Robust Hierarchical A Posteriori Error Estimators for Stabilized Convection–Diffusion Problems

B. Achchab,[1] **A. Agouzal,**[2] **M. El Fatini,**[1] **A. Souissi**[3]

[1] *Université Hassan 1er, LM2CE and LAMSAD, FSJES and ESTB, BP 218, Berrechid, Morocco*

[2] *Université de Lyon; CNRS; Université Lyon 1; Institut Camille Jordan, F-69622 Villeurbanne cedex, France*

[3] *GAN, LMA, FSR et LERMA, EMI, Université Mohammed V-Agdal, Rabat BP 1014, Morocco*

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We construct a hierarchical a posteriori error estimator for a stabilized finite element discretization of convection-diffusion equations with height Péclet number. The error estimator is derived without the saturation assumption and without any comparison with the classical residual estimator. Besides, it is robust, such that the equivalence between the norm of the exact error and the error estimator is independent of the meshsize or the diffusivity parameter. Numer Methods Partial Differential Eq 28: 1717–1728, 2012

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I. INTRODUCTION

Since the pioneering work of Babuska and Reinboldt [1], many works were devoted to hierarchical error estimators [2–9]. The estimator here is based on saturation assumption. In Ref. [10] a robust hierarchical estimator was constructed. The proof of robustness, however, relies on saturation assumption uniformly in coefficient diffusion. In Refs. [11, 12] a direct proof of reliability, efficiency, and robustness is given without saturation assumption and without comparison with the classical residual estimator.

Correspondence to: B. Achchab, LM2CE, FSJES, Université Hassan I, Settat B.P. 784, Morocco (e-mail: achchab@yahoo.fr)
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Here, we deal with the convection-diffusion equations in height Péclet number. Because the standard Galerkin approximation fails for such problems, many cures have been proposed in the literature, including streamline–diffusion (SD) [13, 14], subgrid viscosity (SGV) [15], residual free bubbles (RFB) [16], and face penalty (FP) [17]. For the SD method, a residual error estimator was constructed in Ref. [18] and robustness is recovered uniformly to coefficient of diffusion in Ref. [19]; see also Ref. [20,21] for Orthogonal subscales (OSS), Ref. [22] for (SGV), Ref. [23] for (RFB), or the work of [24] for (FP) stabilizations; see also other works [25, 26].

In this work, we focus our attention on robustness, in the frame of hierarchical error estimator for the convection-diffusion equations, stabilized by the SD method. Combining the works of [11, 12], we obtain a robust error estimates, such that equivalence between the norm of the exact error and error estimator is independent of the meshsize and the diffusivity parameter. To ensure this robustness, we control the convective term by the dual norm related to the energy norm.

The outline of the article is as follows: in Section II, we state the problem, and we recall some technical lemmas, which will be important in the construction of our estimators; in Section III, we present our main results; conclusion is reached in Section 4.

II. SETTING THE PROBLEM

Let Ω to be an open bounded domain of $\mathbb{R}^d$ $(d = 2, 3)$. We consider the model problem:

$$(P)\begin{cases} -\epsilon\Delta u + b\cdot\nabla u + \sigma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega), \sigma \in L^\infty(\Omega)$, and $b \in (W^{1,\infty}(\Omega))^d$. We assume there are two positive constants c_1 and c_2 not depending on ϵ such that

$$\|\sigma\|_\infty \le c_2 \quad \text{and} \quad -\frac{1}{2}\text{div}b + \sigma \ge c_1.$$

Let $(\mathcal{T}_h)_h$ to be a regular triangulation by d-simplex elements of Ω. We denote by $\mathcal{N}$ the set of internal vertices and $\mathcal{E}$ the set of internal edges (faces) of $\mathcal{T}_h$, for each $T \in \mathcal{T}_h$, the set E_T denotes the internal faces of T, whereas n_T is the unit external normal of ∂T and $\Delta(T)$ is the set of elements of $\mathcal{T}_h$ sharing an edge (face) with T. We denote by $[.]_F$ the jump of functions across the face F, and we set $[\frac{\partial u_h}{\partial n_T}]_F = 0$ if F is a boundary edge (face). For each $T \in \mathcal{T}_h$, we denote by $P_k(T)$ the polynomial space of degree at most k. For each $T \in \mathcal{T}_h$ and $E \in E_T$ an internal edge (face) of T with vertices $(Q_1, \ldots, Q_d)$ $((Q_0, Q_1, \ldots, Q_d)$ are the vertices of $T)$, we introduce the d-simplex T_E with vertices $(P, Q_1, \ldots, Q_d)$ where P is the point with barycentric coordinates:

$$\lambda_0(P) = \delta_{E,T} \quad \text{and} \quad \lambda_i(P) = \frac{1-\delta_{E,T}}{d}, \quad \text{for } i = 1, \ldots, d,$$

where $\{\lambda_i\}$ are the standard barycentric coordinates associated with the vertices $\{Q_i\}_{i=0}^d$ and

$$\delta_{E,T} = \min\left\{\frac{1}{d+1}, \frac{\sqrt{\epsilon}}{h_T}\right\}.$$

For each $T \in \mathcal{T}_h$ and all $E \in E_T$, we introduce the unique function $b_{E,T} \in \mathcal{C}^0(T)$ defined by:

$$b_{E,T} \in P_d(T_E), \quad b_{E,T} = 0 \text{ on } T/T_E, \quad b_{E,T}(a_E) = 1,$$

where a_E is the barycentric center of the edge (face) E. We consider the space $P_d^0(T)$ spanned by the functions $\{b_{E,T}, E \in E_T, b_T\}$, where b_T is the standard bubble function. In the following analysis, we denote by $C, C_0, C_1, \ldots$ positive generic constants independent of ϵ but depending only on the minimal angle of $\mathcal{T}_h$ and may be changed in different occurrences. For $\omega \subset \Omega$, we denote by $\|.\|_{\epsilon,\omega}$, the norm defined in $H^1(\omega)$ by

$$\forall v \in H^1(\omega), \quad \|v\|_{\epsilon,\omega}^2 = \epsilon |v|_{1,\omega}^2 + \|v\|_{0,\omega}^2,$$

and we set

$$\||v\||_{\epsilon,\omega}^2 = \|v\|_{\epsilon,\omega}^2 + \|b \cdot \nabla v\|_{-1,\omega},$$

where

$$\|b \cdot \nabla v\|_{-1,\omega} = \sup_{w \in H_0^1(\omega)} \frac{\langle b \cdot \nabla v, w \rangle_{0,\omega}}{\|w\|_{\epsilon,\omega}}.$$

We define the following discrete spaces:

$$V_h = \left\{ v_h \in H_0^1(\Omega); \forall T \in \mathcal{T}_h v_{h|T} \in P_1(T) \right\},$$

and

$$V_h^0 = \left\{ v_h \in H_0^1(\Omega); \forall T \in \mathcal{T}_h v_{h|T} \in P_d^0(T) \right\}.$$

We consider the problem

$$(P_h) \begin{cases} \text{Find } u_h \in V_h \text{ such that:} \\ \forall v_h \in V_h;\ a(u_h, v_h) + \sum\limits_{T \in \mathcal{T}_h} \delta_T (f - b \cdot \nabla u_h - \sigma u_h, b \cdot \nabla v_h) = \int_\Omega f v_h dx. \end{cases}$$

where $a(.,.)$ is the bilinear form defined on $(H_0^1(\Omega))^2$ by:

$$\forall (u, v) \in \left(H_0^1(\Omega)\right)^2, \quad a(u, v) = \int_\Omega \varepsilon \nabla u \cdot \nabla v + (b \cdot \nabla u + \sigma u) v dx,$$

and δ_T are positive reals satisfying

$$\forall T \in \mathcal{T}_h, \quad \delta_T \|b\|_{1,\infty,T} \leq C h_T.$$

To define our estimator, we consider the following problems defined for each element $T \in \mathcal{T}_h$ by:

$$(P_h)_T \begin{cases} \text{Find } w_T \in P_d^0(T) \text{ such that:} \\ \forall v_h \in P_d^0(T),\ \int_T (\epsilon \nabla w_T \cdot \nabla v_h + \sigma w_T v_h) dx & = \dfrac{1}{2} \sum\limits_{E \in E_T} \int_E [\epsilon \nabla u_h \cdot n] v_h d\gamma \\ & + \int_T (b \cdot \nabla u_h + \sigma u_h - f) v_h dx. \end{cases}$$

For each open $\omega \subset \Omega$ and for each $g \in L^2(\Omega)$, we define

$$osc(g, \omega) := \left\{ \sum_{T \in \mathcal{T}_h, T \subset \omega} \alpha_T^2 \|g - g_T\|_{0,T}^2 \right\}^{\frac{1}{2}},$$

where

$$\forall T \in \mathcal{T}, \quad g_T = \frac{1}{\mathrm{mes}_d(T)} \int_T g dx \text{ and } \alpha_T = \min\left\{1, \frac{h_T}{\sqrt{\epsilon}}\right\}.$$

We recall the following Lemmas [11, 12].

Lemma 2.1. *For each $v \in H_0^1(\Omega)$, there is a unique element $\Pi v \in V_h^0$ satisfying*

$$\forall T \in \mathcal{T}_h, \quad \int_T (v - \Pi v) dx = 0 \quad \text{and} \quad \forall S \in \mathcal{E}, \int_S (v - \Pi v) d\sigma = 0. \tag{1}$$

Moreover,

$$\forall T \in \mathcal{T}_h, \quad \|\Pi v\|_{\epsilon,T} \leq C \left(\alpha_T^{-1} \|v\|_{0,T} + \epsilon^{1/4} \alpha_T{}^{-1/2} \sum_{E \in E_T} \|v\|_{0,E} \right),$$

and

$$\forall T \in \mathcal{T}_h, \forall v_h \in P_1(T), \quad \int_T \nabla v_h \cdot \nabla (v - \Pi v) dx = 0.$$

Lemma 2.2. *There is an operator I_h defined from $H_0^1(\Omega)$ to V_h such that for each $u \in H_0^1(\Omega)$, we have*

$$\left\{ \sum_{T \in \mathcal{T}_h} \left(\alpha_T^{-2} \|u - I_h u\|_{0,T}^2 + \sum_{e \in E_T} \frac{\epsilon^{1/2}}{\alpha_T} \|u - I_h u\|_{0,e}^2 \right) \right\}^{\frac{1}{2}} \leq C \|u\|_{\epsilon,\Omega}.$$

From the previous lemmas, we get

Lemma 2.3. *Let $v \in H_0^1(\Omega)$, we have*

$$\|v - I_h v\|_{\epsilon,\Omega} + \|\Pi(v - I_h v)\|_{\epsilon,\Omega} \leq C \|v\|_{\epsilon,\Omega}.$$

and

$$\left(\sum_{T \in \mathcal{T}_h} \alpha_T^{-2} \|v - I_h v - \Pi(v - I_h v)\|_{0,T}^2 \right)^{\frac{1}{2}} \leq C \|v\|_{\epsilon,\Omega}.$$

Proof. Using the results of Lemmas 2.1 and 2.2, we deduce the first inequality. On the one hand, by setting $e = v - I_h v$, we get

$$\|e - \Pi e\|_{0,T} \leq \|e - \Pi e\|_{\epsilon,T}.$$

On the other hand, using the fact that $\int_T (e - \Pi e) dx = 0$, we get

$$\|e - \Pi e\|_{0,T} \leq C h_T \|e - \Pi e\|_{1,T} \leq C \frac{h_T}{\sqrt{\epsilon}} \|e - \Pi e\|_{\epsilon,T},$$

then, we obtain

$$\|e - \Pi e\|_{0,T} \leq C \min\left(1, \frac{h_T}{\sqrt{\epsilon}}\right) \|e - \Pi e\|_{\epsilon,T} = C\alpha_T \|e - \Pi e\|_{\epsilon,T}.$$

We deduce that

$$\left(\sum_{T \in \mathcal{T}_h} \alpha_T^{-2} \|v - I_h v - \Pi(v - I_h v)\|_{0,T}^2\right)^{\frac{1}{2}} \leq C\|e - \Pi e\|_{\epsilon,\Omega} \leq C\|v\|_{\epsilon,\Omega}.$$

∎

III. ROBUST HIERARCHICAL A POSTERIORI ERROR ESTIMATOR

In the coming analysis, we need the following lemmas:

Lemma 3.1. *There is a constant C not depending of ϵ such that:*

$$\forall u \in H_0^1(\Omega), \quad C\| |u| \|_{\epsilon,\Omega} \leq \sup_{v \in H_0^1(\Omega)} \frac{a(u,v)}{\|v\|_{\epsilon,\Omega}}.$$

Proof. Let $u \in H_0^1(\Omega)$. There is a constant C_0 independent of ϵ such that

$$a(u,u) \geq C_0 \|u\|_{\epsilon,\omega}^2.$$

Let $v \in H_0^1(\Omega)$ a weak solution of the problem $-\epsilon \Delta v + v = b \cdot \nabla u$. Using v as a test function, we get

$$\int_\Omega v b \cdot \nabla u dx = \|v\|_{\epsilon,\Omega}^2 = \|b \cdot \nabla u\|_{-1,\Omega}^2,$$

then

$$\begin{aligned} a(u,v) &= \epsilon \int_\Omega \nabla u \cdot \nabla v dx + \sigma \int_\Omega u v dx + \|b \cdot \nabla u\|_{-1,\Omega}^2 \\ &\geq -C_1 \|u\|_{\epsilon,\Omega} \|v\|_{\epsilon,\Omega} + \|b \cdot \nabla u\|_{-1,\Omega}^2 \\ &\geq -\frac{C_1^2}{2} \|u\|_{\epsilon,\Omega}^2 - \frac{1}{2} \|v\|_{\epsilon,\Omega}^2 + \|b \cdot \nabla u\|_{-1,\Omega}^2 \\ &\geq -\frac{C_1^2}{2} \|u\|_{\epsilon,\Omega}^2 + \frac{1}{2} \|b \cdot \nabla u\|_{-1,\Omega}^2. \end{aligned}$$

We set $w = u + \frac{C_0}{C_1^2} v$. One verifies that, on the one hand

$$a(u,w) \geq \frac{C_0}{2} \|u\|_{\epsilon,\Omega}^2 + \frac{C_0}{2C_1^2} \|b \cdot \nabla u\|_{-1,\Omega}^2,$$

and on the other hand

$$\|w\|_{\epsilon,\Omega} \leq \|u\|_{\epsilon,\Omega} + \frac{C_0}{C_1^2}\|b\cdot\nabla u\|_{-1,\Omega}.$$

Hence, there is a constant independent of ϵ such that

$$C|||u|||_{\epsilon,\Omega} \leq \frac{a(u,w)}{\|w\|_{\epsilon,\Omega}},$$

so

$$C|||u|||_{\epsilon,\Omega} \leq \sup_{v\in H_0^1(\Omega)} \frac{a(u,v)}{\|v\|_{\epsilon,\Omega}}.$$

■

Lemma 3.2. *Let $T \in \mathcal{T}_h$, we set $R = b\cdot\nabla u_h + \sigma u_h - f$, we have*

$$\alpha_T\|R\|_{0,T} \leq C(osc(R,T) + \|w_T\|_{\epsilon,T}).$$

Proof. On the one hand, using b_T as a test function, we get

$$\int_T R.b_T dx = \int_T (\epsilon\nabla w_T\cdot\nabla b_T + \sigma w_T b_T)dx \leq \|w_T\|_{\epsilon,T}\|b_T\|_{\epsilon,T}.$$

We have

$$\|b_T\|_{0,T} \leq Ch_T^{d/2} \quad \text{and} \quad \|b_T\|_{1,T} \leq Ch_T^{\frac{d-2}{2}},$$

this implies

$$\|b_T\|_{\epsilon,T} \leq h_T^{d/2}\left(1 + \frac{\sqrt{\epsilon}}{h_T}\right) \leq Ch_T^{d/2}\alpha_T^{-1},$$

then

$$\int_T R\cdot b_T dx \leq Ch_T^{d/2}\alpha_T^{-1}\|w_T\|_{\epsilon,T}.$$

On the other hand,

$$\begin{aligned}\|R_T\|_{0,T} &\leq Ch_T^{-d/2}\left|\int_T R_T b_T dx\right| \leq Ch_T^{-d/2}\left(\int_T |R_T - R|b_T dx + \left|\int_T Rb_T dx\right|\right)\\ &\leq Ch_T^{-d/2}\left(h_T^{d/2}\mathrm{osc}(R,T) + h_T^{d/2}\alpha_T^{-1}\|w_T\|_{\epsilon,T}\right)\\ &\leq C(\mathrm{osc}(R,T) + \alpha_T^{-1}\|w_T\|_{\epsilon,T}.\end{aligned}$$

Using the triangular inequality and the fact that $\alpha_T \leq 1$, we get

$$\alpha_T\|R\|_{0,T} \leq C(\mathrm{osc}(R,T) + \|w_T\|_{\epsilon,T}).$$

■

Lemma 3.3. *Let $T \in \mathcal{T}_h$, we have:*

$$\forall v_h \in P_1(T),\ |\langle R, b \cdot \nabla v_h \rangle_{0,T}| \le C \frac{\|b\|_{1,\infty,T}}{h_T} (\mathrm{osc}(R,T) + \|w_T\|_{\epsilon,T}) \|v_h\|_{\epsilon,T}.$$

Proof. For each $v_h \in P_1(T)$, we have

$$|v_h|_{1,T} \le \frac{1}{\sqrt{\epsilon}} \|v_h\|_{\epsilon,T}, \qquad |v_h|_{1,T} \le C \frac{1}{h_T} \|v_h\|_{0,T} \le C \frac{1}{h_T} \|v_h\|_{\epsilon,T},$$

we get

$$\|b \cdot \nabla v_h\|_{0,T} \le \|b\|_{1,\infty,T} |v_h|_{1,T} \le C \frac{\alpha_T}{h_T} \|b\|_{1,\infty,T} \|v_h|_{\epsilon,T}.$$

Then

$$\begin{aligned} |\langle R, b \cdot \nabla v_h \rangle_{0,T}| &\le \|R\|_{0,T} \|b \cdot \nabla v_h\|_{0,T} \\ &\le C \frac{\alpha_T}{h_T} \|R\|_{0,T} \|b\|_{1,\infty,T} \|v_h|_{\epsilon,T} \\ &\le C \frac{\|b\|_{1,\infty,T}}{h_T} (\mathrm{osc}(R,T) + \|w_T\|_{\epsilon,T}) \|v_h\|_{\epsilon,T}. \end{aligned}$$

■

Our main result is the following theorem, which gives the upper bound of the error:

Theorem 3.4. *For $u_h \in V_h$ be the solution of problem (P), we get the following estimation*

$$|||u - u_h|||_{\epsilon,\Omega} \le C \left(\sum_{T \in \mathcal{T}_h} (\|w_T\|^2_{\epsilon,T} + osc^2(b \cdot \nabla u_h + \sigma u_h - f, T)) \right)^{\frac{1}{2}}.$$

Proof. First, we have

$$\begin{aligned} |||u_h - u|||_{\epsilon,\Omega} &\le \sup_{v \in H_0^1(\Omega)} \frac{a(u_h - u, v)}{\|v\|_{\epsilon,\Omega}} \\ &\le \sup_{v \in H_0^1(\Omega)} \frac{a(u_h, v - I_h v) - \int_\Omega f \cdot (v - I_h v) + a(u_h, I_h v) - \int_\Omega f I_h v dx}{\|v\|_{\epsilon,\Omega}}. \end{aligned}$$

Let $v \in H_0^1(\Omega)$, we set $e = v - I_h v$, and we define $r(T)$:

$$\forall T \in \mathcal{T}_h, \quad r(T) = \int_T (b \cdot \nabla u_h + \sigma u_h - f) dx.$$

Second, we have

$$
\begin{aligned}
& a(u_h, v - I_h v) - \int_\Omega f \cdot (v - I_h v) dx \\
&= \int_\Omega (\sigma u_h + b \cdot \nabla u_h - f) e dx + \epsilon \int_\Omega \nabla u_h \cdot \nabla e dx \\
&= \sum_{T \in \mathcal{T}_h} \left(\frac{1}{2} \sum_{T \in \mathcal{T}_h} \left[\epsilon \frac{\partial u_h}{\partial n_T} \right] e d\gamma + \int_T R e dx \right) \\
&= \sum_{T \in \mathcal{T}_h} \left(\frac{1}{2} \sum_{T \in \mathcal{T}_h} \left[\frac{\epsilon \partial u_h}{\partial n_T} \right] \Pi e d\gamma + \int_T R \Pi e + \int_T R(e - \Pi e) dx \right) \\
&= \sum_{T \in \mathcal{T}_h} \left(\int_T (\epsilon \nabla w_T \nabla \Pi e + \sigma w_T \Pi e) dx + \int_T (R - r(T))(e - \Pi e) dx \right).
\end{aligned}
$$

Using Lemma 3.2, we obtain

$$
a(u_h, v - I_h v) - \int_\Omega f (v - I_h v) dx
$$

$$
\leq C \left(\sum_{T \in \mathcal{T}_h} (\|w_T\|^2_{\epsilon,T} + \mathrm{osc}^2(b.\nabla u_h + \sigma u_h - f, T)) \right)^{\frac{1}{2}} \|v\|_{\epsilon,\Omega}.
$$

Let

$$
\sup_{v \in H^1_0(\Omega)} \frac{a(u_h, v - I_h v) dx - \int_\Omega f \cdot (v - I_h v)}{\|v\|_{\epsilon,\Omega}}
$$

$$
\leq C \left(\sum_{T \in \mathcal{T}_h} (\|w_T\|^2_{\epsilon,T} + \mathrm{osc}^2(b \cdot \nabla u_h + \sigma u_h - f, T)) \right)^{\frac{1}{2}}.
$$

As

$$
a(u_h, I_h v) - \int_\Omega f I_h v dx = \sum_{T \in \mathcal{T}_h} \delta_T (R, b \cdot \nabla I_h v)_{0,T}.
$$

Using Lemma 3.3, we get

$$
\begin{aligned}
\sum_{T \in \mathcal{T}_h} \delta_T (R, b \cdot \nabla I_h v)_{0,T} &\leq C \sum_{T \in \mathcal{T}_h} \delta_T \frac{\|b\|_{1,\infty,T}}{h_T} \|I_h v\|_{\epsilon,T} (\mathrm{osc}(R, T) + \|w\|_{\epsilon,T}) \\
&\leq C \left(\sum_{T \in \mathcal{T}_h} \left(\|w_T\|^2_{\epsilon,T} + \mathrm{osc}^2(b \cdot \nabla u_h + \sigma u_h - f, T) \right) \right)^{\frac{1}{2}} \|I_h v\|_{\epsilon,\Omega} \\
&\leq C \left(\sum_{T \in \mathcal{T}_h} \left(\|w_T\|^2_{\epsilon,T} + \mathrm{osc}^2(b \cdot \nabla u_h + \sigma u_h - f, T) \right) \right)^{\frac{1}{2}} \|v\|_{\epsilon,\Omega}.
\end{aligned}
$$

From the previous inequalities, we obtain

$$|||u-u_h|||_{\epsilon,\Omega} \leq C\left(\sum_{T\in\mathcal{T}_h}\left(\|w_T\|^2_{\epsilon,T} + \mathrm{osc}^2(b\cdot\nabla u_h + \sigma u_h - f, T)\right)\right)^{\frac{1}{2}}.$$

∎

To show the efficiency of our estimator, we need the following Lemmas:

Lemma 3.5. *Let $w_T \in P_d^0(T)$, there is an element $v_T \in H_0^1(\Delta_T)$, such that $\forall K \in \Delta_T$, $(v_T)_{|K} \in P_d^0(K)$ satisfying:*

$$v_T = w_T \ \ \textit{in } T, \textit{ and } \|v_T\|_{\epsilon,\Delta(T)} \leq \|v_T\|_{\epsilon,T}.$$

Proof. Let $v_T \in H_0^1(\Delta)$ be the unique function defined by

$$\forall K \in \Delta_T, (v_T)_{|K} \in P_d^0(K), \quad v_T = w_T \text{ in } T,$$

$$\forall K \in \Delta(T)/T, \quad \int_K v_T dx = 0,$$

anf for each internal face E that is not belonging to T, $\int_E v_T d\gamma = 0$.

Using the inverse inequality, we get

$$\forall K \in \Delta(T), \|v_T\|_{\epsilon,K} \leq C\sum_{E\in E_T} \epsilon^{\frac{1}{4}}\alpha_T^{-\frac{1}{2}}\|w_T\|_{0,E}.$$

As we have

$$\|w_T\|_{0,E} \leq Ch_T^{-\frac{1}{2}}\alpha_T\|w_T\|_{\epsilon,T},$$

we get

$$\sum_{E\in E_T} \epsilon^{\frac{1}{4}}\alpha_T^{-\frac{1}{2}}\|w_T\|_{0,E} \leq Ch_T^{-\frac{1}{2}}\alpha_T\epsilon^{\frac{1}{4}}\alpha_T^{-\frac{1}{2}}\|w_T\|_{\epsilon,T} \leq C\|w_T\|_{\epsilon,T},$$

and then

$$\|v_T\|_{\epsilon,\Delta(T)} \leq C\|w_T\|_{\epsilon,T}.$$

∎

Lemma 3.6. *we have*

$$\forall T \in \mathcal{T}_h, \quad \alpha_T\|R\|_{0,T} \leq C(osc(R,T) + |||u-u_h|||_{\epsilon,T}).$$

Proof. On the one hand, we have

$$\begin{aligned}\|R_T\|_{0,T} &\leq Ch_T^{-d/2}\left|\int_T R_T b_T dx\right| \\ &\leq Ch_T^{-d/2}\left(\int_T |R_T - R| b_T dx + |\int_T R b_T dx|\right) \\ &\leq C(\mathrm{osc}(R,T) + h_T^{-d/2}\left|\int_T R b_T dx\right|.\end{aligned}$$

On the other hand, as $b_T \in H_0^1(T)$, we have

$$\begin{aligned}
\int_T Rb_T dx &= \int_T (-\epsilon\Delta u_h + b\cdot\nabla u_h + \sigma u_h - f)b_T dx \\
&= \int_T (-\epsilon\Delta(u_h - u) + b\cdot\nabla(u_h - u) + \sigma(u_h - u))b_T dx \\
&= \int_T (\epsilon\nabla(u_h - u) + b\cdot\nabla(u_h - u) + \sigma(u_h - u))b_T dx \\
&\le \|b_T\|_{\epsilon,T} |||u - u_h|||_{\epsilon,T} \\
&\le C\alpha_T^{-1} h_T^{d/2} |||u - u_h|||_{\epsilon,T},
\end{aligned}$$

we deduce

$$\|R_T\|_{0,T} \le C\big(\mathrm{osc}(R,T) + \alpha_T^{-1}|||u - u_h|||_{\epsilon,T}\big).$$

Using the triangular inequality and the fact that $\alpha_T \le 1$, we obtain

$$\alpha_T\|R\|_{0,T} \le C(\mathrm{osc}(R,T) + |||u - u_h|||_{\epsilon,T}).$$

■

So, we get the following theorem.

Theorem 3.7. *For each $T \in \mathcal{T}_h$, we have*

$$\|w_T\|_{\epsilon,T} \le C_1|||u - u_h|||_{\epsilon,\Delta(T)} + C_2 osc(f,\Delta(T)),$$

Proof. Let $T \in \mathcal{T}_h$, and v_T the unique function of Lemma 3.5. We have

$$\begin{aligned}
\|w_T\|_{\epsilon,T}^2 &= \frac{1}{2}\sum_{E\in E_T}\int_E \left[\epsilon\frac{\partial u_h}{\partial n_T}\right] w_T dx + \int_T R w_T dx \\
&= \frac{1}{2}\sum_{E\in E_T}\int_E \left[\epsilon\frac{\partial u_h}{\partial n_T}\right] v_T dx + \int_T R w_T dx \\
&= \frac{1}{2}\int_{\Delta(T)} \epsilon\nabla u_h\nabla v_T dx + \int_T R v_T dx \\
&= \frac{1}{2}\int_{\Delta(T)} (\epsilon\nabla u_h\nabla v_T + b\cdot\nabla u_h v_T + \sigma u_h v_T - f)v_T dx dx - \frac{1}{2}\int_{\Delta(T)} R v_T dx + \int_T R v_T dx \\
&= \frac{1}{2}a(u_h - u, v_T) - \frac{1}{2}\int_{\Delta(T)} R v_T dx + \int_T R v_T dx.
\end{aligned}$$

First, using Lemma 3.5, we get

$$a(u_h - u, v_T) \le |||u - u_h|||_{\epsilon,\Delta(T)}\|v_T\|_{\epsilon,\Delta(T)} \le C|||u - u_h|||_{\epsilon,\Delta(T)}\|w_T\|_{\epsilon,T}.$$

Second, using Lemmas 3.5 and 3.6, we get for each $K \in \Delta(T)$,

$$\int_K R v_T dx \leq \|R\|_{0,K} \|v_T\|_{0,K} \leq \alpha_K \|R\|_{0,K} \|v_T\|_{\epsilon,K}$$
$$\leq C(\mathrm{osc}(R, K) + \||u - u_h\||_{\epsilon,K}) \|w_T\|_{\epsilon,T}.$$

Finally, using the previous inequalities, we obtain

$$\|w_T\|_{\epsilon,T} \leq C(\||u - u_h\||_{\epsilon,\Delta(T)} + \mathrm{osc}(f, \Delta(T))).$$

■

IV. CONCLUSION

In this work, we have performed and analyzed an a posteriori error estimators for convection-diffusion equations. This estimator of hierarchical type has the advantage that the robustness is recovered. By an appropriate choice of the finite element spaces and norms, the estimators yield an upper and lower bounds of the error. The estimator is robust with respect to the physical parameters of the problem. Specifically, this a posteriori analysis for the stabilized convection-diffusion problem establishes that the error is bounded uniformly in ϵ. Our main results show that the estimator is efficient and robust.

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