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On the finiteness theorem for rational maps on a variety of general type

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Abstract

The dominant rational maps of finite degree from a fixed variety to varieties of general type, up to birational isomorphisms, form a finite set. This has been known as the Iitaka-Severi conjecture, and is nowdays an established result, in virtue of some recent advances in the theory of pluricanonical maps. We study the question of finding some effective estimate for the finite number of maps, and to this aim we provide some update and refinement of the classical treatment of the subject.

Introduction

Let X be an algebraic variety of general type, over the complex field. The dominant rational maps of finite degree $X \dashrightarrow Y$ to varieties of general type, up to birational isomorphisms $Y \dashrightarrow Y'$, form a finite set. This has been known as the Iitaka-Severi

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conjecture, and is nowdays an established result. The approach introduced by Maehara [13] obtains the full range of applicability in virtue of some recent advances in the theory of pluricanonical maps, originating with the work of Siu [16, 17], and successively due to Tsuji [20], Hacon and McKernan [7], and Takayama [18, 19].

In this paper we study the question of finding some effective estimate for the finite number of maps in the theorem, in the same line as works of Catanese [2], Bandman and Dethloff [1], Heier [8], and the article [6] by the first author. It is worth mentioning that there is another approach to the question, developed by Naranjo and the second author in [14], in a slightly less general setting. To the aim of obtaining effective results, we moreover provide some update and refinement of the classical treatment of the subject, according to the recent progress.

The approach consists, as usual in this kind of questions, of two main steps, that may be called: rigidity and boundedness, from which the finiteness theorem follows. We shortly describe the main points that represent the contribution of the present paper.

We bring the rigidity theorem to a general form (Theorem 2.1), avoiding certain technical restrictions that are in [13], and relying on [7, 16]. We point out that generic rigidity extends to limits (Proposition 2.4). We show that bounds for the degree of a map and of its graph (see §3) are naturally obtained in terms of the canonical volume, the new invariant arising in the asymptotic theory of divisors, cf. Lazarsfeld's book [12]. Finally we discuss the structure of the proof of the finiteness theorem. We propose a new argument, relating the finite number of rational maps and the irreducible components of a certain bunch of (subvarieties of) Chow varieties (Theorem 4.3), in the perspective of obtaining an effective bound in terms of the complexity of these Chow varieties (see Proposition 4.4). Also in this part we use some recent powerful results of minimal model theory, namely [7, 19]. The computations of complexity are explained in §5.

1. Results on pluricanonical maps

We collect here the recent results in the theory of pluricanonical maps that will be used in the paper.

1.1 Volume of a big divisor

Recall that the volume of a divisor is defined as

$$\operatorname{vol}_X(D) = \lim_{m} h^0(X, mD) / (m^n / n!)$$

and for a big divisor also there is the description

$$\operatorname{vol}_X(D) = \lim_{m} (mD)^{[n]} / m^n$$

in terms of moving self-intersection numbers $D^{[n]}$ (cf. Lazarsfeld [12, II.11.4, p. 303]). The following observation will be useful: if $X \dashrightarrow X' \subset \mathbb{P}^m$ is a birational embedding, and D is the pullback on X of the general hyperplane of \mathbb{P}^m , then

$$\deg X' \leq \operatorname{vol}_X(D)$$

1.2 Pluricanonical embeddings

A recent achievement in the theory of pluricanonical maps is the following theorem of uniform pluricanonical birational embedding, cf. Tsuji [20], Hacon-McKernan [7], Takayama [18].

Theorem 1.1

For any dimension n, there is some positive integer r_n such that: for every n-dimensional variety V of general type the multicanonical divisor r_nK_V defines a birational embedding $V \dashrightarrow V' \subset \mathbb{P}^M$.

Remark 1.2 For the theorem we also have a bound

$$\deg V' \leq d_V$$
,

arising from §1.1, take $d_V = \text{vol}(r_n K_V)$. Moreover from elementary geometry we also have a bound

$$M < M_V$$

take for instance $M_V = d_V + n - 1$.

1.3 Pluricanonical maps in a family

Let $q: Y \to T$ be a projective morphism of nonsingular varieties. Consider the relative pluricanonical bundle $\omega_{Y/T}^{\otimes m}$ on Y, and the coherent sheaf $q_*(\omega_{Y/T}^{\otimes m})$ on T. If q is a smooth morphism, the restriction of $\omega_{Y/T}^{\otimes m}$ to the fibre Y_t is the pluricanonical bundle $\omega_{Y_t}^{\otimes m}$, and the natural map $q_*(\omega_{Y/T}^{\otimes m}) \otimes k(t) \to H^0(Y_t, \omega_{Y_t}^{\otimes m})$ is an isomorphism for general $t \in T$. Consider the induced rational map of families

$$Y \dashrightarrow \mathbb{P}(q_*(\omega_{Y/T}^{\otimes m})).$$

If q is a smooth morphism this is, generically, the family of m-canonical maps of fibres. In this setting the following theorem of Kawamata [9] holds:

Theorem 1.3 (semipositivity)

The sheaf $q_*(\omega_{Y/T}^{\otimes m})$ is nef, for a family of connected fibres, provided that T is a curve.

Another fundamental property of relative pluricanonical sheaves is the following theorem of Siu, see [16, 17]:

Theorem 1.4 (invariance of plurigenera)

In a smooth projective family every plurigenus is constant.

In the theory of pluricanonical maps, the need to allow singular varieties arises, and an outstanding role is played by the canonical singularities, cf. [15]. For a possibly singular variety, the plurigenus of any resolution of singularities is independent of the resolution, and is called the plurigenus of the variety.

In this general situation, one still has the following results. In a projective family of varieties, over a smooth curve, every plurigenus is a lower semicontinuous function;

if the varieties in the family have canonical singularities, then every plurigenus is constant; see Takayama [19]. The invariance of plurigenera is also known for flat families (of any dimension) of canonical singularities of general type, see Kawamata [10].

Moreover we point out the following result, for which a proof will be given in the next section.

Theorem 1.5 (invariance of the general type)

In a projective family of varieties, over a smooth curve, the varieties of general type form an open subfamily, in the Zariski topology.

1.4 Extension of differentials

The results in the previous section on the variation of plurigenera are all based on the technique of extension of differentials, from a special fibre to the total space of the family. The following general result, for one-dimensional families, is due to Takayama [19], we quote it in a slightly less general form, suitable for our purposes.

Theorem 1.6

Let $\pi: V \to S$ be a projective morphism of nonsingular varieties, with S a curve, and let Z be an irreducible component of some fibre $V_0 = \pi^{-1}(t_0)$, also a nonsingular variety. Assume moreover that π has connected fibres. Then the restriction map

$$\pi_* \mathcal{O}_V(mK_V)_{t_0} \longrightarrow H^0(Z, \mathcal{O}_Z(mK_Z))$$

is surjective for every m > 0.

Using the extension theorem, we may give a proof of Theorem 1.5.

Proof. Let T be a nonsingular curve, and let $q: Y \to T$ be a family of projective varieties, i.e. a projective morphism with reduced irreducible fibres Y_t of the same dimension, say n. We have to show that: if some fibre Y_0 is of general type then the nearby fibres Y_t are of general type.

Consider some resolution of singularities $\mu: V \to Y$ such that the strict transform $Z = Y_0'$ is smooth. So Z is of general type. Then $\dim H^0(Z, mK_Z) \ge cm^n$ for $m \gg 0$. From Theorem 1.6 we have that the restriction homomorphism

$$\pi_* \mathcal{O}_V(mK_V)_{t_0} \longrightarrow H^0(Z, mK_Z)$$

is surjective.

The image $\pi_*\mathcal{O}_V(mK_V)$ is a torsion free coherent sheaf on the smooth curve T, hence it is a locally free sheaf. So the dimension of $\pi_*\mathcal{O}_V(mK_V)\otimes k(t)$ is constant. In t_0 this dimension is $\geq cm^n$, by what we have seen above.

Moreover as the composite map $V \to Y \to T$ is generically smooth we may assume that in some neighborhood of t_0 for every $t \neq t_0$ the induced map $V_t \to Y_t$ is a resolution of singularities. Since $mK_V|_{V_t} = mK_{V_t}$ we have the inclusion

$$\pi_* \mathcal{O}_V(mK_V) \otimes k(t) \hookrightarrow H^0(V_t, \mathcal{O}_{V_t}(mK_V|_{V_t})) = H^0(V_t, mK_{V_t}).$$

It follows that dim $H^0(V_t, mK_{V_t}) \ge cm^n$ for $m \gg 0$, hence Y_t is of general type. This holds for every t in a neighborhood of t_0 .

2. Rigidity

First step is a theorem of rigidity, which is more precisely a theorem of birational triviality of families.

2.1 The rigidity theorem

Theorem 2.1

Let X be a projective variety of general type, of dimension n. Let T be a smooth variety, and let $Y \to T$ be a smooth family of n-dimensional projective varieties Y_t of general type. Assume that $f: X \times T \dashrightarrow Y$ is a family of dominant rational maps, i.e. that f restricts to a dominant rational map $f_t: X \dashrightarrow Y_t$ for every t. Then there is $g: Y \dashrightarrow Y_0 \times T$, a birational isomorphism over T, defined on every Y_t , such that $g \circ f$ is a trivial family $f_0 \times 1: X \times T \dashrightarrow Y_0 \times T$. Therefore all maps f_t are birationally equivalent.

This is a slightly more general version than the one in Maehara [13], avoiding some technical restrictions. We recall here that two rational maps $f: X \dashrightarrow Y$ and $f': X \dashrightarrow Y'$ are said to be birationally equivalent if there is a birational isomorphism $g: Y \dashrightarrow Y'$ such that $f' = g \circ f$. The proof of the theorem will be given in the next section. We point out the following:

Corollary 2.2

Let X be a projective variety of general type, of dimension n. Let T be a smooth variety, and let $Y \to T$ be a family of n-dimensional projective varieties Y_t of general type. Assume that $f: X \times T \dashrightarrow Y$ is a family of dominant rational maps $f_t: X \dashrightarrow Y_t$. Then almost all maps f_t are birationally equivalent.

Proof. Choose some resolution of singularities $Z \to Y$. By generic smoothness, over some open subset $T' \subset T$ the family $Z \to T$ is smooth and the fibre Z_t is a (nonsingular) birational model of Y_t . The assertion follows from the theorem applied to the restricted family $Z' \to T'$.

Remark 2.3 We think that the rigidity theorem holds more generally if $Y \to T$ is a family of projective varieties Y_t , with canonical singularities and of general type.

2.2 Proof of the rigidity theorem

2.2.1. The variation of higher differentials (cf. [13, §4.1]). Let $f: X \times T \dashrightarrow Y$ be a family of rational maps, as in the statement. Let $m = r_n$ be the integer defined in Theorem 1.1, and consider the exact sequence of sheaves on T

$$0 \to q_*(\omega_{Y/T}^{\otimes m}) \to V \otimes \mathcal{O}_T \to \mathcal{Q} \to 0 \text{ where } V := H^0(X, \omega_X^{\otimes m})$$

in which we know from Theorem 1.4 that $q_*(\omega_{Y/T}^{\otimes m})$ is locally free. Let ν be the rank of this sheaf.

The claim is that the image of this inside $V \otimes \mathcal{O}_T$ is a trivial subsheaf $W \otimes \mathcal{O}_T$. It follows that the composite map $X \times T \dashrightarrow Y \dashrightarrow \mathbb{P}(W) \times T$ is of the form $f_0 \times 1$, which gives the theorem.

In order to prove the claim, we may restrict to a Zariski open subset of T. Suppose indeed that for some $T' \subset T$ the isomorphism $q_*(\omega_{Y/T'}^{\otimes m}) \stackrel{\sim}{\to} W \otimes \mathcal{O}_{T'}$ is proved. Since $q_*(\omega_{Y/T}^{\otimes m})$ is locally free it follows that there is an induced map $q_*(\omega_{Y/T}^{\otimes m}) \to W \otimes \mathcal{O}_T$, and this is injective as a map of bundles, of the same rank, hence it is an isomorphism.

Therefore we may assume that the cokernel Q is locally free. Then the exact sequence above may be seen as the pullback of the universal sequence on the suitable Grassmannian through the morphism

$$T \to \operatorname{Grass}_{\nu}(V)$$
 sending t to $H^0(Y_t, \omega_{Y_t}^{\otimes m})$.

Claim: this map is constant.

In order to prove the claim, it is enough to restrict to curves in T.

2.2.2. One parameter families (cf. [13, §4.2]). Let C be a nonsingular curve, let Y be a nonsingular variety, and assume that $Y \to C$ is a smooth projective family. If $C \hookrightarrow D$ is an embedding into a projective nonsingular curve, complete the family to a flat family $Y' \to D$. Some limit fibre is possibly singular. Then apply semistable reduction: there is a projective nonsingular curve B with a finite map $\varphi : B \to D$, and there is a projective nonsingular variety Z with a morphism $Z \to B$, and with a morphism of families $Z \to Y'$, such that: restricted to $A = \varphi^{-1}(C)$ the family coincides with the pullback of the original family over C, and moreover the new limit fibres have only normal crossings as singularities.

$$\begin{array}{ccccc} Y & \hookrightarrow & Y' & \leftarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ C & \hookrightarrow & D & \xleftarrow{\varphi} & B \end{array}$$

Let $X \times C \dashrightarrow Y$ be a family of rational maps. It is viewed as a family $X \times D \dashrightarrow Y'$, and can be lifted to a family $X \times B \dashrightarrow Z$.

As the construction of the relative canonical sheaf is compatible with pullback, one shows that the induced map $A \to \operatorname{Grass}_{\nu}(V)$, to the appropriate Grassmannian, factorizes as $A \to C$ followed by the induced map $C \to \operatorname{Grass}_{\nu}(V)$. So it is enough to prove that the map on A is constant.

2.2.3. Proof of the claim (cf. [13, §5]). Thus one is reduced to the situation where B is a nonsingular projective curve, Z is a nonsingular projective variety and $Z \to B$ is a family with connected fibres having only normal crossings. Consider the exact sequence $0 \to q_*(\omega_{Z/B}^{\otimes m}) \to V \otimes \mathcal{O}_B \to \mathcal{Q} \to 0$. Let $A \subset B$ be an open subset such that the restriction \mathcal{Q}_A is locally free. Let $T(\mathcal{Q}) \subset \mathcal{Q}$ be a torsion subsheaf such that $\mathcal{Q}/T(\mathcal{Q})$ is locally free. Then consider the exact sequence

$$0 \to \mathcal{K} \to V \otimes \mathcal{O}_B \to \mathcal{Q}/T(\mathcal{Q}) \to 0$$

in which the kernel \mathcal{K} is locally free. Note that $q_*(\omega_{Z/B}^{\otimes m}) \subset \mathcal{K}$ and the two sheaves coincide on A. It follows that there is an induced morphism

$$B \to \operatorname{Grass}_{\nu}(V)$$
 sending t to \mathcal{K}_t

which is $H^0(Z_t, \omega_{Z_t}^{\otimes m})$ for $t \in A$.

The pullback of the Plücker line bundle is $\det \mathcal{K}^{\vee}$. If this is a non-constant map then $\deg \mathcal{K}^{\vee} > 0$ and $\deg \mathcal{K} < 0$.

On the other hand from the semipositivity Theorem 1.3 we have that the sheaf $q_*(\omega_{Z/B}^{\otimes m})$ is nef. Then $\det q_*(\omega_{Z/B}^{\otimes m})$ is nef and $\deg q_*(\omega_{Z/B}^{\otimes m}) \geq 0$. Moreover $q_*(\omega_{Z/B}^{\otimes m}) \subset \mathcal{K}$ and the two coincide on A, and this implies $\deg \mathcal{K} \geq \deg q_*(\omega_{Z/B}^{\otimes m}) \geq 0$, contradiction.

2.3 Generic rigidity and limit maps

Let X be a projective variety, of dimension n, let $Y \to T$ be a family of n-dimensional projective varieties, with T a smooth curve, and let $f: X \times T \dashrightarrow Y$ be a family of dominant rational maps $f_t: X \dashrightarrow Y_t$, defined over all of T. The graph $F:=\Gamma(f)$ with the natural projection $F \to T$ represents the family of graphs $F_t = \Gamma(f_t)$. Moreover $\deg(f_t) =: d$ is constant in the family.

A more general definition is obtained if we admit that some special fibre Y_{t_0} may have multiplicity, i.e. may be of the form d_0Y_0 , multiple of an irreducible. In this situation $\deg(f_t) =: d$ is constant for $t \neq t_0$. Since $\pi_*\Gamma(f_t) = dY_t$ holds for $t \neq t_0$, where π denotes projection to Y, the same holds for every t, and $\pi_*\Gamma(f_{t_0}) = dd_0Y_0$ implies that $\deg(f_{t_0}) = dd_0$. So in particular the limit map has higher degree.

The more general definition above may be expressed equivalently by requiring that the graph $\Gamma(f)$ defines a family of graphs $\Gamma(f_t)$, without requiring that Y defines the family of images, i.e. that every fibre Y_t is reduced.

Now consider a family which is *generically trivial*, as in Corollary 2.2, i.e. which is obtained as

$$X \times T \xrightarrow{h \times 1} V \times T \xrightarrow{g} Y$$

a constant family followed by a birational isomorphism g, defined over $T - \{t_0\}$. In this situation we have $f_t = g_t \circ h$ for $t \neq t_0$, so all these maps are birationally equivalent, of degree $d := \deg(f_t) = \deg(h)$. Concerning the limit map f_{t_0} we can say the following.

Proposition 2.4

Assume that the family f is generically trivial and of constant degree, i.e. that $deg(f_t) = d$ for every t. Then the limit map f_{t_0} is in the same birational equivalence class as the general f_t .

Proof. We have the graphs $F_t = \Gamma(f_t)$ for every t, and $G_t = \Gamma(g_t)$ for $t \neq t_0$, and $H = \Gamma(h)$. In terms of composition of correspondences we have that $F_t \subset H \circ G_t$ holds for $t \neq t_0$ (a little conflict of notations is hidden here).

The family G_t converges to a cycle G_0 in the space of cycles of $V \times Y$. The intersection number $G_t \cdot (\{x\} \times Y) = 1$ is constant in the deformation, and $G_0 \cdot (\{x\} \times Y) = 1$ implies that the part of G_0 that dominates V is an irreducible reduced cycle and is the graph G'_0 of some rational map $g_0 : V \dashrightarrow Y_0$. By continuity we have that $F_{t_0} \subset H \circ G_0$, and more precisely that $F_{t_0} \subset H \circ G'_0$, since F_0 dominates X. But this implies that $f_{t_0} = g_0 \circ h$. As the degree is constant in the family, then g_0 is birational, hence f_{t_0} is birationally equivalent to h and hence to every f_t .

Corollary 2.5

Let X be a projective variety of general type, of dimension n. Let T be a smooth curve, and let $Y \to T$ be a family of n-dimensional projective varieties Y_t of general type. Assume that $f: X \times T \dashrightarrow Y$ is a family of dominant rational maps $f_t: X \dashrightarrow Y_t$, of constant degree. Then all maps f_t are birationally equivalent.

Proof. Immediate from Corollary 2.2 and Proposition 2.4 above. \Box

Remark 2.6 There is a property of the limit map which forces the generically trivial family to be of constant degree. Let us say that a dominant rational map of finite degree $X \dashrightarrow Y$ is primitive if it admits no factorization $X \dashrightarrow Z \dashrightarrow Y$ in which both factor maps have degree > 1. The same argument as in the proof above implies that, if the generic degree is > 1, and if the limit map f_{t_0} is a primitive map, then it is in the same birational equivalence class as the general f_t .

3. Boundedness

Second step, that on a given variety the family of rational maps, of the type being considered, is a bounded family. Here we point out that the canonical volume is a natural tool.

3.1 Bound for the graph of a map

Let X be a nonsingular variety of general type, of dimension n. Choose an embedding $X \hookrightarrow \mathbb{P}^N$ or more generally a birational embedding

$$X \dashrightarrow X' \subset \mathbb{P}^N$$

and let H be the pullback of a general hyperplane. Every rational map of finite degree $X \dashrightarrow Y$ which dominates a variety of general type, followed by the r_n -canonical birational embedding $Y \dashrightarrow Y' \subset \mathbb{P}^M$, may be seen as a rational map $X' \dashrightarrow Y' \subset \mathbb{P}^M$, with bounded embedding dimension $M \le M_X$, see §1.2. This map has an associated graph

$$\Gamma \subset X' \times \mathbb{P}^M$$

and the degree of the graph is bounded too.

Proposition 3.1

Let X and H be as in the setting above. There is some positive integer γ_X such that every associated graph Γ in $\mathbb{P}^N \times \mathbb{P}^M$ has bounded degree:

$$\deg \Gamma \leq \gamma_X$$
.

More precisely we have:

- 1. $\deg \Gamma \leq \operatorname{vol}_X(H + r_n K_X)$ in general;
- 2. $\deg \Gamma \leq (H + r_n K_X)^n$ if $H + r_n K_X$ is nef;
- 3. deg $\Gamma \leq (2r_n)^n \operatorname{vol}(K_X)$ if $H \equiv r_n K_X$.

Proof. 1. Consider the birational embedding $X \dashrightarrow \Gamma \subset \mathbb{P}^N \times \mathbb{P}^M$ and apply §1.1. It follows that $\deg \Gamma \leq \operatorname{vol}_X(H + f^*r_nK_Y)$, and this is $\leq \operatorname{vol}_X(H + r_nK_X)$ as 'volume increases in effective directions': $\operatorname{vol}_X(D) \leq \operatorname{vol}_X(D')$ if $D \leq D'$. 2. This is because $\operatorname{vol}_X(D) = D^n$ if D is big and nef. 3. is just a special case of point 1, using $\operatorname{vol}_X(rD) = r^n \operatorname{vol}_X(D)$.

The proposition implies that every rational map on X, of the type being considered, is birationally equivalent to some map whose graph is a point in the disjoint union of Chow varieties \bigsqcup Chow_{n,γ} $(X' \times \mathbb{P}^{M_X})$, taken over all $\gamma \leq \gamma_X$, and more precisely a point in the Zariski open subset parametrizing rational maps of finite degree.

3.2 Remarks on the degree of a map

The degree of the graph is greater than the degree of the map. We insert here a few remarks on how to bound the degree of a map, which however are not needed in the following.

Proposition 3.2

Let $f: X \dashrightarrow Y$ be a rational map of finite degree.

- 1. If K_X and K_Y are nef, then $\deg(f) K_Y^n \leq K_X^n$;
- 2. if K_X and K_Y are big, then $\deg(f) \operatorname{vol}(K_Y) \leq \operatorname{vol}(K_X)$.

Proof. 1. Use the asymptotic Riemann-Roch theorem (cf. [12, I, 1.4.41]): if D is nef, then $h^0(mD) = \frac{1}{n!}D^n\,m^n + O(m^{n-1})$. Since $f^*K_Y \leq K_X$ consider $h^0(mf^*K_Y) \leq h^0(mK_X)$ and apply aRR, comparing the leading coefficients. 2. Using the volume of a big divisor, expressed in terms of moving self-intersection numbers, it is easy to see that under a rational map of finite degree one has $\deg(f)\operatorname{vol}_Y(D) \leq \operatorname{vol}_X(f^*D)$. Moreover $f^*K_Y \leq K_X$ implies that $\operatorname{vol}_X(f^*K_Y) \leq \operatorname{vol}(K_X)$.

Proposition 3.3

Let X be a variety of general type. There is some positive rational number $\epsilon_X > 0$ such that: if $f: X \dashrightarrow Y$ is a rational map of finite degree which dominates Y of general type, then $\operatorname{vol}(K_Y) \ge \epsilon_X$ and therefore

$$\deg(f) \le \frac{1}{\epsilon_X} \operatorname{vol}(K_X).$$

Proof. It has been shown by Hacon and McKernan ([7, Corollary]) that for every Y of general type of given dimension one has $vol(K_Y) \ge \epsilon$ for some $\epsilon > 0$. Therefore ϵ_X exists. Then use point 2 in the proposition above.

4. Finiteness

The finiteness theorem is the following

Theorem 4.1

Let X be a projective variety of general type, of dimension n. The set of rational maps $X \dashrightarrow Y$ which dominate n-dimensional projective varieties Y of general type, up to birational isomorphisms $Y \dashrightarrow Y'$, is a finite set.

We shortly discuss the original argument of [13] and we propose a variation which possibly leads to some effective bound for the finite number of maps.

4.1 The original argument

The parameter space for rational maps of finite degree $X \dashrightarrow Y' \subset \mathbb{P}^M$, with fixed M, seen at the end of §3.1, is quasi-projective (and highly reducible). Call it T for a while. There is an algebraic subset $Y \subset \mathbb{P}^M \times T$ whose reduced fibres in the projection over T are the dominated varieties Y', and there is a total rational map $X \times T \dashrightarrow Y$.

Then use smoothening stratification: there is a stratification into (locally closed) smooth strata T_{α} and for each restricted family $Y_{\alpha} \to T_{\alpha}$ there is a resolution of singularities $Z_{\alpha} \to Y_{\alpha}$ such that the composite map $Z_{\alpha} \to T_{\alpha}$ is a smooth family. So for every $t \in T$ also there is a resolution of singularities $Z_t \to Y_t$. Moreover the stratification may be refined so that the subset of T parametrizing varieties (admitting a smooth model) of general type is a union of a subcollection of strata T_{β} , thus a constructible subset. cf. [13, §3].

It follows that: the number of equivalence classes of rational maps in Theorem 4.1 is smaller than the number of connected components of strata. If two maps are in the same connected stratum, there is a curve (possibly reducible) connecting the two points and contained in the stratum. This gives a family of rational maps $X \dashrightarrow Z_t$ such that Z_t is smooth of general type. Because of the rigidity Theorem 2.1 they are all birationally equivalent.

Unfortunately such a stratification seems to be rather intractable. A variation of the argument is used by Heier [8] for morphisms $X \to Y$ onto varieties with ample K_Y . In this situation, the observation is that the number of equivalence classes of these morphisms is smaller than the number of connected components of the union of Chow varieties. We show in the next sections that the idea is applicable in a more general situation.

4.2 Parametrization

Let X be a nonsingular projective variety of general type, of dimension n, and choose a birational embedding $X \dashrightarrow X' \subset \mathbb{P}^N$, as in §3.1.

In the Chow variety $\operatorname{Chow}_{n,\gamma}(X' \times \mathbb{P}^M)$ consider the Zariski open subset

$$G_{\gamma}(X' \times \mathbb{P}^M)$$

parametrizing graphs of rational maps $g: X' \dashrightarrow \mathbb{P}^M$ such that the (closed) image $Y' \subset \mathbb{P}^M$ still is of dimension n. For such a map we have

$$\deg_2 \Gamma = \deg(g) \deg Y'$$
 and $p_*(\Gamma) = \deg(g) Y'$.

For any k > 0 with $\gamma \ge k$ define

$$G_{\gamma,k}(X'\times\mathbb{P}^M)$$

by requiring that $\deg_2 \Gamma = k$. For any d > 0 with d|k define

$$G_{\gamma,k,d}(X'\times\mathbb{P}^M)$$

by the condition that $p_*\Gamma$ is of the form dY' (without requiring that Y' be irreducible).

Proposition 4.2

We have that:

- 1. $G_{\gamma,k}(X' \times \mathbb{P}^M)$ is a union of irreducible components of $G_{\gamma}(X' \times \mathbb{P}^M)$,
- 2. $G_{\gamma,k,d}(X' \times \mathbb{P}^M)$ is an algebraic subset of $G_{\gamma,k}(X' \times \mathbb{P}^M)$.

Proof. In a family of graphs Γ_t , parametrized by an irreducible component of $G_{\gamma}(X' \times \mathbb{P}^M)$, the intersection number $\deg_2(\Gamma_t)$ has a constant value k > 0. Thus $G_{\gamma,k}(X' \times \mathbb{P}^M)$ is a union of irreducible components of $G_{\gamma}(X' \times \mathbb{P}^M)$. This proves point (1). The map $G_{\gamma,k}(X' \times \mathbb{P}^M) \longrightarrow \operatorname{Chow}_{n,k}(\mathbb{P}^M)$ such that $t \mapsto p_*(\Gamma_t)$ is an algebraic correspondence. Hence the subset defined by the condition that $p_*(\Gamma_t)$ is of the form dY', i.e. that the associated Chow form is of the form F^d , is the inverse image of a d-ple Veronese variety. This proves point (2).

Let $m=r_n$ be the integer defined in Theorem 1.1. Define in $G_\gamma(X'\times\mathbb{P}^M)$ the subset

$$S_{\gamma}(X' \times \mathbb{P}^M),$$

parametrizing maps such that the image Y' is of general type and m-canonically embedded, and similarly define

$$S_{\gamma,k,d}(X'\times\mathbb{P}^M).$$

Note that the degrees γ, k, d are birationally invariant.

4.3 Refined finiteness theorem

We point out a more precise version of the finiteness theorem, in which the finite number of maps is related to the finite number of irreducible components of a certain bunch of (subvarieties of) Chow varieties, over which we have some control.

Theorem 4.3

Let X be a projective variety of general type, of dimension n. The number of rational maps on X which dominate n-dimensional projective varieties of general type, up to birational equivalence, is bounded by the number of irreducible components of the disjoint union

$$| | G_{\gamma,k,d}(X' \times \mathbb{P}^{M_X})$$

taken over all γ, k, d such that $1 < k \le \gamma \le \gamma_X$ and d > 1 is a divisor of k.

Proof. For every M and γ, k, d , there is a correspondence

$$S_{\gamma,k,d}(X' \times \mathbb{P}^M)/_{\sim} \swarrow \cdots \searrow i.c. G_{\gamma,k,d}(X' \times \mathbb{P}^M)$$

between the set of birational equivalence classes of maps and the set of irreducible components of the algebraic set, in which a class of maps corresponds to an irreducible component if and only if the two meet at some point. Each equivalence class corresponds to one or several components, and the key point is that: two different classes of maps cannot correspond to one and the same irreducible component of the algebraic set. For $M = M_X$, this gives the theorem.

Assume that two points in $S_{\gamma,k,d}(X' \times \mathbb{P}^M)$ are in the same irreducible component of $G_{\gamma,k,d}(X' \times \mathbb{P}^M)$. Then they are connected by some irreducible curve contained in the irreducible component. This gives a family of rational maps $X \dashrightarrow Y'_t \subset \mathbb{P}^M$ of finite degree d > 1. After removing finitely many points if necessary (not the two given points), we may assume that every Y'_t is of general type, by Theorem 1.5. Since the degree of maps is constant in the deformation, then the two given maps are birationally equivalent, by Corollary 2.5.

4.4 Towards an effective estimate

The theorem above also leads to an almost effective result.

Proposition 4.4

In Theorem 4.1 the number of equivalence classes of rational maps has an upper bound of the form $B(n, v_X)$ only depending on the dimension n and the canonical volume $v_X = \text{vol}(K_X)$. Here the function B can be explicitly computed in terms of the function r_n that is defined in Theorem 1.1.

Proof. This is obtained using a general estimate (of Bézout type) for the number of irreducible components of an algebraic subset. If $V \subset \mathbb{P}^C$ is defined by equations of degree $\leq D$, the number of irreducible components of V is bounded above by D^C . The same estimate holds more generally for a locally closed subset in \mathbb{P}^C of the form $U \cap V$, where U is a Zariski open subset and V is an algebraic subset, defined by equations of degree $\leq D$. This is applied to the disjoint union of varieties that is defined in Theorem 4.3.

The following will be shown later on in $\S5.3$:

- the embedding dimension of $\operatorname{Chow}_{n,\gamma}(X' \times \mathbb{P}^M)$ is bounded by some function $C(n,\gamma,M)$,
- the algebraic subset $G_{\gamma,k,d}(X' \times \mathbb{P}^M)$ is defined by equations whose degree is bounded by some function $D(n,\gamma,M,d')$,

where d' may be taken to be $\deg X'$, and moreover these bounds are effectively computable.

It follows that the number of irreducible components of $G_{\gamma,k,d}(X' \times \mathbb{P}^M)$ is bounded upperly by $D(n, \gamma, M, d')^{C(n,\gamma,M)}$. The sum over γ, k, d is bounded by

$$(\gamma_X)^3 D(n, \gamma_X, M_X, d')^{C(n, \gamma_X, M_X)}$$
.

Finally recall that M_X and γ_X and d' are bounded in terms of the canonical volume v_X and the function r_n , see §1.2.

Remark 4.5 The result of Heier [8], although restricted to morphisms, is really effective, based on some effective result of uniform pluricanonical very ampleness, of the same author. Concerning the effectivity of the function r_n , that is still unknown, see Hacon-McKernan [7] and Chen [3].

5. Remarks on the complexity of Chow varieties

In this section we give the proofs of two points that were used in the proof of Proposition 4.4. The complete details are indeed so cumbersome, and we prefer to concentrate on the method, especially on a few points involving the use of Chow varieties, rather than working out explicit formulas. Moreover we believe that the present results will be improved and simplified if a different parametrization is used for rational maps, that we plan to study in a subsequent paper.

5.1 On elimination of variables

As a basic tool we need some estimate of the complexity of the process of elimination of variables. This follows from known estimates of the complexity of Gröbner bases of a polynomial ideal, since such a basis always contains a basis of the resultant ideal (also called the eliminant ideal). We refer to the paper of Dubé [5] for the explicit results. Thus the following holds.

Let f_1, \ldots, f_p be homogeneous polynomials of degree $\leq d$ in the variables x_0, \ldots, x_r with indeterminate coefficients. Then the resultant ideal, after elimination of the variables x_i , is generated by homogeneous polynomials of degree $\leq \delta(r, d)$ in the coefficients of the polynomials f_j , where δ is some suitable integer function, effectively computable.

The following corollaries are easily deduced.

If the indeterminate coefficients of f_1, \ldots, f_p are specialized as homogeneous polynomials of degree $\leq d$ in a new set of variables y_1, \ldots, y_r , then the resultant ideal, after elimination of the variables x_i , is generated by homogeneous polynomials of degree $\leq \delta'(r,d) := d \, \delta(r,d)$ in the variable y_i .

If f_1, \ldots, f_p are multi-homogeneous polynomials of degree $\leq d$ depending on sequences x_1, \ldots, x_k of r+1 variables each, with indeterminate coefficients, then the resultant ideal, after elimination of x_1, \ldots, x_k , is generated by homogeneous polynomials in the coefficients of f_1, \ldots, f_p , of degree at most equal to

$$\delta(k, r, d) := \delta'(r, \delta'(r, \dots, \delta'(r, d)) \dots).$$

5.2 Chow varieties and their equations

Standard references are [4, 11]. Let Z be a subvariety in $\mathbb{P}^r = \mathbb{P}(V)$ of dimension n and degree k. We denote with the symbol V^{\vee} the dual vector space. There is an irreducible polynomial $F_Z(u_0,\ldots,u_n)$, homogeneous of degree k with respect to each variable $u_i \in V^{\vee}$, such that $F_Z(u_0,\ldots,u_n)=0$ if and only if the linear space $u_0(x)=\cdots=u_n(x)=0$ meets Z. This polynomial F_Z , which is unique up to proportionality, is called the associated form of the variety Z, and its coefficients are said to be the coordinates of Z. The associated form of a positive cycle $Z=\sum a_i Z_i$ of pure dimension n is defined as $F_Z=\prod F_Z^{a_i}$.

Let $\mathbb{F}_{n,k,r}$ be the projective space of multihomogeneous polynomials of degree k in u_0, \ldots, u_n . The dimension of this space is a function

$$\varphi(n,k,r)$$

that is effectively computable, by elementary algebra.

If $S \subset \mathbb{P}^r$ is an algebraic subset, then in $\mathbb{F}_{n,k,r}$ the subset $\operatorname{Chow}_{n,k}(S)$ of associated forms F_Z of cycles Z of dimension n and degree k supported in S is an algebraic subset, called the *Chow variety* of S, relative to the pair n, k. Equations for the Chow variety are obtained from the following characterization of associated forms. Assume that S in \mathbb{P}^r is defined by equations $f_{\alpha}(x) = 0$, of degree at most d'.

Necessary and sufficient conditions for a multihomogeneous form $F(u_0, u_1, \ldots, u_n)$ so that it is associated to some *n*-cycle supported in $S \subset \mathbb{P}^r$ are:

for every $u_1, \ldots, u_n \in V^{\vee}$ there are $x_1, \ldots, x_k \in V$ rm such that:

- 1. $F(u_0, u_1, \ldots, u_n)$ and $u_0(x_1) \cdots u_0(x_k)$ are proportional, as polynomials in u_0 ;
- 2. $f_{\alpha}(x_i) = 0$ for j = 1, ..., k and for every α ;
- 3. $u_i(x_j) = 0$ for i = 1, ..., n and j = 1, ..., k;
- 4. for every x_j and every v_0, v_1, \ldots, v_n passing through x_j one has $F(v_0, v_1, \ldots, v_n) = 0$.

We now describe how these conditions can be translated into equations. The proportionality condition 1 is bilinear in the two polynomials. Every linear form passing through a point x can be expressed as $x \cdot s := s(x, -)$ where s is an antisymmetric bilinear form. Writing then $F(x \cdot s_0, x \cdot s_1, \ldots, x \cdot s_n) = \sum \varphi_{\alpha}(F, x) M_{\alpha}(s_0, s_1, \ldots, s_n)$, condition 4 turns out to be equivalent to: $\varphi_{\alpha}(F, x_j) = 0$ for every j and every α . The polynomials φ_{α} are linear in F and of degree k = k(n + 1) with respect to the variable x. The degrees of equations are collected in the following table.

Eliminating x_1, \ldots, x_k from the equations above yields some finite system of equations of the form $P(F, u_1, \ldots, u_n) = 0$. It follows from §5.1 that the function

$$\Delta(n, k, r, d') := \delta(k, r, \max\{d', k(n+1)\})$$

is an upper bound for the degrees of polynomials P with respect to F.

The conditions so that F is an associated form are therefore equivalent to requiring that: for every u_1, \ldots, u_n the sequence F, u_1, \ldots, u_n satisfies the resultant equations P. Writing $P(F, u_1, \ldots, u_n) = \sum P_{\alpha}(F) M_{\alpha}(u)$, we obtain equations $P_{\alpha}(F) = 0$ for every α and every P, which define the Chow variety. Note that the degree of a polynomial P_{α} coincides with the degree of the corresponding P with respect to the variable F. Hence all these degrees are bounded by the same function $\Delta(n, k, r, d')$.

5.3 On Chow forms of graphs and direct images

We now work with cycles Γ in a product variety $X' \times \mathbb{P}^M \subset \mathbb{P}^N \times \mathbb{P}^M$. Define $\overline{M} := (N+1)(M+1)-1$, the dimension N being fixed. Let d' be an integer such that X' admits equations of degree at most d', for instance deg X' is such an integer. Then

 $X' \times \mathbb{P}^M$ also admits equations of degree at most d'. Because of the analysis in the preceding section, the Chow variety $\operatorname{Chow}_{n,\gamma}(X' \times \mathbb{P}^M)$ is embedded in a projective space of dimension

$$C(n, \gamma, M) := \varphi(n, \gamma, \overline{M})$$

and admits equations of degree bounded by

$$D(n, \gamma, M, d') := \Delta(n, \gamma, \overline{M}, d').$$

We describe how the Chow form of $p_*(\Gamma)$ depends on the Chow form of Γ . In the product $(\mathbb{P}^N \times \mathbb{P}^M) \times (\mathbb{P}^M)^* \times \cdots \times (\mathbb{P}^M)^*$ consider the incidence subset Φ consisting of all sequences $(x, y), v_0, \ldots, v_n$ such that

5. $x \otimes y$ belongs to Γ ,

6.
$$v_i(y) = 0$$
 for $i = 0, ..., n$,

and denote by q the natural projection to $(\mathbb{P}^M)^* \times \cdots \times (\mathbb{P}^M)^*$. It is easily seen that Φ is irreducible of dimension (n+1)M-1. The equations of the subscheme $q(\Phi)$ are obtained by eliminating the variables x, y from the equations above. The degree of $q:\Phi\to q(\Phi)$ is equal to the degree of $p:\Gamma\to Y'$, provided of course it is a finite degree. Therefore $q(\Phi)$ is of the same dimension as Φ if and only if Y' is of dimension n. In this case the cycle $q_*(\Phi)$ is a hypersurface in $(\mathbb{P}^M)^* \times \cdots \times (\mathbb{P}^M)^*$, and is defined by some multihomogeneous polynomial $G(v_0, \ldots, v_n)$ of the same degree k with respect to each v_i . This is the Chow form of $p_*(\Gamma)$.

The conditions above may be written as equations involving the Chow form F of the cycle Γ and the other variables. The degrees of these equations are:

$$\begin{array}{c|ccccc}
 & F & v_0, \dots, v_n & x, y \\
\hline
5 & 1 & - & \bar{\gamma}, \bar{\gamma} \\
6 & - & 1 & -, 1
\end{array}$$

where $\bar{\gamma} = \gamma(n+1)$. Eliminating x, y from the equations above one obtains a finite set of equations of the form $G_{\alpha}(F, v_0, \dots, v_n) = 0$. Applying 5.1 one sees that the degrees relative to F are bounded by the same function $D(n, \gamma, M, d')$.

The algebraic subfamily $G_{\gamma}(X', \mathbb{P}^M)$ of those Γ for which Y' still is of dimension n is defined by the condition that all these forms in v_0, \ldots, v_n generate a principal ideal. In an irreducible component of this subset one has therefore a single form $G(F, v_0, \ldots, v_n)$ of the same degree k with respect to each v_i , and whose degree with respect to F is bounded by $D(n, \gamma, M, d')$. The algebraic subfamily $G_{\gamma,k,d}(X', \mathbb{P}^M)$ of those F such that the form $G(F) := G(F, -, \dots, -)$ is a d-th power is therefore defined by equations arising from the quadratic equations $Q_{\nu}(G) = 0$ of the d-ple Veronese variety by specialization as $Q_{\nu}(G(F)) = 0$, and are of degree $\leq 2D(n, \gamma, M, d')$.

We may assume, as a feedback, that the coefficient 2 was already included in the definition of the function D.

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