*Acta Math. Hung.*
**55 (3—4)** (1990), 201—206.

# ON CONVEX-SUSLIN SPACES

J. C. FERRANDO (Valencia)

The linear spaces we use are defined over the field of the real or complex numbers. All the topological spaces which will appear are supposed to be Hausdorff. A topological space $X$ is a polish space, [1], if it is separable and there is a metric $d$ on $X$ compatible with its topology such that the metric space $(X, d)$ is complete. A topological space is a Suslin space, [2], if it is the continuous image of a polish space. A topological space $E$ is a $K$-Suslin space, [3], if there is a polish space $X$ and a mapping $f$ from $X$ into the compact subsets $K(E)$ of $E$ such that $E=\bigcup\{f(x): x\in X\}$ and given any $x\in X$ and any neighbourhood $V$ of the set $f(x)$ in $E$, there is a neighbourhood $U$ of $x$ in $X$ with $f(U)\subset V$. A topological space $E$ is a quasi-Suslin space, [4], if there is a polish space $X$ and a mapping $f$ from $X$ into the family $P(E)$ of all the parts of $E$ such that $E=\bigcup\{f(x): x\in X\}$ and given any sequence $(x_n)_n$ in $X$ converging to $x$ with $z_n\in f(x_n)$ for $n=1, 2, \ldots$ then the sequence $(z_n)_n$ has an adherent point $z\in f(x)$. If $A$ is a subset of a topological space $E$, we denote by $O(A)$ the interior of the set of all the points $x$ of $E$ such that every neighbourhood of $x$ meets $A$ in a set of second category. A subset $A$ in a topological space has the Baire property, [5], if $O(A)\sim A$ is of first category. If $B$ is a subset of a linear space, we denote by $\langle A\rangle$ its linear span. If $B$ is a bounded closed absolutely convex subset of a locally convex space $E$, $E_B$ denotes the normed space over the linear hull of $B$.

In this paper we introduce a class of topological spaces which contains the $K$-Suslin spaces and the semi-reflexive spaces with $\mathscr{C}$-webs, [6], in the category of the locally convex spaces and which partially answers the Grothendieck conjecture, [7].

We say that a sequence $\{A_n, n=1, 2, \ldots\}$ of subsets of a locally convex space $E$ is compacting if $A_n\supset A_{n+1}$ for $n=1, 2, \ldots$ and given $x_n\in A_n$, $n=1, 2, \ldots$ the set $\{x_n, n=1, 2, \ldots\}$ is relatively weakly compact in $E$. A web $\{A_{n_1 n_2 \ldots n_p}; p, n_1, n_2, \ldots, n_p=1, 2, \ldots\}$ in a locally convex space is said to be compacting if given a sequence $(m_p)_p$ of positive integers, the sequence $\{A_{m_1 m_2 \ldots m_p}, p=1, 2, \ldots\}$ is compacting. Every locally convex space $E$ with a compacting web will be called a convex-Suslin space.

PROPOSITION 1. *If $E$ is a locally convex $K$-Suslin space, then $E$ is a convex-Suslin space.*

PROOF. Let $X$ and $f$ be the polish space and the mapping from $X$ into $K(E)$ associated with $E$, resp. Since $X$ is separable there is a web in $X$ formed by the closed balls $B_{n_1 \ldots n_p}$ of radii less than $1/2^p$. The sets $f(B_{n_1 \ldots n_p})$ form a compacting web in $E$.

PROPOSITION 2. *Let $E$ be a semi-reflexive locally convex space. If $E$ has a $\mathscr{C}$-web then $E$ is a convex-Suslin space.*

PROOF. If the sets $A_{n_1 \ldots n_p}$ form a $\mathscr{C}$-web in $E$, $(m_p)_p$ is a sequence of positive integers and $x_p \in A_{m_1 \ldots m_p}$ for $p=1, 2, \ldots$, then $\{x_p, p=1, 2, \ldots\}$ is a bounded set of $E$ and therefore a relatively weakly compact set.

Every non-separable reflexive Fréchet space is a convex-Suslin space which is not $K$-Suslin.

PROPOSITION 3. *Let $E$ be a locally complete space. If $W$ is a compacting web in $E$, then $W$ is a $\mathscr{C}$-web.*

PROOF. Let $W=\{A_{n1, \ldots n_1};\ p, n_1, \ldots, n_p=1, 2, \ldots\}$. Let $(m_p)_p$ be a sequence of positive integers and let $x_p$ be a point of $A_{m_1 \ldots mp}$ for $p=1, 2, \ldots$. If $M=\overline{\Gamma\{x_p, p=1, 2, \ldots\}}$ then $M$ is a Banach disk in $E$. Now $\sum_{p=1}^{\infty} \frac{1}{2^p} x_p$ converges in $E_M$. Therefore $W$ is a $\mathscr{C}$-web.

PROPOSITION 4. *If $E$ is a metrizable locally convex space, then $E'(\mu(E', E))$ and $E''(\mu(E'', E'))$ are convex-Suslin spaces.*

PROOF. If $\{U_n, n=1, 2, \ldots\}$ is a decreasing base of neighbourhoods of the origin in $E$ we set $A_n=U_n^0$ and $A_{n_1 \ldots n_p}=A_{n_1}$ for $p, n_1, \ldots, n_p=1, 2, \ldots$. Then the sets $A_{n_1 \ldots n_p}$ form a compacting web in $E'$. On the other hand, if $\{V_p, p=1, 2, \ldots\}$ is a base of neighbourhoods of the origin in the strong bidual, we set $A_{n_1 \ldots n_p}= =n_1 V_1 \cap n_2 V_2 \cap \ldots \cap n_p V_p$. If $(m_p)_p$ is a sequence of positive integers and $x_p \in A_{m_1 \ldots m_p}$ for $p=1, 2, \ldots$, using a theorem of Grothendieck, [9, p. 394], it follows that $\{x_p, p=1, 2, \ldots\}$ is a $\beta(E', E)$-equicontinuous set of $E''$. Therefore $\{x_p, p=1, 2, \ldots\}$ is a relatively compact subset of $E''(\sigma(E'', E'))$.

As an immediate consequence of the convex-Suslin space definition we have the following results:

PROPOSITION 5. *Let $E$ be a convex-Suslin space. If $F$ is a closed subspace of $E$, then $F$ is convex-Suslin.*

PROPOSITION 6. *Let $E$ and $F$ be locally convex spaces and let $f$ be a weakly continuous mapping from $E$ onto $F$. If $E$ is a convex-Suslin space the same is true for $F$.*

PROPOSITION 7. *Let $E$ be a locally convex space and let $\{E_n, n=1, 2, \ldots\}$ be a sequence of subspaces of $E$ covering $E$. If every $E_n$ is a convex-Suslin space, then $E$ is a convex-Suslin space.*

PROPOSITION 8. *Every countable locally convex hull of the form $E=\bigcup_{n=1}^{\infty} A_n(E_n)$ in which every $E_n$ is a convex-Suslin space, is itself a convex-Suslin space.*

PROPOSITION 9. *Let $\{E_n, n=1, 2, \ldots\}$ be a sequence of locally convex spaces. If every $E_n$ is a convex-Suslin space, then $E=\prod_{n=1}^{\infty} E_n$ is a convex-Suslin space.*

PROOF. Let $\{A^{(n)}_{m_1\dots m_p};\ p, m_1, \dots, m_p=1, 2, \dots\}$ be a compacting web in $E_n$ for $n=1, 2, \dots$ . The family

$$A_{m_1}=p_1^{-1}(A^{(1)}_{m_1}),\quad A_{m_1(m_2m_1')}=p_1^{-1}(A^{(1)}_{m_1m_2})\cap p_2^{-1}(A^{(2)}_{m_1'}),\quad A_{m_1(m_2m_1')(m_3m_2'm_1'')}=$$
$$=p_1^{-1}(A^{(1)}_{m_1m_2m_3})\cap p_2^{-1}(A^{(2)}_{m_1'm_2'})\cap p_3^{-1}(A^{(3)}_{m_1''}),\ \dots$$

where $p_r$ is the projection from $E$ onto $E_r$, is a web in $E$. If

$$x_1\in A_{m_1},\ x_2\in A_{m_1(m_2m_1')},\ x_3\in A_{m_1(m_2m_1')(m_3m_2'm_1'')},\ \dots$$

and if $x_p^r=p_r(x_p)$, then $\{x_p^r,\ p=1, 2, \dots\}$ is a relatively weakly compact subset of $E_r$ for $r=1, 2, \dots$ . Now, since $\{x_p,\ p=1, 2, \dots\}\subset\prod_{r=1}^{\infty}\overline{\{x_n^r,\ n=1, 2, \dots\}}\,E_r$, it follows that $\{x_p,\ p=1, 2, \dots\}$ is a relatively weakly compact set of $E$.

LEMMA. *Let $E$ be a locally convex space and let $\{A_n,\ n=1, 2, \dots\}$ be a compacting sequence in $E$. If $U$ is a neighbourhood of the origin in $E$, there exists a $q\in\mathbf{N}$ such that $q^{-1}A_q\subset U$.*

PROOF. Suppose $A_n\not\subset nU$ for $n=1, 2, \dots$ . Taking $x_n\in A_n\sim nU$, $n=1, 2, \dots$ the set $\{x_n,\ n=1, 2, \dots\}$ is bounded in $E$. Thus, there is a positive integer $q$ with $x_q\in qU$, a contradiction.

THEOREM 1. *Let $E$ be a locally convex Baire space and let $F$ be a convex-Suslin space. If $f$ is a linear mapping from $E$ into $F$ with closed graph, then $f$ is continuous.*

PROOF. We shall see first that if $M$ is a closed convex set of $F$, then the set $f^{-1}(M)$ has the Baire property.

Let $\{A_{n_1\dots n_p};\ p, n_1, \dots, n_p=1, 2, \dots\}$ be a compacting web in $F$ and let $B_{n_1\dots n_p}=$ $=f^{-1}(M\cap A_{n_1\dots n_p})$. We define

$$D=O\big(f^{-1}(M)\big)\sim\bigcup_{n=1}^{\infty}O(B_n),$$

$$D_{n_1}=O(B_{n_1})\sim\bigcup_{n=1}^{\infty}O(B_{n_1n}),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$D_{n_1\dots n_p}=O(B_{n_1\dots n_p})\sim\bigcup_{n=1}^{\infty}O(B_{n_1\dots n_pn}),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Since $f^{-1}(M)=\bigcup_{n=1}^{\infty}B_n$ and $B_{n_1\dots n_p}=\bigcup_{n=1}^{\infty}B_{n_1\dots n_pn}$ for $p=1, 2, \dots$ , the sets $D, D_{n_1}, \dots, D_{n_1\dots n_p}, \dots$ are nowhere dense in $E$ [4, p.4]. Thus, if we put $D(p)=$ $=\bigcup\{D_{n_1\dots n_p};\ n_1, \dots, n_p=1, 2, \dots\}$, the set $H=D\cup\big(\bigcup_{p=1}^{\infty}D(p)\big)$ is of first category in $E$. It is sufficient to prove that $O\big(f^{-1}(M)\big)\sim H\subset f^{-1}(M)$.

If $x \in O(f^{-1}(M)) \sim H$ there is a sequence $(m_p)_p$ of positive integers such that $x \in O(B_{m_1 \dots m_p})$ for $p=1, 2, \dots$. Since $E$ is a Baire space, if we put $C_{n_1 \dots n_p} = f^{-1}(A_{n_1 \dots n_p})$, there exists a sequence $(i_n)_n$ of positive integers such that $\bar{C}_{i_1 \dots i_p}$ has an interior point for $p=1, 2, \dots$. Let $\{U_p, p=1, 2, \dots\}$ be a decreasing sequence of absolutely convex neighbourhoods of the origin with $U_p \subset \bar{C}_{i_1 \dots i_p} - \bar{C}_{i_1 \dots i_p}$ and let $T_1$ be the locally convex topology on $E$ which has as a base of neighbourhoods of the origin, the sequence $\{p^{-1} U_p, p=1, 2, \dots\}$. Since the graph of $f$ is closed in $E \times F(\sigma(F, F'))$, there is a Hausdorff locally convex topology $T_2$, coarser than $\sigma(F, F')$, such that $f\colon E \to F(T_2)$ is continuous. We are going to prove now that the mapping $f\colon E(T_1) \to F(T_2)$ is continuous. fact, if $V$ is a closed absolutely convex neighbourhood of the origin in $F(T_2)$ then, by the lemma, there is a positive integer $q$ with $q^{-1} \bar{A}^{T_2}_{i_1 \dots i_q} \subset 2^{-1} V$. From the continuity of $f\colon E \to F(T_2)$ it follows that $q^{-1} U_q \subset f^{-1}(V)$.

Let $x_p \in (x + p^{-1} U_p) \cap B_{m_1 \dots m_p} \sim H$. It is clear that $x_p \to x$ in $E(T_1)$ and that $\{f(x_p), p=1, 2, \dots\}$ is a relatively compact set in $F(\sigma(F, F'))$. Since $M$ is closed in $F(\sigma(F, F'))$ there is a subnet $\{z_j, j \in J, \geqq\}$ of $(x_p)_p$ and a point $y \in M$ such that $f(z_j) \to y$ in $F(\sigma(F, F'))$, so $f(z_j) \to y$ in $F(T_2)$. But $f\colon E(T_1) \to F(T_2)$ is continuous, therefore $y = f(x)$. We must conclude that $x \in f^{-1}(M)$. Thus $f^{-1}(M)$ has the Baire property.

Finally, if $W$ is a closed absolutely convex neighbourhood of the origin in $F$, $f^{-1}(W)$ is a second category set which has the Baire property. According to the Banach difference theorem, [4, p. 13], it follows that $f^{-1}(W)$ is a neighbourhood of the origin in $E$.

COROLLARY 1. *Let $E$ be a convex-Suslin space and let $F$ be a locally convex Baire space. If $f$ is a linear mapping from $E$ onto $F$ with closed graph, then $f$ is open.*

PROOF. This is shown in a standard way.

COROLLARY 2. *Let $E$ be a locally convex Baire space. If $E$ is convex-Suslin then $E$ is a Fréchet space.*

PROOF. Clearly $E$ is complete [4, p. 105]. If $\{A_{n_1 \dots n_p}; p, n_1, \dots, n_p = 1, 2, \dots\}$ is a compacting web in $E$ there exists a sequence $(m_p)_p$ of positive integers such that $\bar{A}_{m_1 \dots m_p} - \bar{A}_{m_1 \dots m_p}$ is a neighbourhood of the origin for $p=1, 2, \dots$. On the other hand, if $U$ is a closed absolutely convex neighbourhood of the origin in $E$, there is a positive integer $q$ with $q^{-} \bar{A}_{m_1 \dots m_q} \subset 2^{-1} U$. So $q^{-1}(\bar{A}_{m_1 \dots m_q} - \bar{A}_{m_1 \dots m_q}) \subset U$.

THEOREM 2. *Let $E$ be a metrizable locally convex space with the property that every weakly convergent sequence converges in $E$. If $E$ is a convex-Suslin space, then $E$ is a $K$-Suslin space.*

PROOF. Since $E$ is metrizable it is sufficient to prove that $E$ is a quasi-Suslin space [4, p. 67].

If we consider the set $\mathbf{N}$ of positive integers endowed with the discrete topology, then $X = \mathbf{N}^{\mathbf{N}}$ is a polish space. On the other hand, if $\{A_{n_1 \dots n_p}; p, n_1, \dots, n_p = 1, 2, \dots\}$ is a compacting web in $E$ we define $f\colon X \to P(E)$ such that $f((n_p)_p) = \bigcap_{p=1}^{\infty} \bar{A}_{n_1 \dots n_p}$. Clearly, $E = \bigcup \{f(x)\colon x \in X\}$. If $(x^r)_r$, where $x^r = (m^r_p)_p$, is a sequence in $X$ which converges to $x=(m_p)_p$, let $z^r \in f(x^r)$ for $r=1, 2, \ldots$ . Since $(m_p^r)_r \to m_p$ in N there is a $q(p) \in \mathrm{N}$ with $m_p^r = m_p$ for $r \geqq q$. If $(r_p)_p$ is a strictly increasing sequence of positive integers such that $r_p > \max\{q(1), \ldots, q(p)\}$ for $p=1, 2, \ldots$ , then $z^{r_p} \in \bar{A}_{m_1 \ldots m_p}$ for every $p \in \mathrm{N}$. Now for each $p \in \mathrm{N}$ there is a sequence $(y_n^p)_n$ in $E$ with $y_n^p \in A_{m_1 \ldots m_p}$ for $n=1, 2, \ldots$ such that $(y_n^p)_n$ converges to $z^{r_p}$ in $E$. Let $\{U_k,\ k=1, 2, \ldots\}$ be a decreasing base of absolutely convex neighbourhoods of the origin in $E$. For each $p$ of N there exists a positive integer $n_p$ such that $n_p > p$ and $y_{n_p}^p - z^{r_p} \in 2^{-1} U_p$. Since $y_{n_p}^p \in A_{m_1 \ldots m_p}$ for $p=1, 2, \ldots$ the sequence $(y_{n_p}^p)_p$ has a weakly adherent point $z$ in $E$, and because $E$ is metrizable there exists a subsequence of $(y_{n_p}^p)_p$ which converges weakly to $z$. But this means that $z$ is an adherent point of the sequence $(y_{n_p}^p)_p$ in $E$. It is easy now to show that $z$ is also an adherent point of $(z^r)_r$ in $E$ and that $z \in f(x)$, since $z \in \bigcup_{p=1}^{\infty} \bar{A}_{m_1 \ldots m_p}$.

Thus $E$ is a quasi-Suslin space.

COROLLARY. *$l^1(I)$ with* card $I > \aleph_0$ *is not a convex-Suslin space.*

PROOF. It is easy to see that in $l^1(I)$ every weakly convergent sequence converges under the strong topology. Now if $l^1(I)$ were a $K$-Suslin space, it would be separable.

$l^1(I)$ with card $I > \aleph_0$ is a locally convex space with a $\mathscr{C}$-web which is not a convex-Suslin space. On the other hand, in [8] some examples of Suslin spaces without $\mathscr{C}$-webs are given.

THEOREM 3. *Let $E$ be a locally convex Baire space and suppose that $F$ has a web $\{A_{n_1 \ldots n_p};\ p, n_1, \ldots, n_p = 1, 2, \ldots\}$ with the property that given a sequence $(m_p)_p$ of positive integers and given $x_p \in A_{m_1 \ldots m_p}$ for $p=1, 2, \ldots$ there exists a weakly adherent point $x$ of $(x_p)_p$ such that $x \in \bigcap_{p=1}^{\infty} A_{m_1 \ldots m_p}$. If $f$ is a linear mapping from $E$ into $F$ with closed graph, then there exists a sequence $(i_p)_p$ of positive integers such that $f(E) = \bigcap_{p=1}^{\infty} \langle A_{i_1 \ldots i_p} \rangle$.*

PROOF. A similar argument to that used in the proof of Theorem 1 shows that every set $f^{-1}(A_{n_1 \ldots n_p})$ has the Baire property. Since $E$ is a Baire space, there is a sequence $(i_p)_p$ of positive integers such that the sets $f^{-1}(A_{i_1 \ldots i_p})$, $p=1, 2, \ldots$ are of second category in $E$. According to the Banach difference theorem, $f^{-1}(A_{i_1 \ldots i_p}) - f^{-1}(A_{i_1 \ldots i_p})$ is a neighbourhood of 0 for every $p \in \mathrm{N}$. Hence $E = \langle f^{-1}(A_{i_1 \ldots i_p}) \rangle$ for $p=1, 2, \ldots$ .

COROLLARY. *Let $E$ be a locally convex Baire space and let $F$ be a convex-Suslin space with a compacting web $\{A_{n_1 \ldots n_p};\ p, n_1, \ldots, n_p = 1, 2, \ldots\}$ formed by closed sets. If $f$ is a linear mapping from $E$ into $F$ with closed graph, there exists a sequence of positive integers $(i_p)_p$ such that $F(E) = \bigcap_{p=1}^{\infty} \langle A_{i_1 \ldots i_p} \rangle$.*

PROBLEM. Obviously $l^\infty(\mu(l^\infty, l^1))$ is a convex-Suslin space. We do not know if $l^\infty$ is a convex-Suslin space.

## References

[1] N. Bourbaki, *Topologie générale.* Chap. 9. Utilisation des nombres réels en topologie générale (Paris, 1958).
[2] L. Schwartz, Sur le théorème du graphe fermé, *C. R. Acad. Sc. Paris,* **263** (1966), 602—605.
[3] A. Martineau, Sur les théorèmes de S. Banach et L. Schwartz concernant le graphe fermé, *Studia Math.,* **30** (1968), 43—51.
[4] M. Valdivia, *Topics in locally convex spaces,* North Holland (1982).
[5] M. Valdivia, On the closed graph theorem in topological spaces, *Manuscrip. Math.,* **23** (1978), 173—184.
[6] M. De Wilde, Réseaux dans les espaces linéaires à semi-normes, *Mém. Soc. Liège,* **18** (1969).
[7] A. Grothendieck, *Produits tensoriels topologiques et spaces nucléaires,* Mem. Amer. Math. Soc. No. **16** (1955).
[8] M. Valdivia, A class of locally convex spaces without $\mathscr{C}$-webs, *Ann. Inst. Fourier,* **32** (1982), 261—269.
[9] G. Köthe, *Topological Vector Spaces I,* Springer (Berlin, 1983).

*(Received September 17, 1986)*

DEPARTAMENTO DE MATEMATICA APLICADA
UNIVERSIDAD POLITÉCNICA
ESCUELA UNIVERSITARIA DE INFORMATICA
46071-VALENCIA
SPAIN