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DEGREE OF UNDECIDABILITY OF THE COMPLETENESS PROBLEM IN ALGEBRAS OF RECURSIVE FUNCTIONS

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A set $M \subset \mathfrak{A}$ in an algebra $\langle\mathfrak{A}, \Omega\rangle$ is said to be complete if its closure by operations of the signature Ω coincides with $\mathfrak{A}$, i.e., $[M]_\Omega = \mathfrak{A}$. Already in 1967, A. P. Ershov and A. A. Lyapunov [1] called the problem of establishing a criterion for completeness of an arbitrary set $M \subset \mathfrak{A}$, when $\mathfrak{A}$ is a set of recursive functions (algorithmic completeness criterion), one of the central problems in theoretical programming. However, it has become possible to obtain such a criterion (i.e., to describe all precomplete classes) only for the algebra of unary partial recursive functions with the signature of a unique inversion operation [2] and for the algebra of unary general recursive functions and predicates with a closure operation relative to program schemes, using arrays, markers, and the equality predicate [3].

For algebras of recursive functions it is natural to restrict the statement of the problem: find a criterion for completeness of an arbitrary finite or arbitrary effective set $M \subset \mathfrak{A}$. However, the problem remains algorithmically undecidable even in this case [4]. In this note we establish the degree of undecidability of the problem of finite and effective algorithmic completeness. We study only algebras which are not partial [5].

We fix a computable numeration μ of all nonempty finite subsets of the set **N** as well as universal Kleene functions $\varphi_n(x)$ for all unary partial recursive functions and $\psi_n(y, x)$ for the binary ones. Let

$$\mathfrak{A} = \{\varphi_n(x) \mid n \in \mathrm{N}\} \text{ and } \langle\mathfrak{A}', \Omega\rangle$$

be an algebra with $\mathfrak{A}' \subseteq \mathfrak{A}$. The numerations

$$\tau(m) = \{\varphi_n(x) \mid n \in \mu(m)\}$$

and

$$\rho(m) = \{\psi_m(n, x) \mid n \in \mathrm{N}\} = \{\varphi_{h(m,n)}(x) \mid n \in \mathrm{N}\}$$

(here h is some general recursive function) of all finite subsets and all effective subsets of the set $\mathfrak{A}$ allow us to introduce sets characterizing the problems of finite and effective completeness for the algebra:

$$P_{\text{к}}(\langle\mathfrak{A}', \Omega\rangle) = \{m \mid [\tau(m)]_\Omega = \mathfrak{A}'\},$$
$$P_{\text{e}}(\langle\mathfrak{A}', \Omega\rangle) = \{m \mid [\rho(m)]_\Omega = \mathfrak{A}'\}.$$

An algebra $\langle\mathfrak{A}', \Omega\rangle$, $\mathfrak{A}' \subseteq \mathfrak{A}$ of a finite type is effectively generated, i.e., $P_{\text{e}}(\langle\mathfrak{A}', \Omega\rangle) \neq \varnothing$, if and only if the set $\mathfrak{A}'$ has a universal recursive function.

Indeed, if F(n, x) is a universal recursive function for the family $\mathfrak{A}'$, then $F(n, x) = \psi_i(n, x)$ for some i, whence

$$[\rho(i)]_\Omega = \mathfrak{A}'.$$

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Conversely, if $\rho(i)$ is a system of generators for the algebra $\langle\mathfrak{A}', \Omega\rangle$ then, by means of a standard numeration of terms in the signature $\Omega \cup \rho(j)$ [6], we may get a recursive numeration of the functions realized by them.

In particular, this implies that each nonpartial algebra of unary general recursive functions (for instance, those with the signature $\{*\}$, or $\{*, +, \times\}$, etc.) cannot have effective or, a fortiori, finite systems of generators.

The problem of completeness in algebras $\langle\mathfrak{A}, \Omega\rangle$ with $\Omega \subseteq \{*, +, ^{-1}\}$ has been studied in a number of papers (see [4] for a survey). We denote

$$A_1 = \langle\mathfrak{A}; *\rangle,\ A_2 = \langle\mathfrak{A}; *, ^{-1}\rangle, \quad A_3 = \langle\mathfrak{A}; *, +\rangle,$$
$$A_4 = \langle\mathfrak{A}; *, +, ^{-1}\rangle, \quad A_5 = \langle\mathfrak{A}; *, \vee_\alpha, I_\alpha\rangle.$$

The operations of the last listed algebra, the superposition, the α-disjunction, and the α-iteration, were defined in Glushkov's system of algorithmic algebras [7]. The signature of the algebra A_5 admits as α any predicate whose truth and falsity domains are recursively enumerable sets. Each of the algebras A_1-A_5 is finitely generated. The algebras $A_6 = \langle\mathfrak{A}; +, ^{-1}\rangle$, $A_7 = \langle\mathfrak{A}; +\rangle$, $A_8 = \langle\mathfrak{A}; ^{-1}\rangle$ have no finite complete systems.

We state our main results as the following theorems.

<u>THEOREM 1.</u> For $i = 1, \ldots, 5$ each set $P_K(A_i)$ and each set $P_e(A_i)$ is Σ_3-complete in the Kleene–Mostowski hierarchy.

<u>THEOREM 2.</u> Each set $P_e(A_i)$, $i = 6, 7, 8$ is Π_4-complete.

For the set $\mathfrak{B}$ of all unary primitively recursive functions the definition of $P_K(\langle\mathfrak{B}, \Omega\rangle)$ is similar to the previous one, with the function $\varphi_n(x)$ replaced by a universal general recursive function for $\mathfrak{B}$.

<u>THEOREM 3.</u> The set $P_K(\langle\mathfrak{B}; *, +, \iota\rangle)$ is Σ_2-complete.

These theorems also determine Turing degrees of undecidability for the problems of finite and effective completeness because each Σ_n- or Π_n-complete set belongs to the degree $0^{(n)}$ [8].

The results obtained explain difficulties in solving the problem of completeness: its degree of undecidability weakly depends on the signature of the algebra, is the same for the finite as for the effective completeness, and is higher relative to other problems, such as the equivalence of terms in algebras, the totality of terms, or their emptiness.

<u>Proofs of the Theorems.</u> Let $A = \langle\mathfrak{A}, \Omega\rangle$ denote one of the algebras in Theorem 1. For $P_K(A)$ and $P_e(A)$ it is necessary to establish, first, whether it lies in the class Σ_3, and, second, whether some Σ_3-complete set is m-reduced to it. The same scheme of proof applies to theorems 2 and 3. The proof of the former statement is technically identical for all three theorems, so we will only prove it for the first theorem. The second statement is specific for each theorem.

We will show that $P_K(A)$ and $P_e(A)$ lie in Σ_3. Let $\{h_1, \ldots, h_K\}$ be a finite complete system for A. In a standard numeration K of terms in the signature

$$\Omega \cup \{\varphi_n \mid n \in \mathbf{N}\}$$

[6] the set S_m (the set S_m') of numbers of all terms which contain only functions in $\tau(m)$ [respectively, in $\rho(m)$] is recursive (recursively enumerable) for each m. Each term T_i with a number i in the numeration κ determines a program for computing the function $T_i(x)$. For a function f the symbolism $f(a) \downarrow t$ denotes the recursive predicate "the computation of the function f at a ends at most after t computation steps" [8]. In this notation

$$\begin{aligned} m \in P_e(A) \leftrightarrow (\forall 1 \leqslant i \leqslant k)\, (\exists a_i)\, [a_i \in \\ \in S_m' \,\&\, (\forall x)\, [(\exists t_i)\, (h_i(x) \downarrow t_i \,\&\, T_{a_i}(x) \downarrow t_i \,\&\, h_i(x) = \\ = T_{a_i}(x)) \vee \neg\, (\exists q_i)\, (h_i(x) \downarrow q_i \vee T_{a_i}(x) \downarrow q_i)]] \leftrightarrow \\ \leftrightarrow (\forall 1 \leqslant i \leqslant k)\, (\exists a_i)\, [(\exists b_i)\, R_1(a_i, b_i, m) \,\& \\ \&\, (\forall x)\, [(\exists t_i)\, R_2(a_i, x, t_i) \vee (\forall q_i)\, R_3(a_i, x, q_i)]], \end{aligned}$$

where R_1, R_2, R_3 are recursive predicates. Applying the Tarski–Kuratowski algorithm, we obtain $P_e(A) \in \Sigma_3$. For $P_K(A)$ is suffices to replace S_m' by S_m in the formula; the result will remain the same.

Similarly one can prove that

$$P_e^{\prime}(A_i) \in \Pi_4 \quad (i = 6, 7, 8), \qquad P_K(\mathfrak{B}) \in \Sigma_2.$$

We will now establish the m-reducibility to $P_K(A_i)$ ($i = 1, \ldots, 5$) of the index set $\{x|\bar{W}_x$ is finite$\}$, where W_0, W_1, W_2, ... is the Gödel numeration of all recursively enumerable sets [8]. To this end, we will fix two functions, $t_1(x)$ and $t_2(x)$, forming a complete system for A_1 [2] and describe an algorithm which constructs for each $m \in N$ four functions $F_m = \{f_m, g_m, h_m, r_m\}$ forming a complete system in the algebra if and only if the set $\overline{W}_m$ is finite.

One can easily describe an algorithm of constructing pairwise disjoint finite sets $K_0 = \{0, 1\}$, $K_i = \{d_0^i, d_1^i, \ldots, d_i^{\,i}\}$ in such a way that the number d_j^i be divisible by $j + 2$ for all $i \geq 1$, $j = 0, 1, \ldots, i$, and $\cup_i K_i = N$. We put $M_m = \bigcup_{x \in W_m} K_x \bigcup \{0\}$ and use a repetition-free enumeration of the set M_m [8]. If a does not appear in this enumeration, then $f_m(a)$, $g_m(a)$, and $h_m(a)$ are undefined. If however $a \in M_m$, then a would appear in the enumeration as the i-th element. Then $H_m(a) = i$. Furthermore, if the value of $t_1(i)$ is defined, the enumeration of M_m would contain a $t_1(i)$-th element, we denote it by b, then $f_m(a) = b$, otherwise $f_m(a)$ is undefined. The definition of $g_m(a)$ is similar, replacing the function t_1 by t_2.

Suppose that the set W_m is infinite, then M_m is also infinite. Since

$$f_m = h_m * t_1 * h_m^{-1} \text{ and } g_m = h_m * t_2 * h_m^{-1},$$

$\{f_m, g_m\}$ forms a basis of the algebra $\langle C_m; *\rangle$, where

$$C_m = \{f \in \mathfrak{A} | \operatorname{Arg} f \subseteq M_m \,\&\, \operatorname{Val} f \subseteq M_m\}.$$

Furthermore, let

$$D_m = \{f \in \mathfrak{A} | \operatorname{Arg} f \subseteq M_m\} \text{ and } q \in D_m;$$

then the function $q' = q * h_m^{-1}$ lies in C_m and $q' * h_m = q$. Therefore, the functions f_m, g_m, h_m generate the algebra $\langle D_m; *\rangle$. Finally, let ℓ be a general recursive injective function with $\operatorname{Val}\ell = M_m$ and $q \in \mathfrak{A}$, then the function $q' = \ell^{-1} * q$ lies in D_m and $\ell * q' = q$. Thus, the system $\{f_m, g_m, h_m, \ell\}$ is complete in A_1.

Now, put $r_m(x) = x + 1$ for the algebras A_1 and A_2, $r_m(x) = x$ for the algebras A_3 and A_4, $r_m(x) = 0$ for the algebra A_5. If $\bar{W}_m$ is finite, then $\bar{M}_m$ is also finite. In this case we construct from F_m in each algebra a function with the properties of the function ℓ: the superposition of $x + 1$ provides $x + k$; the summation of x provides $k \cdot x$, the branching, along the predicates $x = k$, of the functions of D_m enables us to construct the required function for a sufficiently large k. Thus, F_m is complete in its algebra when $\bar{W}_m$ is finite.

If the set W_m is finite, then M_m is also finite. The functions f_m, g_m, h_m are defined only on a finite set, so in the closure of F_m for A_1 and A_2 a general recursive function may be only of the form $x + c$, and for A_3 and A_4 it may be only of the form $c \cdot x$, where $c \geq 1$; the closure of F_m for A_5 is contained in the proper subalgebra of bounded functions (for each $x \in \operatorname{Arg} f$ either $f(x) = x$ or $f(x) \leq n$ with some n).

Finally, if both W_m and $\bar{W}_m$ are infinite (both M_m and $\bar{M}_m$ are infinite), then F_m is not complete in A_2 and A_1 for the same reason; for A_3 and A_4 F_m is contained in a proper subalgebra of the algebra A_4, $\{f \in \mathfrak{A} \,/\, (\forall j)\, \overline{(\operatorname{Arg} f \cap R_j}$ is infinite$) \bigvee (\exists p \geqslant 1)\, (\exists q \geqslant 1)\, (\operatorname{Arg} f = R_p \,\&\, f(p \cdot x) = q \cdot x)\}$ where $R_j = \{j \cdot x \,|\, x \in N\}$ and each set $R_j \cap M_m$ is infinite (this is what caused us to go from W_m to M_m); for A_5, F_m is contained in its proper subalgebra $\{f \in \mathfrak{A} \,|\, f(\overline{M}_m) \cap M_m$ is a finite set$\}$ [9]. As a result, the system of functions F_m is not complete in its algebra when the set $\bar{W}_m$ is infinite.

Thus, the set $\{x \,|\bar{W}_x$ is finite$\}$ is m-reduced to $P_K(A_i)$, $i = 1, \ldots, 5$. The passage to $P_e(A_i)$ is trivial: by means of the same algorithm, we define an effective sequence $\{f_m, g_m, h_m, r_m, r_m, \ldots\}$, which is complete in the algebra if and only if the set $\bar{W}_m$ is finite.

To conclude the proof of Theorem 2, we will establish the m-reducibility of the Π_4-complete set $E = \{m \mid$ the set W_x is infinite for infinitely many $x \in W_m\}$ [8] to the set $P_e(\langle\mathfrak{A}, \Omega\rangle)$ provided that any set of functions in which there are finitely many functions with an infinite domain is not complete in the algebra $\langle\mathfrak{A}, \Omega\rangle$. The algebras A_6, A_7, A_8 have this property.

Let $\rho(l) = \{\psi_l(0, x), \psi_l(1, x), \ldots\}$ be a system of generators of the algebra $\langle \mathfrak{A}, \Omega \rangle$, for instance, $\psi_\ell(n, x)$ is a universal recursive function for $\mathfrak{A}$. We will use a stepwise algorithm of enumerating (with respect to m) of the set W_m with a parallel enumerating (with respect to y) of the sets W_y for which y has appeared in the enumeration of W_m at one of the previous steps. The value of $\psi_m(n, x)$ is set equal to $\psi_\ell(u(n), x)$ if at some step we enumerate $v(n) + 1$ elements in the set W_m: $y_0, y_1, \ldots, y_{v(n)}$, and x elements in the set $W_{y_{v(n)}}$ is undefined. The functions u(n) and v(n) are defined as follows: we write $n = p + q(q + 1)/2$ with $q \geq 0$, $0 \leq p \leq p$ (such a representation is unique) and put $u(n) = p$, $v(n) = q$. Clearly, if the enumeration of W_m contains a $(j + 1)$-st element y_j and the set W_{y_j} is infinite, then the function $\psi_m(i + j(j + 1)/2, x)$ coincides with $\psi_\ell(i, x)$ for $i = 0, 1, \ldots, j$ and for each $x \in N$; if the set is finite, then each of the functions $\psi_m(i + j(j + 1)/2, x)$ $(i = 0, 1, \ldots, j)$ has a finite domain. This implies that the resulting effective sequence $\rho(m)$ either contains only finitely many functions with finite domains (when the sets W_x are infinite only for finitely many elements $x \in W_m$) or contains every function in the sequence $\rho(\ell)$, when $m \in E$.

To conclude the proof of Theorem 3, we will establish the m-reducibility of the Σ_2-complete set $H = \{x \mid W_x \text{ is finite}\}$ to the set $P_K(\langle \mathfrak{B}; *, +, \iota \rangle)$. To this end, we assign to each $m \in N$ a system of primitively recursive functions $Q_m = \{s_m, q_m, x \dot{-} 1, \overline{\mathrm{sg}}(x)\}$ which is complete if and only if the set W_m is finite.

The primitive recursiveness of the functins j(m, x, t) and σ(m, x, t) defining, respectively, the next command for $\varphi_m(x)$ after t computation steps (0 if $\varphi_m(x) \downarrow t$) and the state of computation of $\varphi_m(x)$ after t steps was proved in [10]. This implies the primitive recursiveness of the function $c(m, t) = c_m(t)$ whose value is the number of distinct numbers appearing in the enumeration (with respect to m) of the set W_m after t steps. Obviously, if W_m is finite, then the function $c_m(t)$ is bounded, for instance, $c_m(t) \equiv 0$ if W_m is empty. It is also clear that

$$c_m(t) \leqslant c_m(t+1) \leqslant c_m(t) + 1.$$

Put

$$s_m(i^2 + j) = s(i^2 + j) - c_m(i),$$
$$q_m(i^2 + j) = q(i^2 + j) + c_m(i)$$

for $i \geq 0$, $j + 0, 1, \ldots, 2i$, where

$$s(x) = x + 1 \text{ and } q(x) = x - [\sqrt{x}]^2$$

form a complete system for the algebra $\langle \mathfrak{B}; *, +, \iota \rangle$ [5].

If the set W_m is infinite, then the function $c_m(t)$ is not bounded: for each $z \in N$ there exists w such that $c_m(t) > z$ for each $t \geq w$. In this case the function $q_m(x)$ does not assume the value z when $z > w^2$. Therefore, each value z is assumed by the function $q_m(x)$ only finitely many times. The system Q_m has no functions of almost large magnitude, i.e., it is not complete in the algebra $\langle \mathfrak{B}; *, +, \iota \rangle$ [11].

If the set W_m is finite, then for some r and $t_1, \ldots, t_r$ the following conditions hold:

$$\begin{aligned}
&\text{if } 0 \leqslant x < t_1^2, \text{ then } s_m(x) = s(x) \text{ and } q_m(x) = q(x),\\
&\text{if } t_1^2 \leqslant x < t_2^2, \text{ then } s_m(x) = s(x) - 1 \text{ and } q_m(x) = \\
&\qquad = q(x) + 1,\\
&\text{if } t_2^2 \leqslant x < t_3^2, \text{ then } s_m(x) = s(x) - 2 \text{ and } q_m(x) = \\
&\qquad = q(x) + 2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\text{if } t_r^2 \leqslant x, \text{ then } s_m(x) = s(x) - r \text{ and } q_m(x) = q(x) + r.
\end{aligned}$$

It is easily seen that the functions

$$s_m^1(x) = s_m(x) + (\iota(s_m(x)) \dot{-} (t_1^2 - 1))$$
$$q_m^1(x) = q_m(x) + \overline{\mathrm{sg}}(\iota(s_m(x)) \dot{-} (t_1^2 - 1))$$

satisfy these conditions:

$$\text{if} \quad 0 \leqslant x < t_2^2, \text{ then } s_m^1(x) = s(x) \text{ and } q_m^1(x) = q(x) + 1,$$
$$\text{if } t_2^2 \leqslant x < t_3^2, \text{ then } s_m^1(x) = s(x) - 1,$$
$$\text{and } q_m^1(x) = q(x) + 2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\text{if } t_r^2 \leqslant x, \text{ then } s_m^1(x) = s(x) - r + 1$$
$$\text{and } q_m^1(x) = q(x) + r.$$

Repeating these operations r times, we deduce that $s_m^r(x) = s(x)$ and $q_m^r(x) = q(x) + r$, and using the function $x - 1$ we also construct q(x). Thus, Q_m is complete in $\langle \mathfrak{B};\ *,\ +,\ \iota \rangle$. The proofs of the theorems are complete.

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