Time-Dependent Rate Coefficients for Diffusion-Influenced Reactions with Centrosymmetric Potentials

Olga K. Dudko[†] and Attila Szabo*[,‡]

Mathematical and Statistical Computing Laboratory, Division of Computational Bioscience, Center for Informational Technology, and Laboratory of Chemical Physics, National Institute of Diabetes and Digestive and Kidney Diseases, National Institutes of Health, Bethesda, Maryland, 20892

Simple closed-form expressions are presented for the time-dependent rate coefficients of diffusion-influenced reactions in the presence of spherically symmetric potentials. For diffusion-controlled contact reactions, our expression reproduces the first two terms in both the short- and long-time expansions of the rate coefficient. At intermediate times, agreement with numerical results for the Debye−Hückel potential is found to be within a few percent for a wide range of parameters. For diffusion-influenced contact reactions (described by the radiation boundary condition), the agreement is even better. When the reactivity depends on the distance between the reactants (e.g., exponentially), our analytic result is less accurate, because it reproduces the two terms in the long-time expansion only to the linear order of the reciprocal of the diffusion coefficient. Our results should prove useful in the analysis of experimental data for diffusion-influenced reactions with centrosymmetric interaction potentials.

Introduction

The kinetics of irreversible diffusion-influenced biomolecular reactions are described using time-dependent rate coefficients, $k(t)$. Simple analytic expressions exist only for free diffusion when the reaction occurs at contact. In this paper, we obtain accurate analytic expressions for both contact- and distance-dependent reactivities when the reactants interact by means of a spherically symmetric potential. These results should prove useful in the analysis of experimental data for essentially irreversible reactions such as fluorescence quenching. In addition, recent theories[1,2] of reversible reactions require as input the Laplace transform of the rate coefficient for the corresponding irreversible reaction. Thus, if the results presented in this paper are used, the time course of reversible reactions can now be readily calculated for centrosymmetric interaction potentials.

For reactions occurring at contact in the presence of spherically symmetric potentials, closed-form expressions for $k(t)$ have been previously proposed by Pedersen and Sibani,[3] Weller,[4] and Flannery.[5] To describe these approximations and to indicate how our work improves upon them, we start with the formally exact relation between the Smoluchowski $k_S(t)$[6] for diffusion-controlled reactions and the Collins−Kimball $k_{CK}(t)$[7] for diffusion-influenced reactions at contact in the presence of a spherically symmetric potential $U(r)$. In Laplace space ($\hat{f}(z) = \int_0^\infty f(t)\, e^{-zt}\, dt$), the relation is[8]

$$\frac{1}{z\,\hat{k}_{CK}(z)} = \frac{1}{k_{RC}} + \frac{1}{z\,\hat{k}_S(z)}, \quad k_{RC} = k_0\, e^{-\beta U(a)} \qquad (1)$$

where a is the contact radius, k_0 is the intrinsic (reaction-controlled) rate constant in the absence of the potential, and β

$= (k_B T)^{-1}$. The Pedersen−Sibani expression for $k_{CK}(t)$ is obtained from eq 1 by using a two-term approximation for $k_S(t)$ which reproduces the two leading terms in the long-time expression of the rate coefficient.[8] The Weller−Flannery expression is based on a two-term expression for $k_S(t)$ which reproduces the first leading term in both the long-time and the short-time expansions.[8] Our work is based on an approximate expression for $k_S(t)$ that reproduces the two leading terms of *both* the long- and short-time expansions. For nonlocal reactions described by a position-dependent sink, we present a somewhat less accurate expression which reproduces the two terms in the long-time expansion exactly only to the linear order in the reciprocal of the diffusion constant.

Derivation of the Closed-Form Expressions

Reactions at Contact. In the presence of an arbitrary spherically symmetric interaction potential $U(r)$ when reaction occurs every time the reactants come in contact (i.e., $r = a$), the long-time limit (or the $z \to 0$ limit in Laplace space) of the Smoluchowski time-dependent rate coefficient is[3]

$$\lim_{t\to\infty} k_S(t) = k_S(\infty) + A_\infty/(\sqrt{\pi t}) \qquad (2a)$$

$$\lim_{z\to 0} z\hat{k}_S(z) = k_S(\infty) + A_\infty \sqrt{z} \qquad (2b)$$

where

$$\frac{1}{k_S(\infty)} = \int_a^\infty \frac{e^{\beta U(r)}}{4\pi D r^2}\, dr \equiv \frac{1}{4\pi D a_e} \qquad (3a)$$

where D is the relative diffusion coefficient and

$$A_\infty = k_S(\infty)^2/(4\pi D^{3/2}) \qquad (3b)$$

* attilas@nih.gov.
[†] Center for Informational Technology.
[‡] National Institute of Diabetes and Digestive and Kidney Diseases.

5892 *J. Phys. Chem. B, Vol. 109, No. 12, 2005*

Dudko and Szabo

The corresponding short-time limit (or $z \to \infty$ in Laplace space) is[8]

$$\lim_{t \to 0} k_S(t) = A_0/(\sqrt{\pi t}) + B_0 \qquad (4a)$$

$$\lim_{z \to \infty} z\hat{k}_S(z) = A_0\sqrt{z} + B_0 \qquad (4b)$$

where

$$A_0 = 4\pi a^2 \sqrt{D} \, e^{-\beta U(a)} \qquad (5a)$$

$$B_0 = 4\pi Da \, e^{-\beta U(a)}[1 - \beta a U'(a)/2] \qquad (5b)$$

where the prime (′) denotes a derivative. For $U(r) = 0$, eqs 2 and 4 are identical.

Our key observation is that the Padé approximant

$$z\hat{k}_S(z) =$$

$$\frac{A_0(A_\infty - A_0)z + [A_\infty B_0 - k_S(\infty) A_0]\sqrt{z} + k_S(\infty)[B_0 - k_S(\infty)]}{(A_\infty - A_0)\sqrt{z} + [B_0 - k_S(\infty)]}$$

$$(6)$$

reproduces the first two terms in the expansion of the Laplace transform of the time-dependent Smoluchowski rate coefficient in *both* the $z \to 0$ (eq 2b) and $z \to \infty$ (eq 4b) limits. Inverting this, we find

$$k_S(t) = k_S(\infty)[1 + \gamma(a_e - a^2 \, e^{-\beta U(a)}/a_e)e^{\gamma^2 Dt} \, \text{erfc}\sqrt{\gamma^2 Dt} + a^2 \, e^{-\beta U(a)}/(\pi Da_e^2 t)^{1/2}] \qquad (7)$$

where erfc(x) is the complementary error function and

$$\gamma = \frac{a \, e^{-\beta U(a)}[1 - \beta a U'(a)/2] - a_e}{a_e^2 - a^2 \, e^{-\beta U(a)}} \qquad (8)$$

For potentials for which $\gamma \geq 0$, eq 7 reproduces the first two leading tearms of both the short- and long-time expansions of $k_S(t)$. However, when $\gamma < 0$, eq 7 reproduces the first two terms in the short-time expansion but only the leading term in the long-time expansion. In this case, it is better to have an expression that reproduces the first two terms in the long-time expansion but only the leading term in the short-time one. Such an expression can be obtained from eq 7 by replacing γ by its absolute value. For the screened Coulomb (Debye−Hückel) potential, γ is always positive when the potential is attractive. When it is repulsive, γ is negative only in a small region of parameter space.

To obtain an analytic expression for the Collins−Kimball time-dependent rate coefficient for a diffusion-influenced reaction, we substitute eq 6 into eq 1. Inverting the resulting expression, we find

$$k_{CK}(t) = k_{CK}(\infty)[1 + \alpha_1 \, e^{\gamma_1^2 t} \, \text{erfc}\sqrt{\gamma_1^2 t} + \alpha_2 \, e^{\gamma_2^2 t} \, \text{erfc}\sqrt{\gamma_2^2 t}] \qquad (9)$$

where

$$\frac{1}{k_{CK}(\infty)} = \frac{1}{k_{RC}} + \frac{1}{k_S(\infty)} \qquad (10a)$$

and

$$\gamma_{1,2} = \frac{\mu \mp \lambda}{A_0(A_\infty - A_0)} \qquad (10b)$$

$$\alpha_{1,2} = \frac{k_{RC}}{2k_S(\infty)} \left\{ 1 \pm \frac{\mu - (A_\infty - A_0)[k_{RC} + k_S(\infty)]}{\lambda} \right\} \qquad (10c)$$

$$2\mu = A_\infty(k_{RC} + B_0) - A_0[k_{RC} + k_S(\infty)] \qquad (10d)$$

$$\lambda^2 = \mu^2 - A_0(A_\infty - A_0)[B_0 - k_S(\infty)][k_{RC} + k_S(\infty)] \quad (10e)$$

Equations 7 and 9 are the key results of this paper for contact reactions. The time integral of $k_{CK}(t)$ can be readily obtained using the identity

$$\int_0^t e^{\gamma^2 \tau} \, \text{erfc}\sqrt{\gamma^2 \tau} \, d\tau = \frac{1}{\gamma^2}\left(e^{\gamma^2 t} \, \text{erfc}\sqrt{\gamma^2 t} + 2\sqrt{\frac{t\gamma^2}{\pi}} - 1 \right)$$

$$(11)$$

Distance-Dependent Reactivity. When the reaction may occur at different distances between the reactants, the time-dependent rate coefficient is (see, e.g., ref 8)

$$k(t) = \int_0^\infty 4\pi r^2 \, \kappa(r) \, p(r, t) \, dr \qquad (12)$$

where $\kappa(r)$ is a sink function and the probability density $p(r, t)$ satisfies

$$\frac{\partial p(r, t)}{\partial t} = \frac{D}{r^2} \frac{\partial}{\partial r} r^2 \, e^{-\beta U(r)} \frac{\partial}{\partial r} e^{\beta U(r)} \, p(r, t) - \kappa(r) \, p(r, t) \quad (13)$$

with the initial condition $p(r, 0) = e^{-\beta U(r)}$.

In the limit $D \to \infty$, $k(t)$ approaches the reaction-controlled rate $k_{RC} = \langle \kappa \rangle = \int_0^\infty 4\pi r^2 \, e^{-\beta U(r)} \, \kappa(r) \, dr$. To find an approximate expression for $k(\infty)$, we assume it has the same structure as eq 10a, that is

$$\frac{1}{k(\infty)} = \frac{1}{\langle \kappa \rangle} + \frac{1}{k_{DC}(\infty)} \qquad (14)$$

where we have introduced a diffusion-controlled (DC) rate coefficient that is the appropriate generalization of the Smoluchowski $k_S(\infty)$. We determine $k_{DC}(\infty)$ by requiring eq 14 to be exact up to order of $1/D$.

To calculate $k(\infty)$ up to order of $1/D$, we take the $t \to \infty$ limit of eq 13 and expand $p(r, \infty)$ as

$$p(r, \infty) = e^{-\beta U(r)} - \frac{f_1(r)}{D} + \cdots \qquad (15)$$
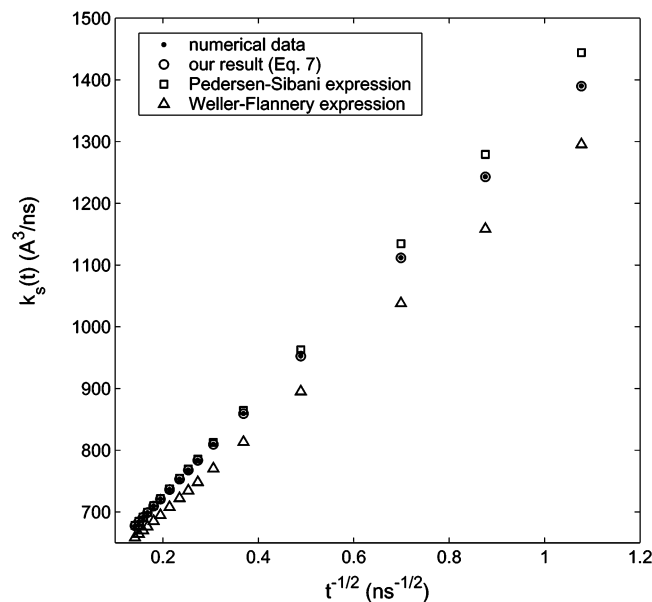
Substituting this into the steady-state version of eq 13, letting $D \to \infty$, and solving the resulting inhomogeneous differential equation, we find that

$$f_1(r) = e^{-\beta U(r)} \int_r^\infty \frac{e^{\beta U(r')}}{r'^2} \int_0^{r'} \kappa(\rho)\rho^2 \, e^{-\beta U(\rho)} \, d\rho \, dr' \quad (16)$$

Substituting eq 15 with $f_1(r)$ in eq 16 into eq 12 and changing the order of integration, we find

$$k(\infty) = \langle \kappa \rangle - 4\pi \int_0^\infty \frac{e^{\beta U(r)}}{Dr^2}[\int_0^r \kappa(\rho)\rho^2 \, e^{-\beta U(\rho)} \, d\rho]^2 \, dr \quad (17)$$

Comparing this with the $1/D$ expansion of eq 14, we find that

Diffusion-Influenced Reactions

J. Phys. Chem. B, Vol. 109, No. 12, 2005 **5893**

Figure 1. The time-dependent Smoluchowski rate coefficient $k_S(t)$ given by previous approximations[2–4] and by eq 7 in this work compared to numerical results in the presence of the Debye−Hückel potential with parameters that describe experimental data.[9] The parameter values are $D = 6.7$ Å²/ns, $a = 7$ Å, $R_D = 9.1$ Å, $\kappa = 0.5$ Å^{-1}.

the steady-state diffusion-controlled rate constant for a nonlocal reaction is approximately given by

$$\frac{1}{k_{DC}(\infty)} = \frac{\int_0^\infty \frac{e^{\beta U(r)}}{4\pi D r^2}\left(\int_0^r \rho^2 \kappa(\rho)\, e^{-\beta U(\rho)}\, d\rho\right)^2 dr}{\left(\int_0^\infty r^2 \kappa(r)\, e^{-\beta U(r)}\, dr\right)^2} \quad (18)$$

Note that eq 18 reduces to eq 3a in the limit of the δ-function sink when $\kappa(r) = k_0 \delta(r - a)/(4\pi r^2)$, which describes a reaction at contact. It can be shown that eq 14 with $k_{DC}(\infty)$ given by eq 18 is exact within the framework of the Wilemski−Fixman approximation.[9–11]

The short-time limit of $k(t)$ can be readily obtained from eqs 12 and 13

$$\lim_{t \to 0} k(t) = \langle \kappa \rangle - \langle \kappa^2 \rangle\, t \cong \langle \kappa \rangle\, e^{-\langle \kappa^2 \rangle t/\langle \kappa \rangle} \quad (19a)$$

Laplace transforming the last term in this relation, we find that

$$\lim_{z \to \infty} \frac{1}{z\hat{k}(z)} = \frac{1}{\langle \kappa \rangle} + \frac{\langle \kappa^2 \rangle}{z\langle \kappa \rangle^2} \quad (19b)$$

The long-time limit of $k(t)$ has the same structure[11] as eqs 1a and 3b. On the basis of these results, we construct the following approximation that reproduces the first two terms in the $z \to 0$ limit and the first two terms in the $z \to \infty$ (eq 19b) limit of the Laplace transform of the rate coefficient $\hat{k}(z)$

$$\frac{1}{z\hat{k}(z)} = \frac{1}{\langle \kappa \rangle} + \frac{1}{k_{DC}(\infty) + k_{DC}(\infty)^2 z^{1/2}/(4\pi D^{3/2}) + \frac{\langle \kappa^2 \rangle z}{\langle \kappa^2 \rangle}} \quad (20)$$

In the limit of the δ-function sink (i.e., reaction at contact), this approximation reduces to the Pedersen−Sibani approximation for $k_{CK}(t)$. Inverting eq 20, we find that the time-dependent rate coefficient for a diffusion-influenced reaction in the
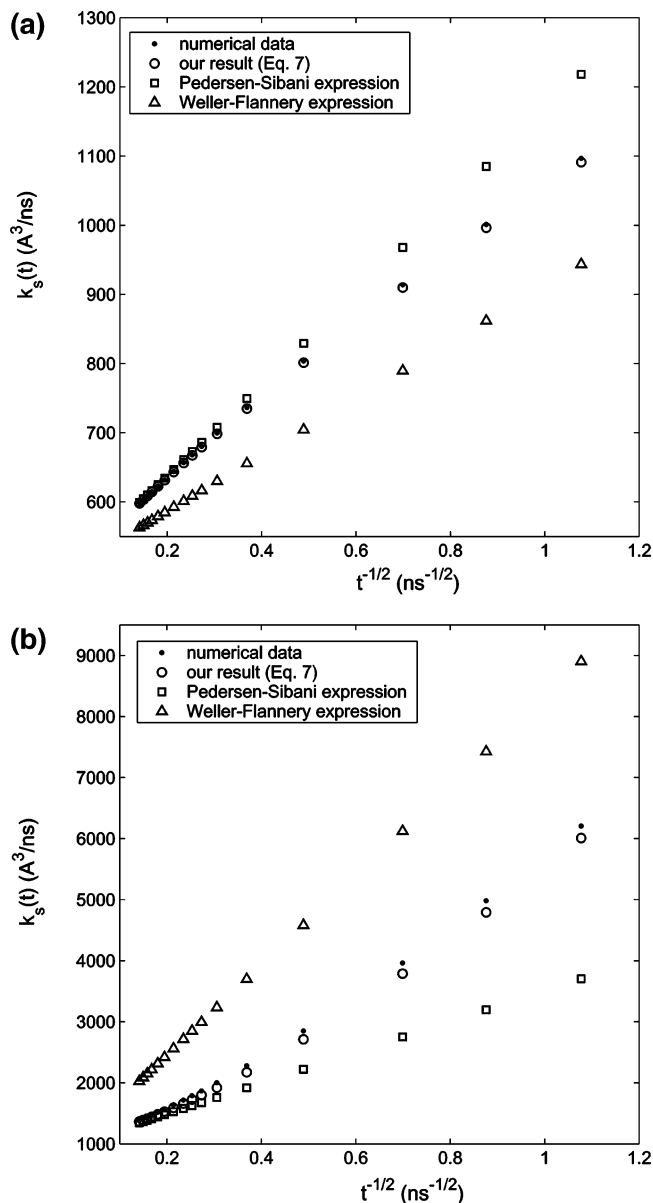
Figure 2. A more demanding test than that performed in Figure 1, with the increased strength and range of the potential. (a) Repulsive Debye−Hückel potential, $R_D = 25$ Å, $\kappa = 0.5$ Å^{-1}; (b) attractive Debye−Hückel potential, $R_D = -25$ Å, $\kappa = 0.1$ Å^{-1}; other parameter values are the same as those in Figure 1.

presence of a sink is given by eq 9 with parameters

$$\frac{1}{k_{CK}(\infty)} = \frac{1}{\langle \kappa \rangle} + \frac{1}{k_{DC}(\infty)} \quad (21a)$$

and

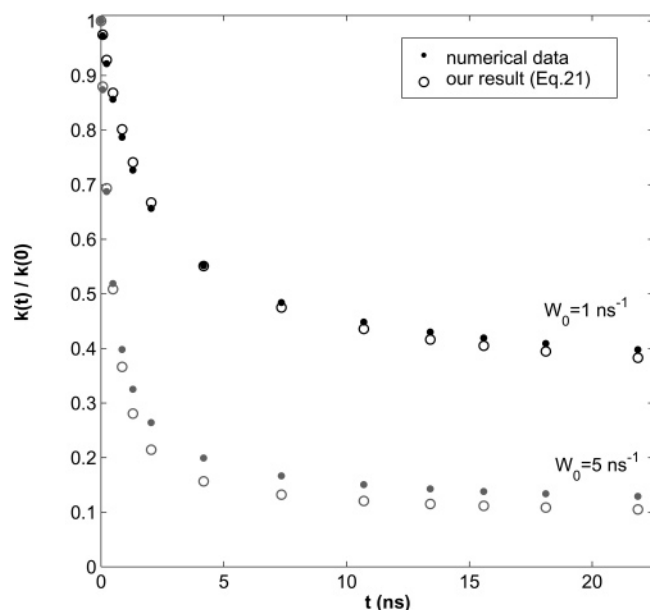$$\gamma_{1,2} = \frac{\mu \mp \lambda}{\langle \kappa \rangle^2/\langle \kappa^2 \rangle} \quad (21b)$$

$$\alpha_{1,2} = \frac{\langle \kappa \rangle}{k_{DC}(\infty)}\left(1 \pm \frac{\mu}{\lambda}\right) \quad (21c)$$

$$2\mu = k_{DC}(\infty)^2/(4\pi D^{3/2}) \quad (21d)$$

$$\lambda^2 = \mu^2 - [\langle \kappa \rangle + k_{DC}(\infty)]\langle \kappa \rangle^2/\langle \kappa^2 \rangle \quad (21e)$$

Comparison with Accurate Numerical Results

The accuracy of our analytical expressions was tested by comparison with accurate numerical results calculated with a

Figure 3. The time-dependent rate coefficient $k(t)$ for distance-dependent reactivity given by eq 9 with the parameters defined in eq 21 compared to numerical results for the attractive Debye−Hückel potential, $R_D = -9.1$ Å; other parameter values as in Figure 1 and for the exponential sink function $\kappa(r) = W_0\, e^{-2(r-a)/L}$, $L = 4$ Å, $W_0 = 1$ and 5 ns^{-1}.

Windows application for spherically-symmetric diffusion problems of Krissinel' and Agmon based on a sophisticated finite-difference approximation to the Smoluchowski equation.[12]

In Figure 1, we compare our eq 7 and the Pedersen−Sibani and Weller−Flannery expressions with the numerical results obtained for a Debye−Hückel potential, $\beta U(r) = (R_D/r)e^{-\kappa(r-a)}/(1 + \kappa a)$, $r \ge a$, and $\beta U(r) = \infty$, $r < a$, where R_D is the Debye length and κ is the inverse of the screening length, for parameters that describe experimental data.[13] We have focused on $k_S(t)$ rather than $k_{CK}(t)$, because the former provides a more stringent test. Our expression reproduces the numerical results to within 0.1% over the whole time range. But because for these parameters the previous approximations are not all that bad, we performed a much more demanding test by increasing the

strength and the range of the potential. It can be seen from Figure 2 that even for such rather extreme parameters our expression is still quite accurate.

In Figure 3, we test our eq 9 with the parameters given by eqs 21 for the rate coefficient $k(t)$ of a nonlocal reaction against numerical results obtained for the attractive Debye−Hückel potential and the exponential sink function $\kappa(r) = W_0\, e^{-2(r-a)/L}$ for two values of W_0. As the strength of the sink increases, the reaction becomes more diffusion-controlled, and the agreement deteriorates primarily because our expression for $k(\infty)$ becomes less accurate. However, it can be seen that, even when $k_{DC}(\infty)$ is about an order of magnitude greater than k_{RC}, our approximation is not too bad.

On the basis of these and other tests, it appears that our analytic expressions for $k_S(t)$ and $k_{CK}(t)$ given by eq 7 and eq 9 (as well as the corresponding Laplace transforms) should be sufficiently accurate for practical purposes. Thus, for contact reactions with spherically symmetric interaction potentials, experimental data can be now analyzed with the same ease as for free diffusion. For nonlocal reactions described by position-dependent sinks, the use of our approximate expression should provide reasonable estimates of the parameters that can then be refined using the numerical solution of the reaction−diffusion equation.[12]

References and Notes

(1) Gopich, I. V.; Szabo, A. *J. Chem. Phys.* **2002**, *117*, 507.
(2) Yang, M.; Lee, S.; Shin, K. J. *J. Chem. Phys.* **1998**, *108*, 117, 8557, 9069.
(3) Pedersen, J. B.; Sibani, P. *J. Chem. Phys.* **1981**, *75*, 5368.
(4) Weller, A. Z. *Phys. Chem.* **1957**, *13*, 335.
(5) Flannery, M. R. *Phys. Rev.* **1982**, *A25*, 3403.
(6) Smoluchowski, M. Z. *Phys. Chem.* **1917**, *92*, 129.
(7) Collins, F. C.; Kimball, G. E. *J. Colloid Sci.* **1949**, *4*, 425.
(8) Szabo, A. *J. Phys. Chem.* **1989**, *93*, 6929.
(9) Wilemski, G.; Fixman, M. *J. Chem. Phys.* **1973**, *58*, 4009.
(10) Doi, M. *Chem. Phys.* **1975**, *11*, 107, 115.
(11) Zhou, H.-X.; Szabo, A. *Biophys. J.* **1996**, *71*, 2440.
(12) Krissinel', E. B.; Agmon, N. *J. Comput. Chem.* **1996**, *17*, 1085.
(13) Cohen, B.; Huppert, D.; Agmon, N. *J. Phys. Chem. A* **2001**, *105*, 7165.