FRATTINI SUBLATTICES OF DISTRIBUTIVE LATTICES

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1. Introduction

The *Frattini sublattice* $\Phi(L)$ of a lattice L is defined to be the intersection of all maximal proper sublattices of L. By using the notion of reducible and irreducible elements, it has been shown in [5], among others, the following result:

$$[\mathrm{Irr}(L)] - \mathrm{Irr}(L) \subseteq \Phi(L) \subseteq L(\vee) \cup L(\wedge),$$

where $\mathrm{Irr}(L)$ stands for the set of all irreducible elements of the lattice L, $[\mathrm{Irr}(L)]$ the sublattice generated by $\mathrm{Irr}(L)$, $L(\vee)$ the set of join-reducible elements and $L(\wedge)$ the set of meet-reducible elements of L. This result can be slightly improved in several ways such as the following:

(i) Let F be the set of join- or meet-reducible elements of a lattice L which are comparable with every element of L. Then,

$$([\mathrm{Irr}(L)] - \mathrm{Irr}(L)) \cup F \subseteq \Phi(L).$$

(ii) If $A \subseteq \Phi(L)$, then $[\mathrm{Irr}(L) \cup A] - \mathrm{Irr}(L) \subseteq \Phi(L)$.

(iii) Let D be a distributive lattice and S the set of those elements s of D having the properties:

(a) there exist $x, y \in D$ different from s such that $s = x \wedge y$;

(b) for each element r of D incomparable with s, we have $r \vee s \ngeq x$ and $r \vee s \ngeq y$.

Dually, let S' be the set of those elements t of D having the following properties:

(a′) there exist $x, y \in D$ different from t such that $t = x \vee y$;

(b′) for each element r of D incomparable with t, we have $r \wedge t \nleq x$ and $r \wedge t \nleq y$.

Then,

$$[\mathrm{Irr}(D) \cup S \cup S'] - \mathrm{Irr}(D) \subseteq \Phi(D).$$

In spite of these improvements, Frattini sublattices of lattices are still not completely determined. Even for finite lattices there is still no efficient way to find their Frattini sublattices. The purpose of this paper is to study the Frattini sublattice of a distributive lattice via the notion of prime sublattices (see Section 2). With our results, Frattini sublattices of finite distributive lattices can easily be determined and those of relatively complemented distributive lattices can be characterized. As an application,

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a solution to problem 2 in [5] is provided. In addition, we also obtain a characterization of maximal sublattices of distributive lattices which, indeed, improves one of Hashimoto's well-known results in [4].

2. Preliminaries

We will use the following terminology. A nonempty sublattice N of a lattice L is called a *prime* sublattice if $L-N$ is either empty or a sublattice of L (note that this notion was, in fact, firstly considered and named as completely isolated sublattice by N. D. Filippov in [2]; further study was carried on in [6]). Evidently, every irreducible element of L is itself a prime sublattice of L. A lattice may not have a proper prime sublattice. However, in distributive lattices proper prime sublattices do usually exist, as every prime ideal, every prime dual ideal, and every non-empty set intersection of a prime ideal and a prime dual ideal is such. For convenience, we shall denote by $P(L)$ and $Q(L)$, the posets of prime ideals and prime dual ideals of L respectively, and consider every lattice as a prime ideal and a prime dual ideal of itself throughout this paper.

A prime sublattice N of L is called a *minimal prime* sublattice of L if N contains no prime sublattice of L other than itself. A sublattice C of L is said to be *convex* if a, $b \in C$, $c \in L$, $a \leq c \leq b$ imply $c \in C$. It will be shown in Section 3 that every minimal prime sublattice of a distributive lattice is convex.

The problem of maximal extension of sublattices in distributive lattice was intensively investigated by Hashimoto. For our purpose, the following two results which can be found in [4] will now be stated.

LEMMA 1 (Hashimoto). *Let a be an element of a distributive lattice L, $a \neq 0, 1$. Let S be a sublattice of L such that $a \notin S$. Then there exist $P \in P(L)$, $Q \in Q(L)$ such that $a \notin P \cup Q \supseteq S$.*

LEMMA 2 (Hashimoto). *Let M be a maximal sublattice of a distributive lattice L. Then either* 1) $M = L - \{0\}$, 2) $M = L - \{1\}$, *or* 3) $M = P \cup Q$, *for some* $P \in P(L)$, $Q \in Q(L)$.

3. The Frattini sublattice of a distributive lattice

In this section, we shall first characterize (Theorem 1) the Frattini sublattice $\Phi(L)$ of a distributive lattice L in terms of minimal prime sublattices of L. Lemma 3, which asserts that the notions of maximal sublattice and minimal prime sublattice are mutually complementary, is a crucial result. Minimal prime sublattices of L, are then, in the second step, characterized by elements of $P(L)$ and $Q(L)$ (Theorem 2). This, in turn, leads to establishing a characterization for maximal sublattices of distributive lattices. With this, the problem of maximal extension of sublattices is considered. Finally, invoking these previous results, we are able to describe an efficient method for determining $\Phi(L)$ completely, when L is finite.

LEMMA 3. *Let M be a subset of a distributive lattice L. Then M is a maximal sublattice of L if and only if $L-M$ is a minimal prime sublattice of L.*

Proof. Assume M is a maximal sublattice of L. Then $M=P\cup Q$ for some $P\in P(L)$, $Q\in Q(L)$ by Lemma 2. Clearly, $L-M=(L-P)\cap(L-Q)$ is a prime sublattice of L. If there is a prime sublattice N of L such that $N\subset L-M$, then we would have $L-N\supset M$, which is a contradiction as $L-N$ is a sublattice. Thus, $L-M$ is a minimal prime sublattice of L.

Let $A=L-M$ be a minimal prime sublattice. Then $M=L-A$ is a sublattice. If M is not maximal, there exists a proper sublattice N of L such that $M\subset N$. By Lemma 1, $M\subset N\subseteq P\cup Q\subset L$, for some $P\in P(L)$, $Q\in Q(L)$. But then it follows that $A=L-M\supset L-N\supseteq(L-P)\cap(L-Q)\neq\emptyset$, which is a contradiction as $(L-P)\cap(L-Q)$ is a prime sublattice of L. Hence M is a maximal sublattice of L.

COROLLARY. *Every maximal sublattice of a distributive lattice is prime.*

Thus, we arrive at the following

THEOREM 1. *Let L be a distributive lattice. Then $\Phi(L)=L-E$ where E is the union of all minimal prime sublattices of L.*

Using this result, we can derive the following corollary which is also a special case of a result in [5].

COROLLARY. *Let L be a distributive lattice. Then*

$$[\mathrm{Irr}(L)]-\mathrm{Irr}(L)\subseteq\Phi(L)\subseteq L(\vee)\cup L(\wedge).$$

Proof. Note that every element of $\mathrm{Irr}(L)$ is a minimal prime sublattice of L. Hence $\Phi(L)\subseteq L(\vee)\cup L(\wedge)$.

Next, let x be any element of $[\mathrm{Irr}(L)]-\mathrm{Irr}(L)$. Assume that x is contained in a minimal prime sublattice M of L. Then certainly M contains no irreducible element of L, for otherwise M would be a singleton and x would be in $\mathrm{Irr}(L)$. Now, since L is distributive, every x can be expressed as

$$x=\bigvee_{a\in A}\left(\bigwedge_{i\in S_a}x_i\right),\qquad x_i\in\mathrm{Irr}(L)$$

where A and S_a are non-empty finite sets of indices. Since M is a prime sublattice and $x\in M$, it follows that for some $a\in A$, $\bigwedge_{i\in S_a}x_i\in M$ which in turn implies that $x_i\in M$ for some $i \in S_a$, contradicting the fact that M contains no irreducible element of L. Thus, we have shown that no element of $[\mathrm{Irr}(L)] - \mathrm{Irr}(L)$ is contained in any minimal prime sublattice of L and so by Theorem 1, $[\mathrm{Irr}(L)] - \mathrm{Irr}(L) \subseteq \Phi(L)$.

In order to prove Theorem 2, we first establish the following

LEMMA 4. *Let L be a distributive lattice and N a prime sublattice of L. Let x be any fixed element of L and $N_x = \{a \in L / a \vee x \in N\}$. Then N_x is either empty or a prime sublattice of L.*

Proof. Assume that N_x is not empty. Then N_x is a sublattice of L. To show that N_x is prime, let $a, b \in L - N_x$. Then $x \vee a$ and $x \vee b$ are not in N and so $x \vee (a \vee b) = (x \vee a) \vee \vee (x \vee b) \notin N$, $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) \notin N$. This, in turn, shows that both $a \vee b$ and $a \wedge b$ are in $L - N_x$. The proof is complete.

Remarks. 1) In general, Lemma 4 does not hold in non-distributive lattice as shown by the following example.

Let L be the lattice of Figure 1 and let $N = \{x, y\}$. Then N is a prime sublattice of L whereas $N_x = \{x, y, z, w\}$ is not a prime sublattice.

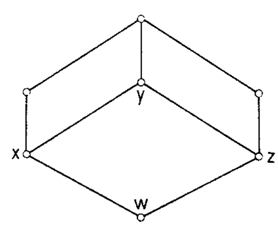

Figure 1

2) In Lemma 4, if $x \in N$, then $N \subseteq N_x$, while if $x \notin N$, then $N_x \subseteq N$. In particular, if N is a minimal prime sublattice of L, $x \notin N$, and $N_x \neq \emptyset$, then $N_x = N$.

3) In Lemma 4, if $N \in P(L)$, then $N_x \in P(L) \cup \{\emptyset\}$; if $N \in Q(L)$, then $N_x \in Q(L) \cup \{\emptyset\}$.

A pair $(P, Q) \in P(L) \times Q(L)$ is said to be *minimal* in $P(L) \times Q(L)$ if the following conditions hold:

1) $P \cap Q \neq \emptyset$,

2) if $(P', Q') \in P(L) \times Q(L)$, $(P', Q') < (P, Q)$, then $P' \cap Q' = \emptyset$.

Dually, (P, Q) is said to be *maximal* in $P(L) \times Q(L)$ if

1) $P \cup Q \subset L$,

2) if $(P', Q') \in P(L) \times Q(L)$, $(P', Q') > (P, Q)$, then $P' \cup Q' = L$.

Let $Z(L)$ be the family of all prime sublattices of L. Let

$$i(P(L) \times Q(L)) = P(L) \times Q(L) - \{(P, Q) / P \cap Q = \emptyset\},$$
$$u(P(L) \times Q(L)) = P(L) \times Q(L) - \{(P, Q) / P \cup Q = L\}.$$

For each pair $(P, Q) \in P(L) \times Q(L)$, we denote

$$f(P, Q) = P \cap Q,$$
$$g(P, Q) = P \cup Q.$$

Then it is not difficult to check that f, g are one-to-one isotone mappings of $i(P(L) \times Q(L))$ and $u(P(L) \times Q(L))$ into $Z(L)$ respectively with the relation

$$g(P, Q) = L - f(L - Q, L - P).$$

We are now in a position to prove the following

THEOREM 2. *A non-empty subset N of a distributive lattice L is a minimal prime sublattice of L if and only if $N = f(P, Q) = P \cap Q$, for some minimal pair (P, Q) in $P(L) \times Q(L)$.*

Proof. Assume that N is a minimal prime sublattice of L. Then $L - N$ is a maximal sublattice of L by Lemma 3. By Lemma 2, $L - N = P \cup Q$, for some $P \in P(L)$, $Q \in Q(L)$. Thus, $N = L - P \cup Q = (L - P) \cap (L - Q)$. Clearly, $(L - Q, L - P)$ is a minimal pair in $P(L) \times Q(L)$.

Suppose that $N = P \cap Q$, where (P, Q) is a minimal pair in $P(L) \times Q(L)$. N is obviously prime. If N is not minimal, then there exists a prime sublattice K of L such that $K \subset P \cap Q$. Let $a \in P \cap Q - K$. If $(a] \cap K = \emptyset$, then $[K)$ is a proper prime dual ideal of L properly contained in Q. Hence $[K) \cap P = \emptyset$, a contradiction. Hence $(a] \cap K \neq \emptyset$ and dually $[a) \cap K \neq \emptyset$. As $[a) \cap K \neq \emptyset$, $K_a \neq \emptyset$. By Lemma 4, K_a is a non-empty prime sublattice of L contained in K, and hence in $P \cap Q$. By a similar argument as that for K, we have $(a] \cap K_a \neq \emptyset$, which implies that $a \in K$, a contradiction. Hence $P \cap Q$ is a minimal prime sublattice of L.

COROLLARY. *Every minimal prime sublattice of a distributive lattice is convex.*

THEOREM 3. *Let M be a subset of a distributive lattice L. Then M is a maximal sublattice of L if and only if $M = g(P, Q) = P \cup Q$, for some maximal pair (P, Q) in $P(L) \times Q(L)$.*

Proof. By Lemma 3 and the fact that

$$g(P, Q) = L - f(L - Q, L - P).$$

As a by-product of Theorem 3, we have

COROLLARY (Hashimoto). *Let P be a maximal ideal and Q a maximal dual ideal of a distributive lattice L. If $P \cup Q \subset L$, then $P \cup Q$ is a maximal sublattice of L.*

G. Birkhoff proposed in [1] the following

PROBLEM 18. Prove or disprove that every proper sublattice S of a lattice can be extended to a maximal proper sublattice.

Birkhoff's conjecture that the answer of the problem may be yes for distributive lattices was disproved by K. Takeuchi [8] with a simple counter example. Generalizing the result of Takeuchi [8], Hashimoto [4] proved that every sublattice of a relatively complemented distributive lattice can be extended to a maximal one. In this connection, we have the following results which can be derived from Theorem 3.

COROLLARY 1. *A distributive lattice L has the property that every proper sublattice of L can be extended to a maximal proper one if and only if for each pair* $(P, Q)\in u(P(L)\times Q(L))$, *there exists a maximal pair* (P^*, Q^*) *in* $P(L)\times Q(L)$ *such that* $P^*\cup Q^*\supseteq P\cup Q$.

In particular, we have

COROLLARY 2. *Let L be a distributive lattice. If the poset* $P(L)$ *is of finite length, then every sublattice of L can be extended to a maximal one.*

Remark. Observe that if L is a relatively complemented distributive lattice, then $P(L)-\{L\}$ is unordered, i.e., $P(L)-\{L\}$ is of length zero. Thus, Hashimoto's result on maximal extension of sublattices is a very special case of Corollary 2.

In the remainder of this section, we shall confine ourselves to considering $\Phi(L)$ for finite distributive lattices L. The following lemma is trivial.

LEMMA 5. *Let L be a finite distributive lattice and N a minimal prime sublattice of L. Then* $N=[a, b]$ *where* $a=\bigwedge N$, $b=\bigvee N$.

THEOREM 4. *Let L be a finite distributive lattice and* $a, b\in L$ *with* $a<b$. *Then* $[a, b]$ *is a minimal prime sublattice of L if and only if* 1) $a\in L(\wedge)-L(\vee)$, 2) $b\in L(\vee)-L(\wedge)$, *and* 3) *for each* $x\in(a, b)$, $x\in L(\vee)\cap L(\wedge)$.

Remark. Note that if $a\in L$, then $\{a\}$ is minimal prime if and only if $a\in\mathrm{Irr}(L)$.

Proof. We have $[a, b]=(b]\cap[a)$, and $[a)\in Q(L)$, $(b]\in P(L)$; thus $[a, b]$ is obviously a prime sublattice of L. Let N be any prime sublattice contained in $[a, b]$. Let $c=\bigwedge N$, $d=\bigvee N$. If $c>a$, then $c\in(a, b]$. By assumption, $c\in L(\vee)$, but then $[c, d]\neq N$ is not prime. Thus, $c=a$. Dually, $d=b$. Hence $N=[c, d]=[a, b]$, proving that $[a, b]$ is minimal prime.

Conversely, assume that $[a, b]$ is a minimal prime sublattice of L. Clearly, $a\notin L(\vee)$. If $a\notin L(\wedge)$, then $a\in\mathrm{Irr}(L)$. Hence $\{a\}\subset[a, b]$, contradicting that $[a, b]$ is minimal. Thus, $a\in L(\wedge)-L(\vee)$. Dually, $b\in L(\vee)-L(\wedge)$. Let $x\in(a, b)$. Since $[a, b]$ is minimal. $x\in L-\mathrm{Irr}(L)=L(\vee)\cup L(\wedge)$. If $x\notin L(\vee)\cap L(\wedge)$, then, say $x\in L(\wedge)-L(\vee)$. Obviously $[x, b]$ is then prime, contradicting the fact that $[a, b]$ is minimal. Thus, $x\in L(\wedge)\cap L(\vee)$, which completes the proof of Theorem 4.

Remark. The above result tells us that to find minimal prime sublattices of a finite distributive lattice L, we need only to check intervals $[a, b]$ where a is join-irreducible but meet-reducible, b is meet-irreducible but join-reducible and each x in (a, b) is both join-and meet-reducible. By excluding all these intervals and $\mathrm{Irr}(L)$, $\Phi(L)$ can easily be found. Equivalently, to determine whether an element a of L is in $\Phi(L)$, let $b_1, \ldots, b_n$ be all the elements in $L(\vee)-L(\wedge)$ which are minimal with respect to the property that $b_i \geq a$, and let $c_1, \ldots, c_m$ be all the elements in $L(\wedge)-L(\vee)$ which are maximal with respect to the property that $c_j \leq a$. Then $a \in \Phi(L)$ if and only if $a \notin \mathrm{Irr}(L)$ and for each $i=1, \ldots, n, j=1, \ldots, m$, the open interval (b_i, c_j) contains an element x which is either join- or meet-irreducible. As an illustration, let us consider the distributive lattice L of Figure 2. Then $[d, a]$ $[d, b]$, $[e, a]$, $[e, c]$, $[m, k]$ $[p, j]$, $[n, j]$ and $[n, k]$ are all the minimal prime sublattices of L. Hence $\Phi(L)$ is the lattice of Figure 3.

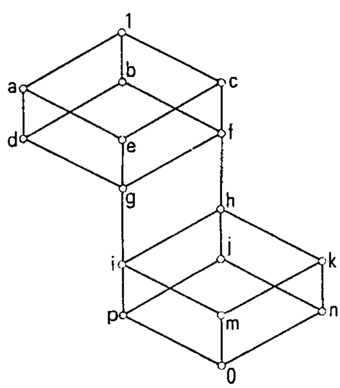

Figure 2

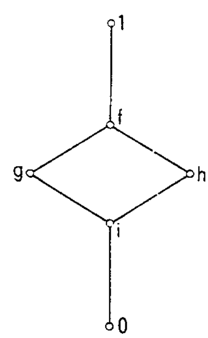

Figure 3

4. Applications

We shall now characterize completely, by means of previous results, the Frattini sublattice of a relatively complemented distributive lattice. From this we shall obtain a corollary answering problem 2 in [5].

THEOREM 5. *Let L be a relatively complemented distributive lattice, $|L|>2$. Then $\Phi(L)=C^*$, where C^* is the sublattice consisting of all bounds (i.e, 0 or 1) in L.*

Proof. If 0 or 1 is in L, then since L is relatively complemented, we have $0 \in L(\wedge)$, $1 \in L(\vee)$, and hence 0 or 1 is in $\Phi(L)$. Thus, $C^* \subseteq \Phi(L)$. Let $x \in L-C^*$. Then there exist $P \in P(L)$, $Q \in Q(L)$ such that $x \in P \cap Q$. Since L is relatively complemented, $P(L)-\{L\}$ and $Q(L)-\{L\}$ are unordered. Thus, (P, Q) is a minimal pair in $P(L) \times Q(L)$. Hence $P \cap Q$ is a minimal prime sublattice. As $x \in P \cap Q$, $x \notin \Phi(L)$. Therefore $\Phi(L)=C^*$.

COROLLARY 1. *Let B be a Boolean lattice, $|B|>2$. Then $\Phi(B)=\{0, 1\}$.*

COROLLARY 2. *Let B^* be a generalized Boolean lattice, $|B^*|>2$. Then* $\Phi(B^*)=\{0\}$.

The following result shows the relation between the Frattini sublattice of a relatively complemented distributive lattice and that of its homomorphic image.

COROLLARY 3. *Let L, K be relatively complemented distributive lattices, $\alpha: L\to K$, an epimorphism. Then*

$$\Phi(L\alpha)\supseteq\Phi(L)\alpha \quad \text{if} \quad |L\alpha|>2,$$

$$\Phi(L\alpha)\subseteq\Phi(L)\alpha \quad \text{if} \quad |L\alpha|\leq 2.$$

The relation between the Frattini sublattice of a finite direct product of relatively complemented distributive lattices and those of its compenents can also be obtained.

COROLLARY 4. *Let $\{L_i\}i=1,\dots,n$ be a finite family of relatively complemented distributive lattices with more than two elements. Then*

$$\left(\Phi\left(\prod_{i=1}^{n} L_i\right)\right)^n \subsetneq \prod_{i=1}^{n} \Phi(L_i).$$

Proof. Since, for each $i=1,\dots,n$, L_i is relatively complemented and distributive, so is the lattice $\prod_{i=1}^{n} L_i=L$. By Theorem 5, $\Phi(L)=C^*$, the bounds of L. If $C^*=\emptyset$, we have nothing to prove. If $|C^*|=1$, say $C^*=\{0\}$, then $0=(0_1,\dots,0_n)$, where for each $i=1,\dots,n$, 0_i is the zero element of L_i. Thus the inclusion is trivial. If $C^*=\{0,1\}$, then $0=(0_1,\dots,0_n)$, $1=(1_1,\dots,1_n)$. Thus $(\Phi(L))^n=\{0,1\}^n\cong\{0_1,1_1\}\times\{0_2,1_2\}\times\cdots\times\{0_n,1_n\}=\prod_{i=1}^{n}\Phi(L_i)$.

Remark. Corollaries 3 and 4 are not true in general. If L is a finite distributive planar lattice L, then we have [7], $\Phi(L\alpha)\subseteq\Phi(L)\alpha$ for any homomorphism α. To see the direct product, consider the lattice L of Figure 4. Then $\Phi(L\times L)$ is the sublattice $\{0,a,b,c,d,e,1\}$ of the lattice of Figure 5 whereas $\Phi(L)=\{x,1\}$.

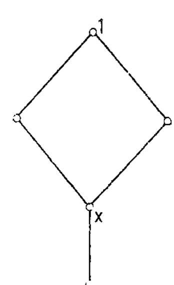

Figure 4

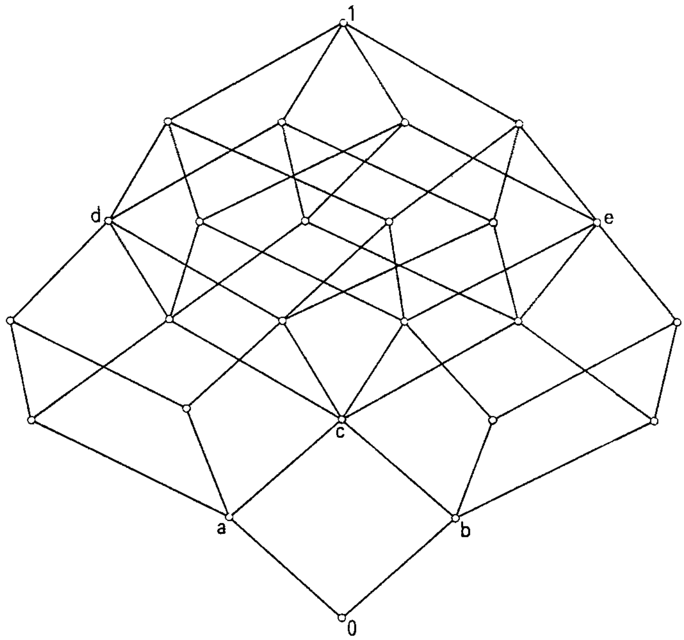

Figure 5

The following problem stated in [5] was, in fact, raised by G. Grätzer:

($*$) Is there any distributive lattice L which is not a chain such that (a) every proper sublattice can be extended to a maximal one, and (b) $\Phi(L)=\emptyset$?

Now, combining the result that any sublattice of a relatively complemented distributive lattice can be extended to a maximal one with Theorem 5, we have solution to the problem ($*$) by taking L to be any relatively complemented distributive lattice without 0 and 1.

COROLLARY 5. Let $\{L_i/i\in I\}$ be a family of relatively complemented distributive lattices without 0 and 1, and with more than two elements. Let $L=\bigoplus_{i\in I} L_i$ be an ordinal sum of $\{L_i/i\in I\}$. Let $L^*=\prod_{j\in J} L_j$, J, a finite subset of I. Then L and L^* have the properties (a) and (b) in ($*$).

COROLLARY 6. Let C be an arbitrary chain. Then there exists a lattice L such that 1) L is distributive, 2) every sublattice of L can be extended to a maximal one, and 3) $\Phi(L)\cong C$.

Proof. To each $i\in C$, assigns a generalized Boolean lattice B_{i*}, $|B_{i*}|>2$. Set $L=\bigoplus_{i\in C} B_{i*}$. Then L satisfies 1) and 2). Moreover, $\Phi(L)=\Phi(\bigoplus_{i\in C} B_{i*})=$
$=\bigoplus_{i\in C}\Phi(B_{i*})\cong C$, which is 3).

To end this paper, we would like to mention some open problems.

It has been shown on the one hand in [5] that every lattice is the Frattini sublattice of a lattice, and on the other hand in [7] that there exist infinitely many finite distributive planar lattices which cannot be represented even by finite modular planar lattices. The following problem, which we are unable to solve, naturally arises:

(1) Is every distributive lattice isomorphic to the Frattini sublattice of some distributive lattice?

(2) Study the Frattini sublattice $\Phi(S)$ of a Stone lattice S.

Let L be a finite modular planar lattice. It was proved in [7] that $\Phi(L)=[\mathrm{Irr}(L)]-\mathrm{Irr}(L)$. This result is not true for finite planar lattices. Thus, it may be of some interest to consider the following

(3) Study the Frattini sublattice of a finite planar lattice.

Finally, in view of Corollary 5, we may ask the following

(4) Let L be a distributive lattice such that

i) L is not a chain,

ii) $0, 1 \notin L$,

iii) L is indecomposable as a non-trivial ordinal sum of lattices, and

iv) L satisfies properties (a) and (b) in $(*)$.

Is L necessarily relatively complemented?

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