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A geometric approach in the study of traveling waves for some classes of non-monotone reaction–diffusion systems

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Abstract

In this paper we further extend a recently developed method to investigate the existence of traveling waves solutions and their minimum wave speed for non-monotone reaction–diffusion systems. Our approach consists of two steps. First we develop a geometrical shooting argument, with the aid of the theorem of homotopy invariance on the fundamental group, to obtain the positive semi-traveling wave solutions for a large class of reaction–diffusion systems, including the models of predator–prey interaction (for both predator-independent/dependent functional responses), the models of combustion, Belousov–Zhabotinskii reaction, SI-type of disease transmission, and the model of biological flow reactor in chemostat. Next, we apply the results obtained from the first step to some models, such as the Beddinton–DeAngelis model and the model of biolocal flow reactor, to show the convergence of these semi-traveling wave solutions to an interior equilibrium point by the construction of a Lyapunov-type function, or the convergence of semi-traveling waves to another boundary equilibrium point by the further analysis of the asymptotical behavior of semi-traveling wave solutions.

Keywords: Non-monotone reaction–diffusion systems; Traveling wave solutions; Homotopy invariance; Fundamental groups; Minimum wave speed

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1. Introduction

In this paper we attempt to develop a method that is efficient to show the existence of traveling wave solutions for fairly large class of non-monotone reaction–diffusion systems. Consider the interaction of two species, such as predator and prey, the disease transmission among the susceptibles and infectives, the nutrient and bacterial in chemostat, or in the problem of combustion and the chemical reaction, etc., in which one species or substance serves as the supplier and the other acts as a consumer. Let $u(x,t)$ and $v(x,t)$ denote respectively the densities of the supplier and consumer at time t and the location $x \in \mathbb{R}^n$. Suppose that the space variation of densities is the result of a diffusion process. Then the model of such dynamical interaction can be formulated by a system of reaction–diffusion equations as follows.

$$\begin{aligned}\frac{\partial u}{\partial t} &= d_1 \Delta u + F(u,v),\\ \frac{\partial v}{\partial t} &= d_2 \Delta v + G(u,v),\end{aligned} \tag{1.1}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x^i}$ is the Laplace operator, $d_1 > 0$ and $d_2 > 0$ are diffusion coefficients, $F(u,v)$ and $G(u,v)$ are the reaction terms.

Motivated by many models in applications, we assume that the reaction functions F and G are twice continuously differentiable and satisfy the following conditions.

A1 $F(0,v) \geq 0$ for all $v \geq 0$, and there is a positive number K such that

$$\begin{aligned}&0 = F(K,0) > F(K,v) \quad \text{for all } v > 0,\\ &0 > F_v(K,0),\\ &0 \leq F(u,0) \qquad \text{for all } u \in [0,K].\end{aligned}$$

A2 $G(u,0) = 0$ for all $u \geq 0$, $G_v(K,0) > 0$ and there are a strictly positive function $\mu(v)$ for $v \geq 0$ and a positive numbers M such that for all $u \in (0,K)$,

$$-\mu(v)v < G(u,v) < Mv \quad \text{for all } v > 0.$$

Here for a function $H(u,v)$, H_u and H_v denote respectively the partial derivatives of H with respect to the variables u and v.

Let us give a brief interpretation of the assumptions **A1–A2** as follows. The number K in **A1** serves as a carrying capacity or maximum amount of resource provided by a system. **A2** provides the upper and lower bounds of the growth rates of consumers or predator populations, which also has a positive basic grow rate when the resource reaches its maximum amount. **A1–A2** are very mild assumptions that can be satisfied by many models, including:

1. Predator–prey system:

$$F(u,v) = ru(K-u) - f(u,v)v, \qquad G(u,v) = \big[\beta f(u,v) - \mu_0 - \lambda v\big]v,$$

where u and v are respectively the densities of prey and predator species, $ru(K-u)$ is the logistic growth rate of the prey, $f(u,v)$ is the functional response to the predator population, the constant β is a convention rate, and $\mu_0 + \lambda v$ (with $\mu_0 > 0$, $\lambda \geq 0$) is the death rate of predator species.

2. SI-type of disease transmission model with permanent immunity:

$$F(u,v) = -\frac{\beta uv}{u+v}, \qquad G(u,v) = \Big[\frac{\beta u}{u+v} - \mu\Big]v,$$

or model with permanent immunity and logistic growth:

$$\begin{aligned} F(u,v) &= -\frac{\beta uv}{u+v}v + r(u+v)(K-u), \\ G(u,v) &= \Big[\frac{\beta u}{u+v} - \mu\Big]v - r(u+v)v. \end{aligned}$$

3. Model of biological flow reactor in chemostat:

$$F(u,v) = -f(u)v, \qquad G(u,v) = \big[f(u) - \mu\big]v.$$

4. Combustion model:

$$F(u,v) = -h(u,v), \qquad G(u,v) = h(u,v),$$

5. Belousov–Zhabotinskii reaction:

$$F(u,v) = -buv, \qquad G(u,v) = \big[1 - v - ru\big]v.$$

The corresponding reaction system of (1.1) is

$$\begin{aligned} u' &= F(u,v), \\ v' &= G(u,v). \end{aligned} \tag{1.2}$$

The implication of **A2** is

$$G_u(K,0) = 0.$$

Under the assumptions **A1** and **A2** it is apparent that the system (1.2) has a boundary equilibrium $E_K = (K,0)$ which is unstable. The main interests of this paper are: (1) To establish a general result on the existence and to identify the minimum wave speed of *semi-traveling wave* solutions connected to the unstable boundary equilibrium E_K; (2) To apply the result obtained in (1) to show the existence of wave fronts connecting the boundary equilibrium E_K and an interior (positive) equilibrium E_* if it exists, or another boundary equilibrium points for models that satisfy some additional conditions. To be specific, we consider a solution $(u(x,t), v(x,t))$ of (1.1) taking the form

$$u(x,t) = U(x\cdot\nu + ct), \qquad v(x,t) = V(x\cdot\nu + ct) \tag{1.3}$$

where $\nu \in \mathbb{R}^n$ is a unit vector denoting the direction of wave propagation, "$\cdot$" is the inner product, and c is the wave speed. The solution (1.3) is called a *semi-traveling wave* solution connected to the unstable equilibrium E_K if $U(\xi)$ and $V(\xi)$ are defined for all $\xi \in \mathbb{R}$ and are strictly positive functions satisfying the limiting condition

$$\lim_{\xi \to -\infty} (U(\xi), V(\xi)) = E_K = (K, 0). \tag{1.4}$$

We note that, under the very general assumption **A1** and **A2**, there is no guarantee of the existence of positive (interior) equilibrium point of the reaction system (1.2). Even there does exist such a positive equilibrium point, it may not be stable. Hence in general we may not expect the existence of a wave front of (1.1), a solution $(U(\xi), V(\xi))$ which converges, as $\xi \to \infty$, to a positive equilibrium. However, it is clear that the confirmation of existence of a semi-traveling wave is an important first step towards the proof of existence of a wave front connecting E_K and an interior equilibrium under addition assumption. Moreover, in the case that (1.2) does not have the interior equilibrium, such as the model of nutrient and bacteria bioreactor, combustion, we can show that the existence of a semi-traveling wave actually gives rise to a wave front that converges to another boundary equilibrium as $\xi \to \infty$.

The existence of traveling wave solutions has been a subject of intensive study for last few decades and there have been numerous research papers published in literature because of its important application to many areas of sciences. In addition, the minimum wave speed has a possible link to the population spreading speed. It is conjectured that the minimum wave speed is identical to the population spreading speed, which is proved to be true for some ecological models [2,15,16,18,24]. A standard approach, such as the monotone iteration or comparison argument, has been established that is very efficient to deal with the traveling wave solutions for the monotone systems. Nevertheless there is still lack of efficient methods in the studies of traveling wave solutions for many important but non-monotone systems. For instance, there is a striking contrast between the research done on the predator–prey models and on the competition models and cooperative models in ecology. The existence of traveling wave solutions for competition models or cooperative models now has been well understood because these models are monotone systems. However, the research on the existence of traveling waves for predator–prey systems is very limited. To date, the earliest research done in this direction was the pioneer work [4,5] by S. Dunbar published in 1983 and 1984. In [4,5] Dunbar gave a complete analysis of the existence of traveling wave solutions (connecting the boundary equilibrium $(K, 0)$ and a stable interior equilibrium $E_* = (u_*, v_*)$ for the classical Lotka–Volterra predator–prey model

$$\begin{aligned} \frac{\partial}{\partial t} u &= d_1 \Delta u + ru(1 - \frac{u}{K}) - uv, \\ \frac{\partial}{\partial t} v &= d_2 \Delta v + [\beta u - \mu] v. \end{aligned} \tag{1.5}$$

It is quite surprising, however, that there was no any extension of Dunbar's work to more general predator–prey models for almost twenty years after the publication of Dunbar's papers, except a few papers [6–8,21] in which singular perturbation method or Conley index theory were applied to some singularly perturbed or abstract predator–prey systems, or a bifurcation approach with the application of Implicit function theorem for sufficiently large wave speed. The singular perturbation or index theory certainly are power tools in the studies of traveling wave solutions. However, many concrete models and relevant model parameters may not fit well with the frame of these abstract approaches. Moreover, these approach may not be able to provide an explicit expression of the minimum wave speed that is important in the point view of application.

Dunbar's method consists of two parts: (1) To construct a Wezarwisk's set, and then with the application of the Wezarwisk invariance principle to catch a global solution inside this Wezarwisk's set. This global solution will serve as a candidate for a traveling wave solution;

(2) To confirm the convergence of this global solution to the interior equilibrium by constructing a Lyapunov function, together with the application of LaSalle's invariance principle. However, the main difficulty of Dunbar's method is the construction of a Wezarwisk's set and the use of Wezarwisk's invariance principle that are far from trivial.

There has been some progress occurred since 2004 on the existence of traveling wave solutions for predator–prey models [9,11,10,17,19] in which the functional response considered depends on only the prey population.

In this paper we shall further extend a geometric method developed in [11] (for the case of $d_1 = 0$), to obtain a global (a semi-traveling wave) solution by constructing a geometrically much simpler set without the use of Wezarwisk's invariance principle. This approach enable us to handle more general reaction–diffusion system under just very mild condition **A1** and **A2**, even without the restriction of $d_1 < d_2$, a condition imposed in Dunbar's paper [5]. One of the main results of this paper can be stated as follows:

Theorem 1.1. *Under the assumptions* **A1** *and* **A2**, *for each* $c \geq 2\sqrt{d_2 M}$, *the system* (1.1) *has a positive semi-traveling wave solution of the form* (1.3) *satisfying the boundary condition* (1.4). *If, in addition,* $G_v(K,0) = M$, *then the condition* $c \geq 2\sqrt{d_2 M}$ *is also necessary, and hence*

$$c_* = 2\sqrt{d_2 G_v(K,0)}$$

is the minimum wave speed.

This paper is organized as follows. Section 2 provides a few basic results that lead to the construction a geometrically simple set. Section 3 presents a key information on the unstable manifold of the boundary equilibrium E_K that will be used in Section 4, with the aid of the theorem of homotopy invariance on the fundamental group to show the existence of semi-traveling wave solutions connected to the boundary equilibrium E_K. In Section 5, we study the convergence of the semi-traveling wave solutions to an interior equilibrium E_* or another boundary equilibrium point for some concrete models. In particular, we have developed a technique of constructing a "Liapunov" function (to be more accurate, a function which is not that positively definite but has a negative derivative in a restricted region) that can be applied to those predator–prey models (such as the Beddington–DeAngelis model) where the functional responses depend on both prey and predator species.

2. Preliminaries

By a scaling if necessary we can let $d_1 = d$ and $d_2 = 1$ in (1.1). Throughout of this paper we suppose that $d_1 = d > 0$. For the case of $d_1 = 0$ we refer the readers to the paper [11]. Upon substituting

$$u(x,t) = U(x \cdot \nu + ct), \qquad v(x,t) = V(x \cdot \nu + ct)$$

into the system (1.1) we obtain the system for $(U(\xi), V(\xi))$ with $\xi = x \cdot \nu + ct$ as

$$\begin{aligned} cU' &= dU'' + F(U,V), \\ cV' &= V'' + G(U,V), \end{aligned} \tag{2.1}$$

where $'$ and $''$ denote the first and second order derivatives, respectively. We look for a positive solution of (2.1) satisfying the limiting boundary condition

$$\lim_{\xi \to -\infty} (U(\xi), V(\xi)) = E_K = (K, 0). \tag{2.2}$$

Instead of using a standard transformation, for a constant $c > 0$ we make the following changes of variables and scaling:

$$\begin{aligned} X(t) &= U(ct), \\ Y(t) &= \frac{1}{c}\big[cU(ct) - dU'(ct)\big], \\ W(t) &= V(ct), \\ Z(t) &= \frac{1}{c}\big[cV(ct) - V'(ct)\big]. \end{aligned} \tag{2.3}$$

Then, upon a straightforward computation, (2.1) is transformed to a 4-dimensional system of first order equations

$$\begin{aligned} X' &= \rho\big[X - Y\big], \qquad \rho = \frac{c^2}{d} \\ Y' &= F(X, W), \\ W' &= c^2\big[W - Z\big], \\ Z' &= G(X, W). \end{aligned} \tag{2.4}$$

Recall by Assumption **A2** that for $0 < X < K$ and $W > 0$,

$$-\mu(W)W < G(X, W) < MW. \tag{2.5}$$

Our first step is to use the inequality (2.5) to make the comparison of the vector field of the system (2.4) with the following planar system

$$W' = c^2(W - Z), \qquad Z' = -\mu(W)W. \tag{2.6}$$

Let us first state the following lemma, its proof can be found in [3, Lemma 2.1].

Lemma 2.1. *The system* (2.6) *has a strictly monotone decreasing solution* $(W_1(t), Z_1(t))$ *defined for* $t \in \mathbb{R}$ *satisfying that* $Z_1(t) > W_1(t) > 0$ *and (see Fig. 1A)*

$$\begin{aligned} &W_1(t) \searrow 0, \qquad Z_1(t) \searrow 0 \qquad \text{as } t \to \infty, \\ &W_1(t) \nearrow \infty, \qquad Z_1(t) \nearrow \infty \qquad \text{as } t \to -\infty. \end{aligned} \tag{2.7}$$

Let $W_1^{-1} : (0, \infty) \to (0, \infty)$ be the inverse of W_1 and define $\sigma : [0, \infty) \to [0, \infty)$ by

$$\sigma(0) = 0, \qquad \sigma(W) = Z_1(W_1^{-1}(W)) \quad \text{for } W > 0, \tag{2.8}$$

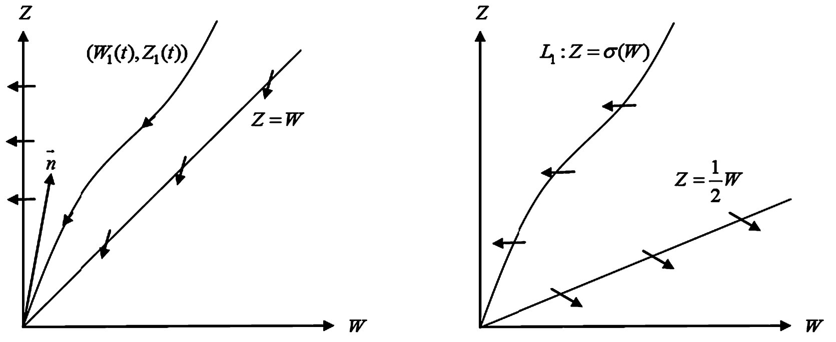

Fig. 1.A *Fig. 1.B*

Fig. 1. A: The orbit $\{(W_1(t), Z_1(t)) : t \in \mathbb{R}\}$ and the vector field of the system (2.6) at the lines $Z = W$ and $W = 0$. $\vec{\eta}$ is an eigenvector of the system (2.6) associated with the equilibrium $(0, 0)$. B: The vector field of the system (2.4) at the surface $Z = \sigma(W)$ [the graph of the orbit $(W_1(t), Z_1(t))$] and at the plane $Z = W/2$ points outside the region bounded by the surface $Z = \sigma(W)$ and at the plane $Z = W/2$.

and let graph of the function $\sigma(W)$ is denoted by L_1 (see Fig. 1B). It is obvious that the graph of the function $\sigma(W)$ for $W > 0$ is the orbit $\{(W_1(t), Z_1(t)) : t \in \mathbb{R}\}$. Hence $\sigma(W) > W$ for $W > 0$ and

$$\frac{d\sigma(W)}{dW} = -\frac{\mu(W)W}{c^2[W - \sigma(W)]}. \tag{2.9}$$

In addition, $\sigma(W)$ is monotone increasing, $\sigma(W) \to \infty$ as $W \to \infty$ and $\sigma(W) \to 0$ as $W \to 0$.

$$\sigma'(0) = \lim_{W \to 0} \frac{\sigma(W)}{W} = \sigma_1 = -\frac{\mu_0}{\xi_0} = \frac{c^2 + \sqrt{c^4 + 4c^2\mu_0}}{2c^2} > 1, \tag{2.10}$$

where $\mu_0 = \mu(0)$ and

$$\xi_0 = \frac{c^2 - \sqrt{c^4 + 4c^2\mu_0}}{2}$$

is an eigenvalue of the linearization of system (2.6) at $W = Z = 0$.

Now we define a wedged region $\Sigma \in \mathbb{R}^4$ as follows (see Fig. 2A).

$$\Sigma = \left\{(X, Y, W, Z) : 0 \le X \le K,\ Y \in \mathbb{R},\ W \ge 0,\ \frac{1}{2}W \le Z \le \sigma(W)\right\}. \tag{2.11}$$

Then the boundary of Σ consists of surfaces P_1–P_4, Q_3–Q_5 with

$$P_1 = \left\{0 < X < K,\ Y \in \mathbb{R},\ \sigma(W) = Z,\ W > 0\right\},$$
$$P_2 = \left\{0 < X < K,\ Y \in \mathbb{R},\ Z = \frac{1}{2}W,\ W > 0\right\},$$
$$P_3 = \left\{X = K,\ Y < K,\ 0 < \frac{1}{2}W \le Z \le \sigma(W)\right\},$$

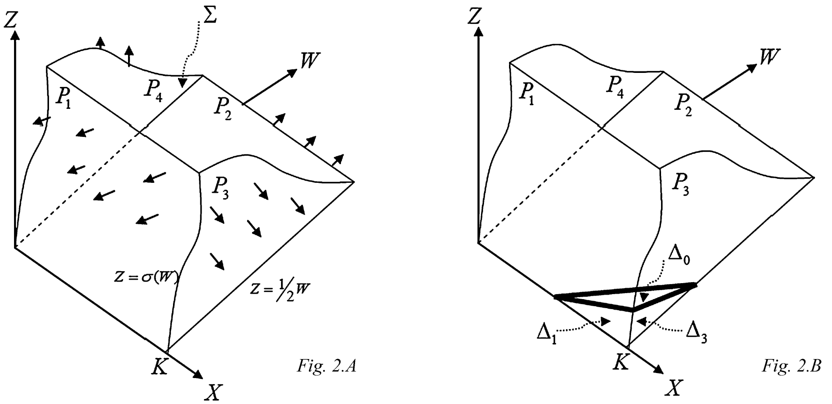

Fig. 2. A: The set Σ and the vector field of the system (2.4) at the boundaries P_1–P_4. B: Part of the unstable manifold E_K^U of the equilibrium E_K in the set Σ bounded in the sides by Δ_1–Δ_3 and at the top by Δ_0. Both A and B are visualization on the 3-dimensional X–W–Z space with the omission of Y coordinate.

$$
\begin{aligned}
Q_3 &= \left\{X = K,\ Y \geq K,\ 0 < \frac{1}{2}W \leq Z \leq \sigma(W)\right\},\\
P_4 &= \left\{X = 0,\ 0 < Y,\ 0 < \frac{1}{2}W \leq Z \leq \sigma(W)\right\},\\
Q_4 &= \left\{X = 0,\ Y \leq 0,\ 0 < \frac{1}{2}W \leq Z \leq \sigma(W)\right\},\\
Q_5 &= \left\{0 \leq X \leq K,\ 0 = W = Z\right\}.
\end{aligned}
\tag{2.12}
$$

The vector filed of (2.4) has a simple property in the boundary of Σ, which can be characterized by the following two lemmas.

Lemma 2.2. *Let $\Phi_t(p)$ be the flow of (2.4), i.e. $\Phi_t(p)$ is a solution of (2.4) satisfying the initial condition $\Phi_0(p) = p \in \mathbb{R}^4$. Then for any $p \in \mathrm{Int}(\Sigma)$ (the interior of Σ), $\Phi_t(p)$ cannot leave Σ from a point in the boundary $Q_3 \cup Q_4 \cup Q_5$ of Σ at any positive time.*

Proof. First we show that for $p \in \mathrm{Int}(\Sigma)$, $\Phi_t(p)$ cannot leave Σ from the plane Q_5. If this is not the case, then there would be a time $t_1 > 0$ such that $\Phi_t(p) = (X(t), Y(t), W(t), Z(t)) \in \mathrm{Int}(\Sigma)$ for $t \in [0, t_1)$ and $\Phi_{t_1}(p) \in Q_5$. Recall that Assumption **A2** implies that $G(X, 0) = 0$ for $0 \leq X \leq K$. From the third and fourth equations of the system (2.4) one immediately deduces that $W(t) = Z(t) = 0$ for all $t \in [0, t_1]$, which leads to a contradiction. Hence $\Phi_t(p)$ cannot leave Σ from the plane Q_5.

Next suppose $p_0 = (X_0, Y_0, W_0, Z_0) \in Q_3$. Then $X_0 = K \leq Y_0$ and $W_0 > 0$. If $Y_0 > K$, then the first equation of (2.4) yields that at p,

$$X' = \rho[X_0 - Y_0] < 0.$$

So that the vector field of (2.4) points interior of Σ and hence $\Phi_t(p)$ cannot exit Σ at p. If $Y_0 = K = X_0$. Then by first two equations of (2.4) and Assumption **A1** we have $X' = 0$ and $Y' = F(K, W_0) < 0$. It follows that

$$X'' = \rho[X' - Y'] = -\rho F(K, W_0) > 0.$$

Now suppose $(X(t), Y(t), W(t), Z(t))$ is a solution of (2.4) with $(X(t_0), Y(t_0), W(t_0), Z(t_0)) = p_0$. Then $X(t)$ satisfies that

$$X'(t_0) = 0, \qquad X''(t_0) > 0.$$

This implies that $X(t)$ has a strict local minimum K in a small neighborhood of t_0. That is, a solution of (2.4) can only reach the point p_0 from out side of Σ, equivalently, the flow $\Phi_t(p)$ cannot exit Σ at p_0 from $\mathrm{Int}(\Sigma)$. Finally to show that $\Phi_t(p)$ cannot exits Σ, at a positive time, from a point in Q_4, let us first consider a system $(2.4)_\epsilon$ obtained from (2.4) by replacing $F(X, W)$ with $F(X, W) + \epsilon$, where ϵ is a positive constant. By Assumption **A1**, $F(0, W) + \epsilon > 0$ for all $W \geq 0$. Then with the use of the same argument for the set Q_3 one concludes that for any $\epsilon > 0$, a solution to the system $(2.4)_\epsilon$ cannot exits Σ, at a positive time, from a point in Q_4. Hence by the continuity of solutions on ϵ and by letting $\epsilon \to 0$ we deduce that $\Phi_t(p)$ cannot exits Σ from a point in Q_4 at a positive time. □

Next let us study the vector field at the sets $P_1, \cdots, P_4$.

Lemma 2.3. *Let $c \geq 2\sqrt{M}$, where c is the wave speed in the system (2.4) and M is the positive number given in the assumption* **A2**. *Then the vector field of (2.4) at any point in $\bigcup_{i=1}^4 P_i$ points outside of Σ. That is, if $\Phi_t(p)$ with $p \in \mathrm{Int}(\Sigma)$ does not stay in Σ for all $t \geq 0$, then it can only leave Σ from a point in $\bigcup_{i=1}^4 P_i$ at some positive time.*

Proof. First it is apparent that

$$X' = \rho[K - Y] > 0 \quad \text{for } (X, Y, W, Z) \in P_3$$

and

$$X' = \rho[0 - Y] < 0 \quad \text{for } (X, Y, W, Z) \in P_4.$$

Hence the vector field of (2.4) points outside of Σ at $P_3 \cup P_4$. Next consider a point $p_1 = (X_1, Y_1, W_1, Z_1) \in P_1$. Then $Z_1 = \sigma(W_1) > W_1$ and $0 < X_1 < K$. By the inequality (2.5) and the third and fourth equations of (2.4) and we obtain

$$\begin{aligned}
W' &= c^2[W_1 - Z_1] = c^2[W_1 - \sigma(W_1)] < 0, \\
\frac{Z'}{W'} &= \frac{G(X_1, W_1)}{c^2[W_1 - \sigma(W_1)]} = -\frac{G(X_1, W_1)}{c^2[\sigma(W_1) - W_1]} \\
&< \frac{\mu(W_1)W_1}{c^2[\sigma(W_1) - W_1]} = \frac{d\sigma(W_1)}{dW}.
\end{aligned} \tag{2.13}$$

(2.13) implies that the vector field at the point $p_1 \in P_1$ points outside of Σ (see Fig. 1B and Fig. 2A). Finally, let $p_2 = (X_2, Y_2, W_2, Z_2) \in P_2$. Then $0 < X_2 < K$ and $Z_2 = \frac{1}{2}W_2$. By the assumption **A2** and the inequality $c \geq 2\sqrt{M}$ we deduce that

$$
\begin{aligned}
&W' = c^2(W_2 - Z_2) = c^2[1 - \frac{1}{2}]W_2 = c^2\frac{W_2}{2} > 0,\\
&\frac{Z'}{W'} = \frac{G(X_2, W_2)}{c^2[W_2 - Z_2]} < \frac{2MW_2}{c^2W_2} \le \frac{1}{2}.
\end{aligned}
\tag{2.14}
$$

It follows from (2.14) that the vector field at the point $p_2 \in P_2$ points exterior of Σ (see Fig. 1B and Fig. 2A). □

Remark 2.1. The construction of the set Σ was first introduced in [11] for the predator–prey systems and then be applied to more general systems in [3] under the condition that the diffusion coefficient $d = d_1 = 0$. In the case of $d = 0$, the system corresponding to the traveling wave system (2.1) is 3-dimensional, which results in a simpler structure of the vector field at the boundary of Σ, and which results in a 2-dimensional local unstable manifold E_K^U of the boundary equilibrium E_K. Thus a shooting argument is more manageable. When $d > 0$, the vector field at the boundaries $X = 0$ and $X = K$ of Σ becomes more complicated and, in particular, the local unstable manifold E_K^U is 3-dimensional. The shooting argument used for $d = 0$ is no longer available, which motivate us to apply the homotopy invariance on the fundamental group to handle the existence of semi-traveling wave solutions.

3. Unstable manifold of the equilibrium E_K

Now we turn to study the unstable manifold of the equilibrium $E_K = (K, K, 0, 0)$ with respect to the system (2.4) [here for the convenience we use the same notation E_K to denote the equilibrium $(K, K, 0, 0)$ of (2.4)]. The linearization of (2.4) at E_K is

$$
\begin{aligned}
&\dot{X} = \rho[X - Y],\\
&\dot{Y} = F_u(K, 0)X + F_v(K, 0)W,\\
&\dot{W} = c^2[W - Z],\\
&\dot{Z} = G_v(K, 0)W.
\end{aligned}
\tag{3.1}
$$

The assumptions **A1** and **A2** imply that

$$
F_u(K, 0) \le 0, \qquad 0 < G_v(K, 0) \le M.
$$

Let

$$
c \ge 2\sqrt{M}.
$$

Then we have $c^2 \ge 4G_v(K, 0)$. Upon a direct computation one is able to verify that the linear system (3.1) has eigenvalues

$$
\begin{aligned}
&\lambda_1 = \frac{c^2 + c\sqrt{c^2 - 4G_v(K, 0)}}{2} > 0,\\
&\lambda_2 = \frac{c^2 - c\sqrt{c^2 - 4G_v(K, 0)}}{2} > 0,
\end{aligned}
$$

$$\lambda_3 = \frac{\rho + \sqrt{\rho^2 - 4\rho F_u(K,0)}}{2} > 0,$$
$$\lambda_4 = \frac{\rho - \sqrt{\rho^2 - 4\rho F_u(K,0)}}{2} \le 0. \tag{3.2}$$

Let $\mathbf{h}_i$ be the eigenvector (or generalized eigenvector) associated with the positive eigenvalues λ_i for $i = 1, 2, 3$. From the unstable manifold theorem [22, p. 107] it follows that the local unstable manifold E_K^U of E_K is tangent to the space spanned by the eigenvectors $\mathbf{h}_1$–$\mathbf{h}_3$. That is, there are a small neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^3$ and a smooth (twice continuously differentiable) function $\mathcal{M} = (\mathcal{M}_1, \cdots, \mathcal{M}_4) : \mathcal{O} \to \mathbb{R}^4$ such that the local unstable manifold E_K^U can be expressed as

$$E_K^U = \Big\{k_1\mathbf{h}_1 + k_2\mathbf{h}_2 + k_3\mathbf{h}_3 + \mathcal{M}(k_1, k_2, k_3) + E_K : (k_1, k_2, k_3) \in \mathcal{O}\Big\}.$$

With the application of Implicit Function Theorem we can obtain the following theorem. A complete proof of this theorem will be given in Appendix A.

Theorem 3.1. *For each fixed $c \ge 2\sqrt{M}$, there exist continuous functions $k_1, k_2, k_3 : [0, 1] \times [0, 1] \to \mathcal{O}$ such that for*

$$p(s,\theta) = \sum_{i=1}^{3} k_i(s,\theta)\mathbf{h}_i + \mathcal{M}\big(k(s,\theta)\big) + E_K \in E_K^U \tag{3.3}$$

with $(s,\theta) \in [0, 1] \times [0, 1]$, where $k(s,\theta) = (k_1(s,\theta), k_2(s,\theta), k_3(s,\theta))$, we have

$$\begin{aligned}
\Delta_0 &= \big\{p(s,\theta) : (s,\theta) \in (0, 1) \times (0, 1)\big\} \subset \mathrm{Int}(\Sigma),\\
\Delta_1 &= \big\{p(s,0) : s \in (0, 1]\big\} \subset P_1,\\
\Delta_2 &= \big\{p(s,1) : s \in (0, 1]\big\} \subset P_2,\\
\Delta_3 &= \big\{p(1,\theta) : \theta \in [0, 1]\big\} \subset P_3,
\end{aligned}$$
$$p(0,\theta) = p_0 = (X_0, Y_0, 0, 0) \quad \textit{for all } \theta \in [0, 1] \textit{ with } 0 < X_0 < K,\ X_0 < Y_0.$$

In addition, for any $p \in \Delta_0$, $\Phi_t(p) \in \Sigma$ for all $t \le 0$ and $\Phi_t(p) \to E_K$ as $t \to -\infty$, where $\Phi_t(p)$ is the flow of (2.4) (see the sets Δ_0–Δ_3 in Fig. 2B).

4. Existence of a positive semi-traveling wave

In this section we show, under the assumption

$$c \ge 2\sqrt{M} \tag{H}$$

and with the aid of Theorem 3.1 and the application of homotopy invariance of the fundamental group, the existence of a positive semi-traveling wave solution $\Phi_t(p) \in \Sigma$ of the system (2.4).

Let us begin with the following lemma.

Lemma 4.1. *Let $p_0 = (X_0, Y_0, 0, 0)$ with $0 < X_0 < K$ and $X_0 < Y_0$. Then there is a neighborhood $\mathcal{V}$ of p_0 such that for each $p \in \mathcal{V}$, the flow $\Phi_t(p)$ of (2.4) must exit the region $\{(X, Y, W, Z) : 0 \leq X < K\}$ from the boundary $\{X = 0\}$ at some positive time. That is, there is a time $t_p > 0$ and $\epsilon > 0$ such that $\Phi_t(p) = (X(t), Y(t), W(t), Z(t)) \in \{(X, Y, W, Z) : 0 \leq X < K\}$ for all $t \in [0, t_p)$, $X(t_p) = 0$, and $X(t) < 0$ for $t_p < t \leq t_p + \epsilon$.*

Proof. Let $\Phi_t(p_0) = (X_0(t), Y_0(t), W_0(t), Z_0(t))$. Then $W_0(0) = Z_0(0) = 0$ implies that $W_0(t) = Z_0(t) \equiv 0$, for the plane $\{W = Z = 0\}$ is invariant to the system (2.4). Hence $(X_0(t), Y_0(t))$ is a solution of the planar system

$$X' = \rho[X - Y], \qquad Y' = F(X, 0). \tag{4.1}$$

Note that $F(X, 0) \geq 0$ for $X \in [0, K)$. Thus it is obvious that, by the analysis of the vector field of (4.1), $X_0(t)$ is strictly decreasing and $(X_0(t), Y_0(t))$ will exits the strip region $\{(X, Y) : 0 \leq X < K\}$ from the left side $\{X = 0\}$ at a finite positive time. Then the lemma is a direct consequence of the continuity of solutions on the initial value. □

Corollary 4.2. *Let $c \geq 2\sqrt{M}$ and let $p(s, \theta)$ be defined in Theorem 3.1. If $s_0 > 0$ is sufficiently small, then for each $\theta \in [0, 1]$, $\Phi_t(p(s_0, \theta))$ must exit Σ at the boundary $P_1 \cup P_2 \cup P_4$ as t increases.*

Proof. By Theorem 3.1 $p(0, \theta) = p_0 = (X_0, Y_0, 0, 0)$ with $0 < X_0 < \min\{K, Y_0\}$ for all $\theta \in [0, 1]$. Let $\mathcal{V} \subset \mathbb{R}^4$ be a small neighborhood $p(0, \theta)$ defined in Lemma 4.1. Note that $p(s, \theta)$ is continuous with respect to (s, θ). Hence $p(s_0, \theta) \in \mathcal{V}$ for all $\theta \in [0, 1]$. The corollary therefore follows from Lemma 4.1. □

Now we are in position to establish one of our main theorem stated below.

Theorem 4.3. *For each $c \geq 2\sqrt{M}$, there is a point $p_* \in E_K^U$ such that $\Phi_t(p_*) \in \mathrm{Int}(\Sigma)$ for all $t \in \mathbb{R}$ and $\Phi_t(p_*) \to E_K$ as $t \to -\infty$.*

Proof. Let Δ_0 be the set defined in Theorem 3.1. We claim that there must be a $p_* \in \Delta_0$ such that $\Phi_t(p_*) \in \mathrm{Int}(\Sigma)$ for all $t \geq 0$. Suppose on contrary that the claim is false. Let $s_0 > 0$ be a sufficiently small number and let

$$\Delta^{s_0} = \left\{p(s, \theta) : (s, \theta) \in [s_0, 1] \times [0, 1]\right\} \subset \Delta_0.$$

Then Lemma 2.3 implies that for each $p(s, \theta) \in \Delta^{s_0}$, there is a first time $t(s, \theta) \geq 0$ such that

$$\Phi_{t(s,\theta)}\big(p(s, \theta)\big) \in \bigcup_{i=1}^{4} P_4. \tag{4.2}$$

Let $\Phi_t\big(p(s, \theta)\big) = \big(X(t, p(s, \theta)), Y(t, p(s, \theta)), W(t, p(s, \theta)), Z(t, p(s, \theta))\big)$ and let

$$\tilde{\Phi}(s,\theta) = \big(X(t(s,p),p(s,\theta)),W(t(s,\theta),p(s,\theta)),Z(t(s,\theta),p(s,\theta))\big),$$
$$(s,\theta)\in[s_0,1]\times[0,1].$$

Moreover, let

$$\begin{aligned}
\tilde{P}_1 &= \Big\{0<X<K,\ \sigma(W)=Z,\ W>0\Big\},\\
\tilde{P}_2 &= \Big\{0<X<K,\ Z=\frac{1}{2}W,\ W>0\Big\},\\
\tilde{P}_3 &= \Big\{X=K,\ 0<\frac{1}{2}W\le Z\le\sigma(W)\Big\},\\
\tilde{P}_4 &= \Big\{X=0,\ \ 0<\frac{1}{2}W\le Z\le\sigma(W)\Big\}.
\end{aligned}\tag{4.3}$$

It is clear that $\tilde{P}_i$ is the projection of P_i onto the X–W–Z space and the set

$$\mathcal{C}=\cup_{i=1}^4\tilde{P}_i\subset\mathbb{R}^3$$

is homotopy to a cylinder in $\mathbb{R}^3$ with the side surfaces of $\mathcal{C}$ in the order $\tilde{P}_1$–$\tilde{P}_3$–$\tilde{P}_2$–$\tilde{P}_4$–$\tilde{P}_1$. By the definitions of $\tilde{\Phi}$, $\tilde{P}_i$ and (4.2), it is obvious that

$$\tilde{\Phi}:[s_0,1]\times[0,1]\to\mathcal{C}=\bigcup_{i=1}^4\tilde{P}_i$$

is a continuous function. Now we define a family of functions $\{\varphi_\epsilon:[0,1]\to\mathcal{C},\ \epsilon\in(0,1]\}$ by

$$\varphi_\epsilon(\xi)=\begin{cases}\tilde{\Phi}(1,\,4\epsilon\xi), & \text{for } 0\le\xi<\frac{1}{4},\\ \tilde{\Phi}\Big(1-\epsilon(1-s_0)(4\xi-1),\,\epsilon\Big), & \text{for } \frac{1}{4}\le\xi<\frac{1}{2},\\ \tilde{\Phi}\Big(1-\epsilon(1-s_0),\,3\epsilon-4\epsilon\xi\Big), & \text{for } \frac{1}{2}\le\xi<\frac{3}{4},\\ \tilde{\Phi}\Big(1-4\epsilon(1-s_0)(1-\xi),\,0\Big), & \text{for } \frac{3}{4}\le\xi\le 1.\end{cases}\tag{4.4}$$

By the continuity of $\tilde{\Phi}(s,\theta)$ one easily concludes that, with regarding the fundamental group on $\mathcal{C}$, the function φ_ϵ is homotopy to the function φ_1 and hence φ_ϵ and φ_1 are in the equivalent class for all $\epsilon\in[0,1)$ (see [14]). Now Theorem 3.1 and the definition of $t(s,\theta)$ imply that

$$\Phi_{t(1,\theta)}(p(1,\theta))=p(1,\theta)\in P_3,\ \ \theta\in[0,1].$$

It follows that $\tilde{\Phi}(1,\theta)\in\tilde{P}_3$ for $\theta\in[0,1]$. Hence

$$\varphi_1\Big(\big[0,\frac{1}{4}\big]\Big)=\big\{\tilde{\Phi}(1,\theta):\ \theta\in[0,1]\big\}\subset\tilde{P}_3.\tag{4.5}$$

Similarly, by Theorem 3.1, Corollary 4.2 and the definition of $\tilde{\Phi}$ we deduce that

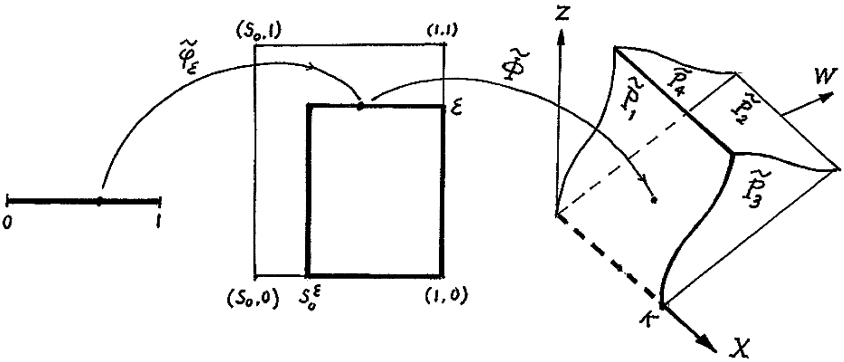

Fig. 3. $s_0^\epsilon = 1 - \epsilon(1 - s_0)$. For $0 \le \epsilon \le 1$, the function $\tilde{\varphi}_\epsilon$ maps the unit interval [0, 1] into the sides of a rectangle with the vertexes $(1, 0)$, $(1, \epsilon)$, (s_0^ϵ, ϵ) and $(s_0^\epsilon, 0)$. The mapping $\varphi_\epsilon : [0, 1] \to \bigcup_{i=1}^4 \tilde{P}_i$ is the composition of $\tilde{\Phi}$ and $\tilde{\varphi}_\epsilon$.

$$\varphi_1\Big(\big(\frac{1}{4}, \frac{1}{2}\big]\Big) = \big\{\tilde{\Phi}(s, 1) : \ s \in [s_0, 1]\big\} \subset \tilde{P}_2$$
$$\varphi_1\Big(\big(\frac{1}{2}, \frac{3}{4}\big]\Big) = \big\{\tilde{\Phi}(s_0, \theta) : \ \theta \in [0, 1]\big\} \subset \tilde{P}_2 \cup \tilde{P}_4 \cup \tilde{P}_1$$
$$\varphi_1\Big(\big(\frac{3}{4}, 1\big)\Big) = \big\{\tilde{\Phi}(s_0, 0) : \ s \in [s_0, 1)\big\} \subset \tilde{P}_1. \tag{4.6}$$

Noticing that $\tilde{P}_3 \cap \tilde{P}_4 = \emptyset$, $\tilde{P}_1 \cap \tilde{P}_2 = \emptyset$ (see Fig. 3), we therefore conclude from (4.5) and (4.6) that φ_1 maps the interval [0, 1] into $\mathcal{C}$ a one loop curve. On the other hand φ_0 maps [0, 1] into a single $\varphi_1(0) = \tilde{\Phi}(1, 0) = p(1, 0) \in \tilde{P}_3$. Hence φ_0 is not in the equivalent class of φ_1. This leads to a contradiction, which confirms the existence of $p_* \in \Delta_0 \in \operatorname{Int}\Sigma$ with $\Phi_t(p_*) \in \Sigma$ for all $t \ge 0$. The convergence of $\Phi_t(p_*)$ to E_K as $t \to -\infty$ follows from Theorem 3.1. $\square$

Next we consider the non-existence of nonnegative, nontrivial semi-traveling wave solution of the system (1.1) connected to E_K. Here a traveling wave is nontrivial if the function $V(t)$ in the system (2.1) is not identical to zero.

Lemma 4.4. *For all $0 < c < 2\sqrt{G_v(K, 0)}$, the system (1.1) does not have a nonnegative, nontrivial semi-traveling wave solution of the system (1.1) connected to E_K.*

Proof. Noticing that the plane $\{(X, Y, 0, 0)\}$ is invariant set of the system (2.4), to proof the lemma, it is sufficient to show that if $(X(t), Y(t), W(t), Z(t))$ is a solution of (2.4) that converges to E_K as $t \to -\infty$ and $(W(t), Z(t)) \ne (0, 0)$ for and $t \in \mathbb{R}$, then $W(t)$, which corresponds to the V-component of the system (2.1), must change sign as $t \to -\infty$. In view of the third and fourth equations in the system (2.4) and with the use of Assumption **A2** we can rewrite the equations for $(W(t), Z(t))^T = N(t)$ as

$$N'(t) = AN(t) + B(t)N(t), \tag{4.7}$$

where

$$A = \begin{bmatrix} c^2 & -c^2 \\ G_v(K, 0) & 0 \end{bmatrix}, \qquad B(t) = \begin{bmatrix} 0 & 0 \\ b(t) & 0 \end{bmatrix} \tag{4.8}$$

with

$$b(t) = \int_0^1 \left[G_v\big(X(t), \theta W(t)\big) - G_v(K, 0)\right] d\theta. \tag{4.9}$$

Notice that $X(t) \to K$ and $W(t) \to 0$ exponentially as $t \to -\infty$, and hence $b(t) \to 0$ exponentially as $t \to -\infty$. By the variation-of-parameters formula, for any $\xi \geq t$, we can express the function $N(t)$ as

$$N(t) = e^{A(t-\xi)} \left[N(\xi) - \int_t^{\xi} e^{-A(s-\xi)} B(s) N(s) ds \right]. \tag{4.10}$$

Now $c < 2\sqrt{G_v(K,0)}$ yields that the matrix A has a pair of complex eigenvalues $\lambda = \alpha \pm i\beta$ with

$$\alpha = \frac{c^2}{2} > 0, \qquad \beta = 4c^2 G_v(K,0) - c^4 > 0.$$

It follows that there is a 2×2 real nonsingular matrix T such that

$$e^{At} = e^{\alpha t} R(t), \quad \text{with} \quad R(t) = T \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} T^{-1}. \tag{4.11}$$

Let $\|N\|$ and $\|H\|$ denote respectively the norm of $N \in \mathbb{R}^2$ and the operator norm of a matrix $H \in \mathbb{R}^{2\times 2}$. Then it is clear that there is a positive number m such that $\|R(t)\| \leq m$ for all t. It therefore follows from (4.10) and (4.11) that

$$\begin{aligned} e^{-\alpha(t-\xi)} \|N(t)\| &= \left\| R(t-\xi)N(\xi) - \int_t^{\xi} R(t-s)B(s)e^{-\alpha(s-\xi)} N(s) ds \right\| \\ &\leq m\|N(\xi)\| + \int_t^{\xi} m\|B(s)\| e^{-\alpha(s-\xi)} \|N(s)\| ds. \end{aligned}$$

The above inequality and the Gronwall inequality yield that for $t \leq \xi$,

$$e^{-\alpha(t-\xi)} \|N(t)\| \leq m\|N(\xi)\| e^{\int_t^{\xi} m\|B(s)\| ds}. \tag{4.12}$$

From (4.12) we therefore obtain, for $t \leq \xi$,

$$\begin{aligned} \frac{1}{\|N(\xi)\|} \left\| \int_t^{\xi} e^{-A(s-\xi)} B(s) N(s) ds \right\| &\leq m \int_t^{\xi} m\|B(s)\| e^{\int_s^{\xi} m\|B(\tau)\| d\tau} ds \\ &= m \left[e^{\int_t^{\xi} m\|B(s)\| ds} - 1 \right] \end{aligned}$$

$$\begin{aligned}&\leq m\left[e^{\int_{-\infty}^{\xi} m\|B(s)\|ds}-1\right]\\&\to 0 \qquad \text{as } \xi\to-\infty\end{aligned} \tag{4.13}$$

uniformly for all $t\leq\xi$ and $N(\xi)\neq 0$. By rewriting (4.10) as

$$N(t)=\|N(\xi)\|e^{\alpha(t-\xi)}R(t-\xi)\left[\frac{N(\xi)}{\|N(\xi)\|}+\frac{1}{\|N(\xi)\|}\int_t^{\xi}e^{-A(s-\xi)}B(s)N(s)ds\right], \tag{4.14}$$

together with the use of (4.11) and (4.13) we therefore conclude that the both components, $W(t)$ and $Z(t)$, oscillate around zero infinitely many times as $t\to-\infty$. □

As a consequence of Theorem 4.3 and Lemma 4.4 we have

Corollary 4.5. *If $G_v(K,0)=M$, then the system (1.1) has a nonnegative, non-trivial semi-traveling wave solution connected to E_K if and only if $c\geq c_*=2\sqrt{G_v(K,0)}$. That is, $c_*=2\sqrt{G_v(K,0)}$ is the minimum wave speed.*

Remark 4.6. It is obvious that Theorem 1.1 is the implication of Theorem 4.3 by a scaling to reduce the diffusion coefficient d_2 to 1.

5. The boundedness of semi-traveling wave solutions

In this section we shall show that the positive semi-traveling wave solutions obtained in the Section 4 is bounded under the following additional assumption:

A3 There are positive numbers B, b and $\gamma>0$ such that

$$\begin{aligned}&F(u,v)\leq B \quad \text{for all } u\in[0,K],\ v\geq 0,\\&bF(u,v)+G(u,v)<0 \quad \text{for all } u\in(0,K],\ v\geq\gamma.\end{aligned}$$

For the convenience of reference let us re-insert the system (2.4) here:

$$\begin{aligned}X'&=\rho\big[X-Y\big], \qquad \rho=\frac{c^2}{d},\\Y'&=F(X,W),\\W'&=c^2\big[W-Z\big],\\Z'&=G(X,W).\end{aligned} \tag{5.1}$$

Lemma 5.1. *For any $c\geq 2\sqrt{M}$, if $\Phi_t(p)=(X(t),Y(t),W(t),Z(t))$ is a solution of (2.4) with $\Phi_t(p)\in\Sigma$ for all $t\geq 0$ and $|Y(0|<2K$, then*

$$|Y(t)|<H_1=3K+\frac{B}{\rho} \quad \textit{for all } t\geq 0,$$

where $B>0$ is the constant defined in the Assumption **A1**.

Proof. The vector filed of (2.4) at the plane $X + Y = H_1$ with $X \in [0, K]$ and $W > 0$ satisfies

$$\begin{aligned} X' + Y' &= \rho[X - Y] + F(X, W) \\ &= \rho(2X - H_1) + F(X, W) \\ &\leq -\rho H_1 + 2\rho K + B \\ &= -\rho K \\ &< 0. \end{aligned} \tag{5.2}$$

Noticing that $X(0) + Y(0) \leq 3K < H_1$, $X(t) \leq K$ and $W(t) > 0$ for all $t \geq 0$, (5.2) immediately implies that $X(t) + Y(t) < H_1$ for all $t \geq 0$. Hence $Y(t) < H_1 - X(t) < H_1$ for all $t \geq 0$.

Next consider that vector filed of (2.4) in the plane $X–Y = H_1$ with the part $X \in [0, K]$ and $W > 0$. One has, by the Assumption **A1**,

$$\begin{aligned} X' - Y' &= \rho[X - Y] - F(X, W) \\ &\geq \rho H_1 - B \\ &= 3\rho K \\ &> 0. \end{aligned}$$

The above inequality yields that if $X(t_1) - Y(t_1) = H_1$ for some $t_1 \geq 0$, then $X(t) - Y(t) > H_1$ for all $t > t_1$. It follows that

$$X'(t) = \rho(X - Y) > \rho H_1 > 0 \quad \text{for all } t > t_1.$$

Hence $X(t) \to \infty$ as $t \to \infty$, contradicting the assumption of $X(t) \leq K$. Thus we conclude that $X(t) - Y(t) < H_1$ for all $t \geq$. That is, $Y(t) > X(t) - H_1 > -H_1$ for all $t \geq 0$. □

Lemma 5.2. *Under the Assumption* **A3**, *there is a positive number γ_1 such that for any solution $\Phi_t(p) = (X(t), Y(t), W(t), Z(t))$ of (2.4), if $\Phi_t(p) \in \Sigma$ for all $t \geq 0$ and $|Y(0| < 2K$, $Z(0) < \gamma_1$, then*

$$Z(t) \leq H_2 = \gamma_1 + 2bH_1 \quad \textit{for all } t \geq 0,$$

and hence $W(t) \leq 2Z(t) \leq 2H_2$ for all $t \geq 0$, where H_1 is the number obtained in Lemma 5.1.

Proof. Since $\sigma^{-1}(Z) \to \infty$ as $Z \to \infty$, there is a $\gamma_1 > 0$ such that $\sigma^{-1}(Z) > \gamma$ for all $Z \geq \gamma_1$. Let

$$H_0 = \gamma_1 + bH_1$$

be a fixed constant. Now consider the vector field of (2.4) at the plane $bY + Z = H_0$ in the part of $X \in (0, K]$, $|Y| < H_1$, $W \geq \sigma^{-1}(Z)$. Since $bY + Z = H_0$ and $|Y| < H_1$, one has

$$Z = H_0 - bY > H_0 - bH_1 = \gamma_1.$$

So that $W \geq \sigma^{-1}(Z) > \gamma$. It therefore follows from the above inequality and Assumption **A3** that

$$bY' + Z' = bF(X, W) + G(X, W) < 0. \tag{5.3}$$

Since $bY(0) + Z(0) < bH_1 + \gamma_1 = H_0$ and $|Y(t)| < H_1$ for all $t \geq 0$, from (5.3) we conclude that $bY(t) + Z(t) < H_0$ for all $t \geq 0$. So that $Z(t) < H_0 - bY(t) \leq H_0 + bH_1 = H_2$ for all $t \geq 0$. □

As an immediate consequence of Lemmas 5.1 and 5.2 we have

Theorem 5.3. *Under the additional assumption* **A3***, a positive semi-traveling wave solution* $\Phi_t(p) \in \Sigma$ *is bounded.*

6. Existence of wave fronts

In this section we show the existence of traveling wave fronts, i.e., the positive traveling wave solutions connecting the boundary equilibrium E_K and an interior equilibrium $E_* = (u_*, v_*)$ or another boundary equilibrium point for a couple of models.

6.1. Beddington–DeAngelis models

Let us first consider the following predator–prey system

$$\begin{aligned} \frac{\partial u}{\partial t} &= d\Delta u + ru(1-u) - \frac{uv}{a + mu + nv}, \\ \frac{\partial v}{\partial t} &= \Delta v + \Big[\frac{\beta u}{a + mu + nv} - \mu - \lambda v\Big]v. \end{aligned} \tag{6.1}$$

Suppose $a = 1$ and both m and n are positive. Then (6.1) is the Beddington–DeAngelis model. If $a = 1$, $m > 0$ and $n = 0$, then (6.1) is a Holling Type II model. (6.1) is a ratio-dependent model if $a = 0$, $m > 0$ and $n > 0$ [20]. It is obvious that the system (6.1) has a constant equilibrium $E_1 = (1, 0)$. If we further suppose

$$\frac{\beta}{a + m} - \mu > 0. \tag{6.2}$$

The corresponding functions

$$\begin{aligned} F(u, v) &= ru(1-u) - \frac{uv}{a + mu + nv}, \\ G(u, v) &= \Big[\frac{\beta u}{a + mu + nv} - \mu - \lambda v\Big]v \end{aligned}$$

satisfy Assumptions **A1**–**A3** with $K = 1$ and $M = G_v(1, 0) = \dfrac{\beta}{a + m} - \mu$.

From Corollary 4.5 it follows that the system (6.1) has a positive semi-traveling wave solution $u(x, t) = U(x \cdot \nu + ct)$, $v(x, t) = V(x \cdot \nu + ct)$ satisfying

$$\lim_{\xi \to -\infty} (U(\xi), V(\xi)) = E_1$$

if and only if $c \geq c_* = 2\sqrt{\dfrac{\beta}{1+m} - \mu}$.

It is also known that the condition (6.2) implies the existence of a unique positive constant equilibrium $E_* = (u_*, v_*)$ of the system (6.1) with $0 < u_* < 1$ (for detail see Lemma 3.4 and the discussion in Case II in Section 3.2 in [3]). In what follows we will restrict our discussion for the Beddington–DeAngelis model, i.e., $a = 1$, $m > 0$ and $n > 0$. We shall show that, under some additional conditions, this semi-traveling solution is a traveling wave front, i.e., $(U(\xi), V(\xi)) \to E_*$ as $\xi \to \infty$.

By the transformation (2.3), the system (2.4) corresponding to the traveling wave solution $(U(\xi), V(\xi))$ is

$$\begin{aligned} X' &= \rho[X - Y], \\ Y' &= rX(1 - X) - \frac{XW}{1 + mX + nW}, \\ W' &= c^2[W - Z], \\ Z' &= \left[\frac{\beta X}{1 + mX + nW} - \mu - \lambda W\right]W. \end{aligned} \tag{6.3}$$

Corollary 4.5 and Theorem 5.3 imply that, for each $c \geq 2\sqrt{G_v(1,0)}$, (6.3) has a positive and bounded semi-traveling wave solution $\Phi_t(p_*) = (X(t), Y(t), W(t), Z(t)) \in \Sigma$ for all $t \in \mathbb{R}$ and $\Phi_t(p_*) \to E_1 = (1, 1, 0, 0)$ as $t \to -\infty$.

Lemma 6.1. *Let $\Phi_t(p_*)$ be defined as above, then*

(a) *there is a positive number C_0 such that*

$$\frac{Z(t)}{W(t)} \leq C_0 \quad \textit{for all } t \geq 0;$$

(b) *$X(t)$ does not converge to 0 as $t \to \infty$.*

Proof. Since $\phi_t(p_*)$ is bounded, there is an $M_0 > 0$ such that $0 < W \leq M_0$ for all $t \geq 0$. Recall that $\Phi_t(p_*) \in \Sigma$ implies that $Z(t) \leq \sigma(W(t))$. The fact that

$$\lim_{W \to 0+} \frac{\sigma(W)}{W} = \sigma'(0) < +\infty$$

then immediately yields that for all $t \geq 0$,

$$\frac{Z(t)}{W(t)} \leq \sup\left\{\frac{\sigma(W)}{W} : W \in (0, M_0]\right\} = C_0 < +\infty.$$

This proves part (a). To prove part (b), suppose in the opposite that $X(t) \to 0$ as $t \to \infty$. Then there is a $t_1 > 0$ such that for all $t \geq t_1$,

$$\frac{\beta X(t)}{1+mX(t)+nW(t)} - \mu - \lambda W \le -\frac{\mu}{2}. \tag{6.4}$$

From the equation for $Z(t)$ in (6.3), result in part (a), and the inequality (6.4) we therefore obtain

$$Z'(t) \le -\frac{\mu}{2}W(t) \le -\frac{\mu}{2C_0}Z(t), \quad \text{for all } t \ge t_1, \tag{6.5}$$

which clearly yields that $Z(t) \to 0$ as $t \to \infty$. Thus $W(t) \to 0$ as $t \to \infty$, for $W(t) \le 2Z(t)$ when $\Phi_t(p_*) \in \Sigma$. In view of the second equation in the system (6.3) we easily see that there is a $t_2 \ge t_1$ such that $Y'(t) > 0$ for all $t \ge t_2$. The assumption of $0 < X(t) \to 0$ as $t \to \infty$ implies that there must be a $t_3 \ge t_2$ such that $X'(t_3) < 0$. Thus, with the use the first equation of (6.3) and the inequality $Y'(t) > 0$ for all $t \ge t_3$ one concludes that $X'(t) = \rho\big[X(t) - Y(t)\big] \le \big[X(t_3) - Y(t_3)\big] = X'(t_3)$ for all $t \ge t_3$. And hence

$$X(t) \le X(t_3) + X'(t_3)(t - t_3) \to -\infty$$

as $t \to \infty$. This contradicts the fact that $X(t) > 0$ for all $t \in \mathbb{R}$. $\square$

To study the convergence of a semi-traveling solutions to the positive equilibrium $E_* = (u_*, v_*)$, let

$$\psi(X, W) = \frac{1}{1+mX+nW}$$

and rewrite the equations of Y' and Z' in (6.3) as

$$\begin{aligned}
Y' &= X\psi(X,W)\Big[r(1-X)(1+mX+nW) - W\Big] \\
&= X\psi(X,W)\Big[r(1-X)(1+mX) - (1-r(1-X)n)W\Big], \\
&= X\psi(X,W)\Big[g(X) - (1+rnX-rn)W\Big] \\
Z' &= W\psi(X,W)\Big[\beta X - (\mu+\lambda W)(1+mX+nW)\Big] \\
&= W\psi(X,W)H(X,W),
\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}
&g(X) = r(1-X)(1+mX), \\
&H(X,W) = \beta X - (\mu+\lambda W)(1+mX+nW).
\end{aligned} \tag{6.7}$$

Notice that $E_* = (u_*, v_*)$ is an interior equilibrium of (6.1). One has

$$g(u_*) - (1+rnu_* - rn)v_* = 0, \qquad H(u_*, v_*) = 0. \tag{6.8}$$

Upon a straightforward calculation we can further rewrite the system (6.3) as

$$
\begin{aligned}
X' &= \rho[X - Y] \\
Y' &= X\psi(X, W)\big[g(X) - g(u_*) - (1 + rnX - rn)W + (1 + rnu_* - rn)v_*\big] \\
&= X\psi(X, W)\big[g(X) - g(u_*) - rn(X - u_*)W - m_*(W - v_*)\big], \\
W' &= c^2[W - Z], \\
Z' &= W\psi(X, W)\big[H(X, W) - H(u_*, v_*)\big] \\
&= W\psi(X, W)\big[\delta(X - u_*) - \theta(X, W)(W - v_*)\big],
\end{aligned} \tag{6.9}
$$

with

$$
\begin{aligned}
m_* &= 1 - r(1 - u_*)n, \\
\delta &= \beta - (\mu + \lambda v_*)m, \\
\theta(X, W) &= \mu n + \lambda\big[1 + mX + n(W + v_*)\big].
\end{aligned}
$$

The first equation in (6.8) yields that

$$
m_* = 1 - r(1 - u_*)n = g(u_*)/v_* > 0. \tag{6.10}
$$

The second equations in (6.7) and (6.8) imply that

$$
\beta - (\mu + \lambda v_*)m = (\mu + \lambda v_*)(1 + nv_*)/u_* > 0. \tag{6.11}
$$

Theorem 6.2. *Suppose that*

$$
g(0) = r \geq g(u_*). \tag{6.12}
$$

Then the system (6.1) has a traveling wave solution connecting the equilibria E_1 and E_* if and only if

$$
c \geq c_* = 2\sqrt{\frac{\beta}{1 + m} - \mu}.
$$

Proof. In view of Theorem 4.3, we only need to consider the case $c \geq c_*$, in which the system (6.9) has a solution $(X(t), Y(t), W(t), Z(t)) = \Phi_t(p_*) \in \Sigma$ for all $t \in \mathbb{R}$ and $\Phi_t(p_*) \to E_1$ as $t \to -\infty$. It is sufficient to show that $\Phi_t(p_*) \to E_*$ as $t \to \infty$. To this end, we define

$$
\begin{aligned}
L(t) = X(t) - u_* \ln X(t) - \big[X(t) - Y(t)\big]\left[1 - \frac{u_*}{X(t)}\right] \\
+ \frac{m_*}{\delta}\left[Z(t) - v_*\frac{Z(t)}{W(t)} - v_* \ln W(t)\right].
\end{aligned} \tag{6.13}
$$

Then following a direct calculation we arrive at, with $X = X(t)$, $Y = Y(t)$, $W = W(t)$ and $Z = Z(t)$,

$$
\begin{aligned}
L'(t) &= \Big[1 - \frac{u_*}{X}\Big]X' - (X' - Y')\Big[1 - \frac{u_*}{X}\Big] - (X - Y)\frac{u_*}{X^2}X' \\
&\quad + \frac{m_*}{\delta}\left(\Big[1 - \frac{u_*}{W}\Big]Z' + v_*\Big[\frac{Z}{W^2} - \frac{1}{W}\Big]W'\right) \\
&= \psi(X,W)\left([X - u_*]\big[g(X) - g(u_*)\big] - rnW[X - u_*]^2\right) \\
&\quad - \psi(X,W)m_*(X - u_*)(W - v_*) - \rho\frac{u_*}{X^2}[X - Y]^2 \\
&\quad + \frac{m_*}{\delta}\psi(X,W)\left(\delta[W - v_*][X - u_*] - \theta(X,W)[W - v_*]^2\right) - \frac{m_*}{\delta}\frac{1}{W^2}[Z - W]^2 \\
&= \psi(X,W)\Big([X - u_*]\big[g(X) - g(u_*)\big] - rnW[X - u_*]^2\Big) - \rho\frac{u_*}{X^2}[X - Y]^2 \\
&\quad - \frac{m_*}{\delta}\left(\psi(X,W)\theta(X,W)[W - v_*]^2 + \frac{1}{W^2}[Z - W]^2\right).
\end{aligned}
\tag{6.14}
$$

It is obvious that $g''(X) < 0$. Hence the assumption (6.12) implies that $g(X) - g(u_*) > 0$ for $X \in (0, u_*)$ and $g(X) - g(u_*) < 0$ for $X \in (u_*, 1)$. Thus from the equality (6.14) it follows that $L'(t) \le 0$ for all t, so that $L(t)$ is decreasing. Let us first confirm that following

Claim. *There is an $x_0 \in (0, u_*]$ such that $X(t) \ge x_0$ for all $t \ge 0$.*

Proof. Suppose that the Claim were false. Then there would be a sequence $0 < t_n \to +\infty$ such that $X(t_n) \to 0$ as $n \to \infty$. Note that the function $x - u * \ln x$ for $x \in (0, u_*)$ is monotone increasing as x decreases and $x - u * \ln x \to +\infty$ when $x \to 0+$. By Theorem 5.3 and Lemma 6.1, there is real number M_1 such that

$$
\frac{m_*}{\delta}\left[Z(t) - v_*\frac{Z(t)}{W(t)} - v_* \ln W(t)\right] \ge M_1, \quad \text{for all } t \ge 0. \tag{6.15}
$$

Since $X(t_n) - u_* \ln X(t_n) \to +\infty$ as $n \to \infty$, there is an integer j such that $X(t_j) < u_*$ and

$$
X(t_j) - u_* \ln X(t_j) + M_1 > L(0). \tag{6.16}
$$

Recall $L(t)$ is decreasing, (6.15) and (6.16) therefore yield that

$$
\begin{aligned}
L(0) \ge L(t_j) &= X(t_j) - u_* \ln X(t_j) - \big[X(t_j) - Y(t_j)\big]\left[1 - \frac{u_*}{X(t_j)}\right] \\
&\quad + \frac{m_*}{\delta}\left[Z(t_j) - v_*\frac{Z(t_j)}{W(t_j)} - v_* \ln W(t_j)\right] \\
&> L(0) - \big[X(t_j) - Y(t_j)\big]\left[1 - \frac{u_*}{X(t_j)}\right].
\end{aligned}
\tag{6.17}
$$

Noticing that $(1 - u_*/X(t_j)) < 0$, from (6.17) we deduce that $X(t_j) - Y(t_j) < 0$, and hence $X'(t_j) = \rho\big[X(t_j) - Y(t_j)\big] < 0$ in view of the first equation in the system (6.3). One therefore concludes that $X'(t) < 0$ for all $t > t_j$. For if this were not the case, then there would exist a $t_* > t_j$ such that $X'(t_*) = 0$ and $X'(t) < 0$ for $t \in (t_j, t_*]$. Hence

$$X(t_*) - u_* \ln X(t_*) > X(t_j) - u_* \ln X(t_j), \quad X(t_*) - Y(t_*) = 0.$$

Therefore,

$$L(0) \geq L(t_*) > X(t_j) - u_* \ln X(t_j) + M_1 > L(0),$$

which leads a contradiction. On the other hand, $X'(t) < 0$ implies that $\lim_{t\to\infty} X(t) = \lim_{n\to\infty} X(t) = 0$, which leads to another contradiction to part (b) of Lemma 6.1. This completes the proof of the claim. □

From the claim one easily sees that $L(t)$ is bounded below and $L'(t)$ is bounded. As a consequence we much that $L'(t) \to 0$ as $t \to \infty$. Hence $(X(t), Y(t), W(t), Z(t)) \to E_*$ as $t \to \infty$. This shows the existence of a traveling wave solution for each wave speed $c \geq c_*$. □

Remark 6.1. For the predator–prey model with predator-independent functional response:

$$\begin{aligned} \frac{\partial u}{\partial t} &= d\Delta u + b(u) - f(u)v, \\ \frac{\partial v}{\partial t} &= \Delta v + \Big[\beta f(u) - \mu - \lambda v\Big] v, \end{aligned} \tag{6.18}$$

where $b(u)$, $f(u)$ are nonnegative functions and $\mu \geq 0$, $\sigma \geq 0$ are constants, the convergence of the semi-traveling waves to an interior equilibrium point E_*, if it exists, can be obtain for more general function $f(u)$ under certain conditions. First, in the light of Theorem 4.3, for any positive number K, the condition

(B1) $b(K) = 0$ and $\beta f(K) - \mu > 0$

grantees the existence of a positive semi-traveling wave connected to the boundary equilibrium $E_K = (K, 0)$ whenever the wave speed c satisfies

$$c \geq 2\sqrt{\max\{\beta f(u) : u \in [0, K]\} - \mu}.$$

If in addition suppose that

(B2) The system (6.18) has an interior equilibrium $E_* = (u_*, v_*)$ with $0 < u_* < K$.
(B3) $f(u)$ is monotone increasing over the interval $(0, K]$, and

$$g(u) > g(u_*) \quad \text{for } u \in (0, u_*), \qquad g(u) < g(u_*) \quad \text{for } u \in (u_*, K],$$

where $g(u)/f(u)$,

then the convergence of this semi-traveling solution to the interior equilibrium E_* can be confirmed by the construction of a Lyapunov function, which is a modification of the Lyapunov function given in [11] for the case of diffusion coefficient $d = 0$. To see this, let $(X(t), Y(t), W(t), Z(t))$ be a positive semi-traveling wave solution of the system (2.4) corresponding to the reaction–diffusion system (6.18). If we let

$$L(t) = \beta \int_{u_*}^{X(t)} \frac{f(s) - f(u_*)}{f(s)} ds + \beta \Big[Y(t) - X(t) \Big] \Big[1 - \frac{f(u_*)}{f(X(t))} \Big]$$
$$+ \Big[Z(t) - v_* \frac{Z(t)}{W(t)} - v_* \ln W(t) \Big],$$

then one is able to verify that $L'(t) \le 0$, and to show the convergence of $(X(t), Y(t), W(t), Z(t)) \to (u_*, u_*, v_*, v_*)$ as $t \to +\infty$.

6.2. Model of biological flow reactor

As a second example, let us consider a model of biological flow reactor introduced in [13] to study the microbial growth in a flow reactor, which takes the form

$$\frac{\partial S}{\partial t} = dS_{xx} - \alpha s_x - f(S)P$$
$$\frac{\partial P}{\partial t} = P_{xx} - \alpha P_x + \big[f(S) - \mu \big] P, \tag{6.19}$$

where $S(x, t)$ and $P(x, t)$ denote respectively the concentrations of nutrient and bacterias in a flow reactor. $d > 0$, $\alpha > 0$ and $\mu > 0$ are constants. It is assumed that the nutrient uptake function $f \in C^2$ satisfies the following conditions:

C $f(0) = 0$ and there is a unique $S_* > 0$ such that $f(S_*) = \mu$ and

$$0 < f(S) < f(S_*) \quad \text{if } 0 < S < S_*, \qquad f(S) > f(S_*) \quad \text{if } S > S_*.$$

Given any constant $S_0 > S_*$, we look for a positive traveling wave solution

$$S(x, t) = U(x + ct), \qquad P(x, t) = V(x + ct)$$

such that

$$\lim_{\xi \to -\infty} (U(\xi), V(\xi)) = (S_0, 0), \qquad \lim_{\xi \to \infty} (U(\xi), V(\xi)) = (s_0, 0), \tag{6.20}$$

for some constant $0 < s_0 < S_*$.

The existence of above type of traveling wave solutions has been confirmed in [12] using an analytic shooting argument. Here we show that how the traveling wave solutions of form (6.20) can be obtained using the Theorem 4.3 of this paper. That is, we have the following

Theorem 6.3. *For each fixed $S_0 > S_*$, let*

$$M = \max\{f(S) : S \in [0, S_0]\} - \mu. \tag{6.21}$$

Then for each $c \ge -\alpha + 2\sqrt{M}$, the system (6.19) has a traveling wave solution $(U(t), V(t))$ satisfying the boundary condition (6.20), where the number s_0 depends on both S_0 and c. In addition, if $\max\{f(S) : S \in [0, S_0]\} = f(S_0)$, then $c_* = -\alpha + 2\sqrt{f(S_0) - \mu}$ is the minimum *wave speed. In addition, the function $U(t)$ is decreasing for $t \in \mathbb{R}$, and there is a t_0 such that the function $V(t)$ is increasing over the interval $(-\infty, t_0]$ and decreasing on $[t_0, \infty)$.*

Proof. First we see that the functions $U(t)$ and $V(t)$ (we use t instead of ξ) satisfy the system

$$\begin{aligned}(c+\alpha)U' &= dU'' - f(U)V,\\ (c+\alpha)V' &= V'' + \big[f(U) - \mu\big]V.\end{aligned} \tag{6.22}$$

It is apparent that (6.22) is a particular type of System (2.1) with c being replaced by $c+\alpha$. Moreover, it is clear that all Assumptions **A1**–**A3** are satisfied and

$$-\mu V \le (f(U) - \mu)V \le MV, \quad U \in [0, S_0] \tag{6.23}$$

with the number M be given in (6.21). Hence from Theorems 4.3 and 5.3 it follows that for each $c+\alpha \ge 2\sqrt{M}$, the system (6.22) has a positive and bounded semi-traveling wave solutions $(U(t), V(t))$ with $0 < U(t) < S_0$ for all $t \in \mathbb{R}$ and $(U(t), V(t)) \to (S_0, 0)$ as $t \to -\infty$. The convergence of $(U(t), V(t)) \to (s_0, 0)$, $t \to \infty$, for some positive number $s_0 < S_*$ can be done by using the following differential–integral forms of $U(t)$ and $V(t)$ (see [12]):

$$\begin{aligned} U'(t) &= -\frac{1}{d}e^{\delta_1 t}\int_t^\infty e^{-\delta_1 s} f(U(s))V(s)ds, & \delta_1 &= \frac{c+\alpha}{d}\\ V'(t) &= e^{\delta_2 t}\int_t^\infty e^{-\delta_2 s}\big[f(U(s)) - \mu\big]V(s)ds, & \delta_2 &= c+\alpha, \end{aligned} \tag{6.24}$$

which describe the behavior of the functions $U(t)$ and $V(t)$. We shall omit the details of convergence analysis and refer the readers to the Ref. [12]. □

Remark 6.2. By using the same approach presented for biological flow reactor model (6.19) we can easily obtain the existence of traveling solutions for the combustion model [1,23], i.e.,

$$F(u, v) = -h(u, v), \qquad G(u, v) = h(u, v).$$

Suppose that

(i) $h(u, v) > 0$ and $h(0, v) = h(u, 0) = 0$ for all $u > 0$, $v > 0$;
(ii) $h_v(U_0, 0) > 0$ for any $U_0 > 0$.

Now for each $U_0 > 0$, if

$$M = \sup\left\{\frac{h(U_0, v)}{v} : \ v > 0\right\}$$

is a real number, then we can conclude, by the same argument used in Section 6.2, that for each $c \ge 2\sqrt{d_2 M}$, the combustion model has a positive traveling wave solution $u(x,t) = U(x \cdot \nu + ct)$, $v(x,t) = V(x \cdot \nu + ct)$ such that

$$(U(-\infty), V(-\infty)) = (U_0, 0), \qquad (U(\infty), V(\infty)) = (0, V_0)$$

for some positive number V_0. Moreover, $U(t)$ is decreasing and $V(t)$ is increasing.

Appendix A. Proof of Theorem 3.1

The linearization of the system (2.4) at the equilibrium $E_K = (K, K, 0, 0)$ is

$$\begin{aligned}
\dot{X} &= \rho[X - Y], \\
\dot{Y} &= F_u(K,0)X + F_v(K,0)W, \\
\dot{W} &= c^2\big[W - Z\big], \\
\dot{Z} &= G_v(K,0)W.
\end{aligned} \tag{A.1}$$

By the Assumptions **A1** and **A2**,

$$F_u(K,0) \le 0, \qquad F_v(K,0) < 0, \qquad 0 < G_v(K,0) \le M.$$

Throughout this section we consider

$$c \ge 2\sqrt{M}.$$

Then $c \ge 2\sqrt{G_v(K,0)}$. Upon a direct computation one is able to verify that the linear system (A.1) has eigenvalues

$$\begin{aligned}
\lambda_1 &= \frac{c^2 + c\sqrt{c^2 - 4G_v(K,0)}}{2} > 0, \\
\lambda_2 &= \frac{c^2 - c\sqrt{c^2 - 4G_v(K,0)}}{2} > 0, \\
\lambda_3 &= \frac{\rho + \sqrt{\rho^2 - 4F_u(K,0)}}{2} > 0, \\
\lambda_4 &= \frac{\rho - \sqrt{\rho^2 - 4F_u(K,0)}}{2} \le 0,
\end{aligned} \tag{A.2}$$

where λ_1 and λ_2 are roots of the quadratic polynomial

$$q(\lambda) = \lambda^2 - c^2\lambda + c^2 G_v(K,0),$$

and λ_3 and λ_4 are roots of the quadratic polynomial

$$h(\lambda) = \lambda^2 - \rho\lambda + \rho F_u(K,0). \tag{A.3}$$

Hence we have

$$\begin{aligned} &\lambda_1 + \lambda_2 = c^2, \qquad && \lambda_1\lambda_2 = c^2 G_v(K,0), \\ &\lambda_3 + \lambda_4 = \rho, && \lambda_3\lambda_4 = \rho F_u(K,0) \end{aligned} \tag{A.4}$$

and

$$h(\lambda) = (\lambda - \lambda_3)(\lambda - \lambda_4). \tag{A.5}$$

First we consider the case $c > 2\sqrt{G_v(K,0)}$. Then it is obvious that $\lambda_1 > \lambda_2$. The corresponding eigenvectors to the positive eigenvalues $\lambda_1, \lambda_2, \lambda_3$, when $\lambda_3 \neq \lambda_i$ for $i = 1, 2$, are

$$\mathbf{h}_i = \begin{bmatrix} h_{1i} \\ h_{2i} \\ h_{3i} \\ h_{4i} \end{bmatrix} = \begin{bmatrix} h_{1i} \\ h_{2i} \\ 1 \\ \dfrac{G_v(K,0)}{\lambda_i} \end{bmatrix}, \quad i = 1, 2, \qquad \mathbf{h}_3 = \begin{bmatrix} 1 \\ \dfrac{F_u(K,0)}{\lambda_3} \\ 0 \\ 0 \end{bmatrix} \tag{A.6}$$

where for $i = 1, 2$,

$$\begin{aligned} &h_{1i} = -\frac{\rho F_v(K,0)}{h(\lambda_i)}, \quad && h_{2i} = \frac{(\lambda_i - \rho)F_v(K,0)}{h(\lambda_i)} \quad && \text{if } \lambda_i \neq \lambda_3, \\ &h_{1i} = 0, && h_{2i} = \frac{F_v(K,0)}{\lambda_3 - \lambda_4} && \text{if } \lambda_i = \lambda_3. \end{aligned} \tag{A.7}$$

Remark. For $i = 1, 2$, if $\lambda_i = \lambda_3$, then $\mathbf{h}_i$ is a generalized eigenvector, i.e., $[A - \lambda_i]^2\mathbf{h}_i = 0$, where A is the coefficient matrix of the linear system (A.1).

From the unstable manifold theorem [22, p. 107] it follows that the local unstable manifold E_K^U of E_K is tangent to the plan spanned by the eigenvectors $\mathbf{h}_1$–$\mathbf{h}_3$. That is, there are a small neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^3$ and a smooth (twice continuously differentiable) function $\mathcal{M} = (\mathcal{M}_1, \cdots, \mathcal{M}_4) : \mathcal{O} \to \mathbb{R}^4$ such that the local unstable manifold E_K^U can be expressed as

$$E_K^U = \Big\{ k_1\mathbf{h_1} + k_2\mathbf{h_2} + k_3\mathbf{h_3} + \mathcal{M}(k_1, k_2, k_3) + E_K : (k_1, k_2, k_3) \in \mathcal{O} \Big\},$$

where the function $\mathcal{M}_i$ satisfies that

$$\mathcal{M}_i(0,0,0) = \frac{\partial \mathcal{M}_i(0,0,0)}{\partial k_j} = 0, \quad i, j = 1, 2, 3. \tag{A.8}$$

Moreover, from the uniqueness of local unstable manifold (i.e. the uniqueness of the function $\mathcal{M}$ and the fact that the plane $\{W = Z = 0\}$ is an invariant set of the system (2.4) one is able to deduce that

$$\mathcal{M}_3(0,0,k_3) = \mathcal{M}_4(0,0,k_3) \equiv 0 \quad \text{for all small } k_3. \tag{A.9}$$

We want to parameterize $k = (k_1, k_2, k_3) = k(s, \theta, \tau)$ for $s \in [0, 1]$, $\theta \in [0, 1]$ and $\tau \in [0, \tau_0]$ with τ_0 small such that $k(s, \theta, \tau) \in \mathcal{O}$ and for each fixed $\tau \in (0, \tau_0]$,

$$\Delta_1^*(\tau) = \left\{ \sum_{i=1}^{3} k_i(s,0,\tau)\mathbf{h_i} + E_K + \mathcal{M}(k(s,0,\tau)),\ s \in (0,1) \right\} \subset P_1,$$

$$\Delta_2^*(\tau) = \left\{ \sum_{i=1}^{3} k_i(s,1,\tau)\mathbf{h_i} + E_K + \mathcal{M}(k(s,1,\tau)),\ s \in (0,1) \right\} \subset P_2,$$

$$\Delta_3^*(\tau) = \left\{ \sum_{i=1}^{3} k_i(1,\theta,\tau)\mathbf{h_i} + E_K + \mathcal{M}(k(1,\theta,\tau)),\ \theta \in [0,1] \right\} \subset P_3. \tag{A.10}$$

For this purpose, we write

$$p(k) = (x(k), y(k), w(k), z(k)) = \sum_{i=1}^{3} k_i \mathbf{h_i} + E_K + \mathcal{M}(k) \in E_K^U.$$

Then by (A.4), (A.6), and (A.7) we have

$$\begin{aligned} x(k) &= h_{11}k_1 + h_{12}k_2 + k_3 + \mathcal{M}_1(k_1,k_2,k_3) + K, \\ y(k) &= h_{21}k_1 + h_{22}k_2 + \frac{F_u(K,0)}{\lambda_3}k_3 + \mathcal{M}_2(k_1,k_2,k_3) + K, \\ w(k) &= k_1 + k_2 + \mathcal{M}_3(k_1,k_2,k_3), \\ z(k) &= \frac{1}{c^2}(\lambda_2 k_1 + \lambda_1 k_2) + \mathcal{M}_4(k_1,k_2,k_3). \end{aligned} \tag{A.11}$$

Let $\sigma(W)$ be the function defined in (2.8) and $\sigma_1 = \sigma'(0)$ be given in (2.10). Then we can write

$$\sigma(W) = \sigma_1 W + \sigma_*(W), \tag{A.12}$$

where $\sigma_*(W)$ satisfies that

$$\sigma_*(W) = \sigma(W) - \sigma'(0)W = 0(W^2) \quad \text{as } W \to 0. \tag{A.13}$$

By the definition of P_i for $i = 1, 2, 3$ and in view of (A.11),

(1) $p(k) \in P_1$ if and only if
$z(k) = \sigma(w(k)) = \sigma_1(w(k)) + \sigma_*(w(k))$ and $0 < x(k) < K$.
(2) $p(k) \in P_2$ if and only if
$z(k) = \dfrac{w(k)}{2}$ and $0 < x(k) < K$.
(3) $p(k) \in P_3$ if and only if $x(k) = K > y(k)$ and $\dfrac{w(k)}{2} \le z(k) \le \sigma(w(k)$, i.e.:

$$\begin{aligned} &h_{11}k_1 + h_{12}k_2 + k_3 + \mathcal{M}_1(k) = 0, \\ &h_{21}k_1 + h_{22}k_2 + \frac{F_u(K,0)}{\lambda_3}k_3 + \mathcal{M}_2(k) < 0, \\ &\frac{w(k)}{2} \le z(k) \le \sigma(w(k). \end{aligned} \tag{A.14}$$

We shall show the existence of values of k_1, k_2, and k_3 that satisfy the conditions (1)–(3) by the Implicit Function Theorem. To this end let us parameterize k_1, k_2 and k_3 as functions on $s \in [0,1]$, $\theta \in [0,1]$, $\tau \in [0,\tau_0]$ with τ_0 a small positive number as follows:

$$\begin{aligned}
k_1 &= k_1(s,\theta,\tau) = s\tau[\alpha_1(s,\tau)(1-\theta)+\theta],\\
k_2 &= k_2(s,\theta,\tau) = s\tau\big[1-\theta+\alpha_2(s,\tau)\theta\big],\\
k_3 &= k_3(s,\theta,\tau) = \tau\big[s\alpha_3(\theta,\tau)+a_0(s-1)\big], \quad s\in(0,1], \ \tau\in(0,\tau_0],
\end{aligned} \tag{A.15}$$

where $0 < a_0 < K$ is a fixed constant.

For the notational simplicity, any real valued function $g(\delta,\lambda)$ will be denoted by $O(|\delta|^j)$ for small δ if there is a positive number m such that

$$|g(\delta,\lambda)| \le m|\delta|^j$$

for all small δ and all λ in a bounded set in $\mathbb{R}^l$, where j and l are positive integers.

With the use of expressions (A.11) and (A.15), we define the functions

$$\begin{aligned}
&F_1(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(\theta,\tau),s,\theta,\tau)\\
&\quad = \frac{1}{s\tau}\big[z(k(s,\theta,\tau)) - \sigma(w(k(s,\theta,\tau)))\big]\\
&\quad = \frac{1}{c^2}\Big(\lambda_2\big[\alpha_1(s,\tau)(1-\theta)+\theta\big] + \lambda_1\big[1-\theta+\alpha_2(s,\tau)\theta\big]\Big)\\
&\qquad - \sigma_1\big[\alpha_1(s,\tau)(1-\theta)+\theta+1-\theta+\alpha_2(s,\tau)\theta\big] + O(\tau),
\end{aligned} \tag{A.16}$$

$$\begin{aligned}
&F_2(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(\theta,\tau),s,\theta,\tau)\\
&\quad = \frac{1}{s\tau}\Big[z(k(s,\theta,\tau)) - \frac{1}{2}(w(k(s,\theta,\tau)))\Big]\\
&\quad = \frac{1}{c^2}\Big(\lambda_2\big[\alpha_1(s,\tau)(1-\theta)+\theta\big] + \lambda_1\big[1-\theta+\alpha_2(s,\tau)\theta\big]\Big)\\
&\qquad - \frac{1}{2}\big[\alpha_1(s,\tau)(1-\theta)+\theta+1-\theta+\alpha_2(s,\tau)\theta\big] + O(\tau),
\end{aligned} \tag{A.17}$$

$$\begin{aligned}
&F_3(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(\theta,\tau),s,\theta,\tau)\\
&\quad = \frac{1}{\tau}\Big[x(k(s,\theta,\tau)) - K\Big]\\
&\quad = s\Big(h_{11}\big[\alpha_1(s,\tau)(1-\theta)+\theta\big] + h_{12}\big[1-\theta+\alpha_2(s,\tau)\theta\big]\Big)\\
&\qquad + s\alpha_3(\theta,\tau) + a_0(s-1) + O(\tau),
\end{aligned} \tag{A.18}$$

where functions α_1, α_2 and α_3 will be chosen so that

$$\begin{aligned}
z\big(k(s,0,\tau)\big) &= \sigma\big(w(s,0,\tau)\big), && s\in(0,1),\ \tau\in(0,\tau_0]\\
z\big(k(s,1,\tau)\big) &= \frac{1}{2}\big(w(s,1,\tau)\big), && s\in(0,1),\ \tau\in(0,\tau_0]\\
x\big(k(1,\theta,\tau)\big) &= K, && \theta\in[0,1],\ \tau\in(0,\tau_0].
\end{aligned} \tag{A.19}$$

In view of (A.16)–(A.18) the equalities in (A.19) is equivalent to that the functions α_1, α_2 and α_3 satisfy

$$\begin{aligned} &F_1\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(0,\tau),s,0,\tau\big)=0, \quad s\in(0,1),\\ &F_2\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(1,\tau),s,1,\tau\big)=0, \quad s\in(0,1),\\ &F_3\big(\alpha_1(1,\tau),\alpha_2(1,\tau),\alpha_3(\theta,\tau),1,\theta,\tau\big)=0, \quad \theta\in[0,1]. \end{aligned} \tag{A.20}$$

Lemma A.1. *Let $\tau_0>0$ be sufficiently small. Then for each $\tau\in(0,\tau_0]$, there exist continuous functions $\alpha_1(\cdot,\tau)$, $\alpha_2(\cdot,\tau)$, $\alpha_3(\cdot,\tau):[0,1]\to\mathbb{R}$ such that the equalities (A.20) are satisfied with*

$$\alpha_1(s,0)\equiv\alpha_1^0=\frac{\lambda_1-c^2\sigma_1}{c^2\sigma_1-\lambda_2}, \qquad \alpha_2(s,0)\equiv\alpha_2^0=1, \qquad s\in[0,1], \tag{A.21}$$

$$\alpha_3(\theta,0)=\alpha_3^0(\theta)=-h_{11}\big[\alpha_1^0(1-\theta)+\theta\big]-h_{12}\Big[1-\theta+\alpha_2^0\theta\Big], \quad \theta\in[0,1]. \tag{A.22}$$

Moreover, $\dfrac{\partial\alpha_i(s,\tau)}{\partial s}$ for $i=1,2$, and $\dfrac{\partial\alpha_3(\theta,\tau)}{\partial\theta}$ are continuous.

Proof. Let $\mathcal{X}=C([0,1]:\mathbb{R}^3)$ and define $\tilde F=(\tilde F_1,\tilde F_2,\tilde F_3):\mathcal{X}\times[0,\tau_0]\to\mathcal{X}$ by

$$\begin{aligned} &\tilde F_1(\alpha_1,\alpha_2,\alpha_3,\tau)(s)=F_1(\alpha_1(s),\alpha_2(s),\alpha_3(0),s,0,\tau), \quad s\in[0,1]\\ &\tilde F_2(\alpha_1,\alpha_2,\alpha_3,\tau)(s)=F_1(\alpha_1(s),\alpha_2(s),\alpha_3(1),s,1,\tau), \quad i=1,2,\ \ s\in[0,1]\\ &\tilde F_3(\alpha_1,\alpha_2,\alpha_3,\tau)(\theta)=F_3(\alpha_1(1),\alpha_2(1),\alpha_3(\theta),1,\theta,\tau), \quad \theta\in[0,1]. \end{aligned} \tag{A.23}$$

Then with the use of (A.16)–(A.18), (A.21)–(A.23), the equality $\lambda_1+\lambda_2=c^2$ [see (A.4)], and following a straightforward calculation we obtain

$$\begin{aligned} \tilde F_1(\alpha_1^0,\alpha_2^0,\alpha_3^0,0)\ &=\frac{1}{c^2}\big[\lambda_2\alpha_1^0+\lambda_1\big]-\sigma_1\big[\alpha_1^0+1\big]=0,\\ \tilde F_2(\alpha_1^0,\alpha_2^0,\alpha_3^0,0)\ &=\frac{1}{c^2}\big[\lambda_2+\lambda_1\alpha_2^0\big]-\frac{1}{2}\big[1+\alpha_2^0\big]=\frac{1}{c^2}\big[\lambda_1+\lambda_2\big]-1=0,\\ \tilde F_3(\alpha_1^0,\alpha_2^0,\alpha_3^0,0)\ &=0. \end{aligned}$$

Moreover the linear operator $\mathcal{L}=(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)=D_{(\alpha_1,\alpha_2,\alpha_3)}\tilde F(\alpha_1,\alpha_2,\alpha_3,0):\ \mathcal{X}\to\mathcal{X}$ is

$$\begin{aligned} \mathcal{L}_1(\alpha_1,\alpha_2,\alpha_3)(s)\ &=\Big[\frac{\lambda_2}{c^2}-\sigma_1\Big]\alpha_1(s),\\ \mathcal{L}_2(\alpha_1,\alpha_2,\alpha_3)(s)\ &=\Big[\frac{\lambda_1}{c^2}-\frac{1}{2}\Big]\alpha_2(s),\\ \mathcal{L}_3(\alpha_1,\alpha_2,\alpha_3)(\theta)\ &=h_{11}\big[\alpha_1(1)(1-\theta)+\theta\big]+h_{12}\big[1-\theta+\alpha_2(1)\theta\big]+\alpha_3(\theta) \end{aligned}$$

for $s\in[0,1]$ and $\theta\in[0,1]$. It is clear that $\mathcal{L}:\mathcal{X}\to\mathcal{X}$ is bounded and invertible. The lemma therefore follows directly from the Implicit Function Theorem. $\square$

Lemma A.2. *Let $\alpha_1(s,\tau)$, $\alpha_2(s,\tau)$, $\alpha_3(\theta,\tau)$ be defined in Lemma A.1, and $k_i(s,\theta,\tau)$ be given as in (A.15). Then for any sufficiently small $\tau_0 > 0$, the set*

$$\Delta_0^*(\tau) = \Big\{ \sum_{i=1}^{3} k_i(s,\theta,\tau)\mathbf{h_i} + \mathcal{M}(k(s,\theta,\tau)) + E_K,\ (s,\theta) \in (0,1)\times(0,1) \Big\} \in \mathrm{Int}(\Sigma)$$

for all $\tau \in (0,\tau_0]$.

Proof. By differentiating F_1 and F_2 with respect to θ at $\tau = 0$, $s \in [0,1]$ and with the use of (A.21), (A.22) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\theta} F_1\big(\alpha_1(s,0),\alpha_2(s,0),\alpha_3(\theta,0),s,\theta,0\big)\\
&= \frac{1}{c^2}\Big(\lambda_2\big[1-\alpha_1(s,0)\big] + \lambda_1\big[\alpha_2(s,0)-1\big]\Big) - c^2\sigma_1\big[\alpha_2(s,0)-\alpha_1(s,0)\big]\\
&= \frac{1}{c^2}\Big(\lambda_2\big[1-\alpha_1^0\big] + \lambda_1\big[\alpha_2^0-1\big]\Big) - c^2\sigma_1\big[\alpha_2^0-\alpha_1^0\big]\\
&= \Big[\frac{\lambda_2}{c^2} - \sigma_1\Big]\big[1-\alpha_1^0\big]\\
&< 0.
\end{aligned} \tag{A.24}$$

$$\begin{aligned}
&\frac{\partial}{\partial\theta} F_2\big(\alpha_1(s,0),\alpha_2(s,0),\alpha_3(\theta,0),s,\theta,0\big)\\
&= \frac{1}{c^2}\Big(\lambda_2\big[1-\alpha_1(s,0)\big] + \lambda_1\big[\alpha_2(s,0)-1\big]\Big) - \frac{1}{2}\big[\alpha_2(s,0)-\alpha_1(s,0)\big]\\
&= \frac{1}{c^2}\Big(\lambda_2\big[1-\alpha_1^0\big] + \lambda_1\big[\alpha_2^0-1\big]\Big) - \frac{1}{2}\big[\alpha_2^0-\alpha_1^0\big]\\
&= \Big[\frac{\lambda_2}{c^2} - \frac{1}{2}\Big]\big[1-\alpha_1^0\big]\\
&< 0.
\end{aligned} \tag{A.25}$$

Similarly for $\tau = 0$ and $\theta \in [0,1]$ we have

$$\begin{aligned}
&\frac{\partial}{\partial s} F_3\big(\alpha_1(s,0),\alpha_2(s,0),\alpha_3(\theta,0),s,\theta,0\big)\\
&= h_{11}\big[\alpha_1(s,0)(1-\theta)+\theta\big] + h_{22}\big[1-\theta+\alpha_2(s,0)\theta\big] + \alpha_3(\theta,0) + a_0\\
&= h_{11}\big[\alpha_1^0(1-\theta)+\theta\big] + h_{22}\big[1-\theta+\alpha_2^0\theta\big] + \alpha_3^0(\theta) + a_0\\
&= a_0\\
&> 0.
\end{aligned} \tag{A.26}$$

The continuity on τ and inequalities (A.24), (A.25) yield that for each fixed sufficiently small $\tau > 0$ and each $s \in (0,1]$, the function $F_i\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(\theta,\tau),s,\theta,\tau\big)$ is strictly decreasing with respect to $\theta \in [0,1]$ for $i = 1,2$. Hence for $\theta \in (0,1)$,

$$F_1\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(\theta,\tau),s,\theta,\tau\big) < F_1\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(0,\tau),s,0,\tau\big) = 0$$
$$F_2\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(\theta,\tau),s,\theta,\tau\big) > F_2\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(1,\tau),s,1,\tau\big) = 0. \tag{A.27}$$

Moreover the inequality (A.26) implies that for all sufficiently small $\tau > 0$ and $\theta \in [0,1]$, $F_3\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(\theta,\tau),s,\theta,\tau\big)$ is increasing with respect to s. So that for $s \in (0,1)$,

$$F_3\big(\alpha_1(s,\tau),\alpha_2(s,\tau),\alpha_3(\theta,\tau),s,\theta,\tau\big) < F_3\big(\alpha_1(1,\tau),\alpha_2(1,\tau),\alpha_3(\theta,\tau),1,\theta,\tau\big) = 0. \tag{A.28}$$

It follows from the definitions of F_i, $i = 1,2,3$, and (A.27), (A.28) that

$$\begin{aligned}
&\frac{1}{2}w(k(s,\theta,\tau) < z(k(s,\theta,\tau) < \sigma(w(k(s,\theta,\tau)), \qquad (s,\theta) \in (0,1)\times(0,1),\\
&K - a_0 = x(k(0,\theta,\tau) < x(k(s,\theta,\tau)) < x(k(1,\theta,\tau)) = K. \qquad (s,\theta) \in (0,1)\times(0,1). \quad \square
\end{aligned} \tag{A.29}$$

Lemma A.3. *For each sufficiently small $\tau > 0$, $y(k(1,\theta,\tau)) < K$ for all $\theta \in [0,1]$.*

Proof. By the expression (A.11) one has

$$\begin{aligned}
y(k(1,\theta,\tau)) &= h_{21}k_1(1,\theta,\tau) + h_{22}k_2(1,\theta,\tau) + \frac{F_u(K,0)}{\lambda_3}k_3(1,\theta,\tau) + \mathcal{M}_2(k(1,\theta,\tau)) + K\\
&= \tau\zeta(\theta) + O(\tau^2) + K,
\end{aligned} \tag{A.30}$$

where

$$\begin{aligned}
\zeta(\theta) &= h_{21}[\alpha_1(1,0)(1-\theta)+\theta] + h_{22}[1-\theta+\alpha_2(1,0)\theta] + \frac{F_u(K,0)}{\lambda_3}\alpha_3(\theta,0)\\
&= h_{21}[\alpha_1^0(1-\theta)+\theta] + h_{22}[1-\theta+\alpha_2^0\theta] + \frac{F_u(K,0)}{\lambda_3}\alpha_3^0(\theta)\\
&= \big[h_{21} - \frac{F_u(K,0)}{\lambda_3}h_{11}\big][\alpha_1^0(1-\theta)+\theta] + \big[h_{22} - \frac{F_u(K,0)}{\lambda_3}h_{12}\big][1-\theta+\alpha_2^0\theta]\\
&= \big[h_{21} - \frac{F_u(K,0)}{\lambda_3}h_{11}\big][\alpha_1^0(1-\theta)+\theta] + \big[h_{22} - \frac{F_u(K,0)}{\lambda_3}h_{12}\big].
\end{aligned} \tag{A.31}$$

From (A.7) and equalities $\rho F_u(K,0) = \lambda_3\lambda_3$ and $\lambda_3 + \lambda_4 = \rho$ (see (A.4)) it follows that for $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3$,

$$\begin{aligned}
h_{2i} - \frac{F_u(K,0)}{\lambda_3}h_{1i} &= \frac{F_v(K,0)}{h(\lambda_1)}\big[\lambda_i - \rho + \frac{\rho F_u(K,0)}{\lambda_3}\big]\\
&= \frac{F_v(K,0)}{h(\lambda_1)}\big(\lambda_i - \rho + \lambda_4\big)
\end{aligned}$$

$$= \frac{F_v(K,0)}{h(\lambda_1)}\big(\lambda_i - \lambda_3\big)$$
$$= \frac{F_v(K,0)}{\lambda_i - \lambda_4}. \tag{A.32}$$

Substituting (A.32) into (A.31), and noticing $\alpha_2^0 = 1$, we arrive at

$$\zeta(\theta) = F_v(K,0)\left[\frac{\alpha_1^0(1-\theta)+\theta}{\lambda_1-\lambda_4} + \frac{1}{\lambda_2-\lambda_4}\right]. \tag{A.33}$$

Moreover, by the expressions of α_1^0, λ_1, λ_2, and σ_1 (see (A.21), (A.2) and (2.10)),

$$|\alpha_1^0| = \frac{\sqrt{c^4+4c^2\mu_0} - \sqrt{c^4-4c^2G_v(K,0)}}{\sqrt{c^4+4c^2\mu_0} + \sqrt{c^4-4c^2G_v(K,0)}} < 1. \tag{A.34}$$

Noticing that $\lambda_1 > \lambda_2 > 0$ and $\lambda_4 \le 0$, (A.34) implies that

$$\frac{\alpha_1^0(1-\theta)+\theta}{\lambda_1-\lambda_4} + \frac{1}{\lambda_2-\lambda_4} > \frac{1}{\lambda_2-\lambda_4} - \frac{1}{\lambda_1-\lambda_4} = \frac{\lambda_1-\lambda_2}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)} > 0. \tag{A.35}$$

Recall that $F_v(K,0) < 0$, from (A.30), (A.33) and (A.34) if therefore follows that for all small τ,

$$y(k(1,\theta,\tau)) < K \quad \text{for } \theta \in [0,1].$$

In view of (A.7), it is apparent that the above argument works as well for the case of $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$. □

As a consequence of Lemmas A.1–A.3 we immediately have

Corollary A.4. *Let $\tau_0 > 0$ be sufficiently small. Then for each $\tau \in (0,\tau_0]$, $\Delta_i^*(\tau) \in P_i$ for $i = 1,2,3$, where $\Delta_i^*(\tau)$ is defined in (A.10).*

Lemma A.5. *Let $\tau_0 >$ be sufficiently small. Then for each $\tau \in (0,\tau_0]$ and $p \in \Delta_0^*(\tau)$, $\Phi_t(p) \in \Sigma$ for all $t \le 0$ and $\Phi_t(p) \to E_K$ as $t \to -\infty$.*

Proof. Let A be the coefficient matrix of the linear system (A.1). Consider first the case that λ_1, λ_2 and λ_3 are distinct. Let $\mathbf{h}_i^*$ be the eigenvectors of the matrix A^T corresponding to the positive eigenvalue λ_i for $i = 1,2,3$. Then

$$\langle \mathbf{h}_i^*, \mathbf{h}_j\rangle = \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{if } i \neq j, \end{cases}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product. Without loss of generality suppose that $\langle \mathbf{h}_i^*, \mathbf{h}_i\rangle = 1$ for $i = 1,2,3$. Define a function $\mathcal{V} : E_K^U \to \mathbb{R}$ by

$$\mathcal{V}(p)=\sum_{i=1}^{3}\langle \mathbf{h}_i^*, p-E_K\rangle^2$$

for $p=\sum_{i=1}^{3}\mathbf{h}_i k_i+\mathcal{M}(k)+E_K$ with $k=(k_1,k_2,k_3)\in\mathcal{O}$. Then it is apparent that $\mathcal{V}(E_K)=0$ and $\mathcal{V}(p)>0$ for all $p\neq E_K$ since $\mathbf{h}_i^*$, $i=1,2,3$ are linearly independent. Noticing that

$$\begin{aligned}
p-E_K &= O(\|k\|^2)\\
\Phi_t'(p)\big|_{t=0} &= A(p-E_K)+O(\|p-E_K\|^2)=A\textstyle\sum_{i=j}^{3}\mathbf{h}_j k_j+O(\|k\|^2),
\end{aligned}$$

it follows that the derivative of $\mathcal{V}(p)$ along the system (2.4) is

$$\begin{aligned}
\dot{\mathcal{V}}(p) &= 2\sum_{i=1}^{3}\langle \mathbf{h}_i^*, p-E_K\rangle\langle \mathbf{h}_i^*, \Phi_t'(p)\big|_{t=0}\rangle\\
&= 2\sum_{i=1}^{3}\langle \mathbf{h}_i^*, \sum_{j=1}^{3}\mathbf{h}_j k_j\rangle\langle \mathbf{h}_i^*, A\sum_{i=j}^{3}\mathbf{h}_j k_j\rangle+O(\|k\|^3)\\
&= 2\sum_{i=1}^{3}\langle \mathbf{h}_i^*, \sum_{j=1}^{3}\mathbf{h}_j k_j\rangle\langle \mathbf{h}_i^*, \sum_{i=j}^{3}\mathbf{h}_j\lambda_j k_j\rangle+O(\|k\|^3)\\
&= 2\sum_{i=1}^{3}\lambda_i k_i^2+O(\|k\|^3). \qquad (A.36)
\end{aligned}$$

Therefore, $\dot{\mathcal{V}}(p)>0$ for all small k. Let $\Delta(0,\tau_0]=\cup_{\tau\in(0,\tau_0]}\Delta_0^*(\tau)$. Then the set

$$\Lambda=\{p\in E_K^U:\mathcal{V}(p)\le\epsilon\}\cap\Delta(0,\tau_0]$$

is negatively invariant. It is obvious that $\Delta(\tau)\subset\Lambda$ for all sufficiently small $\tau>0$. That is, any solution $\Phi_t(p)$ through a point $p\in\Delta(\tau)$ will stays in $\Delta(\tau)\subset\Sigma$ for all $t\le 0$ and $\Phi_t(p)\to E_K$ as $t\to-\infty$. It is clear that the argument is valid for the case of repeated eigenvalue if we replace the eigenvector by the corresponding eigenvector. □

Proof of Theorem 3.1. It is apparent that Theorem 3.1 is an immediate consequence of Corollary A.4 and Lemma A.5 by letting $\Delta_i=\Delta_i^*(\tau)$, with $i=0,1,2,3$, for any fixed, sufficiently small $\tau>0$. □

A final remark. In this section we have restricted our discussion under the condition $c>2\sqrt{G_v(K,0)}$, which implies that λ_1 and λ_2 are distinct eigenvalues. Recall that throughout this paper it is assumed $c\ge 2\sqrt{M}$, and in addition the Assumption **A2** implies that $0<G_v(K,0)\le M$. Hence the condition $c>2\sqrt{G_v(K,0)}$ is automatic if $G_v(K,0)<M$. However, if $G_v(K,0)=M$. Then $c_*=2\sqrt{G_v(K,0)}$ is a minimum wave speed. Notice that if $c=c_*$, then $\lambda_1=\lambda_2$ are repeated eigenvalues. In this case all proofs for Lemmas A.1–A.3, A.5 and Corollary A.4 remain valid if we replace the eigenvector by the corresponding generalized eigenvector. We shall omit the detailed computation for the case of $c=2\sqrt{G_v(K,0)}$ in order to reduce

the length of the paper. The alternative way to handle the case $c = c_* = 2\sqrt{G_v(K, 0)}$ when $G_v(K, 0) = M$ is to pick a sequence $c_n > c_*$ with $c_n \to c_*$ as $n \to \infty$ and show that the corresponding semi-traveling wave solutions converge to a semi-traveling wave corresponding to the minimum wave speed.

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