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On the Modified Westergaard Equations for Certain Plane Crack Problems

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ABSTRACT

An error in Westergaard's equation for a certain class of plane crack problems, originally pointed out by Sih, is briefly discussed anew. The source of the difficulty is traced to an oversight in an earlier work by MacGregor, upon whose work Westergaard based his equations. Several examples of interest illustrating the consequences of the necessary correction to these equations are given.

1. Introduction

The Westergaard equations, which apply for a certain class of plane problems in linear elasticity, were shown to be generally incorrect by Sih in 1966, [1]. Specifically, by use of the well known Goursat–Kolosov complex representation of the plane problem, it was shown that the stress and displacement field equations appropriate to the restricted class of problems alluded to above include a real constant term which is lacking in the Westergaard equations.

In this paper the constant term which, according to Sih's analysis, should be appended to Westergaard's equations, is shown to be the result of an oversight in a lesser known work of MacGregor [2], upon whose work Westergaard based his formulations [3]. The consequences of the corrected equations are then demonstrated for several familiar plane crack problems, and for the approximate plane crack-tip stress and displacement field equations.

The problem of the centrally cracked strip of finite width loaded unaxially in uniform tension is also discussed. A Westergaard type stress function is introduced which provides an approximate closed form solution. This approximate solution has the merits of yielding the Fedderson secant formula for the crack-tip stress intensity factor, and for providing an analytical expression for the crack opening displacement which closely matches experimental data and which is a considerable improvement over the calculation first introduced by Irwin [4].

2. Modified Westergaard Equations

In MacGregor's complex characterization of the plane problem (omitting body force) the holomorphic functions

$$\begin{aligned} J(z) &= \theta(x, y) + \mathrm{i}\Omega(x, y) \\ H(z) &= \theta_0(x, y) + \mathrm{i}\Omega_0(x, y) \end{aligned} \tag{1}$$

are introduced together with their derivatives

$$\begin{aligned} \mathrm{i}J'(z) \equiv \mathrm{i}W(z) &= \Phi(x, y) + \mathrm{i}\Psi(x, y) \\ H'(z) \equiv -K(z) &= -\Gamma(x, y) - \mathrm{i}\Pi(x, y)\,. \end{aligned} \tag{2}$$

The bi-harmonic Airy stress function $U(x, y)$ is represented as a linear combination of the single-valued harmonic functions θ and θ_0 by

$$U(x, y) = y\theta + \theta_0 = U(z, \bar{z}) = \frac{\mathrm{i}(\bar{z} - z)}{2} \operatorname{Im}[\mathrm{i}J(z)] + \operatorname{Re}[H(z)] \tag{3}$$

The complex representation of the plane stress field is then readily shown to be

$$\sigma_{xx} = 2\Phi + y\frac{\partial\Phi}{\partial y} + \frac{\partial\Gamma}{\partial x} = 2\,\mathrm{Re}\,[\mathrm{i}\,W(z)] - y\,\mathrm{Im}\,[\mathrm{i}\,W'(z)] + \mathrm{Re}\,[K'(z)]$$

$$\sigma_{yy} = -y\frac{\partial\Phi}{\partial y} - \frac{\partial\Gamma}{\partial x} = +y\,\mathrm{Im}\,[\mathrm{i}\,W'(z)] - \mathrm{Re}\,[K'(z)]$$

$$\sigma_{xy} = -\Psi - y\frac{\partial\Phi}{\partial x} + \frac{\partial\Gamma}{\partial y} = -\mathrm{Im}\,[\mathrm{i}\,W(z)] - y\,\mathrm{Re}\,[\mathrm{i}\,W'(z)] - \mathrm{Im}\,[K'(z)]\,. \tag{4}$$

For the restricted class of plane problems for which $\sigma_{xy}=0$ at all points along the line $y=0$, which includes plane crack problems in which the internal crack (or cracks) is situated along the x-axis and where the applied loads are symmetrically located with respect to the crack plane, it follows from (4) that

$$\Psi - \frac{\partial\Gamma}{\partial y} = \mathrm{Im}\,[\mathrm{i}\,W(z) + K'(z)] = 0\,. \tag{5}$$

Consequently

$$\Psi = \frac{\partial\theta}{\partial x} = \frac{\partial\Gamma}{\partial y} = -\frac{\partial\Pi}{\partial x}$$

from which it necessarily follows that

$$\frac{\partial\Phi}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial\Gamma}{\partial x}\right)$$

which is satisfied in the most general sense if one chooses

$$\Phi + A \equiv -\frac{\partial\Gamma}{\partial x} \tag{6}$$

or

$$\mathrm{Re}\,[\mathrm{i}\,W(z) + K'(z)] \equiv -A \tag{7}$$

everywhere. Here A is a real constant. The oversight in MacGregor's work rests in the fact that A was omitted or, put another way, was necessarily presumed to be zero. Substituting equations (5) through (7) into (4) and introducing

$$Z(z) \equiv \mathrm{i}\,W(z) \tag{8}$$

one obtains

$$\sigma_{xx} = \mathrm{Re}\,[Z(z)] - y\,\mathrm{Im}\,[Z'(z)] - A$$

$$\sigma_{yy} = \mathrm{Re}\,[Z(z)] + y\,\mathrm{Im}\,[Z'(z)] + A$$

$$\sigma_{xy} = -y\,\mathrm{Re}\,[Z'(z)] \tag{9}$$

which are the equations obtained by Sih when $Z(z) \equiv 2\phi'(z)$. Because the stress components are required to satisfy given boundary conditions the constant A will in general depend on the manner of the applied loading and will vanish only for rather special loading conditions.

The displacement field equations must likewise be corrected. In the Goursat–Kolosov representation the displacement field is specified by the well known form [5]

$$2\mu(u + \mathrm{i}\,v) = \kappa\phi(z) - z\,\overline{\phi'(z)} - \overline{\psi(z)} \tag{10}$$

where $u(x, y)$ and $v(x, y)$ are respectively the x- and y-components of the displacement vector, $\mu = E/2(1+\nu)$ is the shear modulus, E and ν are Young's Modulus and Poisson's Ratio respectively, and $\kappa = [3-\nu/1+\nu]$ for plane stress and $\kappa = [3\text{–}4\nu]$ for plane strain. The holomorphic functions $\phi(z)$ and $\psi(z)$ can be shown to be related to those introduced in equations (1), (2) and (3) by the relations

$$\begin{aligned} \mathrm{i}\,W(z) &= Z(z) = 2\phi'(z) \\ H(z) &= X(z) + z\phi(z) \\ H'(z) &= -K(z) = \psi(z) + \phi(z) + z\phi'(z) \\ X'(z) &= \psi(z)\,. \end{aligned} \tag{11}$$

Adding to (10) its complex conjugate in the case of plane stress, one obtains

$$\begin{aligned} Eu &= (3-\nu)\,\mathrm{Re}\,[\phi(z)] - (1+\nu)\{x\,\mathrm{Re}\,[\phi'(z)] + y\,\mathrm{Im}\,[\phi'(z)] + \mathrm{Re}\,[\psi(z)]\}\,. \\ Ev &= (3-\nu)\,\mathrm{Im}\,[\phi(z)] + (1+\nu)\{x\,\mathrm{Im}\,[\phi'(z)] - y\,\mathrm{Re}\,[\phi'(z)] + \mathrm{Im}\,[\psi(z)]\}\,. \end{aligned} \tag{12}$$

The Goursat–Kolosov equivalent of equations (5) and (7), with the help of (11), read

$$\mathrm{Im}\,[z\phi''(z) + \psi'(z)] = 0$$

$$\mathrm{Re}\,[z\phi''(z) + \psi'(z)] \equiv A$$

or

$$z\phi''(z) + \psi'(z) = A \tag{13}$$

everywhere. Integrating

$$Z\phi'(z) - \phi(z) + \psi(z) = Az + B$$

which is equivalent to the pair of equations

$$\begin{aligned} x\,\mathrm{Re}\,[\phi'(z)] - y\,\mathrm{Im}\,[\phi'(z)] - \mathrm{Re}\,[\phi(z)] + \mathrm{Re}\,[\psi(z)] &= Ax \\ y\,\mathrm{Re}\,[\phi'(z)] + x\,\mathrm{Im}\,[\phi'(z)] - \mathrm{Im}\,[\phi(z)] + \mathrm{Im}\,[\psi(z)] &= Ay\,. \end{aligned} \tag{14}$$

The constant B, which must be real, can be omitted because its retention merely serves to add to the displacement field a term which represents a rigid body displacement. Upon combining (12) with (14)

$$\begin{aligned} Eu &= 2(1-\nu)\,\mathrm{Re}\,[\phi(z)] - (1+\nu)2y\,\mathrm{Im}\,[\phi'(z)] - (1+\nu)Ax \\ Ev &= 4\,\mathrm{Im}\,[\phi(z)] - (1+\nu)\,2y\,\mathrm{Re}\,[\phi'(z)] + (1+\nu)Ay\,. \end{aligned} \tag{15}$$

To avoid confusion with the bar symbol used to denote complex conjugation let

$$2\phi(z) = \int Z(z)\mathrm{d}z \equiv \tilde{Z}(z) \tag{16}$$

where upon

$$\begin{aligned} Eu &= (1-\nu)\,\mathrm{Re}\,[\tilde{Z}(z)] - (1+\nu)\,y\,\mathrm{Im}\,[Z(z)] - (1+\nu)Ax \\ Ev &= 2\,\mathrm{Im}\,[\tilde{Z}(z)] - (1+\nu)\,y\,\mathrm{Re}\,[Z(z)] + (1+\nu)Ay \end{aligned} \tag{17}$$

emerge as the modified Westergaard field equations for plane stress.

3. Applications

To illustrate use of the modified Westergaard equations it is worth while to treat anew the familiar problem of the infinite plate with colinear periodic cracks as shown in Fig. 1. The factor k is any real number.

Using the Kolosov equations [5]

$$\sigma_{xx} + \sigma_{yy} = 2\{\phi'(z) + \overline{\phi'(z)}\} = 4\,\mathrm{Re}\,[\phi'(z)]$$

$$\sigma_{yy} - \sigma_{xx} + 2\mathrm{i}\,\sigma_{xy} = 2\{\bar{z}\phi''(z) + \psi'(z)\}\,. \tag{18}$$

The boundary conditions can be expressed as follows: For all points situated on any crack border

$$\sigma_{yy} + \mathrm{i}\,\sigma_{xy} = 2\,\mathrm{Re}\,[\phi'(z)] + \{\bar{z}\phi''(z) + \psi'(z)\} = 0 \tag{19}$$

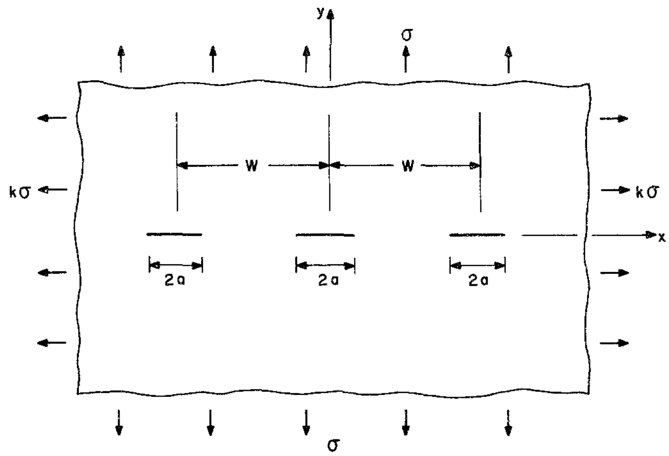

Figure 1.

At $|z| = \infty$

$$\sigma_{yy}(\infty) = \sigma, \quad \sigma_{xx}(\infty) = k\sigma, \quad \sigma_{xy}(\infty) = 0\,. \tag{20}$$

Due to the symmetry of the loading relative to the x-axis (13) must be satisfied. With $\bar{z}=z$ at $y=0$, (13) reduces (19) to

$$2\,\mathrm{Re}[\phi'(z)] = -A \tag{21}$$

for all points on any crack border. In semi-inverse fashion, owing to the periodic and symmetric nature of the crack spacing, the function $2\phi'(z)$ can be chosen to have the form

$$2\phi'(z) \equiv \frac{g(z)}{\left\{\sin^2\left(\frac{\pi z}{W}\right) - \sin^2\left(\frac{\pi a}{W}\right)\right\}^{\frac{1}{2}}} - A \tag{22}$$

where the denominator of the first term has no real part along the crack borders. The function $g(z)$ is presumed to be holomorphic in the region of definition, except possibly at the point $z=\infty$, and must be such that $\mathrm{Im}[g(x)]=0$ along the crack borders. The function $2\phi'(z)$ so defined satisfies boundary condition (21).

From boundary condition (20)

$$\sigma_{yy}(\infty)-\sigma_{xx}(\infty)+2\mathrm{i}\,\sigma_{xy}(\infty) = (1-k)\sigma = 4y\,\mathrm{Im}[\phi''(z)]+2A-4\mathrm{i}\,y\,\mathrm{Re}[\phi''(z)]$$

from which

$$(1-k)\sigma = (\bar{z}-z)2\phi'(z)+2A, \qquad |z|\to\infty\,. \tag{23}$$

Inasmuch as $2\phi'(z)$ must be holomorphic throughout, including the point at infinity, it will therefore be continuous at and in the neighborhood of this point, and for $|z|$ arbitrarily large

$$2\phi'(z) \to \frac{g(z)}{\sin\left(\frac{\pi z}{W}\right)} - A$$

where upon

$$(\bar{z}-z)\left\{\frac{g'(z)}{\sin\left(\frac{\pi z}{W}\right)} - \frac{g(z)}{\sin^2\left(\frac{\pi z}{W}\right)}\cdot\frac{\pi}{W}\cos\frac{\pi z}{W}\right\} + 2A = (1-k)\sigma \tag{24}$$

which can be identically satisfied by choosing

$$g(z) = \sigma\sin\left(\frac{\pi z}{W}\right)\;;\;\; A = \tfrac{1}{2}(1-k)\sigma\,. \tag{25}$$

The condition that $\text{Im}\,[g(x)]=0$ along the crack borders is also seen to be satisfied.

The stress function which solves this problem is thus

$$Z(z) = \frac{\sigma \sin\left(\dfrac{\pi z}{W}\right)}{\left\{\sin^2\left(\dfrac{\pi z}{W}\right) - \sin^2\left(\dfrac{\pi a}{W}\right)\right\}^{\frac{1}{2}}} - \tfrac{1}{2}(1-k)\sigma\,. \tag{26}$$

For uniaxial uniform tension applied in the y-direction, $k=0$ and $A=\sigma/2$. The stress function (26) then assumes a form equivalent to that given by Sanders [6]. When $k=1$, $A=0$, which corresponds to loading by equal uniform biaxial tension. The stress function introduced by Westergaard for this problem in reference [3] is therefore a solution only for this special loading condition.

As another illustration of consequence concerning this particular class of plane crack problems, consider the so-called crack-tip stress and displacement field equations. These can be obtained for opening mode crack surface displacements (mode I) by consideration of the problem of Fig. 1, modified to a single centrally located crack of length $2a$. A stress function which will satisfy the boundary conditions along such a cut has the form

$$2\phi''(z) = Z(z) = \frac{g(z)}{\{z^2-a^2\}^{\frac{1}{2}}} - A\,.$$

Proceeding as in the previous example, it will turn out that $g(z)=\sigma z$ and $A=(1-k)\sigma/2$ so that

$$Z(z) = \frac{\sigma z}{\{z^2-a^2\}^{\frac{1}{2}}} - (1-k)\frac{\sigma}{2}\,. \tag{27}$$

Introducing crack-tip polar coordinates (r, θ) through the coordinate transformation $\zeta=(z-a)=r\,e^{i\theta}$

$$Z(\zeta) = \frac{\sigma(\zeta+a)}{\{(\zeta+a)^2-a^2\}^{\frac{1}{2}}} - (1-k)\frac{\sigma}{2}\,.$$

For $|\zeta|$ very small, i.e., $|\zeta| \ll a$

$$Z(\zeta) \cong \frac{K_{\mathrm{I}}}{\{2\pi\zeta\}^{\frac{1}{2}}} - (1-k)\frac{\sigma}{2} \tag{28}$$

where

$$K_{\mathrm{I}} \equiv \sigma\{\pi a\}^{\frac{1}{2}} \tag{29}$$

is the crack-tip stress intensity factor. Substituting (28) into (9), (16) and (17) one obtains for the plane stress crack-tip stress and displacement fields the approximations

$$\begin{aligned}
\sigma_{xx} &\cong \frac{K_{\mathrm{I}}}{\{2\pi r\}^{\frac{1}{2}}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right] - (1-k)\sigma \\
\sigma_{yy} &\cong \frac{K_{\mathrm{I}}}{\{2\pi r\}^{\frac{1}{2}}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right] \\
\sigma_{xy} &\cong \frac{K_{\mathrm{I}}}{\{2\pi r\}^{\frac{1}{2}}}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right) \\
u &\cong \frac{K_{\mathrm{I}}}{\mu}\left\{\frac{r}{2\pi}\right\}^{\frac{1}{2}}\cos\left(\frac{\theta}{2}\right)\left[\left(\frac{1-\nu}{1+\nu}\right) + \sin^2\left(\frac{\theta}{2}\right)\right] - \frac{\sigma}{E}(1-k)\,r\cos\theta \\
v &\cong \frac{K_{\mathrm{I}}}{\mu}\left\{\frac{r}{2\pi}\right\}^{\frac{1}{2}}\sin\left(\frac{\theta}{2}\right)\left[\left(\frac{2}{1+\nu}\right) - \cos^2\left(\frac{\theta}{2}\right)\right] + \frac{\nu\sigma}{E}(1-k)r\sin\theta\,.
\end{aligned} \tag{30}$$

Again only when $k=1$, i.e., equal uniform biaxial tensile loading, do these equations reduce to the form currently found in the literature [7].

To further illustrate use of the modified Westergaard equations consider the centrally cracked strip (plate) of finite width loaded uniaxially in uniform tension, Fig. 2, of great interest in fracture toughness testing, and which has not been given an exact closed form solution.

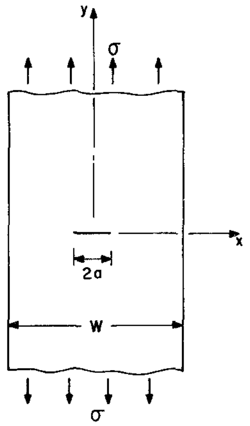

Figure 2.

A widely used approximate solution to this problem was first introduced by Irwin [4], by means of the stress function (26) with $k=1$, which, as has been shown, is the exact solution to the periodic colinear crack problem in an infinite sheet loaded in uniform biaxial tension. To the stress field associated with this stress function Irwin adds a uniform horizontal compressive stress of magnitude σ along the vertical edges of the strip which, interestingly, has the effect of compensating for the missing A term. This combination partially satisfies boundary conditions along the vertical edges, leaving a horizontal stress of varying magnitude which depends on the relative crack size. The crack tip stress intensity factor emanating from this stress function is the so-called tangent formula

$$K_{\mathrm{I}} = \sigma \left\{ W \tan \left(\frac{\pi a}{W} \right) \right\}^{\frac{1}{2}} \tag{31}$$

Subsequently more accurate truncated series (polynomial) representations for K_{I} have obtained by Isida [8] and Scrawley *et al.* [9], which show the tangent formula to be in varying degree of small error, depending on the crack size. Recently a secant formula has been proposed by Fedderson [9]

$$K_{\mathrm{I}} = \sigma \left\{ \pi a \sec \left(\frac{\pi a}{W} \right) \right\}^{\frac{1}{2}} \tag{32}$$

which matches almost identically Isida's K_{I} values, deemed to be the most accurate. Having the added virtue of being concise and therefore relatively simple to use, Fedderson's secant formula has now in some quarters replaced the tangent formula in fracture toughness testing.

There will be some practical interest then in obtaining the corresponding stress function, that is, one which comes acceptably close to solving the problem of Fig. 2 and which yields the secant formula for K_{I}.

It is convenient to let

$$Z(z) \equiv Z^*(z) - A\,. \tag{33}$$

Then

$$\begin{aligned} \sigma_{xx} &= \mathrm{Re}[Z^*(z)] - y\,\mathrm{Im}\,[Z^{*\prime}(z)] - 2A \\ \sigma_{yy} &= \mathrm{Re}[Z^*(z)] + y\,\mathrm{Im}\,[Z^{*\prime}(z)] \\ \sigma_{xy} &= -y\,\mathrm{Re}[Z^{*\prime}(z)] \end{aligned} \tag{34}$$

$$\tilde{Z}(z) = \int (Z^*(z) - A)\,\mathrm{d}z = \tilde{Z}^*(z) - Az \tag{35}$$

and

$$Eu = (1-\nu)\,\mathrm{Re}[\tilde{Z}^*(z)] - (1+\nu)y\,\mathrm{Im}\,[Z^*(z)] - 2Ax$$
$$Ev = 2\,\mathrm{Im}[\tilde{Z}^*(z)] - (1+\nu)\,y\,\mathrm{Re}[Z^*(z)] + 2\nu Ay\,. \tag{36}$$

A stress function which satisfies the crack border boundary condition, partially satisfies the vertical edge boundary condition and yields the secant formula for K_{I} has the form

$$Z(z) = Z^*(z) - A = \frac{\sigma\left\{\dfrac{\pi a}{W}\csc\left(\dfrac{\pi a}{W}\right)\right\}^{\frac{1}{2}}\sin\left(\dfrac{\pi z}{W}\right)}{\left\{\sin^2\left(\dfrac{\pi z}{W}\right) - \sin^2\left(\dfrac{\pi a}{W}\right)\right\}^{\frac{1}{2}}} - \tfrac{1}{2}\sigma\left\{\frac{\pi a}{W}\csc\left(\frac{\pi a}{W}\right)\right\}^{\frac{1}{2}} \tag{37}$$

For $|\zeta| \ll a$, where $\zeta = z - a = r\,\mathrm{e}^{i\theta}$

$$Z^*(\zeta) \cong \frac{\sigma\left\{\dfrac{\pi a}{W}\csc\left(\dfrac{\pi a}{W}\right)\right\}^{\frac{1}{2}}}{\left\{\dfrac{\dfrac{2\pi\zeta}{W}\sin\left(\dfrac{\pi a}{W}\right)\cos\left(\dfrac{\pi a}{W}\right)}{\sin^2\left(\dfrac{\pi a}{W}\right) + \dfrac{2\pi\zeta}{W}\sin\left(\dfrac{\pi a}{W}\right)\cos\left(\dfrac{\pi a}{W}\right)}\right\}^{\frac{1}{2}}}$$

from which

$$Z^*(\zeta) \cong \frac{\sigma\left\{\pi a \sec\left(\dfrac{\pi a}{W}\right)\right\}^{\frac{1}{2}}}{\{2\pi\zeta\}^{\frac{1}{2}}} = \frac{K_{\mathrm{I}}}{\{2\pi\zeta\}^{\frac{1}{2}}}\,. \tag{38}$$

Using (38) and

$$2A = \sigma\left\{\frac{\pi a}{W}\csc\left(\frac{\pi a}{W}\right)\right\}^{\frac{1}{2}} \tag{39}$$

in (34) through (36) will give the crack tip stresses and displacements as in equations (30), except that $(1-k)\sigma$ is replaced by $2A$ as given by (39).

The crack border condition

$$\sigma_{yy} = \sigma_{xy} = 0,\ \ y = 0, \qquad |x| < a$$

is seen to be satisfied by inspection. At $z = \frac{1}{2}W + \mathrm{i}y$

$$Z^{*\prime}\left(\frac{W}{2} + \mathrm{i}y\right) = \mathrm{i}2A\,\frac{\pi}{W}\tanh\left(\frac{\pi y}{W}\right)\left[\frac{\sin\left(\dfrac{\pi a}{W}\right)}{\cosh\left(\dfrac{\pi y}{W}\right)}\right]^2\left\{1 - \left[\frac{\sin\left(\dfrac{\pi a}{W}\right)}{\cosh\left(\dfrac{\pi y}{W}\right)}\right]^2\right\}^{-\frac{3}{2}}$$

which has no real part. Thus $\sigma_{xy}(\frac{1}{2}W, y) = 0$ for all y. On the other hand

$$\sigma_{xx}\left(\frac{W}{2}, y\right) = 2A\left\{\frac{1 - \left[\dfrac{\sin\left(\dfrac{\pi a}{W}\right)}{\cosh\left(\dfrac{\pi y}{W}\right)}\right]^2\left[1 + \dfrac{\pi y}{W}\tanh\left(\dfrac{\pi y}{W}\right)\right]}{\left\{1 - \left[\dfrac{\sin\left(\dfrac{\pi a}{W}\right)}{\cosh\left(\dfrac{\pi y}{W}\right)}\right]^2\right\}^{\frac{3}{2}}} - 1\right\}. \tag{40}$$

(37) would be an exact solution if the right side of (40) were to vanish for all values of y, all other boundary conditions having been satisfied. Results of calculation of (40) are shown in Fig. 3. For small crack sizes, $(\pi a/W) \leqq 0.3$, the right side of (40) gives values very close to zero along the entire vertical edge, having a maximum value of about four percent of the applied load at the crack plane when $(\pi a/W) = 0.3$. For $(\pi a/W) > 0.5$ the resulting horizontal boundary stress exceeds fifteen percent of the applied load at the crack plane. The pattern of this boundary stress distribution is interesting in that through Poisson's Ratio effects it tends to suppress vertical displacement of points situated just above and below the crack plane.

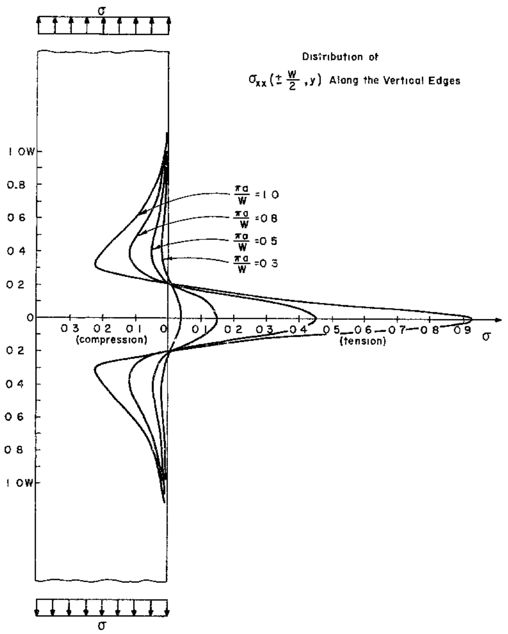

Figure 3.

Owing to the greater relative accuracy of the secant formula for K_I, one might expect that for small to moderate crack sizes, e.g., $(\pi a/W) < 0.5$, the stress function (37) will yield good estimates for other centrally located quantities such as the crack opening displacement, of interest in elastic compliance calibrations. For displacement gage points located along the plate center line, it follows from (36), after some calculation, that

$$\frac{E}{\sigma W} v(o, y) = \left\{ \frac{\pi a}{W} \csc\left(\frac{\pi a}{W}\right) \right\}^{\frac{1}{2}} \left[\frac{2}{\pi} \cosh^{-1} \left[\frac{\cosh\left(\frac{\pi y}{W}\right)}{\cos\left(\frac{\pi a}{W}\right)} \right] - \frac{(1+\nu)}{W} y \left\{ 1 + \left[\frac{\sin\left(\frac{\pi a}{W}\right)}{\sin\left(\frac{\pi y}{W}\right)} \right]^2 \right\}^{-\frac{1}{2}} + \frac{\nu}{W} y \right]. \tag{41}$$

Calculation of (41) is compared with experimental data obtained from Alum. 7075–T6 center cracked sheets, reported in reference [10], and shown in Fig. 4. The data points defining the experimental curve were obtained in the low load or elastic range. The predicted crack opening displacement, eq. (41), is a considerable improvement over Irwin's calculation, and is surprisingly close to the experimental curve in the large crack size range where the vertical edge boundary condition is poorly approximated. The fact that the predicted compliance curve lies

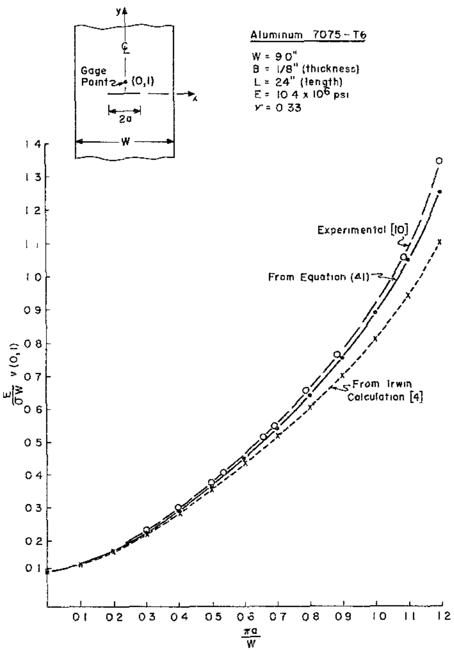

Figure 4

entirely below the experimental curve appears to be explainable by the particular nature of the distribution of the excess of vertical edge boundary stress shown in Fig. 3. Imposition of an identical distribution along the vertical edges, but reversed in sense, (leaving those edges free of traction as they should be) would tend to increase somewhat the vertical displacement from that given by (41) for all points a little above and below the crack plane and would thereby elevate the curve of (41).

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RÉSUMÉ

On discute brièvement une erreur, qui a été signalée à l'origine par Sih, dans l'équation de Westergaard relative à une certaine catégorie de problèmes de fissures planes.

L'origine de la difficulté réside dans une surestimation faite par MacGregor dans un travail antérieur, sur lequel Westergaard a basé ses équations.

On donne plusieurs exemples intéressants qui illustrent les conséquences des corrections qu'il est nécessaire d'apporter à ces équations.

ZUSAMMENFASSUNG

Ein Fehler in Westergaard's Formel für eine gewisse Klasse von Flächenrißproblemen, auf den schon Sih hingewiesen hatte, wird besprochen.

Der Ursprung dieser Schwierigkeit liegt in einem Versehen das McGregor in einer früheren Arbeit unterlaufen ist, auf welcher Westergaard's Formeln aufgebaut sind.

Die Konsequenzen die sich aus den fur diese Formeln erforderlichen Korrekturen ergeben, werden an Hand verschiedener interessanter Beispiele nachgewiesen.