Computers & Industrial Engineering 57 (2009) 304–309

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

Optimal preventive maintenance policy for leased equipment using failure rate reduction

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ARTICLE INFO

Article history:
Received 30 September 2007
Received in revised form 14 September 2008
Accepted 28 November 2008
Available online 7 December 2008

Keywords:
Preventive maintenance
Minimal repair
Failure rate reduction
Lease contract

ABSTRACT

This study proposes a maintenance scheme for leased equipment using failure rate reduction method and derives an optimal preventive maintenance (PM) policy that minimizes expected total cost. Under the proposed maintenance scheme, the lessor (equipment owner) rectifies failures with minimal repairs within the lease period, and the lessor may incur a penalty when repair time exceeds a time limit as specified in the lease contract. To reduce the expected total cost, the lessor may employ PM actions to decrease the number of possible failures. In this study, an efficient algorithm is developed to derive the optimal PM policy and a closed-form solution is obtained for the case where the lifetime distribution of the equipment is Weibull. The expected total cost using the optimal PM policy under the proposed maintenance scheme is then compared with the performance of other policies under various maintenance schemes through numerical examples.

1. Introduction

Most businesses require various types of equipment to manufacture their products or to provide service for customers. Due to rapid technological innovations, increased complexity of equipment, and the cost of professional technicians required to maintain equipment, it may not be economical for these businesses to own certain equipment. Therefore, there is a trend toward leasing instead of buying equipment (Glickman & Berger, 1976; Nisbet & Ward, 2001). For leased equipment, the maintenance of the equipment is usually specified in a lease contract provided by the lessor (equipment owner) to ensure that the equipment could fulfill its intended purpose (Barlow & Hunter, 1960). As a result, the equipment was bundled with maintenance and offered by the lessor under a leased contract (Martin, 1997; Murthy & Asgharizadeh, 1999).

In general, two types of maintenance actions are considered in a lease contract – corrective maintenance (CM) and preventive maintenance (PM). Corrective maintenance rectifies failed equipment back to its operational status, whereas PM improves the operational status of the leased equipment, thereby decreasing the likelihood of equipment failure. There is a vast literature dealing with maintenance policies (Barlow & Hunter, 1960; Chun, 1992; Glickman & Berger, 1976; Jack & Dagpunar, 1994; Jaturonnatee, Murthy, & Boondiskulchok, 2006; Murthy & Yeung, 1995; Nakagawa, 1981; Nguyen & Murthy, 1988; Pham & Wang, 1996; Pongpech & Murthy, 2006; Seo & Bai, 2004; Sheu, Lin, & Liao, 2006; Wang, 2002; Yeh & Chen, 2006; Yeh & Lo, 2001). In practice, minimal repair is the most commonly performed CM when restoring failed equipment (Nakagawa, 1981; Nakagawa & Kowada, 1983). Following minimal repair, the equipment is operational; however, the failure rate remains unchanged. When the time needed for minimal repair exceeds the limit specified in the lease contract, the lessor might incur a penalty since it may cause serious damage to the lessee (equipment user). Therefore, a lessor must undertake some remedial measures to avoid costs incurred by equipment failures.

Most lessors undertake PM to reduce the number of equipment failures within the lease period. Preventive maintenance is a tradeoff between PM costs and failure costs. Usually, as PM is planned, the cost of PM is less than the cost incurred when equipment fails. Numerous PM policies have been proposed and studied under various situations, such as perfect or imperfect maintenance (Brown & Proschan, 1983; Jack & Dagpunar, 1994; Jaturonnatee et al., 2006; Pham & Wang, 1996; Sheu et al., 2006), age reduction or failure rate reduction (Chan & Shaw, 1993; Jaturonnatee et al., 2006; Nakagawa, 1981; Pongpech & Murthy, 2006), and periodical or sequential maintenance (Chun, 1992; Jack & Dagpunar, 1994; Pongpech & Murthy, 2006; Seo & Bai, 2004; Yeh & Chen, 2006; Yeh & Lo, 2001).

Jaturonnatee et al. (2006) developed a sequential PM scheme using failure rate reduction that considers the number of PM actions, PM degree, and time epochs simultaneously. Their

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doi:10.1016/j.cie.2008.11.025

maintenance scheme is very general but not easy to implement in practice. For practical needs, Pongpech and Murthy (2006) reduced Jaturonnatee's scheme to a periodical PM scheme in which the PM actions are carried out periodically with various maintenance degrees. Since this maintenance scheme is a special case of Jaturonnatee's scheme, the resulting performance is not as good as Jaturonnatee's. This study proposes a maintenance scheme, in which preventive maintenance actions are taken to reduce the failure rate of the leased equipment by the fixed amount specified in the lease contract. We employ a different approach to simplify Jaturonnatee's scheme. Under our approach, the PM actions are performed sequentially with a fixed maintenance degree. As we will see later on, the performance of the proposed maintenance scheme is better than Pongpech's scheme and is close to that of Jaturonnatee's scheme.

The remainder of this paper is as follows. The mathematical model of the proposed maintenance scheme is developed in Section 2. In Section 3, the optimal PM policy is examined and an efficient algorithm is proposed for leased equipment with general lifetime distributions. In Section 4, the optimal PM policy is derived and a reduced algorithm is proposed for the Weibull lifetime distribution. The performance of PM is evaluated via numerical examples, the expected total cost is compared with two other maintenance schemes (Jaturonnatee et al., 2006; Pongpech & Murthy, 2006), and some practical applications are given in Section 5. Finally, conclusions are drawn in Section 6.

2. Mathematical formulation

Given that the failure rate of equipment, $h(t)$, is a strictly increasing function (degenerating equipment) over time t with $h(0) = 0$; within the lease period, failed equipment is repaired using minimal repair by the lessor with a fixed repair cost C_m. Following minimal repair, the equipment is operational; however, its failure rate remains the same as that just prior to failure. Assume that any minimal repair requires a random amount of repair time T_m, which follows a general cumulative distribution function G. Each failure incurs a fixed penalty cost C_n to the lessor. Furthermore, if the repair time exceeds a predetermined value τ, then there is a penalty C_τ per unit time to the lessor for the delay in restoring the equipment back to operational condition. That is, the total expected cost to the lessor at each failure is $C_m + C_n + C_\tau \int_\tau^\infty \overline{G}(t)\mathrm{d}t$.

To reduce the number of possible failures, the lessor may perform n PM actions within the lease period. After performing the ith PM action at time epoch t_i, the failure rate of the equipment is reduced by a fixed amount $\delta \geqslant 0$, where $0 < t_1 < t_2 < \cdots < t_n < L$. In practice, the cost of a PM action is a non-negative and non-decreasing function of the maintenance degree $\delta \geqslant 0$. In this paper, we consider the case where the PM cost function $C_{pm}(\delta)$ increases linearly with maintenance degree δ; those is, $C_{pm}(\delta) = a + b\delta$ where $a > 0$ and $b \geqslant 0$ are the fixed cost and the variable cost for each PM action, respectively. Furthermore, it is assumed that the time required for performing minimal repair and PM actions are both very short compared to the leased period and, hence, are negligible.

Without any PM actions, the failure process of equipment is a non-homogeneous Poisson process (NHPP) with intensity $h(t)$, since minimal repairs (Nakagawa, 1981; Nakagawa & Kowada, 1983) rectify failures. Consequently, the expected number of failures within the interval $[0, t]$ is $H(t) = \int_0^t h(u)\mathrm{d}u$. When PM actions are performed, the equipment failure process at each interval $[t_i, t_{i+1}]$ is still an NHPP. After the ith PM action; however, the failure intensity becomes $h(t_i) - i\delta \geqslant 0$ for all $i = 1,2,\ldots,n$ as shown in Fig. 1.

According to NHPP, the expected number of failures within the lease period under the proposed PM scheme becomes

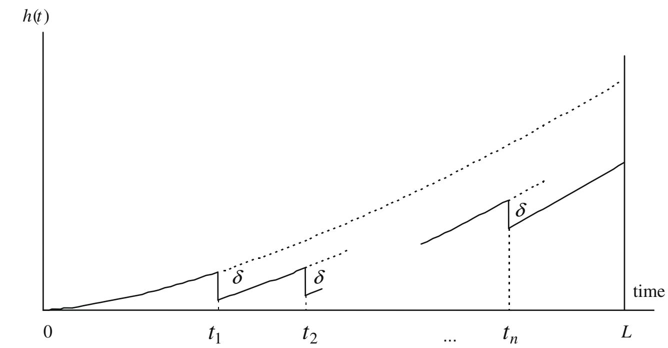

Fig. 1. Preventive maintenance scheme under fixed failure rate reduction.

$$\Lambda \equiv \Lambda(n, \delta, \underline{\mathbf{t}}) = \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} [h(t) - i\delta]\mathrm{d}t = H(L) - \delta \sum_{i=1}^{n} (L - t_i), \tag{1}$$

where $\underline{\mathbf{t}} = (t_1,t_2,\ldots,t_n)$ represent the vector of time epochs to perform PM actions. The expected total cost to the lessor within the lease period includes minimal repair cost, penalty cost, and PM cost. As a result, the expected total cost becomes

$$\begin{aligned} C(n, \delta, \underline{\mathbf{t}}) &= [C_m + C_n + C_\tau G(\overline{\tau})]\Lambda + nC_{pm}(\delta) \\ &= KH(L) + nC_{pm}(\delta) - K\delta \sum_{i=1}^{n} (L - t_i), \end{aligned} \tag{2}$$

where $K = C_m + C_n + C_\tau \int_\tau^\infty \overline{G}(t)\mathrm{d}t$ represents the expected cost for each failure. Without PM actions ($n = 0$), the expected total cost reduces to

$$C_0 \equiv C(0, 0, \underline{\mathbf{t}}; L) = KH(L). \tag{3}$$

The objective of this study is to find an optimal PM policy $(n^*, \delta^*, \underline{\mathbf{t}}^*)$ for the lessor such that the expected total cost in Eq. (2) is minimized. Note that there are $n + 2$ decision variables (including the number of PM actions n, PM degree δ, and time epochs t_i) in the objective function Eq. (2). In the next section, the properties of the optimal PM policy are investigated and an efficient algorithm is developed based on these properties.

3. The optimal PM policy

Observing Eq. (2), it is clear that there is a trade-off between $nC_{pm}(\delta)$ and $K\delta\sum_{i=1}^{n}(L - t_i)$ in finding the optimal policy since $KH(L)$ is a constant. Therefore, if $nC_{pm}(\delta) - K\delta\sum_{i=1}^{n}(L - t_i) \geq 0$ for all $n > 0$, then preventive maintenance is not worthwhile, which implies $n^* = 0$. In this case, the resulting expected cost becomes $C_0 = KH(L)$. On the other hand, when $nC_{pm}(\delta) - K\delta\sum_{i=1}^{n}(L - t_i) < 0$ for all $n > 0$, n^* exists and the optimal policy is derived based on the following mathematical program:

$$\begin{aligned} &\text{Minimize } C(n, \delta, \underline{\mathbf{t}}) = C_0 + nC_{pm}(\delta) - K\delta \sum_{i=1}^{n} (L - t_i) \\ &\text{Subject to } h(t_i) - i\delta \geq 0 \quad \text{for all} \quad i = 1,2,\ldots,n. \end{aligned} \tag{4}$$

Since $h(t)$ is a strictly increasing function of t, the inverse function of the failure rate, h^{-1}, is also a strictly increasing function. Given any $n > 0$ and $\delta > 0$, the following theorem shows the relationship between the optimal time epoch t_i^* and the inverse failure rate function h^{-1}. (Note that all the proofs of the Theorems in this paper are given in the Appendix.)

Theorem 1. *Given any $n > 0$ and $\delta > 0$, if $h(t)$ is a strictly increasing function of t, then $t_i^* = h^{-1}(i\delta)$.*

Theorem 1 shows that the optimal time epoch for performing the ith PM is when $h(t_i) = i\delta$ and hence $t_i^* = h^{-1}(i\delta)$. This result also implies that the failure rate should be reduced to zero after each PM as shown in Fig. 2. Using the result of Theorem 1, the objective function becomes

$$C(n,\delta|\underline{\mathbf{t}}^*) = C_0 + nC_{pm}(\delta) - K\delta\left\{nL - \sum_{i=1}^{n}[h^{-1}(i\delta)]\right\}. \tag{5}$$

Now, there are only two decision variables, n and δ, in Eq. (5) to be determined. To find the optimal (n^*,δ^*), we first consider the case where n is given and determine the optimal maintenance degree δ_n^*. Then, the optimal number of PM actions can be obtained using any search method directly.

Again, if $\sum_{i=1}^{n}[h^{-1}(i\delta)] \geq nL - \frac{nC_{pm}(\delta)}{K\delta}$ for all $n > 0$, then $n^* = 0$ and the resulting expected cost is $C_0 = KH(L)$. Hence, in the following discussion, we will focus on the case where $\sum_{i=1}^{n}[h^{-1}(i\delta)] < nL - \frac{nC_{pm}(\delta)}{K\delta}$. Given any $n > 0$, the following theorem shows that under some reasonable conditions, there exists a unique $\delta_n^* \in \left[0, \frac{h(L)}{n}\right]$ such that the expected total cost is minimized.

Theorem 2. *Given any $n > 0$, the following results hold:*

(i) *If $b - KL \geqslant 0$, then $\delta_n^* = 0$.*
(ii) *If $b - KL < 0$ and $2\left(\frac{\partial\sum_{i=1}^{n}h^{-1}(i\delta)}{\partial\delta}\right) + \delta\left(\frac{\partial^2\sum_{i=1}^{n}h^{-1}(i\delta)}{\partial\delta^2}\right) > 0$, then there exists a unique $\delta_n^* \in \left[0, \frac{h(L)}{n}\right]$ such that the expected total cost is minimized.*

For any prespecified number of PM actions $n > 0$, Theorem 2 shows that if the marginal cost of a PM action b is greater than a constant KL, then the optimal maintenance degree is $\delta_n^* = 0$. In this case, the resulting expected cost is $C_0 + na$, which implies that PM action is not worthwhile and the optimal policy is $(n^*,\delta_n^*) = (0,0)$. On the other hand, if the marginal cost is relatively low, then there exists a unique optimal maintenance degree when the condition $2\left(\frac{\partial\sum_{i=1}^{n}h^{-1}(i\delta)}{\partial\delta}\right) + \delta\left(\frac{\partial^2\sum_{i=1}^{n}h^{-1}(i\delta)}{\partial\delta^2}\right) > 0$ is satisfied. As we will show in the next section, this condition is reasonable since all the Weibull distributions with increasing failure rates satisfy this condition.

Using the result of Theorem 2, the optimal maintenance degree can be easily obtained by any search method. Now, the last decision variable to be determined is the optimal number of PM actions within the lease period. In practice, there is a maximal number $\bar{n}$ of PM actions that can be performed within a finite lease period. Without loss of generality, we may set $\bar{n} = \left[\frac{L}{\tau}\right]$ or any large number and search the optimal value of n from 0 to $\bar{n}$. In summary, the following algorithm can be used to seach for the optimal PM policy $(n^*,\delta^*,\underline{\mathbf{t}}^*)$ for the lessor.

Step 1: If $b - KL \geqslant 0$, then $(n^*,\delta^*,\underline{\mathbf{t}}^*) = (0,0,\mathbf{0})$ and STOP.
Step 2: Set $(n^*,\delta^*,\underline{\mathbf{t}}^*) = (0,0,\mathbf{0})$, $C(n^*,\delta^*,\underline{\mathbf{t}}^*) = C_0$, $\bar{n} = \left[\frac{L}{\tau}\right]$, and n = 1.
Step 3: Search for $\delta_n^* \in \left(0, \frac{h(L)}{n}\right)$ such that $C(n,\delta_n^*|\underline{\mathbf{t}}^*) = \min_{\delta} C(n,\delta|\underline{\mathbf{t}}^*)$.
Step 4: If $C(n,\delta_n^*|\underline{\mathbf{t}}^*) < C(n^*,\delta^*,\underline{\mathbf{t}}^*)$, then set $C(n^*,\delta^*,\underline{\mathbf{t}}^*) = C(n,\delta_n^*|\underline{\mathbf{t}}^*)$ and $(n^*,\delta^*,\underline{\mathbf{t}}^*) = (n,\delta_n^*,\underline{\mathbf{t}}^*)$.
Step 5: If $n = \bar{n}$, then STOP; otherwise, set $n = n + 1$ and go to Step 3.

Although there exists a unique solution for the optimal maintenance degree δ_n^*, the nonlinear search in Step 3 may be time-consuming. In the next section, we show that there is a closed-form solution for δ_n^* when the lifetime distribution is Weibull. This result will significantly increase the efficiency of the above algorithm.

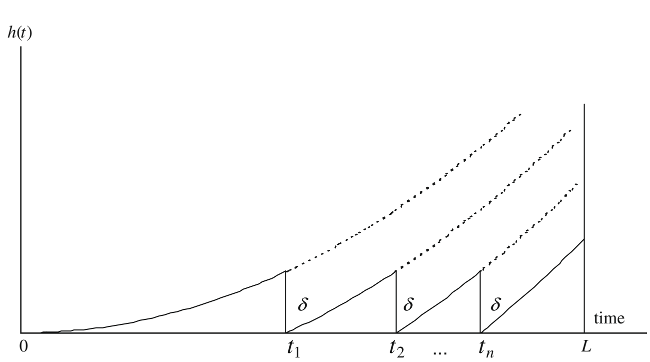

Fig. 2. Optimal PM policy under fixed failure rate reduction.

4. Weibull case

The Weibull case is often discussed in the reliability field due to its flexibility in the shapes of lifetime distributions. This section investigates the proposed PM scheme for the Weibull case and derives a closed-form solution that can be easily applied in practice.

There are two parameters for a Weibull distribution: scale parameter $\alpha > 0$ and shape parameter $\beta > 0$. The failure rate function of a Weibull distribution is $h(t) = \alpha\beta(\alpha t)^{\beta-1}$. If $\beta > 1$, then the failure rate increases in time t. Again, it is assumed that the PM cost function $C_{pm}(\delta) = a + b\delta$, which increases linearly with PM degree δ. Then, from Eq. (4), the mathematical program becomes

$$\text{Minimize } C(n,\delta,\underline{\mathbf{t}}) = K(\alpha L)^{\beta} + n(a + b\delta) - nK\delta L + K\delta\sum_{i=1}^{n} t_i$$
$$\text{Subject to } h(t_i) - i\delta \geq 0 \quad \text{for all} \quad i = 1, 2, \ldots, n. \tag{6}$$

From Theorem 1, we know that $t_i^* = h^{-1}(i\delta)$. For the Weibull case, we have the inverse failure rate function $t = h^{-1}(y) = wy^{\frac{1}{\beta-1}}$ for any $y > 0$, where $w = (\beta^{-1}\alpha^{-\beta})^{\frac{1}{\beta-1}}$. Therefore, $\sum_{i=1}^{n}[h^{-1}(i\delta)]$ in Eq. (5) becomes $w\left(\delta^{\frac{1}{\beta-1}}\right)\left(\sum_{i=1}^{n} i^{\frac{1}{\beta-1}}\right)$. Taking the first derivative of $\sum_{i=1}^{n}[h^{-1}(i\delta)]$ with respect to δ, we have

$$\frac{\partial\sum_{i=1}^{n}[h^{-1}(i\delta)]}{\partial\delta} = w\left(\sum_{i=1}^{n} i^{\frac{1}{\beta-1}}\right)\left(\frac{1}{\beta-1}\right)\left(\delta^{\frac{1}{\beta-1}-1}\right) > 0 \tag{7}$$

for all $\delta > 0$, which means $\sum_{i=1}^{n}[h^{-1}(i\delta)]$ is an increasing function of δ. Furthermore, the second derivative of $\sum_{i=1}^{n}[h^{-1}(i\delta)]$ with respect to δ is

$$\frac{\partial^2\sum_{i=1}^{n}[h^{-1}(i\delta)]}{\partial\delta^2} = \left(\frac{\partial\sum_{i=1}^{n}[h^{-1}(i\delta)]}{\partial\delta}\right)\left(\frac{2-\beta}{\beta-1}\right)(\delta^{-1}). \tag{8}$$

Using Eqs. (7) and (8), the condition $2\left(\frac{\partial\sum_{i=1}^{n}h^{-1}(i\delta)}{\partial\delta}\right) + \delta\left(\frac{\partial^2\sum_{i=1}^{n}h^{-1}(i\delta)}{\partial\delta^2}\right) > 0$ given in Theorem 2 becomes $\left(\frac{\partial\sum_{i=1}^{n}h^{-1}(i\delta)}{\partial\delta}\right)\left(\frac{\beta}{\beta-1}\right) > 0$ which holds for the Weibull case with $\beta > 1$. Therefore, the following theorem holds for the Weibull case.

Theorem 3. *Given any $n > 0$, for the Weibull case, the following results hold.*

(i) *If $b - KL \geqslant 0$, then $\delta_n^* = 0$.*
(ii) *If $b - KL < 0$, then $\delta_n^* = \left\{n\left(L - \frac{b}{K}\right)\left(\frac{1}{w}\right)\left(\frac{1}{\sum_{i=1}^{n} i^{\frac{1}{\beta-1}}}\right)\left(\frac{\beta-1}{\beta}\right)\right\}^{(\beta-1)}$.*

Theorem 3 shows that the optimal maintenance degree has a closed-form solution for any $n > 0$. Therefore, Step 3 in the algorithm given in the previous section can be easily solved, and the optimal number of PM actions can be obtained by searching the optimal value of n from 0 to $\bar{n}$. The above algorithm is reduced under the Weibull case.

- Step 1: If $b - KL \geqslant 0$, then $(n^*, \delta^*, \underline{\mathbf{t}}^*) = (0, 0, \mathbf{0})$ and STOP; otherwise, set $n = 1$.
- Step 2: Set $\delta_n^* = \left\{ n\left(L - \frac{b}{K}\right)\left(\frac{1}{w}\right)\left(\frac{1}{\sum_{i=1}^{n} i^{\frac{1}{\beta-1}}}\right)\left(\frac{\beta-1}{\beta}\right)\right\}^{(\beta-1)}$ and $t_i^* = w(i\delta)^{\frac{1}{\beta-1}}$.
- Step 3: If $n = \bar{n}$, then STOP; otherwise, set $n = n + 1$ and go to Step 2.

The performance of the optimal policy is evaluated through numerical examples for the Weibull case and is compared with two other policies obtained by Jaturonnatee et al. (2006) and Pongpech and Murthy (2006).

5. Numerical examples for the Weibull case

Consider that the lifetime distribution of the equipment is Weibull distributed with scale parameter $\alpha > 0$ and shape parameter $\beta > 1$, i.e., $h(t) = \alpha\beta(\alpha t)^{\beta-1}$. Then, the mean life of the equipment is $\mu = \left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta} + 1\right)$. To evaluate the performance of implementing the optimal PM policy, let the expected total cost without PM actions, C_0, be the baseline and define $\Delta\% = (C_0 - C^*)/C_0 \times 100$, which is the percentage of cost reduction, where C^* is the expected total cost under the optimal PM policy. Using this measure of performance, the optimal PM policy under the proposed maintenance scheme is derived and its performance is evaluated. Furthermore, the performance of the resulting optimal policy is compared with Jaturonnatee's and Pongpech and Murthy's optimal policies.

5.1. Numerical results under fixed failure rate reduction

Let the minimal repair cost $C_m = 100$ and the repair time T_m follows Weibull (2, 0.5). If the repair time exceeds $\tau = 2$, then there is a penalty $C_\tau = 300$ per unit time to the lessor. Then, the total expected cost to the lessor at each failure becomes $K = C_m + C_n + C_\tau \int_\tau^\infty \overline{G}(t)\mathrm{d}t = 221.8 + C_n$. Furthermore, let the PM cost with maintenance degree δ be $C_{pm}(\delta) = 100 + 50\delta$ and $\alpha = 1$. Then, Table 1 summarizes the numerical results for various combinations of β, L, and C_n.

For example, when $\alpha = 1$, $\beta = 3$, $L = 3$, and $C_n = 0$, the mean life of the equipment is $\mu = \left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta} + 1\right) \approx 0.89298$, and the expected total cost is $C_0 = 5988.65$ without any PM actions. However, under the optimal PM policy, we have $n^* = 6$, $\delta^* = 3.1494$, and $t_i^* = h^{-1}(i\delta)$, and the expected total cost becomes $C^* = 2712.31$. That is, within the lease period, 6 PM actions should be carried out at time epochs $t_1^* = 1.025$, $t_2^* = 1.449$, $t_3^* = 1.775$, $t_4^* = 2.049$, $t_5^* = 2.291$ and $t_6^* = 2.510$ with the fixed maintenance degree $\delta^* = 3.1494$. Under this PM policy, the expected total cost can be reduced up to 54.7%.

Furthermore, from Table 1, we have the following observations:

1. As β increases, the optimal number of PM actions n^* increases, and the optimal maintenance degrees δ^* increases.
2. As the lease period L increases, the optimal expected total cost C^* increases, the optimal number of PM actions n^* increases, and the percentage of cost reduction $\Delta\%$ also increases. This result implies that the PM actions have significant impact on the expected cost when the lease period is relatively long.
3. As C_n increases, both the optimal expected total cost C^* and the optimal number of PM n^* increases. This result is reasonable since more PM actions should be carried out to reduce the number of failures when the expected cost for each failure increases.

From Theorem 1, we know that the time epoch for performing PM action is $t_i^* = h^{-1}(i\delta)$. For the Weibull case, Table 2 summarizes the optimal time epochs for PMs and the time intervals between two successive PMs for various values of β.

From Table 2, it can be found that the time intervals between two successive PMs become longer when $\beta < 2$ and are constant when $\beta = 2$. However, when $\beta > 2$, the maintenance time intervals become shorter and shorter, which means the PM action should be perform more frequently as the equipment deteriorates.

5.2. Comparisons

In this section, the performance of the optimal policies under three different maintenance schemes are compared – Policy J (Jaturonnatee, Murthy, and Boondiskulchok), Policy P (Pongpech and Murthy), and Policy Y (the proposed policy). Table 3 lists and compares the above three policies.

Table 4 compares their numerical performances. It shows n^*, C^*, and $\Delta\%$ for each policy when $L = 5$. From Table 4, we have the following observations:

1. When $\beta = 2$, the performances $\Delta\%$ of these three optimal policies perform are almost the same. When $\beta \neq 2$, Policy J has the best performance, however, the performance of Policy Y is better than Policy P and very close to Policy J.
2. The optimal number of PM actions n^* is almost the same among these three policies.

Although Policy J has the best performance, it is not easy to be specified in the leased contract. Policy P is easy to be specified since it is a periodical maintenance policy; however, its performance is not as good as Policy Y. Therefore, the proposed policy not only performs well but also can be easily implemented in practice.

5.3. Practical applications

In the real word, there are many situations where businesses choose to lease instead of buy certain equipment. For example, a

Table 1
Numerical results for various combinations of β, L and C_n.

β	μ	L	C_n	C_0	n^*	C^*	δ^*	$\Delta\%$
1.5	0.90275	0.5	0	78.42	0	78.42	0	0
			200	149.13	0	149.13	0	0
		1	0	221.80	0	221.80	0	0
			200	421.80	1	320.27	0.8131	24.1
		3	0	1152.52	2	630.90	0.9123	47.6
			200	2191.75	3	837.56	0.6805	61.8
		5	0	2479.82	3	924.48	0.8760	62.7
			200	4715.89	5	1256.33	0.5769	73.4
2	0.88623	0.5	0	55.45	0	55.45	0	0
			200	105.45	0	105.45	0	0
		1	0	221.80	0	221.80	0	0
			200	421.80	1	357.94	0.8815	15.1
		3	0	1996.22	3	1015.60	1.3873	49.1
			200	3796.22	5	1377.76	0.9605	63.7
		5	0	5545.04	6	1811.06	1.3642	67.3
			200	10545.04	9	2399.17	0.9763	77.2
3	0.89298	0.5	0	27.73	0	27.73	0	0
			200	52.73	0	52.73	0	0
		1	0	221.80	0	221.80	0	0
			200	421.80	1	393.41	1.0360	6.7
		3	0	5988.65	6	2712.31	3.1494	54.7
			200	11388.65	10	3504.32	2.1929	69.2
		5	0	27725.22	17	7099.88	3.7202	74.4
			200	52725.22	24	8761.85	2.8147	83.4

Table 2
Optimal time epochs for PMs under various value of β ($C_n = 0$, $L = 5$).

β	n^*	δ^*	t_i^*	$\Delta t_i = t_i - t_{i-1}$
1.5	3	0.8760	0.341, 1.364, 3.069	0.341, 1.023, 1.705
2	6	1.3642	0.682, 1.364, 2.046, 2.728, 3.410, 4.092	0.682, 0.682, 0.682, 0.682, 0.682, 0.682
3	17	3.7202	1.114, 1.575, 1.929, 2.227, 2.490, 2.728, 2.946, 3.150, 3.341, 3.521, 3.693, 3.858, 4.015, 4.167, 4.313, 4.454, 4.591	1.114, 0.461, 0.354, 0.298, 0.263, 0.238, 0.219, 0.203, 0.191, 0.181, 0.172, 0.164, 0.158, 0.152, 0.146, 0.141, 0.137

Table 3
Comparisons among three different maintenance policies.

Policy name	J	P	Y
Authors	Jaturonnatee, J., Murthy, D.N.P., & Boondiskulchok, R.	Pongpech, J., & Murthy, D.N.P.	Yeh, R.H., Kao, K.C., & Chang, W.L.
Maintenance	Sequential	Periodical	Sequential
Method	Failure rate reduction	Failure rate reduction with fixed maintenance interval	Failure rate reduction with fixed maintenance degree
Parameters	$2n + 1$	$n + 1$	$n + 2$
Weibull case	–	–	The closed form solution for given n

transportation company may lease trucks or buses from a vehicle leasing company to reduce the initial capital required for starting a business. A manufacturing company may lease machine tools to build a factory. Even a small office may lease a photocopier instead of buying one. In this paper, photocopier leasing is used as an example to evaluate the performance of the proposed optimal preventive maintenance policy.

Using the notations given in this paper, a supplier of photocopiers may provide the following lease contract:

- Leased period $L = 3$ (years)
- Penalty cost $C_n = 100$ ($) per failure
- Maximum repair time $\tau = 2$ (days)
- Penalty cost $C_\tau = 200$ ($) per unit time when the repair time exceeds τ

Suppose that the lifetime distribution of the photocopier is Weibull with scale parameter $\alpha = 0.5$ and shape parameter $\beta = 1.5$. Then, the mean life of the machine is 1.81 (years). The supplier will provide all the required repair and maintenance. Based on past experience, the related costs are estimated as follows: minimal repair cost $C_m = 300$ ($) per failure, repair time T_m is Weibull distributed with parameters $(\alpha, \beta) = (2, 0.5)$, and PM cost is $C_{pm} = 100 + 50\delta$ ($) per PM.

Without PM actions, the expected total cost is $C_0 = 884.02$ ($) using Eq. (3). However, based on the proposed optimal preventive maintenance policy, two PM actions with fixed maintenance degree $\delta^* = 0.3296$ should be carried out within the lease period at $t_1^* = 0.386$ and $t_2^* = 1.545$. As a result, the expected total cost reduces to $C^* = 471.67$ ($), which is a 46.6% saving.

6. Conclusion

Under the failure rate reduction method, this study proposes a maintenance scheme for leased equipment and derives the optimal PM policy for leased equipment. Some structural properties of the optimal policy are obtained, and an efficient algorithm is developed based on these properties. Furthermore, a closed-form solution is obtained for the case where the lifetime distribution of the equipment is Weibull.

From the numerical examples of the Weibull case, we found that the performance of the optimal policy with fixed maintenance degree is better than with fixed maintenance interval (periodical policy). Furthermore, when the lease period is relative long, the PM actions should be performed within the leased period since the expected cost can be reduced significantly.

In this paper, the uniqueness property of the optimal PM policy is obtained for the case when the lifetime distribution is a general distribution. However, for some generalizations of this maintenance scheme, the uniqueness property might not hold or the conditions for the existence of the optimal policy might be very complicated. Some possible generalizations, such as nonlinear maintenance cost, time-dependent penalty cost, or various penalty schemes, are extended issues for the future study in this area.

Table 4
Numerical performances among three different maintenance schemes ($L = 5$).

β	C_τ	C_n	Policy J			Policy P			Policy Y		
			n^*	C^*	Δ%	n^*	C^*	Δ%	n^*	C^*	Δ%
1.5	0	0	2	615.31*	45.0	–	–	–	2	620.90	44.5
	0	200	4	1042.31*	68.9	–	–	–	4	1065.91	68.2
	300	0	3	907.63*	63.4	–	–	–	3	924.48	62.7
	300	200	5	1223.91	74.0	–	–	–	5	1256.33	73.4
2	0	0	4	1280.00*	48.8	3	1300.00	48.0	4	1280.00*	48.8
	0	200	7	2067.71*	72.4	7	2075.00	72.3	7	2067.71*	72.4
	300	0	6	1811.05*	67.3	6	1820.70	67.2	6	1811.06*	67.3
	300	200	9	2399.16*	77.2	9	2404.50	77.2	9	2399.17*	77.2
3	0	0	10	5437.03*	56.5	9	5750.00	54.0	10	5477.39	56.2
	0	200	20	7712.87*	79.4	21	8034.90	78.6	20	7827.84	79.1
	300	0	16	7009.92*	74.7	17	7312.50	73.6	17	7099.88	74.4
	300	200	24	8610.66*	83.7	25	8969.90	83.0	24	8761.85	83.4

–: No results given in the literature (Pongpech and Murthy, 2006).
* Minimum one of the optimal cost among these three policies under same parameters.

Acknowledgement

This research is supported in part by a grant (NSC 96-2221-E011-002) from the National Science Council, Taiwan.

Appendix A

Proof of Theorem 1

Given $n > 0$ and $\delta > 0$, from Eq. (4), we have $\frac{\partial C(\underline{\mathbf{t}}|n,\delta)}{\partial t_i} = K\delta > 0$, for all $i = 1,2,\cdots,n$. This result implies that minimizing $C(\underline{\mathbf{t}}|n,\delta)$ is equivalent to minimizing $\sum_{i=1}^{n} t_i$ since $C_{pm}(\delta)$, K, and L are all constants. Hence, PM should be taken as early as possible and subject to the constraint $h(t_i) - i\delta \geqslant 0$ for all $i = 1,2,\ldots,n$. Since $h(t)$ is a strictly increasing function of t, we have $t_i \geqslant h^{-1}(i\delta)$. Therefore, the minimal $C(\underline{\mathbf{t}}|n,\delta)$ is attained at $t_i^* = h^{-1}(i\delta)$ for $i = 1,2,\ldots,n$.

Proof of Theorem 2

For any $n > 0$, taking the first and second partial derivatives of Eq. (5) with respect to δ, we have

$$\frac{\partial C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta} = n(b - KL) + K\sum_{i=1}^{n} h^{-1}(i\delta) + K\delta\left(\frac{\partial \sum_{i=1}^{n} h^{-1}(i\delta)}{\partial \delta}\right), \quad \text{(A1)}$$

and

$$\frac{\partial^2 C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta^2} = 2K\left(\frac{\partial \sum_{i=1}^{n} h^{-1}(i\delta)}{\partial \delta}\right) + K\delta\left(\frac{\partial^2 \sum_{i=1}^{n} h^{-1}(i\delta)}{\partial \delta^2}\right). \quad \text{(A2)}$$

Since $h^{-1}(i\delta)$ is strictly increasing in $(i\delta)$, we know that $\frac{\partial h^{-1}(i\delta)}{\partial \delta} > 0$ for all i. Therefore, if $b - KL \geqslant 0$, then Eq. (A1) is positive for all δ which implies that $C(n,\delta|\underline{\mathbf{t}}^*)$ is a strictly increasing function of δ for any $n > 0$. In this case, the optimal maintenance degree is $\delta_n^* = 0$.

On the other hand, when $b - KL < 0$, we have $\left.\frac{\partial C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta}\right|_{\delta=0} = n(b - KL) < 0$ and $\left.\frac{\partial C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta}\right|_{\delta=\infty} = \infty$. If $2\left(\frac{\partial \sum_{i=1}^{n} h^{-1}(i\delta)}{\partial \delta}\right) + \delta\left(\frac{\partial^2 \sum_{i=1}^{n} h^{-1}(i\delta)}{\partial \delta^2}\right) > 0$, then Eq. (A2) is positive for all δ. In this case, Eq. (A1) is a strictly increasing function of δ and changes its sign at most once from negative to positive. Therefore, there exists a unique $\delta_n^* > 0$ such that $\left.\frac{\partial C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta}\right|_{\delta=\delta_n^*} = 0$. Furthermore, given any $n > 0$, there is an upper bound for maintenance degree, which is $\frac{h(L)}{n}$. Therefore, if $\left.\frac{\partial C(\delta,\underline{t}^*|n)}{\partial \delta}\right|_{\delta=\frac{h(L)}{n}} \leq 0$, then $\delta_n^* = \frac{h(L)}{n}$; otherwise, $\delta_n^* \in \left[0, \frac{h(L)}{n}\right]$.

Proof of Theorem 3

For the Weibull case with any $n > 0$, Eq. (5) becomes

$$C(n,\delta|\underline{\mathbf{t}}^*) = C_0 + nC_{pm}(\delta) - nK\delta L + K\delta w\left(\delta^{\frac{1}{\beta-1}}\right)\left(\sum_{i=1}^{n} i^{\frac{1}{\beta-1}}\right). \quad \text{(A3)}$$

Taking the first and second partial derivatives of Eq. (A3) with respect to δ, we have

$$\frac{\partial C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta} = n(b - KL) + \frac{\beta K w}{\beta - 1}\left(\sum_{i=1}^{n} i^{\frac{1}{\beta-1}}\right)\left(\delta^{\frac{1}{\beta-1}}\right) > n(b - KL) \quad \text{(A4)}$$

and

$$\frac{\partial^2 C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta^2} = \frac{\beta K w}{(\beta - 1)^2}\left(\sum_{i=1}^{n} i^{\frac{1}{\beta-1}}\right)\left(\delta^{\frac{2-\beta}{\beta-1}}\right) > 0. \quad \text{(A5)}$$

If $b - KL \geqslant 0$, then Eq. (A4) is positive for all δ, which implies that $C(n,\delta|\underline{\mathbf{t}}^*)$ is a strictly increasing function of δ for any $n > 0$. In this case, the optimal maintenance degree is $\delta_n^* = 0$.

When $b - KL < 0$, we have $\left.\frac{\partial C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta}\right|_{\delta=0} = n(b - KL) < 0$ and $\left.\frac{\partial C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta}\right|_{\delta=\infty} = \infty$. Since Eq. (A5) is positive for all δ, Eq. (A4) is a strictly increasing function of δ and changes its sign at most once from negative to positive. Therefore, there exists a unique $\delta_n^* > 0$ such that $\left.\frac{\partial C(n,\delta|\underline{\mathbf{t}}^*)}{\partial \delta}\right|_{\delta=\delta_n^*} = 0$. Therefore, if $\left.\frac{\partial C(\delta,\underline{t}^*|n)}{\partial \delta}\right|_{\delta=\frac{h(L)}{n}} \leq 0$ then we have $\delta_n^* = \frac{h(L)}{n}$. Otherwise, letting Eq. (A4) equal to zero, we have $\delta_n^* = \left\{n\left(L - \frac{b}{K}\right)\left(\frac{1}{w}\right)\left(\frac{1}{\sum_{i=1}^{n} i^{\frac{1}{\beta-1}}}\right)\left(\frac{\beta-1}{\beta}\right)\right\}^{(\beta-1\}}$.

References

Barlow, R. E., & Hunter, L. C. (1960). Optimum preventive mathematical policies. *Operations Research, 8*, 90–100.

Brown, M., & Proschan, F. (1983). Imperfect repair. *Journal of Applied Probability, 20*, 851–859.

Chan, J. K., & Shaw, L. (1993). Modeling repairable systems with failure rates that depend on age and maintenance. *IEEE Transactions on Reliability, 42*, 566–570.

Chun, Y. H. (1992). Optimal number of periodic preventive maintenance operations under warranty. *Reliability Engineering & System Safety, 37*, 223–225.

Glickman, T. S., & Berger, P. D. (1976). Optimal price and protection period decisions for a product under warranty. *Management Science, 22*, 1381–1390.

Jack, N., & Dagpunar, J. S. (1994). An optimal imperfect maintenance policy over a warranty period. *Microelectronics and Reliability, 34*, 529–934.

Jaturonnatee, J., Murthy, D. N. P., & Boondiskulchok, R. (2006). Optimal preventive maintenance of leased equipment with corrective minimal repairs. *European Journal of Operational Research, 174*, 201–215.

Martin, H. H. (1997). Contracting out maintenance and a plan for future research. *Journal of Quality in Maintenance Engineering, 3*(2), 81–90.

Murthy, D. N. P., & Asgharizadeh, E. (1999). Optimal decision making in a maintenance service operation. *European Journal of Operational Research, 116*, 259–273.

Murthy, D. N. P., & Yeung, V. (1995). Modelling and analysis of maintenance service contracts. *Mathematical and Computer Modelling, 22*(10–12), 219–225.

Nakagawa, T. (1981). A summary of periodic replacement with minimal repair at failure. *Journal of the Operational Research Society of Japan, 24*(3), 213–227.

Nakagawa, T., & Kowada, M. (1983). Analysis of a system with minimal repair and its application to replacement policy. *European Journal of Operational Research, 12*, 176–182.

Nisbet, A., & Ward, A. (2001). Radiotherapy equipment – purchase or lease? *The British Journal of Radiology, 74*, 735–744.

Nguyen, D. G., & Murthy, D. N. P. (1988). Optimal reliability allocation for products sold under warranty. *Engineering Optimization, 13*, 35–45.

Pham, H., & Wang, H. (1996). Imperfect maintenance. *European Journal of Operational Research, 94*, 425–438.

Pongpech, J., & Murthy, D. N. P. (2006). Optimal periodic preventive maintenance policy for leased equipment. *Reliability Engineering & System Safety, 91*, 772–777.

Seo, J. H., & Bai, D. S. (2004). An optimal maintenance policy for a system under periodic overhaul. *Mathematical and Computer Modelling, 39*, 373–380.

Sheu, S. H., Lin, Y. B., & Liao, G. L. (2006). Optimum policies for a system with general imperfect maintenance. *Reliability Engineering & System Safety, 91*, 362–369.

Wang, Hongzhou. (2002). A survey of maintenance policies of deteriorating systems. *European Journal of Operational Research, 139*, 469–489.

Yeh, R. H., & Chen, C. K. (2006). Periodical preventive-maintenance contract for a leased facility with Weibull life-time. *Quality & Quantity, 40*, 303–313.

Yeh, R. H., & Lo, H. C. (2001). Optimal preventive-maintenance warranty policy for repairable products. *European Journal of Operational Research, 134*, 59–69.