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BIRATIONAL AUTOMORPHISMS OF MULTIDIMENSIONAL ALGEBRAIC MANIFOLDS

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Introduction

This survey summarizes the results of investigations during two last decades in the field of birational automorphism groups of Fano manifolds. This subject, which is traditional for the Moscow school of algebraic geometry (and has been under active study only here in the second half of this century), has its origin in the works of the eminent Italian geometrician G. Fano. He proposed several very fruitful conceptions [27–30] but did not present them in a complete form: he did not prove any of his correctly guessed theorems. His reasoning has significant gaps and outright mistakes. Thus, the modern stage in multidimensional birational geometry begins with the correction of the latter.

In 1971, the work [12] was published, where the Fano theorem about the rigidity of smooth 3-dimensional quartics was proved. The maximal singularities method, the theory proposed in [12], appeared to be very effective for the study of the birational correspondences of multidimensional algebraic manifolds with negative canonic sheaf (of Fano manifolds). In the subsequent works [4, 5], birational automorphism groups were described and a birational type of several series of Fano 3-folds was found: double spaces and double quadrics of index 1, the double Veronese cone of index 2 (a proof was completed in [24]), the complete intersection of a quadric and a cubic in $\mathbb{P}^5$ (the proof has a gap and was corrected in [20]). The results of these investigations were summed up and presented in detail in the survey [6]. Let us note that no other approaches for the proof of the theorems listed above has appeared till now.

The success of the maximal singularities method in 3-dimensional birational geometry posed the question on its generalization to the multidimensional case and to the case of singular Fano manifolds. Such a generalization was found to be possible and effective: in [17, 34], a theorem on the birational rigidity of smooth 4-dimensional quintics (i.e., on the coincidence of groups of birational and biregular automorphisms and the theorem on the absence of a fibre structure on Fano manifolds of lower dimension) was proved. In [18, 19], the maximal singularities method was used for the study of manifolds of arbitrary dimension and manifolds with simple singularities.

The results of [4–6, 12, 17–20, 34] were briefly summarized in [33]. The aim of this survey is to give a complete and systematic exposition of the mentioned works. Also, we must note that some of the publications presented arguments in a shortened form (the proof of the theorem on the V_6^3 manifold in [20], the description of relations in the group $\mathrm{Bir}V_6^3$ in [5, 6]), and so here they are given in complete form for the first time.

The structure of this work is as follows. In the first chapter, we expound the "general theory" of the maximal singularities method (according to [6, 12, 18]). In the second chapter, we study smooth Fano manifolds of degree not exceeding 4. We follow the exposition in [6, 12] (the double space, the double 3-dimensional quadric, and the 3-dimensional quartic) and [18] (the double spaces and double quadrics of dimension 4 and greater). In the third chapter, birational automorphisms of a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^5$ are described. We follow the exposition in [6, 20]. In the fourth chapter, birational automorphisms of smooth 4-dimensional quintics are described (according to [34]). The fifth chapter contains a description of the birational correspondence of 3-dimensional quartics with a double point of a general structure.

Let us mention the main directions of birational geometry closely connected and sometimes interlaced with the subject of this work (for a historical survey, see [32, 36]). On the one hand, we must mention

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the investigations of 2-dimensional geometry over nonclosed fields. Y. I. Manin [13, 16] began the work in this area, which was continued by V. A. Iskovskikh and his students [7, 9–11] (see also [1, 2]). Then V. G. Sarkisov [21, 22] developed an analog of the maximal singularities method for conic bundles and, using Mori theory, began to study the general case of birational correspondence of algebraic manifolds that have conic bundle structure [23] (see also [26, 35]). These works are, however, out of the range of this survey, as are also the incomplete results of S. T. Hashin and S. L. Tregub about birational automorphisms of some Fano manifolds (such as $V_{10} \subset \mathbb{P}^7$ or a 3-dimensional cubic).

Remark. During preparation of this survey, the authors became aware of J. Collar's preprint [25], in which the absence of a conic bundle structure on general Fano hypersurfaces in a projective space of any dimension (in particular, on a general quartic and on a general quintic in $\mathbb{P}^5$) was proved. Let us note that the maximal singularities method gives stronger results about the birational geometry of manifolds.

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Chapter 1.
Maximal Singularities of a Birational Map

The set of results of the present chapter is the "general theory" of the maximal singularities method. Let V be an arbitrary algebraic manifold with the following properties:

(1) V is a nonsingular projective manifold, $\dim V = m \geq 3$;
(2) $\operatorname{Pic} V = \mathbb{Z}(-K_V)$, where K_V is the canonical divisor of V;
(3) The linear system $|-K_V|$ is free (in particular, it is nonempty, and it defines a morphism $\sigma\colon V \to \mathbb{P}^{\dim|K_V|}$, which is finite automatically everywhere on a manifold of the same dimension m).

Definition. A *test manifold* of dimension m' is a pair (V', H'), where V' is a nonsingular projective manifold, $\dim V' = m'$, H' is a divisor on V'; moreover:

(1) the linear system $|H'|$ is free;
(2) the linear system $|H' + iK_{V'}|$ is empty for $i \geq 2$.

Here are some typical examples of test manifolds:

(A) $V' = V$, $H' = -K_V$ — this is the most important case for us;
(B) $H' = -\frac{1}{r}K_{V'}$, where $K_{V'}$ is divisible by $r \in \mathbb{Z}_+$ in $\operatorname{Pic} V'$ and the system $\left|-\frac{1}{r}K_{V'}\right|$ is free (for example, $V' = \mathbb{P}^{m'}$, H is a hypersurface in $\mathbb{P}^{m'}$);
(C) V' is a bundle of a Fano manifold, i.e., there exists a morphism $p'\colon V' \to S'$, a general fibre of which is a manifold with an ample anticanonical divisor; for H' we take the inverse image on V' under p' of any very ample divisor on S'.

Let us note that $|H' + iK_{V'}| = \varnothing$ in case (B) for $r \geq 2$ and in case (C) for $i \geq 1$.

Set (V', H') be a test manifold of dimension m and $\chi\colon V \to V'$ be a birational map. In this chapter we shall develop a technique for studying χ.

1. A Resolution of Singularities of a Birational Map.

1. Definition 1. A *resolution of singularities* of any map $\chi\colon V \to V'$ is a set of data $R(V,\chi)$ which consists of a collection of manifolds V_i, $0 \leq i \leq N$, and a collection of morphisms $\varphi_{i,i-1}\colon V_i \to V_{i-1}$, $1 \leq i \leq N$, with the following properties:

(1) $V_0 = V$;
(2) $\varphi_{i,i-1}\colon V_i \to V_{i-1}$ is the blowing of a nonsingular irreducible submanifold $B_{i-1} \subset V_{i-1}$, which is the singularity in V_{i-1},
(3) the composition map $\psi = \chi \circ \varphi_{1,0} \circ \cdots \circ \varphi_{N,N-1}\colon V_N \to V'$ can be extended to a regular map (a birational morphism).

Using the notation of Definition 1, let $|\chi|$ be the proper inverse image of the linear system $|H'|$ on V under χ; $\varphi_{i,j} = \varphi_{j+1,j} \circ \cdots \circ \varphi_{i,i-1}\colon V_i \to V_j$, $\varphi_{i,i} = \mathrm{id}_{V_i}$ if $N \geq i > j \geq 0$; $|\chi|_i$, $0 \leq i \leq N$, be the proper inverse image of the linear system $|H'|$ on V_i under $\chi \circ \varphi_{i,0}\colon V_i \to V'$ (thus, the linear system $|\chi|_N$ is free); $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$ be the exceptional divisor of the ith blowing $\varphi_{i,i-1}$; $\nu_i = \operatorname{mult}_{B_{i-1}} |\chi|_{i-1}$, $1 \leq i \leq N$, be the multiplicity of the common divisor of $|\chi|_{i-1}$ along the cycle B_{i-1}.

Proposition 1. *There exists a resolution such that*

(1) $\nu_i > 0$, $\nu_i \geq \nu_j$ *for all* $N \geq j \geq i \geq 1$;

(2) *the following three subschemes of* V_i *are normally flat (and thus have equal multiplicities) along* B_i *for all* i, $0 \leq i \leq N-1$:

a) *the common divisor of the linear system* $|\chi|_i$;

b) *the scheme-theoretic intersection of two common divisors of the linear system* $|\chi|_i$;

c) *the base subscheme of* $|\chi|_i$.

The proof follows from the Hironaka theory [31] (see also [11], or [8, Chapter 1, §1]). According to [8] and [11] we shall call a resolution with properties (1), (2) an *admissible resolution.*

2. An admissible resolution of singularities of the map χ is not uniquely defined. In particular, it is possible to claim some additional conditions on it. These conditions will be used in Chapter 2 when we shall study birational isomorphisms of a quintic. A simple construction will be useful for the construction of a resolution with additional conditions.

Definition 2. Let $U \subset V$ be an open subset. A *restriction of the resolution* $R(V,\chi)$ on U is the following set of data: a collection of manifolds U_i, $0 \leq i \leq N$, where $U_0 = U$, $U_i = \varphi_{i,i-1}^{-1}(U_{i-1})$, and a collection of morphism restrictions $\varphi_{i,i-1}\colon U_i \to U_{i-1}$. If $B_{i-1} \cap U_{i-1} = \varnothing$, then $\varphi_{i,i-1}\colon U_i \to U_{i-1}$ is an isomorphism. Omitting all trivial blowings (i.e., all U_i and $\varphi_{i,i-1}$ such that $\varphi_{i,i-1}\colon U_i \to U_{i-1}$ is an isomorphism), we have a truncated restriction of the resolution, which will be denoted $R(V,\chi)|_U$.

Proposition 2. *If* $R'(U,\chi|_U)$ *is some admissible resolution of singularities of the map* $\chi\colon U \to V'$, *and* $U \subset V$ *is open, then there exist a resolution* $\overline{R}(V,\chi)$ *such that* $\overline{R}(V,\chi)|_U = R'(U,\chi|_U)$.

Proof. This result is a direct consequence of the Hironaka theory [31].

3. Up to the end of this chapter we shall fix an admissible resolution $R(V,\chi)$ of the singularities of the map $\chi\colon V \to V'$. Thus, we shall use the notations introduced above and also introduce some more.

If $F \subset V_i$ is an irreducible cycle, then, for $j \leq i$, by F_j we shall denote $\varphi_{i,j}(F)$ (for example $\varphi_{i,j}(B_i) = B_{i,j}$, and it will be automatically assumed when dealing with F_j, for $F \subset V_i$, that $j \leq i$). For $j \geq i$ let F^j be the proper inverse image of F in V_j, if it is well defined (for example, E_i^j is the proper inverse image of an exceptional divisor $E_i \subset V_i$; as before, it will be automatically assumed when dealing with F^j, for $F \subset V_i$, that $j \geq i$).

Let us discuss some questions on the reducibility of cycles.

Definition 3. An irreducible cycle $H \subset V_i$ of codimension 2 is called *correct* if for $j \leq i$, such that $\operatorname{codim} H_j = 2$, the following conditions hold:

1) H_j does not contain the inverse image of a proper subset of B_{j-1};

2) H_j is in a general position with B_j.

Proposition 3. *Any component of the intersection of two general divisors from the linear system* $|\chi|_i$, $0 \leq i \leq n$, *is a correct cycle. The same for any* B_i, *if* $\operatorname{codim} B_i = 2$.

Proof. This result is a consequence of the normal flatness condition (Proposition 1).

We shall deal with correct cycles only. Thus, the term "correct" will be omitted, and the expression "a cycle $H \subset V_i''$ of codimension 2" will mean "a correct cycle $H \subset V_i$ of codimension 2."

The importance of the correctness condition is stated in

Proposition 4. *Let* $H \subset V_i$ *be a (correct) cycle of codimension 2. Let* $\alpha \leq \beta \leq i$ *and* $\operatorname{codim} H_\alpha = 2$. *Then*

$$\deg(\varphi_{\beta,\alpha}\colon H_\beta \to H_\alpha \ \operatorname{mult}_x H_\alpha \geq \operatorname{mult}_y H_\beta$$

for all $y \in H_\beta$, *where* $x = \varphi_{\beta,\alpha}(y)$.

Proof. Obviously, it is enough to prove the statement for one step of the resolution, i.e., for $\alpha = \beta - 1$ (since the degree is multiplicative). But, in this case (see Definition 3), the morphism $\varphi_{\beta,\alpha}\colon H_\beta \to H_\alpha$ is either everywhere finite or is a monoidal map with H_α as the center along which H_α is normally flat. In both cases, the inequality holds.

Definition 4. A *type of an irreducible cycle* $F \subset V_i$ is a pair (dim F, dim F_0) (where $F_0 = \varphi_{i,0}(F)$ according to the above notations).

2. Noether–Fano Inequalities.

1. The linear system $|\chi|_N$ is the inverse image of the linear system $|H'|$ under the birational morphism $\chi \circ \varphi_{N,0}\colon V_N \to V'$. A divisor of the linear system $|\chi|_N$ is obviously linearly equivalent to the divisor $-nK_V - \sum_{i=1}^{N} \nu_i E_i$ (for simplicity of notations, we denote the complete inverse image by the same symbol: $\varphi_{j,i}^* E_i$, $j \geq i$, will be denoted by E_i), where $n \geq 1$ and $|\chi| \subset |-nK_V|$. The values of n and ν_i, $1 \leq i \leq N$, are not independent: there are some inequalities for them.

Definition 1 (see [8, 11]). Let us define on the set $\{E_i, 1 \leq i \leq N\}$ of exceptional divisors the following structure:

(1) the partial order: $E_i \geq E_j$ if $i \geq j$ and $E_{i,j} \subset E_j$ (let us recall that $E_{i,j} = \varphi_{i,j}(E_i)$;

(2) the oriented graph Γ: an oriented edge connects E_i with E_j if $i > j$ and $B_{i-1} \subset E_j^{i-1}$ (notation $i \to j$).

Let also:

$P(i,j)$ be the set of paths in Γ that go from the vertex E_i to the vertex E_j, if $i \neq j$, and $P(i,i) = 0$ (if $i < j$ then, obviously, $P(i,j) = 0$).

Let $r_{ij} = \#P(i,j)$ if $i \neq j$, $r_{ii} = 1$.

Assume $\delta_i = \operatorname{codim} B_i - 1$.

Proposition 1 (*the strengthened Noether-Fano inequality*). *There exists an index β, $0 \leq \beta \leq N-1$, such that $\nu_{\beta+1} > \delta_\beta n$ and $\sum_j r_{\beta+1,j}\nu_j > \sum_j r_{\beta+1,j}\delta_{j-1} n$ for all $n \geq 2$. If the linear system $|H' + K_{V'}|$ is empty, then such an index exists for all $n \geq 1$.*

Proof. By the definition of a test manifold, the linear system $|H' + tK_{V'}|$ is empty for $t \geq 2$, and so the linear system $\left|-nK_V - \sum_{i=1}^{N} \nu_i E_i + tK_{V_N}\right|$ of divisors on V_N is empty for $t \geq 2$ (this is a consequence of the equality $(\chi \circ \varphi_{N,0})_* c_1(V_N) = C_1(V')$). If $n \geq 2$, then assuming $t = n$ and taking into account the equality $K_{V_N} = K_V + \sum_{i=1}^{N} \delta_{i-1} E_i$, we get that the linear system $\left|\sum_{i=1}^{N} (\delta_{i-1} n - \nu_i) E_i\right|$ is empty. Let us calculate in explicit form the multiplicity of all irreducible components of the divisor $\sum_{i=1}^{N} (\delta_{i-1} n - \nu_i) E_i$. Such components can be only the proper inverse images of the divisors $E_i \subset V_i$ on V_N, i.e., E_i^N. Obviously, $E_i^N = E_i - \sum_{j \to i} E_j$. A simple calculation shows that $\sum_{i=1}^{N} (\delta_{i-1} n - \nu_i) E_i = \sum_{i=1}^{N} \left(\sum_{j \leq i} (\delta_{j-1} n - \nu_j) r_{ij}\right) E_i^N$. From the effectiveness of all E_i^N and the emptiness of the linear system $\left|\sum_{i=1}^{N} (\delta_{i-1} n - \nu_i) E_i\right|$ it follows that there exists an index β such that

$$\sum_{j \leq \beta+1} r_{\beta+1,j} (\delta_{j-1} n - \nu_j) < 0. \tag{$*$}$$

Finally, if $(*)$ holds and $\nu_{\beta+1} \leq \delta_\beta n$, then $E_{\beta+1}$ is not the minimal vertex of Γ. Thus, we can find a vertex E_j such that $(\beta+1) \to j$ (if $E_{\beta+1}$ is minimal, then $(*)$ is equivalent to the inequality $\nu_{\beta+1} > \delta_\beta n$). Let us rewrite $(*)$ as follows:

$$\begin{aligned}\sum_j r_{\beta+1,j}\nu_j = \nu_{\beta+1} + \sum_{(\beta+1)\to\varepsilon} \sum_j r_{\varepsilon,j}\nu_j &> \sum_j r_{\beta+1,j}\delta_{j-1} n \\ &= \delta_\beta n + \sum_{(\beta+1)\to\varepsilon} \sum_j r_{\varepsilon,j}\delta_{j-1} n.\end{aligned}$$

By the supposition, $\nu_{\beta+1} \leq \delta_\beta n$; so we can find an index ε such that $(\beta+1) \to \varepsilon$ and $\sum_j r_{\varepsilon,j}\nu_j > \sum_j r_{\varepsilon,j}\delta_{j-1} n$. Repeating this procedure, if necessary, we get an index such that both inequalities hold.

If $|H' + K_{V'}| = 0$ and $n = 1$, then assuming $t = n = 1$, we get both inequalities for $n = 1$.

Definition 2. If an index β satisfies the conditions of the above proposition, then we shall say that B_β is a *maximal singularity*.

2. Now we shall give a direct geometrical description of some maximal singularities.

Definition 3. An irreducible closed subset $B \subset V$ of codimension ≥ 2 is called a *maximal subset* if and only if the following inequality holds:

$$\operatorname{mult}_B |\chi| > (\operatorname{codim} B - 1)n.$$

Proposition 2. (A) *Let B_β be a maximal singularity such that $\dim B_\beta = \dim B_{\beta,0}$. Then $B_{\beta,0}$ is a maximal subset.*

(B) *Let $B \subset V$ be a maximal subset. Then there exists an index β, $0 \leq \beta \leq N-1$, such that $B \subset B_{\beta,0}$, $\varphi_{\beta,0}\colon V_\beta \to V$ is an isomorphism in a neighborhood of a general point of B_β, $E_{\beta+1}$ is the minimal vertex of Γ, and B_β is the maximal singularity.*

Proof. (A) If $i \leq j$, then, in view of the admissibility of the resolution of singularities, $\nu_i \geq \nu_j$. Thus, if we set $i = \min\{j \mid B_{\beta,j} \subset E_j\}$, then $\nu_i \geq \nu_{\beta+1} > \delta_\beta n$ and $\varphi_{i-1,0}$ is an isomorphism in a neighborhood of a general point in B_{i-1}. Therefore, $\nu_i = \operatorname{mult}_{B_{i-1,0}} |\chi|$ and, moreover, $\dim B_{i-1,0} = \dim B_{i-1}$, $\delta_\beta = \operatorname{codim} B_{\beta,0} - 2$. The proof of (A) is completed.

(B) It is obvious that B is contained in the base subset of the linear system $|\chi|$. So the set $\{i | B \subset B_{i,0}\}$ is nonempty. Let $i_1 = \min\{i \mid B \subset B_{i,0}\}$. Then $\varphi_{i_1,0}$ is an isomorphism in a neighborhood of a general point in B_{i_1}, $\varphi_{i_1,0}^{-1}$ is an isomorphism in a neighborhood of a general point in B, and E_{i_1+1} is the minimal vertex of Γ. Thus, the condition "to be a maximal singularity" for E_{i_1+1} is equivalent to the inequality $\nu_{i_1+1} > \delta_{i_1} n$. By the equimultiplicity of $|\chi|_{i_1}$ along B_{i_1}, we have $\nu_{i_1+1} = \operatorname{mult}_B |\chi|$. But $\delta_{i_1} \leq \operatorname{codim} B - 1$. This implies (B).

Thus, the maximal singularities not losing their dimension under the "descent" on V are just the same as the maximal subsets. Their properties are essentially different from the properties of the maximal singularities that lose their dimension on V.

3. Multiplication in the Chow Ring.

1. Most constructions of this work use the language of the numerical Chow rings.

For an arbitrary nonsingular manifold Z, let $A(Z) = \bigoplus_{i=1}^{\dim Z} A^i(Z)$ be its *Chow ring* of the algebraic cycles modulo numerical equivalency, graded by the codimension of the cycles. We use a capital letter for designation of the cycle and we use the same lower-case letter to denote its class (so, if X, H, W_i are the cycles, then x, h, w_i are their classes). For $x \in A^1(\dots)$ let $|x|$ be the set D of divisors such that $d = x$ (almost always $|x|$ will be a linear system). If $\pi\colon Z_1 \to Z_2$ is a surjection, then we do not make a distinction between $A(Z_2)$ and $\pi^* A(Z_2) \hookrightarrow A(Z_1)$.

Let e_i be the class of E_i in $A^1(V_N)$, b_i be the class of B_i in $A^{\delta_i+1}(V_N)$, $g_i \in A^{\dim V-2}(V_i)$ be the class of a plane in a fibre of the morphism $\varphi_{i,i-1}\colon E_i \to B_{i-1}$, if $\operatorname{codim} B_{i-1} \geq 3$ (let us agree to use the notation g_i only in the case where it has sense, i.e., if $\operatorname{codim} B_{i-1} \geq 3$), $f_i \in A^{\dim V-1}(V_i)$ be the class of a line in the fibre of the morphism $\varphi_{i,i-1}\colon E_i \to B_{i-1}$, $h \in A^1(V) \hookrightarrow A^1(V_N)$ be the class of the anticanonical divisor of the manifold V, $h' \in A^1(V_N)$ be the class of a divisor of the linear system $|\chi|_N$. Let us note that $\operatorname{Pic} V_N = A^1(V_N)$ and the linear system $|\chi|_N$ is complete, so $|h'| = |\chi|_N$. Obviously, $h' = nh - \sum_{i=1}^N \nu_i e_i$ and the divisor class of the linear system $|\chi|_j$ is $nh - \sum_{i=1}^j \nu_i e_i$. Let $\operatorname{Bs}|\dots|$ be the base set of the linear system $|\dots|$. It is easy to see that $\operatorname{Bs}\left|nh - \sum_{i=1}^N \nu_i e_i\right| = 0$.

We shall use the notation conventions for the multiplication in the Chow ring: $(w \cdot v)$ for $w, v \in A(\dots)$, $w^2 = (w \cdot w)$.

We shall give the basic formulas for the multiplication in $A(V_N)$ that will be useful for us. Set $d = h^m > 0$. We shall give formulas only for the cycles of complementary dimension, so that the result of multiplication is an integer.

Proposition 1. *There exist the following multiplication formulas*:

(1)

$$(a\cdot b)=((\varphi_{i,j})_*a\cdot b)$$

if $N\geq i>j\geq 0$, $a\in A^k(V_i)$, $b\in A^{m-k}(V_j)$;

(2) $h^m=d$, $(h^{m-1}\cdot e_i)=(h\cdot f_i)=(h^{m-2}\cdot e_i\cdot e_j)=0$ *for all* $i\neq j$, $(h^{m-2}\cdot e_i^2)=-(h^{m-2}\cdot b_{i-1})$;

(3) $(e_i\cdot f_j)=-\delta_{ij}$ *(the Kronecker symbol)*, $(g_j\cdot h^2)=(g_j\cdot h\cdot e_i)=(g_j\cdot e_i\cdot e_k)$ *for all* $i\neq k$. $(g_j\cdot e_i^2)=-(g_j\cdot b_{i-1})$ *if* $i\neq j$, $(g_j\cdot e_j^2)=1$;

(4) *let* $Z\subset V_i$ *be an irreducible cycle of codimension* 2; *then the value of* $(z\cdot g_j)$ *is equal to*

(a) 0, *if* $j>i$,

(b) $(\operatorname{mult}_{B_{j-1}}Z_{j-1})\deg(\varphi_{i,j}\colon\ Z\dashrightarrow Z_j)$, *if* $j\leq i$ *and* $\operatorname{codim}Z_{j-1}=2$,

(c) $-(\deg Z_j)\deg(\varphi_{i,j}\colon\ Z\to Z_j)$ *for* $j\leq i$, *if* $\operatorname{codim}Z_j=2$ *and* $Z_j\subset E_j$, *where* $\deg Z_j$ *is the intersection number of* Z_j *with the fibre of the morphism* $\varphi_{j,j-1}\colon\ E_j\to B_{j-1}$ *over a general point in* B_{j-1} *(the fibre is a projective space and it intersects* Z_j *along a hypersurface)*,

(d) 0, *if* $j\leq i$ *and* $\operatorname{codim}Z_j\geq 3$;

(5) *let* $Z\subset V_i$, $0\leq i\leq N$, *be an irreducible cycle of codimension* $2k$; *then*

$$\begin{aligned}(z\cdot h^{m-k})&=\deg(\varphi_{i,0}\colon\ Z\to Z_0)(z_0\cdot h^{m-k})\\&=\deg(\sigma\circ\varphi_{i,0}\colon\ Z\to\sigma(Z_0))\deg\sigma(Z_0)\end{aligned}$$

if $\operatorname{codim}Z_0=k$, *where the last degree is calculated with respect to the inclusion* $\sigma(Z_0)\subset\mathbb{P}^{\dim|l|}$, *and* $(z\cdot h^{m-k})=0$, *if* $\operatorname{codim}Z_0>k$.

Proof. (1) is a particular case of the projection formula [14], (2)–(4) are a combination of (1) and the well-known fact that a normal sheaf of an exceptional divisor is the tautological sheaf of the associated bundle (see [15] or [3, Chapter 4, pp. 642–650]), and (5) is a consequence of the projection formula. We shall use the formulas from the multiplication table without any references. Also, we shall use freely the multiplication formulas in the Chow ring of the blowings of 3-folds.

Corollary 1.

(A) $(h^{m-2}\cdot h'^2)=dn^2-\sum i=1^N(h^{m-2}\cdot b_{i-1})\nu_i^2$;

(B) $(h^{m-1}\cdot h')=dn$;

(C) $(f_i\cdot h')=\nu_i$;

(D) $(g_j\cdot h'^2)=\nu_j^2-\sum(g_j\cdot b_{i-1})\nu_i^2$.

Proof. All this is a consequence of the above proposition. Let us only note that the summation in (D) varies over all indices $i\neq j$. But the product $(g_j\cdot b_{i-1})=0$: g_j is defined only for those j for which $\operatorname{codim}B_{j-1}\geq 3$, so $(b_{j-1}\cdot z)=0$ for all $z\in A^{m-2}(V_N)$ (also, it is a consequence of the projection formula). Therefore, it is possible not to exclude the index $i=j$ in the sum. Such "extra" members will be ignored in what follows without any special mention.

2. Definition 1. The class $y\in A^{m-2}(V_N)$ is called *nonnegative* if for all (correct!) cycles $Z\subset V_i$ of codimension 2, $0\leq i\leq N$, $(z\cdot y)\geq 0$.

Lemma 1. (A) *Let* $Z\subset V_i$, $0\leq i\leq N$, *be a (correct) cycle*, $\operatorname{codim}Z=\operatorname{codim}Z_0=2$. *Then the following inequalities hold*:

(1) $(h^{m-2}\cdot z)>0$;

(2) $((h^{m-2}-g_j)\cdot z)\geq 0$ *for* j *such that* $0\leq j\leq N$ *and* $\operatorname{codim}B_{j-1}\geq 3$.

(B) *Let* $Z\subset V_i$, $1\leq i\leq N$, *be a (correct) cycle*, $\operatorname{codim}Z=2$, $\operatorname{codim}Z_0\geq 3$. *Let* E_j *be a minimal vertex in* Γ *such that* $\operatorname{codim}B_{j-1}\geq 3$. *Then*: (1) $(h^{m-2}\cdot z)=0$; (2) $(g_j\cdot z)\leq 0$.

Proof. (A) Let us recall that $|h|=|-K_V|$ is the free linear system which defines an everywhere finite morphism $\sigma: V\to W\subset\mathbb{P}^{\dim|l|}$, $\dim W=\dim V$ (W is a manifold, maybe singular). Now we can see that (1) is a consequence of the projection formula. Let us prove (2). If $j>i$, then $(z\cdot g_j)=0$ (by the projection formula) and, in this case, we can use (1). Let $j\leq i$. Then we have the equalities

$$(z\cdot h^{m-2})=\deg(\varphi_{i,0}\colon\ Z\to Z_0)(z_0\cdot h^{m-2})$$

and

$$(z \cdot g_j) = \deg(\varphi_{i,j}\colon\ Z \to Z_j) \operatorname{mult}_{B_{j-1}} Z_{j-1}.$$

By Proposition, 1.4 we have

$$\operatorname{mult}_{B_{j-1}} Z_{j-1} \le (\operatorname{mult}_{B_{j-1,0}} Z_0) \deg(\varphi_{j-1,0}\colon\ Z_{j-1} \to Z_0).$$

But, in view of the degree multiplicity,

$$\deg(\varphi_{i,0} : Z \to Z_0) = \deg(\varphi_{i,j} : Z \to Z_j) \deg(\varphi_{j,0} : Z_j \to Z_0)$$

and, by using the obvious fact that $\deg(\varphi_{j,j-1}\colon\ Z_j \to Z_0) = 1$, we get that (2) is equivalent to the inequality $(z_0 \cdot h^{m-2}) \ge \operatorname{mult}_{B_{j-1,0}} Z_0$. The morphism $\sigma : V \to W \subset \mathbb{P}^{\dim |l|}$ is finite everywhere, so for all points $x \in Z_0$ we have

$$\operatorname{mult}_x Z_0 \le \operatorname{mult}_{\sigma(x)} \sigma(Z_0) \deg(\sigma\colon\ Z \to Z_0).$$

On the other hand, $(z_0 \cdot h^{m-2}) = \deg \sigma(Z_0) \deg(\sigma\colon\ Z_0 \to \sigma(Z_0))$. Also, it is obvious that $\operatorname{mult}_{\sigma(x)} \sigma(Z_0) \le \deg \sigma(Z_0)$ (the multiplicity of a projective manifold at a point is not greater than its degree). Consequently, for all $x \in Z_0$, $\operatorname{mult}_x Z_0 \le (z_0 \cdot h^{m-2})$. (2) is proved.

(B) (1) is obvious. To prove (2), let us assume that the converse is true: $(g_j \cdot z) > 0$. From this we have that $j \le i$ (otherwise, $(g_j \cdot z) = 0$), $Z_j \not\subset E_j$, $B_{j-1} \subset Z_{j-1}$ (because Z_j has a nonempty intersection with the fibre over any point in B_{j-1}) and $\operatorname{codim} Z_j = 2$. But, in this case, in view of $\dim Z_0 < \dim Z = \dim Z_j$, we can find an index $j_1 \le j-1$ such that $Z_{j_1} \subset E_{j_1}$ and, therefore, $B_{j-1,j_1} \subset Z_{j-1,j_1} = Z_{j_1} \subset E_{j_1}$, i.e., $E_j \ge E_{j_1}$. Thus, the vertex E_j is not minimal; this contradicts the conditions of the lemma.

Corollary 2. *The classes h^{m-2} and $h^{m-2} - g_j$, where E_j is a minimal vertex in Γ and* $\operatorname{codim} B_{j-1} \ge 3$, *are nonnegative.*

Corollary 3. *For all multiplicities ν_i we have the inequality $\nu_i \le \sqrt{d}\, n$.*

Proof. In view of the admissibility of the resolution, it is enough to prove that $\nu_1 \le \sqrt{d}\, n$. If $\operatorname{codim} B_0 = 2$, then we set $y = h^{m-2}$; otherwise, if $\operatorname{codim} B_0 \ge 3$, we set $y = h^{m-2} - g_1$. In both cases, y is nonnegative. So $(h'^2 \cdot y) \ge 0$. In the first case, we have

$$0 \le dn^2 - \nu_1^2 (h^{m-2} \cdot b_0) - \sum_{i=2}^{N} (b_{i-1} \cdot h^{m-2}) \nu_i^2 \le dn^2 - \nu_1^2.$$

In the second case,

$$0 \le dn^2 - \nu_1^2 - \sum_{i=2}^{N} (b_{i-1} \cdot y) \nu_i^2 \le dn^2 - \nu_1^2.$$

In both cases, $\nu_1 \le \sqrt{d}\, n$. The corollary is proved.

Corollary 4. *The codimensions of the maximal subset $B \subset V$ and of the maximal singularity $B_\beta \subset V_\beta$ satisfy the inequalities* $\operatorname{codim} B < \sqrt{d} + 1$, $\operatorname{codim} B_\beta < \sqrt{d} + 1$.

Corollary 5. *Let $Z_i \subset V$, $1 \le i \le k$, be the cycles of codimension 2 such that* $\operatorname{mult}_{Z_i} |\chi| \ge n$ *for all i, $1 \le i \le k$, and* $\operatorname{mult}_{Z_1} |\chi| > n$. *Then*

$$\sum_{i=1}^{k} (z_i \cdot h^{m-2}) \le d - 1.$$

In particular, for the maximal cycle $Z \subset V$ of codimension 2 we have $(z \cdot h^{m-2}) \le d - 1$.

Proof. We proceed as in the proof of Corollary 3: the inequality $(h'^2 \cdot h^{m-2}) \ge 0$ gives

$$dn^2 \ge \sum_{i=1}^{k} (z_i \cdot h^{m-2}) \left(\operatorname{mult}_{Z_i} |\chi|\right)^2.$$

The corollary is proved.

Lemma 1 and Corollary 2 are the simplest techniques for the elimination of maximal singularities (and subsets) with the aid of the test class: the Noether–Fano inequalities give us the lower bounds for multiplicities and the product of the class l' or l'^2; for a specially chosen class it gives us the upper bound. In some cases, this provides enough information for the elimination of the maximal singularity. It is necessary to develop an analogous technique for dealing with the maximal singularities losing dimension on V. This will be done in the next section.

4. The Test Class.

1. Let now B_β be a maximal singularity with the property $\operatorname{codim} B_{\beta,0} \geq 3$. For simplicity of notations, let $r_j = r_{\beta+1,j}$. Let also $I_q = \{i | E_i \leq E_{\beta+1}, \dim B_{i-1} = q\}$, $0 \leq q \leq m-2$, $I = \bigcup_{q=0}^{m-2} I_q$, and $I' = \bigcup_{q=0}^{m-2} I_q$. We shall give the following basic definition.

Definition 1. A *test class* is any numerical class in $A^{m-2}(V_N) \oplus \mathbb{R}$ of the following type:

$$th^{m-2} - \sum_{j \in I'} r_j g_j, \quad \text{where } t \in \mathbb{R}_+.$$

The coefficient t essentially defines the numerical properties of the test class, the study of which is the main part of our method. The properties of the test class are partially described in the following lemma.

Lemma 1. *Let y be a test class.*

(A) *If $Z \subset V_i$ is a (correct) irreducible cycle of codimension 2 and* $\operatorname{codim} Z_0 \geq 3$, *then* $(z \cdot y) \geq 0$.

(B) *If $i \in I_{m-2}$ and* $\operatorname{codim} B_{i-1,0} \geq 3$, *then* $(b_{i-1} \cdot y) \geq r_i$.

Proof. (A) The reasoning here is along the same line as in the difficult case of the positivity lemma in [11, § 7] (see also [8, Ch. 2, § 1]).

Let us note that $(z \cdot h^{m-2}) = 0$. If the last set is nonempty, then let $t = \min\{j \in I' \mid (z \cdot g_j) = 0\}$ (in the other case, there is nothing to prove). We want to prove that $Z_t \subset E_t$ and $(z \cdot g_t) < 0$.

In fact, Z has the type $(m-2, m')$, $m' \leq m-3$. Therefore, the set $K = \{1 \leq j \leq i | Z_j \subset E_j, \operatorname{codim} B_{j-1} \geq 3\}$ is nonempty. Let $t_1 = \max j \in K$. Then $Z_{t_1} \subset E_{t_1}$, $\operatorname{codim} Z_{t_1} = 2$, and so $(z_{t_1} \cdot g_{t_1}) < 0$ (and, therefore, $(z \cdot g_{t_1}) < 0$) and $(z \cdot g_j) = 0$ for all $j < t_1$ (by the projection formula). In particular, $t_1 \leq t$ and, in view of the fact that $Z_{t_1 - 1} \subset B_{t_1 - 1}$, we have $E_{t_1} \leq E_t$, i.e., $t_1 \in I'$; therefore, $t_1 = t$.

Thus, $Z_t \subset E_t$ and $(z \cdot g_t) < 0$. Let $J = \{j \in I' \mid j \neq t, (z \cdot g_j) \neq 0\}$. Hence, if $j \in J$, then $(z \cdot g_j) > 0$ and $i \geq j > t$. We want to prove that

$$r_t \geq \sum_{k \in J} r_k. \tag{$*$}$$

To prove this, let us define a map ω from the set $\{j | t \leq j \leq i\}$ into itself in the following way: $\omega(j) = \max\{i^* | i^* \leq j-1, Z_{i^*} \subset E_{i^*}\}$ for $j \geq t+1$, $\omega(t) = t$. It is obvious that, if $j \neq t$, then $\omega(j) < j$. Let ω^k be (as usual) the kth power of the endomorphism ω. We have $\omega^{(i-t)}(j) = t$ for all j. Also, the map ω has the following properties:

(1) $\varphi_{j-1,\omega(j)}$ is an isomorphism in a neighborhood of a general point in Z_{j-1} if $t < j \leq i$;

(2) $B_{\omega(j)-1} = Z_{\omega(j)-1}$ if $\omega(j) \geq t+1$, in particular, $\operatorname{codim} B_{\omega(j)-1} = 2$;

(3) the vertices $E_{\omega(j)}$ and $E_{\omega^2(j)}$ are connected by an edge in Γ: $\omega(j) \to \omega^2(j)$ for all j, $\omega(j) \geq t+1$.

In fact, property (1) is a consequence of the definition. To prove (2) let us note that $Z_{\omega(j)} \subset E_{\omega(j)}$, and so $Z_{\omega(j)-1} \subset B_{\omega(j)-1}$. But $\operatorname{codim} Z_{\omega(j)-1} = 2$ if $\omega(j) \geq t+1$, and so $Z_{\omega(j)-1} = B_{\omega(j)-1}$. Property (3) is a consequence of (1) and (2): $B_{\omega(j)-1} = Z_{\omega(j)-1} = Z^{\omega(j)-1}_{\omega^2(j)} \subset E^{\omega(j)-1}_{\omega^2(j)}$.

Note, now, that if $j \in J$, then the vertices E_j and $E_{\omega(j)}$ are connected by an edge in Γ: $j \to \omega(j)$. One can prove in the same way as for property (3) above: inasmuch as $(z \cdot g_j) \neq 0$, then Z_j has a nonempty intersection with the fibre over any point in B_{j-1}, and so $B_{j-1} \subset Z_{j-1} = Z^{j-1}_{\omega(j)} \subset E^{j-1}_{\omega(j)}$, i.e., $j \to \omega(j)$.

From all this it follows that for any vertex E_j, $j \in J$, in Γ there exists a special path which connects E_j with E_t. Namely, let $k^*(j) = \min\{k \mid \omega^k(j) = t\}$; then this path is $j \to t$ if $k^*(j) = 1$, and $j \to \omega(j) \to$

$\cdots \to \omega^{k^*(j)}(j) = t$ if $k^*(j) \geq 2$. That is, this path goes from E_j to E_t in a sequence through the vertices $E_{\omega^k(j)}$, $1 \leq k \leq k^*(j)$. Let us denote this path by P_j.

Let us now prove the inequality $(*)$. To this end, let us relate any path in Γ from $E_{\beta+1}$ to E_j, $j \in J$, to its composition with P_j, the path from E_j to E_t. Thus, we get the path from $E_{\beta+1}$ to E_t. A composition with P_j defines an injective map:

$$P(\beta+1, j) \overset{\circ P_j}{\to} P(\beta+1, t),$$

which we denote by $\circ P_j$. Let us prove that the intersection of the images of two different sets $P(\beta+1, j)$, $j \in J$, in $P(\beta+1, t)$ is empty. In fact, let $\lambda, \mu \in J$, $\lambda < \mu$, and

$$\circ P_\lambda\big(P(\beta+1,\lambda)\big) \bigcap \circ P_\mu\big(P(\beta+1,\mu)\big) \neq 0,$$

i.e., there exists a path from $E_{\beta+1}$ to E_t which belongs to both of these subsets. Then, by property (2), this path after the vertex E_μ and untill the vertex E_t goes through vertices E_α such that $\operatorname{codim} B_{\alpha-1} = 2$ (i.e., $\alpha \in I_{m-2}$). On the other hand, this path goes through the vertex E_λ, $\lambda < \mu$, and $\operatorname{codim} B_{\lambda-1} \geq 3$; we have a contradiction. Thus, we have that the intersection of the subsets $\circ P_j\big(P(\beta+1,j)\big) \subset P(\beta+1,t)$ for different $j \in J$ is empty. Therefore,

$$\sum_{j\in J} \# \circ P_j\big(P(\beta+1,j)\big) = \sum_{j\in J} \# P(\beta+1,j) \leq \# P(\beta+1,t).$$

But this is exactly $(*)$.

Now we have

$$\Big(z \cdot -\sum_{j\in I'} r_j g_j\Big) \geq \sum_{k\in J} r_k\big(z \cdot -(g_t + g_k)\big).$$

It remains to prove that each member $-\big(z\cdot(g_t+g_k)\big)$ in the sum is nonnegative. In fact,

$$-(z\cdot g_t) = \deg(\varphi_{i,t}\colon\; Z \to Z_t)(z_t \cdot g_t),$$

where Z_t is a divisor in E_t. If $F_x \subset E_t$ is a fibre of the morphism $\varphi_{t,t-1}\colon\; E_t \to B_{t-1}$ over some point $x \in B_{t-1}$, then $(h_t \cdot g_t) = \deg(Z_t \bigcap F_x)$ (i.e., the degree of the hypersurface $Z_t \bigcap F_x$ in the projective space $F_x \cong \mathbb{P}^{\delta_t - 1}$: Z_t cannot contain the whole fibre F_x because Z is a correct cycle). Then, by Proposition 1.4, $(z\cdot g_k) = \operatorname{mult}_{B_{k-1}} Z_{k-1} \deg(\varphi_{i,k}\colon\; Z \to Z_k) \leq \operatorname{mult}_{B_{k-1,t}} Z_t \deg(\varphi_{i,t}\colon\; Z \to Z_t)$. Therefore, it is enough to prove that $\deg(Z_t \bigcap F_x) \geq \operatorname{mult}_{B_{k-1,t}} Z_t$. But this is obvious: if $z \in B_{k-1,t}$, $z^* = \varphi_{t,t-1}(z) \in B_{t-1}$, then

$$\operatorname{mult}_{B_{k-1,t}} Z_t \leq \operatorname{mult}_z Z_t \leq \operatorname{mult}_z(Z_t \bigcap F_{z^*}) \leq \deg(Z_t \bigcap F_{z^*})$$

(the multiplicity of a point does not exceed the degree of a manifold). (A) is proved.

(B) Consider B_{i-1}, $i \in I_{m-2}$, $\operatorname{codim} B_{i,0} \geq 3$. All previous considerations can be used for studying B_{i-1}. Let t, J, ω, P_j be as above. Then

$$\begin{aligned} -\Big(b_{i-1}\cdot \sum_{j\in I'} r_j g_j\Big) &= -\Big(b_{i-1}\cdot \Big(r_t - \sum_{k\in J} r_k\Big) g_t\Big) + \sum_{k\in J} r_k\big(b_{i-1}\cdot -(g_t+g_k)\big) \\ &\geq \Big(r_t - \sum_{k\in J} r_k\Big)(b_{i-1}\cdot -g_t), \qquad (b_{i-1}\cdot g_t) \leq -1, \end{aligned}$$

as before. We assert that

$$r_t \geq r_i + \sum_{k\in J} r_k. \tag{**}$$

This inequality is proved by the same method as for $(*)$. Let us note that $i \to \omega(i)$: $B_{i-1} = B^{i-1}_{i-1,\omega(i)} \subset E^{i-1}_{\omega(i)}$. Let P_i be a path in Γ that begins in E_i, finishes in E_t, and consecutively goes through all vertices $E_{\omega^k(i)}$, $\omega^k(i) \to \omega^{k+1}(i)$. Let

$$\circ P_i\colon\ P(\beta+1,i) \to P(\beta+1,t)$$

be an (injective) map that each path from $E_{\beta+1}$ to E_i maps into its composition with P_i. Because the part of P_i between E_i and E_t goes only through vertices E_α such that $\operatorname{codim} B_{\alpha-1} = 2$ (i.e., $\alpha \in I_{m-2}$), P_i cannot go through E_j, $j \in J$. Thus, we again have

$$\circ P_i\big(P(\beta+1,i)\big) \bigcap \circ P_j\big(P(\beta+1,j)\big) = \varnothing \text{ when } j \in J.$$

Therefore, $\#P(\beta+1,i) + \sum\limits_{j\in J} \#P(\beta+1,j) \le \#P(\beta+1,t)$, i.e., we have exactly $(**)$. (B) is proved.

Remark. The key assertions in the proof of Lemma 1 are the proofs of inequalities $(*)$ and $(**)$. The specific properties of r_i, $i \in I$, ensure the correctness of these inequalities. On the other hand, only these properties were used in the proof. This means that if r_i^*, $i \in I$, is some other set of positive integer coefficients such that, in every step of the proof, if some assertion is true for $\{r_i\}$, then the same is true for $\{r_i^*\}$, then the class

$$y^* = -\sum_{i\in I'} r_i^* g_i$$

satisfies conditions (A) and (B) of Lemma 1. This trivial remark will be useful for us: sometimes the ordinary test class is too "wasteful" for our means and we have to "correct" it.

Corollary 1. *The test class*

$$y = \sum_{j\in I'} r_j(h^{m-2} - g_j)$$

is nonnegative.

Proof. Let $Z \subset V_i$ be a cycle of codimension 2. If Z has the type $(m-2, m-2)$, then $\big(z\cdot(h^{m-2}-g_j)\big) \ge 0$ for all $j \in I'$ (Lemma 3.1). If Z has the type $(m-2, m')$, $m' \le m-3$, then the inequality $(z \cdot y) \ge 0$ was proved in the previous lemma.

Lemma 2 (*the square inequality*).

$$\Big(\sum_{j\in I} r_j \nu_j^2\Big)\Big(\sum_{j\in I} r_j\Big) > \Big(\sum_{j\in I} r_j \delta_{j-1}\Big)^2 n^2.$$

The proof of this inequality is a direct computation based on the Noether–Fano inequality.

Some steps in the study of birational maps with the aid of the maximal singularities method demand long and hard computations. At the same time, these computations are more or less standard. We shall present here some of these computations for the general case and shall not repeat them in what follows.

Let $y = th^{m-2} - \sum\limits_{j\in I'} r_j g_j$ be a test class. Suppose that there exists some set J of indexes $J \subset \{j \mid 1 \le j \le N\}$ satisfying the following conditions:

(i) if $j \in J$, then $\operatorname{codim} B_{j-1,0} = 2$;

(ii) if $(b_{j-1} \cdot y) < 0$, then $j \in J$.

The compatibility of these conditions is guaranteed by Lemma 1, part (A).

Suppose also that $\operatorname{codim} B_{i-1,0} \ge 3$ for all $i \in I_{m-2}$.

Lemma 3. *Under above hypothesis the following inequality holds*:

$$(h'^2 \cdot y) < \frac{n^2}{\sum\limits_{j\in I} r_j}\left(td\Big(\sum_{j\in I} r_j\Big) - \Big(\sum_{j\in I} r_j\delta_{j-1}\Big)^2\right) - \sum_{j\in J}(b_{j-1}\cdot y)\nu_j^2.$$

Proof. By the multiplication formulas, we have

$$(h'^2 \cdot y) = (h^2 \cdot y)n^2 - \sum_{i=1}^{N}(b_{j-1} \cdot y)\nu_j^2 - \sum_{j\in I'} r_j\nu_j^2.$$

It is obvious that $(h^2 \cdot y) = d \cdot t$. By Lemma 1 and our supposition, $(b_{i-1} \cdot y) \geq r_i$ if $i \in I_{m-2}$. If $i \notin I_{m-2} \bigcup J$, then, by the supposition, $(b_{j-1} \cdot y) \geq 0$; so

$$\sum_{i=1}^{N}(b_{i-1} \cdot y)\nu_i^2 \geq \sum_{i\in I_{m-2}} r_i\nu_i^2 + \sum_{j\in J}(b_{j-1} \cdot y)\nu_j^2,$$

and we have

$$(h'^2 \cdot y) \leq d \cdot tn^2 - \sum_{j\in I} r_j\nu_j^2 - \sum_{j\in J}(b_{j-1} \cdot y)\nu_j^2.$$

Applying the square inequality (Lemma 2) to the middle term, we complete the proof.

Remark. An analogous estimation holds for an arbitrary class $y^* = th^{m-2} - \sum_{i\in I'} r_i^* g_i$, provided the analog of Lemma 1 holds and the above conditions are satisfied.

In difficult cases, we shall use Lemma 3 many times. To simplify the work with the inequalities, let us introduce the following notations:

$$\sum\nolimits_i = \sum_{j\in I_i} r_j, \qquad i = 0, 1, \ldots, m-2.$$

5. The Maximal Singularities Method.

1. We shall describe a general scheme which allows one with the aid of the previously developed techniqe, to study the birational correspondence of the Fano manifolds. First, let us formulate the supposition which plays an important role in this theory.

The Main Hypothesis. *Let χ: $V \to V'$ be a birational automorphism and each of the following conditions be satisfied:*

(i) $n(\chi) \geq 2$;

(ii) $|H' + K_{V'}| = \varnothing$.

Then the linear system $|\chi|$ has a maximal cycle.

If this proposition is satisfied (for fixed V and for some test manifold of the same dimension), then we say that V satisfies the *main hypothesis* (or for V the main hypothesis holds).

A description of the birational isomorphisms of the manifold V is possible (and is even comparatively simple) if V satisfies the main hypothesis. It is precisely the proof of the main hypothesis that is of the greatest difficulty. The main part of this work contains proofs of the main hypothesis for different classes of Fano manifolds. The scheme of our reasonings will always be the same: assume that the main hypothesis is not true. Then we prove that a biraticnal map which does not have maximal cycles also cannot have maximal singularities. But this contradicts Proposition 2.1. Thus, the first thing to do is to eliminate the maximal singularities which lose dimension under the descent on V. The largest class of Fano manifolds which it is possible to study with the aid of the maximal singularities method gives

Theorem 1. *Let V be a manifold of degree $d \leq 4$. Then V satisfies the main hypothesis.*

Proof. Assume the contrary. Let χ: $V \to V'$ be a birational map. $|\chi|$ does not contain maximal cycles and either $n(\chi) \geq 2$ or $|H' + K_{V'}| = \varnothing$. If we take some admissible resolution of singularities of χ, then we have (in the notations of this chapter) that χ has the maximal singularity B_β and

$$\operatorname{codim} B_{\beta,0} > \operatorname{codim} B_\beta.$$

Consider the test class

$$y = \sum_{j\in I'} r_j(h^{m-2} - g_j).$$

By Corollary 4.1, y is nonnegative. Thus, $(h'^2 \cdot y) \geq 0$. On the other hand, Lemma 4.3 is true if $J = \varnothing$ (because y is nonnegative). Thus, we have

$$0 \leq (h'^2 \cdot y)\frac{n^2}{\sum\limits_{j\in I} r_j}\left(d\Bigl(\sum_{j\in I'} r_j\Bigr)\Bigl(\sum_{j\in I} r_j\Bigr) - \Bigl(\sum_{j\in I} r_j\delta_{j-1}\Bigr)^2\right).$$

It is easy to see that the expression in brackets is negative if $d \leq 4$. We get a contradiction. The theorem is proved.

2. If the manifold V satisfies the main hypothesis, then there are no difficulties for the description of its birational isomorphisms. Further reasoning uses the following scheme:

(1) first, we have to find all cycles on V which can be maximal (i.e., all cycles such that the linear subsystem consisting of the divisors which have multiplicity greater than $(\operatorname{codim} B-1)n$ along B is nonempty and movable for some $n \in \mathbb{Z}_+$);

(2) we find the "untwisting" birational automorphisms, i.e., for all B which we found at the previous step $B \subset V$, we construct a birational map $\tau_B \in \operatorname{Bir} V$ for which B is the maximal cycle (these maps appear in a very natural way from the geometrical point of view and are always involutions);

(3) then we "untwist" the maximal cycles, i.e., for the map $\chi\colon\ V \to V'$ with the maximal cycle $B \subset V$, we consider the composition $\chi\tau_B\colon\ V \to V'$ the degree of which is less than $n(\chi)$. The induction by the degree gives us a description of both $\operatorname{Bir}(V, V')$ and $\operatorname{Bir} V$. The group $\operatorname{Bir} V$ is generated by biregular automorphisms and the map τ_B which we construct at step (2).

That is the general structure of our study (the order of the steps does not have to be the same as here).

For simplicity of the exposition, we shall define two classes of Fano manifolds.

Definition.

(i) The manifold V is called *birationally rigid* if

(a) for all test manifolds (V', H') and for all birational maps $\chi\colon\ V \to V'$ there exists a birational automorphism $\widetilde{\chi} \in \operatorname{Bir} V$ such that the linear system $|\chi\circ\widetilde{\chi}|$ of the divisors on V (where $\chi\circ\widetilde{\chi}\colon\ V \to V'$ is a composition) is a subsystem of the anticanonical system on V, i.e., $|\chi \circ \widetilde{\chi}| \subset |-K_V|$ (in other words, $\chi \circ \widetilde{\chi}$ is a "linear projection");

(b) if (V', H') is a test manifold such that $|H' + K_{V'}| = \varnothing$, then there are no birational maps $\chi\colon\ V \to V'$.

(ii) The manifold V is called *birationally superrigid* if it is birationally rigid and $\operatorname{Bir} V = \operatorname{Aut} V$.

Remarks.

(1) The difference between birational superrigidity and birational rigidity consists in the different formulation of (a): in the superrigidity case, it becomes the following: for all test manifolds (V', H') and all birational maps $\chi\colon\ V \to V'$ the linear system $|\chi|$ is a subsystem of the anticanonical system $|-K_{V'}|$.

(2) It is clear that, if for a manifold V which satisfies the main hypothesis, steps (1), (2), and (3) can be fulfilled, then V is birationally superrigid. This is our plan of action.

(3) Assume that a manifold V satisfies the main hypothesis and it also satisfies the following condition: for all cycles $B \subset V$ and for all $n \in \mathbb{Z}_+$ the linear system of divisors on V with the base set $\bigl|nh - ((\operatorname{codim} B - 1)n + 1)B\bigr|$ has a stationary component. Then it is obvious that V is birationally superrigid. It will be our scheme to prove the birational superrigidity.

(4) From part (b) in definition of the birational rigidity, it follows that a birationally rigid manifold does not have the bundle structure on a Fano manifold of lesser dimension, i.e., there does not exist a rational map $V \to W$ such that a general fibre of it is birationally equivalent to the Fano manifold. In particular, there does not exist the structure of a conic bundle on V and V is not a rational manifold.

If we take $(V', -K_{V'})$ as a test manifold, where V' is a Fano manifold with the free anticanonical system $|-K_{V'}|$, then a birationally rigid manifold V is birationally equivalent to V' only if V and V' are birationally isomorphic. If V is superrigid, then all birational isomorphisms $V \to V'$ are biregular.

Thus, the birational superrigidity defines the birational geometry of the given manifold in an almost complete way.

3. In this work, we shall prove the birational rigidity of the following Fano manifolds:

(1) of the double cover $V_2 \to \mathbb{P}^m$ with a smooth branching divisor $W_{2m} \subset \mathbb{P}^m$ which is a hypersurface of degree $2m$, $m \geq 3$;

(2) of the double cover $V_4 \to Q_2 \subset \mathbb{P}^{m+1}$ of a smooth m-dimensional quadric with a smooth branching divisor which is the intersection of Q_2 and the hypersurface of degree $2(m-1)$, $m \geq 3$;

(3) of the 3-dimensional quartic $V_4 \subset \mathbb{P}^4$;

(4) of the full intersection $V_6 \subset \mathbb{P}^5$ of a quadric and a cubic in $\mathbb{P}^5$;

(5) of the 4-dimensional quintic $V_5 \subset \mathbb{P}^5$;

(6) of the 3-dimensional quartic V_4' in $\mathbb{P}^4$ with one double point of a general structure.

Moreover, the birational superrigidity will be proved for the manifolds of types (1) and (2), where $m \geq 4$, and of (3) and (5). We shall give a complete description (generators and relations) of the groups Bir V for the manifolds of type (2), where $m = 3$, and for (4) and (6).

Chapter 2
Manifolds of Small Degree

In this chapter, we shall study those Fano manifolds (from the list at the end of Chapter 1) which have degree not greater than 4, i.e., the double spaces, the double quartics of index 1 and of dimension 3 or more, and also the quartics of dimension 3. These manifolds satisfy the main hypothesis (Theorem I.5.1).

Since they satisfy the main hypothesis, the problem of describing their birational geometry is that of describing the maximal cycles. In Sec. 1, this will be done for the double covers (except for the case of the double quadrics of dimension 3: these manifolds have nontrivial birational automorphisms). The 3-dimensional double quadrics will be studied in Sec. 2. In Sec. 3, we shall study the birational correspondence of the 3-dimensional smooth quartic.

1. Birational Automorphisms of Double Spaces and Double Quadrics.

1. We study the manifold V which is a double cover of a smooth manifold Q, $\sigma\colon V \to Q$, $\deg \sigma = 2$, with the smooth branching divisor $W \subset Q$. We shall study the following two cases:

$Q = \mathbb{P}^m$, W is a smooth hypersurface of degree $2m$;

$Q \subset \mathbb{P}^{m+1}$ is a smooth quadric and W is the intersection of Q and the hypersurface $\widetilde{W} \subset \mathbb{P}^{m+1}$ of degree $2(m-1)$.

We assume that $m \geq 3$, where $m = \dim V$.

Let us note that Pic $V = \mathbb{Z}(-K_V)$ and the anticanonical system $|K_V|$ exactly defines the double cover morphism $\sigma\colon V \to Q$.

Theorem. *The manifold V of the type* (1) *for $m \geq 3$ and of the type* (2) *for $m \geq 4$ is birationally superrigid.*

Proof. Fix a test manifold (V', H') of dimension m and a birational map $\chi\colon V \to V'$ (assume that they exist). By Theorem I.5.1, if $n(\chi) = n \geq 2$ or $|H' + K_{V'}| = \varnothing$, then the linear system $|\chi|$ has a maximal cycle.

We shall show that this is impossible.

2. Since most of our constructions are valid for the case (2), where $m = 3$, we shall also describe the maximal cycles for these manifolds.

Proposition 1.

(i) *If V is of the type (1), $m \geq 3$, or of the type (2), $m \geq 4$, then the linear system $|\chi|$ cannot have maximal cycles.*

(ii) *If V is of the type (2), $m = 3$, and $Z \subset V$ is the maximal cycle of $|\chi|$, then $(z \cdot h) = 1$.*

Proof. Assume the contrary. Then, by Corollary I.3.4, the codimension of the maximal cycle equals 2 and the degree is not greater than $(d-1)$.

Thus, let $\nu = \mathrm{mult}_B\,|\chi| > n$, where $B \subset V$ is a cycle of codimension 2, $(b \cdot h^{m-2}) \leq d-1$, and, if $d = 4$, $m = 3$, then $(b \cdot h) \geq 2$ (part (ii) of Proposition 1). We shall study separately the cases $d = 2$ and $d = 4$.

3. The case of the double space ($d = 2$). We have that $(h^{m-2} \cdot b) = 1$. Let $B = \sigma(\overline{B}) \subset Q$. Then $\deg \overline{B} = 1$. Thus, $\overline{B}$ is an $(m-2)$-dimensional plane in $\mathbb{P}^m$. The double cover $\sigma^{-1}(\overline{B}) \to \overline{B}$ is not connected and $\sigma\colon B \to \overline{B}$ is an isomorphism (note that $\overline{B}$ is not contained in W). Hence, the scheme $W \bigcap \overline{B} \subset \overline{B}$ is not reduced everywhere (i.e., every component of $W \bigcap \overline{B}$ is multiple), because the branching divisor of the cover $\sigma^{-1}(\overline{B}) \to \overline{B}$ must have a singularity at each point.

Lemma 1. *Let $X \subset \mathbb{P}^M$ be a hypersurface and $H \subset \mathbb{P}^M$ be a hyperplane. Then* $\dim \operatorname{Sing}(X \bigcap H) \le \dim \operatorname{Sing} X + 1$.

Proof. Let $(x_0 : x_1 : \dots : x_M)$ be uniform coordinates in $\mathbb{P}^M$ such that H is defined by the equation $x_M = 0$. Let X be defined by the equation $f(x) = 0$. If $\dim \operatorname{Sing}(X \bigcap H) \le 0$, then the lemma is true ($\dim \varnothing = -1$). So, let $\dim \operatorname{Sing}(X \bigcap H) \ge 1$. Let Y be a component of the maximal dimension of $\operatorname{Sing} X \bigcap H$. Then the equalities $\dfrac{\partial}{\partial x_0} = \dots = \dfrac{\partial f}{\partial x_{M-1}} = 0$ hold on Y. But then

$$Y \bigcap \left(\frac{\partial f}{\partial x_M} = 0\right) = Y' \subset \operatorname{Sing} X, \quad \dim Y' \ge \dim Y - 1.$$

(The statement of lemma is true in the case of the full intersections in $\mathbb{P}^M$ and is proved analogously.)

Applying the lemma twice we have that, if $\dim B \ge 3$, then $\operatorname{Sing} W \ne \varnothing$. Thus, $\dim \overline{B} = 1$ or 2, i.e., $\overline{B}$ is a line or a plane ($m = 3$ or 4). Let us consider the second case ($m = 4$).

Let $Q^* \subset \mathbb{P}^4$ be a general hyperplane and $V^* = \sigma^{-1}(Q^*)$ be the associated double cover: $\sigma : V^* \to Q^* = \mathbb{P}^3$ with the smooth branching divisor $W^* = W \bigcap Q^*$. Let $B^* = V^* \bigcap B$, $\overline{B^*} = \overline{B} \bigcap Q^*$, and let $|\chi|^*$ be the restriction of the linear system $|\chi|$ on V^*. Since $\overline{B^*}$ is a line and $\deg(\sigma\colon B^* \to \overline{B^*}) = 1$, B^* is a nonsingular rational curve. Let $\pi : V_1^* \to V^*$ be its blowing, $E^* = \pi^{-1}(B^*)$, e^* be the class of E^* in $A^1(V_1^*)$ and h^* be the class of the inverse image of a plane under σ. Then the linear system $|nh^* - \widetilde{\nu} e^*|$ does not have stationary components (since $\widetilde{\nu} = \operatorname{mult}_{B^*} |\chi|^*$). On the other hand, the general divisor of the linear system $|4h^* - e^*|$ has a nonnegative intersection with all effective curves on V_1^*.

Indeed, let B' be a component of $\sigma^{-1}(\overline{B^*})$ different from B^* and $\widetilde{B}'$ be its proper inverse image on V_1^*. It is easy to see that $\widetilde{B}'$ is the only base curve of the linear system $|h^* - e^*|$. Therefore, it is enough to check that $\big((4h^* - e^*) \cdot \widetilde{b}'\big) \ge 0$. But $(h^* \cdot \widetilde{b}') = 1$, $(e^* \cdot \widetilde{b}') = 4$ (because B^* intersects B' in 4 points). Thus, we prove that $\big((nh^* - \widetilde{\nu} e^*)^2 \cdot (4h^* - e^8)\big) \ge 0$. But a direct computation gives $8n^2 - 2n\widetilde{\nu} - 6\widetilde{\nu}^2 \ge 0$. This contradicts the condition $\widetilde{\nu} > n$. The case $m = 4$ is proved.

In the case $m = 3$, the considerations are analogous. Let us blow B, let π be the map $\pi\colon \widetilde{V} \to V$, and $E = \pi^{-1}(B)$. Then the class $(3h - e)$ is nonnegative, and so

$$\big((nh - \nu e)^2 \cdot (3h - e)\big) = 6n^2 - 2n\nu - 4\nu^2 \ge 0.$$

But this contradicts the condition $\nu > n$. Proposition 1 is proved for $d = 2$.

4. The case of the double quadric ($d = 4$). We have $(b \cdot h^{m-2}) \le 3$. Let $\overline{B} = \sigma(b) \subset Q$. First, note that $\overline{B}$ is not contained in W for $m \ge 4$ ($W = \widetilde{W} \bigcap Q$ is a smooth complete 3-dimensional intersection with $\operatorname{Pic} W \cong \mathbb{Z}$ that is generated by a hyperplane section).

First, let us consider the case $m \ge 4$.

Let Q^* be the section of Q by a general 4-plane, $V^* = \sigma^{-1}(Q^*)$ be the 3-dimensional nonsingular double cover of Q^* which is branched over $W \bigcap Q = W^*$, $B^* = V^* \bigcap B$, $\overline{B^*} = \sigma(B^*)$, h^* be the class of the inverse image of the hyperplane section of Q^*.

The case $(b \cdot h^{m-2}) = (b^* \cdot h^*) = 3$. It is obvious that $\sigma\colon \overline{B^*} \to B^*$ is an isomorphism, $\deg \overline{B^*} = 3$, and $\overline{B^*}$ is a rational normal curve. Let $\pi : V_1^* \to V^*$ be the blowing of B^* and $E^* = \pi^{-1}(B^*)$. As before, $\dim \operatorname{Bs}|nh^* - \widetilde{\nu} e^*| \le 1$ and the intersection of the class $\big((m-1)h^* - e^*\big)$ with all effective 1-cycles on V_1^* is nonnegative (the curve $\overline{B^*}$ is a section by quadrics, and so it is enough to check that $\big(((m-1)h^* - e^*) \cdot \widetilde{b}'\big) \ge 0$, where $\widetilde{B}'$ is the proper inverse image on V_1^* of the second component of B' of the disconnected cover $\sigma\colon \sigma^{-1}(\overline{B^*}) \to \overline{B^*}$; but B' intersects B^* in $3(m-1)$ points). Thus, $\big((nh^* - \widetilde{\nu} e^*)^2 \cdot ((m-1)h^* - e^*)\big) \ge 0$. But a direct computation gives

$$4(m-1)n^2 - 6n\widetilde{\nu} - 3(m-1)\widetilde{\nu}^2 - \widetilde{\nu}^2\big(2 - 2g(B^*) + (K_{V^*} \cdot b^*)\big),$$

where $g(B^*) = 0$ is a genus of the curve B^*, $K_{V^*} = (m-4)h^*$ is a canonical class of V^*. Hence

$$4(m-1)n^2 - 6n\widetilde{\nu} - (6m-13)\widetilde{\nu}^2 \geq 0,$$

which contradicts the supposition $\widetilde{\nu} > n$.

The case $(b^* \cdot h^*) = (b \cdot h^{m-2}) = 2$. In this case, two subcases can appear:

1) $B^* \to \overline{B^*}$ is an isomorphism and deg $\overline{B^*} = 2$; so $\overline{B^*}$ is the plane conic on Q, $\overline{B^*} = P \bigcap Q^*$ (P is the 2-plane which is not contained in Q^*);
2) $B^* \to \overline{B^*}$ is a double cover, $\overline{B^*}$ is a line, $\overline{B}$ is an $(m-2)$-plane on Q (so $m = 4$).

If (1) is true, then let $\pi\colon V_1^* \to V^*$ be the blowing of B^* and $E^* = \pi^{-1}(B^*)$. Since $\dim \mathrm{Bs}|nh^* - \widetilde{\nu}e^*| \leq 1$ and the class $(m-1)h^* - e^*$ is nonnegative, we have

$$\Big((nh^* - \widetilde{\nu}e^*)^2 \cdot \big((m-1)h^* - e^*\big)\Big) \geq 0.$$

But a direct computation gives

$$4(m-1)n^2 - 2(m-1)\widetilde{\nu}^2 - 4n\widetilde{\nu} - \widetilde{\nu}^2\big(2 + 2(m-4)\big) < 0.$$

We get a contradiction.

Let (2) be true. Then B^* is a curve of arithmetic genus 2 which has δ, $0 \leq \delta \leq 2$, double points x_i. It is clear that, if $\delta \geq 1$, then $\sigma(x_i)$ are exactly those points at which the line $\overline{B^*}$ is tangent to the surface W^*. The general hyperplane $\Lambda \subset \mathbb{P}^4$ which contains the line $\overline{B^*}$ is everywhere transversal to W^* (in a finite number of points $\overline{B^*} \bigcap W^*$, we have to make sure this is so; in other points, we can apply the Bertini theorem). Thus, $\Lambda \bigcap Q^* = \Lambda^* \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a nonsingular quadric, and $\sigma^{-1}(\Lambda^*)$ is its nonsingular double cover, branched over the nonsingular curve $W^* \bigcap \Lambda$. Let a and b be the classes of the inverse images of two families of lines on Λ^* (a and b are in $\mathrm{Pic}\, \sigma^{-1}(\Lambda^*)$), b is the class of the inverse image of $\overline{B^*} \subset \Lambda^*$. Then the class of the nonstationary part of the restriction of the linear system $|\chi|$ on $\sigma^{-1}(\Lambda^*)$ is $n(a+b) - \nu b = na - (\nu - n)b$ (because $h|_{\sigma^{-1}(\Lambda^*)} = a + b$). If $\nu > n$, then, multiplying this class by a, we get a negative number, because $a^2 = b^2 = 0$ and $(a \cdot b) = 2$. But class a is represented by a sheaf of curves without base points (by the inverse image of the sheaf of lines on the quadric). We have a contradiction.

The case $(b \cdot h^{m-2}) = 1$. We have that B is an $(m-2)$-plane, and so $m = 4$. Let $S \subset \mathbb{P}^5$ be the general 3-plane which contains $\overline{B}$. Then $S \bigcap Q = \overline{B} \bigcup P(S)$ and $P(S)$ is a plane. Simple reasoning shows that, for general S, the surface $\sigma^{-1}(P(S))$ is nonsingular. Indeed, let S_1 be the general hyperplane in $\mathbb{P}^5$ which contains $\overline{B}$. It is easy to check that $\sigma^{-1}(S \bigcap Q)$ has only 0-dimensional singularities. $S_1 \bigcap Q$ is a cone over the nonsingular quadric $\overline{Q}$ in $\mathbb{P}^3$. $\overline{B}$ corresponds to the line on $\overline{Q}$; the planes which correspond to the lines of the second family form a linear system of divisors on $\overline{Q}$ with the cone apex as a unique base point (this point is outside of the branching divisor for general S_1). The inverse image of the general plane in the family under σ will be nonsingular.

Let us take such S and set $P = \sigma^{-1}(P(S))$. P is a double cover of the plane and has a branching in the nonsingular curve of degree 6, i.e., the canonical divisor of P is 0. Let $l^* \in A^1(P)$ be the class of the inverse image of the line on $P(S)$, $K_P = 0$. Let $\overline{B^*}$ be the line $\overline{B} \bigcap P(S)$, $\sigma^{-1}(\overline{B^*}) = B_1^* \bigcup B_2^*$, $B_1^* = B \bigcap P$. The linear system $|\chi|$ bounded on P has B_i^* as its ν_i^*-multiple component, $\nu_1^* = \widetilde{\nu} > n$, and does not have other stationary components. Thus, the class $nh^* - \nu_1^* b_1^* - \nu_2^* b_2^*$ has a nonnegative intersection with any effective 1-cycle. In particular,

$$\big((nh^* - \nu_1^* b_1^* - \nu_2^* b_2^*) \cdot b_i^*\big) \geq 0, \qquad i = 1, 2.$$

But $(b_1^* \cdot b_2^*) = 3$, $b_i^{*2} + (K_P \cdot b_i^*) = 2g(B_i^*) - 2$, so $b_i^{*2} = -2$, and $n + 2\nu_1^* - 3\nu_2^* \geq 0$ and $n - 3\nu_1^* + 2\nu_2^* \geq 0$. It is easy to see that the inequality $\nu_i^* \leq n$ is the consequence of all this.

Finally, let $m = 3$. We assume that $\nu > n$, $(b \cdot h) = 2$ or 3, $\overline{B} \not\subset W$, and we shall have a contradiction by the same kind of reasoning as before by setting $m = 3$ everywhere. We leave all details to the reader. The only new case we have to discuss is $\overline{B} \subset W$, $\overline{B}$ is a smooth conic, and $\sigma\colon B \to \overline{B}$ is an isomorphism.

Lemma 2. *Through every point $x \in W$ passes not less than 1 line $Z \subset Q$ which is tangent to W at x.*

Proof. It is obvious that $T_x W$ is a plane in $T_x Q \cong \mathbb{P}^3$. But $Q \bigcap T_x Q$ is a quadratic cone with the apex at $T_x Q$. The plane $T_x W$ intersects it at two lines (taking the multiplicity into the account). These lines are contained in $T_x W$, and so they are tangent to W at x.

Let us blow $B \subset V$: π: $\widetilde{V} \to V$, $E = \pi^{-1}(B)$, and let us consider the proper inverse image on V of the 1-dimensional family of curves $C \subset V$ such that $\sigma(C) \subset Q$ is a line that intersects $\overline{B}$ and is tangent to W in $\sigma(C) \bigcap \overline{B}$. The class of the curves from this family is $\frac{1}{2}h^2 - 2f$, where $f \in A^2(\widetilde{V})$ is the class of the fibre of the ruled surface E. Since the family is 1-dimensional and the system $|nh - \nu e|$ is nonstationary, we have

$$\left((nh - \nu e) \cdot \left(\frac{1}{2}h^2 - 2f\right)\right) \geq 0$$

Thus, $2n - 2\nu \geq 0$. We have a contradiction. Proposition 1 is completely proved.

2. The Birational Automorphisms of the Double Quadric of Dimension 3.

1. We shall consider the smooth manifold V which is a double cover of the smooth 3-dimensional quartic σ: $V \to Q \subset \mathbb{P}^4$ with the smooth branching divisor $W \subset Q$. This divisor is the section by the hypersurface $\widetilde{W} \subset \mathbb{P}^4$ of degree 4 (i.e., this is the case $m = 3$ that was omitted in Sec. 1). A given Fano manifold has nontrivial birational automorphisms (the manifolds considered in Sec. 1 do not have such automorphisms). The nonregular birational endomorphisms of V are related to the lines on V, i.e., to the curves $L \subset V$ such that $(l \cdot h) = 1$, $h = -K_V$, $\operatorname{Pic} V = \mathbb{Z}h$.

Let us consider the lines on V in detail.

Proposition 1.

(1) Let $L \subset V$ be a line. Then $\overline{L} = \sigma(L) \subset Q$ is a line in $\mathbb{P}^4$ and there are two possible cases:

(i) *$\overline{L} \not\subset W$, $\overline{L}$ is tangent to W at the points of intersection $\overline{L} \bigcap W$, the curve $\sigma^{-1}(\overline{L})$ is not connected — it has two components $\sigma^{-1}(L) = L \bigcup L'$, both of which are lines;*

(ii) $\overline{L} \subset W$.

(2) There is a 1-dimensional family of lines on W.

Proof. (1) is obvious. The existence of the lines can be proved by computation of the constants; the 1-dimensionality of the family can be proved by methods of the deformation theory. See [8].

2. We relate to any line $L \subset V$ a birational involution τ_L of V, which is constructed in the following way. Let $\overline{L} = \sigma(L)$. The projection from L, π_L: $Q \to \mathbb{P}^2$, with a line as a fibre gives us the rational map $\pi = \pi_L \circ \sigma$: $V \to \mathbb{P}^2$ that is not defined at the points which belong to $\sigma^{-1}(\overline{L})$. Let C_t be the fibre of π over the point $t \in \mathbb{P}^2$. It is the inverse image in V of the line $\pi_L^{-1}(t)$ in Q. For a general $t \in \mathbb{P}^2$, C_t is a smooth elliptic curve.

If $\overline{L} \subset W$, then a general fibre of π intersects L in one point. Let us define the birational automorphism τ_L on fibres as a reflection in C_t with the center at the point $C_t \bigcap L$.

If $\overline{L} \not\subset W$, then let $\sigma^{-1}(\overline{L}) = L \bigcup L'$. For a general $t \in \mathbb{P}^2$, $\#C_t \bigcap L = \#C_t \bigcap L' = 1$. Let us define τ_L on fibres as a reflection in C_t with the center at the point $C_t \bigcap L'$.

Lemma 1. *Let $L \subset V$ be a line, $\overline{L} \subset W$. Then τ_L is a biregular automorphism which is a permutation of the fibres of the double cover σ: $V \to Q$.*

Proof. Let $t \in \mathbb{P}^2$ be a general point, C_t be a fibre of the map π over t. Let $z = C_t \bigcap L$; then the action of the reflection τ_L on C_t is described by the formula $\tau_L(x) + x \sim 2z$. But $\sigma(z) \in W$ by the supposition. Thus, the map σ: $C_t \to \sigma(C_t)$, where $\sigma(C_t) \cong \mathbb{P}^1$, is defined by the linear system $|2z|$ on C_t. This means exactly that $\sigma \circ \tau_L(x) = \sigma(x)$. It is obvious that $\tau_L \neq id_V$; hence the lemma is proved.

Let us consider now a line $L \subset V$ such that $\overline{L} \not\subset W$. Let $\sigma^{-1}(\overline{L}) = L \bigcup L'$ and let ρ: $\widetilde{V} \to V$ at first blow L and then the proper inverse image of L'. Let E and E' be the proper inverse images of the exceptional divisors on $\widetilde{V}$. The projection π: $V \to \mathbb{P}^2$ can be extended to the regular map $\widetilde{\pi}$: $\widetilde{V} \to \mathbb{P}^2$, $\pi = \pi \circ \rho$, the fibres of which will be again denoted by C_t, $t \in \mathbb{P}^2$. For a general $t \in \mathbb{P}^2$, C_t is a smooth elliptic curve. The morphism $\widetilde{\pi}$ has two sections over a general point: the surfaces E and E'. Set $\widetilde{\tau}_L = \rho^{-1} \circ \tau_L \circ \rho \in \operatorname{Bir} \widetilde{V}$.

Lemma 2. *The involution* $\widetilde{\tau}_L$ *can be extended to a biregular automorphism of the invariant open set* $\widetilde{V} \setminus T$, codim $T = 2$, dim $\widetilde{\pi}(T) = 0$. *The action of* $\widetilde{\tau}_L$ *on*

$$\text{Pic } \widetilde{V} \setminus T = \text{Pic } \widetilde{V} = \mathbb{Z}\, h \oplus \mathbb{Z}\, e \oplus \mathbb{Z}\, e'$$

is defined by the relations

$$\widetilde{\tau}_L^*(h) = 9h - 10e - 6e',$$
$$\widetilde{\tau}_L^*(e) = 8h - 9e - 6e',$$
$$\widetilde{\tau}_L^*(e') = e'.$$

Proof. Let $h_1, e_1, e_1' \in \text{Pic } \widetilde{V}$ be the classes of the proper inverse images under $\widetilde{\tau}_L$ of the general divisor of the system $|h|$, of the divisors E and E' respectively. For a general point $t \in \mathbb{P}^2$, the map $\widetilde{\tau}_L$, restricted to the fibre $C_t \subset \widetilde{V}$, moves the point $x \in C_t$ into the point $\widetilde{\tau}_L(x) \in C_t$ such that $\widetilde{\tau}_L(x) + x \sim 2(C_t \bigcap E')$ as divisors on the curve C_t. Since the kernel of the restriction of Pic $\widetilde{V}$ to a general fibre of $\widetilde{\pi}$ is $\widetilde{\pi}^*\,\text{Pic}\,\mathbb{P}^2 = \mathbb{Z}(h-e-e')$ (because $\widetilde{\pi}$ is defined by the linear system $|h-e-e'|$), we have

$$h_1 + h = 4e' + m(h - e - e'),$$
$$e_1 + e = 2e' + m^*(h - e - e'),$$
$$e_1' = e' + m'(h - e - e').$$

Let us restrict $\widetilde{\tau}_L$ to the general $\widetilde{\tau}_L$-invariant surface S from the linear system $|h - e - e'|$. It is easy to check that $K_S = 0$; so the restriction of $\widetilde{\tau}_L$ to S (we shall use the same notation for it) may be extended to a biregular involution of the surface S. Let h_S, e_S, e_S' be the restrictions of the classes h, e, e' to S. By the definition of $\widetilde{\tau}_L$, we have $\widetilde{\tau}_L^*(e_S') = e_S'$ and from the relations

$$(\widetilde{\tau}_L^*(h_S) \cdot e_S') = (h_S \cdot e_S') = 1,$$
$$(\widetilde{\tau}_L^*(e_S) \cdot e_S') = (e_S \cdot e_S') = 2$$

we get that $m = 10$, $m^* = 8$. It is obvious that $m' = 0$.

It is clear that the restriction of h_1 to S is $\widetilde{\tau}_L^*(h_S)$ (because of the generality of S). Thus, $h_1 = 9h - 10e - 6e'$. Note now that $\widetilde{\tau}_L$ is correctly defined on all fibres C_t, $t \in \mathbb{P}^2$, such that they are irreducible and nonsingular in $C_t \bigcap E'$ (because the reflection with the center at the nonsingular point is defined for the rational curves of the arithmetic genus 1). All fibres C_t (except for a finite number of them) are nonsingular in $C_t \bigcap E'$ and do not have as components the fibres of the ruled surfaces E and E'. It remains to prove that only a finite number of C_t are such that $\rho(C_t)$ is a reducible curve: i.e., it remains to prove that the line L has a nonempty intersection with only a finite number of lines in V. Let $Z \subset V$ be a line, $Z \bigcap L \neq \varnothing$. Then for its proper inverse image $\widetilde{Z} \subset \widetilde{V}$ we have $(\widetilde{z} \cdot h_1) = -1$. But the linear system $|h_1|$ does not have stationary components (by construction). Thus, $\widetilde{\tau}_L$ is correctly defined on the complement of a finite set of the fibres of ρ, which is an invariant open set.

We have now $h_1 = \widetilde{\tau}_L^*(h)$, $e_1 = \widetilde{\tau}_L^*(e)$, $e_1' = \widetilde{\tau}_L^*(e')$. The lemma is proved.

3. Now we shall formulate the main result.

Theorem. (A) *The involutions* τ_L, *where* L *goes through the set of all lines in* V *such that* $\sigma(L) \not\subset W$, *freely generates a normal subgroup of the finite index in the group* Bir V, *which we shall denote by* $B(V)$:

$$B(V) = \mathop{*}_{\substack{L \subset V \\ \sigma(L) \not\subset W}} \langle \tau_L \rangle .$$

(B) *The group* Bir V *is a semidirect product of the normal subgroup* $B(V)$ *and the finite (and, in the general case, isomorphic to* $\mathbb{Z}/2\mathbb{Z}$*) group of automorphisms* Aut V:

$$1 \to B(V) \to \text{Bir } V \to \text{Aut } V \to 1.$$

The action of Aut V *on* $B(V)$ *is defined by the relations* $\lambda \tau_L \lambda^{-1} = \tau_{\lambda(L)}$ *for all lines* $L \subset V$.

(C) *The manifold* V *is birationally rigid.*

Proof. Let (V', L') be a test manifold and $\chi: V \dashrightarrow V'$ be a birational map. Set $\nu_B(\chi) = \mathrm{mult}_B |\chi|$ for the irreducible cycle $B \subset V$.

Proposition 2. *Let $n(\chi) \geq 2$ or $|L' + K_{V'}| = \varnothing$. Then only one among the numbers $\nu_L(\chi)$ is strictly greater than $n(\chi)$; all others are strictly less than $n(\chi)$ (here L goes through the set of all lines in V such that $\sigma(L) \not\subset W$).*

We shall deduce the theorem from Proposition 2.

Lemma 3.

(i) *If $\nu_L(\chi) > n(\chi)$, then $\nu_L(\chi\tau_L) < n(\chi\tau_L) < n(\chi)$;*

(ii) *If $\nu_L(\chi) < n(\chi)$, then $\nu_L(\chi\tau_L) > n(\chi\tau_L) > n(\chi)$.*

Proof. In the notations of Sec. 2, let, for all $\chi \in \mathrm{Bir}\ (V, V')$, $|\widetilde{\chi}|$ be the proper inverse image of the linear system $|\chi|$ on $\widetilde{V}$. Note that the class of the divisor $|\widetilde{\chi}|$ in $\mathrm{Pic}\,\widetilde{V}$ is

$$n(\chi)h - \nu_L(\chi)e - \nu_{L'}(\chi)e',$$

where $L \bigcup L' = \sigma^{-1}(\overline{L})$.

Consider the commutative diagram of the birational maps:

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\widetilde{\tau}_L"] \arrow[d, "\rho"] & \widetilde{V} \arrow[d, "\rho"] & \\ V \arrow[r, "\tau_L"] & V \arrow[r, "\chi"] & V' \end{tikzcd}$$

The linear system $|\widetilde{\chi\tau_L}|$ coincides with the proper inverse image of the linear system $|\widetilde{\chi}|$ by $\widetilde{\tau}_L$. But since $\widetilde{\tau}_L \in \mathrm{Aut}\ \widetilde{V} \setminus T$, codim $T = 2$, the class of the divisor of the system $|\widetilde{\chi\tau_L}|$ is obtained by the action of $\widetilde{\tau}_L^*$ on the divisor class $|\widetilde{\chi}|$. Thus,

$$n(\chi\tau_L)h - \nu_L(\chi\tau_L)e - \nu_{L'}(\chi\tau_L)e' = \widetilde{\tau}_L^*(n(\chi)h - \nu_L(\chi)r - \nu_{L'}(\chi)e').$$

Applying Lemma 2, we have the relations

$$\begin{aligned} n(\chi\tau_L) &= 9n(\chi) - 8\nu_L(\chi), \\ \nu_L(\chi\tau_L) &= 10n(\chi) - 9\nu_L(\chi), \\ \nu_{L'}(\chi\tau_L) &= 6n(\chi) - 6\nu_L(\chi) + \nu_{L'}(\chi). \end{aligned}$$

From this we get Lemma 3 directly.

We shall prove the theorem under the supposition of the validity of Proposition 2. Let $n(\chi) \geq 2$ or $|L' + K_{V'}| = \varnothing$. Then, by Proposition 2 and Lemma 3, we can find a (unique) line $L \subset V$ such that $n(\chi\tau_L) < n(\chi)$. If $Z \subset V$ is a line, $\sigma(Z) \not\subset W$, then making the right multiplication of this inequality by τ_Z, we can diminish the degree $n(\cdot)$. This diminishing process is possible while $n(\chi) \geq 2$ or $|L' + K_{V'}| = \varnothing$. In the last case, we have a contradiction because the diminishing process cannot be infinite. In the first case, we have the following: if $B(V) \subset \mathrm{Bir}\ V$ is a subgroup generated by all τ_L, $L \subset V$, $\sigma(L) \not\subset W$, then, for any $\chi \in \mathrm{Bir}(V, V')$, we can find $\widetilde{\chi} \in B(V)$ such that $n(\chi \circ \widetilde{\chi}) = 1$. The birational rigidity of V is proved.

Finally, let $L_1, \ldots, L_k$ be a sequence of lines in V, $\sigma(L_i) \not\subset W$, such that $L_i \neq L_{i+1}$ for all i, $1 \leq i \leq k-1$. Let $\chi_t = \tau_{L_1} \ldots \tau_{L_t}$, $1 \leq t \leq k$, $\chi_0 = \mathrm{id}_V$. We shall prove by induction that $n(\chi_{t+1}) > n(\chi_t)$ if $n(\chi_t) > n(\chi_{t-1})$. Indeed, the last inequality means that L_t is the maximal line in χ_t (because $\chi_t = \chi_{t-1}\tau_{L_t}$ and so $\chi_{t-1} = \chi_t\tau_{L_t}$); so $\nu_{L_t}(\chi_t) > n(\chi_t)$. But, by Proposition 2, $\nu_{L_{t+1}}(\chi_t) < n(\chi_t)$, and so, in view of Lemma 3, $n(\chi_{t+1}) > n(\chi_t)$.

Thus, by using Lemma 3 and Proposition 2 $(k-1)$ times, we have that $\chi_k \notin \mathrm{Aut}\ V$. This proves (A) and (B) (the action of Aut V on $B(V)$ is a conjugation). The theorem is proved.

4. Proof of Proposition 2. By Theorem I.5.1 and by the conditions of Proposition 2, χ has the maximal cycle $B^* \subset V$. In Sec. 1 we considered all cases except for the following one: $B^* = L \subset V$ is a line in V. Thus, let χ have the maximal line L. We shall show that $\overline{L} = \sigma(L) \not\subset W$. Set $n = n(\chi)$, $\nu_B = \nu_B(\chi)$, if $B \subset V$ is an irreducible cycle.

Lemma 4. *Let $Z \subset V$ be a line such that $\overline{Z} = \sigma(Z) \subset W$. Then there exists a 1-dimensional family of lines R on the quadric Q such that they have a nonempty intersection with $\overline{Z}$ and are tangent to W at the points in $\overline{Z} \bigcap R$.*

Proof. Let $S \subset \mathbb{P}^4$ be a general hyperplane, $\overline{Z} \subset S$. Then the quadric $Q \bigcap S$ is nonsingular and the curve $W \bigcap S$ is not connected: $W \bigcap S = \overline{Z} \bigcup W(S)$, $\#\overline{Z} \bigcap W(S) = 4$. Thus, 4 lines on $Q \bigcap S$ are tangent to W at the points in $\overline{Z} \bigcap W$. It remains to note that, for every finite set $\Lambda \subset \overline{Z}$ and general $S \subset \mathbb{P}^4$, $\Lambda \bigcap W(S) = \varnothing$. Thus, the family of lines described in the lemma is, in fact, 1-dimensional. The lemma is proved.

Lemma 5. *Let $Z \subset V$ be a line, $\overline{Z} \subset W$. Then $\nu_Z \leq n$.*

Proof. Let $\pi\colon\ \widetilde{V} \to V$ be the blowing of Z, $E = \pi^{-1}(Z)$, $f \in A^2(\widetilde{V})$ be the class of a fibre of the ruled surface E. Consider the proper inverse image of the family of curves on $\widetilde{V}$ which was described in the previous lemma. The class of a general curve is $\frac{1}{2}h^2 - 2f$. Since the linear system $|nh - \nu_Z e|$ does not have stationary components and the curves of the family sweep the divisor, we have the inequality

$$\left((nh - \nu_Z e) \cdot \left(\frac{1}{2}h^2 - 2f\right)\right) \geq 0.$$

From this follows the lemma.

Let us prove now that the maximal cycle is unique.

Lemma 6. *Let $\nu_L > n$, $L \subset V$ be a line, $\overline{L} \not\subset W$. Then, for all other lines $Z \subset V$ such that $\overline{Z} = \sigma(Z) \not\subset W$, we have $\nu_Z < n$.*

Proof. We shall consider different cases of the arrangement of L and Z.

(i) $\overline{L} = \overline{Z}$, i.e., $\sigma(L) = \sigma(Z)$. Consider the general hyperplane $S \subset \mathbb{P}^4$, $\overline{L} \subset S$, then $\sigma^{-1}(S \bigcap Q) = H \subset V$ is a smooth K3-surface and $S \bigcap Q = Q^*$ is a smooth quadric, $Q^* \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let a^* and b^* be the classes in $\operatorname{Pic} H$ of the inverse images of the lines on Q^* and a^* be the class of the inverse image of $\overline{L}$. Let us restrict the linear system $|\chi|$ on H. The class of the obtained linear system in $\operatorname{Pic} H$ is $n(a^* + b^*)$; the class of its nonstationary part is $na^* + nb^* - \nu_L l - \nu_Z z$, $a^* = l + z$. We have

$$\left((na^* + nb^* - \nu_L l - \nu_Z z) \cdot b^*\right) = 2n - \nu_L - \nu_Z \geq 0.$$

This is exactly what is desired.

(ii) $\overline{L} \neq \overline{Z}$ and $\overline{L} \bigcap \overline{Z} = \varnothing$. Let $S \subset \mathbb{P}^4$ be a (unique) hyperplane which contains $\overline{L}$ and $\overline{Z}$; then $Q \bigcap S$ is a nonsingular quadric which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and in which $\overline{L}$ and $\overline{Z}$ are in one sheaf of lines. Thus, there exists a sheaf of lines that sweep $Q \bigcap S$ and have nonempty intersection with $\overline{L}$ and with $\overline{Z}$. Let $\pi\colon\ \widetilde{V} \to V$ be the blowing of $L \bigcup Z$, $E_1 = \pi^{-1}(L)$, $E_2 = \pi^{-1}(Z)$. Then the class of the proper inverse image of the linear system $|\chi|$ on $\widetilde{V}$ (that is free from stationary components) is $nh - \nu_Z e_2 - \nu_L e_1$. The class of the proper inverse image on $\widetilde{V}$ of the general line Q which intersects $\overline{L}$ and $\overline{Z}$ is $\frac{1}{2}h^2 - f_1 - f_2$ (where f_i is the class of the fibre of E_i). Thus,

$$\left((nh - \nu_L\, e_1 - \nu_Z\, e_2) \cdot \left(\frac{1}{2}h^2 - f_1 - f_2\right)\right) = 2n - \nu_L - \nu_Z \geq 0.$$

(iii) $\overline{L} \neq \overline{Z}$, but $\overline{L} \bigcap \overline{Z} \neq \varnothing$, $\overline{L} \bigcap \overline{Z} \notin W$. It is clear that there exists a plane $P \subset \mathbb{P}^4$ such that $\overline{L} \bigcup \overline{Z} = P \bigcap Q$. Let $\pi\colon\ \widetilde{V} \to V$ at first blow L and then the proper inverse image of Z. Let E_1 be the proper inverse image of the exceptional divisor of the first blowing on $\widetilde{V}$, and E_2 be the exceptional divisor of the second blowing. Consider the class $(h - e_1 - e_2) \in A^1(\widetilde{V})$. Since P is an intersection of the hyperplanes in $\mathbb{P}^4$ (that contain it), then $\left((h - e_1 - e_2) \cdot c\right) \geq 0$ for any curve $C \subset \widetilde{V}$ such that $\sigma \circ \pi(C) \not\subset \overline{L} \bigcup \overline{Z}$. Also, it is clear that $\left((h - e_1 - e_2) \cdot c\right) \geq 0$ for any curve $C \subset E_1$ such that $\pi(C) = L$, and for any curve $C \subset E_2$ such that $\pi(C) = Z$ (because the linear system of hyperplane sections of the quadric Q does not have infinitely close base curves along $\overline{L}$ and $\overline{Z}$). Finally, $(h - e_1 - e_2)$ is nonnegative with respect to all curves $C \subset E_i$,

$i = 1, 2$. Let now L' and Z' be residual components of the nonconnected double covers $\sigma^{-1}(\overline{L}) \to \overline{L}$ and $\sigma^{-1}(\overline{Z}) \to \overline{Z}$, i.e., $\sigma^{-1}(\overline{L}) = L \bigcup L'$, $\sigma^{-1}(\overline{Z}) = Z \bigcup Z'$, and $\widetilde{L}'$ and $\widetilde{Z}'$ be their proper inverse images in $\widetilde{V}$.

We prove now that if $((h - e_1 - e_2) \cdot c) < 0$ for a curve $C \subset \widetilde{V}$, then $C = \widetilde{L}'$ or $C = \widetilde{Z}'$. Set $\delta = \#L \bigcap Z$, therefore, $\delta = 0$ or 1. Now we have

$$((h - e_1 - e_2) \cdot \widetilde{l}') = -2 + \delta,$$
$$((h - e_1 - e_2) \cdot \widetilde{z}') = -2 + \delta.$$

Thus, the class $(3 - \delta)h - e_1 - e_2$ is nonnegative. The system $|nh - \nu_L e_1 - \nu_Z e_2|$ does not have the stationary components. So

$$((nh - \nu_L e_1 - \nu_Z e_2)^2 \cdot ((3 - \delta)h - e_1 - e_2)) \geq 0.$$

The same computations show that the product on the left-hand side of the inequality is

$$4(3 - \delta)n^2 - 2n\nu_L - 2n\nu_Z + 2\delta\nu_L\nu_Z - (4 - \delta)(\nu_L^2 + \nu_Z^2).$$

The trivial, but tiresome, computations show that this expression cannot be nonnegative when $\nu_L > n$, $\nu_Z \geq n$, $\delta = 0, 1$.

(iv) The last case: $\overline{L} \neq \overline{Z}$, $\overline{L} \bigcap \overline{Z} \neq \varnothing$, $\overline{L} \bigcap \overline{Z} \subset W$. Let $\sigma^{-1}(L) = \overline{L} \bigcup L'$, $\sigma^{-1}(\overline{Z}) = Z \bigcup Z'$, $x = L \bigcap Z = \sigma^{-1}(\overline{L} \bigcap \overline{Z}) \in V$. Then x is a point of the intersection of all four curves: L, L', Z, Z'. Let us blow it: $\pi_1\colon\ \widetilde{V}_1 \to V$, $E_1 = \pi_1^{-1}(x) \cong \mathbb{P}^2$. Let $\widetilde{L}$ and $\widetilde{Z}$ be the proper disjoint inverse images of L and Z in $\widetilde{V}_1$, $\pi_2\colon\ \widetilde{V}_2 \to \widetilde{V}_1$ be the blowing of the smooth curve $\widetilde{L} \bigcup \widetilde{Z}$, $E_2 = \pi_2^{-1}(\widetilde{L})$, $E_3 = \pi_2^{-1}(\widetilde{Z})$, $\widetilde{L}'$ and $\widetilde{Z}'$ be the proper inverse images of L' and Z' in $\widetilde{V}_2$, $\pi_3\colon\ \widetilde{V} \to \widetilde{V}_2$ be the blowing of the smooth curve $\widetilde{L}' \bigcup \widetilde{Z}'$, $E_4 = \pi_3^{-1}(\widetilde{L}')$, $E_5 = \pi_3^{-1}(\widetilde{Z}')$. We assert that the class

$$y = h - 2e_1 - e_2 - e_3 - e_4 - e_5$$

is nonnegative.

Indeed, consider the linear system $|y|$. It is the proper inverse image in $\widetilde{V}$ by $\sigma \circ \pi_1 \circ \pi_2 \circ \pi_3$ of the linear system of hyperplane sections of the quadric Q, each of them containing $\overline{L} \bigcup \overline{Z}$. It is easy to check that if the inequality $(y \cdot c) < 0$ is true for the curve $C \subset \widetilde{V}$, then $\pi_1 \circ \pi_2 \circ \pi_3(C) = x$. Thus, to prove the nonnegativity of y, it is enough to state that the class $y|_{\widetilde{E}_1}$ is nonnegative (here $\widetilde{E}_1$ is the proper inverse image of E_1 in $\widetilde{V}$). But the linear system $|y|_{\widetilde{E}_1}|$ of curves in $\widetilde{E}_1$ ($\widetilde{E}_1$ is the plane with four blown points, maybe infinitely close) is the proper inverse image in $\widetilde{E}_1$ of the system of conics in $E_1 \cong \mathbb{P}^2$ that contains these four points. It is enough to prove that there are no lines containing three of them. To this end, we take the general hyperplanes $H_1 \supset \overline{L}$ and $H_2 \supset \overline{Z}$ in $\mathbb{P}^4$. We have the following: L and L' are transversal to $\sigma^{-1}(H_2 \bigcap Q)$, Z and Z' are transversal to $\sigma^{-1}(H_1 \bigcap Q)$, where $\sigma^{-1}(H_1 \bigcap Q)$ and $\sigma^{-1}(H_2 \bigcap Q)$ are nonsingular hypersurfaces. Then the points $\widetilde{L} \bigcap E_1$ and $\widetilde{L}' \bigcap E_1^2$ are not on the line $\sigma^{-1}(H_2 \bigcap Q)^1 \bigcap E_1$, but are on the line $\sigma^{-1}(H_1 \bigcap Q)^1 \bigcap E_1$. The same is true for the pair Z and Z'. The statement is proved.

Thus, the class $y = h - 2e_1 - e_2 - e_3 - e_4 - e_5$ is nonnegative.

Set $\nu_0 = \nu_x$, $\nu_1 = \nu_L$, $\nu_2 = \nu_Z$, $\nu_3 = \nu_{L'}$, $\nu_4 = \nu_{Z'}$. We have

$$((nh - \nu_0 e_1 - \nu_1 e_2 - \nu_2 e_3 - \nu_3 e_4 - \nu_4 e_5)^2 \cdot y) \geq 0.$$

The left-hand side of the inequality is

$$4n^2 - 2\sum_{i=0}^{4} \nu_i^2 + 2(\nu_0 - n)\sum_{i=1}^{4} \nu_i + 2\nu_1\nu_3 + \nu_2\nu_4. \tag{$*$}$$

Note that $\nu_0 \geq \nu_1 > n$ (the restriction of the proper inverse image of the linear system $|\chi|$ from $\widetilde{V}_1$ to $E_1 \cong \mathbb{P}^2$ is a linear system of the plane curves of degree ν_0 with the base point $\widetilde{L} \bigcap E_1$ of multiplicity ν_1).

Assume that Lemma 6 is not true under these conditions, i.e., $\nu_Z = \nu_2 \geq n$. Then, as we already proved, $\nu_3 \leq n$ and $\nu_4 \leq n$. We shall find the maximum of the quadratic form $(*)$ on the set

$$n \leq \nu_0 \leq 2n; \quad n \leq \nu_i \leq 2n, \quad i = 1,2; \quad 0 \leq \nu_i \leq n, \quad i = 3,4.$$

In view of the symmetry, it is easy to check that this maximum is achieved when $\nu_1 = \nu_2 = \theta \geq n$, $\nu_3 = \nu_4 = \mu \leq n$, $\nu_0 = \nu$. Thus, we have:

$$\Xi(\theta,\mu,\nu) = 4n^2 - 2\nu^2 - 4\theta^2 - 4\mu^2 - 4n\theta - 4n\mu + 4\nu\theta + 4\nu\mu + 4\theta\mu$$

Under our conditions we have

$$\frac{\partial \Xi}{\partial \theta} = -8\theta - 4n + 4\nu + 4\mu \leq 0,$$

and so

$$\Xi(\theta,\mu,\nu) \leq \Xi(n,\mu,\nu) = -2\nu^2 - 4\mu^2 - 4n^2 + 4n\nu + 4\nu\mu = -(\nu - 2n)^2 - (\nu - 2\mu)^2 < 0.$$

We have a contradiction. Lemma 6 is proved.

Proposition 2 is proved.

3. Birational Automorphisms of the Smooth 3-Dimensional Quartic.

1. We shall describe the birational correspondences of the smooth hypersurface V of degree 4, $V \subset \mathbb{P}^4$.

Theorem. *The manifold V is birationally superrigid.*

Proof. Let (V', L') be a 3-dimensional test manifold, $\chi\colon\ V \dashrightarrow V'$ be a birational map. By Theorem I.5.1, if $n(\chi) \geq 2$ or $|L' + K_{V'}| = \varnothing$, then χ has the maximal cycle Z. Since the degree of V is 4, Z is a curve (Corollary I.3.4) of degree less than or equal to 3 (Corollary I.3.5).

Lemma 1. *Let $B \subset V$ be a plane curve, i.e., $B \subset P$ for some 2-plane $P \subset \mathbb{P}^4$. Then*

$$\operatorname{mult}_B |\chi| \leq n(\chi).$$

Proof. Let $x \in \mathbb{P}^4$ be a general point, $x \notin V$. Consider the cone C_x with apex x and base B (if B is a line, then C_x is a plane). We have $C_x \bigcap V = B \bigcup B_x$, deg $B_x = 3\deg B$, and $B_x \subset B$ is exactly a hyperplane section of B_x. Thus, B cannot be contained in $|\chi|$ with multiplicity greater than $n(\chi)$. The lemma is proved.

Lemma 2. *Let $B \subset V$ be a rational normal curve of degree 3. Then $\#\nu = \operatorname{mult}_B |\chi| \leq n(\chi)$.*

Proof. Let $\pi\colon\ \widetilde{V} \to V$ be a blowing of B, $E = \pi^{-1}(B)$ be an exceptional divisor. The curve B is sectioned by quadrics in $\mathbb{P}^3$, and in $\mathbb{P}^4$ also, i.e., the linear system $|2h - e|$ is free. The linear system $|nh - \nu e|$, where $n = n(\chi)$, does not have stationary components. Thus,

$$((nh - \nu e)^2 \cdot (2h - e)) = 8n^2 - 6n\nu - 5\nu^2 \geq 0.$$

Therefore, $\nu \leq n$.

Thus, by Lemmas 1 and 2, χ cannot have maximal cycles. Therefore, the cases $n(\chi) \geq 2$ and $|L' + K_{V'}| = \varnothing$ are impossible.

The theorem is proved.

Chapter 3
The Birational Automorphisms of the Complete Intersection of the Quadric and the Cubic in $\mathbb{P}^5$

In this chapter, we shall study the birational geometry of the Fano 3-folds of degree 6. It is well known that such a manifold V can be represented as the complete intersection $V = Q \bigcap \widetilde{Q} \subset \mathbb{P}^5$ of the 4-dimensional quadric Q and the cubic $\widetilde{Q}$. Among all Fano manifolds considered in this work the complete intersection of the quadric and the cubic has the most nontrivial geometry, and so the difficulties of studying them are much greater than in other cases. The maximal singularities method (in its modern form) works well only for manifolds of degree less than or equal to 4. In our case, the degree of the manifold is 6; thus, we have to improve our basic construction. The problem of describing the maximal cycles becomes nontrivial; also the untwisting involutions are related in this case. All this makes the Fano 3-folds of degree 6 the most complex and beautiful object in the 3-dimensional birational geometry (among the objects it is possible to study).

1. Involutions Related to Lines and Conics.

1. Let V be an arbitrary Fano 3-fold of degree 6 (without singularities), $V = Q \bigcap \widetilde{Q} \subset \mathbb{P}^5$. It is well known (by the computation of constants) that there is a 1-dimensional family of lines in V (lines of the space $\mathbb{P}^5$). Let $L \subset V$ be a line.

Proposition 1. *For the normal sheaf $N_{L/V}$ there are two possibilities:*

(a) $N_{L/V} \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L$; *in this case, the line L is called a line of the general type,*

(b) $N_{L/V} \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L$; *in this case, the line L is called a line of the nongeneral type.*

The proof is easy (see [8]).

Proposition 2 (*the Geometrical Criterion of the Generality*). *The line L is a line of nongeneral type only when any one of the following two equivalent conditions holds:*

(i) *there exists a plane $P \subset \mathbb{P}^5$ such that $L \subset P$ and the scheme-theoretic intersection $V \bigcap P$ is not reduced everywhere along L;*

(ii) *let $\sigma\colon \widetilde{V} \to V$ be a blowing of L, $E = \sigma^{-1}(L)$ be an exceptional divisor. Then the restriction to E of the proper inverse image in $\widetilde{V}$ of the section of V by the general hyperplane in $\mathbb{P}^5$, such that this hyperplane contains L, is a divisor not ample on E.*

Proof. From Proposition 1 we have that E is a ruled surface of the type $\mathbb{F}_N$, $N = 1$ or 3 $\big(1$ in case (a) of Proposition 1; 3 in case (b)$\big)$. Let $s \in \operatorname{Pic} E$ be the class of the exceptional divisor, $f \in \operatorname{Pic} E$ be the class of a fibre, $(s^2) = -N$. Let h^* be the class of the restriction to E of the proper inverse image in $\widetilde{V}$ of the general hyperplane section of V which contains L. It is easy to see that $(h^*)^2 = 3$, so $h^* = s + 2f$ if $n = 1$, and $h^* = s + 3f$ if $N = 3$. It is clear that h^* is ample only when $N = 1$. (ii) is proved. Consider σ as a restriction of the blowing $\sigma_L\colon X_L \to \mathbb{P}^5$ of the line L to $\mathbb{P}^5$. The proper inverse images in X_L of the hyperplanes which contain L define a morphism $\pi\colon X_L \to \mathbb{P}^3$. The fibres of π are the proper inverse images of the 2-planes containing L. Let $E_L = \sigma_L^{-1}(L)$ be an exceptional divisor, $E_L \cong \mathbb{P}^1 \times \mathbb{P}^3$. The restriction of π to E_L is a projection on the second factor, so π ties up only those curves that are intersections of the proper inverse images of the planes containing L with E_L. But the system $|h^*|$ on E ties up the curve only when L is of the type (b). This can happen only if there exists a plane $P \subset \mathbb{P}^5$, $L \subset P$, such that, if $\widetilde{P}$ is its proper inverse image in X_L, then $\widetilde{V} \supset \widetilde{P} \bigcap E_L$ (i.e., $\widetilde{P} \bigcap \widetilde{V}$ contains the curve that covers L). But this is exactly (i). The proposition is proved.

Corollary 1. *On the general Fano 3-fold of degree 6, there are no lines of the nongeneral type.*

Proof (*the usual count of constants*). Let S be the set of pairs $(Q, \widetilde{Q})$, where Q is a quadric and $\widetilde{Q}$ is a cubic in $\mathbb{P}^5$, such that there exists projective (1,2) flag, i.e., the pair (L, P), where L is a line, P is a 2-plane, $L \subset P$, and $Q|_P$ and $\widetilde{Q}|_P$ contain L as a component of multiplicity 2 or more. It is easy to check that S is a closed set in $\mathbb{P}\Big(H^0\big(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2)\big)\Big) \times \mathbb{P}\Big(H^0\big(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)\big)\Big)$. This completes the proof.

Corollary 2. *Let V not contain lines of the nongeneral type. Then for any line $L \subset V$ and for any plane $P \supset L$ the intersection of V with a general hyperplane $H \supset P$ is a surface, nonsingular along L, $L \bigcap \operatorname{Sing}(H \bigcap V) = \varnothing$.*

Proof. Let us compare two sets:

$$\bigcup_{x \in L} \left\{ H \in \mathbb{P}^{5^*} \,\middle|\, x \in \operatorname{Sing} H \bigcap V \right\} \text{ and } \left\{ H \in \mathbb{P}^{5^*} | H \supset P \right\}.$$

Both sets are irreducible hypersurfaces in $\mathbb{P}^{5*}$, so it is enough to prove that they are different. Assume the contrary; then for any point $x \in L$ we have: $P \subset T_x V = T_x Q \bigcap T_x \widetilde{Q}$. Thus, $Q|_P$ and $\widetilde{Q}|_P$ are not reduced in $x \in L$ (or Q or $\widetilde{Q}$ contains P). Therefore, L is a line of nongeneral type. We have a contradiction.

Remark 1. The geometric criterion of generality is true for all Fano manifolds which are anticanonically immersed into a projective space. The proof is the same (we did not use any specific properties of V_6^3 in the proof).

We shall study only the general Fano 3-folds of degree 6, and we shall assume that V has the following properties:

(i) V does not contain lines of nongeneral type,

(ii) V does not contain three lines that have a mutual point and are contained in one plane,

(iii) in the representation $V = Q \bigcap \widetilde{Q}$, the quadric Q (which is uniquely defined by V) is nonsingular.

Property (i) was proved earlier, property (iii) is obvious, and property (ii) can be proved in the same way as Corollary 1.

From this point on, we fix the manifold $V = Q \bigcap \widetilde{Q}$ together with its representation as a complete intersection and we assume that properties (i)-(iii) hold.

Remark 2. We introduce properties (ii) and (iii) to avoid too cumbersome computations, which we have to perform if (ii) and (iii) do not hold. These properties simplify essentially the problem of describing maximal cycles (Secs. 3,4) and the "small" maximal singularities (Lemma 5.2). Property (i) is more important. It also simplifies the computations and reduce the number of different cases to consider, but it plays an important role in proving the key Lemma 5.9 (see below). In essence, property (i) is the only thing we need to prove in Lemma 5.9.

2. Let $L \subset V$ be an arbitrary line. We relate to it a birational automorphism of the second order $\alpha_L\colon V \dashrightarrow V$, which is defined in the following way. The projection $\pi\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^3$ from L restricted to V gives the rational map $\pi|_V\colon V \dashrightarrow \mathbb{P}^3$ of degree 2 at a general point. The permutation of the points in the fibres of $\pi|_V$ gives us the involution α_L.

In a more formal way, let $\sigma\colon \widetilde{V} \dashrightarrow V$ be a blowing of L, $E = \sigma^{-1}(L) \subset \widetilde{V}$ be an exceptional divisor. Then the projection $\pi|_V$ can be extended to a morphism $\rho = \pi \circ \sigma\colon V \to \mathbb{P}^3$.

Lemma 1. *The morphism $\rho\colon V \to \mathbb{P}^3$ is finite and has degree 2 everywhere outside a closed subset $W \subset \widetilde{V}$ of codimension 2, and $\rho(W) \subset \mathbb{P}^3$ is a finite subset. The involution $\alpha_L = \sigma^{-1} \circ \alpha_L \circ \sigma \in \operatorname{Bir} \widetilde{V}$ can be extended to a biregular automorphism of $\widetilde{V} \setminus W$ of order 2. Its action on*

$$\operatorname{Pic} \widetilde{V} \setminus W = \operatorname{Pic} \widetilde{V} = A^1(\widetilde{V}) = \mathbb{Z}h \oplus |\mathbb{Z}e$$

is described by the relations

$$\widetilde{\alpha}_L^*(h) = 4h - 5e, \qquad \widetilde{\alpha}_L^*(e) = 3h - 4e.$$

Proof. The projection $\rho\colon \widetilde{V} \dashrightarrow \mathbb{P}^3$ has degree 2 on the complement of a closed set $W \subset \widetilde{V}$ which consists of the curves that are contractible by ρ into a point. We shall prove that there are only a finite number of such curves.

Let $h' = nh - \nu e$ and $e' = mh - \mu e$ be the classes in $\operatorname{Pic} \widetilde{V}$ of the proper inverse images of a general hyperplane section of V and of the divisor $E \subset \widetilde{V}$ by the map $\widetilde{\alpha}_L\colon \widetilde{V} \dashrightarrow \widetilde{V}$. Since ρ is defined by the linear system $|h - e|$, the surfaces of this system are invariant with respect to $\widetilde{\alpha}_L$. Let the surface S be a general divisor of this linear system. Since $K_S = 0$, we have that the restriction of $\widetilde{\alpha}_L$ to S can be extended to a biregular involution of S (which we also denote by $\widetilde{\alpha}_L$). Let h_S and e_S be the classes in $\operatorname{Pic} S$ of the hyperplane section $\sigma(S)$ and of $E \bigcap S$, respectively. In view of the generality of S, we have: $\widetilde{\alpha}_L^*(h_S) = nh_S - \nu e_S$, $\widetilde{\alpha}_L^*(e_S) = mh_S - \mu e_S$. The system of curves $|h_S - e_S|$ is $\widetilde{\alpha}_L$-invariant, and so $n = m+1$, $\nu = \mu + 1$. From the relations

$$\big(\widetilde{\alpha}_L^*(h_S) \cdot (h_S - e_S)\big) = \big(h_S \cdot (h_S - e_S)\big) = 5$$

and

$$(\widetilde{\alpha}_L^*(h_S)^2) = (h_S^2) = 6$$

and taking into account the obvious equalities

$$(h_S \cdot e_S) = 1, \qquad (e_S^2) = -2,$$

we have that either $h' = 4h - 5e$, $e' = 3h - 4e$, or $h' = h, e' = e$.

In the second case, α_L can be extended to a biregular map of V, but this is impossible, because $\rho(E) \subset \mathbb{P}^3$ is a surface of degree 3. Thus, we must consider the first case. The linear system $|h'|$ does not have the stationary components (by the construction), so there are only a finite number of curves $C \subset \widetilde{V}$ such that $(c \cdot h') < 0$. But, if the curve $C \subset \widetilde{V}$ is contractible into a point by ρ, then $((h - e) \cdot c) = 0$ and $(h' \cdot c) < 0$. Thus, ρ contracts only a finite number of curves, $\operatorname{codim} W = 2$, $\widetilde{\alpha}_L \in \operatorname{Aut}(\widetilde{V} \setminus W)$, and $\widetilde{\alpha}_L^*(h) = h'$, $\widetilde{\alpha}_L^*(e) = e'$. The lemma is proved.

Remark. In proving Lemma 1, we also state that any line in V has a nonempty intersection with only a finite number of lines in V (the lines that intersect $L \subset V$ are contractible by the projection from L).

3. For the study of the structure of the group $\operatorname{Bir} V$ we shall need more precise information about the action of α_L (in one special case). Let $P \subset \mathbb{P}^5$ be a plane such that $P \bigcap V$ is the union of the three lines $L \bigcup L_1 \bigcup L_2$ (in particular, $P \subset Q$). Consider the sequence of blowings $\sigma_i\colon\ \widetilde{V}_i \to \widetilde{V}_{i-1}$, $\widetilde{V}_0 = V$, $i = 1, 2, 3$. σ_1 blows $L \subset V$, $\widetilde{E} = \sigma_1^{-1}(L)$, σ_2 blows $\widetilde{L}_1$, the proper inverse image of L_1 in $\widetilde{V}_1$, $\widetilde{E}_1 = \sigma_2^{-1}(\widetilde{L}_1)$, σ_3 blows $\widetilde{L}_2$, the proper inverse image of L_2 in $\widetilde{V}_2$, $\widetilde{E}_2 = \sigma_3^{-1}(\widetilde{L}_2)$. Let $\widetilde{V} = \widetilde{V}_3$, $\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3\colon\ \widetilde{V} \to V$. We shall denote the proper inverse images of the divisors $\widetilde{E}$ and $\widetilde{E}_1$ in $\widetilde{V}$ by the same symbols.

Lemma 2. *The birational map $\alpha_L = \sigma^{-1} \circ \alpha_L \circ \sigma \in \operatorname{Bir} \widetilde{V}$ can be extended to a biregular involution outside of a closed subset $W \subset \widetilde{V}$ of codimension 2, $\widetilde{\alpha}_L \in \operatorname{Aut}(\widetilde{V} \setminus W)$. The action of $\widetilde{\alpha}_L$ on $\operatorname{Pic} \widetilde{V} = A^1(\widetilde{V}) = \mathbb{Z}h \oplus \mathbb{Z}\tilde{e} \oplus \mathbb{Z}\tilde{e}_1 \oplus \mathbb{Z}\tilde{e}_2$ is defined by the relations*

$$\begin{aligned}
\widetilde{\alpha}_L^*(h) &= 4h - 5\tilde{e} - 2\tilde{e}_1 - 2\tilde{e}_2, \\
\alpha_L^*(\tilde{e}) &= 3h - 4\tilde{e} - 2\tilde{e}_1 - 2\tilde{e}_2, \\
\alpha_L^*(\tilde{e}_i) &= \tilde{e}_j, \qquad \{i, j\} = \{1, 2\}.
\end{aligned}$$

The proof is geometrically clear, but the computations are tiresome. Let us introduce some notations:

$\pi\colon\ \mathbb{P}^5 \to \mathbb{P}^3$ is a projection from the line L;

$x = \pi(P) \in \mathbb{P}^3$ is the image of the plane P;

$\sigma_L\colon\ X_L \to \mathbb{P}^5$ is the blowing of the line $L \subset \mathbb{P}^5$ and $E_L = \sigma_L^{-1}(L) \subset X_L$ is its exceptional divisor;

$\rho = \pi \circ \sigma_L\colon\ X_L \to \mathbb{P}^3$ is a locally trivial bundle with a plane as a fibre;

$\widetilde{P} \subset X_L$ is the proper inverse image of the plane P;

$\sigma_P\colon\ X_P \to X_L$ is the blowing of $\widetilde{P}$, $E_P = \sigma_P^{-1}(\widetilde{P})$;

$V^* \subset X_P$ is the proper inverse image of V;

$\sigma_x\colon\ Z \to \mathbb{P}^3$ is the blowing of $x \in \mathbb{P}^3$, $E = \sigma_x^{-1}(x)$.

It is easy to see that the morphism ρ can be extended to a locally trivial bundle $\widetilde{\rho}\colon\ X_P \to Z$ with a plane as a fibre. Let $T \subset V$ be a subset which consists of three points $L \bigcap L_i$, $i = 1, 2$, and $L_1 \bigcap L_2$ (i.e., $T = \operatorname{Sing} P \bigcap V$). Let also

$$\widetilde{\pi} = \sigma^{-1} \circ \pi \circ \sigma\colon\ \widetilde{V} \to Z\,, \quad \pi_P\colon\ \mathbb{P}^5 \to \mathbb{P}^2$$

be projections from P ($\mathbb{P}^2$ is canonically identified with the plane $E = \sigma_x^{-1}(x) \subset Z$).

In these notations, we have that $\widetilde{V} \setminus \sigma^{-1}(T)$ is isomorphic to $V^* \setminus (\sigma_L \circ \sigma_P)^{-1}(T)$ over $V \setminus T$ (because σ and $\sigma_L \circ \sigma_P$ blow in $V \setminus T$ the same set, the smooth curve $(L \bigcup L_1 \bigcup L_2) \setminus T$). Thus, $\widetilde{\pi}$ is regular on $\widetilde{V} \setminus \sigma^{-1}(T)$ and coincides with $\widetilde{\rho}|_{V^*}$. It is easy to see that σ resolves the singularities of the sheaf of sections of V by the hyperplanes which contain P. Thus, $\pi_P \circ \sigma\colon\ \widetilde{V} \to \mathbb{P}^2 (\cong E \subset Z)$ is a regular map. On the other hand, let $\pi_x\colon\ Z \to E$ be an extension of the projection from the point x; then $\pi_P \circ \sigma = \pi_x \circ \widetilde{\pi}$.

For a general point $t \in E$, let $Z_t \subset Z$ be a fibre over t. $Z_t = \pi_x^{-1}(t)$, and $S_t = \pi_P^{-1}(t)$ be a 3-plane in $\mathbb{P}^5$, $S_t \supset P$. Then $S_t \bigcap Q = P \bigcup P(S_t)$ is a pair of planes and $P(S_t) \bigcap \widetilde{Q} = C_t$ is a plane cubic. Let $\widetilde{C}_t$ be the proper inverse image of C_t in V. Then, for a general t, $C_t \bigcap T = \varnothing$ (because T is outside of the line $P \bigcap P(S_t)$ for a general t) and, therefore, $\tilde{\pi}$ is regular on a neighborhood of the curve $\widetilde{C}_t$. But C_t is a fibre of $\pi_P \circ \sigma$ over $t \in E \cong \mathbb{P}^2$. It is obvious that $\#C_t \bigcap L = \#C_t \bigcap L_i = 1$, so $(\tilde{c}_t \cdot \tilde{e}) = (\tilde{c}_t \cdot \tilde{e}_i) = 1$, $i = 1, 2$. Thus, $\widetilde{E}$ and $\widetilde{E}_i$ are sections of the morphism $\pi_P \circ \sigma$. Therefore, $\tilde{\pi}|_{\widetilde{E}_i}: \ \widetilde{E}_i \to E \subset Z$ is an isomorphism in a neighborhood of a general point. Thus, we can find two closed sets $W \subset \widetilde{V}$ and $R \subset Z$ of codimension 2 (the 1-dimensional components of R, if they exist, are in E) such that $\tilde{\pi}\colon\ \widetilde{V} \setminus W \to Z \setminus R$ is a finite morphism of degree 2. This gives us the first statement of the lemma.

We have that $\tilde{\pi}^{-1}(E) = E_1 \bigcup E_2$, and so $\tilde{\alpha}_L$ permutes E_1 and E_2. Therefore, $\tilde{\alpha}_L^*(\tilde{e}_i) = \tilde{e}_j$, $\{i, j\} = \{1, 2\}$.

The class of $\tilde{\alpha}_L^*(h)$ is the class of the proper inverse image of a hyperplane section of V by the map $\alpha_L \circ \sigma\colon\ \widetilde{V} \to V$. Let $H \subset \mathbb{P}^5$ be a general hyperplane, $L \not\subset H$. Then $\pi|_{H \bigcap V}$ is a projection of the surface $H \bigcap V = (H \bigcap Q) \bigcap (H \bigcap \widetilde{Q})$ of degree 6 from the point $H \bigcap L$, i.e., $\pi(H \bigcap V)$ is a surface of degree 5 in $\mathbb{P}^3$. π glues the points $H \bigcap L_i \in P$ into one point x. It is easy to see that, for a general H, $\pi|_{H \bigcap V}$ is an isomorphism from the neighborhoods of $H \bigcap L_i$ onto the components of the neighborhood of the point $\pi|_{H \bigcap V}(H \bigcap L_i)$ and $(\pi_{H \bigcap V})^{-1}(x) = \{H \bigcap L_i | i = 1, 2\}$. Thus, we have $\operatorname{mult}_x \pi(H \bigcap V) = 2$. But $(\pi \circ \sigma)^{-1}(x) = \widetilde{E}_1 \bigcup \widetilde{E}_2$ and the morphism $\pi \circ \sigma_1\colon\ \widetilde{V}_1 \to \mathbb{P}^3$ is defined by the linear system $|h - \tilde{e}|$. Therefore,

$$h + \tilde{\alpha}_L^*(h) = 5(h - \tilde{e}) - 2\tilde{e}_1 - 2\tilde{e}_2.$$

This gives us the first relation of the lemma, and the second follows from the α_L-invariance of the class $(h - \tilde{e})$. Lemma 2 is completely proved.

4. On the manifold V lies a 2-dimensional family of conics. We shall denote by $P(Y)$ a plane which contains the conic Y. It is well known that there exists a 1-dimensional subfamily of the conics Y in V such that $P(Y) \subset Q$ (one can easily prove this by counting constants). It is obvious that $P(Y) \bigcap V = P(Y) \bigcap \widetilde{Q} = Y \bigcup L(Y)$, where $L(Y)$ is a residual line. The conic Y is called a *special conic* if $Y \subset V$ and $P(Y) \subset Q$. Let us note that only a finite number of special conics can be reducible (i.e., can be represented as the union of two lines which, after the assumption of the generality of V, are different and are different from a residual line). The notation $Y \subset V$ will be used only for irreducible conics (except for Sec. 6 below).

To every (irreducible) special conic Y we relate a birational involution of V, which we shall denote by β_Y. The construction of this involution is as follows. Set $P = P(Y)$. Consider the projection $\pi_P\colon\ \mathbb{P}^5 \to \mathbb{P}^2$ from the plane P. Its fibres are 3-planes $S \supset P$, $S \bigcap Q = P \bigcup P(S)$, $P(S)$ is a residual plane. So π_P forms a bundle V over $\mathbb{P}^2$, where the fibres are the plane cubics $C_S = P(S) \bigcap \widetilde{Q}$, which are smooth for a general S. It is clear that C_S intersects $L(Y)$ exactly at one point: $L(Y) \bigcap P(S)$. Now we can define the involution β_Y on the fibres as the reflection with the center $L(Y) \bigcap P(S) \in C_S$ (C_S is an abstract elliptic curve and the center is zero of the group law on C_S).

Consider the resolution of the singularities of the projection $\pi_P|_V$. At first, we blow the conic Y, and then blow the proper inverse image of the residual line $L(Y)$. We shall denote the composition of these two blowings by $\sigma\colon\ \widetilde{V} \to V$. Let $E = \sigma^{-1}(Y)$, let $\widetilde{E}$ be the exceptional divisor of the second blowing $\Big($i.e., the 2-dimensional component of $\sigma^{-1}\big(L(Y)\big)$ $\Big)$. It is obvious that σ resolves the singularities of the linear system of the sections of V by the hyperplanes, containing P. Thus, $\pi_P \circ \sigma$ is the morphism $\widetilde{V} \to \mathbb{P}^2$ and its general fibre is the elliptic curve C_t, $t \in \mathbb{P}^2$. Then $(\tilde{e} \cdot c_t) = 1$ (as noted before), and so $\widetilde{E}$ is a section of $\pi_P \circ \sigma$. We define β_Y on V by stating that $\tilde{\beta}_Y = \sigma^{-1} \circ \beta_Y \circ \sigma\colon\ \widetilde{V} \to V$ is a reflection in each fibre with the centers at the points of $\widetilde{E}$.

Lemma 3. *The involution* $\tilde{\beta}_Y \in \operatorname{Aut}(\widetilde{V} \setminus W)$, *where* W *is a closed set of codimension* 2 *and* $\pi_P \circ \sigma(W) \subset \mathbb{P}^2$ *is a finite set. The action of* $\tilde{\beta}_Y$ *on* $\operatorname{Pic} \widetilde{V} = A^1(\widetilde{V}) = \mathbb{Z}h \oplus \mathbb{Z}e \oplus \mathbb{Z}\tilde{e}$ *is defined by the following relations*:

$$\tilde{\beta}_Y^*(h) = 13h - 14e - 8\tilde{e},$$

$$\tilde{\beta}_Y^*(e) = 12h - 13e - 8\tilde{e},$$

$$\tilde{\beta}_Y^*(\tilde{e}) = \tilde{e}.$$

Proof. Let $h', e', \tilde{e}' \in \operatorname{Pic} \widetilde{V}$ be the classes of the proper inverse images under $\widetilde{\beta}_Y$ of the general divisor of the system $|h|$, of the divisors E and $\widetilde{E}$, respectively. For a general point $t \in \mathbb{P}^2$, the reflection $\widetilde{\beta}_Y$, restricted to the fibre $C_t \subset \widetilde{V}$, moves the point $x \in C_t$ into the point $\widetilde{\beta}_Y(x) \in C_t$, such that $\widetilde{\beta}_Y(x) + x \sim 2(C_t \bigcap \widetilde{E})$ (as the divisors on the curve C_t). Taking into account that the kernel of the restriction of $\operatorname{Pic} \widetilde{V}$ to the general fibre of the morphism $\pi_P \circ \sigma$ is $(\pi_P \circ \sigma)^* \operatorname{Pic} \mathbb{P}^2 = \mathbb{Z}(h - e - \tilde{e})$ (because $\pi_P \circ \sigma$ is defined by the linear system $|h - e - \tilde{e}|$), we have

$$h' + h = 6\tilde{e} + m^*(h - e - \tilde{e}), \quad e' + e = 4\tilde{e} + m(h - e - \tilde{e}), \quad \tilde{e}' = \tilde{e} + \widetilde{m}(h - e - \tilde{e}).$$

Let us restrict $\widetilde{\beta}_Y$ to the general surface S from the linear system $|h - e - \tilde{e}|$. It is easy to see that $K_S = 0$, so $\widetilde{\beta}_Y|_S \in \operatorname{Aut} S$. Let $h_S, e_S, \tilde{e}_S$ be the restrictions of the classes $h, e, \tilde{e}$ to S. Denoting, for simplicity, the restriction $\widetilde{\beta}_Y|_S$ by $\widetilde{\beta}_Y$, by definition, we have, $\widetilde{\beta}_Y^*(\tilde{e}_S) = \tilde{e}_S$. From the relations

$$(\widetilde{\beta}_Y^*(h_S) \cdot \tilde{e}_S) = (h_S \cdot \tilde{e}_S) = 1, \quad (\widetilde{\beta}_Y^*(e_S) \cdot \tilde{e}_S) = (e_S \cdot \tilde{e}_S) = 2$$

we have $m^* = 14$, $m = 12$.

Thus, $h' = 13h - 14e - 8\tilde{e}$. Note now that $\widetilde{\beta}_Y$ is not defined only on the reducible fibres C_t, $t \in \mathbb{P}^2$ (the reflection with the center at a nonsingular point is defined only for the rational curves of arithmetic genus 1 with a double point). There are a finite number of fibres C_t which have a fibre of a ruled surface $\widetilde{E}$ as a component. If $\sigma(C_t)$ is a reducible curve, then one of its components is the line ($\deg \sigma(C_t) = 3$) which intersects $Y \bigcup L(Y)$. There are a finite number of lines in V that intersect $L(Y)$ (see Sec. 2). If the line $L \subset V$ intersects Y, then its proper inverse image $\widetilde{L}$ in $\widetilde{V}$ has a nonnegative intersection with h', $(\tilde{l} \cdot h') = 13 - 14 = -1$. Since $|h'|$ does not have stationary components (by construction), there are only a finite number of such lines.

Thus, $\widetilde{\beta}_Y$ is correctly defined on the complement to W which is a finite set of fibres. $\widetilde{V} \setminus W$ is an invariant open set, and $\dim \pi_P \circ \sigma(W) = 0$. Lemma 3 is proved.

5. It is possible that there exist nontrivial relations between the involutions α_L. Let $P \subset \mathbb{P}^5$ be a 2-plane such that $P \subset Q$ and $P \bigcap V = P \bigcap Q = L_1 \bigcup L_2 \bigcup L_3$ be a union of three (different) lines.

Lemma 4. $(\alpha_{L_1} \circ \alpha_{L_2} \circ \alpha_{L_3})^2 = \mathrm{id}_V$.

Proof. Since the projection from the larger space passes through the projection from the smaller space contained in it, α_{L_i} preserves fibres of the projection $\pi_P|_V$, i.e., the elliptic curves $C_S = P(S) \bigcap \widetilde{Q}$ ($S \subset \mathbb{P}^5$ is a general 3-plane, $S \bigcap Q = P \bigcup P(S)$). It is enough to prove that the relation is true on every C_S for a general S.

We have that C_S is a nonsingular cubic, the points $x_i = C_S \bigcap L_i$ are different and lie on the same line $P \bigcap P(S)$. Let $x \in C_S$ be some point; then $\alpha_{L_i}(x) + x + x_i \sim \sum_{j=1}^{3} x_j$ on C_S. Therefore,

$$\begin{aligned} &\alpha_{L_3}(x) \sim x_1 + x_2 - x_3, \\ &\alpha_{L_2} \circ \alpha_{L_3}(x) \sim x_3 - x_2 + x, \\ &\alpha_{L_1} \circ \alpha_{L_2} \circ \alpha_{L_3}(x) \sim 2x_2 - x, \\ &(\alpha_{L_1} \circ \alpha_{L_2} \circ \alpha_{L_3})^2(x) \sim x. \end{aligned}$$

The lemma is proved.

6. The involution β_Y can be defined, in the same way as was done in Sec. 4, for the reducible special conic $Y = L_1 \bigcup L_2$, $P(Y) \bigcap V = L_1 \bigcup L_2 \bigcup L$. Let $\sigma\colon\ \widetilde{V} \to V$ be a sequence of blowings: first of L_1, then of the proper inverse image of L_2, then of the proper inverse image of L. Let $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}$ be the proper inverse images of the exceptional divisors on $\widetilde{V}$. The projection π_P from the plane $P = P(Y)$ can be extended to a morphism $\pi_P \circ \sigma\colon\ \widetilde{V} \to \mathbb{P}^2$ with an elliptic curve, as a fibre, and with sections $\widetilde{E}$, $\widetilde{E}_1$, $\widetilde{E}_2$. $\widetilde{\beta}_Y = \sigma^{-1} \circ \beta_Y \circ \sigma\colon\ \widetilde{V} \to \widetilde{V}$ is a reflection in each fibre with the center at $\widetilde{E}$.

Lemma 5. *There exists a closed subset* $W \subset \widetilde{V}$, $\operatorname{codim} W = 2$, $\dim \pi_P \circ \sigma(W) = 0$, *such that* $\widetilde{\beta}_Y \in \operatorname{Aut}(\widetilde{V} \setminus W)$. *The action of* $\widetilde{\beta}_Y$ *on* $\operatorname{Pic} \widetilde{V} = \mathbb{Z}h \oplus \mathbb{Z}\tilde{e}_1 \oplus \mathbb{Z}\tilde{e}_2 \oplus \mathbb{Z}\tilde{e}$ *is defined by the relations*

$$\widetilde{\beta}_Y^*(h) = 13h - 14\tilde{e}_1 - 14\tilde{e}_2 - 8\tilde{e},$$
$$\widetilde{\beta}_Y^*(\tilde{e}_i) = 6h - 7\tilde{e}_i - 6\tilde{e}_j - 4\tilde{e}, \quad i, j = 1, 2,$$
$$\widetilde{\beta}_Y^*(\tilde{e}) = \tilde{e}.$$

Moreover,

$$\beta_Y = \alpha_{L_1} \circ \alpha_L \circ \alpha_{L_2} = \alpha_{L_2} \circ \alpha_L \circ \alpha_{L_1}.$$

The proof of Lemma 4, in fact, contains the proof of the last statement. Other statements of the lemma can be proved either by the method of Sec. 4 (i.e., directly) or by computing the action of $(\alpha_{L_1} \circ \alpha_L \circ \alpha_{L_2})^*$ on $\operatorname{Pic} \widetilde{V}$. We leave this to the reader. We shall never use these statements, and β_Y will always mean the involution related to the irreducible conic Y.

2. Formulation of the Theorem and Beginning of the Proof.

1. The main result of this chapter is a description of the generators and of the relations of the group $\operatorname{Bir} V$, namely, we prove that it is generated by the projective automorphisms and involutions α_L and β_Y for all lines and for all special irreducible conics. All relations between these involutions are consequences of the relations defined in Lemma 1.4.

To present the statement of the Theorem, we shall introduce the symbols A_L and B_Y for every line $L \subset V$ and for every special irreducible conic $Y \subset V$. Let G be a quotient group of the free group, generated by A_L and B_Y (for all L, Y), by the normal group, generated by the words

$$\begin{aligned} &A_L^2, \quad \text{for all} \quad L, \\ &B_Y^2, \quad \text{for all} \quad Y, \end{aligned} \qquad (*)$$

$$(A_{L_1} A_{L_2} A_{L_3})^2, \text{ for all ordered triads } (L_1, L_2, L_3)$$
$$\text{such that } L_1 \cup L_2 \cup L_3 = P \cap V, \text{ for some 2-plane } P \subset Q.$$

Let us construct the semidirect product $G \operatorname{Aut} V$ using the action $\operatorname{Aut} V \to \operatorname{Aut} G$, which is defined by the obvious relations

$$\rho A_L \rho^{-1} = A_{\rho(L)}, \qquad \rho B_Y \rho^{-1} = B_{\rho(Y)},$$

where $\rho \in \operatorname{Aut} V$ (it is clear that ρ maps from the set $(*)$ of words into itself, so it generates an automorphism of G). Let ε: $G \operatorname{Aut} V \to \operatorname{Bir} V$ map from A_L into α_L, B_Y into β_Y and be the identity on $\operatorname{Aut} V$.

Theorem.

(A) *The homomorphism* ε *is an isomorphism.*

(B) *The manifold* V *is birationally rigid.*

Remarks.

(i) Part (A) of the theorem states that the involutions α_L and β_Y generate in $\operatorname{Bir} V$ a normal subgroup $B(V)$ of finite index and $\operatorname{Bir} V$ is a semidirect product

$$1 \to B(V) \to \operatorname{Bir} V \to \operatorname{Aut} V \to 1,$$

where ε: $G \to B(V)$ is an isomorphism.

(ii) The set of all planes $P \subset \mathbb{P}^5$ such that $P \bigcap V$ is the union of three lines is obviously finite. Thus, G is close to the direct product of cyclic groups of order 2.

2. Beginning of the Proof of the Theorem. Let (V', L') be a test manifold. Consider $\operatorname{Bir}(V, V')$, the set of birational correspondences χ: $V \to V'$. There is a natural right action of $\operatorname{Bir} V$ on this set. For arbitrary $\chi \in \operatorname{Bir}(V, V')$ the proper inverse image on V of the linear system $|L'|$ is a linear system of divisors $|\chi|$ without stationary components which is sectioned by hypersurfaces of some degree $n(\chi)$.

Let $\mathcal{B} = \{L \subset V \mid L \text{ be a line in } \mathbb{P}^5\} \bigcup \{Y \subset V \mid Y \text{ be a special irreducible conic}\}$. Let $\nu_B(\chi) = \operatorname{mult}_B |\chi|$ for an irreducible cycle $B \subset V$. Now we can formulate the key proposition.

Proposition 1. *Let $n(\chi) \geq 2$ or $|L' + K_{V'}| = \varnothing$.*

(i) *There exists a cycle $B \in \mathcal{B}$ such that $\nu_B(\chi) > n(\chi)$.*

(ii) *For all cycles $B \in \mathcal{B}$, except for one or two, $\nu_B(\chi) \leq n(\chi)$.*

(iii) *If for exactly two cycles B_1 and B_2 the inequality $\nu_{B_i}(\chi) \geq n(\chi)$, $i = 1, 2$, holds, then $B_i \subset V$ are lines and $B_1 \bigcup B_2$ is a reducible special conic.*

3. Deduction of the Theorem from Proposition 1.

Lemma 1. (i) *Let $L \subset V$ be a line. Then:*

$$n(\chi\alpha_L) = 4n(\chi) - 3\nu_L(\chi),$$
$$\nu_L(\chi\alpha_L) = 5n(\chi) - 4\nu_L(\chi).$$

(ii) *Let $Y \subset V$ be an irreducible special conic, $L = L(Y) \subset V$ be a residual line, $L \bigcup Y = P(Y) \bigcap V$. Then*

$$n(\chi\beta_Y) = 13n(\chi) - 12\nu_Y(\chi),$$
$$\nu_Y(\chi\beta_Y) = 14n(\chi) - 13\nu_Y(\chi),$$
$$\nu_L(\chi\beta_Y) = 8n(\chi) - 8\nu_Y(\chi) + \nu_L(\chi).$$

(iii) *Let $P \subset \mathbb{P}^5$ be a plane such that $P \bigcap V = L \bigcup L_1 \bigcup L_2$. Then*

$$\nu_{L_i}(\chi\alpha_L) = 2n(\chi) - 2\nu_L(\chi) + \nu_{L_j}(\chi), \quad \{i, j\} = \{1, 2\}.$$

Proof. Statements (i)–(iii) can be proved by simple computations, using the results of Lemmas 1.1, 1.2, and 1.3, respectively. We shall discuss case (i) and leave cases (ii)-(iii) to the reader.

Using the notations of Secs. 1, 2, let $|\widetilde{\chi}|$ be the proper inverse image in $\widetilde{V}$ of the linear system $|\chi|$. Let us note that the class of the divisor $|\widetilde{\chi}|$ in $\operatorname{Pic} \widetilde{V}$ is $n(\chi)h - \nu_L(\chi)e$. Now consider the diagram of the birational maps:

$$\begin{array}{ccccc}
\widetilde{V} & \xrightarrow{\widetilde{\alpha}_L} & \widetilde{V} & & \\
\sigma\downarrow & & \sigma\downarrow & & \\
V & \xrightarrow{\widetilde{\alpha}_L} & V & \xrightarrow{\chi} & V'
\end{array}$$

It is clear that the linear system $|\widetilde{\chi\alpha_L}|$ in $\widetilde{V}$ coincides with the proper inverse image (by $\widetilde{\alpha}_L$) in $\widetilde{V}$ of the linear system $|\chi|$. But, since $\widetilde{\alpha}_L \in \operatorname{Aut} \widetilde{V} \setminus W$, $\operatorname{codim} W = 2$, the class of the divisor of the system $|\widetilde{\chi\alpha_L}|$ is produced from the class of the divisor of $|\widetilde{\chi}|$ by the action of $\widetilde{\alpha}_L^*$, i.e.,

$$n(\chi\alpha_L)h - \nu_L(\chi\alpha_L)e = \widetilde{\alpha}_L^*(n(\chi)h - \nu_L(\chi)e).$$

Therefore, using Lemma 1.1, which describes the action of $\widetilde{\alpha}_L^*$ on $\operatorname{Pic} \widetilde{V}$, we get (i). (ii) and (iii) can be proved in the same way.

Corollary 1. *Let $\tau = \alpha_L$ or β_Y be an involution. Then it is "untwisting" for χ, i.e., $n(\chi\tau) < n(\chi)$ only when $\nu_L(\chi) > n(\chi)$ or $\nu_Y(\chi) > n(\chi)$, respectively.*

Corollary 2. *Let $n(\chi) \geq 2$ or $|L' + K_{V'}| = \varnothing$, $\tau = \alpha_L$ or β_Y.*

(i) *the equality $n(\chi\tau) = n(\chi)$ is true only for one τ and only in one case, where $\tau = \alpha_L$ and L is a component of the special reducible conic $L \bigcup L'$ and $\nu_{L'}(\chi) > n(\chi)$.*

(ii) *The inequality $n(\chi\tau) > n(\chi)$ is true for all involutions τ, except for one τ^* or two τ_i^*, $i = 1, 2$. In the first case, $n(\chi\tau^*) < n(\chi)$. In the second case, $\tau_i^* = \alpha_{L_i}$, where $L_1 \bigcup L_2$ is a special reducible conic, $n(\chi\tau_i^*) \leq n(\chi)$, $i = 1, 2$, and at least one of these two inequalities is strict.*

Corollaries 1 and 2 are obvious consequences of Proposition 1 and Lemma 1.

Let us proceed with the proof of the theorem (we assume that the conditions of Proposition 1 hold). Let $n(\chi) \geq 2$ or $|L' + K_{V'}| = \varnothing$. Then $\nu_B(\chi) > n(\chi)$ for some $B \in \mathcal{B}$. Let $\tau = \alpha_L$ if $B = L$, and $\tau = \beta_Y$ if $B = Y$. From Corollary 1, $n(\chi\tau) < n(\chi)$. Thus, by twisting on α_L or β_Y we can diminish the degree of $n(\chi)$, while $n(\chi) \geq 2$ or $|L' + K_{V'}| = \varnothing$. In the last case, we get a contradiction, because it is impossible to diminish infinitely the variable with positive, integer values. From this it follows that, if $B(V) \subset \operatorname{Bir} V$ is a subgroup generated by all α_L and β_Y, then for $\chi\colon\ V \dashrightarrow V'$ there exists $\widetilde{\chi} \in B(V)$ such that $n(\chi \circ \chi) = 1$. But this is exactly the birational rigidity. In particular, $\operatorname{Bir} V$ is generated by $B(V)$ and $\operatorname{Aut} V$. $B(V)$ is, obviously, normal and $\operatorname{Aut} V$ is finite.

It remains to prove that the epimorphism $\varepsilon\colon\ G \operatorname{Aut} V \to \operatorname{Bir} V$ has a trivial kernel (i.e., it is an isomorphism). In other words, it remains to prove the completeness of the above list of relations. The key feature is the proof of (ii) and (iii) of Proposition 1: The untwisting process of a birational automorphism is always uniquely defined, except for the case where two maximal lines appear at once. But then they are united in a special conic, and this situation ruled by the relation from Lemma 1.3. Let us go to the formal construction.

4. Completeness of the Set of the Relations (*). We shall denote the words composed from the symbols A_L and B_Y by capital letters and their images in $\operatorname{Bir} V$ by the corresponding small letters. Obviously, the triviality of $\operatorname{Ker}\varepsilon$ means the following: let F be a word composed from A_L and B_Y. If $f = \varepsilon(F) \in \operatorname{Aut} V$, then, using (*), we can transform this word into an empty word (in particular, $f = \mathrm{id}_V$). We shall prove this statement.

To each word $F = T_1 \dots T_{l(F)}$ of length $l(f)$, where $T_i = A_L$ or B_Y, we relate the ordered triad of integer parameters. Let us denote by F_i, $i \leq l(F)$, the left segment of F of length i, $f_i = \varepsilon(F_i) \in \operatorname{Bir} V$. Let $\widetilde{n}(F) = \max\{n(f_i) | 1 \leq i \leq l(F)\}$, $\omega(F) = \#\{i | 1 \leq i \leq l(F),\ n(f_i) = \widetilde{n}(F)\}$. The mentioned triad is $(\widetilde{n}(F), \omega(F),\ l(F))$. We introduce on the set of words the relation $>$: $F > F'$ if and only if one of the following conditions holds: (1) $\widetilde{n}(F) > \widetilde{n}(F')$; (2) $\widetilde{n}(F) = \widetilde{n}(F')$, but $\omega(F) > \omega(F)'$; (3) $\widetilde{n}(f) = \widetilde{n}(F')$, $\omega(F) = \omega(F')$, but $l(F) > l(F')$ (the *lexicographic ordering*). It is easy to see that every chain $F^{(1)} > F^{(2)} > \dots > \cdot F^{(s)} > F^{(s+1)} > \dots$ stops after a finite number of steps. Thus, it is enough to prove that we can transform every nonempty word F such that $f \in \operatorname{Aut} V$, using the relations (*), into the word F', $F' < F$.

If the word F contains the subword $A_L A_L$ or $B_Y B_Y$, then, using the relation $A_L^2 = E$ or $B_Y^2 = E$, respectively, i.e., by eliminating the subword, we get a new word F^*, $\widetilde{n}(F^*) \leq \widetilde{n}(F)$, $\omega(F^*) \leq \omega(F)$, $l(F^*) < l(F)$ (because the image of every left segment from F^* in $\operatorname{Bir} V$ coincides with the image of some left segment of F and the map of the set of left segments of F^* into the set of left segments of F is injective). Thus, $F^* < F$.

We can assume now that F does not contain the subword $A_L A_L$ or $B_Y B_Y$.

Obviously $n(f) = 1$. If $\widetilde{n}(F) = 1$, then $l(F) = 0$ and F is empty. Assume that $\widetilde{n}(F) \geq 2$. Let $s = \min\{i | n(f_i) = \widetilde{n}(F)\} \leq l(F) - 1$. We shall study two cases separately: $T_s = A_L$ and $T_s = B_Y$.

Case 1. $T_s = B_Y$. We have $n(f_{s-1}) = n(f_s\beta_Y) < n(f_s)$ (because of the choice of s). By Corollary 1, $\nu_Y(f_s) > n(f_s)$. By Corollary 2 (ii), if $n(f_{s+1}) \leq n(f_s)$, then $T_{s+1} = T_s = B_Y$. Thus, F contains the subword $B_Y B_Y$. We have a contradiction.

Case 2. $T_s = A_L$. By Corollary 1 and by the choice of s we have $\nu_L(f_s) > n(f_s)$. By the assumption, $T_{s+1} \neq T_s$ and $n(f_{s+1}) \leq n(f_s)$. Thus, (Corollary 2 (ii)), $T_{s+1} = A_{L'}$, where $L \bigcup L'$ is a special reducible conic and $\nu_{L'}(f_s) \geq n(f_s)$. Let $Z \subset V$ be a line in the plane $\langle L \bigcup L' \rangle$, different from L and L'.

Lemma 2.

(i) *Z is the maximal line of the map f_{s-1}, i.e., $\nu_Z(f_{s-1}) > n(f_{s-1})$. Therefore, $n(f_{s-1}\alpha_Z) < n(f_{s-1})$.*

(ii) *The equality*

$$..(f_{s-1}\alpha_Z) - \nu_{L'}(f_{s-1}\alpha_Z) = n(f_s) - \nu_{L'}(f_s) \leq 0$$

holds. Thus, $n(f_{s-1}\alpha_Z\alpha_{L'}) \leq n(f_{s-1}\alpha_Z)$.

(iii) *Z is the maximal line of the map f_{s+1}, i.e., $\nu_Z(f_{s+1}) > n(f_{s+1})$. Therefore, $n(f_{s+1}\alpha_Z) < n(f_{s+1})$.*

Proof. A direct computation, using Lemma 1, gives

$$n(f_{s-1}) = n(f_s\alpha_L) = 4n(f_s) - 3\nu_L(f_s),$$

$$\nu_Z(f_{s-1}) = 2n(f_s) - 2\nu_L(f_s) + \nu_{L'}(f_s).$$

Thus,

$$n(f_{s-1}) - \nu_Z(f_{s-1}) = 2n(f_s) - \nu_L(f_s) - \nu_{L'}(f_s) < 0,$$

and (i) is proved.

We have

$$\nu_{L'}(f_{s-1}\alpha_Z) = 2n(f_{s-1}) - 2\nu_Z(f_{s-1}) + \nu_L(f_{s-1}) = 9n(f_s) - 6\nu(f_s) - 2\nu_{L'}(f_s)$$

and

$$n(f_{s-1}\alpha_Z) = 4n(f_{s-1}) - 3\nu_Z(f_{s-1}) = 10n(f_s) - 6\nu_L(f_s) - 3\nu_{L'}(f_s).$$

Thus,

$$n(f_{s-1}\alpha_Z) - \nu_{L'}(f_{s-1}\alpha_Z) = n(f_s) - \nu_{L'}(f_s) \leq 0.$$

Let us prove (iii). We have

$$n(f_{s+1}) = n(f_s\alpha_{L'}) = 4n(f_s) - 3\nu_{L'}(f_s),$$

$$\nu_Z(f_{s+1}) = 2n(f_s) - 2\nu_{L'}(f_s) + \nu_L(f_s).$$

Therefore,

$$n(f_{s+1}) - \nu_Z(f_{s+1}) = 2n(f_s) - \nu_{L'}(f_s) - \nu_L(f_s) < 0.$$

The lemma is proved.

Let, at first, $\nu_{L'}(f_s) > n(f_s)$. Let us substitute the subword $T_sT_{s+1} = A_LA_{L'}$ for the subword $A_ZA_{L'}A_LA_Z$ using the relations $A_Z^2 = e$ and $A_ZA_{L'}A_L = A_LA_{L'}A_Z$. The length of the word is greater now. Denote the new word by F^*. Let us prove that either $\tilde{n}(F^*) < \tilde{n}(F)$, or $\tilde{n}(F^*) = \tilde{n}(F)$ and $\omega(F^*) < \omega(F)$.

Obviously, $F_t^* = F_t$ when $t \leq s-1$, and

$$f_s^* = f_{s-1}\alpha_Z, \quad f_{s+1}^* = f_{s-1}\alpha_Z\alpha_{L'},$$

$$f_{s+2}^* = f_{s-1}\alpha_Z\alpha_{L'}\alpha_L = f_{s+1}\alpha_Z,$$

and $f_{s+t}^* = f_{s+t-2}$ when $t \geq 3$. From the above lemma, $n(f_t^*) < n(f_s) = \tilde{n}(F)$ when $t = s, s+1, s+2$. F_{t+s}^*, $t \geq 3$, i.e., the left segments of F^* have the same images in $\operatorname{Bir} V$ as the left segments of F_{t+s-2}. Thus, the maximal value of $n(f_s)$ is eliminated, but all other values remain. Therefore, if $\omega(F) \geq 2$, then $\tilde{n}(F^*) = \tilde{n}(F)$ and $\omega(F) - 1 = \omega(F^*)$. If $\omega(F) = 1$, then $\tilde{n}(F^*) < n(F)$. In any case, $F^* < F$.

Let us consider the last case: $\nu_{L'}(f_s) = n(f_s)$. We have

$$n(f_{s+1}) = n(f_s), \quad \nu_{L'}(f_{s+1}) = n(f_{s+1})$$

Using Lemma 2 (iii), Corollary 2 (ii), and the absence of the subwords $A_{L'}A_{L'}$, we have $T_{s+2} = A_Z$. Substitute now the subword $T_sT_{s+1}T_{s+2} = A_LA_{L'}A_Z$ of F for the subword $A_ZA_{L'}A_L$. We obtain a new word, which we shall denote by F^*. From Lemma 2 we have $n(f_t^*) < \tilde{n}(F)$, $t = s, s+1, s+2$; $f_t^* = f_t$, $t \geq s+3$; $l(F) = l(F^*)$. Hence

$$\#\{1 \leq t \leq l(F) \mid n(f_t^*) = \tilde{n}(F)\} = \#\{1 \leq t \leq l(F) \mid n(f_t) = \tilde{n}(F)\} - 1,$$

($\#\varnothing = 0$). Thus, as before, $F^* < F$.

Therefore, using the relations $(*)$, we can transform every word F of symbols A_L, B_Y such that $f \in \operatorname{Aut} V$, into the empty word. The theorem is proved.

3. The Maximal Cycles.

1. Define a birational automorphism $\chi\colon V \to V'$, $n(\chi) \geq 2$, when $|L' + K_{V'}| \neq \varnothing$ and $n(\chi)$ arbitrary, in the other case. Let $n = n(\chi)$, $\nu_B = \nu_B(\chi)$ for every $B \subset V$. We shall prove Proposition 2.1.

In Secs. 5 and 6 the key statement will be proved.

Proposition 1. *There exists a maximal cycle $B \subset V$.*

Now we shall give a description of the maximal cycle.

2. Lemma 1. *Let $B_1, \ldots, B_k$ be different curves on V such that $\nu_{B_i} \geq n$ for all i, $1 \leq i \leq k$, and $\nu_{B_1} > n$. Then $\sum\limits_{i=1}^{k} \deg B_i \leq 5$.*

Proof. This is a special case of Corollary 1.3.5.

Lemma 2. *Let $B \subset V$ be an irreducible curve that is not contained in any 3-plane. Then $\nu_B \leq n$.*

Proof. From the above lemma it is enough to consider the cases $\deg B = 4, 5$. We have the following possibilities:

(A) B is a rational normal curve of degree 4 in some 4-plane;

(B) B is a rational normal curve of degree 5 in $\mathbb{P}^5$;

(C) B is a rational curve of degree 5 in some 4-plane, and it may have the only double point;

(D) B is a smooth elliptic curve of degree 5 in some 4-plane.

Assume, at first, that B is smooth. Let $d = \deg B$ and B be sectioned by hypersurfaces of degree m (obviously, $m \geq d$). Let $\sigma\colon \widetilde{V} \to V$ be a blowing of B, $E = \sigma^{-1}(B)$ be an exceptional divisor, $\nu = \nu_B$. Obviously, the linear system $|mh - e|$ on $\widetilde{V}$ is free and the linear system $|nh - \nu e|$ is not stationary. Thus, $\big((nh - \nu e)^2 \cdot (mh - e)\big) \geq 0$. Computing this product we have the inequality

$$6mn^2 - dm\nu^2 - 2nd\nu - \nu^2(2 - 2g - d) \geq 0,$$

where $g = 0$ or 1 is a genus of the curve B. It is easy to see that if the sum of the coefficients of the above polynomial from n and ν is positive, then $\nu \leq n$, i.e.,

$$m(6 - d) \leq d + 2 - 2g. \qquad (*)$$

If $d = 4$, then we can set $m = 2$ (a rational curve is sectioned by quadrics) and $(*)$ is true. If $d = 5$, then let $m = d \leq d + (2 - 2g)$.

Now let B have a double point x. Blowing it we have: $\sigma_1\colon \widetilde{V}_1 \to V$, $E' = \sigma_1^{-1}(x)$. Then we blow the smooth proper inverse image of B on $\widetilde{V}_1$: $\sigma_2\colon \widetilde{V} \to \widetilde{V}_1$, $E = \sigma_2^{-1}(\widetilde{B})$. If B is sectioned by hypersurfaces of degree m ($m \leq d = 5$), then the system $|mh - 2e' - e|$ is free and the system $|nh - \nu' e' - \nu e|$, $\nu' = \nu_x \geq \nu$, is not stationary. Therefore,

$$\big((nh - \nu' e' - \nu e) \cdot (mh - 2e' - e)\big) \geq 0$$

or

$$6mn^2 - 10n\nu - (5m - 7)\nu^2 - 2(\nu - \nu')^2 \geq 0.$$

From this it follows that $\nu \leq n$ if $m \leq 3$. It is easy to check that B is sectioned by quadrics. The lemma is proved.

Lemma 3. *Let $B = B_1 \bigcap B_2$ be a reducible curve, B_i be irreducible curves and the dimension of the linear hull of B be not less than 4. Let two cases be possible*: (i) $\deg B_i = 2$, $i = 1, 2$; (ii) $\deg B_i = i$, $i = 1, 2$. *Then, if one of two numbers ν_{B_i}, $i = 1, 2$, is strictly greater than n, then the other is strictly less than n.*

Proof. We shall consider case (i), and we shall leave case (ii) to the reader. Let P be the linear hull of B_1. From Lemma 5.3, which will be proved in what follows, a section of V by a general hyperplane $H \supset P$ is a nonsingular surface. Taking into account that H has a nonempty intersection with B_2 and the point $x_H \notin P$ is one of the intersection points (any point in B_2 is the point x_H for some H) we have that the restriction of $|\chi|$ to $V \bigcap H$ has a $\widetilde{\nu}_1 = \nu_{B_1}$-multiple component B_1, a $\widetilde{\nu}_2 = \nu_{B_2}$-multiple base point x_H, and, if B_1 is a special conic, then this restriction may have a ν-multiple component, the line $L(B_1)$. Computing a self-intersection number of the nonstationary part of the system $|\chi|\,|_{H\bigcap V}$, we have

$$6n^2 - 4n\widetilde{\nu}_1 - 2n\widetilde{\nu} - 2\nu_1^2 - 2\nu^2 + 4\nu\widetilde{\nu}_1 - \widetilde{\nu}_2^2 \geq 0$$

(if B is nonspecial, then set $\nu = 0$). The maximum of this quadratic form by ν is attained when $\nu = \widetilde{\nu}_1 - n/2$ and is equal to $13/2n^2 - 6n\widetilde{\nu}_1 - \widetilde{\nu}_2^2 \geq 0$. Exchanging B_1 and B_2 we have $13/2n^2 - 6n\widetilde{\nu}_2 - \widetilde{\nu}_1^2 \geq 0$. Case (i) is a consequence of this fact.

Lemma 4. *Let P be a plane such that $P\bigcap V = B$, let B be a curve of degree not less than 2, and $B = \bigcup_{i=1}^{k} B_i$ be its decomposition into irreducible components (which may be multiple). Then*

$$\sum_{i=1}^{k} \deg B_i \nu_{B_i} \leq n \deg B.$$

Proof. Let $S \supset P$ be a general 3-plane.

Case 1. $P \subset Q$, $\deg B = 3$. We have $S\bigcap Q = P\bigcup P(S)$, $P(S)\bigcap \widetilde{Q}$ is an irreducible cubic, and $\# B\bigcap(P(S)\bigcap\widetilde{Q} = 3$. Let us restrict the linear system $|\chi|$ to $P(S)\bigcap\widetilde{Q}$: it is a linear series of degree $3n$ and its base set has degree $\sum_{i=1}^{k} \deg B_i\nu_{B_i}$. From this follows the proof.

Case 2. $P \not\subset Q$, $B = Q\bigcap P$ is a conic. Then $S\bigcap Q = Q_S$ is an irreducible quadric and $B = P\bigcap Q_S$ is its plane section. Let us restrict $|\chi|$ to the residual curve C, $C\bigcup B = Q_S\bigcap\widetilde{Q}$: we have a linear series of degree $n\deg C$, and its base set has degree $\sum_{i=1}^{k} \#(B_i\bigcap C)\nu_{B_i}$. Since B is an irreducible conic or a pair of lines, the last expression is not less than $1/2\deg C \sum_{i=1}^{k} \deg B_i\nu_{B_i}$. From this follows the proof.

Lemma 5. *Let $x \in V$ be a point such that either there are no lines on V which contain x or there are 6 lines on V containing x. Then $\nu_x \leq 2n$.*

Proof. If there are no lines on V containing x, then, blowing x, we have $\sigma\colon\ \widetilde{V} \to V$, $E = \sigma^{-1}(x)$. Consider the number class $(2h - 3e)$ on $\widetilde{V}$. This class is nonnegative. Indeed, $2h - 3e = (h - e) + (h - 2e)$, $(h-e)$ is nonnegative, and $(h-2e)$ is represented by the proper inverse image of the tangent sheaf to V at x. Thus, the nonnegativity may be violated only with respect to the proper inverse images of the components of $V\bigcap T_xV$. But the multiplicity of each component $V\bigcap T_xV$ in x is not greater than two thirds of its degree (because there are no lines among the components), and so the class $(2h - 3e)$ is nonnegative. Now, $\big((nh - \nu_x e)^2 \cdot (2h - 3e)\big) = 12n^2 - 3\nu_x^2 \geq 0$. Therefore, $\nu_x \leq 2n$.

If there are no less than 5 lines on V containing x, then there are exactly 6 such lines ($V\bigcap T_xV$ is a curve of degree 6 which has 5 lines as components). Blow x. Using the previous notation, let L_i be the lines, $1 \leq i \leq 6$, $\widetilde{L_i}$ be their proper inverse images on $\widetilde{V}$.

Let $H \subset V$ be a section of V by a general hyperplane containing T_xV, $\widetilde{H}$ be its proper inverse image on $\widetilde{V}$. Since $\widetilde{L_i} \subset H$, we have $\widetilde{L}_i \subset \widetilde{H}$ and $\widetilde{H}\bigcap E \ni \widetilde{L}_i\bigcap E$. But $\widetilde{H}\bigcap E$ is a conic, and so 6 points $\widetilde{L}_i\bigcap E$ define it uniquely. Therefore, for a general H, the conic $\widetilde{H}\bigcap E$ is independent of H (if there are multiple components of $V\bigcap T_xV$, then there are infinitely close lines L_i and points $\widetilde{L}_i\bigcap E$, but, in this case, the conic $\widetilde{H}\bigcap E$ is also defined uniquely). H belongs to the sheaf and, in this sheaf, there exists a hyperplane H^* such that the degree of the plane curve $\widetilde{H}^*\bigcap E$ is not less than 3, i.e., $\operatorname{mult}_x H^* = \mu \geq 3$. In other words, the system $|h - \mu e|$, $\mu \geq 3$, is nonempty. Hence

$$\big((nh - \nu_x e)\cdot(h - \mu e)\cdot(h - e)\big) = 6n - \mu\nu_x \geq 0.$$

Thus, $\nu_x \leq 2n$. The lemma is proved.

3. Let us now formulate the statement that plays a crucial role in the description of maximal cycles.

Lemma 6 (*the plane section lemma*).
Let $S \subset \mathbb{P}^5$ be an arbitrary 3-plane and

$$V \bigcap S = \sum_{i=1}^{N} m_i C_i, \qquad \Theta_i^* = \nu_{C_i}.$$

Then

(i) *if $\Theta_i^* > n$, then C_i is a line or a special conic;*
(ii) *if $\Theta_i^* + \Theta_j^* > 2n$, $i \neq j$, then $C_i \bigcup C_j$ is a special reducible conic;*
(iii) *$\Theta_i^* + \Theta_j^* + \Theta_k^* \leq 3$ for all triads of different indices $1 \leq i, j, k \leq r$.*

If $S = T_x V$ for some point x and not all curves C_i which pass through x are lines, then $\nu_x \leq 2n$.

The proof will be given in the next section.

Now, let us prove Proposition 2.1.

Since the maximal cycles exist and the maximal curves are lines and special conics (Lemmas 1–3 and 6), (i) is proved.

Let us prove (ii) and (iii). The situation $\nu_{Y_1} > n$, $\nu_{Y_2} \geq n$ for different special conics is impossible: if $\dim \langle Y_1 \bigcup Y_2 \rangle \geq 4$, then, by Lemma 3; if $\dim \langle Y_1 \bigcup Y_2 \rangle \geq 3$, then by the plane section lemma. The situation $\nu_Y > n$, $\nu_L \geq n$ (or $\nu_Y \geq n$, $\nu_L > n$) is also impossible: if $\dim \langle Y \bigcup L \rangle \geq 4$, then by Lemma 3, if $\dim \langle Y \bigcup L \rangle \leq 3$, then by Lemma 6, and if $L = L(Y)$, then by Lemma 4. Since two lines always lie in some 3-plane, the plane section lemma completes the proof.

4. The Proof of the Plane Section Lemma.

1. We shall use the notation of Lemma 3.6. Let Q_S and $\widetilde{Q}_S$ be the restrictions of the quadric Q and the cubic $\widetilde{Q}$ to the 3-plane S.

Definition 1. A point $x \in S \bigcap V$ is called a *degeneration point* if

$$\min(\operatorname{mult}_x Q_S, \operatorname{mult}_x \widetilde{Q}_S) = 2.$$

This is equivalent to the relation $S = T_x V$.

Lemma 1. *There are only a finite number of degeneration points (when S is fixed). Precisely, if Q_S is a smooth quadric, then there are no such points, if Q_S is an irreducible cone, then there are one or none (and if there is one, then this point is the apex), if Q_S is a pair of planes, then there are two or less (and they are in the intersection line).*

Proof. The point x is a degeneration point if and only if for all hyperplanes $H \supset S$, $\operatorname{mult}_x(H \bigcap V) \geq 2$. Thus, the only nontrivial moment in the proof is the verification of the inequality $\dim \operatorname{Sing}(H \bigcap V) \leq 0$ for all hyperplanes. But this is a consequence of a more general statement.

Lemma 2. *Let a manifold $X \subset \mathbb{P}^M$ be a complete intersection of hypersurfaces, and $H \subset \mathbb{P}^M$ be a hyperplane. Then*

$$\dim \operatorname{Sing}(X \cap H) \leq \dim \operatorname{Sing} X + 1.$$

Proof. Let $X = \bigcap_{i=1}^{N} F_i$, $\deg F_i = k_i \geq 2$. The case $N = 1$ was studied above (Lemma 2.1.1). Let $N \geq 2$. We follow the exposition in the case $N = 1$. Let us note that we can set $\operatorname{codim}_x \operatorname{Sing} X \geq 3$ (in the other case, there is nothing to prove). By taking the section of X by a general $(M - \dim \operatorname{Sing} X - 1)$-plane in $\mathbb{P}^M$ we reduce the proof to the case of smooth complete intersection.

We shall prove that a hyperplane section of the smooth complete intersection $X = \bigcap_{i=1}^{N} F_i \subset \mathbb{P}^M$, $\deg F_i = k_i \geq 2$, may have only zero-dimensional singularities. Let H be defined by the equation $x_M = 0$, where $(x_0 : x_1 : \cdots : x_M)$ are uniform coordinates on $\mathbb{P}^M$, and let F_i be defined by the equation $f_i(x) = 0$, $\deg(f_i) = k_i$, $1 \leq i \leq N$. A point $x \in X \bigcap H$ is singular in $X \bigcap H$ if and only if the rank of the matrix

$J(x) = \left\| \dfrac{\partial f_i(x)}{\partial x_j} \right\|$, $1 \le i \le N$, $0 \le j \le M-1$, is less than N. Assume that there exists a curve $Y \subset X \bigcap H$ such that $Y \subset \operatorname{Sing}(X \bigcap H)$, i.e.,

$$Y \subset \{x \,|\, x \in X \bigcap H, \ \ \mathrm{rk}\ J(x) \le N-1\}.$$

We shall look for a point $z \in Y$ such that the rank of the extended matrix $J^*(x) = \left\| \dfrac{\partial f_i}{\partial x_j} \right\| (z)$, $1 \le i \le N$, $0 \le j \le M$, is also less than N. That will give us a contradiction.

If there is a point $z \in Y$ such that $\mathrm{rk}\, J(z) \le N-2$, then this is the desired point: $\mathrm{rk}\, J^*(z) \le \mathrm{rk}\, J(z) + 1 \le N-1$.

Thus, assume that $\mathrm{rk}\, J(x) = N-1$ for all $x \in Y$. This means that there is only one relation (up to proportionality) between the rows of J. Consider the map

$$\left(\bigoplus_{i=1}^{N} \mathcal{O}(1-k_i) \right) \times \left(\bigoplus_{i=1}^{N} \mathcal{O}(k_i - 1) \right) \xrightarrow{\mu} \mathcal{O} \qquad (*)$$

which maps from the pair of the sets of local sections

$$((s_1, \dots, s_N), (s_1^*, \dots, s_N^*))$$

into the function $\sum_{i=1}^{N} s_i s_i^*$ (we use the natural isomorphism $\mathcal{O}(-k) \otimes \mathcal{O}(k) \underset{\sim}{\to} \mathcal{O}$). The condition on the rank of J means that the set $\{\sigma_j\}$ of sections of the sheaf

$$\bigoplus_{i=1}^{N} \mathcal{O}(k_i - 1),$$

$\sigma_j = \left(\dfrac{\partial f_1}{\partial x_j}, \dots, \dfrac{\partial f_N}{\partial x_j} \right)$, $0 \le j \le M-1$, has rank $N-1$ everywhere on Y. Therefore, the subsheaf $\mathcal{L}$ of the sheaf $\bigoplus_{i=1}^{N} \mathcal{O}(1-k_i)$ which is defined by the relation $\mathcal{L} = \bigcap_{i=1}^{M-1} \operatorname{Ker} \mu(-, \sigma_j)$ is invertible on Y. Consider the section $\sigma_M = \left(\dfrac{\partial f_1}{\partial x_M}, \dots, \dfrac{\partial f_N}{\partial x_M} \right)$. Obviously it defines the injective map of sheaves

$$\mathcal{L} \xhookrightarrow{\mu(-,\sigma_M)} \mathcal{O}.$$

Now, if $\operatorname{codim} \operatorname{Supp} \mathcal{O}/_{\mu(-\sigma_M)(\mathcal{L})} \ne 0$, then the points in Y where the product by σ_M is zero are the singularities of X; we have a contradiction. Thus, the product by σ_M is an isomorphism of the sheaves $\mathcal{L}$ and $\mathcal{O}$ on Y, i.e., $\mathcal{L}_Y$ has a global nonzero section. But $\mathcal{L}_Y$ is a subsheaf of $\bigoplus_{i=1}^{N} \mathcal{O}_Y(1-k_i)$, $k_i \ge 2$. We have a contradiction. Lemma 2 is proved.

For the proof of the last statement of Lemma 1 let us note that, if Q_S is a pair of different planes, then their intersection line cannot be contained in the singular set of the cubic surface $\widetilde{Q}_S$. So there are not more than two points of $\widetilde{Q}_S$ on this line.

Remark. The quadric Q_S cannot be a double plane, because we assumed that Q is nonsingular.

Let $x \in S \bigcap V$ be a degeneration point and $\sigma\colon\ X \to \mathbb{P}^5$ be its blowing, S_x and V_x be the proper inverse images of S and V in X, $E = \sigma^{-1}(x)$. Then $S_x \bigcap V_x$ contains the plane $E \bigcap V_x = E \bigcap S_x$. Let $\widetilde{\sigma}\colon\ \widetilde{X} \to X$ be its blowing, $\widetilde{S}$ and $\widetilde{V}$ be the proper inverse images of S_x and V_x in $\widetilde{X}$. Then $\widetilde{\sigma}\colon\ \widetilde{V} \to V_x$ and $\widetilde{\sigma}\colon\ \widetilde{S} \to S_x$ are isomorphisms and $\widetilde{S} \bigcap \widetilde{V}$ is a curve. Let us denote the proper inverse images of Q_S and $\widetilde{Q}_S$ in $\widetilde{X}$ by Q_x

and $\widetilde{Q}_x$, respectively. This procedure, i.e., the blowing of a degeneration point x and then the blowing of the exceptional plane $E\bigcap S_x$, is called the *double blowing* of a degeneration point.

Let us note that if Q_S is an irreducible cone, then Q_x is a nonsingular surface. If Q_S is a pair of planes, then it is easy to see that the singular set of components of the curve $\widetilde{S}\bigcap\widetilde{V}$ does not intersect the singular set of the surface Q_x.

2. The Scheme of Proof of the Plane Section Lemma. The main difficulty here is that we have to study too many configurations of the curve-components $S\bigcap V$. But the investigation of all these cases may be done in a uniform way. Therefore, we shall not make the tiresome study of all possibilities, but we shall describe a general scheme of investigating a 3-plane section. We shall demonstrate the effectiveness of this scheme on some typical examples, which constitute the greater part of all possibilities.

The most natural way of reasoning is (it seems at first) to section S by a general hyperplane and restrict the linear system $|\chi|$ to this section, i.e., to a K3-surface. It is clear that the stationary components of this restriction may be only among the components of the curve $S\bigcap V$. Taking the intersection of the nonstationary part of this linear system of curves with the components of the 3-plane section, we have a system of linear inequalities. Thus, we have the estimations for the multiplicities of the stationary components (i.e., C_i). But it is very difficult to proceed in this way. The main problem is that a general hyperplane section containing $S\bigcap V$ may have singularities (and it will have them if $S\bigcap V$ has a degeneration point). For the resolution of these singularities we have to make many blowings, and the task becomes too complex. We shall proceed in another way, which is a formal analog of the above one. Here, on the one hand, we eliminate all problems with singularities, and on the other hand, the scheme becomes completely automatic (algorithmic) and allows one monotypically to prove all statements of the plane section lemma. This construction (to be described further) has, probably, a general character. It effectively describes the maximal cycles on a 3-dimensional quartic with a double point (Chapter 5) and effectively eliminates the maximal cycles on a 4-dimensional Fano manifold of degree 8 and index 1, i.e., on a complete intersection of a quadric and a quartic in $\mathbb{P}^6$ (unpublished).

Step 1. Let us construct a special sequence of blowings $\pi_j\colon\ X_j \to X_{j-1}$, $X_0 = \mathbb{P}^5$, where Y_{j-1} is the center and Z_j is the exceptional divisor of π_j (the notations S^j, $V^j\,Q^j$, $\widetilde{Q}^j$ will be used as above).

At first, we proceed with the blowings of the first group: they are π_j, $j \in \mathcal{K}_1 = \{j|1 \le j \le k_1\}$, where $0 \le k_1 \le 2$. If $k_1 = 0$, then $\mathcal{K}_1 = \varnothing$. Namely, if Q_S is nonsingular, then $k_1 = 0$; if Q_S is a cone with x as an apex and x is not a degeneration point, then $k_1 = 1$, $Y_0 = x$, π_1 is the blowing of x; if Q_S is a cone with x as an apex and x is a degeneration point, then $k_1 = 2$, $Y_0 = x$, $Y_1 = Z_1\bigcap S^1 = Z_1\bigcap V^1$, i.e., the pair $\{\pi_1, \pi_2\}$ is a double blowing of x; if Q_S is reducible, then $k_1 = 0$ if there are no degeneration points, and $k_1 = 2$ otherwise (here $\{\pi_1, \pi_2\}$ is a double blowing of any degeneration point as above).

Then there is the sequence of the blowings of the second group: π_j, $j \in \mathcal{K}_2$, $\mathcal{K}_2 = \{j|k_1 + 1 \le j \le k_2\}$. Here Y_{j-1} are points and

$$\min\left(\operatorname{mult}_{Y_{j-1}} Q_S^{j-1}, \operatorname{mult}_{Y_{j-1}} \widetilde{Q}_S^{j-1}\right) = 1$$

and the curve $S^{k_2}\bigcap V^{k_2}$ has only nonsingular components (we can always achieve it).

Then there is the sequence of the blowings of the third and fourth groups: $j \in \mathcal{K}_3\bigcup\mathcal{K}_4$, where $\mathcal{K}_3 = \{j|k_2 + 1 \le j \le k_3\}$, $\mathcal{K}_4 = \{j|k_3 + 1 \le j \le k\}$. Here all Y_{j-1} are curve-components of $S^{j-1}\bigcap V^{j-1}$, $j \in \mathcal{K}_4$ only when $\mathcal{K}_2 \ne \varnothing$ and for some $j \in \mathcal{K}_2$, $\pi_{j-1,i}(Y_{j-1})$ is a curve in the exceptional plane $Z_i\bigcap S^i$ (as usual, $\pi_{j,i}\colon\ X_j \to X_i$ is the composition of the blowings π_t).

After all these blowings have been done, we have $S^k\bigcap V^k = \varnothing$.

This construction is called the *plane section resolution.* Its existence is obvious.

Let us introduce new notations. Let $j = \alpha(i)$, $1 \le i \le r$, if the resolution π_j blows the proper inverse image of C_i^{j-1} (in particular, $\pi_{j-1,0}^{-1}$ is an isomorphism in a neighborhood of a general point in C_i).

Set also $Z_j\bigcap V^j = E_j^*$. If $k_1 = 2$, then, obviously, $E_1^* = E_2^*$. Set $Z_j\bigcap S^j = \widetilde{E}_j$. Note that the canonical class of V^j is

$$-h+2\sum_{j=K_2} e_j^* + \sum_{j=K_3\bigcup K_4} e_j^*, \quad \text{if } k_1=0,$$

$$-h+2\sum_{j=K_1\bigcup K_2} e_j^* + \sum_{j=K_3\bigcup K_4} e_j^*, \quad \text{if } k_1=1,$$

$$-h+2\sum_{j=K_2} e_j^* + \sum_{j=K_1\bigcup K_3\bigcup K_4} e_j^*, \quad \text{if } k_1=2$$

(recall that, in the last case, $e_1^* = e_2^*$).

Step 2. Obviously, the linear system $\left|h-\sum_{j=1}^k e_j^*\right|$ is a free sheaf over V^k. Thus, $\left(h-\sum_{j=1}^k e_j^*\right)^2 = 0$. Let us define the bilinear integer form $\langle\cdot\rangle$ on $A^1(V^k)$ by the following relation:

$$\langle x\cdot y\rangle = \left(x\cdot y\cdot\left(h-\sum_{j=1}^k e_j^*\right)\right)\in\mathbb{Z}$$

(the last product is the standard product on $A^1(V^k)$). This form realizes the idea of the restriction to a general hyperplane section which contains $S\bigcap V$. It is clear that in terms of the form $\langle\cdot\rangle$, the class h is equivalent to $\sum_{j=1}^k e_j^*$.

Consider the proper inverse image of the linear system $|\chi|$ in V^k. The class of its general divisor is $\zeta^* = nh-\sum_{j=1}^k \nu_j^* e_j^*$, where $\nu_j^* = \operatorname{mult}_{Y_{j-1}}|\chi|^{j-1}$. Note that $\nu_j^*=\theta_i$, when $j=\alpha(i)$. We are interested in exactly those coefficients. If $k_1=2$, i.e., if Y_0 is a degeneration point x, then we set $\nu_2^*=0$ ($e_1^*=e_2^*$), $\nu_1^*=\nu_x$.

Let us define the set $\mathcal{K}^*$: $\mathcal{K}^*=\{j|1\le j\le k\}$ if $k_1\ge 1$, and $\mathcal{K}^*=\{j|1\le j\le k\}\setminus\{2\}$ if $k_1=2$.

Consider the free Abelian group $R=\bigoplus_{i\in K^*}\mathbb{Z}e_i^*$. Any $z\in A^1(V^k)$ is numerically equivalent (in the sense of $\langle\cdot\rangle$) to some element of R. In particular, the class ζ^* is equivalent to the element $\zeta=\sum_{i\in K^*}\theta_i e_i^*$, where $\theta_i = n-\nu_i^*$ if $i\ge 3$ or $i\ge 1$ and $k_1\le 1$, and $\theta_1=2n-\nu_1^*=2n-\nu_x$ if $k_1=2$.

The statement of the plane section lemma can be reformulated, in these notations, into a statement about the nonnegativity of θ_i and their sums: $\theta_i^*>n$ is equivalent to $\theta_{\alpha(i)}<0$, $\theta_i^*+\theta_j^*>2n$ is equivalent to $\theta_{\alpha(i)}+\theta_{\alpha(j)}<0$, $\nu_x>2n$ is equivalent to $\theta_1<0$, if x is a degeneration point, and so on.

Since the linear system $|\chi|^k$ is nonstationary and the system $\left|h-\sum_{j=1}^k e_j^*\right|$ is free, we have $\langle\zeta\cdot y\rangle\ge 0$ for any class $y\in A^1(V^k)$ which represents an effective divisor. If we take the classes e_j^*, $1\le j\le k$, as y, then we have the system of linear inequalities

$$\sum_{i\in K^*}\langle e_i^*\cdot e_j^*\rangle > \theta_i\ge 0, \quad j\in\mathcal{K}^*.$$

We shall describe now how to deduce from this system the consequences of the type $\theta_i\ge 0$.

Step 3. The computation of values of the form $\langle\cdot\rangle$ on the base $\{e_i^*|i\in\mathcal{K}^*\}$. The main idea is that the values of the form $\langle\cdot\rangle$, which was constructed using the multiplication rules in $A(V^k)$, may be calculated in a simple way using the data on blown cycles Y_{j-1} in S^{j-1}.

Let, at first, $i<j$, $\{i,j\}\subset\mathcal{K}^*$. Then

$$\langle e_i^*\cdot e_j^*\rangle = (z_i\cdot y_{j-1}) = (\tilde{e}_i\cdot y_{j-1})$$

(the second product is calculated in $A(X_k)$, the third is calculated in $A(S^k)$). Indeed,

$$\langle e_i^* \cdot e_j^* \rangle = \left(\left(h - \sum_{t=1}^{k} e_t^* \right) \cdot e_i^* \cdot e_j^* \right) = (h \cdot e_i^* \cdot e_j^*) - \left(\sum_{t=1}^{k} (e_t^* \cdot e_i^* \cdot e_j^*) \right).$$

Obviously, the first term is zero as are all members of sum, except for one: when $t = j$. Now everything is obvious.

If $i \in \mathcal{K}_2$ or $i = 1$ and $k_1 = 1$, then $\langle e_i^* \cdot e_j^* \rangle = -1$. If $k_1 = 2$, then $\langle e_1^* \cdot e_1^* \rangle = -2$.

If Y_{i-1} is a curve, then the standard computations give

$$\langle e_i^* \cdot e_i^* \rangle = 2p_a(Y_{i-1}) - 2 - \sum_{j \in K_2} (e_j^* \cdot y_{i-1}) - \delta_{k_1,1}(e_1^* \cdot y_{i-1}),$$

where $\delta_{k_1,1}$ is the Kronecker symbol. Now we shall simplify these computations.

Step 4. The reduction (in the case $\mathcal{K}_2 \neq \varnothing$). Let the map p: $R \to R$ be defined by the following relation:

$$p(\varepsilon) = \varepsilon + \sum_{i \in K_2} \langle \varepsilon \cdot e_i^* \rangle \, e_i^*.$$

Obviously, $\langle p(\varepsilon) \cdot e_i^* \rangle = 0$ if $i \in \mathcal{K}_2$. It is easy to prove that also $\langle p(\varepsilon) \cdot e_i^* \rangle = 0$ if $i \in \mathcal{K}_4$. Let $\mathcal{K} = \mathcal{K}^* \setminus (\mathcal{K}_2 \bigcup \mathcal{K}_4)$, $c_i^* = p(e_i^*)$, $i \in \mathcal{K}$,

$$T = \bigoplus_{i = K_2} \mathbb{Z} c_i^*, \quad R = T \bigoplus \left(\bigoplus_{i \in K_2 \bigcup K_4} \mathbb{Z} e_i^* \right).$$

Let $\tilde{p}$ be the projection on the first term of this sum, $\widetilde{\zeta} = \tilde{p}(\zeta) = \sum_{j \in K} \theta_j c_j^*$. It is obvious that all coefficients θ_j that are interesting to us (i.e., $j = \alpha(i)$, $1 \leq i \leq r$) have the index $j \in \mathcal{K}$. It is easy to check that $\left\langle \widetilde{\zeta} \cdot c_j^* \right\rangle \geq 0$ if $j \in \mathcal{K}$.

Let us compute the multiplication table in T. We have

$$\langle p(\varepsilon_1) \cdot p(\varepsilon_2) \rangle = \langle \varepsilon_1 \cdot \varepsilon_2 \rangle + \sum_{i \in K_2} \langle \varepsilon_1 \cdot e_i^* \rangle \langle \varepsilon_2 \cdot e_i^* \rangle .$$

It is an essential simplification, as we shall see. Taking into account the fact that every curve $Y_{i,j}$, $j \geq k_1$, lies on a nonsingular surface, and using the standard formulas for changing the arithmetical genus of a curve after a blowing, we get

$$\langle c_i^* \cdot c_i^* \rangle = 2p_a(Y_{i-1,k_1}) - 2 - \delta_{k_1,1}(e_1^* \cdot y_{i-1}) \ \text{ if } \ i \in \mathcal{K}_3;$$
$$\langle c_i^* \cdot c_i^* \rangle = -k_1, \ \text{ if } \ i \in \mathcal{K}_1 \bigcap \mathcal{K};$$
$$\langle c_i^* \cdot c_1^* \rangle = (e_1^* \cdot y_{i-1}), \ \text{ if } \ i \in \mathcal{K}_3, \ 1 \in \mathcal{K}_1;$$
$$\left\langle c_i^* \cdot c_j^* \right\rangle = (\tilde{e}_i \cdot y_{j-1}) + \sum_{t \in K_2} (y_{i-1} \cdot e_t^*)(y_{j-1} \cdot e_t^*), \ \text{ if } \ \{i, j\} \subset \mathcal{K}_3, \ i < j.$$

In particular, if Y_{i-1,k_1} and Y_{j-1,k_1} are curves on a nonsingular surface (this is so, for example, if the quadric Q_S is irreducible), then $\left\langle c_i^* \cdot c_j^* \right\rangle = (y_{i-1,k_1} \cdot y_{j-1,k_1})$ (the intersection number is calculated with respect to this surface).

To deduce the conditions, we need the system of the linear inequalities $\left\langle \widetilde{\zeta} \cdot c_i^* \right\rangle \geq 0$, $i \in \mathcal{K}$, and the nondegeneracy of the form $\langle \cdot \rangle$ on T (in all cases under consideration). In the assumption of nondegeneracy of the matrix $\| \left\langle c_i^* \cdot c_j^* \right\rangle \|$, let $\Xi = \|\theta_{ij}\|$ be an inverse matrix. Then

$$\theta_j = \sum_{t \in K} \theta_{tj} \left\langle \widetilde{\zeta} \cdot c_t^* \right\rangle .$$

Also

$$\theta_i + \theta_j = \sum_{t \in K} (\theta_{ti} + \theta_{tj}) \left\langle \tilde{\zeta} \cdot c_t^* \right\rangle .$$

Since $\left\langle \tilde{\zeta} \cdot c_t^* \right\rangle \geq 0$, in order to prove the inequality $\theta_j \geq 0$, it is enough to verify that all members in the jth row (column) of Ξ are nonnegative; to prove the inequality $\theta_i + \theta_j \geq 0$ it is enough to verify that all members in the sum of the ith and jth rows of Ξ are nonnegative, and so on.

Thus, the scheme of investigation of a 3-plane section is as follows: We construct the resolution of a plane section and compute the form $\langle \cdot \rangle$, then we simplify the form $\langle \cdot \rangle$ by restricting it to the proper subspace, and, finally, we find the matrix inverse to the matrix that defines $\langle \cdot \rangle$. After that, the verification of the conditions of the plane section lemma is reduced to the checking of the nonnegativity conditions for rows and sums of rows of this matrix.

Remark. This scheme is also effective for manifolds V_6^3 which do not satisfy the generality conditions from 1.1. But then the computations become very complex; the form $\langle \cdot \rangle$ has degenerations, and we have to work with them in a special way. Thus, additional difficulties arise. Working with them adds nothing new to our understanding of the situation. Hence we restrict our study to the general case.

Let us demonstrate the effectiveness of our method.

3. Case 1. Q_S is a nonsingular quadric. Here there are no degeneration points, $\mathcal{K}_1 = \varnothing$, $k_1 = 0$. For every $j \in \mathcal{K}_3$, $Y_{j-1,0}$ is a curve on the quadric $Q_S \cong \mathbb{P}^1 \times \mathbb{P}^1$. We have $\langle c_i^* \cdot c_j^* \rangle = (y_{i-1,0} \cdot y_{j-1,0})$ (intersection on Q_S) and $\langle c_i^* \cdot c_i^* \rangle = 2p_a(Y_{i-1,0}) - 2$, $i, j \in \mathcal{K}_3$. If $i \in \mathcal{K}_3$, then let (α_i, β_i) be the type of the curve $Y_{i-1,0}$ in $Q_S \cong \mathbb{P}^1 \times \mathbb{P}^1$. We have

$$\langle c_i^* \cdot c_j^* \rangle = \alpha_i \beta_j + \alpha_j \beta_i - 2\delta_{ij}(\alpha_i + \beta_i),$$

where δ_{ij} is the Kronecker symbol.

Lemma 1. *The form $\langle \cdot \rangle$ is nondegenerate on $T = \bigoplus_{i \in K = K_3} \mathbb{Z} c_i^*$.*

Proof. Let $\varepsilon = \sum_{i \in K} \varepsilon_i c_i^* \in T^\perp$. Set

$$\alpha(\varepsilon) = \sum_{i \in K} \varepsilon_i \alpha_i, \quad \beta(\varepsilon) = \sum_{i \in K} \varepsilon_i \beta_i.$$

Now

$$\langle \varepsilon \cdot c_j^* \rangle = \beta_j \alpha(\varepsilon) + \alpha_j \beta(\varepsilon) - 2(\alpha_j + \beta_j)\varepsilon_j = 0.$$

Summing up by all $j \in \mathcal{K}$ and taking into account the fact that

$$\sum_{i \in K} \alpha_i = \sum_{i \in K} \beta_i = 3,$$

we have $\alpha(\varepsilon) + \beta(\varepsilon) = 0$. Therefore,

$$\varepsilon_j = \frac{\beta_j - \alpha_j}{2(\beta_j + \alpha_j)} \alpha(\varepsilon)$$

and

$$\alpha(\varepsilon) = \left(\sum_{j \in K} \frac{\alpha_j(\beta_j - \alpha_j)}{2(\beta_j + \alpha_j)} \right) \alpha(\varepsilon).$$

After easy transformations we have

$$\left(\frac{1}{2} - \sum_{j \in K} \frac{\alpha_j^2}{\alpha_j + \beta_j} \right) \alpha(\varepsilon) = 0.$$

Since

$$\frac{\alpha_j^2}{\alpha_j+\beta_j} \geq \frac{\alpha_j}{4}$$

we have

$$\sum_{j\in K} \frac{\alpha_j^2}{\alpha_j+\beta_j} \geq \frac{3}{4}.$$

Thus, $\alpha(\varepsilon)=\beta(\varepsilon)=0$; therefore, $\varepsilon=0$. The lemma is proved.

Set $d_i=\alpha_i+\beta_i$ and $\sigma_i=\alpha_i-\beta_i$ if $i\in\mathcal{K}$, $\omega=1+\sum\limits_{i\in K}\frac{\sigma_i^2}{4d_i}$ if $\omega\geq 1$.

Lemma 2. *The matrix* $\Xi=\|\theta_{ij}\|$, *where*

$$\theta_{ij}=\frac{1}{4}+\frac{\sigma_i\sigma_j}{8\omega d_i d_j}-\frac{2\delta_{ij}}{4d_j},$$

is inverse to $\|\langle c_i^*\cdot c_j^*\rangle\|$.

Proof. A direct computation. We leave to the reader the verification that

(i) $\theta_{ij}\geq 0$ if $i\neq j$;

(ii) $\theta_{ii}\geq 0$ if $d_i\geq 2$;

(iii) $\theta_{ii}+\theta_{ij}\geq 0$ if $i\neq j$.

Obviously, all statements of the plane section lemma are consequences of (i)–(iii) (in the situation under consideration).

4. Case 2. Q_S is an irreducible cone with apex at x and x is not a degeneration point. Here $k_1=1$, $\mathbb{K}_1=\{1\}$, and (after reduction for all $j\in\mathcal{K}_3$) $Y_{j-1,1}$ is a curve on Q_S^1. Q_S^1 is a ruled surface of the type $\mathbb{F}_2$ and $Y_{j-1,1}$ is different from the exceptional section, i.e., $Y_{j-1,0}$ is a curve in $\mathbb{P}^3$. Then

$$T=\mathbb{Z}c_1^*\bigoplus\left(\bigoplus_{i\in K_3}\mathbb{Z}c_i^*\right).$$

We have

$$\langle c_1^*\cdot c_1^*\rangle=-1,\quad \langle c_i^*\cdot c_1^*\rangle=\operatorname{mult}_x Y_{i-1,0}.$$

If $i\neq j$, then on Q_S^1

$$\langle c_i^*\cdot c_j^*\rangle=(y_{j-1,1}\cdot y_{i-1,1}).$$

If $i\in\mathcal{K}_3$, then

$$\langle c_i^*\cdot c_i^*\rangle=2p_a(Y_{i-1,1})-2-\operatorname{mult}_x Y_{i-1,0}.$$

Since the coefficient before c_1^* is of no interest to us, we shall make one more reduction. Set $\tilde{c}_i=c_i^*+\langle c_i^*\cdot c_1^*\rangle\, c_1^*$ and

$$\widetilde{T}=\bigoplus_{i\in K_3}\mathbb{Z}c_i,\quad T=\widetilde{T}\bigoplus\mathbb{Z}c_1^*.$$

Let $\tilde{p}\colon T\to\widetilde{T}$ be the projection onto the first term, $\zeta^*=\tilde{p}(\widetilde{\zeta})=\sum\limits_{i\in K_3}\theta_i\tilde{c}_i$. If $i\in\mathcal{K}_3$, then set $d_i=\deg Y_{i-1,0}$, $\mu_i=\operatorname{mult}_x Y_{i-1,0}$. Then

$$\langle\tilde{c}_i\cdot\tilde{c}_j\rangle=\frac{1}{2}(d_id_j+\mu_i\mu_j)-(2d_i+\mu_i)\delta_{ij},\quad \{i,j\}\subset\mathcal{K}_3.$$

Lemma 3. *The form* $\langle\cdot\rangle$ *is nondegenerate on* $\widetilde{T}$.

The proof is analogous to the proof of Lemma 1, and we leave it to the reader.

Lemma 4. *The matrix* $\Xi = \|\theta_{ij}\|$ *is inverse to* $\| \langle c_i \cdot c_j \rangle \|$.

$$\theta_{ij} = \frac{1}{2}\left(\frac{1}{2} - \frac{\mu_i\mu_j}{\omega(2d_i+\mu_i)(2d_j+\mu_j)} - \frac{2\delta_{ij}}{2d_i+\mu_i}\right),$$

where

$$\omega = \sum_{j\in K_3} \frac{\mu_j d_j}{2d_j+\mu_j}, \quad \omega > 0.$$

Proof. A direct computation.

We again leave to the reader the simple but tiresome checking of the following statements: (i) $\theta_{ij} > 0$ if $i \neq j$; (ii) $\theta_{ii} > 0$ if $d_i \geq 2$; (iii) $\theta_{ii} + \theta_{ij} \geq 0$ if $i \neq j$. The proof is a consequence of these statements.

5. Case 3. Q_S is an irreducible cone with apex at x and x is a degeneration point. Here $k_1 = 2$, $\mathcal{K}_1 = \{1,2\}$, $\mathcal{K} = \{1\} \bigcup \mathcal{K}_3$. If $S \bigcap V$ is the union of 6 lines, then all statements of the plane section lemma are consequences of Lemmas 3.4 and 3.5 (in fact, our construction also works here). Therefore, assume that $\operatorname{mult}_x S \bigcap V = 4$ (if $\operatorname{mult}_x S\bigcap V \geq 5$, then $\operatorname{mult}_x S \bigcap V = 6$). We have that the exceptional conic $\widetilde{E}_1 \bigcap Q^1_S = \widetilde{E}_2 \bigcap Q^2_S$ is not a component of the curve $S^2 \bigcap V^2$. Thus, if $j \in \mathcal{K}_3$, then $Y_{j-1,1}$ is a curve on Q^1_S (i.e., on a surface of the type $\mathbb{F}_2$) which is different from the exceptional section.

Set, as in 4.4, $\mu_i = \operatorname{mult}_x Y_{i-1,0}$ for $i \in \mathcal{K}_3$, $d_i = \deg Y_{i-1,0}$. We have $T = \mathbb{Z}c_1^* \bigoplus \Big(\bigoplus_{i\in K_3} \mathbb{Z}c_i^*\Big)$,

$$\begin{gathered}\langle c_1^* \cdot c_1^* \rangle = -2, \quad \langle c_i^* \cdot c_1^* \rangle = \mu_i,\\ \langle c_i^* \cdot c_j^* \rangle = (y_{i-1,1} \cdot y_{j-1,1}) \text{ on } Q^1_S,\\ \langle c_i^* \cdot c_i^* \rangle = 2p_a(Y_{i-1,1}) - 2.\end{gathered}$$

Let us introduce the classes $\tilde{c}_i = c_i^* + \frac{1}{2}\mu_i c_1^*$, $i \in \mathcal{K}_3$. Then

$$\widetilde{\zeta} = \sum_{i\in K} \theta_i c_i^* = \sum_{i\in K_3} \theta_i \tilde{c}_i + \left(\theta_1 - \sum_{i\in K_3} \frac{\theta_i\mu_i}{2}\right) c_1^*.$$

The multiplication rules in the base $\{c_1^*\} \bigcup \{\tilde{c}_i | i \in \mathcal{K}_3\}$ are very simple: $\langle \tilde{c}_i \cdot \tilde{c}_j \rangle = \frac{1}{2}d_i d_j - 2\delta_{ij} d_i$, $\langle \tilde{c}_i \cdot c_1^* \rangle = 0$. Obviously, $\langle\cdot\rangle$ is nondegenerate on T if and only if it is nondegenerate on $\widetilde{T} = \bigoplus_{i\in K_3} \mathbb{Z}c_i$. The last statement can be proved in the same manner as Lemmas 1 and 3. Then the inverse matrix of $\| \langle c_i \cdot c_j \rangle \|$ is $\Xi = \left\| \frac{1}{4} - \frac{\delta_{ij}}{2d_i} \right\|$, and all statements (i)–(iii) of the plane section lemma can be verified without any difficulty.

To prove the last statement of the plane section lemma we have to prove that $\theta_1 \geq 0$. Let us again use the nondegeneracy of $\langle\cdot\rangle$ on T. It is enough to prove, for all $i \in \mathcal{K}$, that if $\varepsilon = \sum_{j\in K} \varepsilon_j c_j^*$ is a solution of the system $\langle \varepsilon \cdot c_j^* \rangle = \delta_{ij}$, then $\varepsilon_1 \geq 0$. We have

$$\varepsilon = \sum_{j\in K_3} \varepsilon_j \tilde{c}_j + \left(\varepsilon_1 - \sum_{j\in K_3} \frac{\varepsilon_j\mu_j}{2}\right) c_1^*.$$

Let us denote the expression in the parentheses by ε^*. Let, at first, $i \neq 1$. Then $\langle \varepsilon \cdot c_1^* \rangle = 0$, i.e., $\varepsilon^* = 0$, and so $\varepsilon_1 = \frac{1}{2} \sum_{j\in K_3} \varepsilon_j \mu_j$. It is clear that the vector $(\varepsilon_j)_{j\in\mathcal{K}_3}$ is the ith row of the matrix Ξ, i.e.,

$$\varepsilon_1 = \frac{1}{2} \sum_{j\in K_3} \mu_j \left(\frac{1}{4} - \frac{\delta_{ij}}{2d_j}\right) = \frac{1}{2}\left(1 - \frac{\mu_i}{2d_i}\right) > 0$$

(we use the equality $\sum_{j\in\mathcal{K}_3} \mu_j = \mathrm{mult}_x\, S\bigcap V = 4$). Let now $i = 1$, and so $\varepsilon^* = -\frac{1}{2}$. It is easy to see that $\varepsilon_j = \dfrac{1}{2} - \dfrac{\mu_j}{4d_j}$ if $j \in \mathcal{K}_3$. Therefore,

$$\varepsilon_1 = -\frac{1}{2} + \sum_{j\in\mathcal{K}_3} \frac{\varepsilon_j\mu_j}{2} = \frac{1}{2} - \sum_{j\in\mathcal{K}_3} \frac{\mu_j^2}{8d_j} \geq 0.$$

Thus, the last statement of the plane section lemma, i.e., the inequality $\mathrm{mult}_x |\chi| \leq 2n$, is proved in this case.

6. Case 4. Q_S is a pair of planes. In this case, it is difficult to give a general proof as before. The reason for this is the absence of a simple system of parameters which defines the arrangement of curves of a 3-plane section (i.e., defines the form $\langle\cdot\rangle$), such as the type (α, β) in the first case, and the degree and multiplicity at a point $x = (d, \mu)$ in the second and third cases. Even three parameters (the degree of a curve, its multiplicity in a degeneration point, and the number of a plane (i.e., the component of Q_S) which contains it) is not enough. Therefore, we have to use the above scheme in each situation separately. We shall consider some typical cases.

(1) $S\bigcap V$ is the union of six lines A_1, A_2, A_3, B_1, B_2, B_3, $A_i \subset P_1$, $B_i \subset P_2$, where $Q_S = P_1 \bigcup P_2$, and there are no points that belong to three lines or more. There are no degeneration points in this case. $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}_4 = \varnothing$. $\#\mathcal{K}_3 = 6$: the consecutive blowings of these lines give us the resolution of the plane section. We shall enumerate the lines so that $A_i \bigcap B_i \neq \varnothing$, $A_i \bigcap B_j = \varnothing$ if $i \neq j$. Then in the base $\{e_i^*\} = \{c_i^*\}$ (the lines are blown in the order described above), the matrix $\| \langle c_i^* \cdot c_j^* \rangle \|$ is

$$\begin{pmatrix} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{pmatrix}.$$

The inverse matrix is

$$\frac{1}{8} \cdot \begin{pmatrix} -2 & 1 & 1 & 2 & 3 & 3 \\ 1 & -2 & 1 & 3 & 2 & 3 \\ 1 & 1 & -2 & 3 & 3 & 2 \\ 2 & 3 & 3 & -2 & 1 & 1 \\ 3 & 2 & 3 & 1 & -2 & 1 \\ 3 & 3 & 2 & 1 & 1 & -2 \end{pmatrix}.$$

We leave the proof of the Lemma to the reader.

(2) $S\bigcap V$ is the union of two plane cubics with double points: $\mathrm{mult}_x\, Q_S = 2$, $x \in P_1 \bigcap P_2$, $\mathrm{mult}_x\, \widetilde{Q}_S = 2$. $S\bigcap V = C_1 \bigcup C_2$, $\mathrm{mult}_x\, C_i = 2$. x is a degeneration point. We can assume that $\mathcal{K}_2 \bigcup \mathcal{K}_4 = \varnothing$, so $T = \mathbb{Z}c_1^* \bigoplus \mathbb{Z}c_3^* \bigoplus \mathbb{Z}c_4^*$ and the form $\langle\cdot\rangle$ is defined by the matrix

$$\begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

The inverse matrix is

$$\frac{1}{6} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

All statements of the plane section lemma, in this case, can be proved in a straightforward way.

(3) $S \bigcap V$ is the union of six lines, exactly four of which contain a degeneration point x. If we chose the proper order of blowings, then the form $\langle\cdot\rangle$ is defined in the base $\{c_i^* | i \in K\}$ by the matrix

$$\begin{pmatrix} -2 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -2 \end{pmatrix}$$

(the first row and the first column are related to c_1^*). The inverse matrix is defined in the following way:

$$\frac{1}{16} \cdot \begin{pmatrix} 0 & 4 & 4 & 8 & 4 & 4 & 8 \\ 4 & -5 & 3 & 2 & 5 & 5 & 6 \\ 4 & 3 & -5 & 2 & 5 & 5 & 6 \\ 8 & 2 & 2 & -4 & 6 & 6 & 4 \\ 4 & 5 & 5 & 6 & -5 & 3 & 2 \\ 4 & 5 & 5 & 6 & 3 & -5 & 2 \\ 8 & 6 & 6 & 4 & 2 & 2 & -4 \end{pmatrix};$$

this proves the plane section lemma in the case under consideration.

In other situations, the reasoning is the same. The only thing we must note is that, in all cases where the degeneration point exists, it is necessary to check the inequality $\theta_1 \geq 0$, i.e., to check the nonnegativity of all elements in the first row of the matrix inverse to $\| \langle c_i^* \cdot c_j^* \rangle \|$.

The plane section lemma is proved.

5. Elimination of Maximal Singularities Contractible into a Point.

1. Our aim is to prove Proposition 3.1. We shall use all the notations, definitions, and agreements of Chapter 1, which we shall use without any stipulations. We shall assume that there are no maximal cycles: $\nu_B \leq n$ for all curves $B \subset V$ and $\nu_x \leq 2n$ for all points $x \in V$ (the last statement was proved in Secs. 3, 4). Hence, by Proposition 1.2.1, there exists a maximal singularity B_β of the type (1.0). Let us fix it. We shall prove that this situation is impossible and get a contradiction.

2. First, let us describe the initial data in detail. To simplify the notations, we shall assume that the graph of the singularities Γ_β that lie on B_β begins with $E_1, E_2, \ldots$. Moreover, applying Lemma 4.1.1, we can assume that the chosen resolution has the property ($k = 3$) described in this lemma, i.e., it begins with the blowing of points.

Proposition 1. *One of the following cases holds.*

(A) *There is no line L on V that passes through B_0.*

(B) *There is not less than one line on V that passes through B_0 and either* (1) *the graph Γ_β contains only one vertex, the plane E_1, i.e., $I_0 = \{1\}$, or* (2) *$\#I_0 \geq 2$, and for all lines $L \subset V$, $B_0 \in L$, and all $j \in I_0 \setminus \{1\}$ we have $B_{j-1} \notin L^{j-1}$.*

(C) *$\#I_0 \geq 2$, $\{1, 2\} \subset I_0$, there exists only the line $L \subset V$ such that $B_i \in L^i$, $i = 0, 1$, and either*

(1) *$I_0 = \{1, 2\}$, i.e., Γ_β contains only two vertices, the planes E_1 and E_2, and then only curves are blown, or*

(2) *$\#I_0 \geq 3$ and for all $j \in I_0 \setminus \{1, 2\}$ we have $B_{j-1} \notin L^{j-1}$.*

(D) *$\#I_0 \geq 3$, $\{1, 2, 3\} \subset I_0$ and there exists only the line $L \subset V$ such that $B_i \in L^i$, $i = 0, 1, 2$.*

Proof. The proof is the obvious enumeration of all possibilities, keeping in mind that the resolution begins with the blowings of points.

The process of the elimination of a maximal singularity essentially depends on the existence of lines on V which pass through B_0 and their behavior.

3. Elimination of case A. In the proof of Lemma 3.5, it was stated that the class $(2h - 3e_1)$ is nonnegative. By Lemma 1.4.1, this means that the test class

$$y = \sum_{j \in I_0} r_j (2h - 3e_j)$$

is nonnegative. Thus, $(h'^2 \cdot y) \geq 0$. A direct computation, using Lemma 1.4.3, give us the opposite result:

$$(h'^2 \cdot y) < \frac{n^2}{\Sigma_0 + \Sigma_1}(-\Sigma_1^2) < 0.$$

The contradiction eliminates case A.

4. Formulation of the graph lemma.

Lemma 1 (*the graph lemma*).

(1) *In the situation* B, (2) *of Proposition* 1 *either*:

(a) *there exists a subgraph of the graph* Γ_β *of the type* B, (1) *for which the strengthened Fano-Noether inequality holds, i.e., there exists a maximal singularity that satisfies the condition* B, (1), *or*

(b) *there exists the set of positive integer coefficients* r_i^*, $i \in I$, *such that*:

(i) $r_i^* = r_i$ *if* $i \in I_0 \setminus \{1\}$;

(ii) $\sum\limits_{i\in I} r_i^* \nu_i > \sum\limits_{i\in I} r_i^* \delta_{i-1} n$;

(iii) *the class* $\left(-\sum\limits_{i\in I_0} r_i^* e_i\right)$ *is nonnegative with respect to those curves on* V_j, $0 \leq j \leq N$, *that are contractable into a point on* V;

(iv) $\left(\left(-\sum\limits_{i\in I} r_i^* e_i\right) \cdot b_{j-1}\right) \geq r_j^*$ *if* $j \in I_1$;

(v) $r_1^* \leq \sum\limits_{i\in I_0\setminus\{1\}} r_i^*$.

(2) *In the situation of Proposition* 1, C, (2), *either*:

(a) *there exists a subgraph of the graph* Γ_β *of the type* B, (1) *or* C, (1) *for which the strengthened Noether-Fano inequality holds, i.e., there exists a corresponding maximal singularity, or* (b) *there exists a set of positive integer coefficients* r_i^*, $i \in I$, *such that*

(i) $r_i^* = r_i$ *if* $i \in I_0 \setminus \{1,2\}$, *the conditions* (ii)–(v) *of part* 1, (b) *of this lemma hold, and*

(vi) $r_2^* \leq \sum\limits_{i\in I_0\setminus\{1,2\}} r_i^*$.

(3) *In case* D *of Proposition* 1, *we have* $r_1 = r_2 < \sum\limits_{i\in I_0\setminus\{1\}} r_i$.

The proof of the graph lemma is the subject of Sec. 6. Let us note that part (3) is trivial.

5. Elimination of case B.

Lemma 2. *The case* B, (1) *of Proposition* 1 *cannot be realized.*

Proof. Since the class $(h - e_1)$ is nonnegative, we have $\left(h'^2 \cdot (h - e_1)\right) \geq 0$. Therefore, by Lemma 1.4.3, we have $\#I_1 \leq 2$.

Lemma 3. *Let* $P \subset \mathbb{P}^5$ *be an arbitrary plane; then, for a general hyperplane* $H \subset \mathbb{P}^5$ *which contains* P, $V \bigcap H$ *is a nonsingular surface.*

Proof. We have to check the nonsingularity only in the intersection points $P \bigcap V$. By Corollary 1.2 if $L \subset P \bigcap V$ is a line, then $H \bigcap V$ is nonsingular along L for a general H. Then $x \in \operatorname{Sing} H \bigcap V$ if and only if $T_x V \subset H$. If $\dim P \bigcap V = 0$, i.e., $P \bigcap V = \{x_1, \ldots, x_s\}$, $s \leq 6$, then, obviously, $\dim\{H \in \mathbb{P}^{5^*} | H \supset T_{x_i} V \text{ for some } i\} = 1$. But $\dim\{H \in \mathbb{P}^{5^*} | H \supset P\} = 2$. The lemma is proved. In fact, we proved a stronger statement: If there exist points $x_1, \ldots, x_s \in P \bigcap V$ such that $\operatorname{Sing} V \bigcap H \subset \{x_1, \ldots, x_s\}$ for a general $H \supset P$, then $\operatorname{Sing} V \bigcap H = \varnothing$.

Now let $\dim P \bigcap V = 1$. Assume that $P \not\subset Q$. By Corollary 1.2, we have to study only the case where $C = Q \bigcap P$ is an irreducible conic, $C \subset \widetilde{Q}$. Let $S \supset P$ be a general 3-plane; then for a general $H \supset S$, $\operatorname{Sing} H \bigcap V \subset S \bigcap V$, and so $S \bigcap \operatorname{Sing} H \bigcap V \subset \operatorname{Sing} S \bigcap V$ (because S is a hyperplane in H). Therefore, $\operatorname{Sing} H \bigcap V \subset \operatorname{Sing} S \bigcap V$. But if S is general, then $S \bigcap Q$ is a nonsingular quadric, $\dim \operatorname{Sing} S \bigcap V = 0$, and S does not contain degeneration points. This proves the nonsingularity of $H \bigcap V$ for a general $H \supset S$.

Assume, that $P \subset Q$ and $V \bigcap P = \widetilde{Q} \bigcap P$ is a curve of degree 3 without multiple components. For a general 3-plane $S \supset P$ we have $S \bigcap Q = P \bigcup P(S)$ and the line $P(S) \bigcap P$ intersects the curve $\widetilde{Q} \bigcap P$ (i.e., a plane cubic) in three different points. Therefore, S does not contain degeneration points. Hence $\operatorname{Sing} H \bigcap V \subset \operatorname{Sing} S \bigcap V$. The lemma is proved.

Consider now a plane $P \ni B_0$ such that the projectivization of its tangent cone in B_0 is the infinitely close line B_1. Let us restrict the linear system $|\chi|$ to $V \bigcap H$, where $H \supset P$ is a general hyperplane. Then, as was proved above, $V \bigcap H = \widetilde{H}$ is a nonsingular surface. The linear system of curves $|\chi\|_{\widetilde{H}}$ has B_0 as its $(\nu_1 + \nu_2)$-multiple base point and if $\#I_1 = 2$, then the proper inverse image of $|\chi\|_{\widetilde{H}}$ on $\widetilde{H}^1 = \widetilde{H}^2$ has $\#B_2 \bigcap \widetilde{H}^2$ ν_3-multiple base points. Since E_2 is a ruled surface of the type $\mathbb{F}_2$, the exceptional section of which is $E_2 \bigcap E_1^2$, and, on the other hand, $|\chi|^1\big|_{E_1}$ is a linear system of curves of degree $\nu_1 \leq 2n$ which has the ν^*-multiple component $B_1 \subset E_1$, $\nu^* \geq \nu_2 > n$, then B_2 is different from the exceptional section of the surface E_2. Hence, if $\#I_1 = 2$, we have that $\#B_2 \bigcap \widetilde{H}^2 \geq 2$ (the curve $\widetilde{H}^2 \bigcap E_2$ is irreducible and different from the exceptional section because its self-intersection index is 2). One can prove that $\#B_2 \bigcap \widetilde{H}^2 = 2$.

To prove Lemma 2 we shall study the self-intersection of the nonstationary part of the linear system $|\chi\|_{\widetilde{H}}$ and get the information about its base points. We need to enumerate all possible configurations of $P \bigcap V$. We shall restrict our study to several typical cases and leave the other cases to the reader (the reasoning in all situations is the same).

(1) Let $\dim P \bigcap V = 0$. Then $|\chi\|_{\widetilde{H}}$ is nonstationary and its multiplicity in B_0 is $\nu_1 + \nu_2 > 3n$ if $\#I_1 = 1$, and $\nu_1 + \nu_2 > \frac{8}{3}n$ if $\#I_1 = 2$. The index of its self-intersection is $6n^2 < (\frac{8}{3}n)^2$. We get a contradiction.

(2) Let $P \bigcap V = L_1 \bigcup L_2$ be a pair of lines. Let $\mu_i = \nu_{L_i}$. Then the self-intersection index of its nonstationary part is $6n^2 - 2n(\mu_1 + \mu_2) - 2\mu_1^2 - 2\mu_2^2 + 2\mu_1\mu_2$, and its multiplicity in B_0 is $(\nu_1 + \nu_2 - \mu_1 - \mu_2)$. In the case $\#I_1 = 2$, there exist two infinitely close base points of multiplicities $\geq (\nu_3 - \mu_i)$, $i = 1, 2$. Thus, if $\#I_1 = 1$, then

$$6n^2 - 2n(\mu_1 + \mu_2) - 2\mu_1^2 - 2\mu_2^2 + 2\mu_1\mu_2 - (3n - \mu_1 - \mu_2)^2 \geq 0$$

and if $\#I_1 = 2$, then

$$6n^2 - 2n(\mu_1 + \mu_2) - 2\mu_1^2 - 2\mu_2^2 + 2\mu_1\mu_2 - (\nu_1 + \nu_2 - \mu_1 - \mu_2)^2 - (\nu_3 - \mu_1)^2 - (\nu_3 - \mu_2)^2 \geq 0$$
$$(\text{and} \quad \nu_1 + \nu_2 + \nu_3 > 4n).$$

It is easy to check that these inequalities cannot hold (the maximum of the left-hand side is achieved when $\mu_1 = \mu_2 = \mu$ if $\mu_1 + \mu_2 = \text{const}$, and when $\nu_1 = \nu_2 = \nu_3 > \frac{4}{3}n$ if we fix $\nu_1 + \nu_2 + \nu_3$, $\nu_1 + \nu_2$; the quadratic form obtained is negatively defined).

(3) Let $P \bigcap V = C$ be an irreducible cubic curve with a double point in B_0 (recall that for all points $x \in P$ we have $\text{mult}_x P \bigcap V \leq 2$ by the assumption of the generality of V). Let $\mu = \text{mult}_C |\chi|$. As above, we get the inequality

$$6n^2 - 6n\mu - (3n - 2\mu)^2 \geq 0$$

if $\#I_1 = 1$, and

$$6n^2 - 6n\mu - (\nu_1 + \nu_2 - 2\mu)^2 - 2(\nu_3 - \mu)^2 \geq 0$$

if $\#I_1 = 2$ and $\nu_1 + \nu_2 + \nu_3 > 4n$.

Again, a simple computation shows that these inequalities cannot hold.

Considering, in the same way, the other cases for $P \bigcap V$, we complete the proof of Lemma 2.

Lemma 4. *The case* B *of Proposition* 1 *cannot be realized.*

Proof. By Lemma 2, only case (1), (b) of the graph lemma is possible. Let

$$y^* = \sum_{j \in I_0} r_j^* \left(\frac{1}{2}h - e_j\right).$$

We assert that y^* is nonnegative.

Indeed, $(y^* \cdot w) \geq 0$ for all curves $W \subset V_i$ of the type (1,0) (by property (iii) of the numbers r_j^*). Let $W \subset V_i$ be a curve of the type (1,1). By the assumption, there are no maximal cycles; so for $\alpha = \min(\beta, i)$ we have $\deg(\varphi_{\alpha,0}\colon\ W_\alpha \to W_0) = 1$. Taking into account the projection formula, we can substitute W for

W_α, i.e., we can assume that $i \leq 3$. It is enough to prove the nonnegativity for this class of curves. Now if W_0 is not a line, then it is easy to see that for all points $x \in E_1$

$$\operatorname{mult}_{B_0} W_0 + \operatorname{mult}_x W_0^1 \leq \deg W_0$$

(in the opposite case, W_0 would contain a line, as a component, that passes through B_0 in the direction of the infinitely close point x). Thus, $(w \cdot e_1) + (w \cdot e_j) \leq (w \cdot h)$ for all $j \in I_0 \setminus \{1\}$, and, therefore,

$$\left(\left(\frac{1}{2}h - e_1\right) \cdot w\right) + \left(\left(\frac{1}{2}h - e_j\right) \cdot w\right) \geq 0,$$

and the second term is nonnegative. Now if $\left(\left(\frac{1}{2}h - e_1\right) \cdot w\right) \geq 0$, then all summands in the sum which define $(y^* \cdot w)$ are nonnegative. If $\left(\left(\frac{1}{2}h - e_1\right) \cdot w\right) < 0$, we have

$$(y^* \cdot w) = \sum_{j \in I_0 \setminus \{1\}} r_j^* \left(\left(\left(\frac{1}{2}h - e_1\right) \cdot w\right) + \left(\left(\frac{1}{2}h - e_j\right) \cdot w\right)\right) + \left(r_1^* - \sum_{j \in I_0 \setminus \{1\}} r_j^*\right)\left(\left(\frac{1}{2}h - e_1\right) \cdot w\right).$$

The first summand of the sum is nonnegative, the second is also the same (by property (v) of the numbers r_j^* from case (1) of the graph lemma).

Finally, let W_0 be a line. By the definition of case B, $(w \cdot e_j) = 0$ for all $j \in I_0 \setminus \{1\}$, and so

$$(y^* \cdot w) = \frac{1}{2}\left(\sum_{j \in I_0} r_j^*\right)(h \cdot w) - r_1^*(e_1 \cdot w).$$

This expression is again nonnegative by property (v) of the graph lemma — case (1).

Thus, y^* is nonnegative and, in particular, $(h'^2 \cdot y^*) \geq 0$.

We can say more about singular curves, i.e., $(y^* \cdot b_{i-1}) \geq r_i^*$, $i \in I_1$ (property (iv)). Let us compute now the product $(h'^2 \cdot y)$. This may be done in the same way as the product computation for usual test class, because the properties of y^* are analogous. We have

$$(y^* \cdot h'^2) \leq (h^3)n^2\left(\frac{1}{2}\sum_{j \in I_0} r_j^*\right) - \sum_{j \in I} r_j^* \nu_j^2.$$

The inequality (ii) from Lemma 1 gives us the estimation for the last sum, which is analogous to the estimation for the quadratic inequality. Let $\Sigma_i^* = \sum_{j \in I_i} r_j^*$, $i = 0, 1$. We have

$$0 \leq (h'^2 \cdot y^*) \leq \frac{n^2}{\Sigma_0^* + \Sigma_1^*}(-\Sigma_0^{*2} - \Sigma_0^*\Sigma_1^* - \Sigma_1^{*2}) < 0.$$

We get a contradiction. The lemma is proved. Case B is eliminated.

6. Elimination of case D.

Lemma 5. *Case* D *of Proposition* 1 *cannot be realized.*

The **proof** consists of a long chain of estimations. Set

$$y = \sum_{j \in I_0} r_j \left(\frac{1}{2}h - e_j\right).$$

Lemma 6. *If for a curve $C \subset V_i$ the inequality $(c \cdot y) < 0$ holds, then $C_0 = L$ (let us recall that L is the line passing through B_0 in the direction of B_1).*

Proof. We have to repeat the reasoning which proves the nonnegativity of the class y^* in Lemma 4. Instead of property (v) in case (1) of the graph lemma, we shall use case (3) of the same lemma.

Set $J = \{1 \le j \le N | B_{j-1,0} = L\}$ and $\Delta = \sum_{j \in J} (b_{j-1} \cdot h)\nu_j^2$.

Lemma 7.

$$0 \le (h'^2 \cdot y) < \varepsilon\left(-\Sigma_1^2 + \left(\frac{\Delta}{2n^2} - 1\right)(\Sigma_1\Sigma_0 + \Sigma_0^2)\right),$$

where $\varepsilon > 0$ is a nonessential constant.

Proof. Standard calculations give us

$$(h'^2 \cdot y) \le 3n^2\Sigma_0 - \frac{(2\Sigma_0 + \Sigma_1)^2}{\Sigma_0 + \Sigma_1}n^2 - \sum_{j \in J}(b_{j-1} \cdot y)\nu_j^2.$$

Now if $j \in J$, then

$$(b_{j-1} \cdot y) = \sum_{i \in I_0} r_i\big((h - e_i) \cdot b_{j-1}\big) - \frac{1}{2}\left(\sum_{i \in I_0} r_i\right)(h \cdot b_{j-1}) \ge -\frac{1}{2}\Sigma_0(b_{j-1} \cdot h).$$

So

$$\sum_{j \in J}(b_{j-1} \cdot y)\nu_j^2 \ge -\frac{1}{2}\Sigma_0\Delta.$$

Lemma 7 is proved.

Set $\theta = \frac{1}{2}(\nu_1 + \nu_2)$, $\Lambda(t, \varepsilon) = 6n^2 - 2t^2 - \varepsilon^2 + \left(t + \frac{\varepsilon - n}{2}\right)^2$ and $q = \frac{\Sigma_1}{\Sigma_0} > 0$.

Lemma 8. *The following three inequalities hold:*

(i) $\Delta \le \Lambda(\theta, \nu_3)$;

(ii) $\theta > \dfrac{2 + q}{1 + q}n$;

(iii) $\nu_3 > n + \dfrac{2n - \theta}{q}$.

Proof. (i) The reasoning here is completely analogous to the proof of Lemma 5.5 in Chapter 5 of this paper. We shall describe the general idea and the key points, avoiding some details and tiresome, but transparent calculations, which will be presented in detail in Sec. 5 of Chapter 5.

Roughly speaking, the estimation of Δ is performed in the following way. Let us restrict the linear system $|\chi|$ to a general hyperplane section $H \bigcap V$, where $H \supset L$. We shall give the estimation of its self-intersection index, taking into account the fact that each base curve B_{i-1} which is infinitely close to L and which is the d_i-leaf cover of L generates on $H \bigcap V$ d_i infinitely close base points of the linear system $|\chi\|_{H \bigcap V}$ of multiplicity ν_i. This gives us the inequality

$$6n^2 - 2n\nu_L - 2\nu_L^2 - \sum_{i=1}^{3}(\nu_i - \nu_L)^2 \ge \sum_{j \in J \setminus \{\omega\}} d_j\nu_j^2, \qquad (*)$$

where $\omega = \min\{t | t \in J\}$ ($J \ne \varnothing$, because, in the opposite case, $\Delta = 0$ and Lemma 7 gives us a contradiction). Since $\nu_\omega = \nu_L$ and $d_j = (b_{j-1} \cdot h)$, we have

$$\Delta \le 6n^2 - 2n\nu_L - \nu_L^2 - \sum_{i=1}^{3}(\nu_i - \nu_L)^2 \le 6n^2 - 2n\nu_L - \nu_L^2 - 2(\theta - \nu_L)^2 - (\nu_3 - \nu_L)^2.$$

The maximum of the last expression over ν_L is achieved at $\nu_L = \frac{\theta}{2} + \frac{1}{4}(\nu_3 - n)$ and is equal to $\Lambda(\theta, \nu_3)$ (an easy computation).

The main steps of the formal proof of $(*)$ are as follows (see Sec. 5 of Chapter 5). Consider the linear system $|\tau|$ of hyperplane sections of V which contains the line L. Let S be a general element of the system and S^i be its proper inverse image on V_i. Obviously, $\omega > \beta$, $B_{\omega-1} = L^{\omega-1}$, and the map $\varphi_{\omega-1,0}$ is an isomorphism in a neighborhood of a general point $B_{\omega-1}$. The proper inverse image of the system $|\tau|$ on V_ω is, as is easy to see, a free linear system. Therefore, $S^i \not\supset B_i$ for $i \geq \omega$ and hence $s^\omega = s^{\omega+1} = \cdots = s^N$. Let $\mathcal{K} = \{i | B_{i-1}$ is a point, $i \leq \omega - 1$, $B_{i-1} \in L^{i-1}\}$. Obviously $s^\omega = h - \sum_{i \in K} e_i - e_\omega$ and s^ω is a nonnegative class. Therefore,

$$0 \leq (h'^2 \cdot s^\omega) = \left(\left(nh - \sum_{i=1}^{\omega} \nu_i e_i\right)^2 \cdot s^\omega\right) - \sum_{i=\omega+1}^{N} (s^\omega \cdot b_{i-1})\nu_i^2.$$

A direct computation gives us the upper estimation

$$\left(\left(nh - \sum_{i=1}^{\omega} \nu_i e_i\right)^2 \cdot s^\omega\right) \leq 4n^2 - 2n\nu_L - 2\nu_L^2 - \sum_{i \in K} (\nu_i - \nu_L)^2$$

for the first expression. Since $\{1,2,3\} \subset \mathcal{K}$, we have

$$\sum_{j \in J \setminus \{\omega\}} (s^\omega \cdot b_{j-1})\nu_j^2 \leq \sum_{j=\omega+1}^{N} (s^\omega \cdot b_{j-1})\nu_j^2 \leq 4n^2 - 2n\nu_L - 2\nu_L^2 - \sum_{i=1}^{3} (\nu_i - \nu_L)^2.$$

Thus, to prove $(*)$ it is enough to show that $(s^\omega \cdot b_{j-1}) \geq (h \cdot b_{j-1})$. By the projection formula,

$$(s^\omega \cdot b_{j-1}) = \deg(\varphi_{j-1,\omega}\colon\ B_{j-1} \to B_{j-1,\omega})(s^\omega \cdot b_{j-1,\omega})$$

and

$$(h \cdot b_{j-1}) = \deg(\varphi_{j-1,\omega}\colon\ B_{j-1} \to B_{j-1,\omega})(h \cdot b_{j-1,\omega}).$$

Therefore, $(*)$ is a consequence of the following statement.

Lemma 9. *Let $C \subset E_\omega$ be an arbitrary curve which is different from a fibre of the morphism $\varphi_{\omega,\omega-1}$: $E_\omega \to B_{\omega-1}$. Then*

$$(c \cdot s^\omega) \geq d = \deg(\varphi_{\omega,\omega-1}\colon\ C \to B_{\omega-1}).$$

The proof is completely analogous to the proof of the lemma. Let $\sigma\colon\ \widetilde{V} \to V$ be the blowing of the line L, $E = \sigma^{-1}(L)$. Obviously the birational map $\sigma^{-1}\varphi_{\omega,0}\colon\ E_\omega \dashrightarrow E$ is an isomorphism on each fibre of the ruled surfaces over $B_{\omega-1} \setminus T = L \setminus \varphi_{\omega-1,0}(T)$, where T is a finite set of points, i.e., on the complement of a finite set of fibres over the points in T. By Proposition 1.2, the restriction of the proper inverse image of the linear system $|\tau|$ from $\widetilde{V}$ to the surface E is a free linear system of curves $|\tau|^*$ which consists of ample divisors. It is easy to see now that for each irreducible curve $C^* \subset E$ which is different from a fibre of the morphism σ and which is a d^*-leaf cover of L a general curve of the system $|\tau|^*$ intersects C^* at not less than d^* points which do not lie at the fibres over the points in T. Taking as C^* the proper inverse image of C on E, we get the statement of Lemma 9 (and thus statement (i) of Lemma 8).

Let us prove (ii) and (iii). They are direct consequences of the strengthened Noether–Fano inequality (Proposition I.2.1): taking into account the equality $r_1 = r_2$, we can substitute θ for all ν_j (the inequality holds). Therefore, $(\Sigma_0 + \Sigma_1)\theta > (2\Sigma_0 + \Sigma_1)n$. After dividing by Σ_0, we get (ii). If we substitute θ for all ν_j, $j \in I_0$, and ν_3 for all ν_j, $j \in I_1$, we shall get (iii). The lemma is proved.

Let us note that if $n < t \leq 2n$ and $t \geq \varepsilon > n$, then $\dfrac{\partial}{\partial \varepsilon}\Lambda(t, \varepsilon) < 0$, so

$$\Lambda(\theta, \nu_3) \leq \lambda(\theta) = \Lambda\left(\theta, n + \frac{2n - \theta}{q}\right).$$

Therefore,

$$\lambda(t) = 5n^2 - t^2 - (2n - t)^2 \left(\frac{1}{q} + \frac{3}{4q^2}\right).$$

Lemma 10.

(i) *If* $q \geq \frac{1}{2}$ *, then* $\left.\frac{d\lambda}{dt}\right|_{t=\theta} \leq 0$. *In particular,*

$$\lambda(\theta) \leq \lambda\left(\frac{q+2}{q+1}n\right) = \left(4 - \frac{3}{q+1} - \frac{3}{4(q+1)^2}\right)n^2.$$

(ii) *if* $q \leq \frac{1}{2}$ *, then* $\lambda(\theta) \leq \frac{5}{3}n^2$.

(iii) $-q^2 + \left(\frac{\Delta}{2n^2} - 1\right)(q+1) > 0$.

The **proof** consists of a direct computation. Part (iii) is a consequence of Lemma 7.

Now we get a contradiction from the chain of inequalities: from $\theta > n$ we get $\Delta \leq \lambda(\theta) < 4n^2$; therefore, by Lemma 10 (iii), $q < 2$. By Lemma 10 (i)-(ii), $\Delta \leq \lambda(\theta) < 3n^2$, and, by (iii), $q < 1$. Using Lemma 10 once more, we have $\Delta < \frac{37}{16}n^2$, and, by (iii), $q < \frac{1}{2}$. Thus, finally, $\Delta < 2n^2$, and we get a contradiction to Lemma 10 (iii). This eliminates case D.

7. Elimination of case C. This case is a mix of cases B and D. For its elimination we have to unite the reasoning of Secs. 5 and 6. If $L^2 \bigcap B_2 = \varnothing$, then the case C, (1) is impossible (as is case B, (1)). Indeed, let us restrict the linear system $|\chi|$ to a general hyperplane section which contains L and let us estimate the square of its nonstationary part, taking into account the base points B_0 and B_1, and taking into account the obvious fact that the infinitely close base curve B_{j-1}, $j \in I_1$, generates an infinitely close base point which does not lie on L^{j-1}. Therefore,

$$6n^2 - 2(\nu_1 + \nu_2 - n)\nu_L - 4\nu_L^2 - \sum_{i \in I} \nu_i^2 \geq 0.$$

But a direct computation shows that this inequality contradicts the Noether–Fano inequalities:

$$\sum_{i \in I} \nu_i > (\#I + 2)n.$$

If $L^2 \bigcap B_2 \neq \varnothing$, then the elimination process uses the same scheme as in case D: the existence of the infinitely close base point $L^2 \bigcap B_2$ ensures the fulfillment of the inequality $(*)$ (the proof of $(*)$ uses the same reasoning). Inequalities (ii) and (iii) of Lemma 8 hold and are proved analogously (where $q = \frac{1}{2}(\#I_1)$). The only point which deserves mention is that the graph Γ_β is a chain in this case. In fact, we need to show that E_3 is not connected with E_1 by an edge, i.e., $B_2 \not\subset E_1^2$. But if this is not so, then $B_2 = E_1^2 \bigcap E_2$ and the point B_1 is a $(\nu_2 + \nu_3)$-multiple base point of a linear system of curves of degree ν_1 on the plane E_1. But this is impossible, because $\nu_1 \leq 2n < \nu_2 + \nu_3$.

Case C, finally can be eliminated using the scheme of case D and using the reasoning of the proof of Lemma 4 and part (2) of the graph lemma. Indeed, let

$$y^* = \sum_{i \in I_0} r_i^* \left(\frac{1}{2}h - e_i\right).$$

The steps of our proof are the same as in Sec. 6 after substituting r_i^* for r_i everywhere. We shall give only the formulations of statements, because the proofs are essentially the same.

Lemma 11. *If for a curve* $C \subset V_i$ *the inequality* $(c \cdot y^*) < 0$ *holds, then* $C_0 = L$.

The **proof** is analogous to the proof of Lemma 6.

Lemma 12. *Let* $\Sigma_i^* = \sum_{j \in I_i} r_j^*$. *Let us define* J *and* Δ *as in Sec. 6. Then*

$$0 \leq (h'^2 \cdot y^*) < \varepsilon\left(-\Sigma_1^{*^2} + \left(\frac{\Delta}{4n^2} - 1\right)(\Sigma_1^*\Sigma_0^* + \Sigma_0^{*^2})\right),$$

where $\varepsilon > 0$.

The **proof** is an analog of Lemma 7. We shall only explain why Δ must be divided by $4n^2$ (but not by $2n^2$). For $j \in J$, assuming $d_j = \deg(\varphi_{j-1,0}\colon\ B_{j-1} \to L)$, we have

$$(b_{j-1} \cdot y^*) = d_j \left(\frac{1}{2}\Sigma_0^* - r_1^* - r_2^*\right).$$

If $\Sigma_0^* \geq 2(r_1^* + r_2^*)$, then y^* is nonnegative and we get a contradiction as in Lemma 4. Assume the contrary. Then, in view of

$$r_1^* \leq r_2^* + \sum_{i \in I_0 \setminus \{1,2\}} r_i^* \text{ and } r_2^* \leq \sum_{i \in I_0 \setminus \{1,2\}} r_i^*$$

(property (vi) of (2) of the graph lemma), we have

$$r_1^* + r_2^* - \sum_{i \in I_0 \setminus \{1,2\}} r_i^* \leq \frac{1}{2}\Sigma_0^*.$$

Thus,

$$(b_{j-1} \cdot y^*) \geq -\frac{1}{4}\Sigma_0^*(b_{j-1} \cdot h).$$

Now, repeating the proof of Lemma 7, we get the required result.

Let $\theta = \frac{1}{2}(\nu_1 + \nu_2)$, $\Lambda(t) = 6n^2 - 2t^2 + \frac{1}{3}(2t - n)^2$, and $q^* = \frac{\Sigma_1^*}{\Sigma_0^*} > 0$.

Lemma 13. *The following inequalities hold*:

(i) $\Delta \leq \Lambda^*(\theta)$;

(ii) $\nu_1 > \frac{2+q^*}{1+q^*}n$;

(iii) $\nu_2 > n + \frac{1}{q^*}(2n - \nu_1)$.

The proof is completely analogous to the proof of Lemma 8.

Lemma 14.

(i) $\frac{d\Lambda}{dt}|_{t \geq 0} < 0$.

(ii) $-{q^*}^2 + (\frac{\Delta}{4n^2} - 1)(q^* + 1) > 0$.

This is an exact analog of Lemma 10.

Now we can get a contradiction using a chain of inequalities: if $\theta > n$, then $\Delta < 4\frac{1}{3}n^2$; so, by Lemma 14 (ii), $q^* < 3/2$. Now, by Lemma 13, $\nu_1 > \frac{7}{5}n$, and so $\theta > \frac{6}{5}n$ and, finally, $\Delta < 4n^2$. But this contradicts the inequality $q^* > 0$.

Case C is eliminated.

6. The Proof of the Graph Lemma.

1. We shall now prove part (1) of the graph lemma.

Let us note that if there is no index $k \in I_1$ such that $B_{k-1,1} \subset E_1 \cong \mathbb{P}^2$ is a line, then there is nothing to prove: let $r_1^* = \sum_{\substack{j \in I_0,\\ j \to 1}} r_j$, $r_j^* = r_j$ if $j \neq 1$. It is easy to check, using the proof of Lemma I.4.1, that properties (iii) and (iv) hold. Properties (ii) and (v) are obvious.

Hence, let us assume that there exist a line $L \subset E_1$ and an index $k \in I_1$ such that $B_{k-1,1} = L$.

Note now that L is the only line in the plane E_1 with such a property, because the restriction of the linear system $|\chi|^1$ to E_1 is a linear system of plane curves of degree $\nu_1 \leq 2n$ which has L as a component of multiplicity greater than n; hence, it cannot have other components of multiplicity greater than n.

2. Let $\{k_1, \ldots, k_s\} = \{k \in I_1 | B_{k-1,1} = L\}$ and k_i be enumerated in increasing order, i.e., $k_{i+1} > k_i$. Let $k_0 = 1$. Let us define the following sets of indices:

$$\mathcal{N} = \{k_t | 0 \leq t \leq s\}, \quad \mathcal{L}_i = \{j \in I_0 | j \to k_i\}, \quad \mathcal{M}_i = \bigcup_{t=i}^{s} \mathcal{L}_t, \quad \mathcal{K}_i = \bigcap_{t=0}^{i} \mathcal{L}_t.$$

Let us also define new coefficients r_i^* as follows: $r_i^* = r_i$ if $i \notin \mathcal{N}$; $r_{k_i}^* = \sum_{t \in \mathcal{M}_i} r_t$, $0 \le i \le s$. We shall check properties (i)–(v). Property (i) obviously holds. Let us check (iii). Our reasoning will be the same as in the proof of part (A) of Lemma I.4.1. We have to study three different cases.

(1) Let a curve $C \subset V_j$ be such that $\dim C_1 = 0$, i.e., the loss of dimension happens at the αth step of resolution, $\alpha \ge 2$. Then

$$\Big(-\Big(\sum_{i \in I_0} r_i^* e_i\Big) \cdot c\Big) = \Big(-\Big(\sum_{i \in I_0} r_i e_i\Big) \cdot c\Big)$$

by property (i). The last product is nonnegative (Lemma I.4.1).

(2) Assume that $C_1 \subset E_1$ is a curve which is different from the line L. Then there are no curves B_{t-1}, $t \in I_1$, which are projected onto C_1 (see the note above). Thus, if $i \le \beta$, then $\varphi_{i,1}^{-1}$ is an isomorphism in a neighborhood of a general point C_1. If $(e_i \cdot c) \ne 0$ and $i \in I_0$, then $i \le \min(\beta, j)$ and $C_1^i = C_i$, and so

$$B_{i-1} \in C_{i-1} = C_1^{i-1} \subset E_1^{i-1},$$

i.e., $i \to 1$. But

$$r_1^* = \sum_{i \in \mathcal{M}_0} r_i \ge \sum_{i \in \mathcal{L}_0} r_i = \sum_{i \to 1} r_i^*.$$

Therefore, by the inequality $((-e_1 - e_i) \cdot c) \ge 0$ for $i \in I_0$, we have

$$\Big(-\Big(\sum_{i \in I_0} r_i^* e_i\Big) \cdot c\Big) \ge 0$$

(as in the proof of Lemma I.4.1).

(3) Assume, finally, that $C_1 = L$. Let $\alpha = \max\{t \le s | k_t \le j,\ C_{k_t} \subset E_{k_t}\}$. Obviously, $k_\alpha = \max\{t \le j | C_t \subset E_t,\ t \in I\}$ and $\{k_0, \dots, k_\alpha\} = \{t \le j | t \in I,\ C_t \subset E_t\}$.

Assume that $(e_i \cdot c) \ne 0$, $i \in I_0$. If $t \le \alpha - 1$, $k_t < i < k_{t+1}$, then, obviously,

$$B_{i-1} \in C_{i-1} = C_{k_t}^i \subset E_{k_t}^i,$$

i.e., $i \to k_t$. If $i > k_\alpha$, then, analogously, $i \to k_\alpha$. Therefore, $i \in \mathcal{M}_0$ and

$$\sum_{i \in I_0, (e_i \cdot c) > 0} r_i \le \sum_{i \in \mathcal{M}_0} r_i = r_1^*,$$

whence all follows, as in the proof of Lemma 1.4.1. Property (iii) is proved.

3. Let us verify property (iv), i.e., the inequality

$$\Big(\Big(-\sum_{i \in I_0} r_i^* e_i\Big) \cdot b_{j-1}\Big) \ge r_j^*$$

for $j \in I_1$. Let, at first, $j \ne k_i$ for all $1 \le i \le s$, i.e., $j \notin \mathcal{N}$. Then the curve B_{j-1} transforms into a point in E_α, $\alpha \ge 2$, i.e., $B_{j-1,1} = L$. But this contradicts the assumption, so $(b_{j-1} \cdot e_1) = 0$ and, by property (i), and the definition of r_j^* (i.e., $r_j^* = r_j$, $j \in I_0 \setminus \{1\}$), and using the part (B) of Lemma I.4.1, we get the result.

Now let $j = k_\alpha$, $1 \le \alpha \le s$. Let $J = \{i \in I_0 \setminus \{1\} | (b_{j-1} \cdot e_i) \ne 0\}$. Therefore if $i \in J$, then $i < j$ and $(b_{j-1} \cdot e_i) > 0$, i.e., $B_{i-1} \in B_{j-1,i-1}$. If $k_t < i < k_{t+1}$, $t < \alpha$, then $B_{j-1,i-1} = B_{j-1,k_t}^{i-1} \subset E_{k_t}^{i-1}$, so $i \to k_t$. Therefore, $J \subset \mathcal{K}_{\alpha-1} \bigcap \{i | i < j\}$. Since $\mathcal{L}_t \subset \{i | i > j = k_\alpha\}$ for $t \ge \alpha$, we have $\mathcal{L}_t \subset \mathcal{M}_0 \setminus J$. Thus, $\mathcal{M}_\alpha \subset \mathcal{M}_0 \setminus J$. Therefore,

$$\Big(\Big(-\sum_{i \in I_0} r_i^* e_i\Big) \cdot b_{j-1}\Big) = r_1^* - \sum_{i \in J} r_i^* = \sum_{i \in \mathcal{M}_0} r_i - \sum_{i \in J} r_i \ge \sum_{i \in \mathcal{M}_\alpha} r_i = r_j^*,$$

which is exactly what was required.

4. Property (v) is obvious.

Let us now investigate when property (ii) (i.e., the analog of the Noether–Fano inequality for the new coefficients r_i^*) holds. First, let us estimate the difference between r_i^* and r_i (obviously, $i \in \mathcal{N}$). Let $a_t = r_{k_t} - r_{k_t}^*$, $t = 0, \dots, s$.

Lemma 1. *The sequence of integers a_t is nonincreasing and nonnegative. i.e., $a_t \geq a_{t+1} \geq 0$, $0 \leq t \leq s-1$.*

Proof. By definition,

$$r_{k_i} = \sum_{j \to k_i} r_j = \sum_{j \in \mathcal{L}_i} r_j + \sum_{j \in k_i, j \in I_1} r_j = \sum_{j \in \mathcal{L}_i} r_j + \sum_{\substack{j \to k_i, \\ j \in I_1 \setminus \{k_{i+1}\}}} r_j + r_{k_{i+1}}$$

for $0 \leq i \leq s-1$. Using this relation between r_{k_i} and $r_{k_{i+1}}$ several times and taking into account the fact that

$$r_{k_s} = \sum_{j \in \mathcal{L}_s} r_j + \sum_{\substack{j \to k_s, \\ j \in I_1}} r_j,$$

we have

$$r_{k_i} = \sum_{t=i}^{s} \Big(\sum_{j \in \mathcal{L}_t} r_j \Big) + \sum_{t=i}^{s} \Big(\sum_{\substack{j \to k_t, \\ j \in I_1 \setminus \{k_{t+1}\}}} r_j \Big),$$

where we set $k_{s+1} = 0 \notin I$ for convenience of the notations. Let us note that the second sum is nonincreasing with respect to i and the first sum is not less than $\sum_{j \in \mathcal{M}_i} r_j = r^*_{k_i}$; thus, $a_i \geq 0$ for all i. For the proof of the lemma it remains to state that the difference of the first sum and $r^*_{k_i}$ is nonincreasing by i. We have

$$\sum_{t=i}^{s} \Big(\sum_{j \in \mathcal{L}_t} r_j \Big) - \sum_{j \in \mathcal{M}} r_j = \Big(\sum_{t=i+1}^{s} \Big(\sum_{j \in \mathcal{L}_t} r_j \Big) - \sum_{j \in \mathcal{M}_{i+1}} r_j \Big) + \Big(\sum_{j \in \mathcal{L}_i} r_j - \sum_{j \in \mathcal{M}_i \setminus \mathcal{M}_{i+1}} r_j \Big).$$

Since $\mathcal{M}_i = \mathcal{M}_{i+1} \bigcup \mathcal{L}_i$, we have $\mathcal{M}_i \setminus \mathcal{M}_{i+1} \subset \mathcal{L}_i$. Thus, the expression in the last pair of parentheses is nonnegative. The inequality $a_i \geq a_{i+1}$ is proved.

Let us return to the verification of property (ii). If the inequality

$$\sum_{i \in I} (r^*_i - r_i)(\nu_i - \delta_{i-1} n) \geq 0$$

holds, then (ii) also holds. Let us assume that converse is true. Then

$$\sum_{t=0}^{s} (-a_t)(\nu_{k_t} - \delta_{k_t - 1} n) < 0$$

or

$$a_0(\nu_1 - 2n) + \sum_{t=1}^{s} a_t(\nu_{k_t} - n) > 0.$$

Here the first term is nonpositive and all summands in the sum are nonnegative. Therefore, by Lemma 1,

$$a_0 \Big(\sum_{j \in \mathcal{N}} \nu_j - (s+2)n \Big) > 0.$$

Hence

$$\sum_{i=0}^{s} \nu_{k_i} > (s+2)n.$$

But this is exactly the Noether–Fano inequality for the subgraph with vertices E_{k_i}, $0 \leq i \leq s$ (it is easy to see that this subgraph is a chain). In other words if (ii) does not hold, then $B_{k_s - 1}$ is a maximal singularity of the type B, (1). The first proposition of the graph lemma is proved.

5. The Proof of the Second Part of the Graph Lemma. The reasoning is completely analogous to the above. Let us assume that there exist lines $L \subset E_1 \cong \mathbb{P}^2$ and $L' \subset E_2 \cong \mathbb{P}^2$ and indices $k, k' \in I_1$ such that $B_{k-1,1} = L$ and $B_{k'-1,2} = L'$ (if there are no such L' and k', then the reasoning is the same as in Secs. 1-4; if there are no such L and k, then it is the same as the reasoning below with some simplifications). Let us define the following sets $\mathcal{N}_1 = \{k_0, \dots, k_s\}$ as $\mathcal{N}$ in Sec. 2; $\mathcal{N}_2 = \{l_0, \dots, l_p\}$, $l_0 = 2$, $\{l_1, \dots, l_p\} = \{j \in I_1 | B_{j-1,2} = L'\}$, $l_{i+1} > l_i$, $\mathcal{N} = \mathcal{N}_1 \bigcup \mathcal{N}_2$; $\mathcal{L}_i$, $\mathcal{M}_i$, $\mathcal{K}_i$ as in Sec. 2; $\mathcal{L}'_i$, $\mathcal{M}'_i$, $\mathcal{K}'_i$ analogously for E_2:

$$\mathcal{L}'_i = \{j \in I_0 | j \to l_i\}, \quad \mathcal{M}'_i = \bigcup_{t=i}^{p} \mathcal{L}'_t, \quad \mathcal{K}'_i = \bigcup_{t=0}^{i} \mathcal{L}'_t.$$

Let us define the coefficients r_i^* as follows

$$\begin{aligned}
r_i^* &= r_i, && \text{if} \quad i \notin \mathcal{N}, \\
r_{l_i}^* &= \sum_{j \in \mathcal{M}_i} r_j, && \text{for} \quad 0 \le i \le p, \\
r_{k_i}^* &= \sum_{j \in \mathcal{M}_i} r_j, && \text{for} \quad 1 \le i \le s, \\
r_1^* &= r_2^* + \sum_{j \in \mathcal{M}_0 \setminus \{2\}} r_j.
\end{aligned}$$

Properties (i), (v), and (vi) hold. The verification of (iii) and (iv) can be done as in the first case (Secs. 2 and 3). Let us study property (ii). Let $a_i = r_{k_i} - r_{k_i}^*$, $b_j = r_{l_j} - r_{l_j}^*$. By the proof of Lemma 1, $a_i \ge 0$, $b_j \ge 0$, $a_i \ge a_{i+1}$, $b_j \ge b_{j+1}$. We want to prove that $a_0 \ge a_1 + b_0$. Indeed,

$$a_0 = r_1 - r_1^* = r_2 + \sum_{\substack{j \to 1 \\ j \ne 2}} r_j - r_2^* - \sum_{j \in \mathcal{M} \setminus \{2\}} r_j = (r_2 - r_2^*) + \Bigg(\sum_{\substack{j \to 1 \\ j \ne 2}} r_j - \sum_{j \in \mathcal{M}_0 \setminus \{2\}} r_j \Bigg).$$

Repeating again the reasoning of the proof of Lemma 1, we have that the expression in the second pair of parentheses is not less than $r_{k_1} - r_{k_1}^* = a_1$. Finally if for r_i^* an analog of the strengthened Noether–Fano inequality does not hold, then, as in Sec. 4, we have

$$a_0(\nu_1 - 2n) + b_0(\nu_2 - 2n) + \sum_{t=1}^{s} a_t(\nu_{k_t} - n) + \sum_{t=1}^{p} b_t(\nu_{l_t} - n) > 0.$$

If we substitute b_0 for all b_t, $t \ge 1$, and $a_0 - b_0 \ge a_1$ for all a_t, $t \ge 1$, then the inequality will only be stronger. Hence

$$(a_0 - b_0)\Bigg(\sum_{j \in \mathcal{N}_1} \nu_j - (s+2)n \Bigg) + b_0 \Bigg(\sum_{j \in \mathcal{N}_2 \bigcup \{1\}} \nu_j - (p+4)n \Bigg) > 0.$$

If the first summand is positive, then $B_{k_s - 1}$ is a maximal singularity of the type B, (1). If the second summand is positive, then $B_{l_p - 1}$ is a maximal singularity of the type C, (1).

Since the third proposition of the graph lemma is trivial ($B_2 \notin E_1^2$), the graph lemma is completely proved.

Chapter 4
Birational Automorphisms of a Four-Dimensional Quintic

Up to the end of this chapter, V will be a nonsingular hypersurface of degree 5 in $\mathbb{P}^5$. The main task here is to prove the following statement.

Theorem. *V is a birationally superrigid manifold.*

Corollary. *A birational isomorphism of two smooth 4-dimensional quintics is a projective equivalency.*

For proving the theorem we shall fix a test manifold (V', L') and a birational map $\chi\colon V \dashrightarrow V'$ (under the assumption of their existence). Let us note that $(-K_V)$ is a hyperplane section of $V \subset \mathbb{P}^5$. By the notations of Chapter 5, the "degree" of the divisors of the linear system $|\chi|$ is, obviously, the degree of the hypersurfaces in $\mathbb{P}^5$ which section this linear system. We shall show that the assumption of the existence of a maximal singularity will lead us to a contradiction. By Proposition 1.2.1, the theorem is a consequence of this fact.

All notations of Chapter 1 hold.

1. The Choice of a Resolution of Singularities.

In the course of the proof, we shall need for the resolution except admissibility one more property. Let $k \geq 1$ be an arbitrary positive integer.

Lemma 1. *We can choose a resolution $R(V,\chi)$ so that the following condition holds: there exists a set of indices $\mathcal{D} \subset \{1,\dots,N\}$ such that*

(1) $\dim B_{j-1} = 0$ *for $j \in \mathcal{D}$ and if $E_i \geq E_j$ and $i \in \mathcal{D}$, then $j \in \mathcal{D}$;*

(2) *let B_j be a singularity such that $\dim B_j \geq 1$, $\dim B_{j,0} = 0$, then the set $M(j) = \{i \in \mathcal{D} | B_{i,j} \subset E_i\}$ is nonempty and if $m(j) = \max\{i \in M(j)\}$, then either $\dim B_{j,m(j)} \geq 1$ or $\#M(j) = k$.*

Proof. We shall describe the construction of a resolution with the above property. We shall use Definition 1.1.2 and Proposition 1.1.2.

Definition 1. Let $R(V,\chi)$ be an arbitrary (admissible) resolution. The *set of distinguished points of the resolution* $R(V,\chi)$ is the finite set $\mathcal{D}(R) = \{B_{i,0} | \dim B_{i,0} = 0\} \subset V$.

Let $U = V \setminus \mathcal{D}(R)$; then $R(V,\chi)|_U = \overline{R}_1(U, \chi|_U)$ is the resolution of the map $\chi|_U$ and $\mathcal{D}(\overline{R}_1) = \varnothing$ by the construction. Let $\pi_1\colon V^1 \to V$ be a blowing of the set $\mathcal{D}(R)$, $E^{(1)} = \pi_1^{-1}(\mathcal{D}(R))$. Obviously, $V^1 \setminus E^{(1)} \cong U$, and so $\overline{R_1}(U,\chi|_U) = \overline{R}_1(V^1 \setminus E^{(1)}, \chi \circ \pi_1|_{V^1 \setminus E^{(1)}})$ and $\mathcal{D}(\overline{R}_1) = \varnothing$. Now, by Proposition 1.1.2, there exists a resolution $R_1(V^1, \chi \circ \pi_1)$ of the map $\chi \circ \pi_1\colon V^1 \dashrightarrow V'$ such that

$$R_1(V^1, \chi \circ \pi_1)|_{V^1 \setminus E^{(1)}} = \overline{R}_1(V^1 \setminus E^{(1)}, \chi \circ \pi_1|_{V^1 \setminus E^{(1)}}),$$

and, therefore, $\mathcal{D}(R_1) \subset E^{(1)}$. This procedure for constructing the pair $\big(V^1, R_1(V^1, \chi \circ \pi_1)\big)$ from the pair $\big(V, R(V,\chi)\big)$ (which can be applied to any resolution of singularities of a rational map of an arbitrary manifold) shall be called the *blowing of the distinguished points of the resolution $R(V,\chi)$*.

Let $R_1(V^1, \chi\circ\pi_1) = \{\varphi^1_{i,i-1}\colon V^1_i \to V^1_{i-1} | 1 \leq i \leq N_1\}$, where $V^1_0 = V^1$, and let $\varphi^1_{i,i-1}\colon V^1_i \to V^1_{i-1}$ be a blowing of the smooth irreducible center $B^{(1)}_{i-1} \subset V^1_{i-1}$, $E^{(1)}_i = (\varphi^1_{i,i-1})^{-1}(B^{(1)}_{i-1})$ be an exceptional divisor, and $E^{(1)}_0 = E^{(1)} \subset V^1$. Let us represent $\pi_1\colon V^1 \to V$ as the composition of blowings of points, $\pi_1 = \pi_{1,1} \circ \pi_{1,2} \circ \dots \circ \pi_{1,r_1}\colon V^1 \to V$. We declare that the sequence R^* of blowings which consists of the blowings $\pi_{1,1},\dots,\pi_{1,r_1}$ and $\varphi^1_{1,0},\dots,\varphi^1_{N_1,N_1-1}$ is the resolution which has the property, formulated in the lemma, for $k = 1$, where $\mathcal{D} = \{1,\dots,r_1\}$ corresponds the first r_1 blowings $\pi_{1,i}$.

Indeed, let $B^{(1)}_i$ be a singularity such that $\dim B^{(1)}_i \geq 1$ and $\dim \pi_1 \circ \varphi^1_{i,0}(B^{(1)}_i) = 0$, i.e., $\pi_1 \circ \varphi^1_{i,0}(B^{(1)}_i)$ at some point $x \in V$. But $x \neq V \setminus \mathcal{D}(R)$; therefore the restriction of the resolution $R^*|_{V \setminus \mathcal{D}(R)}$ does not have distinguished points (by the construction). Hence, $x \in \mathcal{D}(R)$, i.e., $\varphi^1_{i,0}(B^{(1)}_i) \subset E^{(1)}$. But this is exactly the property of the lemma for $k = 1$.

For an arbitrary k, following the example studied above, we shall apply the procedure of blowing of the distinguished points k times. By blowing the distinguished points of the resolution $R_1(V^1, \chi\circ\pi_1)$, we get V^2, $R_2(V^2, \chi \circ \pi_1 \circ \pi_2)$, where $\pi_2\colon V^2 \to V^1$ is the blowing of $\mathcal{D}(R_1)$, $\mathcal{D}(R_2) \subset E^{(2)} = \pi_2^{-1}(\mathcal{D}(R_1))$. Repeating this k times, we get $\dots, V^i, R_i(V^i, \chi\circ\pi_1\circ\dots\circ\pi_i), \dots, V^k$, $R_k(V^k, \chi\circ\pi_1\circ\dots\circ\pi_k)$ (if $\mathcal{D}(R_j) = \varnothing$ for $j < k$, then all π_i, for $i > j$, are isomorphisms).

Let $R_k(V_k, \chi\circ\pi_1\circ\dots\circ\pi_k) = \{\varphi^k_{i,i-1}\colon V^k_i \to V^k_{i-1} | 1 \leq i \leq N_k\}$, $V^k_0 = V^k$, $\varphi^k_{i,i-1}\colon V^k_i \to V^k_{i-1}$ be a blowing of the smooth irreducible center $B^{(k)}_{i-1}$, $E^{(k)}_i = (\varphi^k_{i,i-1})^{-1}(B^{(k)}_{i-1})$, and $E^{(k)}_0 = E^{(k)} \subset V^k$.

Set $\pi_{i,j} = \pi_{i+1} \circ \cdots \circ \pi_i \colon V^i \to V^j$ if $i > j$. Let $B_i^{(k)}$ be such that $\dim B_i^{(k)} \geq 1$ and $\dim \pi_{k,0} \circ \varphi_{i,0}^k(B_i^{(k)}) = 0$. Set also $m = \max\{j \mid \dim \pi_{k,j} \circ \varphi_{i,0}^k(B_i^{(k)}) = 0\}$. Obviously, $\pi_{k,j+1} \circ \varphi_{i,0}^k(B_i^{(k)}) \subset E^{j+1}$ for all $j < m$, and if $m < k$, then also $\dim \pi_{k,m+1} \circ \varphi_{i,0}^k(B_i^{(1)}) \geq 1$. Let us now represent $\pi_{k,0}$ as the composition of blowings of points: $\pi_{k,0} = \overline{\varphi}_{r,r-1} \circ \cdots \circ \overline{\varphi}_{1,0}$, $\overline{\varphi}_{i,i-1} \colon \overline{V}_i \to \overline{V}_{i-1}$, $\overline{V}_r = V^k$, and let us consider the following resolution of the singularities of the map $\chi\colon \widetilde{R}(V,\chi) = \{\widetilde{\varphi}_{i,i-1}\colon \widetilde{V}_i \to \widetilde{V}_{i-1} \mid 1 \leq i \leq \widetilde{N}\}$, $\widetilde{N} = r + N_k$, $\widetilde{V}_0 = V$, $\widetilde{V}_i = \overline{V}_i$ for $1 \geq i \geq r$, $\widetilde{\varphi}_{i,i-1} = \overline{\varphi}_{i,i-1}$ for $1 \leq i \leq r$, $\widetilde{V}_{r+i} = V_i^k$, $\widetilde{\varphi}_{r+i,r+i-1} = \varphi_{i,i-1}^k$.

Let $\widetilde{E}_i$ be an exceptional divisor of $\widetilde{\varphi}_{i,i-1}$,$\widetilde{B}_{i-1} = \widetilde{\varphi}_{i,i-1}(\widetilde{E}_i)$, ν_i be a multiplicity along $\widetilde{B}_{i-1}$ of the linear system $|\chi|_{i-1}$, i.e., the proper inverse image of $|\chi|$ on $\widetilde{V}_{i-1}$. By our general assumption about the admissibility of the chosen resolutions, we have $\widetilde{\nu}_{r+i} \geq \widetilde{\nu}_{r+j}$ if $i \leq j$.

For the resolution $\widetilde{R}$, the conditions of the lemma hold (see above), and property (2) of Proposition 1.1.1 also holds. But, in general, $\widetilde{R}$ is not admissible. To deduce property (1) from Proposition 1.1.1, it is enough to permute some blowings with disjunct centers. We shall proceed by induction.

Definition 2. The resolution $R(V,\chi) = \{\varphi_{i,i-1}\colon V_i \to V_{i-1} \mid 1 \leq i \leq N\}$ *belongs to the class* r if for the resolution $R(V_r, \chi \circ \varphi_{r,0}) = \{\varphi_{i,i-1}\colon V_i \to V_{i-1} \mid r+1 \leq i \leq N\}$ of the map $\chi \circ \varphi_{i,0}\colon V_r \to V'$ property (1) of Proposition 1.1.1 holds (i.e., $\nu_i \geq \nu_j$ for $r+1 \leq i \leq j$) and the center of each blowing $\varphi_{i,i-1}$ is a point for $i \leq r$.

To complete the proof of the lemma, it is enough to show that the existence of the class r resolution $\widetilde{R}(V,\chi)$, for which the lemma's property and property (2) of Proposition 1.1.1 hold, implies the existence of a class $r-1$ resolution, for which the same properties hold.

Using the above notations, let us consider the singularity $\widetilde{B}_{r-1}$. Let $i_r = \max\{j \mid \widetilde{\nu}_j > \widetilde{\nu}_r\} \bigcup \{r\}$. If $i_r = r$, then there is nothing to prove. In the opposite case, let us note that $\widetilde{B}_{r-1} \notin \widetilde{\varphi}_{i,r-1}(\widetilde{B}_i)$ for $r \leq i \leq i_r$ (it is a consequence of the upper semicontinuity of the point multiplicity with respect to a divisor). Therefore, we can permute the point blowing $\widetilde{B}_{r-1} \in \widetilde{V}_{r-1}$ with the group of blowings $\widetilde{\varphi}_{i,i-1}$, $r+1 \leq i \leq i_r + 1$. The new resolution obviously belongs to the class $r-1$. The lemma is proved.

Definition 3. Using the notations of Lemma 1, we call the singularity B_j such that $\dim B_j \geq 1$ and $\dim B_{j,0} = 0$ a *first genus singularity* if $\dim B_{j,m(j)} \geq 0$, and a *second genus singularity* in the opposite case.

Let us assume, by Lemma 1, that a fixed resolution $R(V,\chi)$ is admissible and the condition of Lemma 1 holds (where $k = 5n^2 + 1$). Let us fix the maximal singularity B_β. We add the notations $\mathcal{D}$, $M(j)$, and $m(j)$ from the statement of Lemma 1 to the notations of Chapter 1 (which we shall use).

2. The Elimination of Maximal Subsets.

1. Proposition 1. *There are no maximal points.*

Proof. By Corollary 1.3.4, the dimension of a maximal subset is positive.

2. Proposition 2. *There are no maximal surfaces.*

Proof. Assume the contrary, i.e., $\widetilde{\nu} = \mathrm{mult}_B |\chi| > n$, where $B \subset V$ is a surface. Let $i = \min\{j \mid B_{j,0} = B\}$. Then $\nu_{i+1} = \widetilde{\nu} > n$. Since

$$0 < (h'^2 \cdot h^2) = 5n^2 - \sum_{i=1}^{N} (b_{i-1} \cdot h^2)\nu_i^2 \leq 5n^2 - (\deg B)\widetilde{\nu}^2,$$

we have $\deg B \leq 4$.

Let us consider a general hyperplane section of V, which we shall denote by V^*. The restriction of the linear system $|\chi|$ to V^* is a nonempty linear system $|\chi|^*$ without stationary components. It has a base curve B^*, which is a hyperplane section of B, and $\deg B^* = \deg B$. B^* is irreducible for a general section. We shall consider all possibilities for B^*.

(1) B^* is a smooth curve. Let $\pi\colon V_1^* \to V^*$ be its blowing, $E^* = \pi^{-1}(B^*)$, e^* be the class of E^* in $A^1(V_1^*)$, $d = \deg B^*$, h^* be a class of the hyperplane section of $V^* \subset \mathbb{P}^4$ in $A^1(V^*)$. It is easy to check that the linear system $|dh^* - e^*|$ is free. On the other hand, the linear system $|nh^* - \widetilde{\nu}e^*|$ does not have stationary components. Therefore, $((dh^* - e^*) \cdot (nh^* - \widetilde{\nu}e^*)^2) \geq 0$. The standard multiplication formulas in

the Chow rings of blowings of 3-folds give us that $h^{*3} = 5$, $(h^{*2} \cdot e^*) = 0$, $(h^* \cdot e^{*2}) = -d$, $e^{*3} = -2g(B^*)+2$, where $g(B^*)$ is a genus of the curve B^*. Thus,

$$5dn^2 - 2n\widetilde{\nu}d - \widetilde{\nu}^2 d^2 + \widetilde{\nu}^2(2g(B^*) - 2) \geq 0.$$

But, $g(B^*) \leq (d-1)(d-2)/2$ and, taking this into account, we get $5dn^2 - 2n\widetilde{\nu}d - 3d\widetilde{\nu}^2 \geq 0$. Hence, $\widetilde{\nu} \leq n$. We have a contradiction.

(2) B^* has singularities. Let, at first, $\deg B^* = 3$, $x^* \in B^*$ be a double point. Let us note that there is a plane which contains B^*. Let $\pi_1\colon\ V_1^* \to V^*$ be the blowing of x^*, $E_1^* = \pi_1^{-1}(x^*)$, e_1^* be the class of E_1^* in $A^1(V_1^*)$, $\widetilde{B}^*$ be the proper inverse image of B^*, i.e., a smooth rational curve. Let $\pi\colon\ V_2^* \to V_1^*$ be the blowing of $\widetilde{B}^*$, $E^* = \pi^{-1}(\widetilde{B}^*)$, e^* be the class of E^* in $A^1(V_2^*)$. It is easy to check that $\mathrm{Bs}|3h^* - 2e_1^* - e^*| = \varnothing$. If $\widetilde{\nu}_1 \geq \widetilde{\nu}$ is a multiplicity of x^* with respect to the linear system $|\chi|^*$, then it is easy to check that the linear system $|nh^* - \widetilde{\nu}_1 e_1^* - \widetilde{\nu}e^*|$ does not have stationary components. Thus,

$$((3h^* - 2e_1^* - e^*) \cdot (nh^* - \widetilde{\nu}_1 e_1^* - \widetilde{\nu}e^*)^2) \geq 0.$$

A direct computation gives

$$0 \leq 15n^2 - 11\widetilde{\nu}^2 - 2\widetilde{\nu}_1^2 - 6n\widetilde{\nu} + 4\widetilde{\nu}\widetilde{\nu}_1 = 15n^2 - 9\widetilde{\nu}^2 - 6n\widetilde{\nu} - 2(\widetilde{\nu} - \widetilde{\nu}_1)^2.$$

This is impossible if $\widetilde{\nu} > n$.

Let $\deg B^* = 4$. The following possibilities exist: (a) B^* is not contained in any plane, but is contained in some 3-plane; in this case, B^* has a unique ordinary double point; (b) B^* is a plane quartic with δ, $1 \leq \delta \leq 3$, double points, including infinitely close points also; (c) B^* is a plane quartic with a triple point.

(a) Let $\pi_1\colon\ V_1^* \to V^*$ be the blowing of the double point $x^* \in B^*$, $E_1^* = \pi_1^{-1}(x^*)$, e_1^* be the class of E_1^*, $\pi\colon\ V_2^* \to V_1^*$ be the blowing of the proper inverse image of $\widetilde{B}^* \subset V_1^*$, $E^* = \pi^{-1}(\widetilde{B}^*)$, e^* be the class of E^*. We have that $\mathrm{Bs}|4h^* - 2e_1^* - e^*| = \varnothing$, and so

$$((4h^* - 2e_1^* - e^*) \cdot (nh^* - \widetilde{\nu}_1 e_1^* - \widetilde{\nu}e^*)^2) \geq 0,$$

where $\widetilde{\nu}_1 = \mathrm{mult}_{x^*} |\chi|^* \geq \widetilde{\nu}$. Calculations give us that

$$20n^2 - 20\widetilde{\nu}^2 - \widetilde{\nu}_1^2 - 4n\widetilde{\nu} + 2\widetilde{\nu}\widetilde{\nu}_1 < 0.$$

(b) We shall restrict ourself to consideration of the cases where all double points of B^* are on B^*, $x_i^* \in B^*$, $1 \leq i \leq \delta$. In other cases where infinitely close double points appear, the reasoning is the same. Let $\pi_1\colon\ V_1^* \to V^*$ be the blowing of the set $\{x_i^* | 1 \leq i \leq \delta\}$, $E_i^* = \pi_1^{-1}(x_i^*)$, e_i^* be the class of E_i^*. Let $\pi\colon\ V_2^* \to V_1^*$ be the blowing of the smooth proper inverse image of $\widetilde{B}^* \subset V_1^*$, $g(\widetilde{B}^*) = 3 - \delta$, $E^* = \pi^{-1}(\widetilde{B}^*)$. We have

$$\mathrm{Bs}\left|4h^* - 2\sum_{i=1}^{\delta} e_i^* - e^*\right| = \varnothing,$$

so

$$0 \leq \left(\left(4h^* - 2\sum_{i=1}^{\delta} e_i^* - e^*\right) \cdot \left(nh^* - \sum_{i=1}^{\delta} \widetilde{\nu}_i e_i^* - \widetilde{\nu}e^*\right)^2\right)$$
$$= 20n^2 - \widetilde{\nu}^2(12 + 2\delta) - 2\sum_{i=1}^{\delta} \widetilde{\nu}_i^2 - 8n\widetilde{\nu} + 4\widetilde{\nu}\sum_{i=1}^{\delta} \widetilde{\nu}_i.$$

But the last expression is negative for $\widetilde{\nu}_i \geq \widetilde{\nu} > n$, $\widetilde{\nu}_i = \mathrm{mult}_{x_i^*} |\chi|^*$.

(c) Let $x^* \in B^*$ be a triple point, $\pi_1\colon\ V_1^* \to V^*$ be its blowing, $E_1^* = \pi_1^{-1}(x^*)$, $\pi\colon\ V_2^* \to V_1^*$ be the blowing of the smooth rational curve $\widetilde{B}^*$ which is the proper inverse image of B^* in V_1^*, $E^* = \pi^{-1}(\widetilde{B}^*)$. Then $\mathrm{Bs}|4h^* - 3e_1^* - e^*| = \varnothing$, and so

$$0 \leq ((4h^* - 3e_1^* - e^*) \cdot (nh^* - \widetilde{\nu}_1 e_1^* - \widetilde{\nu}e^*)^2) = 20n^2 - 15\widetilde{\nu}^2 - 3\widetilde{\nu}_1^2 - 8n\widetilde{\nu} + 6\widetilde{\nu}\widetilde{\nu}_1 < 0$$

for $\nu > n$. Again we have a contradiction. Proposition 2 is proved.

3. Proposition 3.

(A) *The multiplicity of an irreducible plane curve $B \subset V$ with respect to $|\chi|$ is not greater than n.*

(B) *The multiplicity of the linear system $|\chi|$ along a curve $B \subset V$ which is not contained in any plane is not greater than $\sqrt{5/2}n$.*

Corollary 1. *There are no maximal curves.*

Proof of Proposition 1. (A) Let us assume the contrary, i.e., $\widetilde{\nu} = \operatorname{mult}_B |\chi| > n$. Then $\{j | B_{j,0} = B\} \neq \varnothing$ and for $i = \min\{j | B_{j,0} = B\}$ we have that B_i is a curve, $\deg(\varphi_{i,0} \colon B_i \to B) = 1$, and $\nu_{i+1} = \widetilde{\nu} > n$. Indeed, $\{j | B_{j,0} \supset B\} \neq \varnothing$. Let $i = \min\{j | B \subset B_{j,0}\}$. Then $\varphi_{i,0}^{-1}$ is an isomorphism in a neighborhood of a general point B. Therefore, $\nu_i \geq \widetilde{\nu} > n$. Hence, either B_i has the type (1,1), which is exactly what we need, or B_i has the type (2,1) (the type (2,2) is impossible by Proposition 2). But if B_i has the type (2,1), then $B_{i,i'} \subset E_{i'}$ for some $i' \leq i$, and so $B \subset B_{i,0} \subset B_{i'-1,0}$. We have a contradiction.

Let us assume now that $\deg B \geq 2$. Let us denote by P a plane which contains B. Let $x \in \mathbb{P}^5$ be a sufficiently general point, $\langle P, x \rangle = S$ be the 3-plane generated by P and x. Let us consider the cone $Q \subset S$ with x as an apex and B as a base. Since x is general, $Q \bigcap V$ is a curve, $Q \bigcap V = B \bigcup C$, and $\deg C = 4 \deg B$. Then $B \bigcap C = P \bigcap C$ and, therefore, $P \bigcap Q = B$. Hence $\# B \bigcap C = \deg C$ (taking into account multiplicities which equal 1, because of the generality of x). Therefore, the proper inverse image of C on V_{i+1} represents the class $z = 4 \deg B(\frac{1}{5}h^3 - f_{i+1})$ (x is general). It remains to note that the curves C, which were defined above, sweep V. Hence $(h' \cdot z) \geq 0$ and $n \geq \nu_{i+1} = \widetilde{\nu}$. We have a contradiction.

Let us consider now the case where B is a line. Though the above reasoning is valid here (we have to substitute the 2-plane $\langle B, x \rangle$ for the 3-plane $\langle P, x \rangle$), we shall give another proof. Its idea will be generalized below. Let us consider a general 3-plane S which contains B. The intersection $S \bigcap V = S^*$ is a nonsingular curve of degree 5 in $S \cong \mathbb{P}^3$. The restriction of the linear system $|\chi|$ to S^* is a nonempty linear system of curves which has a unique (for a general S) stationary component B of multiplicity ν^*, $\nu^* \geq \widetilde{\nu} > n$. Let us denote by $h^* \in A^1(S^*)$ the class of plane sections $S^* \subset S$ and by $b \in A^1(S^*)$ the class of B. Obviously, $(nh^* - \nu^* b)^2 \geq 0$. But, $(h^* \cdot h^*) = 5$, $(h^* \cdot b) = 1$, $(b \cdot b) = -3$ (the last product is computed using the *adjoining* formula). Thus, $5n^2 - 2n\nu^* - 3{\nu^*}^2 \geq 0$. But this contradicts the inequality $\nu^* > n$. (A) is proved.

(B) Let us assume the contrary. Let $\widetilde{\nu} = \operatorname{mult}_B |\chi| > \sqrt{5/2}n$. Set $i = \min\{j | B_{j,0} = B\}$; then $\nu_{i+1} = \widetilde{\nu}$, $\deg(\varphi_{i,0} \colon B_i \to B) = 1$. Set $y = h^2 - 2g_{i+1}$. We want to prove that y is nonnegative.

Proof. Let $Z \subset V_j$ be a surface. If $j \leq i$, then $(z \cdot y) = (z \cdot h^2) \geq 0$. If $j \geq i+1$ and $\dim Z_0 = 2$, then

$$(y \cdot z) = \deg(\varphi_{j,0} \colon Z \to Z_0)(\deg Z_0 - 2 \operatorname{mult}_{B_i} Z_i) \geq \deg(\varphi_{j,0} \colon Z \to Z_0)(\deg Z_0 - 2 \operatorname{mult}_B Z_0) \geq 0,$$

because, in the opposite case, each section (a chord) of the curve B is contained in Z_0. In this case, the curve B would be a plane curve and Z_0 would be a plane. Finally if $j \geq i+1$ and $\dim Z_0 \leq 1$, then $(y \cdot z) = -2(z \cdot g_{i+1})$. Therefore, the desired inequality is a consequence of Lemma 1.3.1.

Thus, y is nonnegative. Hence $0 \leq (h'^2 \cdot y) \leq 5n^2 - 2\nu_{i+1}^2 < 0$. We have a contradiction. Proposition 3 is proved.

4. Corollary 2. *The maximal singularity B_β has the type* $(2,1)$ *or* $(1,0)$ *or* $(2,0)$.

Proof. In view of Proposition 1.2 and Corollary 1 of Proposition 1.2.2, we have that the maximal singularity cannot have the type (i,i), $i = 0, 1, 2$.

3. Test Class.

Specifying Definition 1.4.1, we shall introduce a family of test classes in the following way. Let $\alpha \in \mathbb{R}_+$ be an arbitrary number; then

$$y(\alpha) = \sum_{j \in I_0} r_j(h^2 - g_j) + \sum_{j \in I_1} r_j(\alpha h^2 - g_j) \in A^2(V_N) \bigotimes \mathbb{R}.$$

Definition 1. We shall define the *cone of negative classes* $N(\alpha) \subset A^2(V_N) \bigotimes \mathbb{R}$ as the intersection of $\{z \in A^2(V_N) \bigotimes \mathbb{R} | (z \cdot y(\alpha) < 0\}$ and $\{z^* \in A^2(V_N) \bigotimes \mathbb{R} |$ for some i, $0 \leq i \leq N$, there exists a (correct) surface $Z \subset V_i$ such that $z^* = \omega z$, where $\mathbb{R} \ni \omega > 0\}$.

Lemma 1. (A) *Let $Z \subset V_i$ be a surface of the type* $(2,1)$ *or* $(1,0)$. *Then* $(z\cdot y(\alpha)) \to 0$, *i.e.*, $z \notin N(\alpha)$.
(B) *For an arbitrary α if $i \in I_2$, then $(b_{i-1}\cdot y(\alpha)) \geq r_i$.*

Proof. Lemma 1 is a consequence of Lemma 1.4.1 and Proposition 2.2.

Remark 1. If $\alpha \leq \zeta$, then $N(\alpha) \supset N(\zeta)$.

Proof. It is an obvious statement.

Lemma 2.

(A) *For any surface $Z \subset V_i$ of the type $(2,2)$, the morphism $\varphi_{\mu,0}$ is an isomorphism in a neighborhood of a general point Z_μ, where $\mu = \min(\beta, i)$; in particular, $Z_j = Z_0^j$ for all j, $0 \leq j \leq \mu$, and* $\deg(\varphi_{\mu,0}\colon\ Z_\mu \to Z_0) = 1$.

(B) *Let $Z \subset V_i$ be a surface such that $\dim Z_0 = 2$ and $(z\cdot(\alpha h^2 - g_j)) < 0$ for some j, $i \geq j \in I_0 \bigcup I_1$, and some $\alpha > 0$. Then $\alpha \deg Z_0 < \operatorname{mult}_{B_{j-1,t}} Z_t$ for all $t < j$.*

Proof. (A) Let us assume the contrary, then $Z_j \subset E_j$ for some j, $1 \leq j \leq \mu$. But, Z_j has the type (2,2), and so B_{j-1} has the same type. Thus, we have the inequality $\nu_j \geq \nu_\beta > n$, which contradicts Proposition 2.2.

(B) From the inequality $(z\cdot(\alpha h^2 - g_j)) < 0$ we get

$$\alpha \deg Z_0 \deg(\varphi_{i,0}\colon\ Z \to Z_0) < \operatorname{mult}_{B_{j-1}} Z_{j-1} \deg(\varphi_{i,j-1}\colon\ Z \to Z_{j-1}).$$

But,

$$\deg(\varphi_{i,0}\colon\ Z \to Z_0) = \deg(\varphi_{i,j-1}\colon\ Z \to Z_{j-1}) \deg(\varphi_{j-1,0}\colon\ Z_{j-1} \to Z_0).$$

Hence

$$\alpha \deg Z_0 \deg(\varphi_{j-1,0}\colon\ Z_{j-1} \to Z_0) < \operatorname{mult}_{B_{j-1}} Z_{j-1},$$

and, by (A), $\alpha \deg Z_0 < \operatorname{mult}_{B_{j-1}} Z_{j-1}$. Finally, the inequality $\operatorname{mult}_{B_{j-1}} Z_{j-1} \leq \operatorname{mult}_{B_{j-1,t}} Z_t$, for all $t < j$, is a consequence of the correctness condition. The lemma is proved.

Corollary 1. *Let $Z \subset V_i$ be a surface such that $h \in N(\alpha)$ for some $\alpha > 0$. Then $\dim Z_0 = 2$, $I_1 \neq \varnothing$, and there exists j, $i \geq j \in I_1$, such that*

$$\alpha \deg Z_0 < \operatorname{mult}_{B_{j-1}} Z_{j-1} \leq \operatorname{mult}_{B_{j-1,t}} Z_t$$

for all $t < j$. If B_β has the type $(*,0)$, then, in addition, we have that $\mathcal{D}\bigcap I_0 \neq \varnothing$, $i > m(\beta)$, and $\alpha \deg Z_0 < \operatorname{mult}_{B_{t-1}} Z_{t-1}$ for all $t \in \mathcal{D}\bigcap I_0$.

Proof. By Lemma 1.3.1, $(z\cdot(h^2 - g_j)) \geq 0$ for all $j \in I'$. Therefore if $(z\cdot y(\alpha)) < 0$, then $I_1 \neq \varnothing$ and $(z\cdot(\alpha h^2 - g_j)) < 0$ for some $j \in I_1$, $i \geq j$. Now we can prove the first statement of the corollary using the above lemma.

Let us assume that B_β has the type $(*,0)$; then $\mathcal{D}\bigcap I_0 \neq \varnothing$ (by Lemma 1.1). Let us note that if $j_1 \in \mathcal{D}\bigcap I_0$ and $j_2 \in I_1$, then $j_2 > j_1$. Indeed, obviously, $j_1 \neq j_2$ and if $j_1 > j_2$, then $B_{\beta,j_1} \subset E_{j_1}$ implies $B_{\beta,j_1-1} \subset B_{j_1-1}$. But B_{j_1-1} is a point, and so $B_{\beta,j_1-1} = B_{j_1-1}$. Taking into account the fact that $B_{\beta,j_2} = B_{j_1-1,j_2}$ and $B_{\beta,j_2} \subset E_{j_2}$, we have $B_{j_1-1,j_2} \subset E_{j_2}$. Therefore, by Lemma 1.1, we have $j_2 \in \mathcal{D}$. We have a contradiction. Hence, $i > j_1$ for all $j_1 \in \mathcal{D}\bigcap I_0$, i.e., $i > m(\beta)$. The last inequality can be proved in the following way. We have $\alpha \deg Z_0 < \operatorname{mult}_{B_{j-1,t-1}} Z_{t-1}$ for all $t < j$, in particular, for all $t \in \mathcal{D}\bigcap I_0$. As above, we have for $t \in \mathcal{D}\bigcap I_0$ that if $B_{\beta,j-1} \subset B_{j-1}$, then $B_{t-1} = B_{\beta,t-1} \in B_{j-1,t-1}$.

4. Maximal Singularities of the Types (2,1) and (1,0).

1. In this section, we shall prove that there are no maximal singularities of the types (2,1) and (1,0).

Lemma 1. $N(1/2) \neq \varnothing$.

Proof. Let us assume the contrary. Then $(h'^2\cdot y(1/2)) \geq 0$ by Proposition 1.1.3. Let us compute this product directly.

$$(h'^2\cdot y(1/2)) = (h^2\cdot y(1/2)) - \sum_{i=1}^{N}(b_{i-1}\cdot y(1/2))\nu_i^2 - \sum_{j\in I'} r_j\nu_j^2.$$

Obviously,

$$(h^2 \cdot y(1/2)) = 5\Big(\sum_{j\in I_0} r_j + 1/2 \sum_{j\in I_1} r_j\Big).$$

By Lemma 3.1 (B), $(b_{i-1} \cdot y(1/2)) \geq r_i$ for $i \in I_2$. If $i \notin I_2$, then $(b_{i-1} \cdot y(1/2)) \geq 0$ because $N(1/2) = \varnothing$; thus,

$$\sum_{i=1}^{N} (b_{i-1} \cdot y(1/2))\nu_i^2 \geq \sum_{i\in I_2} r_i \nu_i^2$$

Combining these estimations and using the square inequality (Lemma 1.4.2), we have $(h'^2 \cdot y(1/2)) < 0$. We have a contradiction.

We shall prove now that if B_β has the type (2,1) or (1,0), then the class $y(1/2)$ is nonnegative. By this reasoning, we can eliminate cases of the type (2,1) and (1,0).

2. Proposition 1. *B_β cannot have the type* $(2,1)$.

Proof. Let us assume that B_β has the type (2,1); then $I_0 = \varnothing$.

Lemma 2. *If B_β has the type* $(2,1)$, *then $y(1/2)$ is nonnegative.*

Proof. Obviously, any singularity B_{j-1} has the type (1,1) for $j \in I_1$. Hence if $Z \subset V_i$ is a surface such that $z \in N(1/2)$, then $\dim Z_0 = 2$ and $2\,\mathrm{mult}_{B_{j-1,0}} Z_0 > \deg Z_0$ for some $j \in I_1$ (by Corollary 3.1 if $B_{j-1,0}$ is a curve). In this case, any section of $B_{j-1,0}$ belongs to Z_0 and Z_0 is a plane. The multiplicity of a plane curve with respect to $|\chi|$ is not greater than n (by Proposition 2.3); therefore, $\nu_j \leq n$. But we know that $\nu_j \geq \nu_\beta > n$. We have a contradiction. Lemma 4.2 and Proposition 1 are proved.

3. Proposition 2. *B_β cannot have the type* $(1,0)$.

Proof. Let us assume that B_β has the type (1,0).

Lemma 3. *If B_β has the type* $(1,0)$, *then $y(1/2)$ is nonnegative.*

Proof. Let us assume that $z \in N(1/2)$ for $Z \subset V_i$. Since $\dim B_{\beta,0} = 0$, we have $\mathcal{D} \bigcap I_0 = \varnothing$. Set $i_1 = \min\{j \subset \mathcal{D} \bigcap I_0\}$. For simplicity of notations, we shall assume that $i_1 = 1$ and $B_{i_1-1} = B_0$ (this assumption is correct, because $\varphi_{i_1-1,0}$ is an isomorphism in a neighborhood of B_{i_1-1} and for all $j \in I$ we have $j \geq i_1$). By Corollary 3.1, we have that $2\,\mathrm{mult}_{B_{\beta,1}} Z_1 > \deg Z_0$ (because $1 \notin I_1 \neq \varnothing$). Thus, there exists a surface $Z_0 \subset V$ such that $\mathrm{mult}_{B_0} Z_0 > 1/2 \deg Z_0$ and the proper inverse image of $Z_1 \subset V_1$ contains a point x such that $\mathrm{mult}_x Z_1 > 1/2 \deg Z_0$, $x \in E_1 \bigcap Z_1$. Moreover, $\mathrm{mult}_{B_0} |\chi| \geq \nu_\beta > 2n$, $\mathrm{mult}_x |\chi|_1 \geq \nu_\beta > 2n$.

Let C be a line in $\mathbb{P}^5$ which passes through B_0 in the direction x. The multiplicity of the intersection of C with Z_0 is greater than $\deg Z_0$, and so $C \subset Z_0$ and $C \subset V$. Let us consider a general 3-plane S which contains C. We have that $S \bigcap V = S^*$ is a nonsingular surface of degree 5 in $S \cong \mathbb{P}^3$. Let us restrict $|\chi|$ to S^*. We get the linear system $|\chi|^*$ which is sectioned on S^* by surfaces in S of degree n. This system has a unique stationary component C of multiplicity $\nu \geq 0$ and not less than 2 base points: a simple point B_0 and an infinitely close point x over B_0 of multiplicities $\nu_1^* > 2n$ and $\nu_2^* > 2n$, respectively. Let $\pi_1\colon\ S_1^* \to S^*$ be a blowing of $B_0 \in S^*$ (S_1^* is isomorphic to $(S^*)^1$, i.e., to the proper inverse image of S^* on V_1), and $E_1^* = \pi_1^{-1}(B_0)$ be an exceptional divisor (a line), $x \in E_1^*$. Let $\pi_2\colon\ S_2^* \to S_1^*$ be a blowing of x, $E_2^* = \pi_2^{-1}(x)$ be an exceptional divisor, e_1^* and e_2^* be the classes of E_1^* and E_2^* in $A^1(S_2^*)$, c be the class of C, and h^* be the class of a plane section of $S^* \subset S \cong \mathbb{P}^3$ in $A^1(S^*) \hookrightarrow A^1(S_2^*)$. Let us note that the class of the proper inverse image of C on S_2^* is $c - e_1^* - e_2^*$. The class of a general element of the nonstationary part of the complete inverse image of the linear system $|\chi|^*$ on S_2^* is $nh^* - \nu_1^* e_1^* - \nu_2^* e_2^* - \nu\tilde{c} = nh^* - \nu c - (\nu_1^* - \nu)e_1^* - (\nu_2^* - \nu)e_2^*$. Therefore,

$$5n^2 - 2n\nu - 3\nu^2 - (\nu_1^* - \nu)^2 - (\nu_2^* - \nu)^2 \geq 0$$

($c^2 = -3$ by the addition formula). But a direct computation gives us that the last expression is negative for $\nu_1^* > 2n$ and $\nu_2^* > 2n$, $\nu \in \mathbb{R}$. We have a contradiction. The lemma is proved.

We have a contradiction to Lemma 1; thus, Proposition 2 is proved.

5. Maximal Singularities of the Type (2,0): An Introduction and Associated Constructions.

1. We come to the main part of the proof. Let us note that B_β has the type (2,0); all other cases are impossible by the above reasoning. Our aim is to obtain a contradiction, i.e., to prove the two inequalities $(h'^2 \cdot y(\alpha)) \geq 0$ and $(h'^2 \cdot y(\alpha)) < 0$ for some α. To this end, we shall need a special technique. The key feature here is a description of $N(\alpha)$ for some α, $1/2 \leq \alpha < 1$. First, let us give an exact description of the problem.

Since $B = \beta$ belongs to the type (2,0), we have $\mathcal{D}\bigcap I_0 = \mathcal{D}\bigcap I = \varnothing$, $\mathcal{D}\bigcap I_0 = \{i_1 < \cdots < i_r = m(\beta)\}$. By the construction of a resolution (Lemma 1.1), φ_{i_t-1,i_t-1} is an isomorphism in a neighborhood of B_{i_t-1}, $1 \leq t \leq r$, $i_0 = 0$. Moreover, for all $j_1 \in \mathcal{D}\bigcap I$ and $j_2 \in I \setminus (\mathcal{D}\bigcap I)$ we have that $j_1 < j_2$ (the reasoning here is analogous to the reasoning of Corollary 3.1: if $j_1 > j_2$, then $B_{j_1-1,j_2} = B_{\beta,j_2} \in E_{j_2}$, and so $j_2 \in \mathcal{D}$ — we have a contradiction). Hence, for simplification of the notations, we can set $i_1 = 1, \ldots, i_r = m(\beta)$, $B_{i_1-1} = B_0 = x_0, \ldots, B_{i_r-1} = B_{r-1} = x_{r-1}$. Therefore, $\mathcal{D}\bigcap I = 1, \ldots, m(\beta)$, i.e., exactly that part of the resolution which is of interest to us begins with r blowings of points $x_0, \ldots, x_{r-1}$, where x_{i+1} is over x_i. Let us sum up all our knowledge about $N(\alpha)$.

Lemma 1. *Let $Z \subset V_i$ be a surface and $z \in N(\alpha)$ for some $\alpha > 0$. Then*

(A) $\dim Z_0 = 2$;

(B) $i > m(\beta)$; $Z_j = Z^j_{j-1} = \cdots = Z^j_0$ *for* $1 \leq j \leq m(\beta)$;

(C) $x_j \in Z_j$ *for* $0 \leq j \leq m(\beta) - 1$ *and* $\operatorname{mult}_{x_j} Z_j > \alpha \deg Z_0$;

(D) $B_{\beta,m(\beta)} \subset Z_{m(\beta)}$ *and* $\operatorname{mult}_{B_{\beta,m(\beta)}} Z_{m(\beta)} > \alpha \deg Z_0$.

Proof. (A) is a consequence of Lemma 3.1. (B) is a consequence of Lemma 3.2 and Corollary 3.1. (C) is a consequence of Corollary 3.1. (D) is also a consequence of Corollary 3.1, because there exists $j \in I_1 \neq \varnothing$ such that $j > m(\beta)$, $\alpha \deg Z_0 < \operatorname{mult}_{B_{j-1,m(\beta)}} Z_{m(\beta)}$. But $B_{\beta,j} \subset E_j$, and so $B_{\beta,m(\beta)} \subset B_{j-1,m(\beta)}$. The lemma is proved.

2. The following lemmas give us a method of describing of $N(\alpha)$.

Let us introduce new notations. Let $X \subset \mathbb{P}^M$ be a manifold, maybe singular, $x \in X$. Let us denote by $T_xX \subset \mathbb{P}^M$ its (closed) tangent cone at x, by $\overline{T_xX} \subset E_x$ its projectivization in the exceptional divisor E_x of the blowing of $x\widetilde{\mathbb{P}^M} \to \mathbb{P}^M$ (we are taking the multiplicities into account: $\deg T_xX = \operatorname{mult}_x X$, $\overline{T_xX} \subset E_x$ is a scheme-theoretic intersection of the proper inverse image of $\widetilde{X} \subset \mathbb{P}^M$ by the x-blowing with E_x). Set also

$$T = T_{x_0}V, \quad \widetilde{V} = V\bigcap T, \quad Q_0 = T_{x_0}\widetilde{V} \subset T.$$

Lemma 2.

(A) *Let $X \subset \mathbb{P}^M$ be an irreducible manifold, $C \subset X$ be an irreducible submanifold, $x \in C$, and*

$$\operatorname{mult}_x C(\operatorname{mult}_x X + 1) > \deg C \operatorname{mult}_x X.$$

Then $C \subset T_xX$.

(B) *Let $Q \subset \mathbb{P}^3$ be an irreducible cone of degree 6 with x as a apex and let $C \subset Q$ be a curve. Then* $\operatorname{mult}_x C = \deg C (\operatorname{mod} m)$.

(C) *Let $X \subset \mathbb{P}^M$ be an irreducible manifold, $x \in X$, and* $\operatorname{mult}_x X = \deg X - 1$. *Then X is contained in a linear subspace of dimension* $\dim X + 1$.

Proof. (A) Considering a section of X by a general linear subspace of codimension $\dim C - 1$ which contains x, we can reduce the problem to the case where C is an (irreducible) curve. The projection from a general subspace of dimension $\operatorname{codim} X - 2$ reduces the problem to the case where X is a hypersurface in $\mathbb{P}^M$. Let $\pi\colon \widetilde{\mathbb{P}^M} \to \mathbb{P}^M$ be the blowing of x, E be the exceptional divisor. Let $\widetilde{X}$, $\widetilde{T_xX}$, $\widetilde{C}$ be the proper inverse images of X, T_xX, C, respectively. Let h be the class of a hyperplane in $A^1(\mathbb{P}^M) = \mathbb{Z}h$. Let us consider an invertible sheaf over $\widetilde{\mathbb{P}^M}$ which is associated with the divisor $\widetilde{T_xX} - E$ and let us denote it by $\mathcal{F}$. Obviously, $\mathcal{H}^0(\widetilde{X}, \mathcal{F} \otimes \mathcal{O}_{\widetilde{X}}) \neq 0$, because $\mathcal{F} \otimes \mathcal{O}_X$ has a nonzero global section which is zero on the proper inverse image of the cycle $X \bigcap T_xX$ in $\widetilde{\mathbb{P}}^M$. But,

$$\deg \mathcal{F}|_{\widetilde{C}} = \Big(\widetilde{c} \cdot \big((\operatorname{mult}_x X)h - (\operatorname{mult}_x X + 1)e\big)\Big) < 0.$$

Therefore, $\widetilde{C} \subset \widetilde{T_x X}$, $C \subset T_x X$.

(B) Obviously, it is enough to prove this statement for an irreducible curve C. If C is a cone generator, then the statement is correct. Let us assume the contrary. Let $\pi\colon \widetilde{Q} \to Q$ be a blowing of the apex x, $E = \pi^{-1}(x)$, $\lambda\colon \widetilde{Q} \to B$ be a morphism, where B is the base equal to a plane section of Q which does not contain x. Let $B^* \to B$ be a desingularization of B, $Q^* = \widetilde{Q} \times_B B^*$ be a desingularization of $\widetilde{Q}$, E^* be the proper inverse image of E on Q^*. It is easy to see that $(e^* \cdot e^*) = -m$. Let C^* be the proper inverse image of C on Q^*. Then $\operatorname{mult}_x C = (c^* \cdot e^*)$. But, $A^1(Q^*) = \mathbb{Z}f \oplus \mathbb{Z}e^*$, where f is the class of a fibre of the ruled surface Q^*. Thus, $c^* = af + be^*$. But, $\deg C = a$ and $\operatorname{mult}_x C = (c^* \cdot e^*) = a - bm$. (B) is proved.

(C) is trivial.

Lemma 3. *Let C be an irreducible reduced curve on an irreducible surface $S \subset \mathbb{P}^3$ and $x \in C$ be a point.*

(A) *If* $\deg S = 2$, *then either* $\deg C \le 2$ *or* $\operatorname{mult}_x C \le \deg C - 2$.

(B) *If* $\deg S = 3$, *then three cases are possible*:

(1) $\deg C \le 4$;

(2) $\deg C = 5$ *or 6 and* $\operatorname{mult}_x C \le \deg C - 2$;

(3) $\operatorname{mult}_x C \le \deg C - 3$.

Proof. (A) Let us assume that $\deg C \le 3$ and $\operatorname{mult}_x C \ge \deg C - 1 > 1/2 \deg C$. If S is a cone with x as an apex, then, by Lemma 2 (B), either deg C= $\operatorname{mult}_x C$, which contradicts the irreducibility of C, or $\operatorname{mult}_x C \le \deg C - 2$. Therefore, x is a simple point of S. But, in this case, by Lemma 2 (A), $C \subset T_x S$, $\deg C \le 2$. We have a contradiction.

(B) Let S be a cone with x as an apex. Then, by Lemma 2 (B), either $\operatorname{mult}_x C = \deg C$ or $\operatorname{mult}_x C \le \deg C - 3$. Hence, let us assume that $\operatorname{mult}_x S \le 2$. If $\deg C = 5$ or 6, $\operatorname{mult}_x C \ge \deg C - 1$ or $\deg C \ge 7$, and $\operatorname{mult}_x C \ge \deg C - 2$, then, by Lemma 2 (A), $C \subset T_x S$, $\deg T_x S = 2$, $\deg C \le 6$. But if $\deg C \ge 5$, then $T_x S$ is an irreducible cone with x as an apex and, by Lemma 2 (B), $\operatorname{mult}_x C \le \deg C - 2$.

We have a contradiction. The lemma is proved.

Let us return now to the quintic we are studying.

Remark 1. Let us recall that $T_{x_0} V = T = T_{x_0}(T)$, and so $\overline{T_{x_0} V} = E_1 = \overline{T_{x_0} T}$. Here we consider $\varphi_{1,0} V_1 \to V$, i.e., the blowing of $x_0 \in V$, as the restriction of the blowing of $x_0 \in \mathbb{P}^5$: $\varphi_{1,0} = \pi_{x_0}|_V$, $\pi_{x_0}\colon \widetilde{\mathbb{P}^5} \to \mathbb{P}^5$.

In other words, the tangent cones to T and V at x_0 are equal; thus, for each submanifold $Y \subset T$, $T_{x_0} Y \subset T$, and $\overline{T_{x_0} Y} \subset E_1$. We must keep this obvious remark in mind for the next lemma.

Lemma 4. *Let $Z \subset Q^* \bigcap \widetilde{V} \subset T$, where $\widetilde{V}$ and T are as above, $Q^* \subset T$ is an irreducible 3-dimensional cone with x_0 as an apex, Z be an (irreducible) surface. Let Z^1 be, as usual, the proper inverse image of Z on V_1. Let $\deg H = d$, $\operatorname{mult}_{x_0} Z = \nu_0^*$, $\operatorname{mult}_{x_1} Z^1 = \nu_1^*$, $\nu_0^* \ge \nu_1^*$. Let us assume that $d \ge 2$. Then*

(A) $\nu_1^* \le d - 1$;

(B) *if* $\deg Q^* = 2$ *or 3, then either* $d \le 5$ *or* $\nu_1^* \le d - 2$;

(C) *if* $\deg Q^* = 3$, *then either* $d \le 6$ *or* $\nu_1^* \le d - 3$.

Proof. (A) Let us note that if $\nu_1^* = d$, then $\nu_0^* = d$, and so Z is a cone with x_0 as an apex. Then $\overline{T_{x_0} Z} = \overline{Z} \subset E_1$ is a (not always irreducible and reduced) curve of degree d with the point x_1 of multiplicity $\operatorname{mult}_{x_1}(E_1 \bigcap Z^1) = d$. Therefore, $\overline{T_{x_0} Z} = Z^1 \bigcap E_1$ is a union of lines, and, therefore, is a line. In other words, $d = 1$. We have a contradiction.

(B) Let $\nu_1^* = d - 1$. Then either $\nu_0^* = d - 1$ and, by Lemma 2 (C), Z is contained in some 3-plane $S \subset T$, $Z \subset S \bigcap V$, $d \le 5$, V cannot contain S, or $\nu_0^* = d$ and Z is a cone with x_0 as an apex. In this case, $\overline{T_{x_0} Z} = \overline{Z} = E_1 \bigcap Z^1$ is an irreducible reduced curve of degree d and $\operatorname{mult}_{x_1}(E_1 \bigcap Z^1) = d - 1$ (if the multiplicity of $E_1 \bigcap Z^1$ in x_1 is d, then, by (A), Z is a plane). But $E_1 \bigcap Z^1 \subset \overline{T_{x_0} Q^*}$ and $\overline{T_{x_0} Q^*}$ is a surface of degree 2 or 3. Now Lemma 3 proves (B).

In case (C), the reasoning is analogous. By (A) and (B), the only case we have to consider is $\nu_1^* = d - 2$, $d \ge 7$. If $\nu_0^* = d - 2$, then let us consider a general 3-plane $S \subset T$ such that $x_0 \in S$. Then $Z \bigcap S \subset Q^* \bigcap S$, where $Z \bigcap S$ is an irreducible curve of degree d, $\operatorname{mult}_{x_0}(Z \bigcap S) = d - 2$, $Q^* \bigcap S$ is an irreducible cone in $S \cong \mathbb{P}^3$ of degree 3 and with x_0 as an apex. But this contradicts Lemma 2 (B).

If $\nu_0^* = d-1$, then Z is contained in some 3-plane and $d \leq 5$. If $\nu_0^* = d$, i.e., Z is a cone with x_0 as an apex, then $Z^1 \bigcap E_1$ is an irreducible reduced curve of degree d which has a point x_1 of multiplicity $\operatorname{mult}_{x_1}(Z^1 \bigcap E_1) \geq d-2$ and $Z^1 \bigcap E_1 \subset \overline{T_{x_0}Q^*}$, where $T_{x_0}Q^*$ is an irreducible surface of degree 3. Thus, the condition $d \geq 7$ contradicts Lemma 3 (B). Lemma 4 is proved.

3. The planes that are contained in V play an essential role in our problem. We shall state their numerical properties in what follows.

Lemma 5. *Let P be a plane, i.e., a 2-plane in $\mathbb{P}^5$ which belongs to V, and p be its class in $A^2(V)$. Then $(p \cdot h^2) = 1$, $(p \cdot p) = 13$, and $p + 3h^2$ is a nonnegative class.*

Proof. The first equality is obvious. To prove the second, let us draw two different 3-planes S_1 and S_2 through P. Then $S_i \bigcap V = P \bigcup G_i$, $i = 1, 2$, where $\deg G_i = 4$. Obviously, $G_1 \bigcap G_2 \subset S_1 \bigcup S_2 = P$, and so $\# G_1 \bigcap G_2 = \#(G_1 \bigcap P) \bigcap (G_2 \bigcap P)$. Two plane quartics intersect at 16 points; therefore, $\# G_1 \bigcap G_2 = 16 = (h^2 - p)^2 = 3 + (p \cdot p)$. Thus, $(p^2) = 13$.

Let us prove the third statement. Let $G \neq P$ be a surface on V. Let Z be a residual surface of the fourth degree in the intersection of V with a general 3-plane which contains P, i.e., $S \bigcap V = Z \bigcup P$. It is easy to see that for a general S the intersection $Z \bigcap G$ is 0-dimensional. Hence, $(z \cdot g) \leq \deg Z \deg G$, i.e., $((h^2 - p) \cdot g) \leq 4(h^2 \cdot g)$. Therefore, $((3h^2 + p) \cdot g) \geq 0$ for any $G \neq P$. But, $((3h^2 + p) \cdot p) = 16 \geq 0$. Thus, $3h^2 + p$ is nonnegative in $A^2(V)$ and by the projection formula it is nonnegative in $A^2(V_N)$.

4. Let us introduce the following notations:

$$\Sigma_i = \sum_{j \in I_i} r_j, \quad \Delta(\alpha) = \sum_{j \notin I} (b_{j-1} \cdot y(\alpha)) \nu_j^2,$$

$$\Sigma(\alpha) = -4\Sigma_0^2 - (7 - 5\alpha)\Sigma_0\Sigma_1 + (5\alpha - 4)\Sigma_1^2 - \Sigma_0\Sigma_2 + (5\alpha - 4)\Sigma_1\sigma_2 - \sum_2^2 - \frac{\Delta(\alpha)}{n^2}(\Sigma_0 + \Sigma_1 + \Sigma_2).$$

Lemma 6. $(h'^2 \cdot y(\alpha)) \leq \Sigma(\alpha)\frac{n^2}{\Sigma_0 + \Sigma_1 + \Sigma_2}$.

Proof. By the multiplication formulas,

$$(h'^2 \cdot y(\alpha)) = (h^2 \cdot y(\alpha))n^2 - \sum_{i=1}^{N} (b_{i-1} \cdot y(\alpha))\nu_i^2 - \sum_{j \in I_0 \bigcup I_1} r_j \nu_j^2.$$

Thus,

$$(h^2 \cdot y(\alpha)) = 5(\Sigma_0 + \alpha\Sigma_1) \text{and} (b_{i-1} \cdot y(\alpha)) \geq \nu_i$$

for $i \in I_2$ (by Lemma 3.1). Thus, we have

$$(h'^2 \cdot y(\alpha)) \leq 5n^2 \left(\sum_0 + \alpha \sum_1 \right) - \delta(\alpha) - \sum_{j \in I} r_j \nu_j^2.$$

Applying to the last sum the square inequality (Lemma 1.4.2), we complete the proof of the lemma.

Corollary 1.

(A) *If $\Sigma_2 \geq 3\Sigma_1$, then $(h'^2 \cdot y(1)) < 0$.*

(B) *If $\Delta(4/5) \geq 0$, then $(h'^2 \cdot y(4/5)) < 0$.*

(C) *If $\Delta(3/4) \geq -\frac{n^2}{4}\Sigma_1$, then $(h'^2 \cdot y(3/4)) < 0$.*

(D) *If $\Delta(7/9) \geq -\frac{2}{9}n^2\Sigma_1$ and $2\Sigma_2 \geq \Sigma_1$, then $(h'^2 \cdot y(7/9)) < 0$.*

(E) *If $\Delta(10/13) \geq -\frac{3n^2}{13}\Sigma_1$ and $2\Sigma_2 \geq \Sigma_1$, then $(h'^2 \cdot y(10/13)) < 0$.*

Proof. The direct computation of $\Sigma(\alpha)$ for $\alpha = 1$ (case (A)), $4/5$ (case (B)), $3/4$ (case(C)), $7/9$ (case (D)), and $10/13$ (case (E)) gives us that $\Sigma(\alpha) < 0$. Let us note that in case (A), $N(1) = \varnothing$; therefore, $\Delta(1) \geq 0$.

Corollary 2.

(A) $\Sigma_2 < 3\Sigma_1$.

(B) $N(4/5) \neq \varnothing$, *hence* $N(\alpha) \neq \varnothing$ *for all* $\alpha \leq 4/5$.

Proof. (A) As was noted above, $N(1) = \varnothing$. But, by part (A) of the above corollary, $h'^2 \in N(1)$ if $\Sigma_2 \geq 3\Sigma_1$. We have a contradiction.

(B) If $N(4/5) = \varnothing$, then $h'^2 \notin N(4/5)$ and $b_{i-1} \notin N(4/5)$ for all i, $1 \leq i \leq N$. Hence $\Delta(4/5) \geq 0$ and $\left(h'^2 \cdot y(4/5)\right) \geq 0$. But this contradicts Corollary 1 (B).

6. The Elimination of Maximal Singularities of the Type (2,0).

1. We shall use the notations and conventions of Sec. 5.

Lemma 1. $\operatorname{mult}_{x_0} \widetilde{V} = \deg Q_0 = 2$ *or* 3. *If* $\operatorname{mult}_{x_0} \widetilde{V} = 3$, *then* $2\Sigma_2 \geq \Sigma_1$.

Proof. Let us note that $\operatorname{mult}_{x_0} \widetilde{V} \leq 5$, because $\deg \widetilde{V} = 5$, and $\operatorname{mult}_{x_0} \widetilde{V} \geq 2$, because $\widetilde{V} = V \bigcap T$, where T is the tangent hyperplane to V at x_0. Thus, $\widetilde{V}$ has a singularity at x_0.

Let $\operatorname{mult}_{x_0} \widetilde{V} = \deg Q_0 = 4$. Let us consider the proper inverse images on V_1 of the plane sections of $\widetilde{V} \subset T$ which contains x_0. Their class in $A^3(V_1)$ is $h^3 - 4f_1$ and they sweep all V. The class of the divisor of the linear system $|\chi|_1$ on V_1, which does not have nonstationary components, is $nh - \nu_1 e_1$. Therefore,

$$\left((h^3 - 4f_1) \cdot (nh - \nu_1 e_1)\right) = 5n - 4\nu_1 \geq 0,$$

i.e., $\nu_1 < 5/4n$. Hence, $\nu_i \leq 5/4n$ for all $i \in I$. Now, by the strengthened Noether–Fano inequality (Proposition 1.2.1), we have

$$\sum_j r_j \delta_{j-1} n < \sum_j r_j \nu_j \leq 5/4n \sum_j r_j.$$

Hence

$$3\Sigma_0 + 2\Sigma_1 + \Sigma_2 < 5/4(\Sigma_0 + \Sigma_1 + \Sigma_2).$$

Thus, $\Sigma_2 \geq 3\Sigma_1$. This contradicts Corollary 5.2 (A).

Let $\operatorname{mult}_{x_0} \widetilde{V} = 5$, i.e., $\widetilde{V}$ be a cone with x_0 as an apex. The proper inverse images on V_1 of the generators of $\widetilde{V}$ represent the class $1/5h^3 - f_1$, so

$$\left((nh - \nu_1 e_1) \cdot (1/5h^3 - f_1)\right) = n - \nu_1 \geq 0.$$

But, $\nu_1 \geq \nu_\beta > n$. We have a contradiction.

Therefore, $\operatorname{mult}_{x_0} \widetilde{V} = 2$ or 3. If $\operatorname{mult}_{x_0} \widetilde{V} = 3$, then let us consider again the proper inverse images on V_1 of the plane sections of $\widetilde{V} \subset T$ which contain x_0. Their class in $A^3(V_1)$ is $h^3 - 3f_1$ and they sweep $\widetilde{V}$. Therefore,

$$\left((nh - \nu_1 e_1) \cdot (h^3 - 3f_1)\right) = 5n - 3\nu_1 \geq 0$$

and $\nu_1 \leq 5/3n$. Hence, $\nu_i \leq 5/3n$ for all $i \in I$. As above, by the strengthened Noether–Fano inequality, we have

$$5/3(\Sigma_1 + \Sigma_2) > 2\Sigma_1 + \Sigma_2, \text{ i.e., } 2\Sigma_2 > \Sigma_1.$$

The next lemma is the first step toward a describing of the set $N(\alpha)$.

Lemma 2. *Let* $Z \subset V_i$ *be a surface. Then*

(A) *if* $z \in N(1/2)$, *then* $Z_0 \subset \widetilde{V}$, *in particular,* $h'^2 \notin N(1/2)$;

(B) *if* $\operatorname{mult}_{x_0} \widetilde{V} = 2$ *and* $z \in N(2/3)$, *then* $Z_0 \subset Q_0 \bigcap \widetilde{V}$ *and* $\deg Z_0 \leq 10$;

(C) *if* $\operatorname{mult}_{x_0} \widetilde{V} = 3$ *and* $z \in N(3/4)$, *then* $Z_0 \subset Q_0 \bigcap \widetilde{V}$ *and* $\deg Z_0 \leq 15$.

Proof. The statements of the lemma are consequences of Lemma 5.1 (C) and Lemma 5.2 (A). Let us note only two things: if $Z \subset V_N$ is a general surface such that $z = h'^2$, then $Z_0 \not\subset \widetilde{V}$ (because such cycles sweep V_N, and so $h'^2 \notin N(1/2)$); the conditions on the Z_0 degree in (B) and (C) are consequences of the fact that V cannot contain Q_0.

2. Now Definition 1.3 begins to work.

Lemma 3. *If B_β is of the first genus and $\#M(\beta) = 1$, i.e., $M(\beta) = \mathcal{D} \bigcap I = \{1\}$, then there exists a plane $P \subset \widetilde{V}$ such that for any surface $Z \subset V_i$, $z \in N(1/2)$, we have $Z_0 = P$. If $\#M(\beta) \geq 2$ (and B_β is of any genus), i.e., $\{1, 2\} \subset M(\beta)$, then V contains a line C which passes through x_0 in the direction of x_1.*

Proof. By Corollary 5.2 (B), $N(1/2) \neq \varnothing$. Therefore, there exists a surface $Z \subset V_i$ such that $z \in N(1/2)$. If $m(\beta) = 1$, then $\dim B_{\beta,1} = 1$ and $\mathrm{mult}_{B_{\beta,1}} Z_1 > 1/2 \deg Z_0$. But,

$$\deg Z_0 \geq \mathrm{mult}_{x_0} Z_0 = \deg \overline{T_{x_0} Z_0} \geq \deg B_{\beta,1}(\mathrm{mult}_{B_{\beta,1}} Z_1),$$

and so $\deg B_{\beta,1} = 1$. Then, for any point $a \in B_{\beta,1}$ a line which passes through x_0 in the direction of a intersects Z_0 with multiplicity greater than $\deg Z_0$; therefore, it is contained in Z_0. All such lines sweep P. Hence $Z_0 = P$.

The proof of the second statement is analogous.

Lemma 4. *Let $m(\beta) \geq 2$, $Z \subset V_i$ be a surface, $z \in N(\alpha)$, $Z_0 \subset Q_0'$, where Q_0' is an irreducible component of Q_0. Then:*

(A) *if $\alpha = 3/4$, then $\deg Z_0 \neq 2, 3, 4, 6, 7, 8$;*
(B) *if $\alpha = 3/4$ and $\deg Q_0' \leq 2$, then only the following cases are possible:*
  (1) $\deg Z_0 \leq 1$;
  (2) $\deg Q_0' = 1$, $\deg Z_0 = 5$, $Z_0 = Q_0' \bigcap V$;
  (3) $\deg Q_0' = 2$, $\deg Z_0 = 9$;
  (4) $\deg Q_0' = 2$, $\deg Z_0 = 10$, $Z_0 = Q_0' \bigcap V$;
(C) *if $\alpha = 7/9$ and $\deg Q_0' = 2$, then $\deg Z_0 = 1$ or 10;*
(D) *if $\alpha = 4/5$ and $\deg Q_0' \leq 3$, then $\deg Z_0 = 1$;*
(E) *if $\alpha = 10/13$ and $\deg Q_0' = 3$, then $\deg Z_0 = 1$ or 14 or 15.*

Proof. By Lemma 5.1 if $h \in N(\alpha)$, then

$$\mathrm{mult}_{x_0} Z_0 > \alpha \deg Z_0, \quad \mathrm{mult}_{x_1} Z_1 > \alpha \deg Z_0, \quad Z_1 = (Z_0)^1$$

$(\mathcal{D} \bigcap I_0 \supset \{1, 2\})$. In each of cases (A)–(E) we use Lemma 5.4 and Lemma 5.2. Let us consider, for example, case (B). We shall prove that the equality $\deg Z_0 = 8$ is impossible. Indeed, we have that $\mathrm{mult}_{x_0} Z_0 > 3/4 \cdot 8 = 6$, i.e., $\mathrm{mult}_{x_0} Z_0 \geq 7$ and, analogously, $\mathrm{mult}_{x_1} Z_1 \geq 7$. But this contradicts Lemma 5.4 (B). Let us consider another example: in case (B), let $\deg Z_0 = 5$. Then, by Lemma 5.2, $Z_0 \subset S$, where $S \subset T$ is a 3-plane. Hence if $\deg Q_0' = 2$, then $Z_0 \subset S \bigcap Q_0'$ and $\deg Z_0 \leq 2$. We have a contradiction. In the other cases, the reasoning is analogous.

Lemma 5. *If B_β is of the second genus, then there is no line $C \subset V$ such that $x_i \in C^i$ for all $i \in M(\beta)$.*

Proof. Let us assume the contrary. Let $S \subset \mathbb{P}^5$ be a general 3-plane which contains C and let $S^* = S \bigcap V \subset S \cong \mathbb{P}^3$ be an associated surface of degree 5. Let us denote the restriction of the linear system $|\chi|$ to S^* by $|\chi|^*$. It is a nonstationary linear system with a unique component C of multiplicity $\nu \geq 0$. The system $|\chi|^*$ has at least $m(\beta)$ base points x_i, $0 \leq i \leq m(\beta) - 1$, where x_i lies over x_{i-1}.

Indeed, $x_0 \in C \subset S^*$. Let $\pi_1 \colon S_1^* \to S^*$ be a blowing of $x_0 \in S^*$, and $E_1^* = \pi_1^{-1}(x_0)$ be an exceptional divisor. Let us note that S_1^* is naturally isomorphic to $(S^*)^1$, i.e., to the proper inverse image of S^* on V_1. Let $C_1 \subset S_1^*$ be the proper inverse image of C on S_1^*, and $|\chi|_1^*$ be the proper inverse image of the linear system $|\chi|^*$ on S_1^*. Then $x_1 \in C_1$ $(C_1 = C^1)$ is a base point of system $|\chi|_1^*$. Repeating this procedure, we get the sequence of blowings $\pi_i \colon S_i^* \to S_{i-1}^*$, $1 \leq i \leq m(\beta)$, the sequence of points $x_{i-1} \in E_{i-1}^*$ with the exceptional divisors $E_i^* = \pi_i^{-1}(x_{i-1}) \subset S_i^*$; $\mathrm{mult}_{x_i} |\chi|_i^* = \nu_{i+1}, x_i \in C_i$, where C_i and $|\chi|_i^*$ are the proper inverse images on S_i^* of C and $|\chi|^*$, respectively. Let l^*, c, e_i^* be the classes in $A^1(S_{m(\beta)}^*)$ of plane sections of $S^* \subset \mathbb{P}^3$, of $C \subset S^*$ and of $E_i^* \subset S_i^*$, respectively. Let us note that the class of the divisor of the linear system $|\chi|_{m(\beta)}^*$ in $A^1(S_{m(\beta)}^*$ is $nh^* - \sum_{i=1}^{m(\beta)} \nu_i e_i^*$ and this system has a unique stationary component $C_{m(\beta)}$ of multiplicity ν whose class is $c - \sum_{i=1}^{m(\beta)} e_i^*$. Therefore, the class of the nonstationary part of $|\chi|_{m(\beta)}^*$ is

$nh^* - \nu c - \sum\limits_{i=1}^{m(\beta)} (\nu_i - \nu)e_i^*$. Let us note that $\nu_i \geq \nu_\beta > n$ and, by Proposition 2.3 (A), $\nu \leq n$. Thus,

$$5n^2 - 2n\nu - 3\nu^2 - \sum_{i=1}^{m(\beta)} (\nu_i - \nu)^2 \geq 0$$

But, $\nu \geq 0$, $m(\beta) = 5n^2 + 1$, and each term in the last sum is not less than 1. We have a contradiction. The lemma is proved.

Lemma 6.

(A) *Let $Z \subset V_i$ be a surface,* $\deg Z_0 = 5$. *Then $z \notin N(3/4)$.*

(B) *Let $Z_1 \subset V_i$, $Z_2 \subset V_j$ be surfaces such that $Z_1, Z_2 \in N(\alpha)$, $\alpha > 0$, and* $\deg Z_{1,0} = \deg Z_{2,0} = 1$. *Then $Z_{1,0} = Z_{2,0}$.*

Proof. (A) Let us assume the contrary. By Lemma 2, Z_0 is contained in a component of Q_0. Using Lemma 5.3 and Lemma 5.2 (C), we have that $Q_0 = Q_0' \bigcup Q_1'$, $\deg Q_0' = 1$, $Z_0 = Q_0' \bigcap V$, $\deg Q_1' \leq 2$. Let us consider the class $y(4/5)$. By Corollary 5.2 (B), $N(4/5) \neq \varnothing$. Therefore, there exists a surface $G \subset V_j$ such that $g \in N(4/5)$. By Lemma 2, we can use Lemma 4 and conclude that G_0 is a plane. Let us note that $G_0 \not\subset Q_0'$ and hence, $G_0 \bigcap Q_0' = C$, where C is a line. Let us note also that B_β cannot be of the first genus in this case (if G_0 is a plane, Q_0' is a 3-plane, $Z_0 = Q_0' \bigcap V$, $G_0 \not\subset Q_0'$, then $(G_0)^i \bigcap (Z_0)^i = (G_0 \bigcap Z_0)^i = C^i$ for $1 \leq i \leq m(\beta)$). But, by Lemma 5.1, $B_{\beta,m(\beta)} \subset G_0^{m(\beta)}$, $B_{\beta,m(\beta)} \subset Z_0^{m(\beta)}$, and so $B_{\beta,m(\beta)} \subset C^{m(\beta)}$. But, if $\dim B_{\beta,m(\beta)} \geq 1$, then this is impossible. Therefore, B_β is of the second genus. Again, $x_i \in G_0^i$, $x_i \in Z_0^i$; so $x_i \in C^i$ for all i, $0 \leq i \leq m(\beta) - 1$, where $m(\beta) = 5n^2 + 1$. But this contradicts Lemma 5.

(B) The reasoning here is analogous to that above. Let us assume that the planes $Z_{1,0}$ and $Z_{2,0}$ are different. Then $C = Z_{1,0} \bigcap Z_{2,0}$ is a line (if $Z_{1,0} \bigcap Z_{2,0}$ is a point x_0, then $Z_{1,0}^1 \bigcap Z_{2,0}^1 = \varnothing$: we have a contradiction). Again $Z_{1,0}^i \bigcap Z_{2,0}^i = Z_{1,i} \bigcap Z_{2,i} = C^i$, and so B_β is of the second genus and $x_i \in C^i$ for all i, $0 \leq i \leq m(\beta) - 1$. But this contradicts Lemma 5. Finally, let us note that, in case (B), α is arbitrary.

The next lemma is the final step in our construction.

Lemma 7. *The following situations cannot be realized:*

(A) *for any surface $Z \subset V_i$ such that $z \in N(3/4)$ we have* $\deg Z_0 = 1$;

(B) *for any surface $Z \subset V_i$ such that $z \in N(7/9)$ we have* $\deg Z_0 = 1$ *and* $2\Sigma_2 \geq \Sigma_1$;

(C) *for any surface $Z \subset V_i$ such that $z \in N(10/13)$ we have* $\deg Z_0 = 1$ *and* $2\Sigma_2 \geq \Sigma_1$;

(D) $\deg Q_0 = 2$, $Q_0 \bigcap \widetilde{V} = G \bigcup P$, $\deg P = 1$, *and G is an irreducible surface of degree* 9;

(E) $\deg Q_0 = 3$, $Q_0 \bigcap \widetilde{V} = G \bigcup P$, $\deg P = 1$, *and G is an irreducible surface of degree* 14;

(F) $\deg Q_0 = 3$, $Q_0 = Q_0' \bigcup Q_0''$, $\deg Q_0' = 2$, *and $G = Q_0' \bigcap \widetilde{V}$ is an irreducible surface of degree* 10.

Proof. By Lemma 6 (B), in cases (A), (B), and (C), there exists a plane $P \subset V$ such that for any surface $Z \subset V_i$, $z \in N(\alpha)$, where $\alpha = 3/4$ (in case (A)), $7/9$ (in case (B)), and $10/13$ (in case (C)). Thus, $Z_0 = P$. In case (F), by Lemma 4 (D) and Corollary 5.2 (B), there exist a plane $P \subset V$ and a surface $Z \subset V_i$ such that $z \in N(4/5)$ and $Z_0 = P$. In all cases (A)–(F), we set $J = \{1 \leq j \leq N | B_{j-1,0} = P\}$. Moreover, in cases (D), (E), and (F), set $\mathcal{L} = \{1 \leq j \leq N | B_{j-1,0} = G\}$. Let, for simplicity, $\mathcal{L} = \varnothing$ in cases (A), (B), and (C). Let us assume that one of the six cases (A)–(F) is realized. If $J = \varnothing$ or $\mathcal{L} = \varnothing$, then in all the following computations we shall replace all sums $\sum\limits_{j \in Y} \dots$ and $\sum_{j \in \mathcal{L}} \dots$ by zero.

Let us use Lemma 5.5. We have

$$0 \leq \big((3h^2 + p) \cdot h'^2\big) = n^2\big((3h^2 + p) \cdot h^2\big) - \sum_{i=1}^{N} \big((3h^2 + p) \cdot b_{i-1}\big)\nu_i^2.$$

Let $d(Z) = \deg(\varphi_{i,0}\colon\ Z \to Z_0)$ for any surface $Z \subset V_i$ of the type (2,2). Let us assume that $Z \subset V_i$ has the type (2,2). Then

$$\big((3l^2 + p) \cdot z\big) = d(Z)\big(z_0 \cdot (3h^2 + p)\big) = d(Z)(z_0 \cdot h^2)\big[\big(z_0(3h^2 + p)\big)/(z_0 \cdot h^2)\big] = (z \cdot h^2)\big[\big(z_0 \cdot (3h^2 + p)\big)/(z_0 \cdot h^2)\big].$$

Therefore, we have the following:
if $Z_0 = P$, then $(z \cdot (3h^2 + p)) = 16(z \cdot h^2)$;
if, in (D), $Z_0 = G$ and $g = 2h^2 - p$, then $(z \cdot (3h^2 + p)) = 16/9(z \cdot h^2)$;
if, in (E), $Z_0 = G$ and $g = 3h^2 - p$, then $(z \cdot (3h^2 + p)) = 16/7(z \cdot h^2)$;
if, in (F), $Z_0 = G$ and $g = 2h^2$, then $(z \cdot (3h^2 + p)) = 16/5(z \cdot h^2)$.

Let $\gamma = 0$ (in cases (A)-(C)), 1/9 (in case (D)), 1/7 (in the case (E)), and 1/5 (in case (F)). Then

$$0 \leq 1/16((3h^2 + p) \cdot h'^2) \geq n^2 - \sum_{j \in \mathcal{I}} (b_{j-1} \cdot h^2)\nu_j^2 - \gamma \sum_{j \in \mathcal{L}} (b_{j-1} \cdot h^2)\nu_j^2.$$

Let us note that if $Z \subset V_i$, $i > \beta$, is a surface of the type (2,2), then

$$(z \cdot g_j) = d(Z) \operatorname{mult}_{B_{j-1}} Z_{j-1} \leq d(Z) \operatorname{mult}_{B_{m(\beta)-1}} Z_{m(\beta)-1} \leq d(Z) \operatorname{mult}_{B_t} Z_t$$

for all $j \in I_1$ and all $t \in M(\beta)$.

If Z_0 is a plane, then $(z \cdot g_j) \leq d(Z) = (z \cdot h^2)$ and, therefore,

$$(z \cdot y(\alpha)) \geq -(1-\alpha)(z \cdot h^2)\Sigma_1.$$

If in case (D), $\deg Z_0 = 9$ and $z \in N(3/4)$, then, by Lemma 3, $m(\beta) > 1$ and, by Lemma 5.4 (B), $\operatorname{mult}_{B_1} Z_1 \leq 7$. Thus, we have

$$(z \cdot (3/4h^2 - g_j)) = 3/4 \cdot 9d(Z) - d(Z) \operatorname{mult}_{B_{j-1}} Z_{j-1} \geq d(Z)(3/4 \cdot 9 - 7) = -1/4d(Z) = -1/4 \cdot 1/9(z \cdot h^2).$$

Hence

$$(z \cdot y(3/4)) \geq -1/4 \cdot 1/9(z \cdot h^2)\Sigma_1.$$

If, in case (E), $\deg Z_0 = 14$ and $z \in N(3/4)$, then, by Lemma 3, $m(\beta) > 1$ and, by Lemma 5.4 (C), $\operatorname{mult}_{B_1} Z_1 \leq 11$. Thus, we have

$$(z \cdot (3/4h^2 - g_j)) = 3/4 \cdot 14d(Z) - d(Z) \operatorname{mult}_{B_{j-1}} Z_{j-1} \geq d(Z)(3/4 \cdot 14 - 11) = -1/2d(Z) = -1/4 \cdot 1/7(z \cdot h^2).$$

Therefore,

$$(z \cdot y(3/4)) \geq -1/4 \cdot 1/7(z \cdot h^2)\Sigma_1.$$

If, in case (F), $\deg Z_0 = 10$ and $z \in N(3/4)$, then, by Lemma 3, $m(\beta) > 1$ and, by Lemma 5.4 (B), $\operatorname{mult}_{B_1} Z_1 \leq 8$. We have, as above,

$$(z \cdot (3/4h^2 - g_j)) \geq -1/4 \cdot 1/5(z \cdot h^2),$$

and, therefore,

$$(z \cdot y(3/4)) \geq -1/4 \cdot 1/5(z \cdot l^2)\Sigma_1.$$

Let us give now an estimation for $\Delta(\alpha)$. By Lemma 2, in cases (D)–(F) if $b_{i-1} \in N(3/4)$, then $i \in J \bigcup \mathcal{L}$. In cases (A)–(C), by the assumption if $b_{i-1} \in N(\alpha)$, then $i \in J$ (where α is as in the statement of the lemma). Therefore,

$$\Delta(\alpha) \geq \sum_{j \in \mathcal{I}} (b_{j-1} \cdot y(\alpha))\nu_j^2 + \sum_{j \in \mathcal{L}} (b_{j-1} \cdot (y\alpha))\nu_j^2 \geq -(1-\alpha)[\sum_{j \in ICal} (b_{j-1} \cdot h^2)\nu_j^2 + \nu \sum_{j \in \mathcal{L}} (b_{j-1} \cdot h^2)\nu_j^2]\Sigma_1.$$

Together with the above inequalities we have that, in cases (A), (D), (E), and (F), $\Delta(3/4) \geq -1/4\Sigma_1 n^2$, in case (B), $\Delta(7/9) \geq -2/9\Sigma_1 n^2$, and, in case (C), $\Delta(10/13) \geq -3/13\Sigma_1 n^2$. Taking into account that $2\Sigma_2 \geq \Sigma_1$, in cases (B) and (C), we have, by Corollary 5.1, $h'^2 \in N(\alpha) \subset N(1/2)$. We get a contradiction with Lemma 2 (A).

3. The End of the Proof of the Quintic Theorem.

Let $\operatorname{mult}_{x_0} \widetilde{V} = 2$. By Corollary 5.2 (B), $N(4/5) \neq \varnothing$. By Lemmas 2 and 4 (E), $Q_0 \bigcap \widetilde{V} = P \bigcup G$, and P is a plane. By Lemma 7 (D), G is reducible. By Lemmas 4 (B) and 6 (A), we have the situation of Lemma 7 (A). We have a contradiction.

Let $\operatorname{mult}_{x_0} \widetilde{V} = 3$. As above, $N(4/5) \neq \varnothing$, i.e., by Lemmas 2 and 4 (D), $Q_0 \bigcap \widetilde{V} = P \bigcup G$ and P is a plane. By Lemma 7 (E), G is reducible. By Lemma 7 (A), there exists a surface $Z \subset V_i$ such that $z \in N(3/4)$ and $Z_0 \neq P$. By Lemmas 4 (A) and 6 (A) (and, of course, by Lemma 2), $\deg Z_0 \geq 9$. We shall study all possibilities consecutively.

Let $\deg Z_0 = 9$, $Q_0 \bigcap \widetilde{V} = P \bigcup Z_0 \bigcup G'$, $\deg G' \leq 5$. By Lemma 5.4 (B) (keeping in mind Lemma 5.1 (C)) if the surface $Z^* \subset V_i$ is such that $z^* \in N(7/9)$, then $Z_0^* = P$. Now, by Lemmas 7 (B) and 1, we get a contradiction.

Let $\deg Z_0 = 10$. By Lemma 7 (F), $Z_0 \neq Q_0' \bigcap \widetilde{V}$, where $Q_0 = Q_0' \bigcup Q_0''$ and $\deg Q_0' = 2$. Thus, Q_0 is irreducible. But then Lemmas 4 (E) and 1 contradict Lemma 7 (C).

Let $\deg Z_0 = 11, 12, 13$. We can use again Lemma 4 (F) to get a contradiction with Lemma 7 (C).

Let us note, finally, that we assumed that $m(\beta) \geq 2$ (the case $m(\beta) = 1$ is impossible by Lemmas 7 (A) and 3).

Therefore, the assumption of the existence of a maximal singularity gives us a contradiction. Hence, by Proposition 1.2.1, we have a proof of the theorem on the birational isomorphisms of 4-dimensional quintics.

Chapter 5
Birational Automorphisms of a 3-Dimensional Quartic with the Simplest Singularity

The object of study in this chapter is the birational geometry of a 3-dimensional quartic $V \subset \mathbb{P}^4$, $\deg V = 4$, which has a double point $x \in V$, $\operatorname{Sing} V = x$, such that the blowing of this point $\varphi\colon V_0 \to V$ resolves singularities, i.e., $\operatorname{Sing} V_0 = \varnothing$ and the exceptional divisor $E = \varphi^{-1}(x)$ is a nonsingular quadric. We assume that the quartic V is general in the following sense: through the singular point x pass exactly 24 lines on V.

In Sec. 1 we shall describe 25 involutions of the manifold V: the reflection with respect to x in $\mathbb{P}^4$ and 24 fibre involutions which are acting in fibres (we can consider V as a fibre bundle with elliptic curves as fibres which are defined by 24 lines which pass through x). We shall prove that these involutions are the free generators of a normal subgroup of a finite index in $\operatorname{Bir} V$ (the group of birational automorphisms of V), so $\operatorname{Bir} V$ is a semidirect product of the normal subgroup $B(V)$ and the group $\operatorname{Aut} V$ of projective automorphisms of the manifold V (this group is finite and, in the general case, trivial). Then we shall prove the birational rigidity of V. The questions about using the "nonsingular" technique in a "singular" case will be considered in Secs. 3 and 5.

1. Reflections with Respect to a Singular Point.

1. Let $\varphi\colon V_0 \to V$ be a blowing of the singular point x, considered as the restriction of the blowing $\widetilde{\varphi}\colon X \to \mathbb{P}^4$ of x, $\widetilde{E} = \widetilde{\varphi}^{-1}(x) \cong \mathbb{P}^3$, $E = \varphi^{-1}(x) = \widetilde{E} \bigcap V_0$ be a nonsingular quadric in $\widetilde{E} \cong \mathbb{P}^3$. Then $\operatorname{Pic} V_0 = \mathbb{Z}h \oplus \mathbb{Z}e$, where h is the class of a hyperplane section and e is the class of the exceptional divisor E. The quadric E is the projectivization of the tangent cone to V at $x \in V \subset \mathbb{P}^4$, and 24 lines on V which contain x are related to 24 intersection points of E with the cubic $\{Q_3 = 0\}$ and with the quartic $\{Q_4 = 0\}$, where Q_3 and Q_4 are uniform components of degrees 3 and 4 of the affine equation of V in the affine coordinate system with x as a center. Let us denote these lines by B_i (this notation will be used up to the end of this chapter), $1 \leq i \leq 24$, and their proper inverse (disjoint) images on V_0 by B_i^*, $1 \leq i \leq 24$, where $\varphi(B_i^*) = \overline{B}_i$. Let $\varphi_i^*\colon V_i^* \to V_0$ be blowing of B_i^* and $E_i^* = \varphi_i^{*^{-1}}(B_i^*)$ be an exceptional divisor.

Lemma 1.

(A) $E_i^* \cong \mathbb{P}^1 \times \mathbb{P}^1$.

(B) *The morphism* $\varphi_i^*\colon E_i^* \to B_i^*$ *is a projection on a factor.*

(C) $\mathcal{O}(-E_i^*) \otimes \mathcal{O}_{E_i^*}$ *is a sheaf of the type* $(1,1)$ *in* $\operatorname{Pic} E_i^* \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Let us consider the blowing of $B_i^* \subset V_0$ as a restriction of the blowing $\widetilde{\varphi}_i^*\colon X_i^* \to X$ of the line $B_i^* \subset X$ with the exceptional divisor $\widetilde{E}_i^* \cong \mathbb{P}^1 \times \mathbb{P}^2 \subset X_i^*$. The divisor E_i^* in $\widetilde{E}_i^* \cong \mathbb{P}^1 \times \mathbb{P}^2$ has the type

(2,1), i.e., it is defined by the equation $\sum_{i=0}^{2} a_i x_i = 0$, where $(x_0 : x_1 : x_2)$ are uniform coordinates in $\mathbb{P}^2$, and $a_i(y_0, y_1)$ are uniform polynomials of degree 2 without common multipliers. If a_i, $0 \leq i \leq 2$, are linearly independent, then, obviously, (A) and (B) hold. If a_i are linearly dependent, $\sum_{i=0}^{2} a_i c_i = 0$, then E_i^* has an exceptional section, the line which is the inverse image of the point $(c_0 : c_1 : c_2)$ under the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$. The self-intersection index of this line is (-2); hence E_i^* is a ruled surface of the type $\mathbb{F}_2$. But this case is impossible, because the quadric $E \subset \widetilde{E} \cong \mathbb{P}^3$, the cubic $\{Q_3 = 0\} \subset \widetilde{E}$, and the quartic $\{Q_4 = 0\} \subset \widetilde{E}$ all pass through the point $B_i^* \bigcap E$ and through the point infinitely close to it which is defined by the exceptional section of E_i^* (i.e., a line). Thus, in contrast to the condition of the lemma, there are ≤ 23 lines on V which pass through x (one more line is an infinitely close line over B_i). Therefore, (A) and (B) are proved.

Let a_i and b_i be the effective generators of the group $\operatorname{Pic} E_i^* \cong \mathbb{Z} \oplus \mathbb{Z}$, which were lifted from factors, i.e., from the classes of the types (1,0) and (0,1), where a_i is the class of the fibre of the projection $\varphi_i \colon E_i^* \to B_i^*$. We have $-E_i^*|_{E_i^*} = \alpha a_i + b_i$. It is easy to compute that ${E_i^*}^3 = 2$; thus, $\alpha = 1$.

2. Let us consider the following birational involution $\tau_0 \colon V \dashrightarrow V$. A general line which passes through x in $\mathbb{P}^4$ intersects V at x (twice) and at two more points: x_1 and x_2. Let $\tau_0(x_i) = x_j$, $\{i, j\} = \{1, 2\}$.

Let $\varphi^* \colon V^* \to V_0$ be the blowing of 24 (nonintersecting) curves $B_i^* \subset V_0$, $E_i^* = (\varpi^*)^{-1}(B_i^*)$. Let h, e, e_i^* be the classes in $\operatorname{Pic} V^*$ of a hyperplane section of $V \subset \mathbb{P}^4$, of $E \subset V_0$, of $E_i^* \subset V^*$, respectively ($\operatorname{Pic} V^* = \mathbb{Z}h \oplus \mathbb{Z}e \oplus \bigoplus_{i=1}^{2} 4\mathbb{Z}e_i^*$).

Lemma 2. *The automorphism τ_0 can be extended to a biregular involution on V^*. Its action on* $\operatorname{Pic} V^*$ *may be defined by the following relations*:

$$\tau_0^* h = 3h - 4e - \sum_{i=1}^{2} 4e_i^*, \qquad \tau_0^* e = 2h - 3e - \sum_{i=1}^{2} 4e_i^*, \qquad \tau_0^* e_i^* = e_i^*.$$

Proof. Let us consider the blowing φ^* as the restriction of the blowing $\widetilde{\varphi}_0 \colon X^* \to X$ of 24 curves B_i^*. Let us note now that the projection from the point x, $\pi \colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$, may be naturally extended to the morphisms $\widetilde{\pi} \colon X \to \mathbb{P}^3$ and $\pi^* \colon X^* \to Y$ with a line as a fibre, where Y is $\mathbb{P}^3$ with 24 blown points $\pi(\overline{B}_i)$, $1 \leq i \leq 24$. Y is naturally isomorphic to the proper inverse image of $\widetilde{E}$ on X^*. Moreover, $\widetilde{\pi}$ and π^* are locally trivial bundles. The involution τ_0 permutes the points of intersection of V^* with fibres of π^*, therefore it is enough to prove that V^* does not contain the fibres of π^*. But, already, V_0 contains only 24 such fibres B_i^*. Thus, let us consider any divisor E_i^*. As was proved above, E_i^* does not contain the fibres of the projection $\pi^* \colon \widetilde{E}_i^* \cong \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$. The statement is proved.

Before describing the action of τ_0 on $\operatorname{Pic} V^*$, let us note that all invariant classes can be lifted from Y. In particular, $\tau_0^* e_i^* = e_i^*$. Then $h + \tau_0^* h$ is an invariant class. Let H be a general hyperplane section of V, $x \notin H$; then $H + \tau_0(H) = \pi^{*-1}(\pi^*(H))$. But, $\pi^*(H)$ is the proper inverse image on Y of a quartic in $\mathbb{P}^3$ which passes through all 24 points $\pi(B_i)$ with multiplicity 1. Thus, we have that $h + \tau_0^* h = 4(h-e) - \sum_{i=1}^{2} 4e_i^*$. Analogously, $e + \tau_0^* e = 2(h - e) - \sum_{i=1}^{2} 4e_i^*$ (the image of the divisor of E in $\mathbb{P}^3$ is the quadric $E \subset \widetilde{E} \cong \mathbb{P}^3$).

3. Each line B_i on V is related to a birational involution. A general plane in $\mathbb{P}^4$ which contains $\overline{B}_i$ intersects V by a curve of degree 4 which is the union of a line and a nonsingular cubic passing through x (because x is a double point on any section of V). The reflection on an elliptic curve with x as a center, i.e., the map that maps a point t into a point q such that $t + q \sim 2x$ on the curve, gives us the involution we need. We shall denote this involution by τ_i.

In a more formal way, let us consider the manifold V_i^*, i.e., the blowing of the curve $B_i^* \subset V_0$. The projection $\pi \colon V \dashrightarrow \mathbb{P}^3$ from the point x can be extended to the morphism $\pi_i^* \colon V_i^* \to Y_i$, where Y_i is the blowing of $\mathbb{P}^3$ at the point $\pi(\overline{B}_i)$. The map π_i^* has degree 2 at a general point and contracts 23 lines into a point. Let $\varepsilon\colon Y_i \to \mathbb{P}^2$ be the extension of the projection $\varepsilon\colon \mathbb{P}^3 \to \mathbb{P}^2$ from the point $\pi(\overline{B}_i)$. Let us consider the composition $\varepsilon \circ \pi_i^*\colon V_i^* \to \mathbb{P}^2$. Its general fibre is an elliptic curve (the residual cubic) C_t, $t \in \mathbb{P}^2$. The proper inverse image of divisor E on V_i^* is a section of $\varepsilon \circ \pi_i^*$ and $(C_t \cdot E) = 1$.

Let $\widetilde{\tau}_i = (\varphi \circ \varphi_i^*)^{-1} \circ \tau_i \circ (\varphi\varphi_i^*) \in \operatorname{Bir} V_i^*$ be the reflection in each fibre with respect to this section.

Lemma 3. *The birational involution $\widetilde{\tau}_i$ of the manifold V_i^* may be extended to a biregular automorphism of the invariant open subset $V_i^* \setminus W$, where* $\operatorname{codim} W = 2$ *and $\varepsilon \circ \pi_i^*(W) \subset \mathbb{P}^2$ is a finite set of points. The action of $\widetilde{\tau}_i$ on* $\operatorname{Pic} V_i^* = \mathbb{Z}h \oplus \mathbb{Z}e \oplus \mathbb{Z}e_i^*$ *is defined by the following relations*:

$$\widetilde{\tau}_i^* h = 11h - 6e - 12e_i^*, \quad \widetilde{\tau}_i^* e = e, \quad \widetilde{\tau}_i^* e_i^* = 10h - 6e - 11e_i^*.$$

Proof. Let us denote by $h', \widetilde{e}, e'$ the classes of the proper inverse images under $\widetilde{\tau}_i$ of a general divisor of the system $|h|$, of divisors of E, and of E_i^*, respectively. Since $\widetilde{\tau}$ acts in the fibres of the morphism $\varepsilon \circ \pi_i^*\colon V_i^* \to \mathbb{P}^2$, let us restrict $\widetilde{\tau}_i$ to a general fibre C_t, $t \in \mathbb{P}^2$, of this morphism. By the definition of the reflection, $\widetilde{\tau}_i$ maps a point $z \in C_t$ into a point $\widetilde{\tau}_i(z) \in C_t$ such that $z + \widetilde{\tau}_i(z) \sim 2(E \bigcap C_t)$ on C_t. Since the kernel of the restriction of $\operatorname{Pic} V_i^*$ to a general fibre of the morphism $\varepsilon \circ \pi_i^*$ is $\mathbb{Z}(h - e - e_i^*)$, we have

$$\begin{aligned} h' &= 6e - h + m(h - e - e_i^*), \\ \widetilde{e} &= e + \widetilde{m}(h - e - e_i^*), \\ e' &= 4e - e_i^* + m'(h - e - e_i^*). \end{aligned}$$

Let us now restrict $\widetilde{\tau}_i$ to the inverse image of a general line in $\mathbb{P}^2$, i.e., to a nonsingular surface S which is the result of the blowing of $\mathbb{P}^3$ at a unique double point x. Obviously, $K_S = 0$, and so $\widetilde{\tau}_i|_S \in \operatorname{Aut} S$. Let us denote, for simplicity, the restriction of $\widetilde{\tau}_i$ to S by the same symbol and the restrictions of the classes $h, e, e_i^*, h', \widetilde{e}, e'$ to S by the symbols $h_S, e_S, E_s^*, h'_S, e_S, e'_S$, respectively. Now we have that $\widetilde{\tau}_i^*(e_S) = e_S$, $\widetilde{\tau}_i^*(h_S) = h'_S$. The indices of self-intersection are easy to compute:

$$(h_S^2) = 4, \quad (e_S^2) = -2, \quad (h_S \cdot e_S) = 0, \quad (e_S \cdot e_S^*) = 1, \quad (h_S \cdot e_S^*) = 1, \quad (e_S^{*^2}) = -2.$$

From the relation

$$\big(\widetilde{\tau}_i^*(h_S) \cdot \widetilde{\tau}_i^*(e_S)\big) = (h'_S \cdot e_S) = (h_S \cdot e_S) = 0$$

we get that $m = 12$, i.e., $h' = 11h - 6e - 12e_i^*$.

Let us note now that the birational automorphism $\widetilde{\tau}_i$ is well defined on those fibres of the morphism $\varepsilon \circ \pi_i^*\colon V_i^* \to \mathbb{P}^2$ which are irreducible and nonsingular at the point $C_t \bigcap E$ (the reflection with a nonsingular point as a center is defined for a cubic curve with a double point). All fibres C_t are nonsingular at $C_t \bigcap E$ except two which are related to two lines on the quadric $E \subset \widetilde{E}$ passing through the point $E \subset B_i^*$. If C_t is reducible, then $\varphi \circ \varphi_i^*(C_t)$ has a line as a component which intersects $\varphi(B_i^*)$. But, as it easy to see, the proper inverse images of these lines on V_i^* have a negative intersection with the class h'. On the other hand, the linear system $|h'|$ does not have nonstationary components by definition. Hence, $\varphi(B_i^*)$ intersects only a finite number of lines on V and $\widetilde{\tau}_i$ is well defined on the complement to a finite set of fibres of the morphism $\varepsilon \circ \pi_i$.

Now, obviously, $\widetilde{\tau}_i^* e = e$ and the last relation of the lemma is an easy consequence of the $\widetilde{\tau}_i$-invariancy of the class $h - e - e_i^*$. The lemma is proved.

2. Formulation of the Theorem and Plan of the Proof.

1. Theorem. (A) *The involutions τ_i, $1 \le i \le 24$, are free generators of the normal subgroup $B(V)$ in* $\operatorname{Bir} V$:

$$B(V) = \mathop{*}_{i=0}^{24} \langle \tau_i \rangle\,;$$

(B) *The group* $\operatorname{Bir} V$ *is a semidirect product*

$$1 \to B(V) \to \operatorname{Bir} V \to \operatorname{Aut} V \to 1$$

of the normal subgroup $B(V)$ and the (finite) group of the projective automorphisms $\operatorname{Aut} V$.

(C) *The action of* $\operatorname{Aut} V$ *on* $B(V)$ *can be described in the following way. Each automorphism* $\rho \in \operatorname{Aut} V$ *has* x *as a stationary point and induces a permutation of* 24 *lines* $\overline{B}_i$, $1 \le i \le 24$. *We shall also denote the corresponding permutation of the indices of the lines, i.e., the set* $\{1, \dots, 24\}$, *by* ρ. *Then* $\rho\tau_0\rho^{-1} = \tau_0$ *and* $\rho\tau_i\rho^{-1} = \tau_{\rho(i)}$, $1 \le i \le 24$.

(D) *The manifold* V *is birationally rigid.*

Corollary. *V is not equivalent to any smooth Fano 3-fold.*

The proof of the corollary is a direct consequence of the classification of a Fano 3-fold (see [8]).

2. Beginning of the Proof of the Theorem. Let us first note that, by the definition of a test manifold (at the beginning of the first chapter), V' is nonsingular. Nevertheless, the birational automorphisms of V belong to the class of birational correspondences $V \dashrightarrow V'$: we can take V_0 for V', and the proper inverse image of a hyperplane section of V on V_0 for H'.

Thus, let us fix a test 3-fold. Let us denote by $\operatorname{Bir}(V, V')$ the set of all birational correspondences $\chi\colon V \dashrightarrow V'$ (the natural action of the group $\operatorname{Bir} V$ on $\operatorname{Bir}(V, V')$ is defined).

For an arbitrary $\chi \in \operatorname{Bir}(V, V')$ let us consider the proper inverse image of the linear system $|H'|$ on V. It is a linear system without stationary components, which is sectioned on V by hypersurfaces of degree ≥ 1. Let us denote this system by $|\chi|$ and the degree of the hypersurfaces which section it by $n(\chi)$.

Let us relate to each $\chi \in \operatorname{Bir}(V, V')$ the set of 25 nonnegative integers $\nu_i^*(\chi)$, $0 \le i \le 24$, where $\nu_i^*(\chi) = \operatorname{mult}_{\overline{B}_i} |\chi|$ for $1 \le i \le 24$, i.e., the multiplicity of a general divisor of $|\chi|$ along the ith line. A number $\nu_0^*(\chi)$, which has the meaning of the multiplicity of $|\chi|$ at the point x, will be defined as follows: The class of the proper inverse image of a general divisor of the system $|\chi|$ on V_0 in $\operatorname{Pic} V_0 = \mathbb{Z}h \oplus \mathbb{Z}e$ is $n(\chi) - nu_0^*(\chi)e$.

Proposition 1. *If $n(\chi) \ge 2$, then among the numbers $\nu_i^*(\chi)$, $0 \le i \le 24$, is exactly one greater than $n(\chi)$, and all others are strictly less than $n(\chi)$. If $|H' + K_{V'}| = \varnothing$, then this statement is true for any $n(\chi) \ge 1$.*

3. The deduction of the theorem from Proposition 1.

Lemma 1.

1) *If $\nu_i^*(\chi) > n(\chi)$, then $\nu_i^*(\chi\tau_i) < n(\chi\tau_i) < n(\chi)$.*
2) *If $\nu_i^*(\chi) < n(\chi)$, then $\nu_i^*(\chi\tau_i) > n(\chi\tau_i) > n(\chi)$.*

Proof. We shall consider the case $i \ge 1$ (the case $i = 0$ is easier and we leave it to the reader; the reasoning is completely analogous).

Let $|\chi|^*$ be the proper inverse image of the linear system $|\chi|$ on V_i^*. Obviously, the class of a divisor of the system $|\chi|$ in $\operatorname{Pic} V_i^*$ is $n(\chi)h - \nu_0^*(\chi)e - \nu_i^*(\chi)e_i^*$.

Let us consider the following commutative diagram of birational maps:

$$\begin{array}{ccccc} V_i^* & \xrightarrow{\widetilde{\tau}_i} & V_i^* & & \\ {\scriptstyle \varphi\circ\varphi_i^*}\downarrow & & {\scriptstyle \varphi\circ\varphi_i^*}\downarrow & & \\ V & \xrightarrow{\tau_i} & V & \xrightarrow{\chi} & V' \end{array}$$

Obviously, the linear system $|\chi\tau_i|^*$ coincides with the proper inverse image of the linear system $|\chi|^*$ on V_i^* under $\widetilde{\tau}_i$. But, since $\widetilde{\tau}_i \in \operatorname{Aut} V_i^* \setminus W$, $\operatorname{codim} W = 2$, the class of a divisor of the system $|\chi\tau_i|^*$ may be produced from the class of a divisor of the system $|\chi|^*$ by the $\widetilde{\tau}_i$ action, i.e.,

$$n(\chi\tau_i)h - \nu_0^*(\chi\tau_i)e - \nu_i^*(\chi\tau_i)e_i^* = \widetilde{\tau}_i^*(n(\chi)h - \nu_0^*(\chi)e - \nu_i^*(\chi)e_i^*).$$

Opening the parentheses of the right-hand side of the equation and using the relations of Lemma 1.3, we get

$$n(\chi\tau_i) = 11n(\chi) - 10\nu_i^*(\chi),$$

$$\nu_i^*(\chi\tau_i) = 12n(\chi) - 11\nu_i^*(\chi), \quad i \ge 1.$$

The lemma is a direct consequence of these equalities.

Let us return to the proof of the theorem. Let us first note that the situation $\nu_i^*(\chi) > n(\chi) = 1$ is impossible (because the degree 1 cannot be diminished).

Then, by Proposition 1 and Lemma 1, for each χ, $n(\chi) \geq 2$, there exists a unique $i \in \mathbb{Z}$, $0 \leq i \leq 24$, such that $n(\chi\tau_i) < n(\chi)$. Thus, if $B(V) \subset \operatorname{Bir} V$ is the subgroup generated by involutions τ_i, $0 \leq i \leq 24$, then for each χ, $n(\chi) \geq 2$, there exists $\widetilde{\chi} \in B(V)$ such that $n(\chi \circ \widetilde{\chi}) = 1$, i.e., V is birationally rigid.

Now the theorem is a consequence of the following statement (other considerations are trivial): If $i_1, \ldots, i_k$ is a sequence of integers such that $0 \leq i_\alpha \leq 24$, where $1 \leq \alpha \leq k$, and $i_\alpha = i_{\alpha+1}$ for $1 \leq \alpha \leq k-1$, then $\tau_{i_1} \ldots \tau_{i_k} \notin \operatorname{Aut} V$. The proof of this statement is an easy induction by k, using Proposition 1 (the maximal cycle is unique) and Lemma 1 (we used analogous reasoning in the proof of the theorem about 3-dimensional double quadrics, Chap. 2, Sec. 2). The theorem is proved.

3. Maximal Singularities and Maximal Subsets.

Let us fix up to the end of this chapter a birational map $\chi\colon V \dashrightarrow V'$, where $n(\chi) = n \geq 2$ or $n(\chi) = 1$ and $|H' + K_{V'}| = \varnothing$. We shall prove Proposition 2.1.

Let us first consider the question of using the technique of Chap. 1 here. Generally, it is possible to use it in the same way. We shall note especially the differences which appear because of the existence of a singular point x.

Let us fix an admissible resolution $R(V_0, \chi\circ\varphi) = \varphi_{i,i-1}\colon V_i \to V_{i-1} | 1 \leq i \leq N$ of the map $\chi\circ\varphi\colon V_0 \dashrightarrow V'$ (and not $\chi\colon V \dashrightarrow V'$), where $\varphi\colon V_0 \to V$ is the blowing of x, as above. Let again $B_{i-1} \subset V_{i-1}$ be the center of the ith blowing, and let $E_i \subset V_i$ be its exceptional divisor. Set, for simplicity, $E = E_0$, $B_{-1} = x \in V = V_{-1}$, $\varphi = \varphi_{0,-1}$. As usual if $N \geq i > j \geq -1$, then $\varphi_{i,j} = \varphi_{j+1,j} \circ \cdots \circ \varphi_{i,i-1}\colon V_i \to V_j$, $\varphi_{i,i} = \mathrm{id}_{V_i}$. We shall denote the proper inverse image of the linear system $|\chi|$ on V_i by $|\chi|_i$. The free system $|\chi|_N$ defines the birational morphism $V_N \to V'$. The class of the divisor of the system $|\chi|_N$ in $A^1(V_N)$ is $h' = nh - \sum_{i=0}^{N} \nu_i e_i$, $n = n(\chi)$. The class of the divisor of the system $|\chi|_i$ in $A^1(V_i)$ is $h'_i = nh - \sum_{j=0}^{i} \nu_j e_j$. From the admissibility it follows that $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_i \geq \nu_{i+1} \geq \ldots$. It is easy to check that, for $i = \min\{1 \leq j \leq N | \varphi_{j,-1}(E_j) \ni x\}$, $2\nu_0 \geq n u_i$. Let $\nu_i^* = \nu_i^*(\chi)$, $0 \leq i \leq 24$. Let us note that $\nu_0 = \nu_0^*$. Let the class of a line in E_i, $1 \leq i \leq 24$, be $f_i \in A^2(V_i)$. For images and inverse images we shall use the same notations, only the indices will change from -1 to N.

The graph of singularities Γ is the graph of all exceptional divisors: it contains a (minimal) vertex E_0; $i \to j$ only when $i > j$ and $B_{i-1} \subset E_j^{i-1}$; the sets $P(i,j)$ and numbers r_{ij} are defined in the same way as in Chap. 1.

The canonical divisor K_{V_N} of the manifold V_N is

$$K_{V_N} = -h + \sum_{i=0}^{N} \delta_{i-1} e_i,$$

where $h \in A^1(V)$ is a hyperplane section of $V \subset \mathbb{P}^4$ and $\delta_{-1} = 1$. The Noether–Fano inequalities are defined as follows.

Proposition and Definition 1. *There exists an index β, $-1 \leq \beta \leq N-1$, such that $\nu_{\beta+1} > \delta_\beta n$ and*

$$\sum_i r_{\beta+1,i}\nu_i > \sum_i r_{\beta+1,i}\delta_{i-1} n.$$

For such an index β B_β is called the *maximal singularity* (the proof is, literally, the same as in Chap. 1). The definition of a maximal subset is changed a little.

Definition 2. An irreducible closed subset $B \subset V$ is called a *maximal subset* if and only if one of the following conditions holds:

(1) $B = x$, $\nu_0^* > n$;

(2) $B = t$, t is a point, $t \neq x$, $\operatorname{mult}_t |\chi| > 2n$;

(3) B is a curve and $\operatorname{mult}_B |\chi| > n$.

The statement of Proposition 1.2.2 also holds in this case. The proof is analogous, except for one case. Let B_β be a maximal singularity and $B_{\beta,-1} = x$. Then, as was noted above, $\nu_{\beta+1} \leq 2\nu_0$ and $\nu_{\beta+1} > 2n$, and so $\nu_0 > n$, as was required.

The multiplication formulas in the Chow ring (Proposition 1.3.1) hold. Let us note that by the degree of the curve $F \subset E$ we understand its degree with respect to the inclusion $F \subset E \subset \widetilde{E} \cong \mathbb{P}^3$.

Lemma 1.

(1) $e^3 = 2$.

(2) *Let $F \subset E$ be a curve. Then $-(f \cdot e) = \deg F$ (by the above definition).*

The proof is obvious.

An analog of Corollary 1.3.1 can be formulated as follows.

Corollary 1. $(h'^2 \cdot e) = 2\nu_0^2 - \sum_{j=1}^{N}(b_{j-1} \cdot e)\nu_j^2$.

Formulas for $(h'^2 \cdot h)$ and $(h'^2 \cdot e_j)$, $j \geq 1$, remain the same ($h^3 = d = 4$).

Lemma 1.3.1 also holds in this case and its proof is analogous, substituting e_j for g_j and taking into account that $\dim B_{j-1} = 0$, $0 \leq j \leq N$.

Corollary 1.3.2 holds.

Corollary 1.3.3 now may be formulated as follows: if $i \geq 1$ and $\nu_0 < \sqrt{2}n$, then $\nu_i \leq 2n$.

No other changes in statements and proofs are required.

The technique of the test class for the elimination of maximal singularities of the type (1.0) shall be discussed in Sec. 5.

4. Maximal Subsets of a Quartic.

1. Proposition 1.

(A) *If $B \subset V$ is a maximal subset, then either $B = x$, i.e., B is a singular point, or $B = B_i$, i.e., one of the 24 lines passing through x.*

(B) *If one of the numbers ν_i^*, $0 \leq i \leq 24$, is strictly greater than n, then all others are strictly less than n.*

Proof. Let us first note that if B is a maximal subset, then $x \in B$.

Lemma 1.

(A) $\mathrm{mult}_t |\chi| \leq 2n$, *where $t \neq x$ is a point.*

(B) $\mathrm{mult}_B |\chi| \leq n$, *where $B \subset V$ is a curve, $x \notin B$.*

Proof. (A) see Sec. 3; (B) see Chap. 1, Sec. 3, for example. The reasoning of Chap. 1 may be used word for word because $x \notin B$.

Lemma 2. *If a curve B is maximal, then $\deg B \leq 3$.*

The proof is the same as in Chap. 1:

$$0 \leq (h'^2 \cdot h) \leq 4n^2 - \deg B(\mathrm{mult}_B |\chi|)^2.$$

But, $\mathrm{mult}_B |\chi| > n$, by the assumption. The lemma is proved.

Let us first consider the case where B is a plane curve.

Lemma 3 (*the plane section lemma*). *Let $P \subset \mathbb{P}^4$ be a plane which contains x. Let the plane curve $V \subset P$ be a union of components $C_1, \dots, C_k$ of degrees $d_1, \dots, d_k$. Let $\theta_i^* = \mathrm{mult}_{C_i} |\chi|$. Then,* (A) *if $d_i \geq 2$, then $\theta_i^* \leq n$;* (B) *if $d_i = d_j = 1$, $i \neq j$, $x \in C_i \bigcap C_j$, then $\theta_i^* + \theta_j^* \leq 2n$.*

Corollary 1. *A plane curve B of degree ≥ 2 is not a maximal subset.*

Corollary 2. *If $\nu_{i_0}^* > n$, $1 \leq i_0 \leq 24$, then $\nu_i^* < n$ for $i \neq i_0$, $1 \leq i \leq 24$.*

Proof of Corollary 2. Two lines passing through x are contained in one plane. Thus, we can use the plane section lemma (B).

The proof of the plane section lemma requires a special technique and will be given in Sec. 6.

2. To prove part (A) of Proposition 1, it remains to state that a nonplane curve $B \ni x$ is not a maximal subset. By Lemma 2, we can consider only the case $\deg B = 3$, i.e., when B is a rational normal curve which is contained in some hyperplane.

A rational normal curve is the intersection of quadrics which contain it. Let $B' \subset V_0$ be the proper inverse image of the curve B on V_0, $\pi'\colon V' \to V_0$ be its blowing, $E' = (\pi')^{-1}(B')$ be the exceptional divisor. The linear system $|2h - e - e'|$ of divisors on V' is free. Let $\nu' = \operatorname{mult}_B |\chi|$; then the linear system $|nh - \nu_0 e - \nu' e'|$ of divisors on V' contains the proper inverse image on V' of the linear system $|\chi|$ and, therefore, does not have nonstationary components. Hence,

$$\left((2h - e - e') \cdot (nh - \nu_0 e - \nu' e')^2\right) \geq 0.$$

But, a direct computation gives us that

$$\left((2h - e - e') \cdot (nh - \nu_0 e - \nu' e')^2\right) = 8n^2 - 6n\nu' - 4\nu'^2 - \nu_0^2 - (\nu' - \nu_0)^2 < 0$$

for $\nu' \geq n$. We have a contradiction. Case (A) of Proposition 1 is proved.

3. Lemma 4. *If one of the two numbers ν_0^* and ν_i^*, $1 \leq i \leq 24$, is strictly greater than n, then the other is strictly less than n.*

Proof. Let us consider the proper inverse image $|\chi|_i^*$ of the linear system $|\chi|$ on V_i^* (see Sec. 1). The class of the divisor of $|\chi|_i^*$ is $nh - \nu_0 e - \nu_i^* e_i^*$. Hence, the linear system $|nh - \nu_0 e - \nu_i^* e_i^*|$ does not have nonstationary components. The linear system $|h - e - e_i^*|$ is free. Thus,

$$\left((h - e - e_i^*) \cdot (nh - \nu_0 e - \nu_i^* e_i^*)^2\right) \geq 0.$$

A direct computation gives

$$4n^2 - \nu_0^2 - \nu_i^{*2} - 2n\nu_i^* - (\nu_0 - \nu_i^*)^2 \geq 0.$$

The lemma is a consequence of these inequalities.

It remains to note that case (B) of Proposition 1 is a consequence of Corollary 2 and Lemma 4.

5. Elimination of a Maximal Singularity of the Type (1,0).

1. Proposition 1. *If $\nu_i^* \leq n$, $0 \leq i \leq 24$, then the maximal singularity B_β of the type* $(1,0)$ *does not exist.*

Proof. Let us assume that there exists a maximal singularity of this type and let us fix it. Let us first note that $B_{\beta,-1} = x$: the case $B_{\beta,-1} \neq x$ is impossible (in this case, we can use the methods of Chap. 1 and get a contradiction in the same way, as in Chap. 1, Sec. 5).

Let us denote by Γ_β a subgraph of the graph of singularities which is defined by the following condition: E_j is a vertex in Γ_β if and only if $E_{\beta+1} \geq E_j$, i.e., $B_{\beta,j} \subset E_j$. Let $I_t = \{1 \leq j \leq N | E_j \leq E_{\beta+1}, \dim B_{j-1} = t\}$, $t = 0,1$. Let us note that $0 \notin I_0$. Let $I = I_0 \bigcup I_1 \bigcup \{0\}$ be the set of indices of the vertices of Γ_β. Let $r_j = r_{\beta+1,j}$ for $j \in I$, $\Sigma_t = \sum\limits_{j \in I_t} r_j$ for $t = 0,1$.

Lemma 1.

(1) *If $j \leq \beta + 1$, then $B_{j-1,-1}$ is a point. Moreover if $j \in I$, then $B_{j-1,-1} = x$.*

(2) *Let $j \leq \beta + 1$, B_{j-1} be a curve, $j_1 = \min\{t | \dim B_{j-1,t} = 1\}$. Then* $\deg(\varphi_{j-1,j_1}\colon B_{j-1} \to B_{j-1,j_1} = 1$, *$B_{j_1-1}$ is a point and $B_{j-1,j_1} \subset E_{j_1} \cong \mathbb{P}^2$ is a line.*

Proof. (1) is a consequence of the fact that V does not contain any maximal subsets, by the definition and by Proposition 4.1;

(2) is proved in the same way as Lemma 1.4 in [6, Chap 2].

Let $i_1 = \min\{j \in I_0 \bigcup I_1\}$. Obviously, $\varphi_{i_1-1,0}$ is an isomorphism in a neighborhood of B_{i_1-1}. Therefore, for simplicity of notations, we can set $i_1 = 1$. Let us note that the restriction of the linear system $|\chi|_0$ to E is a linear system of the type (ν_0, ν_0) of curves on the quadric $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ which cannot have a stationary component of multiplicity greater than n, because $\nu_0 \leq n$. But, $\operatorname{mult}_{B_0} |\chi| = \nu_1 > n$, and so B_0 is a point and, by Lemma 1 if $j \in I \setminus \{0\}$, then $B_{j-1,0} = B_0$.

Definition 1. A *test class* is a number class

$$y = \sum_{j \in I_0} r_j (h - e_j) - r_0 \quad e_0 \in A^1(V_N).$$

Lemma 2.

(A) *Let* $C \subset V_i$, $0 \le i \le N$, *be a curve such that* $\dim C_{-1} = 0$; *then* $(y \cdot c) \ge 0$.

(B) *If* $i \in I_1$, *then* $(b_{i-1} \cdot y) \ge r_i$.

The proof of statements (A) and (B) is the same as the proof of statements (A) and (B) of Lemma 1.4.1.

2. Let us study now the behavior of the test class y with respect to curves which are not contractible into a point on V.

Lemma 3. *Let* $C \subset V_i$, $-1 \le i \le N$, *be an irreducible curve such that* $\dim C_{-1} = 1$. *Let us assume that the curve* C_{-1} *is different from the line passing through* x *in the direction of an infinitely close point* B_0. *Then* $(y \cdot c) \ge 0$.

Proof. Obviously if $x \notin C_{-1}$, then $(y \cdot c) > 0$ (because, in this case, $(c \cdot e_j) = 0$ for $j \subset 0 \bigcup I_0$). Hence, we can assume that $x \in C_{-1}$. Let, at first, $\deg C_{-1} \ge 2$. Then for any point $z \in E$ we have the inequality

$$\operatorname{mult}_x C_{-1} + \operatorname{mult}_z C_0 \le \deg C_{-1} \tag{$*$}$$

(in the opposite case, the line in $\mathbb{P}^4$ which passes through x in the direction of z intersects C_{-1} in two points; one of them is simple and the other is infinitely close, and the sum of their multiplicities is greater than $\deg C_{-1}$, so C_{-1} has this line as a component). But, for all points B_{j-1}, $j \in I_0$, $B_{j-1,-1} = x$. On the other hand, $\varphi_{j,-1}$ is an isomorphism in a neighborhood of a general point C_j for all $j \le \beta$ (because V does not have maximal subsets). Hence,

$$\operatorname{mult}_{B_{j-1}} C_{j-1} \le \operatorname{mult}_{B_{j-1,0}} C_{j-1,0}$$

for $j \in I_0$, $1 \le j \le i$, and, by multiplication formulas, taking $(*)$ into account, we have $((h - e_0 - e_j) \cdot c) \ge 0$ for all $j \in I_0$. Let us give an expression for the product $(y \cdot c)$ in the following way:

$$(y \cdot c) = \sum_{j \in I_0} r_j\big((h - e_0 - e_j) \cdot c\big) + \Big(\big(\sum_{j \in I_0} r_j\big) - r_0\Big)(e_0 \cdot c) \ge (\Sigma_0 - r_0)(e_0 \cdot c).$$

Lemma 4 (*the graph lemma*). $\Sigma_0 \ge r_0$.

The proof of the graph lemma will be given in Sec. 7.

Since $(e_0 \cdot c) \ge 0$, the inequality $(y \cdot c) \ge 0$ is proved.

Let now C_{-1} be a line passing through x but $B_0 \notin C_0$. Since $B_{j-1,0} = B_0$ for all $j \in I$, the point B_{j-1} for $j \in I_0$ is not contained in the proper inverse image of C_{-1} on V_{j-1}, and so $(c \cdot e_j) = 0$ for $i \ge j \in I_0$. If $j > i$, then $(c \cdot e_j) = 0$ by the projection formula. Thus, $(c \cdot e_j) = 0$ for all $j \in I_0$; hence,

$$(y \cdot c) = \Sigma_0 (h \cdot c) - r_0 (e_0 \cdot c) \ge r_0\big((h - e_0) \cdot c\big) \ge 0$$

by the graph lemma. Lemma 3 is proved.

Corollary 1. $(h'^2 \cdot y) \ge 0$.

Corollary 2. *If the point* B_0 *is different from the points of* $\{B_i^* \bigcap E\}$, $1 \le i \le 24$, *i.e.,* $B_0 \notin B_i^*$, $1 \le i \le 24$, *then* $(y \cdot c) \ge 0$ *for all curves* $C \subset V_i$, $-1 \le i \le N$.

Definition 2. If $B_0 \notin B_i^*$, $1 \le i \le 24$, then this case will be called a *simple case*; if $B_0 \in B_i^*$ for some (unique) i, then this case will be called a *difficult case*.

Let us introduce a new notation. In the simple case, we set $\Delta = 0$, $J = \varnothing$.

In the difficult case, let $B = \overline{B}_i$ be a line which passes through x in the direction B_0, $B_0 \in B^0$. Let, for simplicity, $i = 1$; then $B = \overline{B}_1$. Let $\nu^* = \nu_1^*$ and

$$J = \big\{0 \le j \le N \;\big\|\; B_{j-1,-1} = B\big\} \text{ and } \Delta = \sum_{j \in J} (b_{j-1} \cdot h)\nu_j^2.$$

Let in all cases

$$\Sigma = -\Sigma_1^2 + \left(\frac{\Delta}{n^2} - 2\right) r_0(\Sigma_0 + \Sigma_1 + r_0/2).$$

Let us prove several estimations.

Lemma 5 (*The square inequality*).

$$2r_0\nu_0^2 + \sum_{j\in I\setminus\{0\}} r_j\nu_j^2 > \frac{1}{\Sigma_0+\Sigma_1+r_0/2}(2\Sigma_0+\Sigma_1+r_0)^2n^2.$$

The proof is a direct computation based on the proposition and Definition 3.1.

Lemma 6.

$$(h'^2\cdot y) \le \Sigma n^2\frac{1}{\Sigma_0+\Sigma_1+r_0/2}.$$

Proof. By the multiplication formulas, we have

$$(h'^2\cdot y) = 4n^2\Sigma_0 - \sum_{i=0}^{N}(b_{i-1}\cdot y)\nu_i^2 - \sum_{j\in I_0} r_j\nu_j^2 - 2r_0\nu_0^2.$$

Now if $j \in J\bigcup I$, then $(b_{j-1}\cdot y)\ge 0$ by Lemmas 2 and 3. Then, obviously, $J\bigcap I=\varnothing$, $(b_{j-1}\cdot y)=0$ for $i\in I_0\bigcup\{0\}$ (by reason of dimensionality) and $(b_{i-1}\cdot y)\ge r_i$ for $i\in I_1$ (by Lemma 2 (B)). Thus,

$$(h'^2\cdot y) \le 4n^2\Sigma_0 - 2r_0\nu_0^2 - \sum_{j\in I_0\bigcup I_1} r_j\nu_j^2 - \sum_{j\in J}(b_{j-1}\cdot y)\nu_j^2.$$

Let us consider the last sum. Let $J\ne\varnothing$ (the difficult case). Then

$$(b_{j-1}\cdot y) = \sum_{i\in I_0} r_i\bigl(b_{j-1}\cdot(h-e_i)\bigr) - r_0(b_{j-1}\cdot e_0) \ge -r_0(b_{j-1}\cdot e_0) = -r_0(b_{j-1}\cdot h).$$

Thus, $\sum_{i\in J}(b_{j-1}\cdot y))\nu_j^2 \ge -r_0\Delta$. Using this estimation and the square inequality, we shall get the proof.

Corollary 3. $\Delta > 2n^2$ (*in particular*, $J\ne\varnothing$).

Proof. Let us assume the contrary: $\Delta\le 2n^2$. Then $\Sigma \le -\Sigma_1^2 < 0$, and, by the above lemma, $(h'^2\cdot y) < 0$. But this contradicts Corollary 1.

Corollary 3. *The simple case is not realized.*

3. Our aim is to prove the inequality $\Sigma < 0$. To this end, using the definition of Σ, we have to prove an upper estimation for Δ and a lower one for Σ_1.

Lemma 7.

$$\Delta \le 4n^2 - 2n\nu^* - \nu_0^2 - (\nu_0 - nu^*)^2 - (\nu_1-\nu^*)^2.$$

Proof. Let us consider the system $|\tau|$ of hyperplane sections of V containing the line B. Let S be a general element of $|\tau|$, S^i be its proper inverse image on V_i. Since $J\ne\varnothing$, let us take $i_1=\min\{i\in J\}$. Then $i_1>\beta$, $B_{i_1-1}=B^{i_1-1}$, and the map $\varphi_{i_1-1,0}$ is an isomorphism in a neighborhood of a general point B_{i_1-1}. The proper inverse image of $|\tau|$ on V_{i_1} is, obviously, a free linear system. Hence, its general divisor and all proper inverse images S^i, $i>i_1$, of a general divisor, does not contain the cycles B_i, $i\ge i_1$. Thus, $s^{i_1}=s^{i_1+1}=\cdots=s^N$ (recall that s^i is the class of S^i in $A^1(V_i)\hookrightarrow A^1(V_N)$). For simplicity, we shall represent the set of all exceptional divisors as the union of three subsets:

the first: $\{E_i|B_{i-1}$ is a point, $B_{i-1}\in B^{i-1}\}=\{E_i|i\in J_1\}$;

the second: $\{E_i|B_{i-1}$ is a point, $B_{i-1}\notin B^{i-1}\}=\{E_i|I\in J_2\}$;

the third: $\{E_i|B_{i-1}$ is a curve$\}=\{E_i|i\in J_3\}$.

Obviously, $s^{i_1}=h-e_0-\sum_{i\in J_1}e_i - e_{i_1}$ and s^{i_1} is nonnegative. Thus,

$$0 \le (h'^2\cdot s^{i_1}) = \left(\left(nh-\sum_{i=0}^{N}\nu_ie_i\right)^2\cdot\left(h-e_0-\sum e_i - e_{i_1}\right)\right)$$
$$= \left(\left(hn-\sum_{i=0}^{i_1}\nu_ie_i\right)^2\cdot s^{i_1}\right) + \sum_{i=i_1+1}^{N}(e_i^2\cdot s^{i_1})\nu_i^2. \qquad (**)$$

Let us note that

$$(e_i^2 \cdot s^{i_1}) = -1 \text{ if } i \in J_1; \quad = 0 \text{ if } i \in J_2; \quad = -(b_{i-1} \cdot s^{i-1}) = -\#B_{i-1} \bigcap S^{i-1} \text{ if } i \in J_3.$$

Then

$$\begin{aligned}(e_{i_1}^2 \cdot s^{i_1}) &= -\left(b_{i_1-1} \cdot \left(h - e_0 - \sum_{i \in J_1} e_i\right)\right) - (e_{i_1}^3) \\ &= -2 + 2g(B) - \left(b_{i_1-1} \cdot \left(-h + e_0 + \sum_{i=1}^{i_1-1} \delta_{i-1} e_i + h - \sum_{i \in J_1} e_i\right)\right) \\ &= -\left(2 + \sum_{i \in J_1} 1 + \sum_{i \in J_3} (\#B_{i_1-1,i} \bigcap E_i)\right).\end{aligned}$$

But, $\sum_{i \in J_1} 1 \geq 1$ and $\#B_{i_1-1,i} \bigcap E_i = \#B_{i_1-1,i-1} \bigcap B_{i-1}$ if $i \in J_3$.

Finally, $(e_i \cdot e_j \cdot s^{i_1}) = 0$ for $i < j < i_1$, and

$$(e_i \cdot e_{i_1} \cdot s^{i_1}) = (b_{i_1-1} \cdot e_i) = \#B_{i_1-1,i} \bigcap E_i = \#B_{i_1-1,i-1} \bigcap B_{i-1}$$

(in particular, i_1 equals 1 if $i \in J_1$, and equals 0 if $i \in J_2$).

Let us estimate the first term in $(**)$:

$$\begin{aligned}\left(nh - \sum_{i=0}^{i_1} \nu_i e_i)^2 \cdot s^{i_1}\right) &= 4n^2 - 2n\nu^* - 2\nu^{*^2} - 2\nu_0^2 + 2\nu_0 \nu^* - \sum_{i \in J_1} (\nu^* - \nu_i)^2 \\ &\quad - \sum_{i \in J_3} \left[\#(B_{i_1-1,i} \bigcap E_i)(\nu^{*^2} - 2\nu^* \nu_i) + \#(B_{i-1} \bigcap S^{i-1}) \nu_i^2\right].\end{aligned}$$

Let us note now that

$$\#B_{i-1} \bigcap S^{i-1} \geq \#B_{i-1} \bigcap C^{i-1} = \#B_{i-1} \bigcap B_{i_1-1,i-1} = \#E_i \bigcap B_{i_1-1,i} \geq 0$$

(recall that $C^{i-1} \subset S^{i-1}$), and so the expression in the square parentheses is nonnegative and

$$\begin{aligned}\left(\left(nh - \sum_{i=0}^{i_1} \nu_i e_i\right)^2 \cdot s^{i_1}\right) &\leq 4n^2 - 2n\nu^* - 2\nu^{*^2} - 2\nu_0^2 + 2\nu^* \nu_0 \\ &\quad - \sum_{i \in J_1} (\nu^* - \nu_i)^2 \leq 4n^2 - 2n\nu^* - \nu_0^2 - (\nu_0 - \nu^*)^2 - (\nu_1 - \nu^*)^2 - \nu^{*^2},\end{aligned}$$

because $1 \in J_1$.

Let us consider the second sum in $(**)$. Obviously,

$$-\sum_{i=i_1+1}^{N} (e_i^2 \cdot s^{i_1}) \nu_i^2 = \sum_{i=i_1+1}^{N} (b_{i-1} \cdot s^{i_1}) \nu_i^2 \geq \sum_{i \in J \setminus \{i_1\}} (b_{i-1} \cdot s^{i_1}) \nu_i^2,$$

because s^{i_1} is a nonnegative class. Then

$$(b_{i-1} \cdot s^{i_1}) = (b_{i-1,i_1} \cdot s^{i_1}) \deg(\varphi_{i-1,i_1} : B_{i-1} \to B_{i-1,i_1})$$

(recall that b_{i-1,i_1} is the class of B_{i-1,i_1} in $A^2(V_{i_1})$).

Lemma 8. *Let $Z \subset E_{i_1}$ be an arbitrary curve different from a fibre of the morphism $\varphi_{i_1,i_1-1} : E_{i_1} \to B_{i_1-1}$. Then*

$$(z \cdot s^{i_1}) \geq d = \deg(\varphi_{i_1,i_1-1}\colon\ Z \to B_{i_1-1}).$$

Proof. Let us consider the following diagram of birational maps:

$$\begin{tikzcd} V_1^* \arrow[dr, "\varphi_1^*"'] & & V_{i_1} \arrow[ll, dashed, "\lambda"'] \arrow[dl, "\varphi_{i_1,0}"] \\ & V_0 & \end{tikzcd}$$

(the map $\varphi_1^*\colon\ V_1^* \to V_0$ which blows the line $B^0 = B_1^*$ was defined in Sec. 1). By the definition of i_1, we have that $\lambda(E_{i_1}) = E_1^*$ and the map λ can be included in the diagram:

$$\begin{tikzcd} E_{i_1} \arrow[r, "\lambda"] \arrow[d, "\varphi_{i_1,i_1-1}"'] & E_1^* \arrow[d, "\varphi_1^*"] \\ B_{i_1-1} \arrow[r, equal] & B^0 = B_1^* \end{tikzcd}$$

i.e., the action of λ agrees with the standard structures of the ruled surface on E_{i_1} and E_1^* (because $\varphi_{i_1-1,0}$ is an isomorphism in a neighborhood of a general point B_{i_1-1}). Then λ is an isomorphism on each fibre except for a finite number of them, i.e., there exists a finite set of points $t_i \in B^0 = B_{i_1-1}$, $1 \leq i \leq k$, such that

$$\lambda\colon\ E_{i_1} \setminus \bigcup_{i=1}^{k} F_{t_i} \longrightarrow E_1^* \setminus \bigcup_{i=1}^{k} F_{t_i}$$

is an isomorphism over $B^0 \setminus \{t_1, \ldots, t_k\}$ (where F_t is a fibre over $t \in B^0 = B_{i_1-1}$). The proper inverse image of the system $|\tau|$ on V_1^* sections on E_1^* a 2-dimensional free linear system of curves $|\tau|^*$ of the type (1,1) ($E_1^* \cong \mathbb{P}^1 \times B_1^* \cong \mathbb{P}^1 \times \mathbb{P}^1$). $|\tau|^*$ is free, because the proper inverse image of $|\tau|$ on V_1^* is free. The curve $\lambda(Z)$ has the type $(d,*)$ and is irreducible. The set $\lambda(Z) \bigcup_{i=1}^{k} F_{t_i}$ is finite; hence, a general curve of the linear system $|\tau|^*$ intersects $\lambda(Z)$ outside the fibres F_{t_i}. But the intersection index of a curve of the type (1,1) and a curve of the type $(d,*)$ is not less than d. Thus, the curve $\lambda(Z)$ intersects the proper inverse image of a general divisor of $|\tau|$ on V_1^* at not less than d points. In neighborhoods of these points λ^{-1} is an isomorphism; thus, we prove the inequality $(z \cdot s^{i_1}) \geq d$.

Corollary 5. $(b_{i-1} \cdot s^{i_1}) \geq (b_{i-1} \cdot h)$ *for* $i \in J$.

By the estimations proved above, Lemma 7 is a consequence of (**).

The formal computations considered above have a clear geometrical sense. Let S be a general surface of $|\tau|$, then S^0 is nonsingular. The restriction of the linear system $|\chi|_0$ to S^0 has a unique stationary component, i.e., the line $B_1^* = B^0$. It is easy to compute that the square of the free part of the system is $4n^2 - 2n\nu^* - 2{\nu^*}^2 + 2\nu^*\nu_0 - 2\nu_0^2$. The last expression must be greater than the sum of the squares of multiplicities of all base points (simple and infinitely close). The multiplicity of the base point B_0 is $\nu_1 - \nu^*$ and, moreover, each curve of multiplicity ν_i infinitely close to B^0 which is a $d_i = (b_{i-1} \cdot h)$-leaf cover of B^0 generates $\geq d_i\nu_i$-multiple base points. Lemma 7 is a consequence of these considerations.

Lemma 9.

$$\Sigma_1 \geq \frac{1}{\nu_1 - n}((2n - \nu_1)\Sigma_0 + (n - \nu_0)r_0).$$

Proof. If $i \in I \setminus \{0\}$, then $\nu_i \leq \nu_1$. Therefore, by the Noether–Fano inequality (Proposition 3.1), we have

$$r_0 n + 2\Sigma_0 n + \Sigma_1 n \leq \sum_{i \in I} r_i \nu_i \leq r_0\nu_0 + (\Sigma_0 + \Sigma_1)\nu_1.$$

Taking into account the fact that $\nu_1 > n$, we get the lemma.

Set $\nu_0 + \nu_1 = 3\theta$, $n/2 < \theta < n$, and

$$\Lambda(t) = 4n^2 - 2n\nu^* - t^2 - (t - \nu^*)^2 - (2t - \nu^*)^2.$$

Lemma 10.
(A) $\Delta \leq \Lambda(\theta)$;
(B) $\Lambda(t)$ *is a decreasing function of* t, *for* $t > n/2$;
(C) $\Sigma_1 \geq \frac{n-\theta}{2\theta-n}(2\Sigma_0 + r_0)$.

The **proof** is a simple computation; in (C) we have to use the graph lemma.

Lemma 11. $\theta > \dfrac{3}{4}n$.

Proof. Let us assume the contrary. Then, by Lemma 10 (C), $\Sigma_1 \geq \Sigma_0 + \dfrac{1}{2}r_0$. Then,

$$\Lambda(\theta) \leq \Lambda\left(\frac{n}{2}\right) = \frac{5}{2}n^2 + n\nu^* - 2\nu^{*^2} \leq \frac{21}{8}n^2.$$

But,

$$\Sigma \leq -\Sigma_1^2 + \frac{5}{8}r_0\left(\Sigma_1 + \Sigma_0 + \frac{1}{2}r_0\right).$$

It is easy to check that if $\Sigma_1 \geq \Sigma_0 + 1/2r_0$ (taking into account that $\Sigma_0 \geq r_0$), then the right-hand expression is negative, so $\Sigma < 0$. We have a contradiction to Corollary 1.

We can complete the proof of Proposition 1.

By Lemma 11, $\theta > \dfrac{3}{4}n$. Therefore,

$$\Lambda(\theta) \leq \Lambda(3/4n) = \frac{5}{8}n^2 + \frac{5}{2}n\nu^* - 2\nu^{*^2} \leq \frac{45}{32}n^2 < 2n^2.$$

But this contradicts Corollary 3.

4. We can complete the proof of Proposition 2.1.

Let us assume that each $\nu_i^* \leq n, 0 \leq i \leq 24$. Then Proposition 1 and Proposition 4.1 give us a contradiction to Proposition 3.1. Using Proposition 4.1 (B), we can finish the proof.

6. The Proof of the Plane Section Lemma.

1. We shall study the multiplicities of the linear system $|\chi|$ along components of a general plane section V which contains the point x. Let $P \subset \mathbb{P}^4$ be a plane, $x \in P$. The 4-degree plane curve $V \bigcap P$ is the union of components $C_1, \dots, C_k$, $1 \leq k \leq 4$, of degrees $d_1, \dots, d_k$ and multiplicities $m_1, \dots, m_k$; thus, $\sum_{i=1}^k m_i d_i = 4$. We shall denote by μ_i the multiplicity of the curve C_i at x; thus, $\sum_{i=1}^k m_i \mu_i = \mu, 2 \leq \mu \leq 4$.

2. We shall now construct a sequence of monoidal transformations $\{\pi_j\colon\ X_j \to X_{j-1} | 1 \leq j \leq M\}$, where $X_0 = \mathbb{P}^4$. Let Z_{j-1} be a nonsingular irreducible center of the jth blowing π_j, $Y_j = \pi_j^{-1}(Z_{j-1})$ be an exceptional divisor, V^j and P^j be the proper inverse images of the quartic V and the plane P on X_j (in this section we shall denote the proper inverse image of a cycle F on X_j by F^j, because there is no confusion here). These objects have the following properties:

(1) $Z_0 = x$ (hence $V^1 = V_0$), $Z_j = x_j \in P^j \bigcap V^j$ are points for all j, $1 \leq j \leq M_0 - 1$, $M_0 < M$;

(2) the proper inverse images $C_i^{M_0}$ of the curves C_i on P^{M_0} are nonsingular, they do not intersect each other, and $P^{M_0} \bigcap V^{M_0}$ is a divisor on P^{M_0} with normal intersections (although maybe with multiple components);

(3) each blowing π_j, $M_0 \leq j \leq M$, is the blowing of a (not necessarily nonsingular) irreducible component of the curve $P^{j-1} \bigcap V^{j-1}$, so $\pi_j\colon\ P^j \to P^{j-1}$ is an isomorphism for all j, $M_0 + 1 \leq j \leq M$. The order of blowings is defined in the following way. Let us denote by Y_j' the proper inverse image of the exceptional curve $Y_j^* = Y_j \bigcap P^j$, $1 \leq j \leq M_0$, on P^{M_0}. Let

$$P^{M_0} \bigcap V^{M_0} = \sum_{i=1}^{k} m_i C_i^{M_0} + \sum_{i=1}^{M_0} x_j Y_j'.$$

Then

$$M - M_0 = \sum_{i=1}^{k} m_i + \sum_{i=1}^{M_0} x_j$$

and the set $\{M_0 < j \le M\}$ is the union of $k + M_0$ parts: the αth part is $\{M_\alpha < j \le M_{\alpha+1}\}$, $-1 \le \alpha \le k + M_0 - 1$, $M_{-1} = 0$, M_0 was defined above. $M_{\alpha+1} = M_\alpha + x_{M_0-\alpha}$ if $0 \le \alpha \le M_0 - 1$, and $M_{\alpha+1} = M_\alpha + m_{(\alpha-M_0+1)}$ if $M_0 \le \alpha \le M_0 + k - 1$. Moreover, π_j, $M_\alpha < j \le M_{\alpha+1}$, blows the component Y'_α of the curve $P^{j-1} \bigcap V^{j-1}$ if $\alpha \le M_0 - 1$, and blows the component $C_\alpha^{M_0}$ if $\alpha \ge M_0$ (we use the fact that the surface P^{M_0} is not changed by the blowings π_j, $j \ge M_0 + 1$ if $M_{\alpha+1} = M_\alpha$ for some α, $\alpha \le M_0 - 1$, i.e. if the αth part is empty, then the component Y'_α is not blown).

Let $\alpha(j) = \alpha$ if $M_\alpha < j \le M_{\alpha+1}$, and let $\beta(j) = \alpha(j) - M_0 + 1$.

In other words, at first, we blow those components $P^{M_0} \bigcap V^{M_0}$ which are the proper inverse images of the exceptional curves and, in inverse order, beginning with Y'_{M_0} and finishing at Y'_1. Moreover, each component is blown in succession as many times as is enough to eliminate it from the intersection $P^j \bigcap V^j$ (with each blowing, the multiplicity of a component diminishes by 1). Let us note that if $\alpha(j) \ge M_0$, then π_j blows the proper inverse images of the curves C_i, i.e., the components of a plane section and the curves infinitely close to them (which appear in the case of multiple components). Moreover, the blowing of the proper inverse image of C_i is exactly π_j, $\beta(j) = i$. Thus, if $\alpha(j) = M_0 - 1$, then π_j blows the proper inverse image on P^{M_0} of the exceptional curve Y_1^* (Y'_1 is a component of the curve $P_{M_0} \bigcap V^{M_0}$ of multiplicity $x_1 = \mu - 2 \ge 0$).

As a result of all these blowings, we have that $P^M \bigcap V^M = \varnothing$. Thus, the restriction to V of the sequence of blowings $\pi_j\colon\ V^j \to V^{j-1}$ gives us the resolution of the singularities of the sheaf of hyperplane sections which contain $P \bigcap V$. Let $E_j^* = Y_j \bigcap V^j$; then the class of the divisor of the proper inverse image on V^M of this sheaf is $h - \sum_{j=1}^M e_j^*$. Thus, $\mathrm{Bs}\left|h - \sum_{j=1}^M e_j^*\right| = \varnothing$ and $\left(h - \sum_{j=1}^M e_j^*\right)^2 = 0$.

Let $|\chi|^j$ be the proper inverse image on V^j of the linear system $|\chi|$ and $\zeta_j = \mathrm{mult}_{Z_{j-1}} |\chi|^{j-1}$, $1 \le j \le M$. Thus, $nh - \sum_{j=1}^M \zeta_j e_j^*$ is the class of a general divisor of the linear system $|\chi|^M$. Let us note that if $\alpha \ge M_0$, then $\zeta_{M_\alpha+1} = \mathrm{mult}_{C_{(\alpha-M_0+1)}} |\chi| = \theta^*_{\alpha-M_0+1}$ (in the notation of the plane section lemma). Let $v \in A^1(V^M)$ be the class of an arbitrary effective divisor. Obviously,

$$\left(\left(h - \sum_{i=1}^M e_i^*\right) \cdot \left(nh - \sum_{i=1}^M \zeta_i e_i^*\right) \cdot v\right) \ge 0,$$

because $\left|nh - \sum_{i=1}^M \zeta_i e_i^*\right|$ does not have stationary components. Let us note that, by the above reasoning,

$$\left(h \cdot \left(h - \sum_{i=1}^M e_i^*\right)\right) = \sum_{i=1}^M \left(e_i^* \cdot \left(h - \sum_{i=1}^M e_i^*\right)\right).$$

If we take e_i^*, $1 \le i \le M$, for the class v, then we get a system of linear (for ζ and n) inequalities. The plane section lemma is a consequence of these considerations.

3. Let us consider a free Abelian group $R = \bigoplus_{i=1}^M \mathbb{Z} e_i^*$ and let $\omega\colon\ R \to A^1(V^M)$ be a homomorphism which maps $e_i^* \in R$ into $e_i^* \in A^1(V^M)$. Let us define a $\mathbb{Z}$-bilinear form on R, $\langle , \rangle\colon\ R \times R \to \mathbb{Z}$, in the following way:

$$\langle y, z \rangle = \left(\omega(y) \cdot \omega(z) \cdot \left(h - \sum_{i=1}^M e_i^*\right)\right).$$

Since $\left\langle \sum_{i=1}^M (n - \zeta_i) e_i^* \cdot e_j^* \right\rangle \ge 0$ for all j, the plane section lemma holds if the following lemma holds.

Lemma 1. *Let $y = \sum_{i=1}^M \lambda_i e_i^*$ and $\langle y, e_j^* \rangle \ge 0$ for all j. Then*

(1) $\lambda_i \ge 0$ *if* $\alpha(i) \ge M_0$ *and* $d_{\beta(i)} \ge 2$;

(2) $\lambda_i + \lambda_j \ge 0$ *if* $i \ne j$, $\alpha(i) \ge M_0$, $\alpha(j) \ge M_0$ *and* $d_{\beta(i)} = d_{\beta(j)} = \mu_{\beta(i)} = \mu_{\beta(j)} = 1$.

Proof. Let us compute the multiplication table for R. For each exceptional divisor E_i^* the class of its restriction $E_i^*|_{P^i}$ in $\operatorname{Pic} P^i \hookrightarrow \operatorname{Pic} P^M = \operatorname{Pic} P^{M_0} = \mathbb{Z}h \bigoplus \bigoplus_{i=1}^{M} \mathbb{Z}y_i^*$ is defined, where $h \in \operatorname{Pic}\mathbb{P}^2$ is the class of a line and y_i^* is the class of the exceptional line of the blowing $\pi_i\colon\ P^i \to P^{i-1}$, $1 \le i \le M_0$. Let us denote the class of the restriction by $p(e_i^*) \in \operatorname{Pic} P^{M_0}$, $p(e_i^*) = y_i^*$ for $1 \le i \le M_0$. Let us compare the forms $(\cdot)$ and $\langle,\rangle$, where $(\cdot)$ is the usual intersection form on $\operatorname{Pic} P^M$.

Lemma 2 (*the multiplication table in* R).

(1) $\langle e_i^*, e_j^*\rangle = \big(p(e_i^*)\cdot p(e_j^*)\big)$ *if* $i = j$.

(2) $\langle e_i^*, e_i^*\rangle = \big(p(e_i^*)\cdot p(e_i^*)\big)$ *if* $\alpha(i) < M_0 - 1$ *and* $i \ne 1$.

(3) $\langle e_i^*, e_i^*\rangle = \big(p(e_i^*)\cdot p(e_i^*)\big) - 1$ *if* $\alpha(i) = M_0 - 1$ *or* $i = 1$.

(4) *Let* $c_i^{M_0} = d_i h - \mu_i y_1^* - \sum_{j=2}^{M_0} x_{i,j} y_j^*$ *be the class of the curve* $C_i^{M_0}$ *in* $\operatorname{Pic} P^{M_0}$; *then, if* $\beta(t) = i$, *then*

$$\langle e_t^* \cdot e_t^*\rangle = 2p_a(C_i^{M_0}) - 2 - \sum_{j=0}^{M_0} x_{i,j},$$

where $p_a(\dots)$ *is the arithmetical genus of a curve.*

Proof. Let us assume that $i > j$. Then

$$\langle e_i^*, e_j^*\rangle = (h\cdot e_i^*\cdot e_j^*) - \sum_{t=1}^{M} (e_t^*\cdot e_i^*\cdot e_j^*)$$

and in the above expression there is only one nonzero term: $(e_i^*\cdot e_i^*\cdot e_j^*) = (z_{i-1}\cdot e_j^*)$ (the product is taken in $A(V^M)$). But, $E_j^* = Y_j|_{V^j}, Z_{i-1} \subset V^{i-1}$, and so $(z_{i-1}\cdot e_j^*) = (z_{i-1}\cdot y_j)$ (the last product is taken in $A(X_M)$. On the other hand, $Z_{i-1}P^{i-1}$, $Y_j|_{P^j} = E_j^*|_{P^j}$, and so $(z_{i-1}\cdot y_j) = \big(z_{i-1}\cdot p(e_j^*)\big)$ (the last product is taken in $A(P^M)$). Finally, $\big(z_{i-1}\cdot p(e_j^*)\big) = \big(p(e_i^*)\cdot p(e_j^*)\big)$, because if Z_{i-1} is a point, then both sides of the equality are equal to zero ($i > j$) if Z_{i-1} is a curve, then $Z_{i-1} = E_i^*|_{P^i}$, i.e, $z_{i-1} = p(e_i^*)$ (the surface P^{i-1} does not change under the blowing of the curve Z_{i-1}). (1) is proved.

Now $\langle e_i^*, e_i^*\rangle = (h\cdot e_i^*\cdot e_i^*) - \sum_{t=1}^{i-1}(e_t^*\cdot z_{i-1}) - (e_i^{*^3})$, where $(e_i^{*^3}) = 2$ if $i = 1$, $(e_i^{*^3}) = 1$ if $2 \le i \le M_0$, and

$$(e_i^{*^3}) = 2 - 2g(Z_{i-1}) + \left(\left(-h + e_1^* + 2\sum_{t=2}^{M_0} e_t^* + \sum_{t=M_0+1}^{i-1} e_t^*\right)\cdot z_{i-1}\right)$$

if $i > M_0$. Therefore, $\langle e_1^*, e_1^*\rangle = -2$, $\langle E_i^*, e_i^*\rangle = -1$ for $2 \le i \le M_0$, and

$$\langle e_i^*, e_i^*\rangle = -2 + 2g(Z_{i-1}) - \sum_{t=2}^{M_0}(e_t^*\cdot z_{i-1})$$

for $i \ge M_0 - 1$. (2)–(4) are easy consequences of these considerations.

Remark 1. We do not need to consider the case of multiple components C_i, because the classes e_ζ^* and e_ξ^*, such that $\beta(\zeta) = \beta(\xi) \ge 1$, intersect each other (in the sense of the form $\langle,\rangle$) as if these intersections are constructed from different components with the same data. In other words, instead of considering the case of the r-multiple component C, we shall consider the case of the 1-multiple components with equal data, i.e., from this moment we shall consider all $m_i = 1$.

4. Let us note that Lemma 1 gives nothing about λ_j for $\alpha(j) \le M_0 - 1$. Therefore, we shall reduce its proof to a problem of linear algebra. In this problem, we shall work only with coefficients important to us. We shall make this reduction in two stages.

The first reduction. (This is a formal analog of the contraction of the lines $Y_2^*, \dots, Y_{M_0}^*$.) Let $R_1 = \bigoplus_{\alpha(i) \subset M_0-1, i\ne 1^*} \mathbb{Z}e_i^*$, R_i be an orthogonal complement to R_1. $p_1\colon\ R_1 \to R_1$ is defined by the formula $p_1(\varepsilon) = \varepsilon + \sum_{t=2}^{M_0}\langle \varepsilon, e_t^*\rangle\, e_t^*$. Let $c_i^* = p_1(e_{M_i}^*)$, $i \ge M_0 + 1$ (recall that $M_{i+1} = M_i + 1$, $i \ge M_0$, because all components C_i are 1-multiple), $e_{1,0}^* = e_1^*, e_{1,i}^* = p_1(e_{M_{\alpha^*}+i}^*)$ for $\alpha^* = M_0 - 1$, $1 \le i \le x_1 = \mu - 2$. Let $y \in R$ be the class from Lemma 1. It is easy to check that the following statement holds.

Lemma 3. *$\langle p_1(y), c_i^*\rangle \geq 0$ for $1 \leq i \leq k$; $\langle p_1(y) \cdot e_{1,i}^*\rangle \geq 0$ for $0 \leq i \leq \mu - 2$.*

Set

$$R^* = \bigoplus_{i=1}^{k} \mathbb{Z}c_i^* \bigoplus \bigoplus_{i=0}^{\mu-2} \mathbb{Z}e_{1,i}^*$$

$R = R_1 \bigoplus R^*$, $R^* \hookrightarrow R_1^\perp$, $p_2\colon\ R \to R^*$ is the projection on the second term. Obviously, $\langle p_2 \circ p_1(y), c_i^*\rangle = \langle p_1(y), c_i^*\rangle \geq 0$ and $\langle p_2 \circ p_1(y), e_{1,i}^*\rangle = \langle p_1(y), e_{1,i}^*\rangle \geq 0$. Set now

$$y' = p_2 \circ p_1(y) = \sum_{i=1}^{k} \theta_i c_i^* + \sum_{i=0}^{\mu-2} \zeta_i e_{1,i}^*,$$

where $\theta_i = \lambda_j$ for $i = \beta(j)$.

Let us compute the multiplication table for R^*. To this end, we shall compare the form $\langle , \rangle$ with the form $(\cdot)$ of the intersection of curves on a plane with the blown point x, i.e., on P^1, in the following natural way. To each curve C_i, $1 \leq i \leq k$, we shall relate its proper inverse image C_i^1 on P^1. Let e_1^* be the class of the exceptional line on P^1. Let us note that the class of the curve C_i^1 in $A^1(P^1) \hookrightarrow A^1(P^M)$, $i = \beta(j)$, is

$$c_i^1 = p(e_j^*) + \sum_{t=2}^{M_0} \big(p(e_j^*) \cdot p(e_t^*)\big) p(e_t^*)$$

using our usual notation. By Lemma 2, $\big(p(e_j^*) \cdot p(e_t^*)\big) = \langle e^* i, e_t^*\rangle$ and, taking into account the fact that this equality always holds, if one of the terms in the product belongs to R_1, we have that $\langle c_i^*, c_j^*\rangle = \langle c_i^1, c_j^1\rangle = d_i d_j - \mu_i \mu_j$ if $i \neq j$. Then, using the well-known formulas for changing the arithmetical genus of a surface under monoidal transformations, we have that $\langle c_i^*, c_i^*\rangle = 2p_a(C_i^1) - 2$. Using analogous considerations and Lemma 2, we can compute all other products: $\langle e_{1,i}^*, e_{1,j}^*\rangle = (e_1^* \cdot e_1^*) = -1$ if $i \neq j$, $\langle e_{1,i}^*, e_{1,i}^*\rangle = -2$, $\langle c_i^*, e_{1,i}^*\rangle = (c_i^1 \cdot e_1^*) = \mu_i = \operatorname{mult}_x C_i$.

5. Let us eliminate the dependence on the coefficients ζ_j.

The second reduction. Let $R_2 = \bigoplus_{i=0}^{\mu-2} \mathbb{Z}e_{1,i}^*$, $p_3 : R^* \to R^*$ be defined by the formula

$$p_3(\varepsilon) = \varepsilon + \frac{1}{\mu} \sum_{i=0}^{\mu-2} \langle \varepsilon, e_{1,i}^*\rangle\, e_{1,i}^*,$$

$R' = \bigoplus_{i=1}^{k} \mathbb{Z}c_i^*$. Obviously, for each $\varepsilon \in R'$, $\langle e_{1,i}^*, \varepsilon\rangle = \langle e_{1,j}^*, \varepsilon\rangle$, therefore, $p_3(R') \subset R_2^\perp$. Let $c_i = p_3(c_i^*)$, $T = \bigoplus_{i=1}^{k} \mathbb{Z}c_i$, $R^* = R_2 \bigoplus T$, and $p_4\colon\ R^* \to T$ be the projection on the second summand in the last sum. For $y' \in R^*$ we have that $y^* = p_4 \circ p_3(y') = \sum_{i=1}^{k} \theta_i c_i$. Then, as above, $\langle y^*, c_i\rangle = \langle p_3(y'), c_i\rangle \geq 0$, $1 \leq i \leq k$.

Thus, the multiplication table is defined by the following formula:

$$\langle c_i, c_j\rangle = d_i d_j - \frac{1}{\mu}\mu_i \mu_j - \delta_{ij}(3d_i - \mu_i),$$

where δ_{ij} is the Kronecker symbol.

6. For each $\varepsilon = \sum_{i=1}^{k} \varepsilon_i c_i \in T$ we set:

$$d(\varepsilon) = \sum_{i=1}^{k} \varepsilon_i d_i, \qquad \mu(\varepsilon) = \sum_{i=1}^{k} \varepsilon_i \mu_i.$$

Lemma 4. *The form $\langle , \rangle$ is nondegenerate on T.*

Proof. Let $v = \sum_{i=1}^{k} v_i c_i \in T^{\perp}$, i.e., $\langle v, c_i \rangle = 0$ for all i. We have

$$0 = \langle v, c_i \rangle = d_i d(v) - \frac{1}{\mu}\mu_i \mu(v) - v_i(3d_i - \mu_i).$$

Hence, summing over all i. we have that $d(v) = 0$ (we use the equalities $\sum_{i=1}^{k} d_i = 4$ and $\sum_{i=1}^{k} \mu_i = \mu$). Now, $\mu(v) = \sum_{i=1}^{k} \mu_i v_i$,

$$v_i = -\frac{1}{\mu(3d_i - \mu_i)}\mu_i \mu(v),$$

i.e.,

$$\mu(v)\left(1 + \sum_{i=1}^{k} \frac{\mu_i^2}{\mu(3d_i - \mu_i)}\right) = 0.$$

Since $m_i \leq d_i$, the expression in the parentheses is strictly positive and $\mu(v) = 0$. Hence all $v_i = 0$. The lemma is proved.

Let us prove Lemma 1. Let us denote by $\Xi = \|\theta_j^{(i)}\|$ the matrix, which is inverse to the matrix $\| \langle c_i, c_j \rangle \|$. Its columns $\theta_j^{(i)}$ are solutions in $T \bigotimes \mathbb{Q}$ of the system of equations $\left\langle \sum_{t=1}^{k} \theta_t^{(i)} c_t, c_j \right\rangle = \delta_{ij}$. For $y^* = p_4 \circ p_3 \circ p_2 \circ p_1(y) = \sum_{t=1}^{k} \theta_t c_t$ we have the representation $\theta_i = \sum_{t=1}^{k} \langle y^*, c_t \rangle\, \theta_i^{(t)}$. Then $\theta_i + \theta_j = \sum_{t=1}^{k} \langle y^*, c_t \rangle \left(\theta_i^{(t)} + \theta_j^{(t)}\right)$. Taking into account the inequalities $\langle y^*, c_t \rangle \geq 0$, and (see above) the equalities $\lambda_j = \theta_i$, $i = \beta(j)$ for $j > M_0$, we have that Lemma 1 is a consequence of the following estimations.

Lemma 5.

(1) $\theta_j^{(i)} > 0$ *if* $i \neq j$.

(2) $\theta_i^{(i)} > 0$ *if* $d_i \geq 2$.

(3) $\theta_i^{(i)} + \theta_j^{(i)} > 0$ *if* $d_i = d_j = \mu_i = mu_j = 1$, $i \neq j$.

Proof. Let $\theta^{(i)} = \sum_{t=1}^{k} \theta_t^{(i)} c_t$. Let us solve directly the system of linear equations $\left\langle \theta^{(i)}, c_j \right\rangle = \delta_{ij}$. We have

$$\left\langle theta^{(i)}, c_j \right\rangle = d(\theta^{(i)}) d_j - \frac{1}{\mu}\mu(\theta^{(i)}\mu_j - \theta_j^{(i)}(3d_j - \mu_j) = \delta_{ij}.$$

Summing over all j, we have $d(\theta^{(i)}) = 1$. Then,

$$\theta_j^{(i)} = \frac{1}{3d_j - \mu_j}\left(d_j - \frac{1}{\mu}\mu(\theta^{(i)}\mu_j - \delta_{ij}\right)$$

and

$$1 = d(\theta^{(i)}) = \sum_{j=1}^{k} \theta_j^{(i)} d_j = \sum_{i=1}^{k} \frac{1}{3d_j - \mu_j}\left(d_j^2 - \frac{1}{\mu}\mu(\theta^{(i)})\mu_j d_j\right) - \frac{d_i}{3d_i - \mu_i}.$$

Let $r = \sum_{i=1}^{k} \frac{d_i \mu_i}{3d_i - \mu_i} \neq 0$. Taking into account that $d_j^2 = \frac{1}{3}\left(d_j(3d_j - \mu_j) + d_j \mu_j\right)$, we have the estimation

$$\mu(\theta^{(i)}) = \mu\left[\frac{1}{3} + \frac{1}{r}\left(\frac{1}{3} - \frac{d_i}{3d_i - \mu_i}\right)\right] \leq \frac{1}{3}\mu$$

(because $\mu_i \geq 0$). Hence,

$$\theta_j^{(i)} \geq \frac{1}{3d_j - \mu_j}\left(d_j - \frac{1}{3}\mu_j - \delta_{ij}\right).$$

But, $\mu_j \leq d_j$, and we immediately get (1), (2) and (3).

This completes the proof of the plane section lemma.

7. The Proof of the Graph Lemma.

1. Let us assume that $\Sigma_0 < r_0$. By definition, r_0 is the number of paths in Γ from $E_{\beta+1}$ to E_0. Hence, $r_0 = \sum\limits_{I \ni t \to 0} r_t > \sum\limits_{t \in I_0} r_t$. Consequently, there exists some $j_1 \in I_1$ such that $j_1 \to 0$, i.e., the curve $B_{j_1-1} \subset E^{j_1-1}$. Let $\varepsilon = \min\{j^* \leq j_1 | j^* \to 0,\ \dim B_{j^*-1} = 1\}$ (it is possible that $\varepsilon \notin I$, but $\varepsilon \leq j_1 \leq \beta$).

2. Lemma 1. *If* $1 \leq i \leq \varepsilon$, *then* $(b_{i-1} \cdot e^{i-1}) \leq 1$.

The **proof** uses induction by i. If $i = 1$, then B_{i-1} is a point and $(b_{i-1} \cdot e^{i-1}) = 0$. Let us assume that the lemma is proved for $j < i \leq \varepsilon$. Let us consider B_{i-1}. If B_{i-1} is a point, then (as above) $(b_{i-1} \cdot e^{i-1}) = 0$. Thus, let B_{i-1} be a curve. Let $j = \max\{j^* \leq i-1 | B_{i-1,j^*} E_{j^*}\}$. Then, $B_{i-1,j} \subset E_j$. But, $\varphi_{i-1,j}$ is an isomorphism in a neighborhood of a general point B_{i-1}, and so $B_{i-1} = B_{i-1,j}^{i-1}$. Therefore, $b_{i-1} = b_{i-1,j} - \sum\limits_{\alpha=j+1}^{i-1} m_\alpha f_\alpha$, where all $m_\alpha \geq 0$, and $m_\alpha > 0$ only if $B_{i-1,\alpha-1} \bigcap B_{\alpha-1} = B_{i-1,j}^{\alpha} \bigcap B_{\alpha-1} \neq \varnothing$. Then $e^{i-1} = e^j - \sum\limits_{\alpha=j+1}^{i-1} n_\alpha e_\alpha$, where all $n_\alpha \geq 0$ and $n_\alpha > 0$ only if $B_{\alpha-1} \subset E^{\alpha-1}$ (then $n_\alpha = 1$). Therefore,

$$(b_{i-1} \cdot e^{i-1}) = (b_{i-1,j} \cdot e^j) - \sum_{\alpha=j+1}^{i-1} m_\alpha n_\alpha \leq (b_{i-1,j} \cdot e^j).$$

Let us prove that $(b_{i-1,j} \cdot e^j) \leq 1$.

(1) Let B_{j-1} be a point. Then if $B_{j-1} \notin E^{j-1}$, then $E_j \bigcap E^j = \varnothing$. But, $B_{i-1,j} \in E_j$; so $B_{i-1,j} \bigcap E^j = \varnothing$. If $B_{j-1} \in E^{j-1}$, then $e^j = e^{j-1} - e_j$. But, $B_{i-1,j}$ is a line in $E_j \cong \mathbb{P}^2$ (Lemma 5.1), i.e., $b_{i-1,j} = f_j$ and $(e^j \cdot b_{i-1,j}) = -(e_j \cdot f_j) = 1$. The inequality is proved in this case.

(2) Let B_{j-1} be a curve. Let us first note that $j \leq \varepsilon - 1$, and so $B_{j-1} \not\subset E^{j-1}$. By the induction assumption, $(b_{j-1} \cdot e^{j-1}) \leq 1$. Hence, either $B_{j-1} \bigcap E^{j-1} = \varnothing$ (and then $B_{i-1,j} \bigcap E^j = \varnothing$) or B_{j-1} intersects E^{j-1} transversally at one point. In the last case, $e^{j-1} = e^j$ and $B_{i-1,j}$ is a section of a ruled surface $E_j \to B_{j-1}$ (Lemma 5.1). Hence, by the projection formula,

$$(b_{i-1,j} \cdot e^j) = (b_{i-1,j} \cdot e^{j-1}) = (b_{i-1,j-1} \cdot e^{j-1}) = (b_{j-1} \cdot e^{j-1}) = 1.$$

The lemma is proved.

3. Let us consider the morphism $\varphi_{\varepsilon,0}\colon E^\varepsilon \to E$. Let h_i^* be the class of the restriction of a general divisor of the linear system $|\chi|_i$ to E^i and x_i^* be the class of the restriction of the divisor E_i to E^i, i.e., the class of the intersection $E_i \bigcap E^i$ in $A^1(E^i) \hookrightarrow A^1(E^\varepsilon)$.

For each i, $1 \leq i \leq \varepsilon - 1$, there are two possibilities:

(1) $B_{i-1} \bigcap E^{i-1} = \varnothing$; then $x_i^* = 0$, $h_i^* = h_{i-1}^* = h_{i-1}^* - \nu_i x_i^*$, and $\varphi_{i,i-1}\colon E^i \to E^{i-1}$ is an isomorphism;

(2) $B_{i-1} \bigcap E^{i-1}$ is a point (let us denote it by $x_{i-1} \in E^{i-1}$); then $\varphi_{i,i-1}\colon E^i \to E^{i-1}$ is the blowing of x_{i-1} with the exceptional divisor $X_i^* = E_i \bigcap E^i$ and $h_i^* = h_{i-1}^* - \nu_i x_i^*$.

Finally, $B_{\varepsilon-1} \subset E^{\varepsilon-1}$, and so the map $\varphi_{\varepsilon,\varepsilon-1}\colon E^\varepsilon \to E^{\varepsilon-1}$ is an isomorphism. Obviously, $B_{\varepsilon-1,0}$ is a point, and so $B_{\varepsilon-1} = (X_q^*)^{\varepsilon-1}$, $1 \leq q \leq \varepsilon - 1$, i.e., $B_{\varepsilon-1}$ is the proper inverse image of the exceptional line X_q^* and $h_\varepsilon^* = h_{\varepsilon-1}^* - \nu_\varepsilon x_\varepsilon^*$, where $x_\varepsilon^* = (x_q^*)^{\varepsilon-1}$ is the class of the curve $(X_q^*)^{\varepsilon-1}$ in $A^1(E^{\varepsilon-1} \cong A^1(E^\varepsilon)$.

Let a and b be the classes of lines from two sheafs over the quadric $E \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \widetilde{E} \cong \mathbb{P}^3$. By the condition, the linear system of curves $\left|\nu_0 a + \nu_0 b - \sum\limits_{i=1}^{\varepsilon} \nu_i x_i^*\right|$ is nonempty. The class of a general divisor from this system on E^q is $\nu_0 a + \nu_0 b - \sum\limits_{i=1}^{q} \nu_i x_i^* - \nu_\varepsilon x^* q$, because $(\varphi_{\varepsilon \cdot q})_*(x_i^*) = 0$, for $q < i < \varepsilon$, and $(\varphi_{\varepsilon,q})_*(x_\varepsilon^*) = x_q^*$.

Let now $1 \leq i_1 < i_2 < \cdots < i_k = q$ be indices such that $i_{\alpha-1} = \max j \leq i_\alpha - 1 | \varphi_{i_\alpha-1,j}(x_{i_\alpha-1}) \in X_j^*$. In other words, $x_{i_\alpha-1}$ is the first infinitely close point among the points x_j, $j \geq i_\alpha - 1$, which lies over $x_{i_{\alpha-1}-1}$.

Let us construct a sequence of blowings of the points $\overline{\varphi}_{\alpha,\alpha-1}\colon\ \overline{E}_\alpha \to \overline{E}_{\alpha-1}$, $1 \leq \alpha \leq k$, $\overline{E}_0 = E$, with the following properties. Let us denote the center of the αth blowing $\overline{\varphi}_{\alpha,\alpha-1}$ by $y_{\alpha-1}$, and the exceptional divisor by $Y_\alpha^* \subset \overline{E}_\alpha$. Then

(1) $y_\alpha \in Y_\alpha^*$ for all $1 \leq \alpha \leq k-1$;

(2) the isomorphism $\omega = \omega_0 = \mathrm{id}_E\colon\ E \underset{\sim}{\to} \overline{E}_0$ may be extended to a birational morphism $\omega_\alpha\colon\ E^{i_\alpha} \to E_\alpha$, i.e., the following diagram is commutative

$$\begin{array}{ccccccc}
E_0 & \leftarrow \cdots \leftarrow & E^{i_1} & \leftarrow \cdots \leftarrow & E^{i_\alpha} & \leftarrow \cdots \leftarrow & E^{i_k} \\
\downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega_1} & & \downarrow{\scriptstyle\omega_\alpha} & & \downarrow{\scriptstyle\omega_k} \\
\overline{E}_0 & \leftarrow \cdots \leftarrow & \overline{E}_1 & \leftarrow \cdots \leftarrow & \overline{E}_\alpha & \leftarrow \cdots \leftarrow & \overline{E}_k
\end{array}$$

(3) $\omega_{\alpha-1}(\varphi_{i_\alpha-1,i_\alpha-1}(x_{i_\alpha-1})) = y_{\alpha-1}, \omega_\alpha(X_{i_\alpha}^*) = Y_\alpha^*, \omega_{\alpha-1} \circ \varphi_{i_\alpha-1,i_\alpha-1}$ is an isomorphism of neighborhoods of the exceptional divisors $X_{i_\alpha}^*$ and Y_α^*; in particular, $\omega_\alpha\colon\ X_{i_\alpha}^* \to Y_\alpha^*$ is an isomorphism.

The construction of the sequence of surfaces.

Let us assume that $\overline{E}_0, \ldots, \overline{E}_\alpha$ and morphisms $\omega_0, \ldots, \omega_\alpha$ are constructed. By the assumption, ω_α is an isomorphism of neighborhoods of the exceptional lines $X_{i_\alpha}^*$ and Y_α^*. Let $y_\alpha = \omega_\alpha \circ \varphi_{i_{\alpha+1}-1,i_\alpha}(x_{i_{\alpha+1}-1})$. By the definition of numbers i_α, $\varphi_{i_{\alpha+1}-1,i_\alpha}$ is an isomorphism in a neighborhood of $x_{i_{\alpha+1}-1}$; therefore, $\omega_\alpha \circ \varphi_{i_{\alpha+1}-1,i_\alpha}\colon\ E^{i_{\alpha+1}-1} \to \overline{E}_\alpha$ is an isomorphism of a neighborhood of the points $x_{i_{\alpha+1}-1}$ and y_α. Hence if $\overline{\varphi}_{\alpha+1,\alpha}\colon\ \overline{E}_{\alpha+1} \to \overline{E}_\alpha$ is the blowing of the point y_α, then $\omega_{\alpha+1} = \overline{\varphi}_{\alpha+1,\alpha}^{-1} \circ \omega_\alpha \circ \varphi_{i_{\alpha+1},i_\alpha}\colon\ E^{i_{\alpha+1}} \to \overline{E}_{\alpha+1}$ is a birational morphism which has the properties we need.

4. Let us consider the image of the system $\left|\nu_0 a + \nu_0 b - \sum_{i=1}^q \nu_i x_i^* - \nu_\varepsilon x_q^*\right|$ on $\overline{E}_q$. It is a nonempty system of curves and the class of its general divisor is $\nu_0 a + \nu_0 b - \sum_{\alpha=1}^k \nu_{i_\alpha} y_\alpha^* - \nu_\varepsilon y_k^*$. Here $\nu_{i_\alpha} > n$, $\nu_\varepsilon > n$; thus, the graph lemma is a consequence of the following statement.

Lemma 2. *Using the above notation, the linear system*

$$\left|\nu_0 a + \nu_0 b - \sum_{\alpha=1}^k \theta_\alpha y_\alpha^*\right|, \tag{$*$}$$

where $\nu_0 \leq n$, $\theta_\alpha > n$, for $1 \leq \alpha \leq k-1$ and $\theta_k > 2n$, is empty.

Proof. Let us consider the graph Γ^* of exceptional divisors of the blowings $\overline{\varphi}_{\alpha+1,\alpha}$, $0 \leq \alpha \leq k-1$: $Y_{\alpha_1}^*$ and $Y_{\alpha_2}^*$ are connected by an edge (the notation $\alpha_1 \to \alpha_2$, the above graph Γ will not be considered any more), which is oriented from α_1 to α_2 only if $\alpha_1 > \alpha_2$ and $y_{\alpha_1-1} \in (Y_{\alpha_2}^*)^{\alpha_1-1}$ (where, as usual, the upper index designates a proper inverse image). Let $r_{\mu,\xi}$ be the number of paths in Γ^* from Y_μ^* to Y_ξ^*, $r_{\mu,\mu} = 1$.

Let us assume that the linear system $(*)$ is nonempty. Let $C \subset E$ be a general curve of the system, C^α be, as usual, its proper inverse image on $\overline{E}_\alpha$, $\rho_\alpha = \mathrm{mult}_{y_{\alpha-1}} C^{\alpha-1}$, $1 \leq \alpha \leq k$. Then the class of a general divisor of the linear system $\left|\nu_0 a + \nu_0 b - \sum_{\alpha=1}^\xi \theta_\alpha y_\alpha^*\right|$, $\xi \leq k$, is $c^\xi + \sum_{\alpha=1}^\xi (\rho_\alpha - \theta_\alpha) Y_\alpha^*$. Since C^ξ does not contain the exceptional lines as components, the divisor $G_\xi = \sum_{\alpha=1}^\xi (\rho_\alpha - \theta_\alpha) Y_\alpha^*$ is effective for all $\xi \leq k$. In other words, each irreducible component of this divisor must have a nonnegative multiplicity. But

$$\sum_{\alpha=1}^\xi (\rho_\alpha - \theta_\alpha) Y_\alpha^* = \sum_{\alpha=1}^\xi \left(\sum_{\mu=1}^\alpha r_{\alpha,\mu}(\rho_\mu - \theta_\mu)\right) (Y_\alpha^*)^\xi.$$

The proper inverse images of $(Y_\alpha^*)^\xi$ are exactly the irreducible components of the divisor G_ξ; so we have the inequality

$$\sum_{\alpha=1}^{\xi} r_{\xi,\alpha}(\rho_\alpha - \theta_\alpha) \geq 0 \qquad (**)$$

for all $\xi \leq k$. In particular, for $\xi = 1$ we have $\rho_1 \geq \theta_1$, for $\xi = 2$ we have $\rho_1 + \rho_2 \geq \theta_1 + \theta_2 > 2n$. Since $E \subset \widetilde{E} \cong \mathbb{P}^3$, let us consider the line A in $\mathbb{P}^3$ which passes through the point y_0 in the direction of the point y_1. This line intersects C in two points; one of them is a simple point, the other is infinitely close. The sum of the multiplicities of these points is $\rho_1 + \rho_2 > 2n \geq 2\nu_0 = \deg C$, and so A is a component of C. There are two lines on E which pass through y_0. Let A and B be these lines. Let $v \leq n$ be the multiplicity of A in the linear system $(*)$ and $w \leq n$ be the multiplicity of B. Let

$$\zeta = \max\{\alpha | y_{\alpha-1} \in A^{\alpha-1}\} = \max\Big\{\alpha | Y_\alpha^* \bigcap A^\alpha \neq \varnothing\Big\}, \qquad \zeta \geq 2.$$

5. Let us now study the structure of the graph Γ^*.

Lemma 3.

(A) $\alpha \to (\alpha - 1)$ *for all* α, $2 \leq \alpha \leq k$.

(B) *Each vertex of* Γ^* *can be the starting point of not more than 2 oriented edges. If* $\alpha \to \alpha_1$ *and* $\alpha_1 \leq \alpha - 2$, *then* $(\alpha - 1) \to \alpha_1$.

(C) *The initial segment of* Γ^* *from* Y_1^* *to* Y_ζ^* *is a chain* $1 \leftarrow 2 \leftarrow \cdots \leftarrow \zeta$, *i.e., there are no other edges between* Y_i^*, $1 \leq i \leq \zeta$, *except* $i \to (i-1)$.

Proof. (A) is trivial. Property (B) is the general property of all blowings of surfaces: for each sequence of blowings of points, not more than two components of exceptional divisors may have a nonempty intersection. The second statement of (B) is a consequence of (A). (C) is obvious.

Corollary 1. $r_{k,1} = r_{k,i}$, $1 \leq i \leq \zeta - 1$.

Proof. By Lemma 3 (B) if $\xi \to \alpha$, where $\xi \geq \zeta$ and $\alpha \leq \zeta$, then $\alpha = \zeta$ or $\alpha = \zeta - 1$. The corollary is an easy consequence of this fact.

We have that $C = vA + wB + D$, where D does not contain A and B as components, $\deg D = 2\nu_0 - v - w$. Let $\gamma_\alpha = \operatorname{mult}_{y_{\alpha-1}} D^{\alpha-1}$.

Lemma 4.

(A) $\gamma_\alpha \leq \gamma_{\alpha-1}$, *for all* α.

(B) $\gamma_1 \leq \min(\nu_0 - v, \nu_0 - w)$.

(C) $\gamma_1 + v + w = \rho_1$.

(D) $\gamma_i + v = \rho_i$, *for* $2 \leq i \leq \zeta$.

(E) $\gamma_i = \rho_i$, *for* $\zeta + 1 \leq i \leq k$ *if* $k \geq \zeta + 1$.

Proof. (A) is obvious. (B) holds because D is a curve of the type $(\nu_0 - v, \nu_0 - w)$. Thus, if $\gamma_1 > \min(\nu_0 - v, \nu_0 - w)$, then D contains A or B as a component. (C) is obvious. (D) is a consequence of the following: $y_1 \notin B^1$, and so $y_i \notin B^i$ for $\zeta - 1 \geq i \geq 1$. (E), by the definition of ζ if $i \geq \zeta$, then $y_i \notin A^i$.

Now if $\xi = k$, then, by $(**)$ (set $\gamma = \gamma_1$):

$$r_{k,1}(v + w + \gamma) + \sum_{i=2}^{\zeta} r_{k,i}(v + \gamma) + \sum_{i=\zeta+1}^{k} r_{k,i}\gamma \geq \sum_{i=1}^{k} r_{k,i}\theta_i > \sum_{i=1}^{k} r_{k,i}n + n.$$

But, $v + \gamma \leq n$, $w + \gamma \leq n$, $v + w + \gamma \leq n + v$. Thus, using these estimations and taking into account Corollary 1 and the equality $r_{k,k} = 1$, we have

$$r_{k,1}v > \sum_{i=\zeta+1}^{k} r_{k,i}v + n.$$

Lemma 5. *Let Γ be an arbitrary oriented graph, $\Xi_1, \ldots, \Xi_t$ be its set of vertices. We shall denote by $i \succ j$ the oriented edge from Ξ_i to Ξ_j. Let this graph has the following properties:* (1) $i \succ (i-1)$ *and if* $i \succ i'$, *then* $i > i'$; (2) *each vertex is the starting point of not more than 2 edges and, if* $i \succ i'$, $i' \leq i-2$, *then* $(i-1) \succ i'$. *Let* $r(\widetilde{\Gamma}, i, j)$ *be the number of paths in* $\widetilde{\Gamma}$ *from* i *to* j, $r(\widetilde{\Gamma}, i, i) = 1$. *Then* $r(\widetilde{\Gamma}, t, 1) \leq 1 + \sum_{i=3}^{t} r(\widetilde{\Gamma}, t, i)$.

The **proof** is by induction by t (i.e., by the number of vertices). If $t = 1$, then the lemma is obvious. Let us assume that the lemma is proved for all graphs with the number of edges $\leq t-1$. Thus, $2 \succ 1$. Let $t_1 = \max\{i | i \succ 1\}$; then

$$r(\widetilde{\Gamma}, t, i) = \sum_{i \succ 1} r(\widetilde{\Gamma}, t, i) = \sum_{i=2}^{t_1} r(\widetilde{\Gamma}, t, i).$$

Thus,

$$r(\widetilde{\Gamma}, t, 1) - 1 - \sum_{i=3}^{t} r(\widetilde{\Gamma}, t, i) = r(\widetilde{\Gamma}, t, 2) - 1 - \sum_{i=t_1+1}^{t} r(\widetilde{\Gamma}, t, i).$$

Let, at first, $t_1 \geq 3$. Let us note that the subgraph of $\widetilde{\Gamma}$ with the set of vertices $\{2, \ldots, t_1\}$ is a chain. Hence, $r(\widetilde{\Gamma}, t, 1) = r(\widetilde{\Gamma}, t, i)$ for all i, $1 \leq i \leq t_1 - 1$ (this statement is a consequence of properties (1) and (2), and the proof of it is analogous to the proofs of Lemma 3 and Corollary 1). Let us consider the graph $\widetilde{\Gamma}_1$ which is a result of the deletion of vertices $1, \ldots, (t_1 - 2)$ from $\widetilde{\Gamma}$ (and, of course, of the edges, which begin or end at these vertices). For $\widetilde{\Gamma}_1$ properties (1) and (2) hold; it has $t - t_1 + 2 \leq t - 1$ vertices, and so the lemma holds for it, i.e.,

$$r(\widetilde{\Gamma}_1, t, t_1 - 1) \leq 1 + \sum_{i=t_1+1}^{t} r(\widetilde{\Gamma}_1, t, i).$$

But, obviously, $r(\widetilde{\Gamma}_1, t, i) = r(\widetilde{\Gamma}, t, i)$, for $i \geq t_1 - 1$, and $r(\widetilde{\Gamma}, t, t_1 - 1) = r(\widetilde{\Gamma}, t, 2)$. This is exactly what we want to get.

If $t_1 = 2$, then $r(\widetilde{\Gamma}, t, 2) \leq \sum_{i=3}^{t} r(\widetilde{\Gamma}, t, i)$, because $r(\widetilde{\Gamma}, t, 2) = \sum_{i \succ 2} r(\widetilde{\Gamma}, t, i)$. The lemma is proved.

Thus, $r_{k,1} = r_{k,\zeta-1} \leq \sum_{i=\zeta+1}^{k} r_{k,i} + 1$, hence,

$$\sum_{i=\zeta+1}^{k} r_{k,i} v + v > \sum_{i=\zeta+1}^{k} r_{k,i} v + n,$$

i.e., $v > n$. We have a contradiction. Lemma 2 and the graph lemma are proved.

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