finite temperature. The system-bath interaction was assumed to be linear in the bath coordinates. The square linear system-bath interactions lead to dephasing due to the frequency fluctuation of system vibration.

Since working with the operator form of the GQLE is an arduous task, we resort to the following approximation which has originally been introduced and developed to simulate and analyze quantum dissipative dynamics.[53,54] To implement the formalism, we introduce product separable quantum states of the system and bath oscillator at $t = 0$

$$|\psi\rangle = |\phi\rangle\{|\alpha_j\rangle\} \qquad j = 1, 2, ..., N \quad (9)$$

where $|\phi\rangle$ represents an arbitrary initial state of the Brownian particle and $\{|\alpha_j\rangle\}$ describes the initial coherent state of the jth bath oscillator. In the present development, $\{|\alpha_j\rangle\}$ can be constructed as

$$\{|\alpha_j\rangle\} = \exp\left(-\frac{1}{2}|\alpha_j|^2\right)\sum_{n_j=0}^{\infty}\frac{\alpha_j^{n_j}}{\sqrt{n_j!}}|n_j\rangle \quad (10)$$

and $|\alpha_j\rangle$ is expressed in terms of the quantum expectation values of the shifted coordinate and momentum of the j th oscillator

$$\left\{\frac{\omega_j^2}{c_j}\langle\hat{q}_j(0)\rangle_Q - \langle f(\hat{x}(0))\rangle_Q\right\} = \sqrt{\frac{\hbar}{2\omega_j}}(\alpha_j + \alpha_j^*) \quad (11)$$

and

$$\langle\hat{p}_j(0)\rangle_Q = \sqrt{\frac{\hbar}{2}}(\alpha_j - \alpha_j^*) \quad (12)$$

respectively. Here $\langle\cdots\rangle_Q$ is the quantum mechanical average value. To construct an approximate c-number version of the Langevin equation, eq 2, we follow Ray et al.[53,54] and carry out a quantum mechanical averaging of the operator eq 2 to get

$$\begin{aligned} m\langle\dot{\hat{x}}\rangle_Q &= \langle\hat{p}\rangle_Q \\ \langle\dot{\hat{p}}\rangle_Q &= -\langle V_0'(\hat{x},t)\rangle_Q - \langle f'(\hat{x}(t))\int_0^t dt'\gamma(t-t') \times \\ &\qquad f'(\hat{x}(t'))\hat{p}(t')\rangle_Q + \langle f'(\hat{x}(t))\hat{\eta}(t)\rangle_Q \end{aligned} \tag{13}$$

where the quantum mechanical average $\langle\cdots\rangle_Q$ has taken over the above-mentioned initially product separable state $|\psi\rangle$. At this point, we employ the so-called factorization assumption

$$\langle f'(\hat{x}(t))\hat{\eta}(t)\rangle_Q \approx \langle f'(\hat{x}(t))\rangle_Q\langle\hat{\eta}(t)\rangle_Q \tag{14}$$

which is strictly valid for the Markovian case. This assumption is widely used in the context of quantum optics and condensed matter physics.[45–47] With the factorization assumption, eq 13 reads

$$\begin{aligned} m\langle\dot{\hat{x}}\rangle_Q &= \langle\hat{p}\rangle_Q \\ \langle\dot{\hat{p}}\rangle_Q &= -\langle V_0'(\hat{x},t)\rangle_Q - \langle f'(\hat{x}(t))\int_0^t dt'\gamma(t-t') \times \\ &\qquad f'(\hat{x}(t'))\hat{p}(t')\rangle_Q + \langle f'(\hat{x}(t))\rangle_Q\langle\hat{\eta}(t)\rangle_Q \end{aligned} \tag{15}$$

with $\langle\hat{\eta}(t)\rangle_Q$ now being a classical noise term, which because of the quantum mechanical averaging in general is a nonzero number and is given by

$$\langle\hat{\eta}(t)\rangle_Q = \sum_{j=1}^{N}\left[\left\{\frac{\omega_j^2}{c_j}\langle\hat{q}_j(0)\rangle_Q - \langle f(\hat{x}(0))\rangle_Q\right\}\frac{c_j^2}{\omega_j^2}\cos(\omega_j t) + \frac{c_j}{\omega_j}\langle\hat{p}_j(0)\rangle_Q\sin(\omega_j t)\right] \tag{16}$$

To realize $\langle\hat{\eta}(t)\rangle_Q$ as an effective c-number noise, we now introduce the ansatz[53,54] that the momenta $\langle\hat{p}_j(0)\rangle_Q$ and the shifted coordinates $\{(\omega_j^2/c_j)\langle\hat{q}_j(0)\rangle_Q - \langle f(\hat{x}(0))\rangle_Q\}$ of the bath oscillators are distributed according to a canonical distribution of the Gaussian form as

$$P_j\left(\left\{\frac{\omega_j^2}{c_j}\langle\hat{q}_j(0)\rangle_Q - \langle f(\hat{x}(0))\rangle_Q\right\},\langle\hat{p}_j(0)\rangle_Q\right) = N\exp\left\{-\frac{\langle\hat{p}_j(0)\rangle_Q^2 + \frac{c_j^2}{\omega_j^2}\left\{\frac{\omega_j^2}{c_j}\langle\hat{q}_j(0)\rangle_Q - \langle f(\hat{x}(0))\rangle_Q\right\}^2}{2\hbar\omega_j\left(\bar{n}_j(\omega_j)+\frac{1}{2}\right)}\right\} \tag{17}$$

so that for any quantum mechanical mean value, $\langle\hat{O}\rangle_Q$ of the bath operators, its statistical average $\langle\cdots\rangle_S$ is

$$\langle\langle\hat{O}\rangle_Q\rangle_S = \int\left[\langle\hat{O}\rangle_Q P_j d\left\{\frac{\omega_j^2}{c_j}\langle\hat{q}_j(0)\rangle_Q - \langle f(\hat{x}(0))\rangle_Q\right\}d\langle\hat{p}_j(0)\rangle_Q\right] \tag{18}$$

In eq 17, $\bar{n}_j(\omega_j)$ indicates the average thermal photon number of the jth oscillator at temperature T and is given by

$$\bar{n}_j(\omega_j) = \frac{1}{\exp\left(\frac{\hbar\omega_j}{k_BT}\right) - 1} \tag{19}$$

and N is the normalization constant. The distribution P_j given by eq 17 and the definition of statistical average (eq 18) together imply that the c-number noise $\langle\hat{\eta}(t)\rangle_Q$ given by eq 16 must satisfy

$$\langle\langle\hat{\eta}(t)\rangle_Q\rangle_S = 0 \tag{20}$$

and

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = \frac{1}{2}\sum_{j=1}^{N}\frac{c_j^2}{\omega_j^2}\hbar\omega_j\coth\left(\frac{\hbar\omega_j}{2k_BT}\right)\cos(\omega_j(t-t')) \tag{21}$$

It is worth emphasizing at this juncture that in spite of the c-number noise $\langle\hat{\eta}(t)\rangle_Q$ yielding the same anticommutator as that of the operator noise term $\hat{\eta}(t)$, being a c-number term, the commutator of $\langle\hat{\eta}(t)\rangle_Q$ vanishes. In this regard, our treatment is not fully quantum mechanical, rather a semiclassical one, where we have treated the system quantum mechanically and the reservoir degrees of freedom quasi-classically. This semiclassical scheme helps us in handling complicated operator quantum Langevin equation in the same footing as that of the classical Langevin equation retaining the quantum effects in the leading orders. It is worth noting here that the method of projection operators of Zwanzig[38] is in principle applicable to any system, but the operator is too complicated for practical applications.

Equations 20 and 21 imply that the c-number noise $\langle\hat{\eta}(t)\rangle_Q$ is such that it is zero centered and satisfies the standard FDR expressed via eq 6. Furthermore, one can easily recognize that the ansatz (eq 17) is a canonical thermal Wigner distribution function for a shifted harmonic oscillator which always remains a positive definite function. A special advantage of using this distribution function is that it remains valid as a pure state nonsingular distribution function even at $T = 0$. At the same time, the distribution of quantum mechanical mean values of the bath oscillators reduces to the classical Maxwell−Boltzmann distribution in the thermal limit $\hbar\omega \ll k_BT$. Furthermore, this procedure allows us to avoid operator ordering prescription of eq 6 for deriving the noise properties of the bath oscillators and to identify $\langle\hat{\eta}(t)\rangle_Q$ as a classical looking noise with quantum mechanical context. It should be noted that the Wigner distribution function and Wigner representation of the quantum master equation are widely used in the literature, particularly in the context of electron transfer,[55] reaction dynamics,[56,57] quantum optics,[58] and spectroscopy,[50] very successfully.

Now to obtain a finite result in the continuum limit, we impose some conditions on the coupling function and on the number N of the bath oscillators to ensure that the memory kernel $\gamma(t)$ is indeed dissipative in nature. A sufficient condition[40] for $\gamma(t)$ to be dissipative is that it is positive definite and decreases monotonically with time which can be achieved if $N \to \infty$ and if both $c_j\omega_j^2$ and ω_j are sufficiently smooth functions of j. As $N \to \infty$, we substitute the sum by an integral over ω weighted by a density of state $\rho(\omega)$. Thus, to obtain a finite result in the continuum limit, the coupling function $c_j = c(\omega)$ is chosen as

$$c(\omega) = \frac{c_0}{\omega\sqrt{\tau_c}} \tag{22}$$

Consequently, $\gamma(t)$ reduces to

$$\gamma(t) = \frac{c_0^2}{\tau_c}\int d\omega\rho(\omega)\cos(\omega t) \tag{23}$$

where c_0 is some constant and $1/\tau_c$ is the cutoff frequency of the bath oscillators. $\rho(\omega)$ is the density of the modes of the heat bath and is assumed to be Lorentzian

$$\rho(\omega) = \frac{2}{\pi}\frac{1}{\tau_c(\omega^2 + \tau_c^{-2})} \tag{24}$$

Using the above forms of $c(\omega)$ and $\rho(\omega)$, we have the expression for $\gamma(t)$ as

$$\gamma(t) = \frac{c_0^2}{\tau_c}\exp\left(-\frac{|t|}{\tau_c}\right) = \frac{\gamma}{\tau_c}\exp\left(-\frac{|t|}{\tau_c}\right) \tag{25}$$

where $\gamma = c_0^2$. For $\tau_c \to 0$, eq 25 reduces to

$$\gamma(t) = 2\gamma\delta(t) \tag{26}$$

and the noise correlation function eq 21 becomes

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = \frac{\gamma}{2\tau_c}\int_0^\infty d\omega\hbar\omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)\times \cos(\omega(t-t'))\rho(\omega) \tag{27}$$

Equation 27 is an exact expression for the two time noise correlation functions. We now make the following assumption. As $\hbar\omega\coth((\hbar\omega)/(2k_BT))$ is a much more smooth function of ω, at least for not too low temperature, the integral in eq 27 can be approximated as[59]

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S \approx \frac{\gamma}{2\tau_c}\hbar\omega_0\coth\left(\frac{\hbar\omega_0}{2k_BT}\right)\int_0^\infty d\omega\rho(\omega)\cos(\omega(t-t')) \tag{28}$$

where ω_0 is the average bath frequency at temperature T. The above equation clearly displays that the presence of the memory kernel $\gamma(t)$ and temperature T can be exploited to mimic quantum dissipation. Irrespective of the preparation of the bath, the above relation, in the classical regime, reduces to the non-Markovian Einstein equation. We point out that recently Hänggi and Ingold have computed the exact noise correlation function [see eq 35 in ref 41].

In the limit $\tau_c \to 0$, we obtain

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = 2D_0\delta(t-t') \tag{29}$$

where

$$D_0 = \frac{\gamma}{2}\hbar\omega_0\left[\bar{n}(\omega_0) + \frac{1}{2}\right] \tag{30}$$

with

$$\bar{n}(\omega_0) = \frac{1}{\exp\left(\dfrac{\hbar\omega_0}{k_BT}\right) - 1} \tag{31}$$

We want to emphasize here that the form of D_0, given by eq 30, can be obtained if and only if one extracts the $\hbar\omega\coth((\hbar\omega)/(2k_BT))$ term out of the integral. Clearly, at very high temperature, $2k_BT \gg \hbar\omega$, the integrand in eq 27 reduces to $2k_BT\cos(\omega(t-t'))\rho(\omega)$, and in this case eq 29 is strictly valid with $D_0 = (\gamma/2)k_BT$, which is the classical result. On the other hand, with our above assumption, D_0 will be given by eq 30. Here also, for $k_BT \gg \hbar\omega_0$, D_0 reduces to its classical expression, namely, $D_0 = (\gamma/2)k_BT$. Here it is important to note that the above assumption that we have made is valid only at high temperature. When we restrict ourselves with the noise correlation function, given by eq 27, our methodology becomes valid for any arbitrary temperature. However, the noise correlation function, eq 29, makes our treatment quasi-quantum when the system is treated quantum mechanically and the bath quasi-classicallly. In this regard, the latter development, namely, eq 28 to eq 31, does not account for the dynamics which are fully quantum mechanical. Nevertheless, the ansatz (eq 17), which is the canonical thermal Wigner distribution function for a shifted harmonic oscillator and always remains positive definite, contains some quantum information of the bath comprised of a set of quantum mechanical harmonic oscillators. We again emphasize that with the noise correlation function eq 27 one incorporates the full quantum information from the harmonic bath.

We now summarize our developments as follows. Writing $x = \langle\hat{x}\rangle_Q$ and $p = \langle\hat{p}\rangle_Q$, the c-number GQLE, eq 15 reads

$$\begin{aligned} m\dot{x} &= p \\ \dot{p} &= -\langle V_0'(\hat{x},t)\rangle_Q - \langle f'(\hat{x}(t))\int_0^t dt'\gamma(t-t')\times \\ &\qquad f'(\hat{x}(t'))\hat{p}(t')\rangle_Q + \langle f'(\hat{x}(t))\rangle_Q\eta(t) \end{aligned} \tag{32}$$

where $\eta(t) = \langle\hat{\eta}(t)\rangle_Q$ and is a classical-like noise term, and with the noise properties given in eqs 5 and 27, eq 32 can be regarded as the c-number analogue of the GQLE, valid at any arbitrary temperature. In the high temperature approximation, $k_BT \gg \hbar\omega$, and with $\tau_c \to 0$, eq 32 reads

$$\begin{aligned} m\dot{x} &= p \\ \dot{p} &= -\langle V_0'(\hat{x},t)\rangle_Q - \gamma\langle[f'(\hat{x})]^2\hat{p}\rangle_Q + \langle f'(\hat{x})\rangle_Q\eta(t) \end{aligned} \tag{33}$$

where the c-number noise term $\eta(t)$, apart from being zero centered, satisfies eq 29. We now add $V_0'(\hat{x},t)$, $\gamma[f'(x)]^2p$, and $f'(x)\eta(t)$ on both sides of eq 33 and rearrange it to obtain

$$\begin{aligned} m\dot{x} &= p \\ \dot{p} &= -V_0'(\hat{x},t) + Q_V - \gamma[f'(x)]^2p + \\ &\qquad Q_1 + f'(x)\eta(t) + Q_2 \end{aligned} \tag{34}$$

where

$$\begin{aligned} Q_V &= V_0'(x,t) - \langle V_0'(\hat{x},t)\rangle_Q \\ Q_1 &= \gamma[f'(x)]^2 p - \langle [f'(\hat{x})]^2 \hat{p}\rangle_Q \\ \text{and}\quad Q_2 &= \eta(t)[\langle f'(\hat{x})\rangle_Q - f'(x)] \end{aligned} \tag{35}$$

Referring to the quantum nature of the system in the Heisenberg picture, we now write the system operator $\hat{x}$ and $\hat{p}$ as

$$\begin{aligned} \hat{x} &= x + \delta\hat{x} \\ \hat{p} &= p + \delta\hat{p} \end{aligned} \tag{36}$$

where $\hat{x}(=\langle\hat{x}\rangle_Q)$ and $\hat{p}(=\langle\hat{p}\rangle_Q)$ are the quantum mechanical mean values and $\delta\hat{x}$ and $\delta\hat{p}$ are the operators which are quantum fluctuations around the respective mean values. By construction, $\langle\delta\hat{x}\rangle_Q = \langle\delta\hat{p}\rangle_Q = 0$, and they follow the commutation relation $[\delta\hat{x}, \delta\hat{p}] = i\hbar$. Using eq 36 in $V_0'(\hat{x}, t)$, $[f'(\hat{x})]^2 p$, and $f'(\hat{x})$, by a Taylor series expansion in $\delta\hat{x}$ around x, one can express Q_V, Q_1, and Q_2 as

$$Q_V = -\sum_{n\geq 2} \frac{1}{n!} V_0^{(n+1)}(x,t)\langle\delta\hat{x}^n\rangle_Q \tag{37}$$

$$Q_1 = -\gamma[2pf'(x)Q_f + pQ_3 + 2f'(x)Q_4 + Q_5] \tag{38}$$

and

$$Q_2 = \eta(t)Q_f \tag{39}$$

respectively, where

$$\begin{aligned} Q_f &= -\sum_{n\geq 2} f^{(n+1)}(x)\langle\delta\hat{x}^n\rangle_Q \\ Q_3 &= \sum_{m\geq 1}\sum_{n\geq 2} \frac{1}{m!}\frac{1}{n!} f^{(m+1)}(x) f^{(n+1)}(x)\langle\delta\hat{x}^m\delta\hat{x}^n\rangle_Q \\ Q_4 &= \sum_{n\geq 1}\frac{1}{n!} f^{(n+1)}(x)\langle\delta\hat{x}^n\delta\hat{p}\rangle_Q \\ \text{and}\quad Q_5 &= \sum_{m\geq 1}\sum_{n\geq 1} \frac{1}{m!}\frac{1}{n!} f^{(m+1)}(x) f^{(n+1)}(x)\langle\delta\hat{x}^m\delta\hat{x}^n\delta\hat{p}\rangle_Q \end{aligned} \tag{40}$$

From the above set of expressions, it is evident that Q_V represents quantum correction due to nonlinearity of the system potential, and Q_1 and Q_2 represent the quantum corrections due to the nonlinearity of the system-bath coupling function. Using the above quantum correction terms, one arrives at the dynamical equations for the system variable as

$$\begin{aligned} m\dot{x} &= p \\ \dot{p} &= -V_0'(x,t) + Q_V - \gamma[f'(x)]^2 p - 2\gamma p f'(x)Q_f - \gamma pQ_3 - \\ &\quad 2\gamma f'(x)Q_4 - \gamma Q_5 + f'(x)\eta(t) + Q_f\eta(t) \end{aligned} \tag{41}$$

The above quantum Langevin equation is characterized by a classical term V_0', as well as its correction Q_V. The terms containing γ are nonlinear dissipative terms where Q_f, Q_3, Q_4, and Q_5 are due to associated quantum contribution in addition to the classical nonlinear dissipative term $\gamma[f'(x)]^2 p$. The last term in the above equation refers to a quantum multiplicative noise term in addition to the usual classical contribution $f'(x)\eta(t)$. It is therefore easy to recognize the classical limit of the above equation derived earlier by Lindenberg and Seshadri.[60] Moreover, the quantum dispersions due to potential and coupling terms in the Hamiltonian are entangled with nonlinearity.

In the overdamped limit, the adiabatic elimination of the fast variable is usually done by simply putting $\dot{p} = 0$. This adiabatic elimination provides the correct equilibrium distribution only when the dissipation is state independent. However, in state-dependent dissipation, which is a manifestation of the nonlinear nature of the system−bath coupling function $f(x)$, the conventional adiabatic elimination of the fast variable does not provide correct results. To obtain a correct equilibrium distribution, one follows the approach proposed by Sancho et al.[61] The methodology of Sancho et al. consists of a systematic expansion of the relevant variables in powers of γ^{-1} and rejection of the terms smaller than $O(\gamma^{-1})$. In this limit, the transient correction terms Q_4 and Q_5 do not affect the dynamics of the system which varies over a much slower time scale. So the equations governing the dynamics of the system variables are

$$\begin{aligned} m\dot{x} &= p \\ \dot{p} &= -V_0'(x,t) + Q_V - \gamma([f'(x)]^2 + \\ &\quad 2f'(x)Q_f + Q_3)p + f'(x)\eta(t) + Q_f\eta(t) \end{aligned} \tag{42}$$

For the linear system−bath coupling, $f(x) = x$ and $f'(x) = 1$. From eq 40 we see that Q_f and Q_3 vanish as these terms involve third- or higher-order derivatives. Equation 42 reduces to

$$m\ddot{x} + \gamma\dot{x} + V_0'(x,t) = \eta(t) + Q_V \tag{43}$$

The Fokker-Planck equation corresponding to the Langevin equation, eq 43, is given by

$$\frac{\partial P(x,p,t)}{\partial t} = -\frac{\partial}{\partial x}(pP) + \frac{\partial}{\partial p}[\gamma p + V_0'(x,t) - Q_V] + D_0\frac{\partial^2 P}{\partial p^2} \tag{44}$$

For general system-reservoir coupling, the Fokker-Planck equation corresponding to the Langevin equation eq 42 which involves space-dependent dissipation reads as

$$\frac{\partial P(x,p,t)}{\partial t} = -\frac{\partial}{\partial x}(pP) + \frac{\partial}{\partial p}[\gamma h(x)p + V_0'(x,t) - Q_V] + D_0 g^2(x)\frac{\partial^2 P}{\partial p^2} \tag{45}$$

where

$$h(x) = [f'(x)]^2 + 2f'(x)Q_f + Q_3 \tag{46}$$

and

$$g(x) = f'(x) + Q_f \tag{47}$$

For linear system-bath coupling, eq 45 reduces to eq 44.

It is to be noted here that we have treated the system part quantum mechanically and the dissipation part semiclassically. This situation is equivalent to the case that Caldeira and Leggett[62] have discussed for the linear-linear system-bath coupling case. The nonlinear-linear system-bath coupling with the same condition is also considered by Steffen and Tanimura[51] where they assume square-linear coupling. In this work, the dissipative part of the quantum Fokker-Planck equation is obtained as

$$\gamma\frac{\partial}{\partial p}\left(p + k_{\mathrm{B}}T\frac{\partial}{\partial p}\right)P$$

and

$$4x^2\gamma\frac{\partial}{\partial p}\left(p + k_{\mathrm{B}}T\frac{\partial}{\partial p}\right)P + x\gamma\hbar^2\frac{\partial^2 P}{\partial x\partial p}$$

for linear-linear and square-linear system-bath coupling, respectively. In our case, for linear-linear coupling, the dissipative part reduces to the same form at high temperature, that is, with $D_0 = \gamma k_{\mathrm{B}}T$, as is evident from eq 44. Also for square-linear coupling, the first term of the dissipative part of the quantum Fokker-Planck equation of Steffen and Tanimura has the same x-dependence as that of our model. The cross derivative term, that is, $\partial^2 P/\partial x\partial p$, does not appear in our case as we resort to eq 28 and set $\tau_c \rightarrow 0$. One may consider a correlated noise (τ_c finite) instead of considering white noise processes as discussed in ref 63 to get the cross-derivative term. We hope to address these aspects in our future work.

III. Calculation of Quantum Dispersion: Realization of Damped Forced Quantum "Kubo Type" Oscillator

In this section, we discuss the emergence of quantum dissipation terms in our present development. On the basis of the quantum nature of the system in the Heisenberg picture, we already have written the system's operators $\hat{x}$ and $\hat{p}$ in terms of their mean value and quantum fluctuations (see eq 36).

From eq 35, a Taylor series expansion around x, one obtains

$$Q_V(x, \langle\delta\hat{x}^n\rangle) = -\sum_{n\geq 2}\frac{1}{n!}V_0^{(n+1)}(x,t)\langle\delta\hat{x}^n(t)\rangle \tag{48}$$

where $V_0^{(n+1)}(x, t)$ is the $(n + 1)$-th derivative of the potential $V_0(x, t)$. Now putting eq 36 in eq 2 and for linear system−bath coupling, one can easily show that the quantum correction equation is given by

$$m\delta\ddot{\hat{x}}(t) + m\int_0^t \mathrm{d}t_1\gamma(t - t_1)\delta\dot{\hat{x}}(t_1) + V_0''(x,t)\delta\hat{x}(t) + \sum_{n\geq 2}\frac{1}{n!}V_0^{(n+1)}(x,t)\langle\delta\hat{x}^n(t)\rangle = \delta\hat{\eta}(t) \tag{49}$$

where $\delta\hat{\eta}(t) = \hat{\eta}(t) - \eta(t)$.

To obtain the quantum correction term, we need to consider the explicit form of the potential $V_0(x, t)$ which is assumed to be

$$\begin{aligned} V_0(x,t) &= \frac{1}{2}m\Omega_0^2x^2 + V_{\mathrm{e}}(x,t) \\ &= V_{\mathrm{h}}(x) + V_{\mathrm{e}}(x,\xi(t),t) \end{aligned} \tag{50}$$

where $V_{\mathrm{h}}(x) = (1/2)m\Omega_0^2x^2$ is the confining harmonic potential and $V_{\mathrm{e}}(x, \xi(t), t)$ is the time-dependent fluctuating part of the potential. The strength of the fluctuation is assumed to be weak so that the perturbation holds good. The fluctuations in the potential are governed by an Ornstein-Uhlenbeck noise process

$$\dot{\xi}(t) = -\frac{\xi(t)}{\tau} + \frac{\sqrt{2}\sigma^2}{\tau}\zeta(t) \tag{51}$$

where $\zeta(t)$ is the zero mean δ-correlated Gaussian noise. The stochastic process $\xi(t)$ can be characterized by the following set of equations. The probability function $\tilde{\rho}(\xi)$, the variance σ^2, and the correlation function of the noise are given by

$$\tilde{\rho}(\xi) = \sqrt{2\pi\sigma^2}\exp\left(-\frac{\xi^2}{2\sigma^2}\right) \tag{52}$$

where

$$\sigma^2 = \int \xi^2\tilde{\rho}(\xi)\mathrm{d}\xi \tag{53}$$

and

$$\langle\xi(t)\xi(t')\rangle = \sigma^2\exp\left(-\frac{|t - t'|}{\tau}\right) \tag{54}$$

respectively.

We restrict our calculation to the leading order of the quantum correction factors, that is, when

$$Q_V(x, \langle\delta\hat{x}^n\rangle) = -\frac{1}{2!}V_0'''(x,t)\langle\delta\hat{x}^2(t)\rangle \tag{55}$$

Equation 55 reveals the fact that the nonvanishing quantum dispersion term arises only when the system potential includes nonlinear terms. The lowest-order nonlinearity, that is, cubic nonlinearity, gives rise to $\langle\delta\hat{x}^2\rangle$, the quartic nonlinearity gives rise to the $x\langle\delta\hat{x}^2\rangle$ term, and so on. Thus, the cubic nonlinearity alters the zero of the system's potential by resulting in a term $\langle\delta\hat{x}^2\rangle$ (which can be set to zero by appropriate scaling of origin), and the quartic nonlinearity modifies the frequency of the system by incorporating the quantum correction.

We now proceed to calculate $\delta\hat{x}$ perturbatively. To do this, we set $V_{\mathrm{e}}(x, \xi(t), t) = 0$, so that the quantum dispersion obeys the following equation

$$m\delta\ddot{\hat{x}}(t) + m\int_0^t \mathrm{d}t_1\gamma(t - t_1)\delta\dot{\hat{x}}(t_1) + m\Omega_0^2\delta\hat{x}(t) = \delta\hat{\eta}(t) \tag{56}$$

The solution of eq 56 is

$$\delta\hat{x}(t) = G(t)\delta\hat{x}(0) + H(t)\delta\dot{\hat{x}}(0) + \int_0^t \mathrm{d}t'H(t - t')\delta\hat{\eta}(t') \tag{57}$$

where $G(t)$ and $H(t)$ are the inverse Laplace transformation of $\tilde{G}(t)$ and $\tilde{H}(t)$, respectively, with

$$\tilde{G}(s) = \frac{s + \tilde{\gamma}(s)}{s + s\tilde{\gamma}(s) + \omega_0^2} \quad (58)$$

$$\tilde{H}(s) = \frac{1}{s + s\tilde{\gamma}(s) + \omega_0^2} \quad (59)$$

and

$$\tilde{\gamma}(s) = \int_0^\infty \gamma(t) \exp(-st)\mathrm{d}t \quad (60)$$

being the Laplace transforms of the frictional kernel $\gamma(t)$. Squaring eq 57 and taking the quantum statistical average, we obtain the relevant quantum correction $\langle \delta\hat{x}^2(t)\rangle$ to be

$$\begin{aligned}\langle \delta\hat{x}^2(t)\rangle &= G^2(t)\langle \delta\hat{x}^2(0)\rangle + \frac{H^2(t)}{m}\langle \delta\hat{p}^2(0)\rangle \\ &+ \frac{G(t)H(t)}{m}\langle(\delta\hat{x}(0)\delta\hat{p}(0) + \delta\hat{p}(0)\delta\hat{x}(0))\rangle \\ &+ 2\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t-t') \times H(t-t'')\langle \delta\hat{\eta}(t')\delta\eta(t'')\rangle\end{aligned} \quad (61)$$

A standard choice of the initial conditions corresponding to the minimum uncertainty state is.[53,64]

$$\begin{aligned}\langle \delta\hat{x}^2(0)\rangle &= \frac{\hbar}{2m\omega_0} \\ \langle \delta\hat{p}^2(0)\rangle &= \frac{m\hbar\omega_0}{2} \\ \langle(\delta\hat{x}(0)\delta\hat{p}(0) + \delta\hat{x}(p)\delta\hat{x}(0))\rangle &= \hbar\end{aligned} \quad (62)$$

We would like to know the exact form of the function $G(t)$ and $H(t)$ to calculate $\langle \delta\hat{x}^2(t)\rangle$. From the definition of $G(t)$ and $H(t)$, we have

$$H(t) = \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \tilde{H}(s) \exp(st)\mathrm{d}s \quad (63)$$

$$G(t) = \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \tilde{G}(s) \exp(st)\mathrm{d}s \quad (64)$$

Using the residue theorem, one can easily show that for the Ohmic dissipative bath and for the underdamped case

$$G(t) = \exp\left(-\frac{\gamma t}{2}\right)\left[\cos(\omega' t) + \frac{\gamma}{2\omega'}\sin(\omega' t)\right] \quad (65a)$$

$$H(t) = \exp\left(-\frac{\gamma t}{2}\right)\left[\frac{1}{\omega'}\sin(\omega' t)\right] \quad (65b)$$

where $\omega' = \pm[(\omega_0^2 - \gamma^2/4)]^{1/2}$. It is pertinent to point out here that for the overdamped case ω' becomes imaginary and $H(t)$ and $G(t)$ will be modified accordingly. In what follows, we restrict ourselves in the underdamped case (where $\omega_0 > (\gamma/2)$). Now for the Ohmic heat bath, the double integral in eq 61 can be written as

$$\begin{aligned}2\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t-t')H(t-t'')\langle \delta\hat{\eta}(t')\delta\hat{\eta}(t'')\rangle &= \\ \frac{\gamma}{\pi}\int_0^\infty \mathrm{d}\omega\hbar\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) &\times \\ \left|\frac{1 - e^{[-(\gamma/2 - i\omega)t]}\left[\cos\omega' t + (\gamma/2 - i\omega')\frac{\sin\omega' t}{\omega'}\right]}{\omega^2 - \omega_0^2 + i\gamma\omega}\right|^2\end{aligned} \quad (66)$$

From eq 66, we observe that the time dependence of the mean fluctuation in displacement is complicated, but it reduces to a simple form for large time compared to γ^{-1} and is given by

$$\langle \delta\hat{x}^2\rangle_{\mathrm{eq}} = \frac{\gamma}{\pi}\int_0^\infty \mathrm{d}\omega\left\{\hbar\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) \times \frac{1}{\omega^2 - \omega_0^2 + i\gamma\omega}\right\} \quad (67)$$

In the classical limit (i.e., when $\hbar \to 0$) one can find an expression for the classical equipartition result.

$$\langle \delta\hat{x}^2\rangle_{\mathrm{eq}} = \frac{k_BT}{\omega_0^2} \quad (68)$$

In the weak damping regime ($\gamma < \omega$), one obtains from eq 67

$$\langle \delta\hat{x}^2\rangle_{\mathrm{eq}} = \frac{\hbar}{2\omega_0}\coth\left(\frac{\hbar\omega_0}{2\pi k_BT}\right) \quad (69)$$

and consequently the quantum correction factor reduces to

$$Q_V(x, \langle \delta\hat{x}^n\rangle) = -x\xi(t)\langle \delta\hat{x}^2\rangle_{\mathrm{eq}} \quad (70)$$

where we have neglected higher-order quantum correction terms and $V_0(x, \xi(t), t) = (1/2)m\Omega_0^2x^2 + (1/12)x^4\xi(t)$. With this choice of $V_0(x, \xi(t), t)$ eq 43 converts to

$$m\frac{d^2x}{dt^2} + \gamma\dot{x} + m[\Omega_0^2 + \langle \delta\hat{x}^2\rangle_{\mathrm{eq}}\xi(t)]x(t) = \eta(t) \quad (71)$$

Equation 71 is the forced underdamped linear oscillator with a random frequency where the random force $\xi(t)$ is a Gaussian variable with zero mean and Ornstein-Zernike correlation

$$\langle \xi(t)\xi(t')\rangle = \sigma^2 \exp(-\lambda|t - t'|) \quad (72)$$

where the fluctuations of external parameter [frequency in eq 71] are expressed by multiplicative noise. The latter was widely used as a model to understand different phenomena in physics, for example, on-off intermittency, dye laser, polymers in random field, etc., and also in biology to study population dynamics and in economics (stock market prices).[65]

From the very form of the potential $V_0(x, t)$, we may anticipate that the harmonic oscillator, apart from its coupling with reservoir, is driven nonlinearly by an external noise $\xi(t)$ via the interaction term $H_{\mathrm{int}} = (1/12)x^4\xi(t)$, where $\langle \delta\hat{x}^2\rangle_{\mathrm{eq}}$ are the coupling parameters between the system and the external noise and σ is the strength of the noise [see eqs 71 and 72]. $\langle \delta\hat{x}^2\rangle_{\mathrm{eq}}$ is quantum mechanical in origin which reduces to $(k_BT)/(\omega_0^2)$ in the classical limit. At a given temperature, $(k_BT)/(\omega_0^2)$ may be taken as the coupling constant classically. In other words, the classical coupling constant will be a multiple of $(k_BT)/(\omega_0^2)$. On the contrary, the coupling constant will be a multiple of $\langle\delta\hat{x}^2\rangle_{eq}$ in the quantum regime. This is one of the main outcomes of the present study. However, if we live in a classical world, a Kubo type oscillator may be realized microscopically by perturbing the system with the interaction term $H_{int} = (1/2)x^2\xi(t)$, while in the quantum domain the quadratic interaction will not give us a quantum Kubo type oscillator. To realize the Kubo type oscillator microscopically in a quantum regime, one must have a biquadratic coupling. This forms an important observation in our current analysis.

A simple insertion of the term $H_{int} = (1/2)gx^2\xi(t)$ in the classical Hamiltonian of a harmonic oscillator results in the following equation of motion

$$\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + m[\Omega_0^2 + g\xi(t)]x(t) = \eta(t) \tag{73}$$

where g is the coupling constant and classically $g = (k_BT)/(\omega^2)$. However, the introduction of an interaction term which is quadratic in the operator $\hat{x}_j$ will not count any quantum dispersion term as is evident from the expression of Q_V, eq 48. Actually, from eq 48 it is clear that the nonvanishing quantum dispersion term arises only when the system potential includes the nonlinear term, and only the quartic nonlinearity can modify the frequency of the oscillator. It is evident from the very construction of the c-number description of the quantum Brownian motion that both classical and quantum harmonic oscillators evolve under the same effective potential with $Q_V = 0$. However, if the oscillator begins to move under nonlinear forces, the effective potential in quantum dynamics begins to differ from the classical potential, through the term Q_V, as the quantum dispersion terms begin to play their role through Q_V which is zero in the case of linear restoring force.

In many real systems, two types of random forces $\xi(t)$ and $\eta(t)$ may be correlated. In such a situation, we may define an effective noise and an associated effective temperature T_{eff} that depends on the degree of correlation λ between $\xi(t)$ and $\eta(t)$. For such a case, we may easily extend our study of quantum stochastic resonance for two correlated noises to observe the effect of correlation. We hope to address this issue in one of our forthcoming communications.

IV. Stochastic Resonance

We are now in a position to analyze stochastic resonance in quantum systems. We consider that our quantum Brownian particle is externally driven by a periodic sinusoidal force $a\sin(\Omega t)$. Consequently, the equation of motion, eq 71, which is the quantum analogue of the forced damped Kubo type oscillator, read (by setting $m = 1$) as

$$\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + [\Omega_0^2 + \varepsilon(t)]x(t) = \eta(t) + a\sin(\Omega t) \tag{74}$$

where

$$\varepsilon(t) = \langle\delta\hat{x}^2\rangle_{eq}\xi(t) \tag{75}$$

The modified noise $\varepsilon(t)$ is then defined by its statistical properties

$$\begin{aligned}\langle\varepsilon(t)\rangle &= 0\\ \langle\varepsilon(t)\varepsilon(t')\rangle &= D\exp[-\lambda|t - t'|]\end{aligned} \tag{76}$$

where

$$D = \sigma^2\{\langle\delta\hat{x}^2\rangle_{eq}\}^2 \tag{77}$$

The second-order differential, eq 75, can now be written as a pair of first-order differential equations

$$\frac{dx}{dt} = v \tag{78}$$

$$\frac{dv}{dt} = -v\gamma - \Omega_0^2x - \varepsilon(t)x + \eta(t) + a\sin(\Omega t) \tag{79}$$

After averaging over each realization of both external and internal noises, the above equations take the form

$$\frac{d}{dt}\langle x(t)\rangle = \langle v\rangle \tag{80a}$$

$$\frac{d}{dt}\langle v\rangle = -\gamma\langle v\rangle - \Omega_0^2\langle x(t)\rangle - \langle\varepsilon(t)x(t)\rangle + a\sin(\Omega t) \tag{80b}$$

If $\varepsilon(t)$ is a δ-correlated white noise, the term $\langle\varepsilon(t)x(t)\rangle$ vanishes. However, for the external Ornstein-Uhlenbeck noise process, the average would not be zero. To determine the average $\langle\eta(t)x(t)\rangle$, we adopt the Shapiro-Loginov theorem[66]

$$\frac{d}{dt}\langle\varepsilon(t)x(t)\rangle = \left\langle\varepsilon(t)\frac{dx}{dt}\right\rangle - \lambda\langle\varepsilon(t)x(t)\rangle \tag{81}$$

where λ is the correlation rate of the external noise $\varepsilon(t)$. Multiplying eq 78 by $\varepsilon(t)$, one gets after averaging

$$\left\langle\varepsilon(t)\frac{dx}{dt}\right\rangle = \langle\varepsilon(t)v(t)\rangle \tag{82}$$

Using eq 82 in eq 81, we get

$$\frac{d}{dt}\langle\varepsilon(t)x(t)\rangle = \langle\varepsilon(t)v(t)\rangle - \lambda\langle\varepsilon(t)x(t)\rangle \tag{83}$$

By a similar argument, starting from eq 79 we get

$$\left\langle\varepsilon(t)\frac{dv}{dt}\right\rangle = -\gamma\lambda\langle\varepsilon(t)v(t)\rangle - \Omega_0^2\langle\varepsilon(t)x(t)\rangle - \langle\varepsilon(t)^2x(t)\rangle \tag{84}$$

Equation 84 contains the higher-order term $\langle\eta(t)^2x(t)\rangle$, and we resort to a decoupling procedure[67]

$$\langle\varepsilon(t)^2x(t)\rangle \approx \langle\varepsilon(t)^2\rangle\langle x(t)\rangle = D\langle x(t)\rangle \tag{85}$$

Strictly speaking, the above decoupling is exactly valid for dichotomous noise processes. However, as van Kampen showed [see ref 67], the error incorporated by decoupling for correlated noise is of the order of τ^2. Thus, for noise process with short correlation time, the above decoupling does not include a significant error.

Thus, we have from eq 84

$$\frac{d}{dt}\langle\varepsilon(t)v(t)\rangle = -\gamma\lambda\langle\varepsilon(t)v(t)\rangle - \Omega_0^2\langle\varepsilon(t)x(t)\rangle - D\langle x(t)\rangle - \lambda\langle\varepsilon(t)v(t)\rangle \tag{86}$$

Thus we obtain a coupled system of four equations for the four variables

$$\frac{d}{dt}\langle x(t)\rangle = \langle v(t)\rangle \tag{87a}$$

$$\frac{d}{dt}\langle v(t)\rangle = -\gamma\langle v(t)\rangle - \Omega_0^2\langle x(t)\rangle - \lambda\langle\varepsilon(t)x(t)\rangle + a\sin(\Omega t) \tag{87b}$$

$$\frac{d}{dt}\langle\varepsilon(t)x(t)\rangle = -\lambda\langle\varepsilon x(t)\rangle + \langle\varepsilon(t)v(t)\rangle \tag{87c}$$

$$\frac{d}{dt}\langle\varepsilon(t)v(t)\rangle = -(\lambda+\gamma)\langle\varepsilon(t)v(t)\rangle - \Omega_0^2\langle\varepsilon(t)x(t)\rangle - D\langle x(t)\rangle \tag{87d}$$

The above sets of equations are equivalent to the fourth-order ordinary differential equation for $\langle x(t)\rangle$

$$\left\{\frac{d^4}{dt^4} + 2(\lambda+\gamma)\frac{d^3}{dt^3} + (2\Omega_0^2 + \lambda^2 + 3\lambda\gamma + \gamma^2)\frac{d^2}{dt^2} + [2\Omega_0^2(\lambda+\gamma) + \gamma(\lambda+\gamma)\lambda]\frac{d}{dt} + [\Omega_0^2(\Omega_0^2 + \lambda^2 + \lambda\gamma) - D]\right\}\langle x(t)\rangle = (\Omega_0^2 + \lambda^2 + \lambda\gamma - \Omega^2)\times a\sin(\Omega t) + (2\lambda+\gamma)a\cos(\Omega t) \tag{88}$$

We seek a solution of eq 88 in the form

$$\langle x(t)\rangle = \langle x(t)\rangle_0 + \langle x(t)\rangle_\Omega \tag{89}$$

where the output signal $\langle x(t)\rangle_\Omega$ is induced by the external field and $\langle x(t)\rangle_0$ is determined by internal dynamics. Let us assume the solution of $\langle x(t)\rangle_\Omega$ in the form

$$\langle x(t)\rangle_\Omega = A\sin(\Omega t + \varphi) \tag{90}$$

Then one easily obtains

$$A = \left[\frac{\alpha_1^2 + \alpha_2^2}{\alpha_3^2 + \alpha_4^2}\right]^{1/2} \text{ and } \varphi = \tan^{-1}\left[\frac{\alpha_1\alpha_3 + \alpha_2\alpha_4}{\alpha_1\alpha_4 - \alpha_2\alpha_3}\right] \tag{91}$$

where $c_0^2 = \gamma$ and

$$\begin{aligned} \alpha_1 &= (2\lambda+\gamma)a\Omega \\ \alpha_2 &= (\Omega^2 - \Omega_0^2 - \lambda^2 - \lambda\gamma)a \\ \alpha_3 &= (\Omega^2 - \Omega_0^2)(\Omega^2 - \Omega_0^2 - \lambda^2) - \\ &\qquad \sigma - (3\lambda\gamma + \gamma^2) + \lambda\gamma\Omega_0^2 \\ \alpha_4 &= \Omega(\lambda+\gamma)[2(\Omega^2 - \Omega_0^2) + \gamma\lambda] \end{aligned} \tag{92}$$

In the absence of friction γ, eq 91 reduces to

$$A = a\sqrt{\frac{(\Omega_0^2 - \Omega^2 - \lambda^2)^2 + 4\lambda^2\Omega^2}{\{(\Omega^2 - \Omega_0^2)(\Omega_0^2 - \Omega^2 - \lambda^2) - D\}^2 + 4\lambda^2\Omega^2(\Omega^2 - \Omega_0^2)^2}} \tag{93a}$$

$$\varphi = \tan^{-1}\left[\frac{2\Omega\lambda D}{\{(\Omega^2 - \Omega_0^2)(\Omega_0^2 - \Omega^2 - \lambda^2) - D\}^2 - D(\Omega_0^2 - \Omega^2 - \lambda^2)}\right] \tag{93b}$$

In the absence of any noise, $\lambda = 0 = D$, eq 91 reduces to $A = a$ and $\varphi = 0$ which is the result for the corresponding forced underdamped equation. For Gaussian white noise which corresponds $\sigma \to \infty$ and $\lambda \to \infty$ with a constant ratio, (σ/λ), then

$$\begin{aligned} A &= \frac{a}{\Omega^2 - \Omega_0^2} \\ \varphi &= 0 \end{aligned} \tag{94}$$

as it should be since the white noise $\langle x(t)\rangle$ satisfies the equation

$$\frac{d^2\langle x\rangle}{dt^2} + \Omega_0^2\langle x\rangle = a\sin(\Omega t) \tag{95}$$

If the external force $\xi(t)$ is δ-correlated, thereby making $\varepsilon(t)$ δ-correlated, then from eq 74 we see that the average of x obeys the equation

$$\frac{d^2\langle x(t)\rangle}{dt^2} + \gamma\frac{d\langle x(t)\rangle}{dt} + \Omega_0^2\langle x(t)\rangle = a\sin(\Omega t) \tag{96}$$

Thus, quantum mechanics does not play any role when the external noise is a Gaussian white noise for an oscillator. However, for Ornstein-Uhlenbeck noise, the output signal eq 93a shows nonmonotonic dependence on the noise strength D and the correlation rate λ, which is the signature of stochastic resonance. The amplitude of the output signal indeed reaches a maximum when $D = (\Omega^2 - \Omega_0^2)(\Omega_0^2 - \Omega^2 - \lambda^2)$ and $\lambda = (\Omega_0^2 - \Omega^2)$.

It has been already mentioned that Gittermann[35] has shown that SR may be obtained in a linear system like a Kubo type oscillator. From our microscopic realization of the Kubo type oscillator and the associated SR, it becomes clear that though it is a linear system there exists inherent hidden nonlinearity. This is apparent from the "$H_{int}[= (1/12)x^4\xi(t)]$" part of the Hamiltonian, which is an essential ingredient for realizing the fluctuating part of the quantum oscillator. This observation bolsters our belief that the presence of nonlinearity is essential for SR.

We are now in a position to discuss the nature of output signal, A, from a numerical point of view. It is clear from the expression, (eq 93b), that we have a monotonically varying A

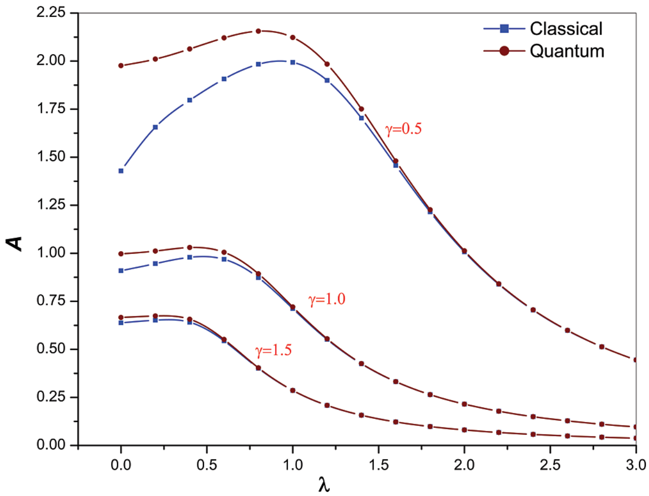

Figure 1. Plot of A (output signal) as a function of λ (correlation time) for various γ (dissipation constant) [$\sigma = 0.1$, all other parameters are set to be unity].

with the monotonicity being governed by λ. From our development it is evident that the amplitude, A, exhibits a maxima at $\lambda^2 = \Omega_0^2 - \Omega^2$. For other values of λ, A decreases from its maximum value. Here we recall that the expression (eq 93b) is physically valid in the dissipationless situation. The nonmonotonicity is not apparent from the expression of A after the inclusion of the dissipation effect. To examine this further, in Figure 1 we have plotted A against varying λ for different values of the dissipation constant γ for both the quantum and classical situations. From the figure, it is clear that the output signal exhibits nonmonotonic dependence on the inverse of correlation time, λ, which is the signature of SR. Figure 1 clearly displays a signature of the fact that the quantum effects amplify the amplitude of SR since the quantum effects modify the effective strength of the external noise. It should be emphasized here that according to our description of the present model the system (Kubo type oscillator) is monitored by three forces, viz., the random force $\eta(t)$ originating from the coupling of the system with the heat bath and the two external forces: (i) $\varepsilon(t)$, a random force that appears multiplicatively, and (ii) the periodic driving force. From the figure, we have observed that the resonance intensity decreases with γ as it should. The results of the quantum model merged asymptotically with its classical counterpart with increasing the value of λ. Actually, when λ is large, the corresponding correlation time $1/\lambda$ is small, implying that the multiplicative noise makes its appearance as a δ-correlated noise, that is, $\langle \xi(t)x(t)\rangle = 0$, and then our quantum model reduces to the classical one.[37]

V. Conclusions

In the course of understanding of stochastic resonance in a linear quantum system, a Kubo type oscillator (an oscillator with fluctuating frequency), we first proposed a system-reservoir model to obtain an approximate c-number generalized quantum Langevin equation which apart from being a c-number integrodifferential equation incorporates the quantum effects. In the present formulation, we have treated the system quantum mechanically and the reservoir quasiclassically. We then calculate the quantum correction terms in leading order and thereby realize quantum Kubo type oscillator and obtain the coupling constant in both classical and quantum regimes. We also demonstrate that the quantum effect plays its role only when the external noise possesses "finite correlation time". We then numerically study the resonance behavior of the forced damped Kubo type oscillator, which being a linear system exhibits a stochastic resonance phenomena. For small λ, the quantum effects increase the amplitude of resonance. Our numerical illustration also shows that the quantum and classical results become identical asymptotically in the limit of large λ because δ-correlated noise appears as a multiplicative noise in the evolution equation. Indeed, under such a situation the averaging over δ-correlated multiplicative noise contributes zero in average dynamics. Furthermore, the microscopic realization of the Kubo type oscillator reveals the fact that though it is a linear system there is an embedded nonlinearity in its very microscopic construction. We thus conclude that nonlinearity is an essential ingredient of stochastic resonance.

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