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On Some Extensions of Bernstein's Theorem

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A well-known theorem of Bernstein asserts that a C^2 function which satisfies the minimal surface equation on the whole of $\mathbb{R}^2$ must be linear. This result was extended to a class of equations corresponding to parametric elliptic functionals by Jenkins [9], and to the entire class of $C^2(\mathbb{R}^2)$ functions whose graphs have quasiconformal Gauss map by Simon [11]. For higher dimensions the result was shown to be true for solutions of the minimal surface equation which are defined over the whole of $\mathbb{R}^n$, $n \leqq 7$, by Simons [13], using (in part) an argument of Fleming [7] and De Giorgi [3]. The result was shown to be false for $n > 7$ by Bombieri De Giorgi and Giusti [2].

We here wish to discuss the higher dimensional case ($n \geqq 3$) for the same class of equations treated in the case $n = 2$ by Jenkins [9]; specifically, we consider the non-parametric Euler-Lagrange equation of a $C^{2,\alpha}$ parametric elliptic functional, with integrand not depending explicitly on the spatial variables. Our main result is that a Bernstein theorem always holds for such equations in case $n = 3$. The result is also shown to hold up to $n = 7$ provided the integrand of the associated parametric functional is close enough, in the C^3 topology, to the area integrand.

These results will be proved by first obtaining pointwise curvature estimates of a kind that were established for solutions of the minimal surface equation by Heinz [8] and extended to higher dimensions by Schoen, Simon, Yau [10] and Simon [12]. The main results appear in Theorem 1 and its corollaries.

§ 1. Notation

$n \geqq 2$ denotes a fixed integer;

$$x_0 = (x_{0_1}, \ldots, x_{0_{n+1}}) \in \mathbb{R}^{n+1}, \qquad x_0' = (x_{0_1}, \ldots, x_{0_n}) \in \mathbb{R}^n;$$
$$B_\rho(x_0) = \{x \in \mathbb{R}^{n+1} : |x - x_0| < \rho\};$$
$$D_\rho(x_0') = \{x \in \mathbb{R}^n : |x - x_0'| < \rho\};$$

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$\mathscr{H}^k$ denotes k-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$;

$\mathscr{L}^{n+1}$, $\mathscr{L}^n$ denote Lebesgue measure in $\mathbb{R}^{n+1}$, $\mathbb{R}^n$ respectively;

$\mathscr{F}$ will denote the set of functions F which are defined on $\mathbb{R}^{n+1}$, which have locally Hölder continuous second partial derivatives on $\mathbb{R}^{n+1} \sim \{0\}$ and which satisfy the conditions:

(i) $F(\mu p) = \mu F(p), \quad \mu > 0, \ p \in \mathbb{R}^{n+1}$,

(i) $F(p) \geqq |p|, \quad p \in \mathbb{R}^{n+1}$,

(iii) $\sum_{i,j=1}^{n+1} F_{p_i p_j}(p)\, \xi_i \xi_j \geqq |p|^{-1} |\xi'|^2, \quad \xi' = \xi - \frac{p}{|p|}\left(\xi \cdot \frac{p}{|p|}\right)$,
$\xi \in \mathbb{R}^{n+1}, \ p \in \mathbb{R}^{n+1} \sim \{0\}$.

Associated with a given $F \in \mathscr{F}$ we have the positive parametric elliptic functional $\mathbf{F}$ defined as follows:

If M is an oriented C^2 hypersurface in $\mathbb{R}^{n+1}$, with continuous unit normal ν and with $\mathscr{H}^n(M) < \infty$, then

$$\mathbf{F}(M) = \int_M F(\nu(x))\, d\mathscr{H}^n(x).$$

Notice that this definition depends on the choice of unit normal ν; we therefore always take "oriented hypersurface M" to mean a pair M, ν where ν is a continuous unit normal for M.

Corresponding to the parametric functional $\mathbf{F}$ we have the non-parametric functional Ψ defined for $C^2(D_\rho(x_0'))$ functions u by

$$\Psi(u) = \mathbf{F}(M_u),$$

where $M_u = \operatorname{graph} u$ (with unit normal $(-Du, 1)/\sqrt{1+|Du|^2}$). One easily checks that we can write

$$\Psi(u) = \int_{D_\rho(x_0')} F(-Du(x), 1)\, d\mathscr{L}^n(x).$$

The Euler-Lagrange equation for extremals of this functional is

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} F_{p_i}(-Du(x), 1) = 0. \tag{1}$$

Given a C^2 oriented hypersurface M with

$$\mathscr{H}^n(M \cap K) < \infty \qquad \text{for each compact } K \subset \mathbb{R}^{n+1}, \tag{2}$$

we will let $[\![M]\!]$ denote the associated rectifiable current; that is, for any smooth n-form ω with compact support in $\mathbb{R}^{n+1}$ we define

$$[\![M]\!](\omega) = \int_M \omega,$$

where the integral on the right is taken in the usual sense of differential geometry. (Actually, to be strictly precise we should write $\int_M i^{\#}\omega$ on the right, where i is the inclusion map $M\subset\mathbb{R}^{n+1}$.) The boundary of $[\![M]\!]$, denoted $\partial[\![M]\!]$, is the $(n-1)$-dimensional current defined by

$$\partial[\![M]\!](\omega)=[\![M]\!](d\omega),$$

whenever ω is a smooth $(n-1)$-form with compact support in $\mathbb{R}^{n+1}$. This current of course corresponds to the oriented set theoretic boundary ∂M in case M is a compact manifold-with-boundary (by Stokes' theorem). We define the *regular set*, $\operatorname{reg} M$, of M by

$$\operatorname{reg} M=\{x\in\bar{M}\sim\operatorname{spt}\partial[\![M]\!]\colon \bar{M} \text{ is a } C^2 \text{ hypersurface in some neighbourhood of } x\}.$$

(Here $\operatorname{spt}\partial[\![M]\!]=\text{support of }\partial[\![M]\!]=\mathbb{R}^{n+1}\sim\bigcup W$, where the union is taken over all open $W\subset\mathbb{R}^{n+1}$ such that $\partial[\![M]\!](\omega)=0$ whenever ω has compact support in W.) The *singular set*, $\operatorname{sing} M$, of M is defined by

$$\operatorname{sing} M=\bar{M}\sim\operatorname{reg} M.$$

Notice that we always have $M\subset\operatorname{reg} M$ (whenever M is an oriented C^2 hypersurface); if

$$\mathscr{H}^n(\operatorname{sing} M\sim\operatorname{spt}\partial[\![M]\!])=0,\tag{3}$$

then after redefinition on a set of $\mathscr{H}^n$-measure zero, we can arrange that

$$M=\operatorname{reg} M,\qquad \operatorname{sing} M=\bar{M}\sim M.\tag{4}$$

$\mathscr{U}$ will henceforth denote the collection of oriented C^2 hypersurfaces M satisfying (2), (3), and (4). $\mathscr{U}(x_0,\rho)$ will denote the set of $M\in\mathscr{U}$ with $x_0\in\bar{M}$ and $\operatorname{spt}\partial[\![M]\!]\subset\mathbb{R}^{n+1}\sim B_\rho(x_0)$.

We say $M\in\mathscr{U}$ is **F**-minimizing in A (A any open subset of $\mathbb{R}^{n+1}$) if for each bounded open V with $\bar{V}\subset A$ we have

$$\mathbf{F}(M\cap V)\leqq\mathbf{F}(N)$$

whenever $N\in\mathscr{U}$ satisfies $\bar{N}\subset A$ and $\partial[\![M\cap V]\!]=\partial[\![N]\!]$.

$\mathscr{F}_j$, where $j\geqq 0$, will denote the collection of $F\in\mathscr{F}$ with the property that

$$\mathscr{H}^j(\operatorname{sing} M\sim\operatorname{spt}\partial[\![M]\!])=0$$

for every $M\in\mathscr{U}$ which is **F**-minimizing in $\mathbb{R}^{n+1}$. One of the main results established in [1] is that $\mathscr{F}_{n-2}=\mathscr{F}$ whenever $n\geqq 2$.

For $F\in\mathscr{F}$, $\mathscr{M}_F(x_0,\rho)$ will denote the collection of $M\in\mathscr{U}(x_0,\rho)$ such that $M\cap B_\rho(x_0)$ is **F**-minimizing in $B_\rho(x_0)$ and such that $\bar{M}\cap B_\rho(x_0)=\partial U_M\cap B_\rho(x_0)$ for some open set $U_M\subset\mathbb{R}^{n+1}$. (In dealing with minimizing hypersurfaces in $\mathscr{M}_F(x_0,\rho)$ instead of rectifiable currents, we are losing no generality by virtue of the regularity theorem (see [1] Theorem (1.2)) and a well-known decomposition argument ([5, 4.5.17]).)

$\mathcal{M}'_F(x_0, \rho)$ will denote the collection of $M \in \mathcal{U}(x_0, \rho)$ which can be represented in the non-parametric form $x_{n+1} = u(x_1, \ldots, x_n)$, $(x_1, \ldots, x_n) \in D_\rho(x'_0)$, where u is a C^2 function satisfying (1) on $D_\rho(x'_0)$. Notice that each $M \in \mathcal{M}'_F(x_0, \rho)$ is **F**-minimizing in $D_\rho(x'_0) \times \mathbb{R}$ ([1] Lemma (2.1)), and hence $\mathcal{M}'_F(x_0, \rho) \subset \mathcal{M}_F(x_0, \rho)$.

§ 2. Main Results

Theorem 1. *Suppose $F \in \mathcal{F}_0$. Then there is a constant $c > 0$ such that for any $M \in \mathcal{M}_F(x_0, \rho)$ one has*

$$\sum_{i=1}^{n} \kappa_i^2(x_0) \leqq c/\rho^2, \tag{5}$$

where $\kappa_1(x_0), \ldots, \kappa_n(x_0)$ are the principal curvatures of M at x_0.

If $F \in \mathcal{F}_1$ then there is a constant c such that (5) *holds for each $M \in \mathcal{M}'_F(x_0, \rho)$. In particular, by letting $\rho \to \infty$ in* (5), *we deduce that any C^2 function u which satisfies* (1) *on the whole of $\mathbb{R}^n$ must be linear.*

Before giving the proof of this theorem we wish to discuss some consequences. As we mentioned in the introduction, we always have $\mathcal{F}_{n-2} = \mathcal{F}$; hence we can immediately deduce the following corollary.

Corollary 1. *If $n = 2$ then for any $F \in \mathcal{F}$ there is a constant c such that* (5) *holds for any $M \in \mathcal{M}_F(x_0, \rho)$.*

If $n = 3$ then for any $F \in \mathcal{F}$ there is a constant c such that (5) *holds for any $M \in \mathcal{M}'_F(x_0, \rho)$. In particular, if u is a $C^2(\mathbb{R}^3)$ solution of* (1), *then u is linear.*

To describe some further consequences of Theorem 1, we introduce the *area integrand* $A \in \mathcal{F}$, defined by $A(p) = |p|$, $p \in \mathbb{R}^{n+1}$. According to [1], Part II, there is an $\eta > 0$ such that if $F \in \mathcal{F}$ and if

$$\begin{aligned} &|F(v) - A(v)| + |F_p(v) - A_p(v)| + \sum_{i,j=1}^{n+1} |F_{p_i p_j}(v) - A_{p_i p_j}(v)| \\ &\quad + \sum_{i,j,k=1}^{n+1} |F_{p_i p_j p_k}(v) - A_{p_i p_j p_k}(v)| < \eta \end{aligned} \tag{6}$$

for all $v \in \mathbb{R}^{n+1}$ with $|v| = 1$, then $F \in \mathcal{F}_{(n-6\frac{1}{2})_+}$, where $(n - 6\frac{1}{2})_+ = \max\{n - 6\frac{1}{2}, 0\}$. (Actually, for each $\varepsilon > 0$ there is η such that (6) implies $F \in \mathcal{F}_{(n-7+\varepsilon)_+}$.) Thus we can deduce the following corollary from Theorem 1.

Corollary 2. *In case $n \leqq 6$ there is an $\eta > 0$ with the following property: If $F \in \mathcal{F}$ satisfies* (6) *then there is a constant c such that* (5) *holds for each $M \in \mathcal{M}_F(x_0, \rho)$.*

In case $n \leqq 7$ there is an $\eta > 0$ with the following property: If $F \in \mathcal{F}$ satisfies (6) *then there is a constant c such that* (5) *holds for each $M \in \mathcal{M}'_F(x_0, \rho)$. In particular, any $C^2(\mathbb{R}^n)$ function satisfying* (1) *must be linear.*

We now wish to prove Theorem 1. First we remark (see e.g. [1], Theorem (1.2)) that to prove an inequality of the form (5), it suffices to show that for each $\varepsilon > 0$ there exists a constant $\theta \in (0, 1)$ (depending on ε and F) such that

$$M \cap B_{\theta\rho}(x_0) \subset \{x : \operatorname{dist}(x, H_M) < \varepsilon\theta\rho\}$$

for some hyperplane $H_M \subset \mathbb{R}^{n+1}$. Theorem 1 is thus an immediate consequence of the following lemma.

Lemma 1. *Let $\varepsilon > 0$ and $F \in \mathscr{F}_0$. Then there exists $\theta \in (0,1)$ with the following property: If $M \in \mathscr{M}_F(x_0, \rho)$, then for some hyperplane H_M we have $x_0 \in H_M \subset \mathbb{R}^{n+1}$ and*

$$M \cap B_{\theta\rho}(x_0) \subset \{x : \operatorname{dist}(x, H_M) < \varepsilon\theta\rho\}. \tag{7}$$

In case $F \in \mathscr{F}_1$, there is a $\theta \in (0,1)$ such that (7) *holds (for suitable H_M with $x_0 \in H_M \subset \mathbb{R}^{n+1}$) whenever $M \in \mathscr{M}'_F(x_0, \rho)$.*

Proof. If the first part of the lemma is false, then there is an $\varepsilon > 0$, $F \in \mathscr{F}_0$ and a sequence $\{M_r\} \subset \mathscr{M}_F(x_0, \rho)$ such that

$$M_r \cap B_{(1/r)\rho}(x_0) \not\subset \left\{x : \operatorname{dist}(x, H) < \varepsilon \frac{1}{r} \rho\right\} \tag{8}$$

for every hyperplane H with $x_0 \in H \subset \mathbb{R}^{n+1}$. However, letting U_r be such that $\partial U_r \cap B_\rho(x_0) = \bar{M}_r \cap B_\rho(x_0)$, we know that from standard compactness, semicontinuity, and regularity theorems (see e.g. [1], I.1(33), Theorems (1.1), (1.2) and Remark (1) following Theorem (1.2)) we can deduce the following. There is an open $U \subset B_\rho(x_0)$ and an $M \in \mathscr{M}_F(x_0, \rho)$ such that $\bar{M} \cap B_\rho(x_0) = \partial U \cap B_\rho(x_0)$ and such that

$$\mathscr{L}^{n+1}((U_k \sim U) \cup (U \sim U_k)) \to 0 \quad \text{as } k \to \infty \tag{9}$$

for some subsequence $\{k\} \subset \{r\}$ [1]. But we are here assuming that $F \in \mathscr{F}_0$, hence $(\bar{M} \sim M) \cap B_\rho(x_0) = \varnothing$. Then $x_0 \in M$ and we have a tangent hyperplane H_M of M at x_0. By (9) and [1, Remark (3) following Theorem (1.2)] we know there is a $\theta_0 \in (0,1)$ such that

$$M_k \cap B_{\theta\rho}(x_0) \subset \{x : \operatorname{dist}(x, H_M) < \varepsilon\theta\rho\}$$

for all $\theta < \theta_0$ and for all sufficiently large k. This contradicts (8).

We now turn to the proof of the second part of the lemma. If this part is false, then there is an $\varepsilon > 0$, $F \in \mathscr{F}_1$ and a sequence $\{M_r\} \subset \mathscr{M}'_F(x_0, \rho)$ such that again (8) holds for every hyperplane H. Then let u_r be a $C^2(D_\rho(x'_0))$ solution of (1) such that graph $u_r = M_r$, and let

$$U_r = \{x : x_{n+1} < u_r(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in D_\rho(x'_0)\} \cap B_\rho(x_0).$$

We have (cf. the proof of the first part of the lemma) a subsequence $\{k\} \subset \{r\}$ and an open $U \subset B_\rho(x_0)$ satisfying (9), and an $M \in \mathscr{M}_F(x_0, \rho)$ satisfying $\bar{M} \cap B_\rho(x_0) = \partial U \cap B_\rho(x_0)$. Now we are given that $F \in \mathscr{F}_1$, hence $\mathscr{H}^1((\bar{M} \sim M) \cap (B_\rho(x_0)) = 0$. We now recall (see [1] I.2(18)) that if

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1 We know $x_0 \in \bar{M}$ because $\mathscr{H}^n(M_k \cap B_\sigma(x_0)) \to \mathscr{H}^n(M \cap B_\sigma(x_0))$ a.e. $\sigma \in (0, \rho)$, while $\liminf_{r \to \infty} \mathscr{H}^n(M_r \cap B_\sigma(x_0)) > 0$ for each $\sigma \in (0, p)$ (see e.g. [1], Theorem (1.1) and I.1(28))

$$v^r=(v_1^r,\dots,v_{n+1}^r)=(-Du_r,1)/\sqrt{1+|Du_r|^2},$$

then

$$v_{n+1}^r\sum_{i,j,l=1}^{n+1}F_{p_ip_j}(v^r)\,\delta_i^r v_l^r\delta_j^r v_l^r+\sum_{i,j=1}^{n+1}\delta_i^r(F_{p_ip_j}(v^r)\,\delta_j^r v_{n+1}^r)=0,$$

where $\delta^r=(\delta_1^r,\dots,\delta_n^r)$ denotes the gradient operator on M_r. Because of the convergence of $M_k\cap B_\rho(x_0)$ to M described in [1, Remark (3) following Theorem (1.2)], and because of the Harnack inequality given in Lemma (2.7) of [1], we know that if v denotes the unit normal of M pointing out of U then for each component M_* of M we have either $v_{n+1}\equiv 0$ or $v_{n+1}>0$. Let us consider first the case when $v_{n+1}>0$ at each point of a component M_*. Let π denote the projection of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ defined by $\pi(x_1,\dots,x_{n+1})=(x_1,\dots,x_n)$. Then $\pi(M_*)$ is an open subset of $D_\rho(x_0')$. Also (cf. [1] Lemma (1.1))[2], because $\mathscr{H}^{n-1}((\overline{M}_*\sim M_*)\cap B_\rho(x_0))=0$, we can find an open set $V\subset B_\rho(x_0)$ such that

$$\overline{M}_*\cap B_\rho(x_0)=\partial V\cap B_\rho(x_0)=\partial\overline{V}\cap B_\rho(x_0), \tag{10}$$

and such that the outward unit normal of V has positive $(n+1)^{\text{st}}$ component at each point of $M_*\cap B_\rho(x_0)$. Next we want to show that if $x_0\in\overline{M}_*\sim M_*$, then x_0' is an *interior* point of the set $\pi(\overline{M}_*\cap B_\rho(x_0))$. Indeed, if $x_0\in\overline{M}_*\sim M_*$ and $x_0'\notin$interior $(\pi(\overline{M}_*\cap B_\rho(x_0)))$, then it is not difficult to see (with the aid of (10)) that we would then have a whole vertical line segment contained in $\overline{M}_*\cap B_\rho(x_0)$. Since we are given $\mathscr{H}^1((\overline{M}_*\sim M_*)\cap B_\rho(x_0))=0$, it would then follow that there is a vertical line segment contained in M_*, thus contradicting the fact that $v_{n+1}>0$ on M_*. Thus if $x_0\in\overline{M}_*\sim M_*$ then we have $x_0'\in$interior $(\pi(\overline{M}_*\cap B_\rho(x_0)))$. Then it follows that for suitably small $\sigma\in(0,\rho)$, we have the representation

$$M_*\cap(D_\sigma(x_0')\times\mathbb{R})=\operatorname{graph}u\cap(D_\sigma(x')\times\mathbb{R}),$$

where u is a C^2 function defined on $\overline{D_\sigma(x_0')}\sim K$, and where K is compact with $\mathscr{H}^1(K)=0$. (K in fact is simply $\pi((\overline{M}_*\sim M_*)\cap(\overline{D}_\sigma(x_0')\times\mathbb{R}))$.) Because K satisfies $\mathscr{H}^1(K)=0$ we can assume (possibly by choosing a smaller σ) that $\partial D_\sigma(x_0')\cap K=\emptyset$. Furthermore u must clearly satisfy the Euler-Lagrange equation on $D_\sigma(x_0')\sim K$. Then, by Theorem A of the appendix, we know that u can be extended to be a $C^2(\overline{D_\sigma(x_0')})$ function. Hence, we can deduce that $\overline{M}_*\cap B_\sigma(x_0)$ is a C^2 hypersurface. This is of course trivially true (for sufficiently small σ) if $x_0\in M_*$ to begin with.

We now consider the possibility that the component M_* of M is such that $v_{n+1}\equiv 0$ on M_*. In view of the fact that $\mathscr{H}^{n-1}((\overline{M}_*\sim M_*)\cap B_\rho(x_0)=0$, it is quite easy to check that in this case we can write $\overline{M}_*=A\times\mathbb{R}$ for some closed subset $A\subset\mathbb{R}^n$. But the singular set of $A\times\mathbb{R}$ consists of a union of vertical lines, hence since $\mathscr{H}^1((\overline{M}_*\sim M_*)\cap B_\rho(x_0)=0$, we deduce that $\overline{M}_*\cap B_\rho(x_0)$ is a C^2 hypersurface.

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[2] Notice that $M_*\subset M$ and $\mathscr{H}^n(M\cap B_\sigma(x_1))\leq c\sigma^n$ whenever $x_1\in\overline{M}$ and $\sigma\in(0,\rho-|x_0-x_1|)$ (by [1], I.1(33)); this fact is needed if one wishes to apply Lemma (1.1) of [1] to M_*

Thus we have shown that *each* component M_* of M is such that $\bar{M}_* \cap B_\sigma(x_0)$ is a C^2 hypersurface for suitable $\sigma > 0$. From the fact that $\mathscr{H}^{n-1}((\bar{M} \sim M) \cap (D_\rho(x'_0) \times \mathbb{R})) = 0$, no two of these hypersurfaces $\bar{M}_* \cap B_\sigma(x_0)$ can intersect non-tangentially. Furthermore no distinct two of these hypersurfaces can make contact which is tangential at each point, and no more than a finite number of components M_* can intersect a given compact subset of $B_\rho(x_0)$. (See Lemma (2.4) of [1] and the latter part of the proof of Corollary (3.1) of [1].)

We thus finally deduce $x_0 \in M$, and the proof is completed as for the first part of the lemma.

Appendix

Here, using an argument essentially due to Finn [6], we wish to consider removability of singularities of solutions of divergence-form equations

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(Du) = 0 \tag{A1}$$

on a C^2 uniformly convex domain $\Omega \subset \mathbb{R}^n$. Here $A_i = A_i(p)$ are $C^{1,\alpha}$ functions of $p \in \mathbb{R}^n$ such that

$$|A_i(p)| \leqq c_1, \qquad p \in \mathbb{R}^n, \tag{A2}$$

$$\sum_{i,j=1}^{n} A_{ip_j}(p)\, \xi_i \xi_j > 0, \qquad p \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \sim \{0\}. \qquad \text{A3)}$$

Notice that such structural conditions clearly hold in case $A_i = F_{p_i}$, with F as in (1).

The theorem we want to prove is the following.

Theorem A. *Suppose K is a compact subset of Ω with $\mathscr{H}^{n-1}(K) = 0$ and suppose $u \in C^2(\bar{\Omega} \sim K)$ satisfies* (A1) *on $\Omega \sim K$. Then u extends to a $C^2(\bar{\Omega})$ solution of* (A1).

Remark. In the case of the minimal surface equation, a slightly more general theorem was proved by DeGiorgi and Stampacchia [4]. (In [4] an analogous theorem to that above, for the minimal surface equation, was proved in case Ω was an *arbitrary* domain in $\mathbb{R}^n$.)

Proof of Theorem A. Let $\varepsilon > 0$ be given. Because $\mathscr{H}^{n-1}(K) = 0$ we can use the definition of Hausdorff measure to find real numbers $\delta_1, \ldots, \delta_N > 0$ and points $x^{(1)}, \ldots, x^{(N)} \in K$ such that $K \subset \bigcup_{i=1}^{N} D_{\delta_1}(x^{(i)})$ and $\sum_{i=1}^{N} \delta_i^{n-1} < \varepsilon$. Furthermore we know from standard theory of elliptic equations that we can find a $C^2(\Omega) \cap C^1(\bar{\Omega})$ function w such that $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(Dw) = 0$ and $w = u$ on $\partial\Omega$. Writing this equation in weak form we have

$$\int_\Omega A_i(Dw)\, D_i \zeta\, dx = 0, \qquad \zeta \in C_0^1(\Omega). \tag{A4}$$

Provided $\zeta=0$ in a neighbourhood of K, we also have

$$\int_\Omega A_i(Du)D_i\zeta\,dx=0. \tag{A5}$$

We now let γ_i, $i=1,\dots,N$, be C^1 functions such that $\gamma_i=0$ on $D_{\delta_i}(x^{(i)})$, $\gamma_i\equiv 1$ on $\mathbb{R}^n\sim D_{2\delta_i}(x^{(i)})$, $|D\gamma_i|\leqq 2/\delta_i$ on $\mathbb{R}^n$ and $\gamma_i\in[0,1]$ on $\mathbb{R}^n$. Then we choose $\zeta=\left(\prod_{i=1}^N \gamma_i\right)$ arctan $(u-w)$ in (A4), (A5). Substracting (A4) and (A5) we then have

$$\int_\Omega (1+(u-w)^2)^{-1}\sum_{i=1}^n (A_i(Du)-A_i(Dw))(D_iu-D_iw)\prod_{j=1}^N \gamma_j\,dx$$
$$=-\int_\Omega\sum_{i=1}^n\left[(A_i(Du)-A_i(Dw))\sum_{k=1}^N\left\{\left(\sum_{\substack{j=1\\j\neq k}}^N\gamma_j\right)D_i\gamma_k\right\}\right]\arctan(u-w)\,dx.$$

Since

$$A_i(Du)-A_i(Dw)=\int_0^1\frac{d}{dt}A_i(Dw+t(Du-Dw))\,dt$$
$$=\int_0^1\sum_{j=1}^n A^t_{ip_j}(D_ju-D_jw)\,dt,$$

where

$$A^t_{ip_j}=A_{ip_j}(Dw+t(Du-Dw)),$$

this gives, by virtue of (A2),

$$\int_0^1\int_\Omega (1+(u-w)^2)^{-1}\sum_{i,j=1}^n A^t_{ip_j}(D_iu-D_iw)(D_ju-D_jw)\left(\prod_{k=1}^N\gamma_k\right)dt\,dt$$
$$\leqq n\pi c_1\sum_{k=1}^N\int_\Omega\prod_{\substack{j=1\\j\neq k}}^N\gamma_j|D\gamma_k|\,dx.$$

Since $\sum_{i=1}^N\delta_i^{n-1}<\varepsilon$, we can let $\varepsilon\to 0$ (note that then $\max\delta_i\to 0$). Thus we obtain, after using (A3) and Fatou's theorem on the left,

$$\int_0^1\int_{\Omega\sim K}(1+(u-w)^2)^{-1}\sum_{i,j=1}^n A^t_{ip_j}(D_iu-D_iw)(D_ju-D_jw)\,dx=0.$$

Hence, by (A3), $Du=Dw$ a.e. on $\Omega\sim K$. But $u=w$ on $\partial\Omega$, and hence we deduce $u=w$ on $\Omega\sim K$. Since $w\in C^2(\Omega)$ this completes the proof.

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