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A novel interval grey prediction model considering uncertain information

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Abstract

Current studies on grey systems are mainly focused on known and deterministic information, rather than uncertain one. Different from previous schemes, this paper proposes an innovative prediction model based on grey number information, which extends its application dealing with uncertain information. By exploiting the geometric features of grey numbers on a two-dimensional surface, all grey numbers can be converted into real numbers without losing any information by means of proposed algorithms. Then a prediction model is established based on those real number sequences. In addition, a general case simulation is carried out to verify the effectiveness and practicability of the proposed model.

1. Introduction

In systems theory, a system can be defined with a color that represents the magnitude of clear information about the system. For instance, a black-box system means that its internal characteristics or mathematical equations that describe its dynamics are completely unknown. To the contrary, if the description of the system is completely known, it can be named as a white system. Intermediately, it is called a grey system. In reality, every system can be modeled as a grey system because there are always some uncertainties in a system. Due to noise existing inside and outside of a system and the limitations of our cognitive abilities, the information we can achieve about the system is always incomplete and limited in scope [1]. The concept of grey system and its theory was first proposed by Deng [2] to deal

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with systems having partially unknown parameters. In contrast to conventional statistical models, grey models require only a limited amount of data to estimate the behavior of unknown parts [3]. Over the past two decades, grey system theory has been developed rapidly and caught the attention of many researchers. Moreover, it has been widely and successfully applied in various fields such as social, economic, financial, agricultural, industrial, and military domains [4].

One of the most important researches in grey system theory is the grey prediction model where the GM (1,1) model is the core [5]. Traditionally, the prediction accuracy of this model is not stable and thus some improvements models have been presented recently. They are mainly focused on: (i) changing raw data sequences in order to enhance the smoothness [6,7]; (ii) improving on the approaches of calculating the parameters of GM(1, 1) and optimizing the conformation of background values [8–10]; (iii) amending residual errors of model [11–13]; (iv) reforming and proposing some expanding models of GM(1,1)[14]; (v) studying the condition of modeling [15]; and (vi)establishing combined prediction models [16].

Generally speaking, these above studies about the grey prediction model are established on the foundation of real number sequences, rather than grey numbers. More specifically, if there is uncertain information, previous studies often converted the information to corresponding real number representations, and then establishes prediction models in a real number space. However, in the process of transformation, some important information may be lost. Admittedly, the amount of data is very limited in a grey system, and the lost information should have a significant effect on the results of a prediction model. For example, assume an interval grey number is $\otimes \in [8, 14]$, if the '$\otimes$' is replaced by a real number '11', obviously the other possible information (i.e.$[8, 11)$ or $(11, 14]$) is ignored. Basically, a traditional grey prediction model is shown in Fig. 1.

In grey system, a whitenization weight function is employed to describe the degree of preference of a grey number to take values in its range. A symmetric triangle is one of the most common geometry forms of whitenization weight functions (it is also named as a triangular whitenization weight function of the central point). Hence this paper will study the grey prediction model of interval grey number in which whitenization weight function is symmetric triangular. By mining and analyzing geometric features of interval grey numbers on a two-dimensional surface, this paper proposes a new grey prediction scheme. This scheme can convert all interval grey numbers into real numbers by means of some proposed algorithms without any information missing. The research results have potential application both from theoretical and practical sides.

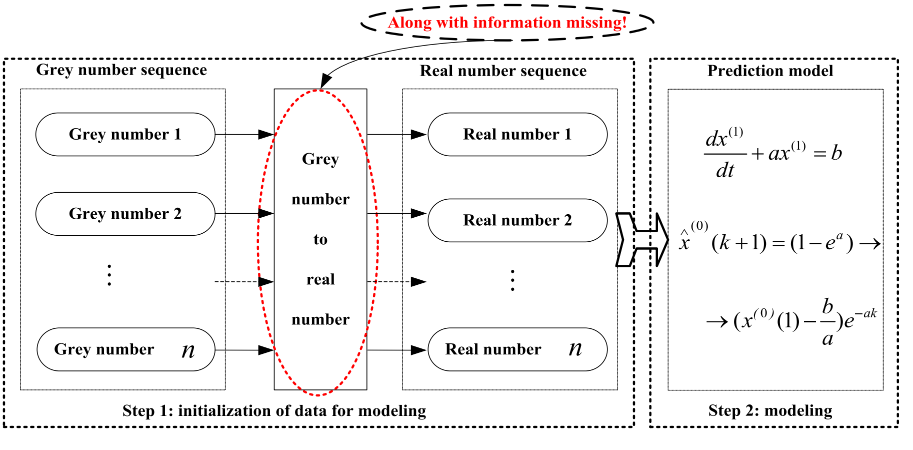

Fig. 1. Traditional grey prediction model.

The remaining parts of the paper are organized as follows: Section 2 introduces the original GM (1, 1) model. Section 3 describes interval grey numbers and whitenization weight functions. In Section 4 we propose the interval grey prediction approach. In Section 5 one case study on Chinese flat-screen TV demand forecasting is presented and conclusions follow in Section 5.

2. Reviewing the GM (1, 1) model

The GM (1, 1) model is called "Grey Model First Order One Variable", which has been widely used. Generally, this model is composed of a dynamic model expressed by an ordinary differential equation with time-varying coefficients. In order to smooth the randomness of primitive data, we convert these data into another form by an operator, named Accumulating Generation Operator (AGO) [2]. Let $X^{(0)}$ be a non-negative sequence $X^{(0)}=(x^{(0)}(1), x^{(0)}(2),\ldots,x^{(0)}(n))$, $n\geq 4$. When this sequence is subjected to the AGO, the following sequence $X^{(1)}=(x^{(1)}(1), x^{(1)}(2),\ldots,x^{(1)}(n))$ is obtained. $X^{(1)}$ is said to be an AGO sequence of $X^{(0)}$, written as $X^{(1)}=\text{AGO}\ X^{(0)}$, where

$$x^{(1)}(k)=\sum_{m=1}^{k} x^{(0)}(m), \quad k=1,2,\ldots,n \tag{1}$$

It is obvious that $X^{(1)}$ is monotonically increasing (see Fig. 2).

By using the transformation, the original data sequence can be formed as a new regular (smooth) data sequence, and then the prediction model can be represented by an ordinary differential equation. Furthermore, the generated mean sequence $Z^{(1)}$ of $X^{(1)}$ is defined as $Z^{(1)}=(z^{(1)}(2), z^{(1)}(3),\cdots,z^{(1)}(n))$, where $z^{(1)}(k)$ is the mean value of adjacent data,

$$z^{(1)}(k)=0.5x^{(1)}(k)+0.5x^{(1)}(k-1), \quad k=2,3,\ldots,n \tag{2}$$

The least square estimate sequence of the grey difference equation of GM (1, 1) is defined as following:

$$x^{(0)}(k)+az^{(1)}(k)=b \tag{3}$$

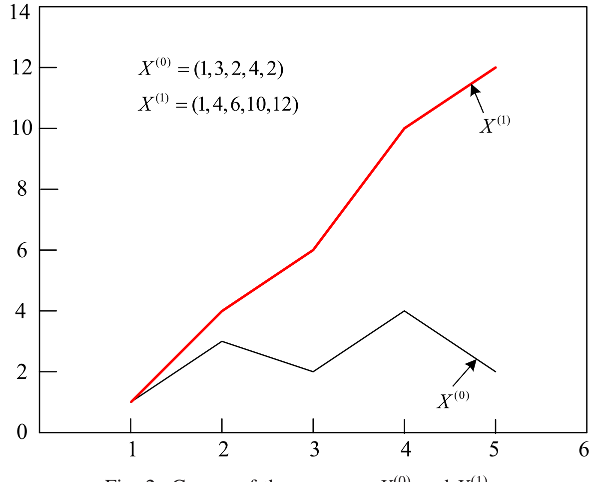

Fig. 2. Curves of the sequence $X^{(0)}$ and $X^{(1)}$.

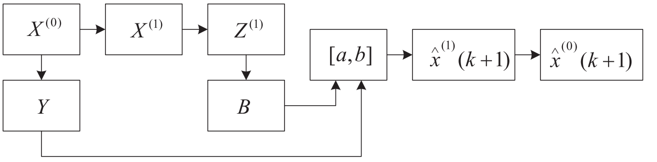

Fig. 3. Flowchart of GM (1, 1) model.

The whitening equation is therefore, as follows:

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b \tag{4}$$

In the Eq. (3), $[a,b]^T$ is a sequence of parameters that can be formed as

$$[a,b]^T = (B^T B)^{-1} B^T Y \tag{5}$$

where

$$Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix} \quad B = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{bmatrix}$$

According to Eq. (4), the solution of $x^{(1)}(t)$ at time k is

$$\hat{x}^{(1)}(k+1) = \left[x^{(0)}(1) - \frac{b}{a}\right] e^{-ak} + \frac{b}{a} \tag{6}$$

To obtain the predicted value of the primitive data at time $(k+1)$, the IAGO (Inverse Accumulated Generating Operation) is used to establish the following grey model.

$$\hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k) = (1-e^{a}) \left[x^{(0)}(1) - \frac{b}{a}\right] e^{-ak} \tag{7}$$

The flowchart of GM (1, 1) is as shown in Fig. 3.

3. Interval grey numbers and whitenization weight functions

In grey system theory, grey numbers are the elementary "cells". A grey number is such a number whose exact value is unknown but a range where the value lies in is known [1]. In applications, there are several kinds of grey numbers, such as interval grey numbers, continuous grey numbers, discrete grey numbers, and essential grey numbers etc. Amongst them, interval grey numbers are the most important one. A grey number having both a lower bound and an upper bound is called an interval grey number, denoted as $\otimes \in [a,b]$ $(a<b)$. In essence, an interval grey number is different from an interval. It is just an uncertain number within an interval. We use the concept of whitenization weight function to describe the probable degree of a grey number in the interval. The function includes typical whitenization weight function (see Fig. 4(a)), lower or upper whitenization weight function (see Fig. 4(b) and (c)), and triangular whitenization weight function(see Fig. 4(d)).

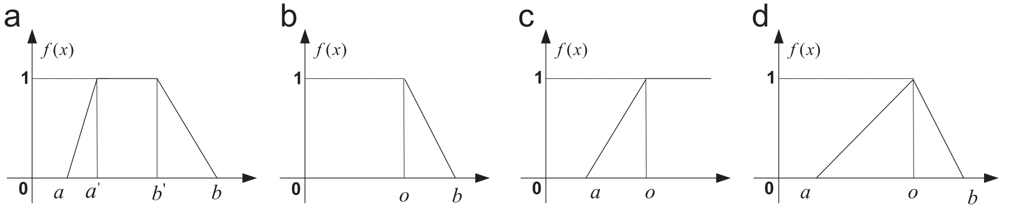

Fig. 4. Whitenization weight function of a grey number.

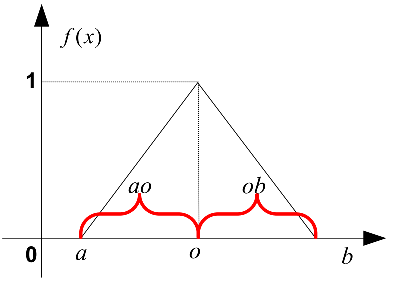

Fig. 5. Whitenization weight function of central point.

In Fig. 4(d), if the length of ao equals that of ob (i.e. $ao = ob$, see Fig. 5), this sort of whitenization weight function is named as a 'whitenization weight function of central point' (WWFCP).

WWFCP means that the value of central point (i.e. 'o') between a and b has the maximum likelihood to represent the whitenization value of an interval grey number. In this paper, we will study an interval prediction model based on a grey number sequence with their WWFCPs.

Definition 1. for an interval grey number $\otimes \in [a, b](a < b)$, the difference value between a and b is called the length of the interval grey number $\otimes$, denoted as $l = b - a$.

Definition 2. the most likely value of an interval grey number is called the whitenization value of the interval grey number, denoted as $\tilde{\otimes}$.

For a WWFCP based interval grey number $\otimes \in [a, b](a < b)$, its whitenization value is $\tilde{\otimes} = (a + b)/2$.

4. Prediction model via interval grey numbers

A prediction model based on interval grey numbers is more complicated than that based on real numbers. The causes are as follows:

(1) Algebraic algorithms of interval grey numbers have some demerits. For example, subtraction, multiplication and division of grey numbers will make the results more uncertain than before.
(2) Although the lower/upper bounds of interval grey number are real numbers, applying the two bounds to establish the lower/upper bounds of the prediction model is unreasonable. Firstly, because the determination of a grey number's lower/upper bounds has the characteristic of

subjectivity; in other words, In essence, the numbers cannot be accurately achieved. Secondly, if we use the sequences of lower and upper bounds to establish a prediction model respectively, the simulation/prediction values of a lower bound may be not less than the corresponding upper one (see case study of Section 4). Thus, it violates the definition of an interval grey number.

(3) Theoretically, interval grey numbers are not some points but some segments on a two-dimensional surface; it is impossible to apply an exponential curve to approximate those segments, which however is widely used in the GM (1, 1) model.

In order to address those issues mentioned above, it is necessary to develop a novel method that can convert grey numbers into real ones without losing any information (see Fig. 6).

Definition 4. a sequence which consists of some interval grey numbers is called an interval grey number sequence, denoted as $X(\otimes)=(\otimes(t_1), \otimes(t_2), \ldots, \otimes(t_n))$.

Definition 5. an interval grey number sequence $X(\otimes)$ is projected on a two-dimensional Cartesian coordinate system; each grey number in the sequence $X(\otimes)$ has a lower boundary point and an upper one; by connecting all those points, we can obtain a figure, called a grey number band of the sequence $X(\otimes)$ (see Fig. 7); a grey number band between $\otimes(t_k)$ and $\otimes(t_{k+1})(k=1,2,\ldots,n-1)$ is called a grey number layer(see Fig. 8).

As discussed previously, the main contribution of us is the transformation from grey numbers to real numbers without any information missing. In this paper we realize the transformation by means of exploiting and analyzing the characteristics of interval grey numbers on a two-dimensional Cartesian coordinate system. Assume that an interval grey number sequence $X(\otimes)=(\otimes(t_1), \otimes(t_2), \ldots, \otimes(t_n))$, where $\Delta t_k=t_k-t_{k-1}=1$ and $\otimes(t_k)\in[a_k, b_k]$, $k=1,2,\ldots,n$, we can calculate the area $s(p)$ of the

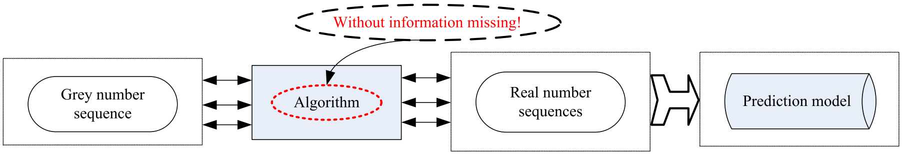

Fig. 6. Idea of grey number prediction models.

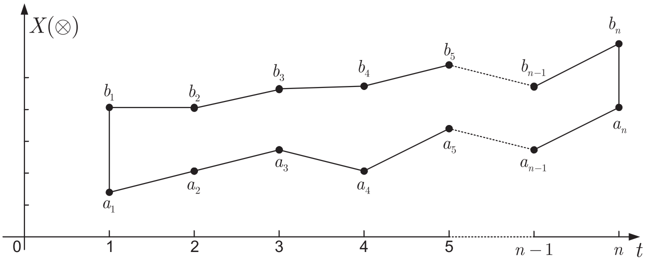

Fig. 7. Grey number band.

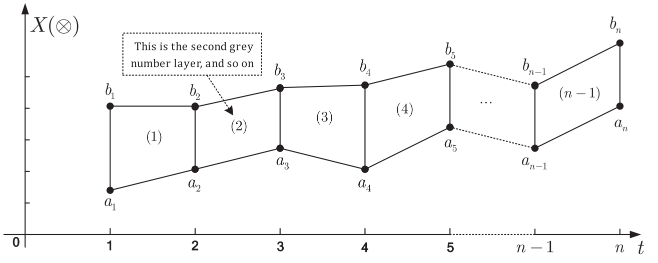

Fig. 8. Grey number layers.

p-th grey number layer (i.e. $a_p a_{p+1} b_{p+1} b_p$) ($p = 1, 2, \ldots, n-1$) according to Definition 2 and Fig. 8 as

$$s(p) = \frac{(b_p - a_p) + (b_{p+1} - a_{p+1})}{2} \cdot (t_{p+1} - t_p) \tag{8}$$

for $\Delta t_k = t_k - t_{k-1} = 1$, so,

$$s(p) = \frac{(b_p - a_p) + (b_{p+1} - a_{p+1})}{2} \tag{9}$$

Formula (9) represents the area of the p-th grey number layer, and it equals to a generated mean value of consecutive neighbors of the length of $\otimes(t_{p+1})$ and $\otimes(t_p)$. All grey numbers in sequence $X(\otimes)$ will be converted into real numbers through Formula (9), and the formed real number sequence is denoted as $S = (s(1), s(2), \ldots, s(p-1))$. Then we can establish a GM (1, 1) model based on the sequence S and the time response equation of GM(1,1) with the sequence S is defined as follows:

$$\hat{s}^{(1)}(k+1) = \left(s(1) - \frac{b}{a}\right)e^{-ak} + \frac{b}{a} \tag{10}$$

The restored values can be given by

$$\hat{s}^{(1)}(k+1) = (1 - e^{a})\left[s(1) - \frac{b}{a}\right]e^{-ak} \tag{11}$$

By the formula (11), the area $s(k)$ of the k-th grey number layer can be simulated and forecasted. Moreover, we can calculate the length of the grey number $\otimes(t_{p+1})$ by

$$s(p) = \frac{(b_p - a_p) + (b_{p+1} - a_{p+1})}{2} \Rightarrow b_{p+1} - a_{p+1} = 2s(p) - (b_p - a_p)$$

when $p = 1$,

$$\hat{b}_2 - \hat{a}_2 = 2\hat{s}(1) - (b_1 - a_1) = b_2 - a_2$$

when $p = 2$,

$$\hat{b}_3 - \hat{a}_3 = 2\hat{s}(2) - (b_2 - a_2)$$

when $p = 3$,

$$\hat{b}_4 - \hat{a}_4 = 2\hat{s}(3) - (\hat{b}_3 - \hat{a}_3) = 2\hat{s}(3) - 2\hat{s}(2) + (b_2 - a_2)$$

$$\vdots$$

when $p = k-1$,

$$\hat{b}_k - \hat{a}_k = 2\hat{s}(k-1) - 2\hat{s}(k-2) + \cdots + (-1)^k(b_2 - a_2) \tag{12}$$

According to Formula (11), we have

$$\hat{s}(k-1) = (1-e^{a})\left[s(1)-\frac{b}{a}\right]e^{-a(k-2)}$$

$$\hat{s}(k-2) = (1-e^{a})\left[s(1)-\frac{b}{a}\right]e^{-a(k-3)}$$

It is obvious that Formula (12) is a geometric progression whose common ratio q is $-e^{a}(q=-e^{a})$. Through the summation formula of geometric progression, the value of $\hat{b}_k-\hat{a}_k$ can be given by

$$\hat{b}_k-\hat{a}_k = \frac{2\times(1-e^{a})\left[s^{(0)}(1)-\frac{b}{a}\right]e^{-a(k-2)}\times\left[1-(e^{a})^{k-2}\right]}{1+e^{a}}+(-1)^{k}(b_2-a_2) \tag{13}$$

Although the length of grey number $\otimes(t_k)$ can be simulated or forecasted by Formula (13), the lower bound a_k and upper bound b_k are not still unknown. In order to solve this problem, it is necessary to establish another formula between a_k and b_k. In Fig. 9, the numbers, such as ①, ②, … indicate the order of grey number layers in the grey number band. The lines such as A_1A_2,B_1B_2, etc. are used to denote the middle position lines of corresponding grey number layers. The points such as A,B,… are the middle points of corresponding lines.

Theorem 1. *in Fig. 9, the vertical coordinate of the middle point of the p-th grey number layer's middle position line is* $(a_p+a_{p+1}+b_p+b_{p+1})/4$.

Proof. when $p=1$, it can be seen from Fig. 9 that points a_1,A_1 and a_2 pass through the same line. The coordinates of a_1 and a_2 are $(1,a_1)$ and $(2,a_2)$ respectively. Then the function of this line is

$$x=(a_2-a_1)t+2a_1-a_2 \tag{14}$$

For the abscissa of A_1 being 1.5, the vertical coordinate of A_1 equals $(a_1+a_2)/2$. Since the line A_1A_2 is the middle position line of the 1-th grey-number layer, according to the theorem of middle position line of trapezoids, we can obtain the length of A_1A_2

$$l_{A_1A_2} = \frac{(b_1-a_1)+(b_2-a_2)}{2} \tag{15}$$

Since the point A is the middle point of the line A_1A_2, the vertical coordinate of A is

$$\frac{(a_1+a_2)}{2}+\frac{(b_1-a_1)+(b_2-a_2)}{2\times2}=\frac{(a_1+a_2+b_1+b_2)}{4}$$

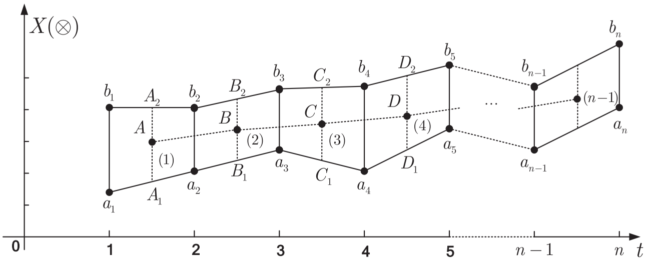

Fig. 9. Grey number layer and its middle position line.

By this means, we can obtain the vertical coordinates of the middle point of the p-th grey number layer's middle-position line, which is $(a_p + a_{p+1} + b_p + b_{p+1})/4$.

According to Theorem 3, the real number sequence consisting of the entire verticals of the middle points of grey number layers' middle position line is

$$W = (w(1), w(2), \ldots, w(n-1))$$

where

$$w(k) = \frac{(a_k + a_{k+1} + b_k + b_{k+1})}{4}, \quad k = 1, 2, \ldots, n-1,$$

Now, establish a GM (1, 1) model based on the sequence W, and the time response formula of GM (1, 1) with the sequence W is as follows,

$$\hat{w}^{(1)}(k+1) = (w(1) - \frac{\beta}{\alpha})e^{-\alpha k} + \frac{\beta}{\alpha} \tag{16}$$

The restored values can be given by

$$\hat{w}^{(0)}(k+1) = (1 - e^{\alpha})\left[w(1) - \frac{\beta}{\alpha}\right]e^{-\alpha k} \tag{17}$$

According to Theorem 3

$$\hat{w}^{(1)}(p) = \frac{(\hat{a}_p + \hat{a}_{p+1} + \hat{b}_p + \hat{b}_{p+1})}{4} \Rightarrow \hat{a}_{p+1} + \hat{b}_{p+1} = 4\hat{w}(p) - (\hat{a}_p + \hat{b}_p)$$

when $p = 1$,

$$\hat{a}_2 + \hat{b}_2 = 4\hat{w}(1) - (\hat{a}_1 + \hat{b}_1) = a_2 + b_2$$

when $p = 2$,

$$\hat{a}_3 + \hat{b}_3 = 4\hat{w}(2) - (a_2 + b_2)$$

when $p = 3$,

$$\hat{a}_4 + \hat{b}_4 = 4\hat{w}(3) - (\hat{a}_3 + \hat{b}_3) = 4\hat{w}(3) - 4\hat{w}(2) + (a_2 + b_2)$$

$$\vdots$$

when $p = k-1$,

$$\hat{a}_k + \hat{b}_k = 4\hat{w}(k-1) - 4(k-2) + \cdots + (-1)^k(a_2 + b_2) \tag{18}$$

since

$$\hat{w}(k-1) = (1 - e^{\alpha})\left[w(1) - \frac{\beta}{\alpha}\right]e^{-\alpha(k-2)}$$

$$\hat{w}(k-2) = (1 - e^{\alpha})\left[w(1) - \frac{\beta}{\alpha}\right]e^{-\alpha(k-3)}$$

So, it is obvious that the Eq. (14) is a geometric progression whose common ratio q is $-e^{\alpha}$ (i.e. $q = -e^{\alpha}$). Through the summation formula of geometric progressions, the value of $\hat{a}_n + \hat{b}_n$ can be given by

$$\hat{a}_n + \hat{b}_n = \frac{4 \times (1 - e^{\alpha})\left[w(1) - \frac{\beta}{\alpha}\right]e^{-\alpha(k-2)} \times \left[1 - (-e^{\alpha})^{k-2}\right]}{1 + e^{\alpha}} + (-1)^k(a_2 + b_2) \tag{19}$$

combining Eq. (13) with (19), we can get

$$\begin{cases} \hat{b}_k - \hat{a}_k = \frac{2\times(1-e^{a})\left[s(1)-\frac{b}{a}\right]e^{-a(k-2)}\times\left[1-(e^{a})^{k-2}\right]}{1+e^{a}} + (-1)^k(b_2-a_2) \\ \hat{a}_k + \hat{b}_k = \frac{4\times(1-e^{\alpha})\left[w(1)-\frac{\beta}{\alpha}\right]e^{-\alpha(k-2)}\times\left[1-(-e^{\alpha})^{k-2}\right]}{1+e^{\alpha}} + (-1)^k(a_2+b_2) \end{cases} \tag{20}$$

According to Eq. (20), the time response formula of $\hat{a_k}$ and $\hat{b_k}$ with the grey number $\otimes(t_k)$ can be achieved by

$$\begin{cases} \hat{a}_k = \frac{F_w\times e^{-\alpha(k-2)}\times[1-(-e^{\alpha})^{k-2}]-F_s\times e^{-a(k-2)}\times[1-(e^{a})^{k-2}]}{2} + (-1)^k a_2 \\ \hat{b}_k = \frac{F_s\times e^{-a(k-2)}\times[1-(e^{a})^{k-2}]+F_w\times e^{-\alpha(k-2)}\times[1-(-e^{\alpha})^{k-2}]}{2} + (-1)^k b_2 \end{cases} \tag{21}$$

where $F_s = \frac{2\times(1-e^{a})\left[s(1)-\frac{b}{a}\right]}{1+e^{a}}$, and $F_w = \frac{4\times(1-e^{\alpha})\left[w(1)-\frac{\beta}{\alpha}\right]}{1+e^{\alpha}}$.

The WWFCP of grey number $\otimes(t_k)$ is as shown in Fig. 5. Then the whitenization value of the $\otimes(t_k)$ can be given by

$$\tilde{\otimes}(t_k) = \frac{\hat{a}_k + \hat{b}_k}{2} \tag{22}$$

Definition 6. We call the Eqs. (21) and (22) an interval grey number prediction model based on the triangular whitenization weight function of central points (abbreviated to IGPM_T: ***I***nterval ***G***rey number ***P***rediction ***M***odel based on the ***T***riangular whitenization weight function of central points).

When $k \le n$, $\hat{a}_k$ and $\hat{b}_k$ are the simulation values of the model
When $k > n$, $\hat{a}_k$ and $\hat{b}_k$ are the prediction values of the model
The steps of IGPM_T are as follows:

Step1: transform interval grey number sequences (i.e.$X(\otimes)$) to real number sequences (i.e. S and W);
Step2: establish models respectively based on real number sequences;
Step3: establish an interval grey number prediction model on the basis of 'Step2';
Step4: simulate or forecast the lower/upper bound of a grey number $\otimes(t_k)$ in $X(\otimes)$;
Step5: calculate the whitenization value of the $\otimes(t_k)$.

It is important to note that applying the IGPM_T to simulate or forecast an interval grey number with a counter is very complicated. Instead, we accomplish the calculations by computer program, and the programming flowchart for IGPM_T is illustrated in Fig. 10.

5. Case study

Currently high-quality, high-definition and low-power flat-panel TVs have become much popular in China. In order to occupy an advantageous position in the fiercely competitive market, companies need to forecast the demand of plat-panel TVs in the next year. However, flat-panel TVs are just emerged in the market lately and it is difficult to obtain accurate statistic information of its sale over the past five years. Market experts infer a probable range of flat-panel TV's sale from 2004 to 2008 according to some known information (see Table 1). Now, we use the new

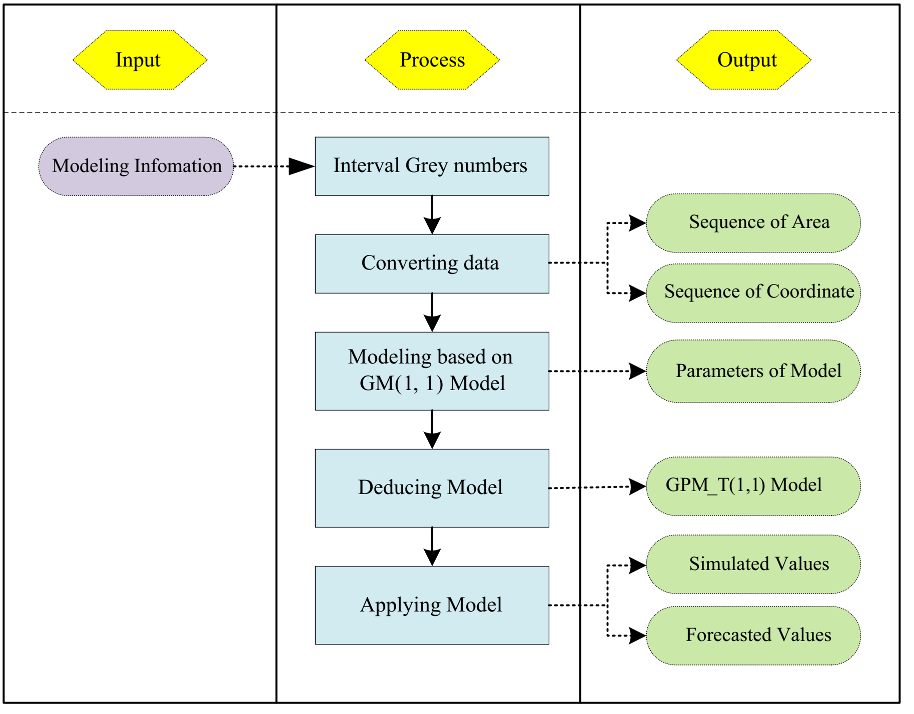

Fig. 10. Programming flowchart for IGPM_T.

proposed model and the traditional GM (1, 1) model to forecast the flat-panel TV's sale volume in 2009 respectively, and make a comparison with the real values.

5.1. Modeling by the new proposed model

Annual sale of flat-panel TVs should be a unique value rather than a range. Due to lack of information, we do not know exact values. Thereby, this is a prediction problem based on interval grey numbers. We use IGPM_T model to solve it, define

$$X(\otimes) = (\otimes(t_1), \otimes(t_2), \otimes(t_3), \otimes(t_4), \otimes(t_5))$$

where

$$\otimes(t_1) \in [21.4, 63.6], \quad \otimes(t_2) \in [196.3, 241.6], \quad \otimes(t_3) \in [345.2, 393.7]$$
$$\otimes(t_4) \in [745.5, 796.9] \text{ and } \otimes(t_5) \in [1284.4, 1341.2]$$

Step1: Transformation (according to Formula 9 and Theorem 1 respectively)

$$S = (43.75, 46.90, 49.95, 54.10)$$
$$W = (130.72, 294.20, 570.32, 1042.00)$$

Step2: Establishing GM (1, 1) based on sequences:S and W (according to Formula 11 and 17 respectively)

$$\hat{s}^{(0)}(k+1) = (1-e^{-0.0718}) \times [43.75 + 584.1835] \times e^{0.0718k}$$
$$\hat{w}^{(0)}(k+1) = (1-e^{-0.6015}) \times [130.72 + 221.9091] \times e^{0.6015k}$$

Table 1
Probable range of flat-panel TV's sale volume from 2004 to 2008 in China.

Year	2004	2005	2006	2007	2008
Interval grey number	$\otimes(t_1)\in$ [21.4, 63.6]	$\otimes(t_2)\in$ [196.3, 241.6]	$\otimes(t_3)\in$ [345.2, 393.7]	$\otimes(t_4)\in$ [745.5, 796.9]	$\otimes(t_5)\in$ [1284.4, 1341.2]

Step3: Establishing the model of IGPM_T based on 'Step2' (according to Eq. 21)

$$\begin{cases} \hat{a}_k = \frac{411.8481\times e^{0.6015(k-2)}\times[1-(-e^{-0.6015})^{k-2}]-45.0532\times e^{0.0718(k-2)}\times[1-(e^{-0.0718})^{k-2}]}{2} + (-1)^k \times 196.3 \\ \hat{b}_k = \frac{45.0532\times e^{0.0718(k-2)}\times[1-(e^{-0.0718})^{k-2}]+411.8481\times e^{0.6015(k-2)}\times[1-(-e^{-0.6015})^{k-2}]}{2} + (-1)^k \times 241.6 \end{cases} \tag{23}$$

Step4: Simulating and forecasting the lower/upper limits of $X(\otimes)$

5.2. Modeling by traditional GM (1, 1) model

The sequence of lower limits: $L=(21.4,\ 196.3,\ 345.2,\ 745.5,\ 1284.4)$

$$\hat{l}_{(0)}(k+1)=(1-e^{-0.5999})\times[21.4+222.5974]\times e^{0.5999k} \tag{24}$$

The sequence of upper limits: $U=(63.6,\ 241.6,\ 393.7,\ 796.9,\ 1341.2)$

$$\hat{u}_{(0)}(k+1)=(1-e^{-0.5623})\times[63.6+245.5691]\times e^{0.5623k} \tag{25}$$

5.3. Comparison of simulated results

Simulation errors and average precisions of the proposed model and the traditional model are illustrated in Tables 2 and 3. We employ matlab8.0 to draw two sets of figures to compare their simulation errors and average precisions as follows (Figs. 11 and 12).

It can be seen from Figs. 11 and 12, the new model is better than the traditional GM (1, 1) model in both simulation errors and average precisions.

5.4. Comparison of predictive results of flat-panel TV's sale volume in 2009

(1) Use the new model to forecast $\hat{a}_6$ and $\hat{b}_6$ (that is the lower and upper bounds in 2009)

$$\hat{a}_6=2266.48,\ \hat{b}_6=2326.77\ and\ \otimes(t_6)\in[2266.48, 2326.77]$$

$$\tilde{\otimes}(t_6)=\frac{\hat{a}_6+\hat{b}_6}{2}=\frac{2266.48+2326.77}{2}=2296.63$$

(2) Use traditional GM(1, 1) model to forecast $\hat{a}_6'$ and $\hat{b}_6'$ (that is the lower and upper bounds in 2009)

$$\hat{a}_6'=4025.71,\ \hat{b}_6'=3880.67\ and\ \otimes'(t_6)\in[4025.71, 3880.67]$$

$$\tilde{\otimes}'(t_6)=\frac{\hat{a}_6'+\hat{b}_6'}{2}=\frac{4025.71+3880.67}{2}=3953.19$$

Table 2
Simulation values of flat-panel TV's sale by the model of IGPM_T.

Year		2005		2006		2007		2008	
Items		Lower limit a_2	Upper limit b_2	Lower limit a_3	Upper limit b_3	Lower limit a_4	Upper limit b_4	Lower limit a_5	Upper limit b_5
Sale Volumes $k=2,3,4,5$	Original values $\otimes(t_k)\in[a_k,b_k]$	196.3	241.6	345.2	393.7	745.5	796.9	1284.4	1341.2
	Simulation values $\hat{\otimes}(t_k)\in[\hat{a}_k,\hat{b}_k]$	196.3	241.6	338.68	386.84	672.64	724.90	1210.54	1266.18
	Simulation errors $e_a(k)=\lvert a_k-\hat{a}_k\rvert$ $e_b(k)=\lvert b_k-\hat{b}_k\rvert$	0.0000	0.0000	6.52	6.86	72.86	72.00	73.86	75.02
	Average errors (%) $\Delta_a(k)=e_a(k)/a_k$ $\Delta_b(k)=e_b(k)/b_k$	0.0000	0.0000	1.8888	1.7424	9.7733	9.0350	5.7501	5.5935
	Average precision (%) $r_a(k)=1-\Delta_a(k)$ $r_b(k)=1-\Delta_b(k)$	100.0000	100.0000	98.1112	98.2576	90.2267	90.965	94.2499	94.4065
The average relative error: $\Delta=\frac{1}{4}\sum_{k=2}^{5}\Delta_a(k)+\frac{1}{4}\sum_{k=2}^{5}\Delta_b(k)=4.22\%$									

Table 3
Simulation values of flat-panel TV's sale based on the GM (1, 1) of the sequence of lower/upper limits.

Year		2005		2006		2007		2008	
Items		Lower limit a_2	Upper limit b_2	Lower limit a_3	Upper limit b_3	Lower limit a_4	Upper limit b_4	Lower limit a_5	Upper limit b_5
Sale Volumes $k=2,3,4,5$	Original values $\otimes(t_k)\in[a_k,b_k]$	196.3	241.6	345.2	393.7	745.5	796.9	1284.4	1341.2
	Simulation values $\hat{\otimes}(t_k)\in[\hat{a}_k,\hat{b}_k]$	200.54	233.32	365.37	409.39	665.68	718.34	1212.80	1260.44
	Simulation errors $e_a(k)=\lvert a_k-\hat{a}_k\rvert$ $e_b(k)=\lvert b_k-\hat{b}_k\rvert$	4.24	8.28	20.17	15.69	79.82	78.56	71.60	80.76
	Average errors (%) $\Delta_a(k)=e_a(k)/a_k$ $\Delta_b(k)=e_b(k)/b_k$	2.1600	3.4272	5.8430	3.9853	10.7069	9.8582	5.5746	6.0215
	Average precision (%) $r_a(k)=1-\Delta_a(k)$ $r_b(k)=1-\Delta_b(k)$	97.8400	96.5728	94.157	96.0147	89.2931	90.1418	94.4254	93.9785
The average relative error: $\Delta=\frac{1}{4}\sum_{k=2}^{5}\Delta_a(k)+\frac{1}{4}\sum_{k=2}^{5}\Delta_b(k)=5.95\%$									

(3) Difference of predictive values between the two models and the authority data

The authority data of flat-panel TV's sale volume in 2009 is 25 million [17],

It can be seen from Table 4, the new model has higher accuracy than traditional GM (1, 1) model. As a matter of fact, the predictive result of traditional GM (1, 1) model is unreasonable because its lower bound is greater than its upper bound ($\otimes'(t_6) \in [4025.71, 3880.67]$, and $\hat{a}_6' = 4025.71 > \hat{b}_6' = 3880.67$). This violates the definition of interval grey number. Hence, we cannot employ traditional GM (1, 1) model to forecast the flat-panel TV's sale volume. The example demonstrates the validity and practicability of the novel model.

6. Conclusions

In contrast to real numbers, interval grey numbers with triangular whitenization weight functions of central points have more complicated data structure. Thus, it is more difficult to establish prediction models based on interval grey numbers. This paper analyzes the geometric features of interval grey numbers and their whitenization weight functions on a two-dimensional surface, and proposes two novel important concepts (namely, grey number band and grey

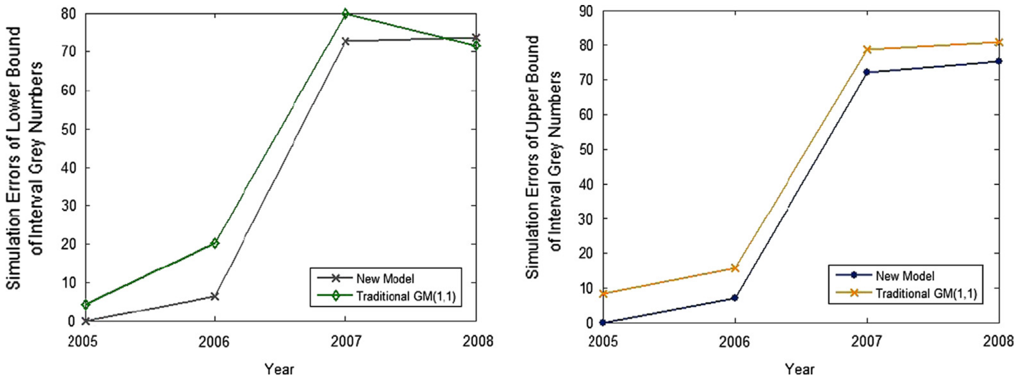

Fig. 11. Comparison of simulation errors between new model and traditional GM(1, 1).

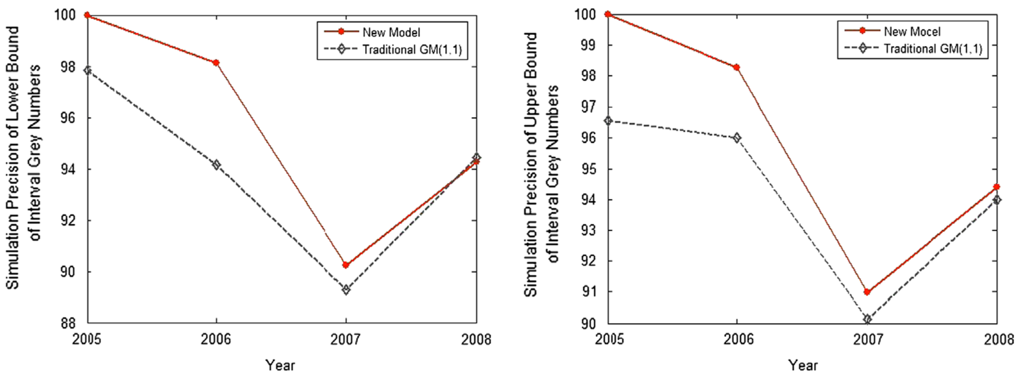

Fig. 12. Simulation precisions' comparison of new model and traditional GM(1, 1).

Table 4
Comparison of predicted values and the prediction accuracy.

Different methods	Predictive value (million)	Authority data (million)	Predictive error	Predictive precision
New model	2296.63	2500	−203.37	91.87%
Traditional GM(1, 1)	3953.19		1453.19	58.13%

number layer). They can convert grey numbers into real ones without any information loss by calculating the areas of grey number layers and the middle point's vertical coordinates of the grey layers' median lines. On the basis of GM (1, 1) model, this paper proposed the simulation and prediction models of lower/upper limits of a grey number, and it solves a complicated prediction problem about interval grey number. Moreover, it has an important contribution of extending the range of grey system theory. In the appendix, we give the programs based on the model so that readers can be convenient to use it.

However, the model presented in this paper also has some limitations. For example the model only considers the simplest and most special whitenization weight function (namely, other triangular whitenization weight functions). It does not take into account a more general situation (viz.). We will effectively solve those questions in our future work.

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