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A Horseshoe with Positive Measure

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Let $f\colon M \to M$ be a diffeomorphism satisfying Smale's Axiom A. For Ω a basic set of f (see [2]) one defines

$$W^s(\Omega) = \{x \in M : f^n(x) \to \Omega \text{ as } n \to +\infty\}.$$

If f is C^2, then $W^s(\Omega)$ has Lebesgue measure zero unless Ω is an attractor [1]. Here we show that the C^2 assumption is necessary by proving

Theorem. *There is a C^1 horseshoe with positive Lebesgue measure.*

First we adopt some notation for Cantor sets. Let I be a closed interval and $\alpha_n > 0$ numbers with $\sum_{n=0}^{\infty} \alpha_n \leqq \ell(I)$. Let $\underline{a} = a_1 a_2 \ldots a_n$ denote a sequence of 0's and 1's of length $n = n(\underline{a})$; we permit the empty sequence $\underline{a} = \emptyset$ with $n(\underline{a}) = 0$. Define $I_\phi = I = [a, b]$, $I_\phi^* = \left[\frac{a+b}{2} - \frac{\alpha_0}{2}, \frac{a+b}{2} + \frac{\alpha_0}{2}\right]$ and $I_{\underline{a}}^* \subset I_{\underline{a}}$ recursively as follows. Let $I_{\underline{a}0}$ and $I_{\underline{a}1}$ be the left and right intervals remaining when the interior of $I_{\underline{a}}^*$ is removed from $I_{\underline{a}}$; let $I_{\underline{a}k}^*$ $(k = 0, 1)$ be the closed interval of length $\alpha_{n(\underline{a}k)}/2^{n(\underline{a}k)}$ and having the same center as $I_{\underline{a}k}$. The Cantor set K_I is given as

$$K_I = \bigcap_{m=0}^{\infty} \bigcup_{n(\underline{a})=m} I_{\underline{a}}.$$

This is the standard construction of the Cantor set except that we allow ourselves some flexibility in the lengths of intervals removed. The measure of K_I is

$$m(K_I) = \ell(I) - \sum_{n=0}^{\infty} \alpha_n.$$

Suppose one is given another interval J and $\beta_n > 0$ with $\sum_{n=0}^{\infty} \beta_n \leqq \ell(J)$. One can then construct $J_{\underline{a}}, J_{\underline{a}}^*$ and K_J as above. Let us assume now that $\frac{\beta_n}{\alpha_n} \to \gamma \geqq 0$ as $n \to \infty$. Pick a sequence $\delta_n \to 0$ and for each $\underline{a}$ let $g\colon I_{\underline{a}}^* \to J_{\underline{a}}^*$ be a C^1 orientation preserving diffeomorphism so that

(i) $f'(x) = \gamma$ for x an endpoint of $I_{\underline{a}}^*$,

(ii) $f'(I_{\underline{a}}^*) \subset$ interval spanned by $\gamma \pm \delta_n$ and $\frac{\beta_n}{\alpha_n} \pm \delta_n$.

Then g extends from $\bigcup_{\underline{a}} I_{\underline{a}}^*$ by continuity to a homeomorphism $g\colon I \to J$; g is C^1 with derivative γ at each point of K_I.

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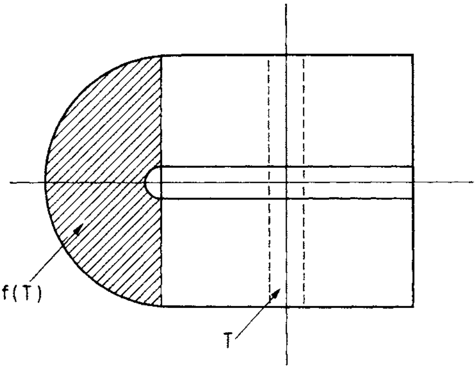

Fig. 1

We will now construct a horseshoe with positive measure. Choose $\beta_n > 0$ with

$$\sum_{n=0}^{\infty} \beta_n < 2 \quad \text{and} \quad \frac{\beta_{n+1}}{\beta_n} \to 1 \quad \left(\text{e.g. } \beta_n = \frac{1}{(n+100)^2}\right).$$

Let $J = [-1, 1]$,

$$I = \left[\frac{\beta_0}{2}, 1\right] \quad \text{and} \quad \alpha_n = \frac{\beta_{n+1}}{2}.$$

Then

$$\sum_{n=0}^{\infty} \alpha_n < \ell(I) \quad \text{and} \quad \gamma = \lim \frac{\beta_n}{\alpha_n} = \lim \frac{2\beta_n}{\beta_{n+1}} = 2.$$

So one gets a C^1 diffeomorphism $g: I \to J$ as above. One defines a diffeomorphism f of the square $S = J \times J$ into $\mathbf{R}^2$ by

(i) $\quad f(x, y) = (g(x), g^{-1}(y)) \quad$ for $(x, y) \in I \times J$,

(ii) $\quad f(x, y) = (g(-x), -g^{-1}(y)) \quad$ for $(x, y) \in (-I) \times J$ and

$$f(T) \cap (J \times J) = \emptyset \quad \text{where } T = \left(-\frac{\beta_0}{2}, \frac{\beta_0}{2}\right) \times J.$$

The mapping f can be extended to the sphere exactly as in Smale [2], pp. 770–773. Then $\Omega = \bigcap_{n=-\infty}^{+\infty} f^n(S) = K_J \times K_J$ has Lebesgue measure $m(\Omega) = m(K_J)^2 = (2 - \sum_{n=0}^{\infty} \beta_n)^2 > 0$. We have just duplicated Smale's example of a horseshoe, using a specific map $f: J \times J \to \mathbf{R}^2$.

References

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