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STABILITY AND EFFICIENT DESIGN OF CYLINDRICAL SHELLS OF METAL COMPOSITES SUBJECT TO COMBINATION LOADING

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A study is made of the stability of boron–aluminum shells under a combination of axial compression and uniform external pressure. An approximate theoretical model is constructed to describe the deformation of a layer of a fiber composite consisting of elastoplastic components. The model is used to derive the equations of state of multilayered shells reinforced by different schemes. The nonlinear equation describing the subcritical state is solved by the method of discrete orthogonalization with the use of stepped loading. The homogeneous problem is also solved by discrete orthogonalization. It is shown that shells can be efficiently designed for combination loading by plotting the envelope of the boundary curves for specific reinforcement schemes. The envelope is convex for elastic shells and is of variable curvature for elastoplastic shells.

Several requirements must be met when calculating the stress–strain state and stability of laminated shells made of composites. First of all, it is necessary to choose a suitable model of deformation of the composite at the level of an elementary layer, that model then being used to construct a nonlinear theory of shells comprised of a set of layers differing in their physico-mechanical properties or directions of reinforcement. Secondly, the design method should make it possible to find efficient variants for the distribution of the components by layer and the reinforcement scheme in order to increase the load-carrying capacity of the shells. These problems have been solved with a fairly high degree of accuracy in numerous studies [5, 6, 9, 11] for composites whose components are deformed elastically. However, substantially less progress has been made in solving the same problems for composites with elastoplastic components. This situation is attributable mainly to the complexity of constructing a satisfactory model of the deformation of a composite beyond the elastic limit. Theoretical models in which the composite is regarded as an anisotropic elastoplastic body without allowance for its structural features have a limited range of application. In a number of cases, it has been possible to relate the corrected characteristics of an inelastic composite with the properties of its components. This method has been used in particular to describe the deformation of fiber composites [1, 3, 4, 11]. However, one shortcoming of the approaches employed in these studies is the failure to account for the accumulation of plastic strains in the components due to the nonuniformity of the stress–strain state at the microstructural level. The use of numerical methods can resolve this problem but also seriously complicates the design of laminated structures. The most efficient method of calculating corrected characteristics is probably a method which uses simplifying hypotheses but still provides satisfactorily agreement with the empirical data. It was shown in [1, 2] that good results in this regard can be obtained by using the hypotheses of the method of thin sections [9] to construct the relations linking the averaged stresses and strains for a unidirectional monolayer. The stability of pinned laminated cylindrical boron–aluminum shells was calculated in [2, 3] in describing the deformation of a monolayer on the basis of this approximation. External pressure and axial compression were examined separately with allowance for the effect of temperature. The stability of cylindrical and conical boron–aluminum shells under such loads was studied in [3, 4] with different boundary conditions.

Below, we present the solution of a problem on the stability of cylindrical shells of metal composites subject to a combination of axial compression and external pressure. We should mention that the case of combination loading has also received little attention in studies of problems on the stability of elastically deformed composite shells.

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1. We assume that the shell is composed of N layers of reinforced fibrous material. The local coordinate system for each layer is oriented so that axis 1 is directed along the fibers, axis 2 is directed perpendicular to the fibers, and axis 3 is perpendicular to the plane determined by the first two axes. The coordinate trihedral 1, 2, 3 is right-handed. This system is rotated about the axis 3 relative to the coordinate system of the shell x, y, z, the direction of that axis coinciding with the direction of the z axis. Axis 1 and the x axis form the angle φ. We are examining balanced packets of layers formed by successively laying fibers at the angles φ and $-\varphi$. We derive the equations of state for the shell on the basis of the relations between the stresses and the strains in an elementary layer. These relations, written in increments, have the form

$$d\varepsilon = [\, a_{ij} \,]\, d\sigma , \tag{1.1}$$

where

$$\varepsilon = [\, \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12} \,]^T; \quad \sigma = \left[\sigma_{11}, \sigma_{22}, 0, \sigma_{23}, \sigma_{13}, \sigma_{12}\right]^T,$$

$$i, j = 1, \dots, 6.$$

Representing the equations of state in this form is acceptable for elastic composites and elastoplastic composites if the appropriate theory is used in each layer.

It is assumed that all of the increments are infinitesimal and that the constitutive equations can be linearized. In a quasistatic loading process, the coefficients a_{ij} depend on the stress state and the loading history prior to the current loading. A quasistatic loading process can be regarded as consisting of small load steps. In each step, we calculate the small but finite increments of the stresses and strains with constant values for the coefficients a_{ij}, which depend on the stresses and strains summed over all of the preceding steps. If the matrix $[\, a_{ij} \,]$ is not degenerate, then it can be used to perform all of the same operations as the elastic compliance matrix. The coefficients of $[\, a_{ij} \,]$ are the effective "instantaneous" compliances of a reinforced layer and are obtained on the basis of the hypothesis of thin sections [8] with $\sigma_{33} = 0$. In accordance with [9], we use the following equilibrium equations and strain compatibility equations within a given layer:

$$d\sigma_{11} = \xi\, d\sigma_{11}^{(a)} + \eta\, d\sigma_{11}^{(s)}, \quad d\varepsilon_{11} = d\varepsilon_{11}^{(a)} = d\varepsilon_{11}^{(s)},$$

$$d\sigma_{ij} = d\sigma_{ij}^{(a)} = d\sigma_{ij}^{(s)}, \quad d\varepsilon_{ij} = \xi\, d\varepsilon_{ij}^{(a)} + \eta\, d\varepsilon_{ij}^{(s)} \quad (i, j = 1; 2; 3; \;\; i = j \neq 1). \tag{1.2}$$

Here, ξ is the volume content of fibers; $\eta = 1 - \xi$ is the volume content of binder. The indices a and s denote the stresses and strains in the fibers and the binder.

If the relationship between the increments of the stresses and strains in the components is represented in the form (1.1) with the corresponding indices, then we can use Eqs. (1.2) to find the values of the "instantaneous" effective compliances

$$a_{11} = \left[\frac{\xi}{a_{11}^{(a)}} + \frac{\eta}{a_{11}^{(s)}}\right]^{-1}, \quad a_{ij} = \xi\, a_{ij}^{(a)} + \eta\, a_{ij}^{(s)} - \xi\eta \frac{(a_{1i}^{(a)} - a_{1i}^{(s)})(a_{1j}^{(a)} - a_{1j}^{(s)})}{a_{11}^{(a)} a_{11}^{(s)}} a_{11},$$

$$(i, j = 1, \dots; \;\; i = j \neq 1), \quad a_{ij} = a_{ji}. \tag{1.3}$$

Linearized constitutive equations for the components of the material can be obtained by using one of the theories of plasticity. Below, we assume that the relations of the theory of small elastoplastic strains is valid for the fibers and the matrix. We represent the relationship between the increments of the components of the strain and stress tensors in the form [8]

$$d\varepsilon_{ij} = \left[\frac{3}{2}\frac{1}{E_s}\delta_{in}\delta_{jm} + \frac{1}{3}\left(\frac{1}{3K} - \frac{3}{2}\frac{1}{E_s}\right)\delta_{ij}\delta_{nm} + \frac{9}{4}\left(\frac{1}{E_t} - \frac{1}{E_s}\right)\frac{\hat{\sigma}_{ij}\hat{\sigma}_{nm}}{\sigma_u^2}\right] d\delta_{nm}$$

at $d\sigma_u > 0$;

$$d\varepsilon_{ij} = \frac{1+\nu}{E}\left(\delta_{in}\delta_{jm} - \frac{\nu}{1+\nu}\delta_{ij}\delta_{nm}\right)d\sigma_{nm} \tag{1.4}$$

at $d\sigma_u < 0$;

$$E_s = \frac{\sigma_u}{\varepsilon_u}; \quad E_t = \frac{d\sigma_u}{d\varepsilon_u}.$$

Here, E_t and E_s represent the tangent and secant moduli on the $\sigma_u \sim \varepsilon_u$ curve; δ_{ij} is the Kronecker symbol. Equation (1.4) can be used to obtain expressions for the coefficients a_{ij} if we change over to matrix notation (1.1). The carbon fibers, boron fibers, and whisker crystals, which are often used as reinforcing elements, have anisotropic properties. The relations between the strain and stress increments for these materials can also be written in the form (1.1), but the expressions for a_{ij} will be equal to the corresponding components of generalized Hooke's law for an anisotropic body with a specific type of symmetry [8]. We should point out that average values of the volume contents of fibers and matrix in an elementary layer are used in Eqs. (1.3).

The accuracy of Eqs. (1.3) can be checked in the case of elastic components by comparing them with more accurate formulas obtained in [5, 6, 9]. It turns out that the compliances a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, and a_{33} calculated in accordance with (1.3) agree well with the results obtained from those formulas, while the compliances in shear turn out to be too high (by 10–20%). This is not an impediment to the use of Eqs. (1.3), since the experimentally determined mechanical characteristics of composites [5] also differ from the calculated values by 10–20%.

We obtain the resolvent system of equations for a laminated shell on the basis of the Timoshenko hypothesis, as was done in [5, 6]. We use the method of successive loadings to linearize the nonlinear equilibrium equations and the nonlinear relations between the increments and the strains in [6].

Taking into account the symmetry of the structure of the shell with respect to its layers and the fact that the transverse shear stresses are significantly smaller than the longitudinal shear stresses, we regard the packet as a whole as being orthotropic. The increments of the integral force factors will be related to the increments of the strains of the reference surface by a linear transformation

$$\begin{bmatrix} dT \\ dM \end{bmatrix} = \begin{bmatrix} A & E \\ E & D \end{bmatrix} \cdot \begin{bmatrix} d\varepsilon \\ d\kappa \end{bmatrix}, \tag{1.5}$$

where T, M, ε, and κ are vectors whose components are respectively the forces $\overline{T}_{11}$, $\overline{T}_{22}$, S, $\overline{T}_{23}$, and $\overline{T}_{13}$, moments M_{11}, M_{22}, M_{12}, and M_{21}, strains $\tilde{\varepsilon}_{11}$, $\tilde{\varepsilon}_{22}$, $\tilde{\varepsilon}_{12}$, $\tilde{\varepsilon}_{23}$, and $\tilde{\varepsilon}_{13}$, and curvatures and torsion κ_{11}, κ_{22}, τ_1, τ_2. The expressions for the coefficients of matrices A, E, and D have the same form as for the matrices in [6], but here the dependence of those coefficients on the stresses and strains is taken into account in accordance with the chosen model of deformation of the unidirectional layer.

Since the elementary layers are assumed to be thin, during the loading process the intensity of the stresses and strains in the components is monitored only on the middle surface of each layer. The subcritical state of the shell is axisymmetric, but the existence of edge effects also causes the stresses and strains to change along the generatrix. In this direction, we choose a large number of points at which to determine the "instantaneous" compliances of the layers. Then the stiffnesses of the layers and the corrected stiffnesses of the shell are calculated. The values of stiffness between the points are found by using the quadratic approximation. This approximation must be used because the resolvent system of equations is reduced to normal form and is solved by the Runge–Kutta method, with the vector-solutions being converted to orthogonal form by Godunov's method.

Stepped loading can be used to find the limit load, at which the deflections of the shell begin to increase rapidly. This is particularly important for shells that are deformed beyond the elastic limit. However, in the case of reinforced shells, it is also very important to find the load at which the initial axisymmetric state may branch to a nonaxisymmetric trajectory. This load can be found by solving the initial system of equations after it has been linearized by Euler's method. In this case, the increments of the strains and stresses are infinitesimal, and the boundary-value problem becomes homogeneous. This problem is also solved by the method of discrete orthogonalization.

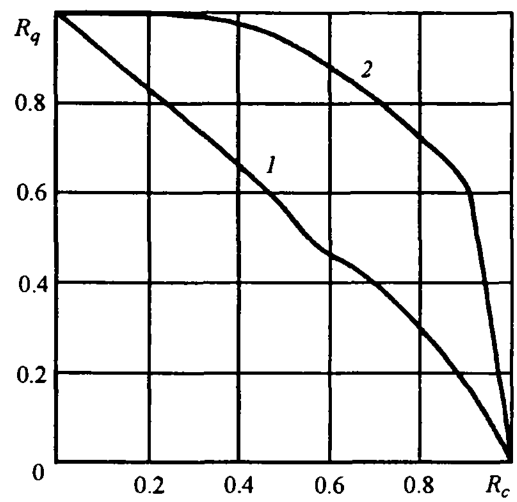

Fig. 1

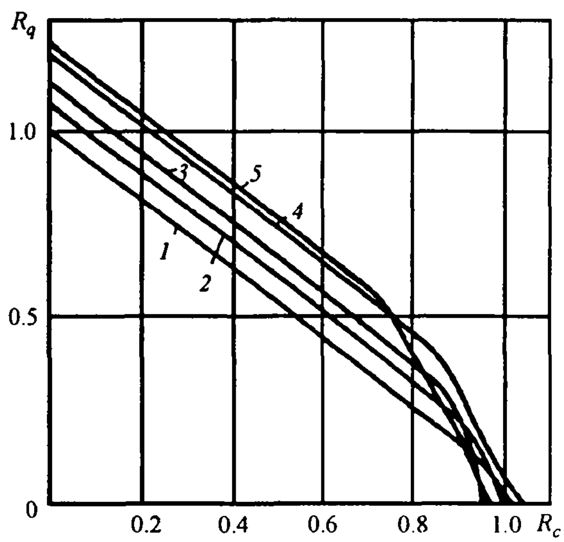

Fig. 2

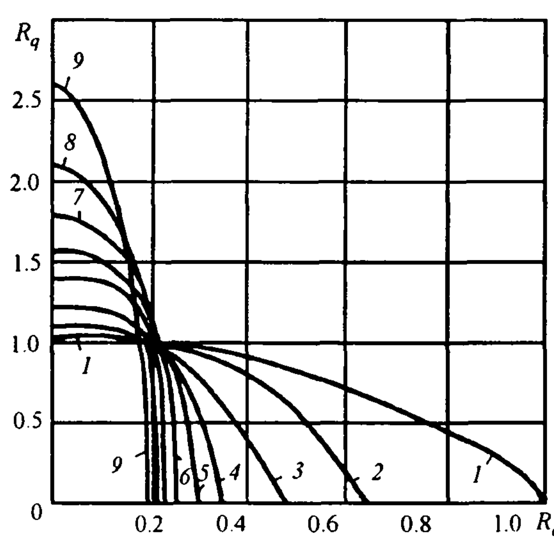

Fig. 3

2. The equations and methods of solution presented above were used to study the stability of cylindrical boron–aluminum shells made by orienting unidirectional layers at angles $\pm\varphi$ relative to the generatrix. The volume content of fibers was taken equal to $\xi = 0.5$. The boron fibers deform elastically, having the elastic modulus $E^{(a)} = 0.377 \cdot 10^6$ MPa and Poisson's ratio $\nu^{(a)} = 0.1$. The binder, composed of alloy D16T, has a nonlinear stress–strain curve in uniaxial compression. This curve is approximated well by the expression [13]

$$\varepsilon_{11} = c_1 \sigma_{11} + c_2 \sigma_{11}^7, \tag{2.1}$$

where $c_1 = 13.456 \cdot 10^{-6}$ MPa^{-1} and $c_2 = 1.107 \cdot 10^{-20}$ MPa^{-7}.

During the initial stage of deformation, the elastic modulus of the alloy $E^{(s)} = 0.7432 \cdot 10^5$ MPa and Poisson's ratio $\nu^{(s)} = 0.32$. Figure 1 shows the boundary curves for a cylindrical shell made of this material with $Z = 10$. Curve *1* was calculated with the assumption that Hooke's law is valid, while curve *2* was calculated with allowance for the plasticity of the material in accordance with the theory of small elastoplastic strains. It can be seen that in the elastic case the boundary curve differs

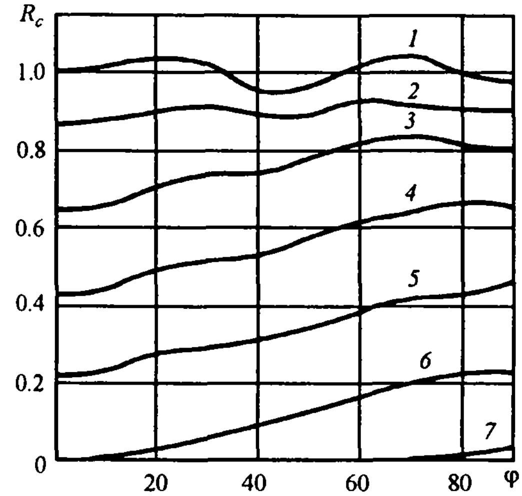

Fig. 4

negligibly from the straight line $R_c + R_q = 1$. Deformation of the material beyond the elastic limit changes the form of the stability curve: it becomes very convex. This fact has long been known from numerous theoretical and experimental studies [7].

Figures 2 and 3 show the form of the boundary curves for boron–aluminum shells. Calculations were performed for a pinned shell with the following parameters:

$$\frac{R}{L} = 0.5; \quad \frac{R}{t} = 25; \quad N = 6; \quad t = 0.00133 \text{ m}.$$

At $\varphi = 0$, the critical value of the axial force is equal to $T_{11,c}^{(0)} = 4.098$ MN/m and the intensity of the external pressure $q_c^{(0)} = 23.7$ MPa without allowance for the plasticity of the binder; accordingly, $T_{11,cn}^{(0)} = 2.362$ MN/m and $q_{cn}^{(0)} = 9.3$ MPa with allowance for its plasticity. In Fig. 2, which shows curves describing the interaction for elastic shells, the parameter R_c is the ratio of the acting axial force to $T_{11,c}^{(0)}$ and the parameter R_q is the ratio of the intensity of external pressure to the value of $q_c^{(0)}$. Curves *1, 2, 3, 4*, and *5* correspond to angles of reinforcement $\varphi = 0$, 30, 50, 70, and 90°. With an increase in the angle φ, the straight section of the curves initially undergoes a discontinuity for decreasing values of R_c. However, this relation is irregular in character, since the dependence of the critical value of the axial load is also irregular. The irregular nature of the indicated relation is evident from Fig. 4, which shows the change in the parameter R_c relative to the angle φ for a prescribed value of the parameter R_q. Curves *1, 2, 3, 4, 5, 6*, and *7* were obtained for $R_q = 0$, 0.2, 0.4, 0.6, 0.8, 1, and 1.2, respectively. In the absence of external pressure, the largest values of the critical axial force are seen when the fibers are laid at angles of 20 and 70°. The application of external pressure disturbs this relation, shifting the maximum in the direction of large angles. At $R_q > 0.6$, the maximum of the axial load is shifted toward $\varphi = 90°$.

Allowing for the plasticity of the binder when calculating the stability of boron–aluminum shells significantly alters the form of the boundary curves (Fig. 3). Here, R_c and R_q are dimensionless parameters characterizing specific critical values of the axial force and the intensity of external pressure in relation to the above-cited values of $T_{11,cn}^{(0)}$ and $q_{cn}^{(0)}$. Curves *1–9* in Fig. 3 were calculated for $\varphi = 0$, 20°, and higher values separated by increments of 10°. Comparing the boundary curves in Figs. 2 and 3, we see that allowing for the plasticity of the binder appreciably affects the form of these curves. The anisotropy of the material is also very pronounced in this case. This is due to the fact that for an axial load or external pressure acting separately, the optimum reinforcement direction coincides with the generatrix of the cylinder in the first case and with the directrix of the cylinder in the second case [2, 3]. Figures 5 and 6 illustrate some interesting features of the interaction of the loads discovered in determining the critical states. Figure 5 shows the dependence of the critical axial loads on the

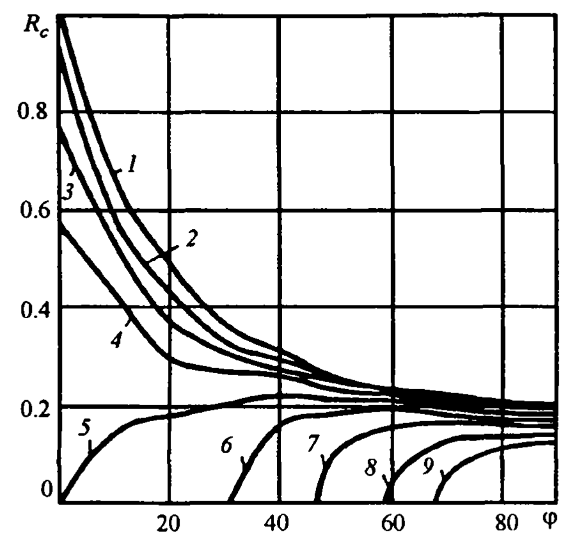

Fig. 5

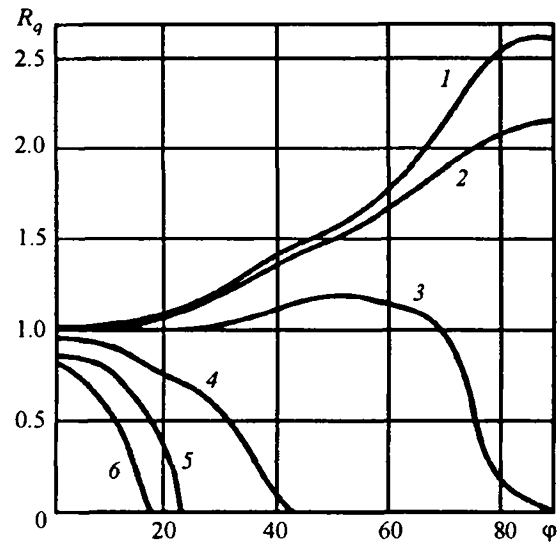

Fig. 6

reinforcement angle φ for a specified external pressure, while Fig. 6 shows the relation for critical external pressure with a specified axial load. The ith curve was calculated for $R_q = 0.25\ (i-1)$ in Fig. 5 and $R_c = 0.1\ (i-1)$ in Fig. 6. It can be seen that at $R_q < 1$ the optimum reinforcement is the same as in the absence of external pressure. If $R_q = 1$, then the largest axial load will be obtained at $\varphi \approx 40°$. When $R_q > 1$, the maximum of the curves shifts toward $\varphi = 90°$.

When the critical axial loads are calculated, it is found that even at $R_c > 0.2$ the optimum reinforcement becomes that which corresponds to pure axial compression ($\varphi = 0°$). However, the maximum occurs at $\varphi \approx 50°$ when $R_c = 0.2$. This is shown by the boundary curves in Fig. 3. The envelope of these curves consists of sections corresponding to different reinforcement angles. There may be different optimum reinforcement schemes for angles $40 \le \varphi \le 80°$ within the range $R_q < 1.7$ and $R_c > 0.25$. The layers must be arranged so that the fibers are oriented along the directrix when $R_q > 1.7$ and $R_c < 0.2$ and along the generatrix when $R_q < 1$ and $R_c < 0.25$. When the plasticity of the binder is ignored, the envelope of the boundary curves for the given shell (Fig. 2) consists of two sections. It is necessary that $\varphi = 90°$ when $R_q > 0.5$ and $R_c < 0.67$, while the angle $\varphi = 70°$ will be the most efficient angle when $R_q < 0.5$ and $R_c > 0.67$.

Thus, the above analysis of the stability of a specific boron–aluminum shell showed that the most efficient reinforcement schemes for combination loading depend on the ratio of the acting loads and differ appreciably for elastic shells and shells which deform beyond the elastic limit. As regards the curves obtained in this study, we note that they were constructed without distinguishing between the limit loads and the critical loads at which branching occurs.

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