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Ordinal utility and economic price indices*

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Summary. In economic price index theory, a reference level of utility is needed for measuring the change in the cost of living between a base period and a comparison period. A reference level function can be used to derive this reference utility level from the utilities attained at the base and at the comparison prices. Depending on the scale type of the underlying utility function, the reference level function has to satisfy certain invariance conditions. In this paper, these conditions are formulated as functional equations for interval scales and for ordinal utility scales. By solving these equations, we characterize the class of admissible reference level functions for the respective scale type.

1. Introduction

Price indices are used to reflect changes in the cost of living between a base period and a comparison period. Whereas the *statistical* approach to price index theory (see Eichhorn and Voeller [8]) only takes into account the prices and the commodity bundles consumed in a base and a comparison period, the *economic* approach uses the ratio of the minimal expenditures necessary to reach some reference level of utility at the respective prices in comparison and base periods as an indicator of the change in the cost of living (see, for instance, Konüs [9], Diewert [6], Pollak [11]).

Given a utility function defined on the commodity space (the set of nonnegative n-dimensional vectors excluding the origin), the reference level of utility referred to above has to be determined. This can be done by using a *reference level function* which makes the reference level dependent on the utilities attained in base and comparison periods (a formal definition is given in the next section—see also

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Pfingsten [10] and Bossert and Pfingsten [5]). Depending on the scale type of the underlying utility function, this reference level function has to satisfy certain invariance conditions.

In this paper, we formulate these conditions for interval scales and for ordinal utility scales as functional equations and derive their general solutions. These solutions characterize the respective class of admissible reference level functions for the respective utility scale. In particular, it turns out that only four functions are admissible reference level functions for ordinal utility scales.

2. Price indices and utility scales

Let $\mathfrak{R}_{++}(\mathfrak{R}_+, \mathfrak{R})$ denote the set of positive (nonnegative, all) real numbers. N is the set of positive integers. $\mathfrak{R}^n_{++}(\mathfrak{R}^n_+, \mathfrak{R}^n)$ is the n-fold ($n \in N$) Cartesian product of $\mathfrak{R}_{++}(\mathfrak{R}_+, \mathfrak{R})$. $\mathbf{0}^n$ is the origin of $\mathfrak{R}^n$. A utility function on the commodity space $X^n := \mathfrak{R}^n_+ \setminus \{\mathbf{0}^n\}$ is a mapping $U: X^n \mapsto \mathfrak{R}$. We assume that U is continuous and increasing in the sense that $x_i > y_i$ for all $i = 1, \ldots, n$ implies $U(x) > U(y)$.

For simplicity of presentation, we assume that U is onto. (Without that assumption, due to the continuity and increasingness of U, the image of U would be a proper interval, which would leave our results unchanged.)

Given a utility function U, the *cost function* associated with U is defined by

$$C(p, u) = \min_x \{p \cdot x \mid U(x) = u\} \qquad \text{for all } p \in \mathfrak{R}^n_{++}, u \in \mathfrak{R},$$

where $p \in \mathfrak{R}^n_{++}$ is a commodity price vector, $p \cdot x$ is the inner product of p and x, and $u \in \mathfrak{R}$ denotes a utility level to be attained. The cost function dual to a utility function is a standard tool in mathematical economics (see, for instance, Diewert [7]) which gives the minimal cost necessary to achieve *any* possible level of utility at *any* possible price vector. Given our assumptions, a solution to the above cost minimization problem exists for all $p \in \mathfrak{R}^n_{++}$, $u \in \mathfrak{R}$.

If p^0 and p^1 are the price vectors in a base and a comparison period, respectively, the *cost of living index* is defined by the ratio

$$\frac{C(p^1, u^r)}{C(p^0, u^r)}$$

where $u^r \in \mathfrak{R}$ is a given *reference level* of utility. In general (more precisely, whenever C is not of the form $C(p, u) = \bar{c}(p) \cdot d(u)$ with appropriately chosen functions $\bar{c}$ and d—see Diewert [7, p. 551]), the value of the cost of living index depends on the choice of u^r. Consequently, it is desirable to develop a systematic way of obtaining this reference level.

An intuitively appealing suggestion is to require the reference level of utility to depend on the utilities actually attained in the base and the comparison periods, that is, to use a *reference level function* $F: \mathfrak{R}^2 \mapsto \mathfrak{R}$ to determine u^r. Consequently, for $(u, v) \in \mathfrak{R}^2$, $F(u, v)$ is the reference level of utility, if $u = U(x^0)$ and $v = U(x^1)$ are the utilities attained in the base and the comparison period, where x^0 and x^1 are the commodity bundles consumed in these periods, respectively.

Clearly, if utility functions are used, it is important to specify the underlying scale type. If utility is measured as an *interval scale* (see Roberts [12, p. 65]), U is unique up to strictly increasing *affine* transformations, that is, for all $a \in \mathfrak{R}_{++}$, $b \in \mathfrak{R}$, the function $\hat{U}: X^n \mapsto \mathfrak{R}$ defined by

$$\hat{U}(x) = aU(x) + b \qquad \text{for all } x \in X^n$$

can be used instead of U, and only these. In this case, $\hat{U}$ is an *admissible transformation* of U for this scale type.

Ordinal utility scales (see Roberts [12, p. 65]) are unique up to *arbitrary* strictly increasing transformations, that is, a function $\hat{U}: X^n \mapsto \mathfrak{R}$ is an admissible transformation of the ordinal utility scale U if and only if there exists a strictly increasing function $\Phi: \mathfrak{R} \mapsto \mathfrak{R}$ such that

$$\hat{U}(x) = \Phi(U(x)) \qquad \text{for all } x \in X^n.$$

Depending on the scale type used, a reference level function must satisfy certain invariance conditions in order to ensure that the cost of living index is insensitive with respect to admissible transformations of the utilities. The implications for possible reference level functions are studied in the remainder of the paper.

3. Admissible reference level functions

As a minimal requirement on a reference level function F, it should be ensured that, if U is replaced by an admissible transformation (given a certain scale type), the value of the corresponding cost of living index is unchanged. More precisely, if an admissible transformation is applied to the utility values u and v attained in base and comparison periods, the corresponding value of F must coincide with the value of the same transformation at $F(u, v)$. In the case of an interval scale, this means that F must satisfy the functional equation

$$F(au + b, av + b) = aF(u, v) + b \qquad \text{for all } a \in \mathfrak{R}_{++},\ b \in \mathfrak{R},\ (u, v) \in \mathfrak{R}^2. \tag{1}$$

If U is an ordinal scale, we obtain the (stronger) condition

$$F(\Phi(u), \Phi(v)) = \Phi(F(u, v)) \qquad \text{for all strictly increasing } \Phi\colon \mathfrak{R} \mapsto \mathfrak{R},\ (u, v) \in \mathfrak{R}^2. \tag{2}$$

Note that in (1) and (2), it is required that *the same* scale is relevant after applying the reference level function F. This is necessary in our context to ensure that the cost of living index is unaffected by an admissible transformation of U. Consequently, we will *not* be concerned with formulations such as, for example,

$$F(au + b, av + b) = g(a, b)F(u, v) + h(a, b)$$

with functions $g, h\colon \mathfrak{R}_{++} \times \mathfrak{R} \mapsto \mathfrak{R}$ instead of (1) and an analogous modification of (2). See Aczél and Roberts [3] who deal, among other things, with equations of that type.

In the following theorems, we provide the general solutions of (1) and (2) which characterize the classes of admissible reference level functions for interval respectively ordinal utility scales.

THEOREM 1. *$F\colon \mathfrak{R}^2 \mapsto \mathfrak{R}$ satisfies (1) if and only if there exist constants $\alpha, \beta \in \mathfrak{R}$ such that*

$$F(u, v) = \begin{cases} \alpha u + (1 - \alpha)v & \text{if } u \geqslant v \\ \beta v + (1 - \beta)u & \text{if } u < v \end{cases} \qquad \text{for all } (u, v) \in \mathfrak{R}^2. \tag{3}$$

Proof. It is clear that all functions of the form (3) satisfy (1). It remains to show that the functions given by (3) are the only solutions of (1). By inserting $b = 0$ in (1), we obtain

$$F(au, av) = aF(u, v) \qquad \text{for all } a \in \mathfrak{R}_{++},\ (u, v) \in \mathfrak{R}^2; \tag{4}$$

choosing $a = 1$ in (1) leads to

$$F(u + b, v + b) = F(u, v) + b \qquad \text{for all } b \in \mathfrak{R},\ (u, v) \in \mathfrak{R}^2. \tag{5}$$

The solution of the system of equations (4) and (5) can be found in Aczél and Dhombres [2, p. 250]—see also Aczél [1, p. 234] and Bossert [4] for similar results.

For completeness, we provide a derivation of the solution here. From (4), $F(0,0)=0$. Combined with (5), this implies $F(u,u)=u$ for all $u \in \mathfrak{R}$. Now let $u > v$. From (5),

$$F(u, v) = F(u - v + v, 0 + v) = F(u - v, 0) + v. \tag{6}$$

(4) implies

$$F(u - v, 0) = F((u - v) \cdot 1, (u - v) \cdot 0) = (u - v)F(1, 0). \tag{7}$$

Combining (6) and (7), we obtain

$$F(u, v) = (u - v)F(1, 0) + v = \alpha u + (1 - \alpha)v$$

with $\alpha := F(1, 0)$. Analogously, for $u < v$, it follows

$$F(u, v) = (1 - \beta)u + \beta v$$

with $\beta := F(0, 1)$. Combining all these observations, we obtain (3). q.e.d.

THEOREM 2. *The only functions* $F: \mathfrak{R}^2 \mapsto \mathfrak{R}$ *satisfying* (2) *are given by*

(*i*) $F(u, v) = u$ *for all* $(u, v) \in \mathfrak{R}^2$,
(*ii*) $F(u, v) = v$ *for all* $(u, v) \in \mathfrak{R}^2$,
(*iii*) $F(u, v) = \min\{u, v\}$ *for all* $(u, v) \in \mathfrak{R}^2$,
(*iv*) $F(u, v) = \max\{u, v\}$ *for all* $(u, v) \in \mathfrak{R}^2$.

Proof. That the functions F given in (i) through (iv) satisfy (2) can be verified by substitution. By Theorem 1, all functions F satisfying (2) must be of the form (3). Choosing $\Phi(u) = u^3$ for all $u \in \mathfrak{R}$ in (2), it follows

$$\alpha u^3 + (1 - \alpha)v^3 = [\alpha u + (1 - \alpha)v]^3 \tag{8}$$

for all $(u, v) \in \mathfrak{R}^2$ such that $u \geqslant v$ and

$$\beta v^3 + (1 - \beta)u^3 = [\beta v + (1 - \beta)u]^3 \tag{9}$$

for all $(u, v) \in \mathfrak{R}^2$ such that $u < v$. Comparing the coefficients of uv^2 in (8) yields

$$0 = 3\alpha(1 - \alpha)^2$$

which is satisfied if and only if $\alpha = 0$ or $\alpha = 1$. Similarly, $\beta = 0$ or $\beta = 1$ is necessary and sufficient for (9). This gives us the four possibilities

(i) $\alpha = 1$ and $\beta = 0$,
(ii) $\alpha = 0$ and $\beta = 1$,
(iii) $\alpha = 0$ and $\beta = 0$,
(iv) $\alpha = 1$ and $\beta = 1$.

Inserting (i)–(iv) in (3) completes the proof. q.e.d.

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