Quantum Lévy Propagators†

Bruce J. West

Army Research Office, Research Triangle Park, North Carolina, 27709

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The general quantum propagator for a particle in a potential-free region of the vacuum is shown to have a form analogous to that of a classical Lévy stochastic process and not that of a Gaussian wave packet.

1 Introduction

It is a pleasure to contribute a paper for Harvey Scher's festschrift. Our research interests first overlapped in the early seventies when Harvey was working on problems of conduction in amorphous semiconductors with Elliott Montroll and Michael Shlesinger, and I was working with Elliott on how to model the evolution of systems that do not have differential equations of motion. We were all interested in the exotic nature of the statistical processes of Paul Lévy and the incredibly complex physical phenomena that these statistics promised to explain. At that time Elliott published a brief paper on the quantum analogue of the Lévy distribution in a festschrift volume for Josef Jauch.[1] With the recent interest in Lévy distributions on the part of the physics community, and its success in modeling the kind of phenomena that Harvey has investigated throughout his career, I think it would be interesting to offer a sequel to Montroll's paper here.

In the past decade, the understanding of complex physical phenomena has more and more required the application of infinitely divisible statistical distributions such as those of Lévy.[2,3] As the name implies one can partition processes described by such distributions into smaller and smaller parts without changing the statistical character of the phenomenon. In the physical sciences this property is called scaling. Diffusion, whether ordinary or anomalous, has such a scaling property. It is interesting to investigate the consequences of such a partitioning when applied to quantum phenomena. In this regard the Lévy−Khintchine formula has been applied to the investigation of relativistic quantum mechanics.[4] In addition, the quantum dynamics of the reduced density matrix for certain complex systems has been shown to be Lévy stable.[5]

It has been suggested by a number of scientists that the Fokker−Planck equation should be replaced by a fractional diffusion equation in classical systems containing long-term memory or long-range interactions.[6] So too, we argue here, the Schrödinger equation ought to be replaced to describe such phenomena at the quantum level. It may be argued that the Fokker−Planck equation is a coarse-grained description of the evolution of the probability density over space and time and describes the possible futures of a phenomenon. On the other hand, the Schrödinger equation is a fundamental description of the evolution of a system, more like the dynamical description using Hamilton's equations in classical mechanics, than it is like the averaged description of the Fokker−Planck equation.

Herein we explore the possibility that the Schrödinger equation does not provide a unique description of the evolution of quantum mechanical observables. We deduce the equation of evolution for the quantum mechanical wave function from the following conditions: (1) the wave function is a probability amplitude and is therefore mean-square integrable at all times; (2) the space-time evolution of the wave function is determined by a unitary propagator; and (3) the quantum propagator satisfies a chain condition. These constraints in themselves have in the past been used to construct the Schrödinger equation as an approximate expression of the chain condition. We show that in free space the chain condition can be solved exactly and leads to a propagator that is the quantum analogue of the Lévy transition probability in classical probability theory.

It is demonstrated that the quantum Lévy propagator is not the solution of the Schrödinger equation, nor indeed it is not the solution to any integer differential representation of the evolution of the quantum system. The quantum Lévy propagator satisfies a fractional-derivative equation of motion that incorporates long-range interactions into the free particle dynamics.

2 Properties of Quantum Wave Functions

The initial state of a quantum mechanical system is described by the state vector $\langle\Psi|$, such that in configuration space the initial wave function is defined by $\Psi(x_0,t_0) \equiv \langle\Psi|x_0,t_0\rangle$. After a time interval $t - t_0$, the system evolves to the new state $\Psi(x,t)$ in such a way that the two wave functions are tied together by a propagator

$$\Psi(x,t) = \int_\Omega K(x,t|x_0,t_0)\Psi(x_0,t_0)\mathrm{d}x_0 \qquad (1)$$

The kernel $K(x,t|x_0,t_0)$ describes the propagation of the pieces of the initial wave function from the region of x_0 at time t_0 to x at time t, and Ω denotes the spatial domain available to the process. We find it convenient to use the notation of Feynman and Hibbs[7] for the propagator

$$K(1,0) \equiv K(x_1,t_1|x_0,t_0) \qquad (2)$$

and suppressing the domain of integration we rewrite eq 1 as

$$\Psi(1) = \int K(1,0)\Psi(0)\mathrm{d}x_0 \qquad (3)$$

Note that the time and the position are arbitrary in eq 1 so that we can segment space and time into intermediate locations and times and iterate eq 3 to obtain

$$\Psi(1) = \int \mathrm{d}x_2 \int K(1,2)K(2,0)\Psi(0)\mathrm{d}x_0 \quad t_1 > t_2 > t_0 \qquad (4)$$

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The physical interpretation of eq 4 is that the evolution of the probability amplitude away from the initial state $\Psi(0)$ to the measured state $\Psi(1)$ is independent of all possible intermediate states in that evolution. A piece of the wave function in the vicinity of x_0 at time t_0 propagates to the vicinity of x_2 in the time interval $t_2 - t_0$, and then propagates to the vicinity of x_1 in the time interval $t_1 - t_2$. Comparing the two equations of evolution, eqs 3 and 4, we see that the kernel satisfies the chain condition

$$K(1,0) = \int \mathrm{d}x_2 K(1,2)K(2,0) \quad t_1 > t_2 > t_0 \tag{5}$$

from which it is clear that the intermediate time is immaterial in the propagation of the wave function. Equation 5 is analogous to the BCKS chain condition relating the transition probabilities in classical probability theory.[1,3]

The interpretation of the wave function as a probability amplitude requires that the normalization be conserved over time

$$\int |\Psi(1)|^2 \mathrm{d}x_1 = \int |\Psi(0)|^2 \mathrm{d}x_0 \quad t_1 > t_0 \tag{6}$$

Thus, substituting eq 3 into eq 6 we obtain, denoting the complex conjugate of a function by an astrix superscript,

$$\int \mathrm{d}x_1 \int \mathrm{d}x_2 \int \mathrm{d}x_3 K(1,2)K^*(1,3)\Psi(2)\Psi^*(3) = \int |\Psi(2)|^2 \mathrm{d}x_2 \tag{7}$$

For eq 7 to be true for arbitrary initial states we must have[7]

$$\int \mathrm{d}x_1 K(1,2)K^*(1,3) = \delta(x_2 - x_3) \tag{8}$$

The propagator is therefore unitary as well as satisfying the chain condition eq 5.

It is not possible to solve the chain condition, eq 5, in complete generality. However, if we restrict our discussion to free space with no potentials, then we may introduce the idea of translational invariance such that

$$K(x,t|x_0,t_0) = K(x - x_0, t - t_0) \tag{9}$$

and introduce

$$K(10) \equiv K(x_1 - x_0, t_1 - t_0) \tag{10}$$

Equation 9 implies that the evolution of the wave function in free space depends only on the distance between points in space and time and not on the absolute origin of the coordinate system in space and time. Therefore, the evolution equation becomes

$$\Psi(1) = \int K(10)\Psi(0)\mathrm{d}x_0 \tag{11}$$

the chain condition can be written

$$K(10) = \int \mathrm{d}x_2 K(12)K(20) \tag{12}$$

and the unitary condition becomes

$$\int \mathrm{d}x_1 K(12)K^*(13) = \delta(x_2 - x_3) \tag{13}$$

These three equations, (11−13), and normalization define the basic properties of quantum mechanical processes.

3 Nonanalytic Quantum Propagator

The algebraic form of the free particle quantum propagator can be obtained by solving the chain condition (eq 12) subject to the unitary constraint (eq 13). Obtaining the solution to eq 12 is most readily done by introducing the Fourier transform of the kernel over the unbounded domain $-\infty \le x \le \infty$

$$\hat{K}(k,t) = \int_{-\infty}^{\infty} \mathrm{e}^{ikx} K(x,t)\mathrm{d}x \tag{14}$$

and expressing the convolution of two functions as the product of their Fourier amplitudes to obtain from eq 12

$$\hat{K}(k,t_1 - t_0) = \hat{K}(k,t_1 - t_2)\hat{K}(k,t_2 - t_0) \tag{15}$$

Therefore, expressing eq 15 in terms of logarithms

$$\ln[\hat{K}(k,t_1 - t_0)] = \ln[\hat{K}(k,t_1 - t_2)] + \ln[\hat{K}(k,t_2 - t_0)]$$

it is clear that the solution factors into a function of k, say $f(k)$, and a function of time. The function of time must be linear in order for the intermediate time, t_2, to vanish from the equation. Thus, we obtain the solution[3]

$$\hat{K}(k,t) = \mathrm{e}^{f(k)t} \tag{16}$$

The condition of unitarity requires that

$$|\hat{K}(k,t)|^2 = 1 \tag{17}$$

which can only be valid if $f(k)$ has no real part, that is, the Fourier transform of the quantum propagator has the form

$$\hat{K}(k,t) = \mathrm{e}^{ig(k)t} \tag{18}$$

where $g(k)$ is real. In order for the kernel to retain the chain property (eq 12) at all spatial scales it must be infinitely divisible, as we discussed in the Introduction. If we scale the Fourier variable k by a constant factor b, then in order for the propagator to be infinitely divisible the function $g(k)$ must be homogeneous, that is,

$$g(bk) = b^{\alpha} g(k) \quad b > 0 \tag{19}$$

The homogeneity requirement (eq 19) implies that[6]

$$g(k) = b(\alpha)|k|^{\alpha} \tag{20}$$

where $b(\alpha)$ is a real function dependent on the exponent α. Thus, we have for the Fourier transform of the propagator

$$\hat{K}_L(k,t) = \mathrm{e}^{ib(\alpha)|k|^{\alpha}t} \tag{21}$$

and the free-particle quantum kernel is the inverse Fourier transform of eq 21

$$K_L(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ib(\alpha)|k|^{\alpha}t}\mathrm{e}^{-ikx}\mathrm{d}k \tag{22}$$

Note that the propagator given by eq 22 is the quantum analogue of the classical Lévy transition probability.[1,3]

4 The Equations of Evolution

The equation of evolution for the usual free-particle propagator is obtained using the Schrödinger equation to be

$$\frac{\partial K(10)}{\partial t_1} + i\frac{\hbar}{2m}\frac{\partial^2 K(10)}{\partial x_1^2} = \delta(10) \tag{23}$$

for a free particle of mass m and $\delta(10) \equiv \delta(x_1 - x_0)\delta(t_1 - t_0)$. The Fourier, in space, and Laplace, in time, transforms of the solution to eq 23 are given by

$$\tilde{K}(k,s) = (s - ib(2)k^2)^{-1} \tag{24}$$

where $b(2) = \hbar/2m$. The inverse Fourier−Laplace transform of eq 24 yields the Gaussian wave packet

$$K(10) = \begin{cases} [4\pi i(t_1 - t_0)b(2)]^{-1/2} \exp\left[-\dfrac{i(x_1 - x_0)^2}{4(t_1 - t_0)b(2)}\right] & t_1 \geq t_0 \\ 0 & t_1 < t_0 \end{cases} \tag{25}$$

see for example, Feynman and Hibbs.[7] It is clear that the Gaussian propagator given by eq 25 is also obtained from the general free-particle quantum kernel (eq 22) in the case $\alpha = 2$. This is one of the few cases where the integration in eq 22 can be carried out explicitly.

Now we do things a little backward since we already have the solution for the general free-particle kernel (eq 22) and we want its Fourier−Laplace transform so as to be able to construct the equation of evolution. The Fourier−Laplace transform of eq 22 is

$$\tilde{K}_L(k,s) = (s - ib(\alpha)|k|^{\alpha})^{-1} \tag{26}$$

and it is clear that $K_L(10)$, unlike the $\alpha = 2$ case, does not satisfy a partial differential equation of evolution. Instead the inverse Laplace transform of $\tilde{K}_L(k,s)$ in eq 26 yields the equation of evolution for the characteristic function

$$\frac{\partial \hat{K}_L(k,t)}{\partial t} + ib(\alpha)|k|^{\alpha}\, \hat{K}_L(k,t) = \delta(t) \tag{27}$$

We first constructed an equation of the form of eq 27 for the classical characteristic function[8] and found a fractional diffusion equation for the probability density, whose solution was the Lévy distribution. Finally, taking the inverse Fourier transform of $\hat{K}_L(k,t)$ we obtain the equation of evolution for the Lévy propagator

$$\frac{\partial K_L(10)}{\partial t} + iR^{\alpha}K_L(10) = \delta(10) \tag{28}$$

where R^{α} is the Reisz fractional derivative operator[6,9] given by

$$R^{\alpha}K_L(10) = \frac{b(\alpha)}{\pi}\Gamma(\alpha + 1)\sin[\pi\alpha/2]\int_{-\infty}^{\infty}\frac{K_L(x',t)}{|x - x'|^{\alpha+1}}dx' \tag{29}$$

where $x = x_1 - x_0$ and $t = t_1 - t_0$ in eq 29. Thus, eq 28 is the general equation of motion for the quantum propagator of a free particle and whose solution, eq 22, does not have a simple analytical form except for a few special values of the exponent α such as $\alpha = 1$ and 2.

Montroll determined the functional form for the parameter $b(\alpha)$ by means of a simple scaling argument.[1] Assume that if b is a length scale, m is the free particle mass then we can write

$$b(\alpha) = b^{\lambda}\hbar^{\mu}m^{\nu} \tag{30}$$

where to make the parameter quantum mechanical it must also depend on Planck's constant. We also know that when $\alpha = 2$, $b(2) = \hbar/2m$. Thus, we have $\mu = 1$, $\nu = -1$, and $\lambda = \alpha - 2$ so that

$$b(\alpha) = \frac{\hbar b^{\alpha-2}}{2m} \tag{31}$$

so that $b(\alpha)|k|^{\alpha}$ has the dimensions of inverse time. Therefore we have the result that there is a fundamental length in quantum theory except for the singular case $\alpha = 2$, where the Schrödinger equation is valid.

5 Some Further Speculations

The free particle quantum propagator describes the propagation of matter waves through free space, not unlike the lattice Greens' functions used to transport electrons on a lattice. The free particle Gaussian propagator implies that the vacuum is without structure and passively provides a background for the particle motion. On the other hand, the free particle Lévy propagator implies that the vacuum contains long-range interactions. This difference in structure between the two kernels is similar to the distinction between a "bare" and "dressed" kernel in quantum many body theory.

References and Notes

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