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A note on $\mathbb{Z}/p\mathbb{Z}$-actions on K3 surfaces in odd characteristic p

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Introduction

The purpose of this paper is to study $\mathbb{Z}/p\mathbb{Z}$-actions on K3 surfaces defined over an algebraically closed field k of characteristic $p \geqq 3$ with a fixed curve of arithmetic genus $p_a \geqq 2$.

Over the complex number field $\mathbb{C}$, Nikulin [3] determined all finite commutative automorphism subgroups of K3 surfaces whose actions on the space of the global 2-forms are trivial. We call such a group (resp. an element) an N-automorphism group (resp. an N-automorphism). In particular Nikulin showed that the order of an N-automorphism is one of the following; $2, 3, 4, 5, 6, 7, 8$. This result followed from the fact that if g is an N-automorphism of order n of K3 surface X, where n is coprime to p if $p \geqq 3$, $\langle g \rangle$ has only finite fixed points on X and hence the quotient variety $X/\langle g \rangle$ becomes a K3 surface with only rational double points of type A.

On the other hand, in the case that the order of g is equal to the characteristic p, the phenomenon are entirely different while g is always an N-automorphism with fixed points. (See (1.1).) In fact there is a case that g has a fixed curve. (See (1.3), (1.6).)

In this paper, first we construct a K3 surface in characteristic 11 with an automorphism of order 11, which is not in Nikulin's list. In this example, the fixed locus is a rational curve of arithmetic genus 1 and the quotient surface is a rational surface. (See (1.3).)

Next we completely describe K3 surfaces X in odd characteristic p having an automorphism g of order p whose fixed locus contains an irreducible curve C of arithmetic genus $p_a \geqq 2$. As a result we see that the pair (p, p_a) is one of the following: $(5,2)$, $(3,2)$, $(3,3)$, $(3,4)$. (See (1.5).) In each case, we find the defining equation of X in a suitable weighted projective space. (See (1.6).) Moreover we see that the fixed curve is a singular rational curve and the fixed locus of g is equal to this curve. The key idea of proof is to study the projective model $\Phi_{|C|}(X)$ of X associated with the complete linear system $|C|$ by using the theory of projective models of K3 surfaces due to Saint-Donat [4] and to study the induced action of g on $\Phi_{|C|}(X)$, which is a projective transformation.

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1. The statement of the main theorems

Throughout this paper we assume that k is an algebraically closed field of characteristic $p \neq 2$. By a K3 surface over k, we mean a smooth projective surface over k whose irregularity is zero and whose canonical sheaf is trivial. For a K3 surface Y, we call a subgroup (resp. an element) of $\mathrm{Aut}(Y/k)$ an N-automorphism group (resp. an N-automorphism) if its action on the space of the global 2-forms is trivial. For $g \in \mathrm{Aut}(Y/k)$, we put $Y^g := \{y \in Y; gy = y\}$.

Lemma (1.1) *Let Y be a K3 surface over k with* $\mathrm{char}(k) = p \geqq 3$. *If g is an automorphism of order p of Y, g is an N-automorphism and Y^g is not empty.*

Proof. Since $\dim_k H^0(Y, \Omega_Y^2)$ is one, we have $g^*\omega_Y = \alpha\omega_Y$ for some $\alpha \in k$ where ω_Y is a base of $H^0(Y, \Omega_Y^2)$. But since $(g^*)^p = 1$, we have $\alpha^p = 1$ and $\alpha = 1$. If Y^g is empty, the quotient variety $Y/\langle g \rangle$ is smooth and then we have $p\chi(\mathcal{O}_{Y/\langle g \rangle}) = \chi(\mathcal{O}_Y) = 2$. But this is impossible since $p \geqq 3$. □

Over the complex number field $\mathbb{C}$, Nikulin determined all the finite commutative N-automorphism groups of K3 surfaces. (See [3].) Especially Nikulin obtained the following result.

Theorem (1.2) (Nikulin [3].) *Let Y be a K3 surface over $\mathbb{C}$. Let f be an N-automorphism of order $n \neq 1$ of Y. Let m be the number of the fixed points of f. Then the pair (n, m) is one of the following*:

$$(2,8),\quad (3,6),\quad (4,4),\quad (5,4),\quad (6,2),\quad (7,3),\quad \textit{and}\quad (8,2). \quad \square$$

But in characteristic $p \geqq 3$, there is an example which is not in the list of (1.2).

Proposition-Example (1.3) *Let k be an algebraically closed field with* $\mathrm{char}(k) = 11$. *Let $\varphi: Y \to \mathbb{P}^1$ be the non-singular complete elliptic surface defined by $y^2 = x^3 + (t^{11} - t)$. Then Y is a K3 surface. Y has an N-automorphism g of order 11 defined by $g: (x, y, t) \mapsto (x, y, t+1)$. Moreover $Y^g = Y_\infty =$ (the fiber of φ over $t = \infty$) and the quotient variety $Y/\langle g \rangle$ is birationally equivalent to the rational elliptic surface defined by $y^2 = x^3 + T$.*

Proof. By Néron [2, pp. 481], the singular fiber of φ are exactly $Y_i := \varphi^{-1}(t = i)$, where $i = 0, 1, \ldots, 10, \infty$, and every Y_i is a rational curve with an ordinary cusp. Then we have $c_2(Y) = 24$, and then Y is a K3 surface. The other part of (1.3) is obvious. □

Remark (1.4) In characteristic $\neq 11$, this example corresponds to the following elliptic K3 surface $\tilde{Y}$ defined by $y^2 = x^3 + (t^{11} - 1)$. $\tilde{Y}$ has an automorphism g of order 11 defined by $g:(x, y, t) \mapsto (x, y, \zeta t)$ where ζ is a primitive 11-th root of unity. Then $\tilde{Y}^g = \tilde{Y}_\infty =$ (the fiber of φ over $t = \infty$) and the quotient variety $\tilde{Y}/\langle g \rangle$ is birationally equivalent to the rational elliptic surface defined by $y^2 = x^3 + T$. But in this case g is not an N-automorphism. $\square$

Our main theorems are as follows.

Theorem (1.5) *Let X be a K3 surface over k with* $\mathrm{char}(k) = p \geqq 3$. *If g is an (N-)automorphism of order p of X and X^g contains an integral (i.e., irreducible and reduced) curve C with arithmetic genus $p_a(C) \geqq 2$, we have;*
(1) $(p, p_a(C)) = (5, 2), (3, 2), (3, 3),$ *or* $(3, 4)$, *and*
(2) $X^g = C$. $\square$

Theorem (1.6) *In each case of* (1.5), (X, C, g) *is as follows.*
(1) *If* $(p, p_a(C)) = (5, 2)$, *then,*
(1-a). *X is the minimal resolution of Y which is defined in* $\mathbb{P}(1{:}1{:}1{:}3)$ *by* $z^2 = F(x_0, x_1, x_2)$ *where*

$$F(x_0, x_1, x_2) = \sum_{i=0}^{6} c_i x_0^i x_1^{6-i} + (x_2^5 - x_0^4 x_2)x_1 \quad \text{with} \quad c_1 \neq 0,$$

(1-b). *C is naturally isomorphic to the curve defined by $z^2 = F(0, x_1, x_2)$ in* $\mathbb{P}(1{:}1{:}3)$, *which is a rational curve of arithmetic genus* 2, *and,*

(1-c). *g is naturally induced from the action* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ on Y.

(2) *If* $(p, p_a(C)) = (3, 2)$, *then,*
(2-a). *X is the minimal resolution of Y which is defined in* $\mathbb{P}(1{:}1{:}1{:}3)$ *by* $z^2 = F(x_0, x_1, x_2)$ *where*

$$F(x_0, x_1, x_2) = \sum_{i=0}^{6} a_i x_0^i x_1^{6-i} + (b x_1^3 + c x_1^2 x_0 + d x_1 x_0^2 + e x_0^3)(x_2^3 - x_0^2 x_2) + (x_2^3 - x_0^2 x_2)^2$$

with

$$(a_1, c) \neq (0, 0),\ a_0 c^2 - b a_1 c + a_1^2 \neq 0, \quad \text{and} \quad b^2 - 4a_0 \neq 0.$$

(2-b). *C is naturally isomorphic to the curve defined by $z^2 = F(0, x_1, x_2)$ in* $\mathbb{P}(1{:}1{:}3)$, *which is a rational curve of arithmetic genus* 2, *and,*

(2-c). *g is naturally induced from the action* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *on Y.*

(3) *If* $(p, p_a(C)) = (3, 3)$, *then,*

(3-a). *X is the minimal resolution of* $\bar{X}$ *which is defined in* $\mathbb{P}^3$ *by*

$$F(x_0, x_1, x_2, x_3) = \sum_{i=0}^{4} x_0^{4-i} g_i(x_1, x_2) + (x_0^2 x_3 - x_3^3)x_1 = 0$$

satisfying that the system of equations

$$g_3(x_1, x_2) = (\partial g_4/\partial x_2)(x_1, x_2) = 0$$

does not have solutions except for $(0,0)$, *and* $g_4(x_1, x_2)$ *is not divided by* x_1, *where* $g_i(x_1, x_2)$ *is a homogeneous polynomial of* x_1, x_2 *of degree* i,
(3-b). C *is naturally isomorphic to the curve defined by* $F(0, x_1, x_2, x_3) = 0$ *in* $\mathbb{P}^2$, *which is a rational curve of arithmetic genus* 3, *and*,

(3-c). g *is naturally induced from the action* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ *on* $\bar{X}$.

(4) *If* $(p, p_a(C)) = (3, 4)$, *then*,
(4-a). X *is the minimal resolution of* $\bar{X}$ *which is defined in* $\mathbb{P}^4$ *by the following complete intersection*:

$$\begin{aligned} & ax_0^2 + (bx_1 + cx_2 + dx_3)x_0 + (x_2^2 - x_1 x_3) = 0 \\ & ex_0^3 + x_0^2 \bar{f}_1(x_1, x_2, x_3) + x_0 \bar{f}_2(x_1, x_2, x_3) + \bar{f}_3(x_1, x_2, x_3) \\ & \quad + (x_0^2 x_4 - x_4^3) = 0, \end{aligned}$$

satisfying that $\bar{f}_3(1, T, T^2)$ *is not contained in* $k[T^3]$ *and the system of equations*

$$x_2^2 = x_1 x_3, \ \mathrm{rank} \begin{pmatrix} bx_1 + cx_2 + dx_3 & -x_3 & -x_2 & -x_1 \\ \bar{f}_2 & \dfrac{\partial \bar{f}_3}{\partial x_1} & \dfrac{\partial \bar{f}_3}{\partial x_2} & \dfrac{\partial \bar{f}_3}{\partial x_3} \end{pmatrix} \leqq 1$$

has no solutions except for $(0,0,0)$, *where* $\bar{f}_i$ *denotes a homogeneous polynomial of* x_1, x_2, x_3 *of degree* i.
(4-b). C *is naturally isomorphic to the curve defined by* $x_2^2 - x_1 x_3 = 0$, $\bar{f}_3(x_1, x_2, x_3) - x_4^3 = 0$ *in* $\mathbb{P}^3$, *which is a rational curve of arithmetic genus* 4.

(4-c). g *is naturally induced from the action* $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$ *on* $\bar{X}$. □

2. Projective models of K3 surfaces

In order to prove our main theorems, we fully use the theory of projective models of K3 surfaces by Saint-Donat. In this section, we briefly recall some basic facts about projective models of K3 surfaces. For proof and more details, we refer the reader to Saint-Donat [4].

Let X be an arbitrary K3 surface over k with $\mathrm{char}(k) = p \neq 2$. Let C be an arbitrary integral curve on X. Then we have,

$$p_a(C) = (C^2/2) + 1 = \dim|C|. \tag{2.1}$$

Theorem (2.2) *If $p_a(C) \geqq 2$ or equivalently $C^2 > 0$, then,*
(1) *the complete linear system $|C|$ is base point free,*
(2) *$h^0(\mathcal{O}_X(nC)) = (1/2)n^2C^2 + 2$ for any positive integer n.* □

To the end of this section, we assume that an integral curve C satisfies the condition in (2.2). Thus we have the following morphism,

$$\Phi_{|C|}: X \to \mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{O}_X(C))^\vee) \tag{2.3}$$

where

$$N = p_a(C) = \dim|C|.$$

Put $\bar{X} = \operatorname{Im} \Phi_{|C|}(X)$. We can easily see that $\dim \bar{X} = 2$ and $\deg \bar{X}$ is at least $N - 1 = p_a(C) - 1$ since $\bar{X}$ is not contained in any hyperplanes in $\mathbb{P}^N$. Moreover, since $2p_a(C) - 2 = C^2 = (\deg \bar{X})(\deg \Phi_{|C|})$, we see that either the following (1) or (2) can occur:
(1) $\Phi_{|C|}$ is of degree 2 and $\deg \bar{X} = p_a(C) - 1$,
(2) $\Phi_{|C|}$ is birational and $\deg \bar{X} = 2p_a(C) - 2$.
We note that $\Phi_{|C|}$ may not be a finite morphism.

Let $\mathscr{E}_C$ be the set of integral curves Δ such that $C \cdot \Delta = 0$ and let $\mathscr{E}_C^\lambda$ $(\lambda = 1, \ldots, n)$ be the connected components of $\mathscr{E}_C$. Then by the Hodge index theorem and (2.1), we see at once that for $\Delta \in \mathscr{E}_C$, $\Delta^2 = -2$, i.e., Δ is a smooth rational curve and $\mathscr{E}_C$ is a finite set (maybe empty). Then each $\mathscr{E}_C^\lambda (\lambda = 1, \ldots, n)$ has one of the *A-D-E* configurations by putting a suitable weight for each irreducible component of $\mathscr{E}_C^\lambda$. Thus we have the contraction morphism $\nu_C: X \to Y$, where Y is a normal surface with n rational double points $p_\lambda (\lambda = 1, \ldots, n)$ corresponding to $\mathscr{E}_C^\lambda$. We note that the induced morphism $\nu_C: X - \bigcup_{\lambda=1}^{n} \mathscr{E}_C^\lambda \to Y - \bigcup_{\lambda=1}^{n} p_\lambda$ is an isomorphism, and in particular $C \simeq \nu_C(C)$. By construction, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "\Phi_{|C|}"] \arrow[dr, "\nu_C"'] & & \bar{X} \subset \mathbb{P}^N, \\ & Y \arrow[ur, "\theta_C"'] & \end{tikzcd} \tag{2.4}$$

where θ_C is a finite morphism.

Theorem (2.5) *If $\Phi_{|C|}$ is birational, then,*

(1) *the natural* map $S^* H^0(X, \mathcal{O}_X(C)) \to \bigoplus_{n \geqq 0} H^0(X, \mathcal{O}_X(nC))$ *is surjective,*

(2) *the morphism θ_C is an isomorphism, i.e., $\bar{X}$ is normal.* □

3. Proof of (1.5)

Let X be a K3 surface over k with $\operatorname{char}(k) = p \geqq 3$. Assume that X has an automorphism g of order p such that X^g contains an integral curve C with $p_a(C) \geqq 2$. In this section we use the same notation as in Sect. 2.

Since $N = p_a(C) \geqq 2$, we have the morphism $\Phi_{|C|}: X \to \bar{X} \subset \mathbb{P}^N = \{[x_0 : x_1 : \cdots : x_N]\}$. Since $g(C) = C$, g induces the natural isomorphisms; $g_Y: Y \xrightarrow{\sim} Y$, and $g^*: H^0(X, \mathcal{O}_X(C)) \xrightarrow{\sim} H^0(X, \mathcal{O}_X(C))$, and hence $g_* := {}^t g^* (\mathrm{mod}\, k^\times) \in \mathrm{Aut}(\mathbb{P}^N)$ induces an isomorphism $\bar{g} := g_*|_{\bar{X}}: \bar{X} \to \bar{X}$. By construction, we have the following commutative diagram:

$$\begin{tikzcd}
\Phi_{|C|}: X \arrow[rr] \arrow[dr, "\nu_C"] \arrow[dd, "g"'] & & \bar{X} \arrow[dd, "\bar{g}"] & \subset \mathbb{P}^N \arrow[dd, "g_*"] \\
& Y \arrow[ur, "\theta_C"'] \arrow[dd, "g_Y"] & & \\
\Phi_{|C|}: X \arrow[rr] \arrow[dr, "\nu_C"'] & & \bar{X} & \subset \mathbb{P}^N \\
& Y \arrow[ur, "\theta_C"'] & &
\end{tikzcd}$$

, i.e.,

$$\begin{tikzcd}
x \arrow[r, "\Phi_{|C|}"'] \arrow[d, "g"'] & (\varphi_0(x) : \varphi_1(x) : \cdots : \varphi_N(x)) \arrow[d, "\bar{g}"] \\
y = g(x) \arrow[r, "\Phi_{|C|}"'] & (\varphi_0(y) : \varphi_1(y) : \cdots : \varphi_N(y)),
\end{tikzcd}$$

where $\langle \varphi_0, \varphi_1, \ldots, \varphi_N \rangle$ is a basis of $H^0(X, \mathcal{O}_X(C))$ and $\mathrm{div}(\varphi_0) = C$. Let us put $\bar{C} = \mathrm{Im}\, \Phi_{|C|}(C) = \bar{X} \cap \{x_0 = 0\}$. Let $(a_{ij})_{0 \leqq i,j \leqq N}$ be the matrix representation of g_* such that ${}^t(a_{ij})$ is the matrix representation of g^* with respect to the basis $\langle \varphi_0, \varphi_1, \ldots, \varphi_N \rangle$ of $H^0(X, \mathcal{O}_X(C))$. We note that since $(g^*)^p = \mathrm{id}$, every eigenvalue of the action g^* on $H^0(X, \mathcal{O}_X(nC))$ is one and every eigenvalue of (a_{ij}) is also one.

Lemma (3.1) *The matrix (a_{ij}) is written as follows:*

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \alpha_1 & 1 & 0 & & & \cdot \\ \alpha_2 & 0 & 1 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & 1 & 0 \\ \alpha_N & 0 & \cdot & \cdot & 0 & 1 \end{pmatrix} \quad \textit{where} \quad (\alpha_1, \alpha_2, \ldots, \alpha_N) \neq 0$$

in k^N. Moreover $X^g = C$.

Proof. By construction, $g_*|_{\bar{C}} = \mathrm{id}$. By the exact sequence, $0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$, we see that the restriction map $H^0(X, \mathcal{O}_X(C)) \to H^0(C, \mathcal{O}_C(C))$ is surjective and its kernel is $k\varphi_0$. Then $\bar{C}$ generates the hyperplane $H = \{X_0 = 0\}$. Hence we have $g_*|_H = \mathrm{id}$, and since every eigenvalue of (a_{ij}) is one, we have, $(a_{ij}) \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}$ for all ${}^t(x_1, x_2, \ldots, x_N) \in k^N$.

Then (a_{ij}) is written as follows:

$$(a_{ij})=\begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \alpha_1 & 1 & 0 & & & \cdot \\ \alpha_2 & 0 & 1 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & 1 & 0 \\ \alpha_N & 0 & \cdot & \cdot & 0 & 1 \end{pmatrix}$$

for some $(\alpha_1,\ldots,\alpha_2,\ldots,\alpha_N)\in k^N$. Suppose that $(\alpha_1,\alpha_2,\ldots,\alpha_N)=(0,0,\ldots,0)$. Then $\bar{g}=\mathrm{id}$. If $\Phi_{|C|}$ is birational, g is also the identity map on X. But this contradicts our assumption. If $\Phi_{|C|}$ is of degree 2, either $\operatorname{order} g=0 \pmod 2$ or $g=\mathrm{id}$ holds since for a general point $Q\in\bar{X}$, $\Phi_{|C|}^{-1}(Q)$ consists of two points. But this also contradicts our assumption. Hence

$$(\alpha_1,\alpha_2,\ldots,\alpha_N)\neq(0,\ldots,0).$$

In order to prove $X^g=C$, it is enough to show $\bar{X}^{\bar{g}}=\bar{C}$. (Note $\mathscr{E}_C\cap C=\phi$.) Suppose $\bar{X}^{\bar{g}}\neq\bar{C}$. Then since obviously $\bar{X}^{\bar{g}}\supset\bar{C}$ there is a point $(p_0:p_1:\cdots:p_N)\in\bar{X}^{\bar{g}}\setminus\bar{C}$, and we have,

$$\begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \alpha_1 & 1 & 0 & & & \cdot \\ \alpha_2 & 0 & 1 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & 1 & 0 \\ \alpha_N & 0 & \cdot & \cdot & 0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_N \end{pmatrix}=\begin{pmatrix} p_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_N \end{pmatrix}$$

for this point. But this equality does not happen since $(\alpha_1,\ldots,\alpha_N)\neq(0,\ldots,0)$ and $p_0\neq 0$. Then $\bar{X}^{\bar{g}}=\bar{C}$. $\square$

Remark (3.2) Since

$$g^*=\begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \cdot & \cdot & \alpha_N \\ 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & 0 & 1 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & 1 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 1 \end{pmatrix}$$

where $(\alpha_1,\alpha_2,\ldots,\alpha_N)\neq 0$ by (3.1), we have,

$$\begin{aligned} H^0(X,\mathcal{O}_X(C))^{g^*}&:=\{\varphi\in H^0(X,\mathcal{O}_X(C)); g^*\varphi=\varphi\} \\ &=\left\{a_0\varphi_0+\cdots+a_N\varphi_N; \sum_{i=1}^{N} a_i\alpha_i=0, a_i\in k\right\}, \end{aligned}$$

and $\dim_k H^0(X,\mathcal{O}_X(C))^{g*}=N$. Then we can choose a basis $\langle\varphi_0,\varphi_1,\dots,\varphi_N\rangle$ such that $\varphi_0,\varphi_1,\dots,\varphi_{N-1}\in H^0(X,\mathcal{O}_X(C))^{g*}$ and $\operatorname{div}(\varphi_0)=C$. We use this basis from now on to the end of this paper. We note that with respect to this basis,

$$(a_{ij})=\begin{pmatrix}1&0&\cdot&\cdot&\cdot&0\\0&1&0&&&\cdot\\0&0&1&\cdot&&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&\cdot&&\cdot&1&0\\1&0&\cdot&\cdot&0&1\end{pmatrix}. \quad \square$$

Lemma (3.3) *If* $N=p_a(C)\geqq 3$, *then* $(p,p_a(C))$ *is either* $(3,3)$ *or* $(3,4)$, *and* $\Phi_{|C|}$ *is birational.*

Proof. Consider the rational map $\Phi_{|C|g*}:X\dashrightarrow\mathbb{P}^{N-1}$ defined by $x\mapsto(\varphi_0(x):\varphi_1(x):\cdots:\varphi_{N-1}(x))$. Put $Z:=\operatorname{im}\Phi_{|C|g*}(X)$. Then we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\Phi_{|C|}"'] \arrow[dr, dashed, "\Phi_{|C|g^*}"'] & \bar{X} \subset \mathbb{P}^N \arrow[d, dashed, "\pi"'] \\ & Z \subset \mathbb{P}^{N-1} \end{tikzcd} \quad \text{projection from } (0:0:\cdots:0:1).$$

Since $g^*|_{H^0(X,\mathcal{O}_X(C))^{g*}}=\mathrm{id}$, the induced map $g_*|_Z:Z\to Z$ is also identity and we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "\Phi_{|C|}"] \arrow[dr, dashed, "\Phi_{|C|g^*}"] \arrow[dd, "g"'] & & \bar{X} \arrow[dl, dashed, "\pi"'] \arrow[dd, "\bar{g}"] \\ & Z & \\ X \arrow[rr, "\Phi_{|C|}"'] \arrow[ur, dashed, "\Phi_{|C|g^*}"] & & \bar{X} \arrow[ul, dashed, "\pi"'] \end{tikzcd} \tag{3.4}$$

Put $D:=(\operatorname{Im}\Phi_{|C|g*})(C)$.

Claim (3.5) *We have* $\dim D=1$, $\dim Z=2$, *and* $\deg Z\geqq N-2$.

Proof. Since $\varphi_1|_C,\dots,\varphi_{N-1}|_C$ where $N-1\geqq 2$ are linearly independent in $H^0(C,\mathcal{O}_C(C))$, we have $\dim D=1$ and D generates the hyperplane $(x_0=0)$ in $\mathbb{P}^{N-1}$. Since $\varphi_i(i=0,1,2,\dots,N-1)$ are linearly independent in $H^0(X,\mathcal{O}_X(C))$, Z generates $\mathbb{P}^{N-1}$. Then $Z\neq D$. Since Z is irreducible, we have $\dim Z=2$. So $\deg Z\geqq N-2$. This completes the proof of (3.5). $\square$

Let L be a general plane of codim 2 in $\mathbb{P}^{N-1}$. We may assume the following (1), (2), (3) by (3.5).

(1). $\#(L\cap Z)$ is finite and $\#(L\cap Z)\geqq N-2$.

(2). For each $z\in L\cap Z$, $\#(\pi^{-1}(z))$ is finite and $\pi^{-1}(z)\backslash\bar{C}$ is not empty.

(3). $\tilde{L} := \pi^{-1}(L)$ is a plane of codim 2 in $\mathbb{P}^{N-1}$ such that $\#(\tilde{L} \cap \bar{X})$ is finite and,

$$\#(\tilde{L} \cap \bar{X}) = \deg \bar{X} = \begin{cases} 2N-2, & \text{if } \Phi_{|C|} \text{ is birational,} \\ N-1, & \text{if } \deg \Phi_{|C|} = 2. \end{cases}$$

Put $n_i := \#(\pi^{-1}(z_i) \backslash \bar{C})$ where $L \cap Z = \{z_1, \ldots, z_n\}$. Note that $n \geqq N-2$. Since by (3.4) and (3.1) $\langle \bar{g} \rangle$ induces a permutation group on $\pi^{-1}(z_i) \backslash \bar{C}$ which has no fixed points, then $n_i \geqq p$ for any $i = 1, \ldots, n$.

Since $\tilde{L} \cap \bar{X}$ contains at least $\sum_{i=1}^{n} n_i$ points and $n_i \geqq p$, we have,

$$p(N-2) \leqq \sum_{i=1}^{n} n_i \leqq \begin{cases} 2N-2, & \text{if } \Phi_{|C|} \text{ is birational,} \\ N-1, & \text{if } \deg \Phi_{|C|} = 2. \end{cases}$$

Since $p \geqq 3$ and $N \geqq 3$, then (p, N) is either $(3,3)$ or $(3,4)$, and $\Phi_{|C|}$ is birational. □

To finish the proof of (1.5) it is enough to show the following lemma.

Lemma (3.6) *If* $p_a(C) = 2$, *then* $p = \mathrm{char}(k) = 3$ *or* 5.

Proof. By (2.4), we have the following diagram:

$$\begin{tikzcd} X \arrow[rr, "\Phi_{|C|}"] \arrow[dr, "\nu_C"'] & & \bar{X} = \mathbb{P}^2 \\ & Y \arrow[ur, "\theta_C"'] & \end{tikzcd} \tag{3.7}$$

where Y is a K3 surface with only rational double points as its singularities, and θ_C is a finite morphism ramified over a reduced sextic B in $\mathbb{P}^2$. Let the defining equation of B be:

$$\sum_{i+j+h=6} c_{ijh} x_0^i x_1^j x_2^h = 0.$$

Since B must be stable under the action of $g_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, we have:

$$\sum_{i+j+h=6} c_{ijh} x_0^i x_1^j (x_2 - x_0)^h = \sum_{i+j+h=6} c_{ijh} x_0^i x_1^j x_2^h.$$

By expanding the left hand side, we see that this equality is equivalent to the following:

$$\sum_{i+h=6-j} c_{ijh} \sum_{a_h=1}^{h} (-1)^{a_h} \binom{h}{a_h} x_0^{i+a_h} x_2^{h-a_h} = 0 \tag{3.8}$$

for all $j = 0, 1, 2, \ldots, 5$.

Suppose $p \geqq 7$. Then by (3.8), we can directly show that $c_{ijh} = 0$ for all $h \neq 0$.

Then the defining equation of B becomes

$$\sum_{i+j=6} c_{ij0} x_0^i x_1^j = 0, \quad \text{i.e.,} \quad \prod_{i=1}^{6} (a_i x_0 + b_i x_1) = 0$$

for suitale a_i, b_i.

Hence the defining equation of Y in $\mathbb{P}(1:1:1:3)$ is as follows:

$$z^2 = \prod_{i=1}^{6} (a_i x_0 + b_i x_1).$$

So $\nu_C(C) = \theta_C^{-1}((x_0 = 0))$ contains the point $\theta_C^{-1}([0:0:1])$ at which Y is singular. But this is absurd since $C \cap \mathscr{E}_C = \phi$. □

4. Proof of (1.6)

In this section we use the same notation as in Sect. 1, Sect. 2, and Sect. 3 (3.6) freely.

4.A. *Proof of* (1.6)(1)

First assume that a pair $(X, C, \langle g \rangle)$ satisfies the condition of (1.5) and $(p, p_a(C)) = (5, 2)$. Then we have the diagram (3.7), and by (3.8), the defining equation of B is as follows:

$$F(x_0, x_1, x_2) = \sum_{i=0}^{6} c_i x_0^i x_1^{6-i} + (x_2^5 - x_0^4 x_2)(dx_1 + ex_0) = 0. \tag{4.1}$$

Note that $\bar{C} = (x_0 = 0) = \mathbb{P}^1$. By construction $\tilde{C} := \theta_C^{-1}(\bar{C})$ must be integral.

Claim (4.2) *The following conditions are equivalent to one another:*
(1) $\tilde{C}$ *is integral,* (2) $d \neq 0$.
Moreover under these conditions, $\tilde{C}$ *is a rational curve with* $p_a(\tilde{C}) = 2$.

Proof. Note that everything is defined in characteristic 5. Since the defining equation of the scheme theoretic intersection $\bar{C} \cap B$ in $\bar{C}$ is $F(0, X_1, x_2) = c_0 x_1^6 + dx_2^5 x_1 = 0$ and $\tilde{C}$ is a double cover of $\bar{C}$ ramified over the subscheme $\bar{C} \cap B$, (4.2) is obvious. □

Then by changing homogeneous coordinate x_1 by $(x_1 - ex_0)/d$, we may assume that the defining equation of B is as follows:

$$F(x_0, x_1, x_2) = \sum_{i=0}^{6} c_i x_0^i x_1^{6-i} + (x_2^5 - x_0^4 x_2)x_1 = 0. \tag{4.3}$$

Also by construction $\tilde{C} \cap \operatorname{Sing} Y$ must be empty.

Claim (4.4) *The following conditions are equivalent to each other:*
(1) $\tilde{C} \cap \operatorname{Sing} Y = \phi$, (2) $\tilde{C} \cap \operatorname{Sing} B = \phi$, (3) $c_1 \neq 0$.

Proof. It is enough to show that (2) and (3) are equivalent to one another since it is obvious that (1) and (2) are equivalent to one another. By the Jacobian criterion, we have:

$$\operatorname{Sing} B = \left\{ P = [x_0:x_1:x_2]; F\bigg|_P = \frac{\partial F}{\partial x_0}\bigg|_P = \frac{\partial F}{\partial x_1}\bigg|_P = \frac{\partial F}{\partial x_2}\bigg|_P = 0 \right\}$$

$$= \left\{ P = [x_0:x_1:x_2]; \begin{array}{l} \sum_{i=1}^{6} c_i i x_0^{i-1} x_1^{6-i} - 4x_0^3 x_1 x_2 = 0 \\ \sum_{i=0}^{5} c_i(6-i) x_0^i x_1^{5-i} + (x_2^5 - x_0^4 x_2) = 0 \\ x_0^4 x_1 = 0 \end{array} \right\}. \tag{4.5}$$

Then $\bar{C} \cap \operatorname{Sing} B = \{P = [0:x_1:x_2]; c_1 x_1^5 = 0, c_0 x_1^5 + x_2^5 = 0\}$. Hence (2) and (3) are equivalent. □

By (4.2), (4.4), the defining equation of B must be:

$$F(x_0, x_1, x_2) = \sum_{i=0}^{6} c_i x_0^i x_1^{6-i} + (x_2^5 - x_0^4 x_2) x_1 = 0 \tag{4.6}$$

with $c_1 \neq 0$.

Then to finish the proof (1.6) (1), it is enough to show the following claim.

Claim (4.7) *Let Y be the double cover of $\mathbb{P}^2$ ramified over the sextic B defined by (4.6), i.e., $Y = (z^2 = F(x_0, x_1, x_2))$ in $\mathbb{P}(1:1:1:3)$. Then Y is integral and has only rational double points as its singularities.*

Proof. Integrability is obvious. Note that

$$\operatorname{Sing} Y = \left\{ P = [x_0:x_1:x_2;z]; F\bigg|_P = \frac{\partial F}{\partial x_0}\bigg|_P = \frac{\partial F}{\partial x_1}\bigg|_P = \frac{\partial F}{\partial x_2}\bigg|_P = 0 \right\}.$$

Then by (4.4), (4.5), we have:

$$\operatorname{Sing} Y = \begin{cases} \phi, & \text{if } \ c_6 \neq 0, \\ \{P = [1:0:\beta;0]; \beta^5 - \beta + c_5 = 0\}, & \text{if } \ c_6 = 0. \end{cases}$$

Assume $c_6 = 0$. Choose a local parameter system (s, t, u) of

$$\mathbb{A}^3 = \left\{ \left[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{z}{x_0^3} \right] \right\} \subset \mathbb{P}(1:1:1:3)$$

around P such that

$$s = \frac{x_1}{x_0}, t = \frac{x_2}{x_0} - \beta, u = \frac{z}{x_0^3}.$$

Then in $\widehat{\mathcal{O}}_{\mathbb{A}^3,P}$, the local equation of Y is as follows:

$$u^2 = s(-t + t^5 + c_4 s + c_3 s^2 + c_2 s^3 + c_1 s^4 + c_0 s^5).$$

Put $v = -t + t^5 + c_4 s + c_3 s^2 + c_2 s^3 + c_1 s^4 + c_0 s^5$. Then (v, s, u) is also a local parameter system of $\widehat{\mathcal{O}}_{\mathbb{A}^3,P}$ and the local equation of Y in $\widehat{\mathcal{O}}_{\mathbb{A}^3,P}$ is $u^2 = sv$. Hence P is a rational double point of type A_1. □

4.B. *Proof of* (1.6)(2)

First assume that a pair $(X, C, \langle g\rangle)$ satisfies the condition on (1.5) and $(p, p_a(C)) = (3, 2)$. Then we have a diagram (3.7), and by (3.8), the defining equation of B is as follows:

$$F(x_0, x_1, x_2) = \sum_{i=0}^{6} a_i x_0^i x_1^{6-i} + (x_2^3 - x_0^2 x_2)(bx_1^3 + cx_1^2 x_0 + dx_1 x_0^2 + ex_0^3) + f(x_2^3 - x_0^2 x_2)^2 = 0. \tag{4.8}$$

Note that $\bar{C} = (x_0 = 0) = \mathbb{P}^1$. By construction $\tilde{C} := \theta_C^{-1}(\bar{C})$ must be integral.

Claim (4.9) *The following conditions are equivalent to one another*:
(1) $\tilde{C}$ *is integral,* (2) $b^2 - a_0 f \neq 0$.
Moreover, under these conditions $\tilde{C}$ *is a rational curve with* $p_a(\tilde{C}) = 2$.

Proof. Note that everything is defined in characteristic 3. Then the proof is similar to (4.2). □

Also by construction $\tilde{C} \cap \operatorname{Sing} Y$ must be empty.

Claim (4.10) *The following conditions are equivalent to each other*:
(1) $\tilde{C} \cap \operatorname{Sing} Y = \phi$, (2) $\bar{C} \cap \operatorname{Sing} B = \phi$, (3) $f \neq 0$, $(a_1, c) \neq (0, 0)$, *and* $a_0 c^2 - ba_1 c + fa_1^2 \neq 0$.

Proof. Similar to (4.4), □

Then we may assume $f = 1$, and the defining equation of B must be:

$$F(x_0, x_1, x_2) = \sum_{i=0}^{6} a_i x_0^i x_1^{6-i} + (x_2^3 - x_0^2 x_2)(bx_1^3 + cx_1^2 x_0 + dx_1 x_0^2 + ex_0^3) + (x_2^3 - x_0^2 x_2)^2 = 0, \tag{4.11}$$

with $b^2 - a_0 \neq 0$, $(a_1, c) \neq (0, 0)$, and $a_0 c^2 - ba_1 c + a_1^2 \neq 0$.

Then to finish the proof (1.6) (2), it is enough to show the following claim.

Claim (4.12) *Let* Y *be the double cover of* $\mathbb{P}^2$ *ramified over the sextic* B *defined by* (4.11), *i.e.,* $Y = (z^2 = F(x_0, x_1, x_2))$ *in* $\mathbb{P}(1{:}1{:}1{:}3)$. *Then* Y *is integral and has only rational double points as its singularities.*

Proof. Since $\operatorname{Sing} B \cap (x_0 = 0) = \phi$ by (4.10), we have:

$$\operatorname{Sing} Y = \left\{ P = [1{:}x_1{:}x_2; z];\ F|_P = \frac{\partial F}{\partial x_0}\bigg|_P = \frac{\partial F}{\partial x_1}\bigg|_P = \frac{\partial F}{\partial x_2}\bigg|_P = 0 \right\}$$

$$= \left\{ P = [1{:}\alpha{:}\beta; 0];\ \begin{array}{c} \alpha \text{ is a } \textit{multiple} \text{ root of} \\ \sum_{i=0}^{6} a_i x^{6-i} - (g(x))^2 = 0, \\ \beta \text{ is a root of } \beta^3 - \beta = g(\alpha) \end{array} \right\}$$

where $g(x) := bx^3 + cx^2 + dx + e$.

Then for $P = [1:\alpha:\beta;0] \in \operatorname{Sing} Y$, we can put:

$$\sum_{i=0}^{6} a_i x^{6-i} - (g(x))^2 = (x-\alpha)^m f(x)$$

where $m \geqq 2$ and $f(x)$ is a polynomial of x such that $f(\alpha) \neq 0$.

Choose a local parameter system (s, t, u) of

$$\mathbb{A}^3 = \left\{\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{z}{x_0^3}\right]\right\} \subset \mathbb{P}(1:1:1:3) \text{ around } P \text{ such that}$$

$$s = \frac{x_1}{x_0} - \alpha, t = \frac{x_2}{x_0} - \beta, u = \frac{z}{x_0^3}.$$

Then in $\hat{\mathcal{O}}_{\mathbb{A}^3,P}$, the local equation of Y is as follows:

$$u^2 = t^m f(t+\alpha) + (t\bar{g}(t) + s^3 - s)^2$$

where $\bar{g}(t)$ is a polynomial of t.

Note that $f(t+\alpha)$ is unit in $\hat{\mathcal{O}}_{\mathbb{A}^3,P}$. Put

$$v = t\bar{g}(t) + s^3 - s,$$
$$z_1 = t, \quad z_2 = u + v, \quad z_3 = (u - v)/f(t+\alpha).$$

Then (z_1, z_2, z_3) is also a local parameter system of $\hat{\mathcal{O}}_{\mathbb{A}^3,P}$ and the local equation of Y in $\hat{\mathcal{O}}_{\mathbb{A}^3,P}$ is $z_1^m = z_2 z_3$. Hence P is a rational double point of type A_{m-1}. □

4.C. *Proof of* (1.6)(3)

First assume that a pair $(X, C, \langle g \rangle)$ satisfies the condition of (1.5) and $(p, p_a(C)) = (3, 3)$. Then by (3.3) we have the following diagram:

$$\begin{tikzcd} \Phi_{|C|}: X \arrow[rr] \arrow[dr, "\nu_C"'] & & \bar{X} \subset \mathbb{P}^3, \\ & Y \arrow[ur, "\theta_C"'] & \end{tikzcd}$$

where $\Phi_{|C|}$ is birational and $\bar{X}$ is a quartic surface in $\mathbb{P}^3$.

Since $\bar{X}$ is stable under the action of

$$g_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

the defining equation of B is as follows:

$$\begin{aligned} F(x_0, x_1, x_2, x_3) = g_4 + g_3 x_0 + g_2 x_0^2 + g_1 x_0^3 + a x_0^4 \\ + (x_0^2 x_3 - x_3^3)(\alpha x_1 + \beta x_2 + \gamma x_0) = 0, \end{aligned} \tag{4.14}$$

where g_i denotes a polynomial of x_1, x_2 such that $\deg g_i = i$.

Since $\bar{C} = \bar{X} \cap (x_0 = 0) = \{[x_0:x_1:x_2]; g_4 - x_3^3(\alpha x_1 + \beta x_2) = \}$ in $\mathbb{P}^2 = (x_0 = 0)$, must be integral, we have $(\alpha, \beta) \neq (0,0)$. We may assume $\alpha \neq 0$. Then changing homogeneous coordinate x_1 by $(x_1 - \beta x_2 - \gamma x_0)/\alpha$ we may assume that the defining equation of $\bar{X}$ is as follows:

$$F(x_0, x_1, x_2, x_3) = g_4 + g_3 x_0 + g_2 x_0^2 + g_1 x_0^3 + a x_0^4 + (x_0^2 x_3 - x_3^3) x_1 = 0. \tag{4.15}$$

Claim (4.16) *The following conditions are equivalent to one another:*
(1) $\bar{C} = \bar{X} \cap (x_0 = 0)$ *is integral,*
(2) g_4 *is not divided by* x_1.
Moreover under these conditions. $\bar{C}$ *is a rational curve with* $p_a(\bar{C}) = 3$.

Proof. The defining equation of $\bar{C}$ in $\mathbb{P}^2 = (x_0 = 0)$ is $g_4 - x_3^3 x_1 = 0$. Since we work in characteristic 3, we easily see that (1) and (2) are equivalent to one another. Assume these conditions. Then $\bar{C}$ is an integral quartic curve, and hence $p_a(\bar{C}) = 3$. Moreover $\bar{C}$ is rational since $\bar{C}$ is a purely inseparable covering of a rational curve. □

Also by condition $(\operatorname{Sing} \bar{X}) \cap (x_0 = 0)$ must be empty.

Claim (4.17) *The following conditions are equivalent to one another:*
(1) $(\operatorname{Sing} \bar{X}) \cap (x_0 = 0) = \phi$,
(2) *the system of equation* $\partial g_4/\partial x_2 = g_3 = 0$ *has no solutions except for* $(0,0)$.

Proof. By the Jacobian criterion, we have:

$$\begin{aligned} \operatorname{Sing} \bar{X} &= \left\{ P = [x_0 : x_1 : x_2 : x_3];\, F|_P = \left.\frac{\partial F}{\partial x_i}\right|_P = 0 \text{ for all } i \right\} \\ &= \left\{ P = [0 : x_1 : x_2 : x_3];\, g_3|_P = \left.\left(\frac{\partial g_4}{\partial x_1} - x_1^3 \right)\right|_P = \left.\frac{\partial g_4}{\partial x_2}\right|_P = 0 \right\} \\ &\cup \left\{ Q = [1 : 0 : \beta : \gamma];\, g(0, \beta) = \frac{\partial g}{\partial x_1}(0, \beta) + (\gamma - \gamma^3) = \frac{\partial g}{\partial x_2}(0, \beta) = 0 \right\} \end{aligned} \tag{4.18}$$

where $g := \sum_{i=1}^{4} g_i(x_1, x_2) + a$. Then (1) and (2) are equivalent. □

To finish the proof of (1.6)(3), it is enough to show the following claim.

Claim (4.19) *The quartic surface* $\bar{X}$ *defined in* (1.6)(3-*a*) *is integral and has only rational double points as its singularities.*

Proof. By (4.17), under the condition of (4.19) we have $\operatorname{Sing} \bar{X} \cap (x_0 = 0) = \phi$ and in particular $\bar{X}$ is integral. Let Q be a singular point of $\bar{X}$. By (4.17), (4.18), we have:

$$Q = [0, \beta, \gamma] \in \mathbb{A}^3 = (x_0 \neq 0) = \left\{ \left[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0} \right] \right\} \subset \mathbb{P}^3.$$

Choose a local parameter system (s,t,u) of $\hat{\mathcal{O}}_{\mathbb{A}^3,Q}$ such that

$$s=\frac{x_1}{x_0},\quad t=\frac{x_2}{x_0}-\beta,\quad u=\frac{x_3}{x_0}-\gamma.$$

Then by (4.18), the local equation of $\bar{X}$ in $\hat{\mathcal{O}}_{\mathbb{A}^3,Q}$ is calculated as follows:

$$\begin{aligned}F&=g(s,t+\beta)+\{(u+\gamma)-(u+\gamma)^3\}s\\&=s\{u-u^3+h(s,t)\}+t^m k(t)=0,\end{aligned}$$

where $h(s,t)$ is a polynomial of s,t such that $h(0,0)=0$, and $k(t)$ is a polynomial of t such that $k(0)\neq 0$, and m is a positive integer such that $2\leqq m\leqq 4$. Put $z_1=\dfrac{s}{k(t)}$, $z_2=u-u^3+h(s,t)$, and $z_3=t$. Then (z_1,z_2,z_3) is also a local parameter system of $\hat{\mathcal{O}}_{\mathbb{A}^3,Q}$, and the defining equation of $\bar{X}$ in $\hat{\mathcal{O}}_{\mathbb{A}^3,Q}$ is as follows:

$$z_1z_2-z_3^m=0.$$

Hence Q is a rational double point of type A_{m-1}. $\square$

4.D. *Proof of* (1.6)(4)

First assume that a pair $(X,C,\langle g\rangle)$ satisfies the condition of (1.5) and $(p,p_a(C))=(3,4)$. By (3.3) we have the following diagram:

$$\begin{tikzcd}\Phi_{|C|}:X \arrow[rr] \arrow[dr, "\nu_C"'] & & \bar{X}\subset\mathbb{P}^4 \\ & Y \arrow[ur, "\theta_C"'] & \end{tikzcd}\quad\text{and } \Phi_{|C|} \text{ is birational.}$$

Claim (4.20) *$\bar{X}\subset\mathbb{P}^4$ is a complete intersection of an irreducible quadratic hypersurface Q and an irreducible cubic hypersurface T; $\bar{X}=Q\cap T$. The quadratic hypersurface Q is uniquely determined by $\bar{X}$. Let f (resp. g) be the defining equation of T (resp. Q). If $\bar{X}=Q\cap\tilde{T}$ for another cubic hypersurface $\tilde{T}$, then the defining equation $\tilde{f}$ of $\tilde{T}$ is written as follows*:

$$\tilde{f}=af+(a_0x_0+a_1x_1+a_2x_2+a_3x_3+a_4x_4)g.$$

Proof. A quadratic hypersurface $\tilde{Q}=(\tilde{g}=0)$ (resp. a cubic hypersurface $\tilde{T}=(\tilde{f}=0)$) contains $\bar{X}$ if and only if $\tilde{g}\in I:=\mathrm{Ker}(S^2H^0(X,\mathcal{O}_X(C))\to H^0(X,\mathcal{O}_X(2C)))$ (resp. $\tilde{f}\in J:=\mathrm{Ker}(S^3H^0(X,\mathcal{O}_X(C))\to H^0(X,\mathcal{O}_X(3C)))$. By (2.2)(2) and (2.4)(1), we see that $\dim_k I=1$ and $\dim_k J=6$. Then the quadratic hypersurface $\tilde{Q}$ containing $\bar{X}$ is unique. We put this quadratic $Q=(g=0)$. Since $\dim_k J=6$ and x_ig $(i=0,1,2,3,4)$ are linearly independent in J, there is a cubic form f such that $\langle x_0g,x_1g,\dots,x_4g,f\rangle$ is a basis of J. Put $T=(f=0)$. Then $\bar{X}=Q\cap T$ since $\bar{X}\subset Q\cap T$ and $\deg\bar{X}=\deg(Q\cap T)=6$. Moreover Q and T are irreducible since $\bar{X}$ is irreducible and $\deg\bar{X}=6$. This completes the proof of (4.20). $\square$

Claim (4.21) *The defining equation of $\bar{X}\subset\mathbb{P}^4$ is written as follows*:

$$\begin{cases}ax_0^2+(bx_1+cx_2+dx_3)x_0+x_2^2-x_1x_3=0,\\ f_3(x_0,x_1,x_2,x_3)+(x_0^2x_4-x_4^3)=0,\end{cases}$$

where f_3 is a cubic form of x_0, x_1, x_2, x_3. Put

$$f_3 := ex_0^3 + x_0^2\bar{f}_1(x_1,x_2,x_3) + x_0\bar{f}_2(x_1,x_2,x_3) + \bar{f}_3(x_1,x_2,x_3).$$

Proof. By (4.20), $\bar{X} = (g=0)\cap(f=0)$ in $\mathbb{P}^4$ where g (resp. f) is a quadratic (resp. cubic) form of $x_0, x_1, \ldots, x_4$. Put $Q=(g=0)$ and $T=(f=0)$. Since for

$$g_* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_*(\bar{X}) = \bar{X},$$

we have $g_*(Q) \supset \bar{X}$ and $g_*(T) \supset \bar{X}$. Since the defining equation of $g_*(Q)$ (resp. $g_*(T)$) is $g(x_0, x_1, \ldots, x_4 - x_0)$ (resp. $f(x_0, x_1, \ldots, x_4 - x_0)$) and every eigenvalue of g_* is one, we have

$$g(x_0, x_1, \ldots, x_4 - x_0) = g(x_0, x_1, \ldots, x_4)$$

and,

$$f(x_0, x_1, \ldots, x_4 - x_0) = f(x_0, x_1, \ldots, x_4) + \left(\sum_{i=0}^{4} a_i x_i\right) g(x_0, x_1, \ldots, x_4).$$

Then we see at once that g is a quadratic form of $x_0, \ldots, x_3$, i.e., $g := ax_0^2 + (bx_1 + cx_2 + dx_3)x_0 + \bar{g}_2(x_1,x_2,x_3)$. Let us put $f = f_3 + x_4 f_2 + x_4^2 f_1 + Ax_4^3$, where f_i is a homogeneous polynomial of $x_0, \ldots, x_3$ of degree i. Then we have,

$$f(x_0, x_1, x_2, x_3, x_4 - x_0) = f(x_0, x_1, \ldots, x_4) - x_0 f_2 + (-2x_0x_4 + x_0^2)f_1 - Ax_0^3$$

and then,

$$x_0(-f_2 + (-2x_4 + x_0)f_1 - Ax_0^2) = \left(\sum_{i=0}^{4} a_i x_i\right) g(x_0, \ldots, x_3).$$

Since g is irreducible, we have $a_1 = a_2 = a_3 = a_4 = 0$, and $-f_2 + (-2x_4 + x_0)f_1 - Ax_0^2 = a_0 g$. Then we have $f_1 = 0$, and $a_0 g = -f_2 - ax_0^2$, since $a_0 g$ is a polynomial of $x_0, \ldots, x_3$. Hence $f = f_3 - A(x_0^2x_4 - x_4^3) - Ax_4 g$. Therefore we can take as f, $f = f_3 - A(x_0^2x_4 - x_4^3)$ and $T = (f=0)$. Suppose $A = 0$. Then $\bar{X} = (g(x_0, \ldots, x_3) = 0)\cap(f_3(x_0, \ldots, x_3) = 0)$ in $\mathbb{P}^4$, and $Z = (g=0)\cap(f_3=0)$ in $\mathbb{P}^3$. But this is absurd because $\dim \bar{X} = 2$ and $\dim Z = 2$. Hence $A \neq 0$ and we can take $f = f_3 + (x_0^2x_4 - x_4^3)$.

Then $D = Z \cap (x_0 = 0) = (\bar{g}_2 = 0) \subset \mathbb{P}^2 = (x_0 = 0) \subset \mathbb{P}^3$. Since D generates the plane $(x_0 = 0)$ (cf. (3.5)), $\bar{g}_2$ must be irreducible. So we may assume $\bar{g}_2 = x_2^2 - x_1x_3$. This completes the proof of (4.21). $\square$

Then the defining equation of $\bar{C} := \bar{X} \cap (x_0 = 0)$ in $\mathbb{P}^3 = (x_0 = 0)$ is:

$$\bar{f}_3(x_1, x_2, x_3) - x_4^3 = 0, \quad x_2^2 - x_1x_3 = 0. \tag{4.22}$$

By construction, $\bar{C}$ must be integral.

Claim (4.23) *The following conditions are equivalent to one another:*
(1) $\bar{C}$ *defined by* (4.22) *is integral,*
(2) $\bar{f}_3(1, T, T^2)$ *is not contained in* $k[T^3]$.
Moreover under these conditions $\bar{C}$ *is a rational curve with* $p_a(\bar{C}) = 4$.

Proof. Clearly $\bar{C}$ is birationally isomorphic to the affine plane curve defined by $\bar{f}_3(1, T, T^2) - S^3 = 0$. Then $\bar{C}$ is integral if and only if $\bar{f}_3(1, T, T^2) - S^3$ is an irreducible polynomial. Then the proof of (4.23) is similar to (4.9). □

Also by construction $(\operatorname{Sing} \bar{X}) \cap (x_0 = 0)$ must be empty.

Claim (4.24) *The following conditions are equivalent to one another:*
(1) $(\operatorname{Sing} \bar{X}) \cap (x_0 = 0)$ *is empty,*
(2) *the system of equations*

$$x_2^2 = x_1 x_3, \quad \operatorname{rank}\begin{pmatrix} bx_1 + cx_2 + dx_3 & -x_3 & -x_2 & -x_1 \\ \bar{f}_2 & \dfrac{\partial \bar{f}_3}{\partial x_1} & \dfrac{\partial \bar{f}_3}{\partial x_2} & \dfrac{\partial \bar{f}_3}{\partial x_3} \end{pmatrix} \leqq 1$$

has no solutions except for $(0, 0, 0)$.

Proof. It is enough to note the following (4.25):

$\operatorname{Sing} \bar{X}$

$$= \left\{ P = [x_0 : x_1 : x_2 : x_3 : x_4];\ \begin{array}{c} g|_P = 0, \quad f_3|_P = 0, \\ \operatorname{rank}\begin{pmatrix} \partial g/\partial x_i \\ \partial f_3/\partial x_i \end{pmatrix}_{i=0,\ldots,4} (P) \leq 1 \end{array} \right\}$$

$$= \left\{ P = [0 : x_1 : x_2 : x_3 : x_4];\ \begin{array}{c} x_2^2 = x_1 x_3, \quad x_4^3 = \bar{f}_3(x_1, x_2, x_3), \\ \operatorname{rank}\begin{pmatrix} bx_1 + cx_2 + dx_3 & -x_3 & x_2 & -x_1 \\ \bar{f}_2 & \dfrac{\partial \bar{f}_3}{\partial x_1} & \dfrac{\partial \bar{f}_3}{\partial x_2} & \dfrac{\partial \bar{f}_3}{\partial x_3} \end{pmatrix} \leqq 1 \end{array} \right\}$$

$$\cup \{ Q = [1 : d : c : b : \gamma];\ a + bd - c^2 = 0, \quad f_3(1, d, c, b) + \gamma - \gamma^3 = 0 \}. \quad \square$$

Remark (4.26) Under the condition of (4.23), the condition of (4.24)(2) is satisfied by almost all $(b, c, d, \bar{f}_2)$. □

To finish the proof of (1.6)(4), it is enough to show the following:

Claim (4.27) *The surface* $\bar{X}$ *defined in* (1.6)(4-*a*) *is integral and has only rational double points as its singularities.*

Proof. Similar to the proof of (4.19) by using (4.24). □

References

1. Katsura, T.: Generalized Kummer surfaces and their unirationality in characteristic p. J. Fac. Sci. Univ. Tokyo Sect. IA **34**, 1–41 (1987)
2. Néron, A.: Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. Publ. Math. Inst. Hautes Etud. Sci. **21** (1964)
3. Nikulin, V.: Finite groups of automorphisms of Kählerian surfaces of type K3. Trans. Mosc. Math. Soc. **38**, 71–135 (1980)
4. Saint-Donat, B.: Projective models of K3 surfaces. Am. J. Math. **96**, 602–639 (1974)

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