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A Vertex Ranking Algorithm for the Fixed-Charge Transportation Problem

S. SADAGOPAN[1] AND A. RAVINDRAN[2]

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Abstract. In many distribution problems, the transportation cost consists of a fixed cost, independent of the amount transported, and a variable cost, proportional to the amount shipped. In this paper, we propose an efficient algorithm for solving the fixed-charged transportation problem, based on Murty's extreme point ranking scheme. An improved lower bound on the fixed costs developed in the paper and dynamic updating of upper bound on linear costs and ranking limits are demonstrated to improve the computational requirements of Murty's scheme significantly. The ideas developed are illustrated with the aid of an example. Finally, a stopping criterion with an ϵ-optimum solution is introduced using Balinski's approximation scheme.

Key Words. Fixed charge problems, transportation problems, vertex ranking algorithm, nonconvex problems.

1. Introduction

The fixed-charge transportation problem can be stated as a distribution problem in which there are m suppliers (warehouses or factories) and n customers (destinations or demand points). Each of the m suppliers can ship to any of the n customers at a shipping cost per unit c_{ij} (unit cost for shipping from supplier i to customer j) plus a fixed cost d_{ij}, assumed for opening this route. Each supplier $i = 1,2, \ldots, m$ has a_i units of supply, and each customer $j = 1,2, \ldots, n$ demands b_j units. The objective is to determine which routes to be opened and the size of the shipment so that the total cost of meeting demand, given the supply constraints, is minimized. Many distribution problems in practice can only be modeled as fixed charge

[1] Assistant Professor, Industrial and Management Engineering Program, Indian Institute of Technology, Kanpur, India.
[2] Professor, School of Industrial Engineering, Purdue University, West Lafayette, Indiana.

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transportation problems. For example, railroads and trucks have invariably used freight rates which consist of a fixed cost and a variable cost.

Mathematically, the fixed-charge transportation problem is the following Problem A.

$$\text{A:} \quad \text{minimize} \quad Z = C(x) + D(y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij}x_{ij} + d_{ij}y_{ij}),$$

$$\text{subject to} \quad \sum_j x_{ij} = a_i, \qquad i = 1, \ldots, m,$$

$$\sum_i x_{ij} = b_j, \qquad j = 1, \ldots, n,$$

$$x_{ij} \geq 0, \qquad \text{for all } (i, j),$$

$$y_{ij} = 0, \qquad \text{if } x_{ij} = 0,$$

$$y_{ij} = 1, \qquad \text{if } x_{ij} > 0.$$

Without loss of generality, we assume that

$$\sum a_i = \sum b_j$$

and

$$a_i,\ b_j,\ c_{ij},\ d_{ij} \geq 0.$$

Despite its similarity to a standard transportation problem, Problem A is significantly harder to solve because of the discontinuity in Z introduced by the fixed costs $D(y)$. Problem A is a special case of the general fixed-charge problem, which was first formulated by Hirsch and Dantzig (Ref. 1), who established that:

(i) the feasible region of Problem A is a bounded convex set;
(ii) the objective function Z is a concave function;
(iii) there exists an extreme point optimum for Problem A;
(iv) a local minimum to Problem A need not be the global minimum;
(v) for a nondegenerate problem with all positive fixed costs, every extreme point of the feasible region is a local minimum.

Because of the complexity involved in examining many local minima, earlier attempts to solve this problem consisted of finding an approximate solution. The earliest one was proposed by Balinski (Ref. 2), who observed that there exists an optimal solution to the relaxed version of A (formed by ignoring the integer restrictions on y_{ij}), with the property that

$$y_{ij} = x_{ij}/m_{ij},$$

where

$$m_{ij} = \min(a_i, b_j).$$

So, the relaxed problem would be simply a standard transportation problem with unit transportation costs as $c_{ij} + d_{ij}/m_{ij}$. Other well-known heuristic approaches are the ones by Kuhn and Baumol (Ref. 3), Cooper and Drebes (Ref. 4), Denzler (Ref. 5), Steinberg (Ref. 6), and Walker (Ref. 7). In addition, a number of exact approaches have been proposed by Murty (Ref. 8), Gray (Ref. 9), and Kennington (Ref. 10).

One approach to solving Problem A involves a mixed integer programming formulation (Refs. 10 and 11). By defining y_{ij}'s as 0–1 binary variables and introducing the constraints

$$x_{ij} \leq My_{ij},$$

where M is a larger number, y_{ij}'s are automatically forced to 1 whenever x_{ij}'s are positive. Another approach, due to Murty (Ref. 8), is our primary concern in this paper.

2. Extreme Point Ranking Algorithm

A direct approach, due to Murty (Ref. 8), develops a pattern of search among the extreme points of Problem A for determining the global minimum of the fixed charge problem. His pattern of search is essentially a ranking of the extreme points in increasing order of the variable (linear) costs. The fixed costs are then added to each ranked vertex in determining the optimal solution. The following theorems form the basis of Murty's ranking algorithm for the fixed-charge problem:

Theorem 2.1. Let $\{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$ be a sequence of extreme points of the fixed-charge problem ranked in increasing order of variable costs $C(x)$. Let S_j be the set of all adjacent extreme points of $x^{(j)}$. Then,

$$x^{k+1} \in \left\{\bigcup_{j=1}^{k} S_j - \bigcup_{j=1}^{k} x^{(j)}\right\}.$$

In other words, the next extreme point in the sequence must be adjacent to one of the previously ranked extreme points.

Theorem 2.2. Let Z_u be an upper bound on the total cost Z, and let D_l be a lower bound on the fixed cost $D(y)$ for any feasible solution to

Problem A. Then,

$$C_u = Z_u - D_l$$

is an upper bound on the variable cost $C(x)$ for any feasible x.

Theorem 2.1 enables one to generate the extreme points in a systematic manner by examining only the adjacent extreme points, which can be generated by a simplex pivot. Theorem 2.2 gives a stopping criterion on the number of extreme points to be ranked, because it is sufficient to continue ranking up to the bound C_u. As soon as the last ranked vertex $x^{(k)}$ has a cost

$$C(x^{(k)}) > C_u,$$

the ranking is terminated. The extreme point with the minimum total cost among the ranked extreme points becomes the global minimum to Problem A.

Improvements to Murty's extreme point ranking method for solving the general nonconvex program have been attempted by the author (Ref. 12). Based on these, we develop an improved extreme point ranking algorithm for solving the fixed-cost transportation problem.

3. Improved Vertex Ranking Algorithm

We make the following improvements to Murty's vertex ranking method:

(a) better lower bound on the fixed costs;

(b) dynamic update of upper bound on the total cost during the ranking procedure;

(c) dynamic update of ranking limits.

Finally, we also discuss termination with an *ϵ-optimum solution* or an approximate solution with a maximum error of ϵ, using a lower bound on the total cost.

3.1. Determination of a Lower Bound D_l for Fixed Cost. Murty suggested that a value for D_l be found by summing the k smallest fixed costs, where

$$k = \max(m, n).$$

But this completely ignores feasibility and is not necessarily a good lower bound. Here, we provide a better method for finding a value of D_l in which feasibility is taken into account and degeneracy poses no problem. Consider the following problem, which is a reformulation of Problem A ignoring the linear costs and using Balinski's upper bounds on x_{ij}'s (Problem B).

$$\begin{aligned}
\text{B:}\quad &\text{minimize} && \sum_{(i,j)} d_{ij} y_{ij}, \\
&\text{subject to} && \sum_j x_{ij} = a_i, \qquad i = 1, \ldots, m, \\
& && \sum_i x_{ij} = b_j, \qquad j = 1, \ldots, n, \\
& && 0 \leq x_{ij} \leq m_{ij} y_{ij}, \\
& && y_{ij} \in (0, 1),
\end{aligned}$$

where

$$m_{ij} = \min(a_i, b_j).$$

Following Balinski (Ref. 2), consider a relaxation to Problem B, where we ignore the integer restrictions on y_{ij}. Let the relaxed problem be called Problem C. Then, we have the following result.

Theorem 3.1. In the optimal solution to problem C, the constraints

$$x_{ij} \leq m_{ij} y_{ij}$$

will be satisfied as strict equations.

Theorem 3.1 provides a simple method of solving Problem C. Substituting

$$x_{ij} = m_{ij} y_{ij},$$

we get the following problem (Problem D).

$$\begin{aligned}
\text{D:}\quad &\text{minimize} && \underline{c}(x) = \sum_{(i,j)} (d_{ij}/m_{ij}) x_{ij}, \\
&\text{subject to} && \sum_j x_{ij} = a_i, \qquad i = 1, \ldots, m, \\
& && \sum_i x_{ij} = b_j, \qquad j = 1, \ldots, n, \\
& && x_{ij} \geq 0.
\end{aligned}$$

Problem D is a standard transportation problem and can be solved easily. Moreover, since Problem D is a valid relaxation to Problem B, we get the following result.

Theorem 3.2. The optimal objective function value of Problem D gives a valid lower bound D_l for the fixed costs of Problem A.

3.2. Dynamic Update of Bound. In Murty's method, an initial upper bound Z_u on the total cost Z is found by simply solving a standard transportation problem ignoring the fixed costs. This is updated once at every stage of ranking. Use is made only of the information provided by the newly ranked extreme point at that stage; the other adjacent extreme points, which have not been ranked (but nevertheless generated and stored for future use as prospective candidates), are not used. Since these extreme points are also feasible to Problem A, they can be used to obtain better upper bounds on the total cost Z, and accelerate the termination process.

3.3. Dynamic Update of Ranking Limits

Theorem 3.3. Let $\{\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(k)}\}$ be a sequence of extreme points of Problem D arranged in increasing order of cost $\underline{c}(x)$. Then, at stage k, we can update the limits on ranking as follows:

$$\bar{C} = Z_u - \underline{c}(\underline{x}^{(k)}),$$

$$\bar{D} = Z_u - c(x^{(k)}),$$

where $\bar{C}$ is the limit to which we must rank the extreme points of Problem A with fixed costs set equal to zero and $\bar{D}$ is the limit to which we must rank the extreme points of D using $\underline{c}(\underline{x})$.

Proof. By dynamically updating the upper bound, Z_u represents the total cost of the incumbent solution at stage k. By definition of incumbent solution, any candidate for global optimum must have cost $<Z_u$. Also by definition, all extreme points with fixed costs at least $\underline{c}(\underline{x}^{(k)})$ have total cost $\geq Z_u$. Thus, it is sufficient to analyze the extreme points with variable cost $\leq Z_u - \underline{c}(\underline{x}^{(k)})$. Similarly, the other assertion can be proved.

Incorporating these improvements we propose the following algorithm.

Algorithm

Step 1. Initialize, $k = 0$.

Step 2. Set $k = k + 1$.

Step 3. Generate $x^{(k)}$ and $\underline{x}^{(k)}$. In the process of generating $x^{(k)}$ and $\underline{x}^{(k)}$, set the best solution generated so far as the incumbent solution and its total cost $= Z_u$. Compute $\bar{C}$ and $\bar{D}$. If

$$c(x^{(k)}) + \underline{c}(\underline{x}^{(k)}) > Z_u,$$

stop. Otherwise, return to Step 2.

Note that we are essentially ranking the extreme points of the same convex polytope with two different linear objective functions. Thus, we are searching over the extreme points from two different starting points and sharing the information generated. Though the list of extreme points stored at any time is likely to be slightly longer, as we are solving two problems simultaneously, the overall impact is to bring down the total number of stages of ranking considerably, thereby saving significant computation.

4. Illustrative Example

We shall now illustrate the impact of the suggested improvements using a numerical example (see Table 1).

Table 1. Data for the example.

$d_{ij} = 900$, $c_{ij} = 760$	1000, 71	700, 283	800, 594	$a_i = 50$
900, 594	300, 64	700, 170	600, 564	$a_i = 15$
600, 594	200, 69	400, 79	0, 202	$a_i = 5$
$b_j = 25$	$b_j = 20$	$b_j = 15$	$b_j = 10$	

Stage 1. $k = 1$. We have

$$c(x^{(1)}) = 26155,$$

and the incumbent solution has total cost

$$Z_u = 30350.$$

Also,

$$\underline{c}(\underline{x}^{(1)}) = 2550.$$

Using Murty's method, D_l (lower bound on fixed costs) $= 900$. Hence, we have to rank all extreme points with variable costs up to $30350 - 900 =$

29450. Hence, 21 stages of ranking must be carried out with a maximum list length of 13 extreme points.

In the improved method, we would have, at Stage 1,

$$\bar{C} = 30350 - 2550 = 27800,$$

$$\bar{D} = 30350 - 26155 = 4195.$$

Note that it is not necessary to store extreme points with $c(x) > 27800$. Since

$$c(x^{(1)}) + \underline{c}(\underline{x}^{(1)}) < Z_u,$$

we proceed further.

Stage 2. $k = 2$. We have

$$c(x^{(2)}) = 26835, \qquad Z_u = 30350 \text{ (as before)}, \qquad \underline{c}(\underline{x}^{(2)}) = 2600.$$

$$\bar{C} = 30350 - 2600 = 27750,$$

$$\bar{D} = 30350 - 26835 = 3515,$$

$$c(x^{(2)}) + \underline{c}(\underline{x}^{(2)}) < Z_u.$$

Proceeding further up to Stage 6, we get

$$k = 6, \qquad c(x^{(6)}) = 27365, \qquad Z_u = 30350, \qquad \underline{c}(\underline{x}^{(6)}) = 3000.$$

Since

$$c(x^{(6)}) + \underline{c}(\underline{x}^{(6)}) > Z_u,$$

we terminate the algorithm.

The incumbent solution is optimal. The optimal solution is

$$x_{11} = 25, \qquad x_{23} = 15, \qquad x_{34} = 5,$$

$$x_{12} = 20, \qquad x_{14} = 5,$$

$$\text{variable costs} = 26950,$$

$$\text{fixed costs} = 3400.$$

Thus, it is sufficient to carry out 12 stages $(6+6)$ of ranking. The maximum list of extreme points (for both problems together) is only 15. Note that, for a marginal increase in storage requirements, we save considerable computational effort.

We solved the same problem by increasing all the fixed costs by 500, 1000, 1500, and 2500. McKeown's computer code (Ref. 13) for generating extreme points was incorporated in our algorithm. The results are shown in Table 2.

Table 2. Numerical results for the example

Ratio of fixed to variable costs at optimum	Problem number	Number of stages of ranking		Maximum list length	
		Murty's method	This method	Murty's method	This method
0.126	(original)	21	12	13	15
0.219	2	24	14	16	13
0.312	3	26	14	16	13
0.404	4	26	14	16	15
0.590	5	27	16	16	21

Murty's method is known to blow up, except when variable costs are predominantly high (Ref. 14). This method seems to work even when fixed costs are not negligible, as Table 2 shows. Though any definite conclusion needs an extensive computational study, this method does demonstrate significant improvement over Murty's method.

5. Concluding Remarks

In conclusion, the authors would like to address the question of finding an approximate optimal solution which is at an ϵ-distance from the true minimum. In order to determine the ϵ-optimum solution, we need a lower bound on the total cost Z. This can be obtained by using Balinski's approximation described in Section 1. Balinski (Ref. 2) has shown that the optimal objective value of the following transportation problem gives a lower bound on the minimum value of Z:

$$\text{minimize} \quad \sum_{(i,j)} [c_{ij} + d_{ij}/m_{ij}]x_{ij},$$

$$\text{subject to} \quad \sum_j x_{ij} = a_i, \qquad i = 1, \ldots, m,$$

$$\sum_i x_{ij} = b_j, \qquad j = 1, \ldots, n,$$

$$x_{ij} \geq 0.$$

Let the lower bound be denoted by Z_1. Then, we can modify our algorithm by terminating (i) whenever $(Z_u - Z_1) < \epsilon$ or (ii) when all the ranking is completed. It is not known at present what is the quality of the lower bound with respect to the true minimum.

Extensions to the solution of the general fixed-charge problems using the vertex ranking procedure described in this paper are given in Sadagopan (Ref. 12).

References

1. HIRSCH, W. M., and DANTZIG, G. B., *The Fixed Charge Problem*, The Rand Corporation, Report No. RM-1383, 1954.
2. BALINSKI, M. L., *Fixed Cost Transportation Problems*, Naval Research Logistics Quarterly, Vol. 8, No. 1, 1961.
3. KUHN, H. W., and BAUMOL, W. J., *An Approximation Algorithm for the Fixed Charge Transportation Problem*, Naval Research Logistics Quarterly, Vol. 9, No. 1, 1962.
4. COOPER, L., and DREBES, C., *An Approximate Solution Method for the Fixed Charge Problem*, Naval Research Logistics Quarterly, Vol. 14, No. 1, 1967.
5. DENZLER, D. R., *An Approximate Algorithm for the Fixed Charge Problem*, Naval Research Logistics Quarterly, Vol. 16, No. 3, 1969.
6. STEINBERG, D. I., *The Fixed Charge Problem*, Naval Research Logistics Quarterly, Vol. 17, No. 2, 1970.
7. WALKER, W. E., *A Heuristic Adjacent Extreme Point Algorithm for the Fixed Charge Problem*, Management Science, Vol. 22, No. 5, 1976.
8. MURTY, K. G., *Solving the Fixed Charge Problem by Ranking the Extreme Points*, Operations Research, Vol. 16, pp. 268–279, 1968.
9. GRAY, P., *Exact Solution to the Fixed Charge Problem*, Operations Research, Vol. 19, pp. 1529–1538, 1971.
10. KENNINGTON, J., *The Fixed Charge Transportation Problem: A Computational Study with a Branch-and-Bound Code*, AIIE Transactions, Vol. 8, No. 2, 1976.
11. KENNINGTON, J., and UNGER, E., *A New Branch-and-Bound Algorithm for the Fixed Charge Transportation Problem*, Management Science, Vol. 22, No. 10, 1976.
12. SADAGOPAN, S., *On Ranking the Extreme Points of Convex Polyhedra*, Purdue University, MS Thesis, 1977.
13. MCKEOWN , P. G., Private Communication, 1978.
14. MCKEOWN, P. G., *A Vertex Ranking Procedure for Solving the Linear Fixed Charge Problem*, Operations Research, Vol. 23, No. 6, 1975.