*Weissenberg* centripetal pump effect. Both finding are, however, surprising in view of the high level of drag reduction achieved with this soap solution, e. g., compare the results in figs. 2 and 4, and its markedly Spinnbarkeit properties. We suspect that a similar, but more subtle, disentanglement mechanism is also operative under these high shear rate conditions so that the localized network structure giving rise to viscoelastic behavior becomes disrupted. Otherwise, the behavior which gives rise to drag reduction in this type of soap solution must be fundamentally different from the behavior one usually correlates with drag reduction and normal stress behavior in polymeric solutions.

*Summary*

An unusual stress controlled turbulent flow phenomenon observed in aqueous solutions of association soap-type colloids is described. In polymer solutions which are not shear degrading, a more or less constant level of drag reducing activity is reached at progressively higher flow rates. In contrast, drag reduction activity increases in a soap solution with increasing flow rate until a critical shear stress $(\tau_w)_c$ is attained. For $\tau_w > (\tau_w)_c$, the activity decreases steadily until the turbulent flow behaviour of the soap solution becomes indistinguishable from that of the soap-free electrolyte solution. Maximum drag reduction activity is again obtained when the pumping conditions are reduced so that the condition $\tau_w = (\tau_w)_c$ is again attained.

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# Flow of Elastico-Viscous Liquids in Channels under the Influence of a Periodic Pressure Gradient Part II

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*With 2 figures in 6 details*

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The flow of an incompressible elastico-viscous liquid in a straight channel of rectangular section under the influence of a periodic (axial) pressure gradient is investigated. For large values of the frequency of the forcing agent, an approximate solution is obtained which determines (in terms of tabulated functions) the variation in the mean-square velocity over that part of the section of the channel which is remote from the corners. The results show that, while the general nature of the motion is similar to that of a purely viscous *Newton*ian liquid, in that a high peak of average velocity occurs near the channel wall, the change of shape of the mean-square velocity profiles with the parameters measuring the elasticity of the liquid could be sufficiently marked to permit comparision with experimental curves.

## 1. Introduction

In the first paper of the same main title (*Jones* and *Walters* 1966) consideration was given to the flow of an incompressible elastico-viscous liquid in a channel of circular cross-section under the influence of a periodic pressure gradient. For large values of the frequency of the forcing agent, it was shown that the main effects of elasticity of the type considered was to increase the mean-square velocity in the region close to the channel wall, while having no great effect on its magnitude over the rest of the region of flow, and also to decrease the thickness of the boundary region in which the velocity changes rapidly. The liquids considered in the investigation were the two simple elastico-viscous prototypes designated as $A'$ and $B'$ by *Walters* (1962).

The equation of state of liquid $A'$ can be written in the form[1])

$$P_{ik} = P'_{ik} - P'' g_{ik}, \qquad [1]$$

$$P'_{ik}(x, t) = 2 \int_{-\infty}^{t} \Psi(t - t') \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^r}{\partial x^k} E^{(1)}_{mr}(x', t')\, dt', \qquad [2]$$

where $P_{ik}$ is the stress tensor, $P''$ an arbitrary isotropic pressure, $g_{ik}$ the metric tensor of a fixed coordinate system $x^i$, $E^{(1)}_{ik}$ the rate-of-strain tensor, and

$$\Psi(t - t') = \int_0^{\infty} \frac{N(\tau)}{\tau} e^{-(t-t')/\tau} d\tau. \qquad [3]$$

In these equations, $N(\tau)$ is the distribution function of relaxation times $\tau$ (*Walters* 1960) and $x'^i$ $(= x'^i(x, t, t'))$ is the position at time $t'$ of the element that is instantaneously at the point $x^i$ at time $t$. The equations of state of liquid $B'$ are given by [1] and

$$P'^{ik} = 2 \int_{-\infty}^{t} \Psi(t - t') \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^r} E^{(1)mr}(x', t')\, dt'. \qquad [4]$$

In the present paper, we shall consider the theoretical aspects of the flow of liquids $A'$ and $B'$ through a straight channel of rectangular cross-section under the influence of a periodic pressure gradient. The work was suggested by *Drake*'s recent theoretical treatment of the associated viscous flow problem (*Drake* 1965). Now the flow features associated with a periodic pressure gradient seem to have been investigated experimentally first by *Richardson* and *Tyler* (1929). They found consistently that, in the case of channels of square section and for sufficiently high frequencies, a high peak of average velocity occurred near the boundary wall. Here again, as in Part 1, we shall, in the main, be concerned with the problem of the detection of, and the effect of elasticity on, this type of boundary-layer behaviour. There does not appear to be anywhere in the literature an account of experiments similar to those of *Richardson* and *Tyler* for non-*Newton*ian liquids to which the predictions contained here might be applied. Since such a flow could be readily attained and controlled in practice, and as the results show that the prototypes considered could exhibit marked non-*Newton*ian behaviour, it is greatly to be hoped that such experimental work may soon be undertaken.

## 2. Mathematical formulation

A number of papers have appeared in the literature in which the problem of the *steady* rectilinear flow of non-*Newton*ian liquids in channels of arbitrary sections have been discussed (*Ericksen* 1956; *Green* and *Rivlin* 1956; *Stone* 1957; *Oldroyd* 1958, 1965; *Giesekus* 1961; *Walters* 1962). In each case the associated stress distribution has been determined and a general condition found for that type of simple flow to be possible; if the condition is not satisfied some secondary flow transverse to the assumed primary streamlines is inevitable (*Green* and *Rivlin* 1956; *Langlois* and *Rivlin* 1963; *Jones* 1964; *Camilleri* and *Jones* 1965, 1966). Liquids of type $A'$ and $B'$ have been shown to be capable of steady rectilinear flow in all circumstances (*Walters* 1962). A similar exact analysis does not seem to have been carried out for "semi-steady" oscillatory flows [although *Frater* (1966) has considered some aspects of this type of flow], and is one that is not attempted here except for the simple prototypes $A'$ and $B'$.

Referred to suitably chosen *Cartes*ian coordinates $x, y, z$, the wall of the channel may be represented by the equation $f(x, y) = 0$. The velocity components are taken to be

[1]) Covariant suffixes are written below, and contravariant suffixes above; the summation convention is for repeated suffixes, with the exception of the coordinate labels $x, y, z$.

of the form 0,0, $W(x, y, t)$ (which satisfies the equation of continuity without any restriction on the function $W(x, y, t)$]; it is, of course, to be expected that such a motion is not always possible for non-*Newton*ian liquids. The only non-vanishing rate-of-strain components are

$$E_{zx}^{(1)} = E_{xz}^{(1)} = \frac{1}{2}\frac{\partial W}{\partial x}, \quad E_{yz}^{(1)} = E_{zy}^{(1)} = \frac{1}{2}\frac{\partial W}{\partial y}. \qquad [5]$$

The displacement functions $x', y', z'$ can, in this simple flow, be written down immediately in the form

$$x' = x,\ y' = y,\ z' = z - \int_{t'}^{t} W(x, y, t'')\, dt''. \qquad [6]$$

For prototype $A'$, from eqs. [2] and [6], we obtain

$$P'_{xx} = -2\int_{-\infty}^{t} \Psi(t-t')\frac{\partial W(x,y,t')}{\partial x}\int_{t'}^{t}\frac{\partial W(x,y,t'')}{\partial x} \times dt'\,dt'',$$

$$P'_{yy} = -2\int_{-\infty}^{t} \Psi(t-t')\frac{\partial W(x,y,t')}{\partial y}\int_{t'}^{t}\frac{\partial W(x,y,t'')}{\partial y} \times dt'\,dt'', \quad P'_{zz} = 0,$$

$$P'_{xy} = -\int_{-\infty}^{t} \Psi(t-t')\left[\frac{\partial W(x,y,t')}{\partial x}\frac{\partial}{\partial y} + \frac{\partial W(x,y,t')}{\partial y}\frac{\partial}{\partial x}\right]\int_{t'}^{t} W(x,y,t'')\,dt'\,dt'',$$

$$P'_{yz} = \int_{-\infty}^{t} \Psi(t-t')\frac{\partial W(x,y,t')}{\partial y}\,dt',$$

$$P'_{zx} = \int_{-\infty}^{t} \Psi(t-t')\frac{\partial W(x,y,t')}{\partial x}\,dt'. \qquad [7]$$

For prototype $B'$, eqs. [4] and [6] give

$$P'_{xx} = P'_{yy} = P'_{xy} = 0,$$

$$P'_{zz} = 2\int_{-\infty}^{t} \Psi(t-t')\left[\frac{\partial W(x,y,t')}{\partial x}\frac{\partial}{\partial x} + \frac{\partial W(x,y,t')}{\partial y}\frac{\partial}{\partial y}\right]\int_{t'}^{t} W(x,y,t'')\,dt'\,dt'', \qquad [8]$$

$P'_{yz}$ and $P'_{zx}$ being as defined in [7].

The stress equations of motion require, in the absence of body forces,

$$P'' = P'(x, y, t) - zP(t), \qquad [9]$$

and so

$$\frac{\partial P_{xx}}{\partial z} = \frac{\partial P_{yy}}{\partial z} = \frac{\partial P_{zz}}{\partial z} = -\frac{\partial P''}{\partial z} = P(t), \qquad [10]$$

$$\frac{\partial P'_{xx}}{\partial x} + \frac{\partial P'_{xy}}{\partial y} = \frac{\partial P'}{\partial x}, \quad \frac{\partial P'_{xy}}{\partial x} + \frac{\partial P'_{yy}}{\partial y} = \frac{\partial P'}{\partial y},$$

$$\frac{\partial P'_{zx}}{\partial x} + \frac{\partial P'_{yz}}{\partial y} + P(t) = \varrho\frac{\partial W}{\partial t}, \qquad [11]$$

where $P$ – the longitudinal gradient of pressure excess above the hydrostatic – is a function of $t$ only, independent of position, and $\varrho$ is the density of the liquid. Using the results [7], the third equation of motion [11] may be written as

$$\int_{-\infty}^{t} \Psi(t-t')\,\nabla^2 W(x,y,t')\,dt' + P(t) = \varrho\frac{\partial W(x,y,t)}{\partial t}, \qquad [12]$$

where $\nabla^2$ is the (two-dimensional) *Laplac*ian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$; and this equation of motion suffices to determine the velocity distribution $W(x, y, t)$ under prescribed initial and boundary conditions. The first two equations of motion [11] require, in addition (on eliminating $P'$ between them) that $\Delta = 0$, where

$$\Delta \equiv \frac{\partial^2}{\partial x\,\partial y}(P'_{xx} - P'_{yy}) - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)P'_{xy}. \qquad [13]$$

Thus only when the condition $\Delta = 0$ is automatically satisfied will axial flow (steady or otherwise) be generally possible without the aid of artificial lateral body forces; if $\Delta$ is non-zero the occurrence of some secondary flow normal to the $z$ direction is inevitable.

It is easily seen that the condition $\Delta = 0$ is automatically satisfied in the case of liquid $B'$ (a liquid of the *Roberts* (1953) type, with a normal stress difference equivalent to an extra simple tension along the streamlines in steady simple shearing); liquid $B'$ is therefore, in a *privileged* position so far as axial flow is concerned.
Whether or not this is true of liquid $A'$ will require rather closer study[1]).

For liquid $A'$, on substituting the results [7] into [13], the expression for $\Delta$ takes the

[1]) It has been shown that, as far as available experimental results are concerned, liquid $A'$ is not a suitable prototype to explain the more common types of non-*Newton*ian behaviour (*Walters* 1962).

form

$$\Delta = \int_{-\infty}^{t} \Psi(t-t') \left\{ \left[ \frac{\partial \nabla^2 W(x,y,t')}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \nabla^2 W(x,y,t')}{\partial y} \frac{\partial}{\partial x} \right] + \left[ \frac{\partial W(x,y,t')}{\partial y} \frac{\partial \nabla^2}{\partial x} - \frac{\partial W(x,y,t')}{\partial x} \frac{\partial \nabla^2}{\partial y} \right] \right\} \times \int_{t'}^{t} W(x,y,t'')\, dt'\, dt''. \quad [14]$$

Now, for the simple type of motion investigated here the pressure gradient is periodic with frequency $n/(2\pi)$, so that we can write

$$P(t) = Rl\,[p\, e^{\mathrm{int}}], \quad [15]$$

where $p$ is a real constant. Under this forced regime, and from the fact that, if $Z_1$ and $Z_2$ are any two complex numbers,

$$(Rl\, Z_1)\,(Rl\, Z_2) = \frac{1}{2} Rl\,(Z_1 Z_2) + \frac{1}{2} Rl\,(Z_1 Z_2^*), \quad [16]$$

$Z^*$ denoting the complex conjugate of $Z$, we see that equations [7] and [11] admit a solution of the form[1])

$$W = Rl\,[w(x,y)\, e^{\mathrm{int}}],\ P'_{yz} = Rl\,[p'_{yz}(x,y)\, e^{\mathrm{int}}],$$
$$P'_{zx} = Rl\,[p'_{zx}(x,y)\, e^{\mathrm{int}}],$$
$$P'_{\alpha\beta} = Rl\,[p'^{(0)}_{\alpha\beta}(x,y) + P'^{(2)}_{\alpha\beta}(x,y)\, e^{2\,\mathrm{int}}], \quad [17]$$

where $p'_{yz}(x,y)$, $p'_{zx}(x,y)$, $p'^{(0)}_{\alpha\beta}(x,y)$, $p'^{(2)}_{\alpha\beta}(x,y)$ incorporate appropriate phase factors and Greek suffices take the values $x$, $y$ only. Eq. [12] reduces, in this case, to

$$\nabla^2 w \int_0^\infty \Psi(\xi)\, e^{-\mathrm{in}\,\xi} d\xi - \mathrm{in}\,\varrho\, w = -p, \quad [18]$$

and is to be solved subject to the boundary condition

$$w = 0 \quad \text{on} \quad f(x,y) = 0. \quad [19]$$

On substituting [17] into [14], and using the result [16], we see that $\Delta$ consists of a time independent term together with a term proportional to $\exp(2\,i\,n\,t)$; in a usual notation

$$\Delta = Rl \left[ \frac{1}{n} \frac{\partial(\nabla^2 w, w^*)}{\partial(x,y)} \int_0^\infty \Psi(\xi) \sin n\xi\, d\xi + \frac{2}{n} \frac{\partial(\nabla^2 w, w^*)}{\partial(x,y)} \int_0^\infty e^{-\frac{3}{2}\mathrm{in}\,\xi}\, \Psi(\xi) \sin \frac{1}{2} n\xi\, d\xi\, e^{2\,\mathrm{int}} \right]. \quad [20]$$

Again, taking into account eq. [18] and observing that $Rl(Z_1 Z_2^*) = Rl(Z_1^* Z_2)$ we can reduce the expression for $\Delta$ to the simpler form

$$\Delta = \frac{i\,\varrho\, AB}{A^2 + B^2} \frac{\partial(w, w^*)}{\partial(x,y)}, \quad [21]$$

where

$$A = \int_0^\infty \frac{N(\tau)\, d\tau}{1 + n^2 \tau^2},\quad B = n \int_0^\infty \frac{\tau\, N(\tau)\, d\tau}{1 + n^2 \tau^2}. \quad [22]$$

It is observed that the condition $\Delta = 0$ is satisfied automatically (i. e. without any restriction on the velocity field) if

(i) $w$ (and therefore $w^*$) is a function of $x$ (or $y$) only – in which case the contours $w$ = constant in the $x, y$ plane are straight lines corresponding to simple periodic shearing flow; or

(ii) the function $w$ is a function of $(x^2 + y^2)$ – in which case the contours $w$ = constant are concentric circles corresponding to flows in a channel of circular cross-section (*Jones* and *Walters* 1966); or

(iii) $n = 0$ (in which case $B = 0$) – corresponding to steady flow under a uniform pressure gradient (*Walters* 1962); or

(iv) $N(\tau) = \eta_0\, \delta(\tau)$ (in which case, again, $B = 0$) – corresponding to the flow of a *Newton*ian viscous liquid.

In most other cases the condition $\Delta = 0$ is *not* automatically satisfied, and so axial flows in channels are in general impossible. Liquid $A'$ is *not*, therefore, in a privileged position (as is $B'$) so far as "semi-steady" oscillatory flows are concerned and, in common with other types of elastico-viscous liquids (*Frater* 1966), can, in general, be expected to develop a secondary flow normal to the $z$-direction.[1])

The following theory is an exact treatment for prototype $B'$ and will be approximately true for prototype $A'$, the departure from axial flow which must arise in the case of liquid $A'$ being made negligibly small by sufficiently reducing the amplitude of oscillation.

From here onwards we confine attention to a channel of rectangular cross-section (of sides 2a, 2b) so that the boundary condition

[1]) We notice that a purely periodic motion has associated with it stresses with steady components as well as periodic components with twice the frequency of the velocity field.

[1]) In the case of a periodic forcing agent of frequency $n/(2\pi)$, it is observed that for non-linear materials the equations of state would not, in general, admit a simple solution in which $W$, $P'_{zx}$, $P'_{yz}$ vary sinusoidally with frequency $n/(2\pi)$, because the equations are not usually linear in these quantities. It is not, therefore, surprising that axial "semi-steady" flows are not always possible.

[19] reduces to

$$w = 0 \quad \text{on} \quad x = 0, 2a\ (0 \leqslant y \leqslant 2b)\,, \qquad w = 0 \quad \text{on} \quad y = 0, 2b\ (0 \leqslant x \leqslant 2a)\,. \qquad [23]$$

It is convenient in the study of eqs. [18] and [23] to introduce the finite *Fourier* sine transform of $w(x, y)$, defined by

$$w_F(x, m) = \int_0^{2b} w(x, y) \sin\left(\frac{m \pi y}{2b}\right) dy\,. \qquad [24]$$

Proceeding in the usual manner, the *Fourier* transform of eq. [18] is

$$\frac{\partial^2 w_F}{\partial x^2} - \left(\frac{m^2 \pi^2}{4 b^2} - k^2\right) w_F = \frac{2 b p k^2}{i \varrho n m \pi} [1 - (-1)^m]\,, \qquad [25]$$

where

$$k^2 = -\,\mathrm{i}n\,\varrho \Big/ \int_0^\infty \Psi(\xi)\, e^{-\mathrm{i}n\xi} d\xi$$

$$= -\,\mathrm{i}n\varrho \Big/ \int_0^\infty [N(\tau)\, d\tau / (1 + i n \tau)]\,, \qquad [26]$$

and is to be solved subject to the boundary conditions

$$w_F(x, m) = 0 \quad \text{on} \quad x = 0,\ 2a\,. \qquad [27]$$

The appropriate solution is

$$w_F(x, m) = \frac{2 p b^3 k^2}{i \varrho n \pi m p_m^2} \left[1 - \frac{\cosh (p_m (x - a)/b)}{\cosh (p_m a/b)}\right] \times [(-1)^m - 1]\,, \qquad [28]$$

where

$$p_m^2 = \frac{1}{4} m^2 \pi^2 - k^2 b^2\,. \qquad [29]$$

An expression may be obtained for $w(x, y)$ by using the inversion formula

$$w(x, y) = \frac{1}{b} \sum_{m=1}^{\infty} w_F(x, m) \sin\left(\frac{m \pi y}{2b}\right). \qquad [30]$$

On combining [28] and [30] ,we obtain

$$w(x, y) = \frac{-4 p k^2 b^2}{i \varrho n \pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m + 1)\, p_{2m+1}^2} \times \left\{1 - \frac{\cosh [p_{2m+1}(x - a)/b]}{\cosh [p_{2m+1} a/b]}\right] \sin \frac{(2m + 1) \pi y}{2b}\,. \qquad [31]$$

As eq. [31] stands, it is far too complicated for any general deductions to be made about the behaviour of the flow. It is, however, possible to simplify this equation considerably if we make some assumptions about the frequency of the forcing agent $P(t)$. For sufficiently small values of $n$ (i. e. of $/\ k\ /$) we obtain the approximate formula

$$W = \frac{16 p b^2}{\pi^3 \eta_0} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m + 1)^3} \times \left\{1 - \frac{\cosh\left[\frac{1}{2}(2m + 1)(x - a)\pi/b\right]}{\cosh\left[\frac{1}{2}(2m + 1)\pi a/b\right]}\right\} \times \sin \frac{(2m + 1)\pi y}{2b} \cos nt\,, \qquad [32]$$

where $\eta_0$ $(= \int_0^\infty N(\tau)\, d\tau)$ can be regarded as the limiting viscosity at small rates of shear. Thus, for a slowly varying pressure gradient, the flow is identical with that of a *Newton*ian viscous liquid (of viscosity $\eta_0$), the flow at each instant approaching what would be produced by a steady pressure gradient having the same instantaneous value.

**4. Flow at high frequencies**

To obtain an approximate form for the function $w(x, y)$ of eq. [31] for large values of the frequency, we can follow closely the treatment given by *Drake* (1965) in his discussion of the associated problem in viscous flow theory. The method of approach is to use two integral representations, which enables the infinite-series expression on the right-hand side of [31] to be summed, and which leads to an expression for $w(x, y)$ involving single integrals only.

It is first convenient to transfer the origin of coordinates to the centre of the channel, so that the channel walls are represented by $x = \pm a$, $y = \pm b$; and henceforth $x$, $y$ will refer to this new origin. The expession for $w\ (x, y)$ now takes the form

$$w = -\frac{4 p k^2 b^2}{i \varrho n \pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m + 1)\, p_{2m+1}^2} \times \left[1 - \frac{\cosh (p_{2m+1} x/b)}{\cosh (p_{2m+1} a/b)}\right] \cos \frac{(2m + 1)\pi y}{2b}\,, \qquad [33]$$

and, from symmetry, only the region $x, y \geqslant 0$ need be considered. Comparison of eqs. [26], [29], [33] with eqs. [4], [5] of *Drake* (1965) shows that the steps of *Drake*'s analysis may be followed closely in the elastico-viscous case also. Hence only the essential steps will be given here, and the reader is referred to *Drake* (1965) for fuller explanations.

First we confine attention to regions in which $|kx|$ is large so that, on neglecting exponentially small terms, eq. [33] gives the following asymptotic form for $w(x, y)$:

$$w \sim -\frac{4\,p\,k^2}{i\,\varrho\,n\,\pi}\int_0^{a-x} I(\alpha)\,d\alpha,$$

$$I(\alpha) = \sum_{m=0}^{\infty} \frac{(-1)^m\,b}{(2\,m+1)\,p_{2m+1}}\,e^{-\alpha p_{2m+1}/b} \times \cos\left(\frac{1}{2}(2\,m+1)\frac{\pi\,y}{b}\right). \quad [34]$$

Next, we take $|k\,b|$ as large, and again, on neglecting exponentially small terms, it can be shown (*Drake* 1965) that

$$I \sim \frac{1}{2\,i\,k}\int_0^{ik(b-y)} K_0\,[(B^2-\alpha^2 k^2)^{1/2}]\,dB\,, \quad [35]$$

where $K_0$ is the modified *Bessel* function of the second kind of order zero. The results [34] and [35] lead to the final symmetrical expression

$$w \sim \frac{2\,p}{i\,\varrho\,n\,\pi}\int_0^{ik(a-x)} dA\int_0^{ik(b-y)} K_0\,[(A^2+B^2)^{1/2}]\,dB\,. \quad [36]$$

In the derivation of [36] both $|k\,x|$ and $|k\,b|$ have been assumed to be large Again, [36] is, by using its symmetry properties, also valid if $|k\,y|$ and $|k\,a|$ are both large. It follows, therefore, that for fixed large values of both $|k\,a|$ and $|k\,b|$ expression [36] together with [17] determines the velocity at all points in the channel except in a small region near the centre (where $x, y = 0$). However, at the centre, and for large values of both $|k\,a|$ and $|k\,b|$, it is easily shown from eqs. [17] and [33] that

$$W \sim \frac{p}{\varrho\,n}\sin n\,t, \quad [37]$$

and so $W$ is a phase lag $^1/_2\,\pi$ behind the pressure gradient; this result is independent of the shape of the channel wall and also of the rheological properties of the liquid and so would be true for all (linear) elastico-viscous liquids.

Expression [36] is still, except in certain special cases, inconvenient for computational purposes. In a region of flow where $|k(b-y)|$ is large (i. e. in a region remote from the corner) the function $w(x, y)$ can be expressed in terms of tabulated functions (*Drake* 1965) to give

$$w \sim \frac{p}{i\,\varrho\,n}\left\{1 - e^{-ik(a-x)} - e^{-ik(b-y)} \times \operatorname{erf}\left[\left(\frac{i\,k}{2}\right)^{1/2}\frac{(a-x)}{(b-y)^{1/2}}\right]\right\}. \quad [38]$$

Hence, eqs. [17] and [33] give, when both $|k\,x|$ and $|k\,(b-y)|$ are large

$$W \sim Rl\left\{\frac{p}{i\,\varrho\,n}\left[1 - e^{-ik(a-x)} - e^{-ik(b-y)} \times \operatorname{erf}\left[\left(\frac{i\,k}{2}\right)^{1/2}\frac{(a-x)}{(b-y)^{1/2}}\right]\right]e^{int}\right\}. \quad [39]$$

Similarly, for large values of both $|k(a-x)|$ and $|k\,y|$ the velocity $W$ is given by [39] under the transformation $x \leftrightarrow y$, $a \leftrightarrow b$.

For large values of both $|k\,x|$ – $|k(b-y)|$, the mean square velocity $\overline{W^2}$ $(= {}^1/_2\,[(R\,l\,W)^2 + (Im\,W)^2])$ is [in keeping with the notation of *Drake* (1965)] given, from [39], by

$$\overline{W^2} = \frac{p^2}{2\,\varrho^2\,n^2}\{[1 - e^{-\beta X}\cos(\alpha\,X) - e^{-\beta Y}\Phi\cos(\alpha Y) - e^{-\beta Y}\,\Psi\sin(\alpha Y)]^2 + [e^{-\beta X}\sin(\alpha X) + e^{-\beta Y}\Phi\sin(\alpha\,Y) - e^{-\beta Y}\,\Psi\cos(\alpha Y)]^2\,, \quad [40]$$

where

$$X = \left(\frac{n}{2\,\nu_0}\right)^{1/2}(a-x),\; Y = \left(\frac{n}{2\,\nu_0}\right)^{1/2}(b-y)\,,$$

$$K = \alpha - i\,\beta = \left(\frac{n}{2\,\nu_0}\right)^{-1/2} k,$$

$$\nu_0 = \frac{1}{\varrho}\int_0^{\infty} N(\tau)\,d\tau = \frac{\eta_0}{\varrho}, \quad [41]$$

and $\Phi\,(X, Y, K)$, $\Psi\,(X, Y, K)$ are real variables defined by

$$\Phi + i\,\Psi = \operatorname{erf}\,[(\beta + i\,\alpha)^{1/2}\,X/(2\,Y)^{1/2}]\,. \quad [42]$$

In the case of a *Newton*ian liquid of viscosity $\eta_0$, i. e. when $N(\tau) = \eta_0\,\delta(\tau)$ so that $k^2 = -\mathrm{i}\,n\,\varrho/\eta_0$ (in which case $\alpha = \beta = 1$), the above expression for $\overline{W^2}$ reduces to that given by *Drake* (1965).

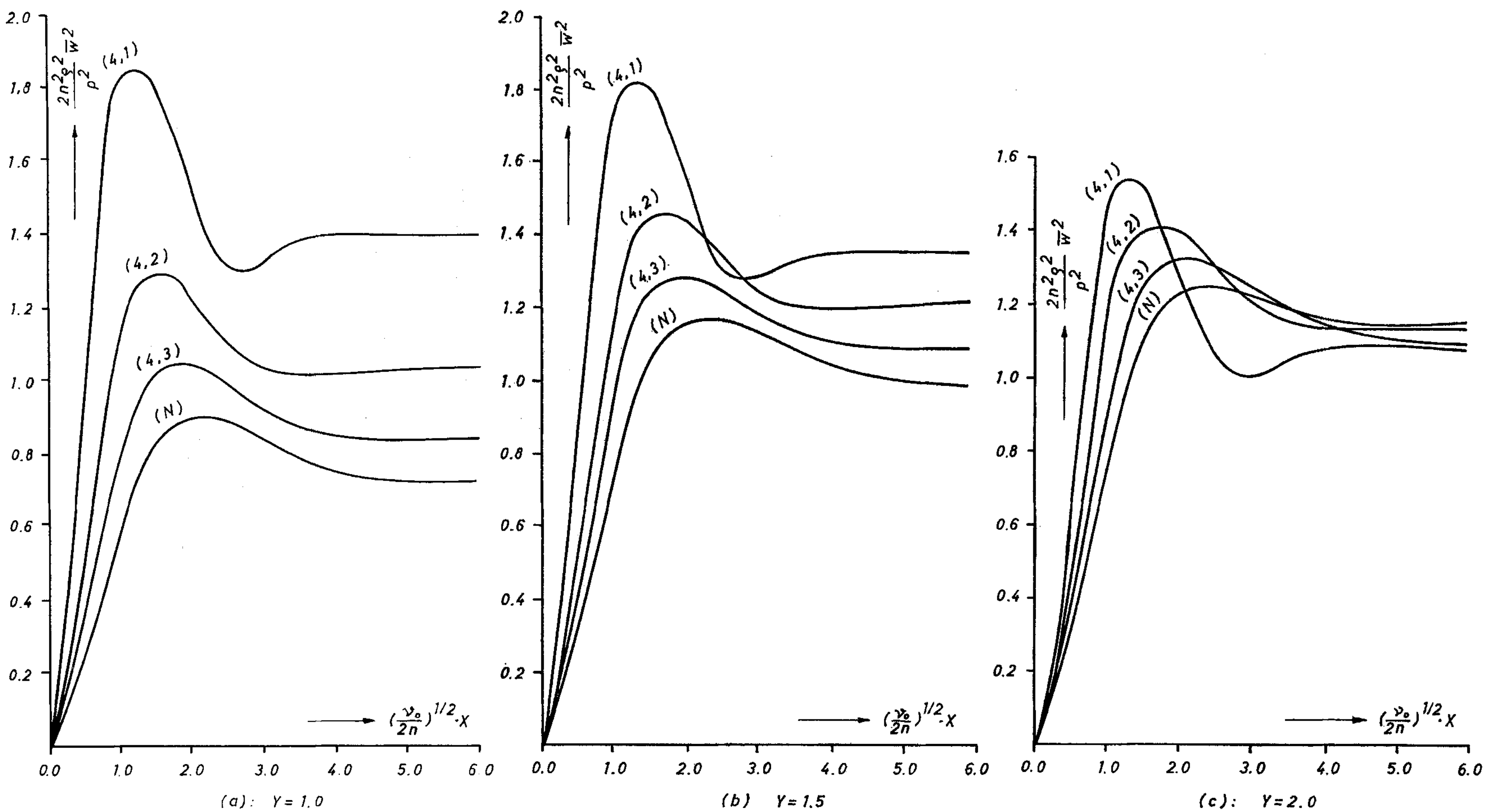
**4. Numerical calculations**

In order to make specific theoretical predictions, we consider the particular spectra $(N(\tau), \tau)$ introduced in the discussions of Part I of the present series (*Jones* and *Walters* 1966). In order to avoid making extensive calculations it is decided to focus attention on the two spectra which have so far proved most useful in characterizing real elastico-viscous solutions. The first is the discrete spectrum (I) of *Oldroyd* (1951), described by three constants $(\eta_0, \lambda_1, \lambda_2)$ and represented by

$$N(\tau) = \eta_0\frac{\lambda_1}{\lambda_2}\,\delta(\tau) + \eta_0\frac{\lambda_1-\lambda_2}{\lambda_1}\,\delta(\tau-\lambda_1);$$

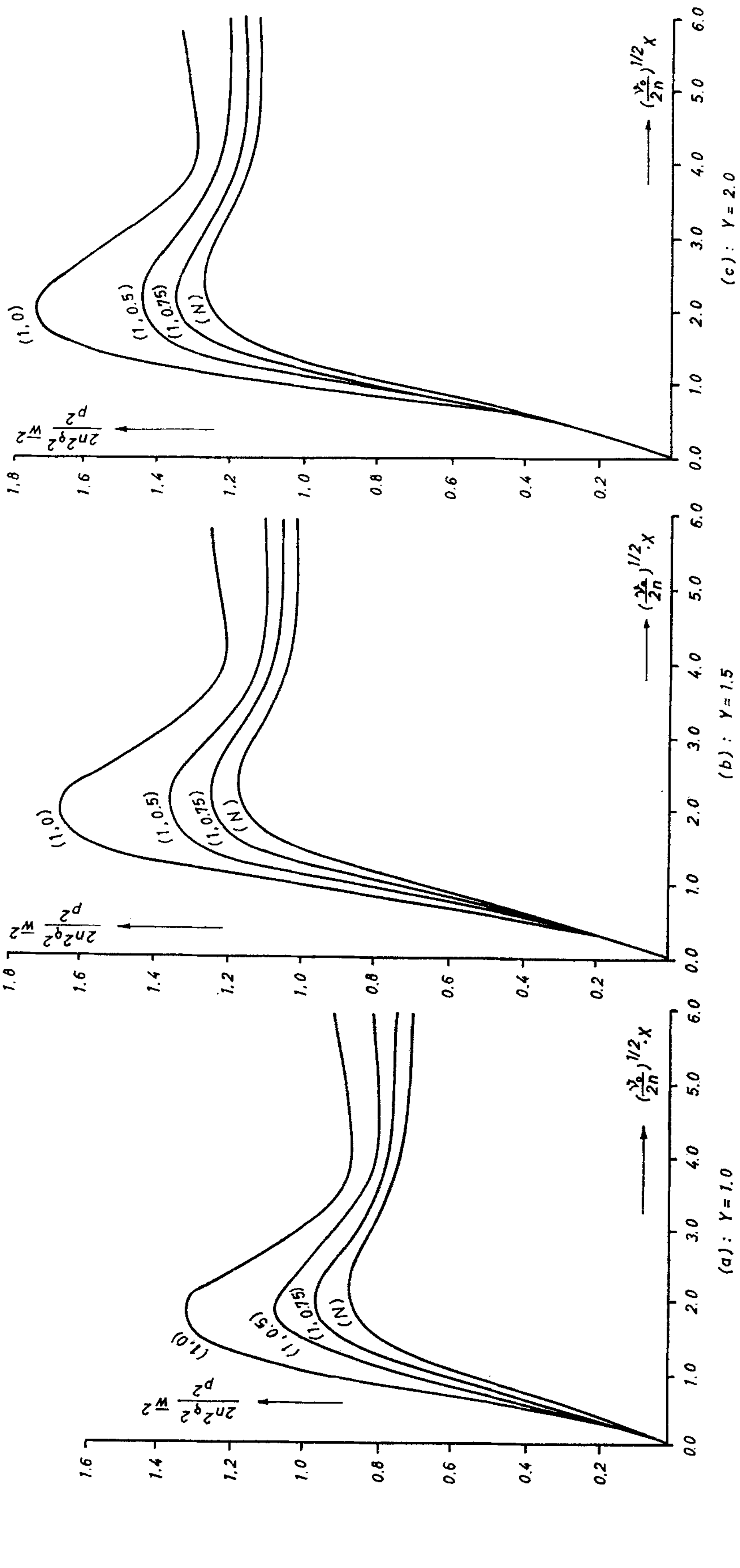
and for this prototype

$$K^2 = -2\,i\,(1 + i\,n\,\lambda_1)/(1 + i\,n\,\lambda_2)\,.$$




1a 1b 1c

Fig. 1a, b, c. Curves showing the variation of meansquare velocity profiles with the parameters ($n\,\lambda_1$, $n\,\lambda_2$) of spectrum (I); the profiles ($N$) correspond to those in the case of a *Newton*ian liquid




2a 2b 2c

Fig. 2a, b, c. Curves showing the variation of meansquare velocity profiles with the parameters ($n$ $\beta$, $\sigma$) of spectrum (IV); the profiles ($N$) correspond to those in the case of a *Newtonian* liquid

The second is the continuous spectrum (IV) of *Walters* (1960), again described by three constants ($\eta_0$, $\sigma$, $\beta$), and represented by

$$\left.\begin{aligned} N(\tau) &= \sigma\, \eta_0\, \delta(\tau) + \frac{1}{\beta}(1-\sigma)\,\eta_0\,(0 \leqslant \tau \leqslant \beta)\,, \\ N(\tau) &= 0 \qquad\qquad\qquad\qquad (\tau > \beta)\,; \end{aligned}\right\}$$

and for this prototype

$$K^2 = 2\,n\,\beta/[i\,n\,\beta\,\sigma + (1-\sigma)\log(1 + i\,n\,\beta)]\,.$$

Now although values of the error function for complex values of its arguments are available from a set of tables, their use will require an amount of work which may not be acceptable in practice if a large number of ($\overline{W}^2$, $X$, $Y$) curves have to be computed. For the purpose of numerical calculation, it is more convenient to compute the function $\overline{W}^2$ directly with the aid of a digital computer; the calculation in the present paper were carried out on an IBM 1620 electronic computer. However, it appears that there are some discrepancies between our numerical solutions in the special case in which $N(\tau) = \eta_0\,\delta(\tau)$ (given schematically by the label $N$ in figs. 1 and 2) and those given by *Drake* (1965); these discrepancies are small so that *Drake*'s findings are in the main not in question.

Figs. 1 ($a, b, c$) and 2 ($a, b, c$) show the variation in mean-square velocity over that part of the section of the channel remote from the corners. The curves are seen to have the same general features as those for flow of a *Newton*ian liquid, each showing a high peak of $\overline{W}^2$ close to the boundary wall. However, it is seen that the change of shape of the ($\overline{W}^2$, $X$, $Y$) curves with the elastic parameters is sufficiently marked to permit comparison with experimental curves. As in the case of a channel of circular cross-section (*Jones* and *Walters* 1966), the effects of elasticity are to increase the value of $\overline{W}^2$ close to the boundary wall, while having no great effect on its magnitude over the rest of the region of flow, and to decrease somewhat the thickness of the boundary region in which the velocity changes rapidly.

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