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THE PONTRJAGIN CLASS OF A SASAKIAN SUBMERSION

A formula expressing the relation between the Pontrjagin classes of a submanifold M of a Riemannian manifold $\tilde{M}$ and those of $\tilde{M}$ is wellknown. On the other hand, some properties of a Riemannian submersion, especially a Riemannian submersion with only totally geodesic fibres, are often compared to some properties of a submanifold of a Riemannian manifold. For example, there are equations such as the co-Gauss equation and the co-Codazzi equation in a Riemannian submersion. Such was the motive of the present study. First the relation between the Pontrjagin classes of the total manifold $\tilde{M}$ and those of the base manifold M in a Riemannian submersion $\pi:\tilde{M}\to M$ was the object of investigation. Unfortunately no clear-cut formula usable in differential geometry was obtained in general cases, and the study must be limited to special cases. Thus we chose a Sasakian submersion where the submersion is not trivial and the base manifold M is a Kaehler manifold. The final result is the relation between the Pontrjagin classes of $\tilde{M}$ and the Chern classes of M.

1. Introduction

A Sasakian submersion is a Riemannian submersion $\pi:(\tilde{M},\tilde{g},\tilde{\xi})\to(M,g,J)$, where $(\tilde{M},\tilde{g},\tilde{\xi})$ is a Sasakian manifold and (M,g,J) is a Kaehler manifold and such that each fiber is a geodesic of $(\tilde{M},\tilde{g})$ tangent to the unit Killing vector $\tilde{\xi}$ at each point q.

Let us first recollect some properties of a Sasakian submersion ([4], [5], [8]). We consider a Sasakian manifold $(\tilde{M},\tilde{g},\tilde{\xi})$ of dimension $n+1$, where $n=2m$([1], [7]). $\tilde{\xi}$ satisfies

$$(*)\qquad \tilde{K}_{TCB}{}^{A}\tilde{\xi}^{T}=\tilde{g}_{CB}\tilde{\xi}^{A}-\delta_{C}^{A}\tilde{\xi}_{B},\qquad \tilde{\xi}_{B}=\tilde{g}_{BA}\tilde{\xi}^{A},$$

where $\tilde{K}_{DCB}{}^{A}$ and $\tilde{g}_{CB}$ are the local components of the curvature tensor and the metric tensor, respectively, of the manifold. We use the indices running as follows,

$$A,B,C,\dots,R,S,T,\dots=0,1,\dots,n,$$

$$h,i,j,\dots,r,s,t,\dots=1,\dots,n$$

and adopt the summation convention. We assume that the Sasakian manifold under consideration admits a Riemannian submersion satisfying the following conditions. The unit Killing vector field $\tilde{\xi}$ is the vertical vector field of the submersion, and the base manifold (M,g,J) is a Kaehler manifold of complex

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dimension m. Covering M with a set of suitable coordinate neighborhoods U we can assign to each point q of $\pi^{-1}U$ and $p = \pi q$ the local coordinates $x^0, x^1, \dots, x^n$ and $x^1, \dots, x^n$, respectively, where x^0 acts as a parameter on the fiber $\pi^{-1}p$. Then the Riemannian submersion is called a Sasakian submersion. With the use of such local coordinates we get $\tilde{\xi}^h = 0$ for the local components $\tilde{\xi}^A$ of $\tilde{\xi}$. Furthermore, with a suitable choice of the parameter x^0 we get

$$\tilde{\xi}^0 = 1, \qquad \tilde{g}_{00} = 1.$$

We define $\Gamma_j = \tilde{g}_{j0}$. Then we get $\Gamma_j = \tilde{g}_{jA}\tilde{\xi}^A = \tilde{\xi}_j$. The local components g_{ji} of the Riemannian metric g of M are given by $g_{ji} = \tilde{g}_{ji} - \Gamma_j\Gamma_i$ and the contravariant components g^{ji} of g defined by $g^{ti}g_{tj} = \delta_j^i$ satisfy $g^{ji} = \tilde{g}^{ji}$, where $\tilde{g}^{BA}$ are the contravariant components of $\tilde{g}$. We have, moreover,

$$\tilde{g}^{00} = 1 + \Gamma_j\Gamma_i g^{ji} = 1 + \Gamma^t\Gamma_t, \qquad \tilde{g}^{0i} = \tilde{g}^{i0} = -\Gamma^i \tag{1.1}$$

where $\Gamma^i = g^{it}\Gamma_t$. Since $\tilde{\xi}$ is a Killing vector field, Γ_i satisfies $\partial_0\Gamma_i = 0$, where $\partial_A = \partial/\partial x^A$. The complex structure J of the base manifold comes from ([4], [5], [8])

$$J_{ji} = \tfrac{1}{2}(\partial_j\Gamma_i - \partial_i\Gamma_j), \qquad J_i{}^h = J_{it}g^{th}. \tag{1.2}$$

Remark 1. In [5] we introduced a (1, 2)-tensor field $\tilde{R}$ on $\tilde{M}$ whose local components are

$$R_{ji}{}^{\kappa} = D_j\Gamma_i^{\kappa} - D_i\Gamma_j^{\kappa}, \qquad D_j = \partial_j - \Gamma_j^{\tau}\partial_{\tau}$$

and some vanishing ones. In the present study, where the dimension of the fiber is one, Γ_i^{κ} and $R_{ji}{}^{\kappa}$ should be replaced by Γ_i and R_{ji}, respectively; hence R_{ji} corresponds to $2J_{ji}$ ([4], [8]).

In order to find the relation between the curvature tensor of $\tilde{M}$ and the curvature tensor of M we take their covariant associates with local components $\tilde{K}_{DCBA} = \tilde{K}_{DCB}{}^T\tilde{g}_{TA}$ and $K_{kjih} = K_{kji}{}^t g_{th}$. As the dimension of the fiber is one, we can decompose the one of $\tilde{M}$ into nine parts

$$\begin{aligned}\tilde{K}_{\mathrm{HHHH}} &+ \tilde{K}_{\mathrm{HHHV}} + \tilde{K}_{\mathrm{HHVH}} + \tilde{K}_{\mathrm{HVHH}} + \tilde{K}_{\mathrm{VHHH}} \\ &+ \tilde{K}_{\mathrm{HVHV}} + \tilde{K}_{\mathrm{HVVH}} + \tilde{K}_{\mathrm{VHHV}} + \tilde{K}_{\mathrm{VHVH}}.\end{aligned}$$

On the other hand, putting $\tilde{\xi}^A = \delta_0^A$ in (*) we get $\tilde{K}_{0CB}{}^A = \tilde{g}_{CB}\delta_0^A - \delta_C^A\tilde{g}_{B0}$, hence $\tilde{K}_{0CBA} = \tilde{g}_{CB}\tilde{g}_{0A} - \tilde{g}_{CA}\tilde{g}_{0B}$. As the Riemannian metric tensor $\tilde{g}$ is decomposed into two parts, $\tilde{g}_{\mathrm{HH}} + \tilde{g}_{\mathrm{VV}}$, namely into the completely horizontal part and the completely vertical part, $\tilde{K}_{\mathrm{VHHH}}, \tilde{K}_{\mathrm{HVHH}}, \tilde{K}_{\mathrm{HHVH}}, \tilde{K}_{\mathrm{HHHV}}$ vanish. The leading local components of the remaining parts are ([4], [5])

$$\begin{aligned}(\tilde{K}_{\mathrm{HHHH}})_{kjih} &= K_{kjih} - J_{kh}J_{ji} + J_{ki}J_{jh} + 2J_{kj}J_{ih}, \\ (\tilde{K}_{\mathrm{HVHV}})_{k0i0} &= -J_k{}^tJ_{it} = -g_{ki},\end{aligned}$$

$$(\tilde{K}_{\mathrm{HVVH}})_{k00h} = g_{kh},$$
$$(\tilde{K}_{\mathrm{VHHV}})_{0ji0} = g_{ji},$$
$$(\tilde{K}_{\mathrm{VHVH}})_{0j0h} = -g_{jh}.$$

The other local components are obtained when the rule of decomposition $\tilde{U} = \tilde{U}_{\mathrm{H}} + \tilde{U}_{\mathrm{V}}$ of a covariant vector into the horizontal part and the vertical part is recalled, which can be written in the present case as follows,

$$(\tilde{U}_{\mathrm{H}})_i = \tilde{U}_i - \Gamma_i \tilde{U}_0, \qquad (\tilde{U}_{\mathrm{H}})_0 = 0,$$
$$(\tilde{U}_{\mathrm{V}})_i = \Gamma_i \tilde{U}_0, \qquad (\tilde{U}_{\mathrm{V}})_0 = \tilde{U}_0.$$

Thus we have

$$(1.3) \qquad \tilde{K}_{kjih} = K_{kjih} - J_{kh}J_{ji} + J_{ki}J_{jh} + 2J_{kj}J_{ih} + \Gamma_k g_{ji}\Gamma_h - g_{ki}\Gamma_j\Gamma_h + g_{kh}\Gamma_j\Gamma_i - \Gamma_k g_{jh}\Gamma_i,$$

$$(1.4) \qquad \tilde{K}_{0jih} = g_{ji}\Gamma_h - g_{jh}\Gamma_i, \tilde{K}_{kj0h} = g_{kh}\Gamma_j - \Gamma_k g_{jh},$$

$$(1.5) \qquad \tilde{K}_{0j0h} = -g_{jh}$$

which are the co-Gauss and the co-Codazzi equations [2] in the case of a Sasakian submersion. As (M, g, J) is a Kaehler manifold, the curvature tensor of M satisfies

$$(1.6) \qquad J_t{}^h K_{kji}{}^t = K_{kjt}{}^h J_i{}^t.$$

The tensor-valued 2-form on $\tilde{M}$ whose local expression is $\tilde{K}_{DCB}{}^A \, dx^D \, dx^C$ is called the curvature form of $\tilde{M}$. In the present paper $\tilde{K}_B{}^A$ means $\tilde{K}_{DCB}{}^A \, dx^D \, dx^C$ and does not mean the Ricci tensor. Similarly $\tilde{K}$ means the $(n+1)$-rowed square matrix $[\tilde{K}_B{}^A]$. Here we consider matrices over the graded algebra generated by 1- and 2-forms on M, especially $(n+1)^2$ 2-forms $\tilde{K}_B{}^A$ $(A, B = 0, 1, \dots, n)$. Then we have

$$(1.7) \qquad \det(1 + z\tilde{K}) = 1 + \tilde{\kappa}_2 z^2 + \tilde{\kappa}_4 z^4 + \cdots + \tilde{\kappa}_n z^n,$$

where z is an unknown and $\tilde{\kappa}_{2k}$ is a $4k$-form, hence vanishes if $4k > n + 1$. The kth Pontrjagin class $\tilde{p}_k = [\tilde{\mathfrak{p}}_k]$ of $\tilde{M}$ is represented by the $4k$-form [3]

$$(1.8) \qquad \tilde{\mathfrak{p}}_k = (2\pi)^{-2k}\tilde{\kappa}_{2k}.$$

Here and in the sequel a cohomology class represented by a closed form α is denoted by $[\alpha]$.

Let Ω be the curvature form of the Kaehler manifold (M, g, J) in complex form. Hence $\Omega = [\Omega_{\beta\alpha}]$ is an m-rowed square matrix and we can put

$$(1.9) \qquad \det(1 - \zeta\Omega) = 1 + \sigma_1\zeta + \sigma_2\zeta^2 + \cdots + \sigma_m\zeta^m.$$

Then the kth Chern class $c_k = [\mathfrak{c}_k]$ of M is represented by the $2k$-form [3]

$$(1.10) \qquad \mathfrak{c}_k = (2\pi i)^{-k}\sigma_k.$$

Our main aim is to prove the formula

$$(1.11)\qquad \sum_{k=0}^{\infty}(4\pi^2)^k\tilde{p}_k=\pi^*\left((1+4[j]^2)^{m+1}\left|\sum_{k=0}^{\infty}c_k(2\pi i\mu)^k\right|^2\right)$$

where j is the Kaehler form $J_{kj}\,\mathrm{d}x^k\,\mathrm{d}x^j$ and

$$(1.12)\qquad \mu=2/(1-2i[j]).$$

For this purpose we must express $\det(1+\tilde{K})$ in terms of structures in the base manifold (M,g,J). This is done in Section 2 and the formula $\det(1+\tilde{K})=\pi^*\det(1+K^*)$ is obtained. K^* is an n-rowed square matrix which is still a little complicated. In order to get a simpler formula

$$\det(1+\tilde{K})=\pi^*((1+4j^2)\det(1+K+2jJ)),$$

where J is the matrix of the complex structure and K is the matrix of the curvature form on M, we use a property of square matrices without eigenvalues, namely the expansion formula

$$\det(1+Az)=\exp\left[\alpha_1 z-\tfrac{1}{2}\alpha_2 z^2+\tfrac{1}{3}\alpha_3 z^3-\cdots\right],$$

$$\alpha_k=\operatorname{trace}A^k,$$

which is proved in Section 3. In Section 4 we make use of the identity

$$1+Az+Bz=(1+Az)(1+(1+Az)^{-1}Bz),$$

calculate $\det(1+(1+Az)^{-1}Bz)$ with the use of the result obtained in Section 3 and get the formula stated above. Section 5 is devoted to the proof of the main theorem.

Remark 2. The way taken in Section 2 seems to be somewhat roundabout, but this is preferred in the present paper since a similar way may more generally be taken in the case of a Riemannian submersion than in Sasakian submersions.

Remark 3. The series such as

$$\alpha_1 z-\tfrac{1}{2}\alpha_2 z^2+\tfrac{1}{3}\alpha_3 z^3-\cdots$$

are essentially finite in our case as the dimension of M is finite. For the same reason we can replace z with 1 and write

$$\det(1+A)=\exp\left[\alpha_1-\tfrac{1}{2}\alpha_2+\tfrac{1}{3}\alpha_3-\cdots\right].$$

2. Reduction of $\det(1+\tilde{K})$ to $\pi^*\det(1+K^*)$

We use the following notation:

$$K_{kji}{}^h\,\mathrm{d}x^k\,\mathrm{d}x^j=K_i{}^h,\qquad K_{kjih}\,\mathrm{d}x^k\,\mathrm{d}x^j=K_{ih},$$

$$J_{ki}\,\mathrm{d}x^k=j_i,\qquad J_k{}^h\,\mathrm{d}x^k=j^h,$$

$$J_{kj}\, dx^k\, dx^j = j, \qquad \Gamma_k\, dx^k = \gamma,$$
$$g_{ki}\, dx^k = g_i, \qquad dx^h = \delta^h.$$

We can treat γ as a 1-form on U since we have $\partial_0 \Gamma_i = 0$. Hence all forms given above are forms on U or can be treated as forms on U. As the curvature tensor of M satisfies the Bianchi identity and (1.6) we have

$$K_{it}\delta^t = K_i{}^t g_t = 0, \tag{2.1}$$

$$K_{it} j^t = K_i{}^t j_t = 0. \tag{2.2}$$

Besides, we have

$$j^t j_t = 0, \qquad j^t J_t{}^h = -\delta^h, \qquad j^t g_t = -g_t j^t = j. \tag{2.3}$$

Now from the definition of $\tilde{K}_B{}^A$ and $\tilde{K}_{BA} = \tilde{K}_B{}^T \tilde{g}_{TA}$ we have, in view of (1.3), (1.4), (1.5),

$$\begin{aligned}
\tilde{K}_{ih} &= K_{ih} + 2j_i j_h + 2j J_{ih} \\
&\qquad + 2(dx^0 + \gamma)(g_i \Gamma_h - g_h \Gamma_i), \\
\tilde{K}_{0h} &= 2g_h(dx^0 + \gamma), \\
\tilde{K}_{i0} &= 2(dx^0 + \gamma) g_i = -\tilde{K}_{0i}, \qquad \tilde{K}_{00} = 0.
\end{aligned}$$

These formulas, and some others which appear consecutively, are not written exactly in the sense that in the second or third members forms in U and forms in $\pi^{-1}U$ are mingled. Exact expressions are obtained when every form in U is pulled back into $\pi^{-1}U$ by π^*. This process is shortened except in the final result (2.4), as no confusion can occur.

Thus, for the elements of the matrix $\tilde{K}$, we obtain

$$\begin{aligned}
\tilde{K}_i{}^h &= \tilde{K}_{it} g^{th} + \tilde{K}_{i0} \tilde{g}^{0h} \\
&= K_i{}^h + 2j_i j^h + 2j J_i{}^h - 2(dx^0 + \gamma)\delta^h \Gamma_i, \\
\tilde{K}_i{}^0 &= \tilde{K}_{it} \tilde{g}^{t0} + \tilde{K}_{i0} \tilde{g}^{00} \\
&= -(K_{it} + 2j_i j_t + 2j J_{it})\Gamma^t \\
&\qquad - 2(dx^0 + \gamma) g_i \Gamma^t \Gamma_t + 2(dx^0 + \gamma)\Gamma^t g_t \Gamma_i \\
&\qquad + 2(dx^0 + \gamma) g_i (1 + \Gamma^t \Gamma_t) \\
&= -(K_{it} + 2j_i j_t + 2j J_{it})\Gamma^t \\
&\qquad + 2(dx^0 + \gamma) g_i + 2\, dx^0 \gamma \Gamma_i, \\
\tilde{K}_0{}^h &= \tilde{K}_{0t} g^{th} = -2(dx^0 + \gamma)\delta^h, \\
\tilde{K}_0{}^0 &= \tilde{K}_{0t} \tilde{g}^{t0} = 2(dx^0 + \gamma)\Gamma^t g_t = 2\, dx^0 \gamma,
\end{aligned}$$

hence,

$$\det(1+\tilde{K}) = \begin{vmatrix} 1+\tilde{K}_0{}^0 & \tilde{K}_i{}^0 \\ \tilde{K}_0{}^h & \delta_i^h + \tilde{K}_i{}^h \end{vmatrix}$$

$$= \begin{vmatrix} 1+\tilde{K}_0{}^0 + \Gamma_s \tilde{K}_0{}^s & \tilde{K}_i{}^0 + \Gamma_s(\delta_i^s + \tilde{K}_i{}^s) \\ \tilde{K}_0{}^h & \delta_i^h + \tilde{K}_i{}^h \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \Gamma_i + 2(\mathrm{d}x^0+\gamma)g_i \\ -2(\mathrm{d}x^0+\gamma)\delta^h & \delta_i^h + K_i{}^h + 2j_i j^h + 2jJ_i{}^h \\ & \qquad - 2(\mathrm{d}x^0+\gamma)\delta^h\Gamma_i \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2(\mathrm{d}x^0+\gamma)g_i \\ -2(\mathrm{d}x^0+\gamma)\delta^h & \delta_i^h + K_i{}^h + 2j_i j^h + 2jJ_i{}^h \end{vmatrix}.$$

Let us define the following n-rowed square matrix K^*,

$$K^* = [K^*{}_i{}^h],$$

$$K^*{}_i{}^h = K_i{}^h + 2jJ_i{}^h + 2j_i j^h.$$

Then we get, in virtue of $(\mathrm{d}x^0+\gamma)^2 = 0$,

$$\det(1+\tilde{K}) = \pi^* \det(1+K^*). \tag{2.4}$$

3. A property of a square matrix with no eigenvalues

We want to present here a property of an n-rowed square matrix over a graded algebra

$$\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_1 + \mathscr{A}_2 + \cdots;$$

for example, the graded algebra generated by 1- and 2-forms on some manifold M. Such a matrix has in general no eigenvalues.

If

$$A = \begin{bmatrix} a_1{}^1 \dots a_n{}^1 \\ \vdots \qquad \vdots \\ a_1{}^n \dots a_n{}^n \end{bmatrix} \qquad a_i{}^h \in \mathscr{A}_1, \tag{3.1}$$

is such a matrix, there exists a sequence $(s_1, \dots, s_n)$ such that

$$\det(1+Az) = 1 + s_1 z + s_2 z^2 + \cdots + s_n z^n, \qquad s_k \in \mathscr{A}_k. \tag{3.2}$$

We show that there exists another expansion formula

$$\log\det(1+Az) = \alpha_1 z - \tfrac{1}{2}\alpha_2 z^2 + \tfrac{1}{3}\alpha_3 z^3 - \cdots, \tag{3.3}$$

where the second member is essentially a finite series and

$$\alpha_k = \operatorname{trace} A^k.$$

If there exist eigenvalues $\lambda_1, \dots, \lambda_n$, we have an immediate proof. s_k is then the kth fundamental symmetric function of $\lambda_1, \dots, \lambda_n$ and

$$\text{trace } A^k = \sum_{i=1}^{n} (\lambda_i)^k.$$

From $\det(1 + Az) = (1 + \lambda_1 z)\dots(1 + \lambda_n z)$ we get

$$\begin{aligned}\log\det(1 + Az) &= \sum_{i=1}^{n} \log(1 + \lambda_i z)\\ &= \sum_{i=1}^{n} \left[\lambda_i z - \tfrac{1}{2}(\lambda_i z)^2 + \tfrac{1}{3}(\lambda_i z)^3 - \cdots\right]\\ &= \alpha_1 z - \tfrac{1}{2}\alpha_2 z^2 + \tfrac{1}{3}\alpha_3 z^3 - \cdots.\end{aligned}$$

Now we consider the case where A has no eigenvalues. An expansion formula of the type (3.2) is still valid, but we must consider that $s_1, \dots, s_n$ are determined directly by (3.2) and the polynomial $1 + s_1 z + \cdots + s_n z^n$ admits in general no factorization. Nevertheless, we can put

$$\log\det(1 + Az) = \xi_1 z - \tfrac{1}{2}\xi_2 z^2 + \tfrac{1}{3}\xi_3 z^3 - \cdots,$$

where the coefficients ξ_k $(k = 1, 2, \dots)$ are determined by A as polynomials $P_k(a_1{}^1, a_1{}^2, \dots, a_n{}^{n-1}, a_n{}^n)$ in the elements of A. Hence, we get $\xi_k = \alpha_k = \text{trace}$ A^k. (3.3) is also valid when the matrix A is a little more general; namely, when the elements of A are elements of $\mathscr{A}$.

Thus we have the following lemma.

LEMMA 3.1. *Let $\mathscr{A}$ be a commutative graded algebra and A be a square matrix where the elements belong to $\mathscr{A}$. Then the following expansion formula is valid*

$$\det(1 + Az) = \exp\left[\alpha_1 z - \tfrac{1}{2}\alpha_2 z^2 + \tfrac{1}{3}\alpha_3 z^3 - \cdots\right],$$
$$\alpha_k = \text{trace } A^k.$$

4. Reduction of $\det(1 + K^*)$ to $(1 + 4j^2)\det(1 + K + 2jJ)$.

We first notice that

$$\begin{aligned}(4.1)\qquad \det(1 + Az + Bz) &= \det(1 + Az)\det(1 + (1 + Az)^{-1}Bz)\\ &= \det(1 + Az) \times\\ &\quad \times \det(1 + Bz - ABz^2 + A^2Bz^3 - A^3Bz^4 + \cdots)\end{aligned}$$

and put for the elements of A and B

$$A_i{}^h = K_i{}^h + 2jJ_i{}^h, \qquad B_i{}^h = 2j_i j^h.$$

Then we have in view of (2.1), (2.2) and (2.3)

$$(AB)_i^{\ h} = A_t^{\ h} B_i^{\ t} = 2(K_t^{\ h} + 2jJ_t^{\ h}) j_i j^t$$
$$= -4jj_i \delta^h,$$
$$(A^2 B)_i^{\ h} = (K_t^{\ h} + 2jJ_t^{\ h})(-4jj_i \delta^t) = -8j^2 j_i j^h$$
$$= -4j^2 B_i^{\ h}.$$

We thus obtain

$$(1 + Az)^{-1} Bz = (1 + 4j^2 z^2)^{-1} (Bz - ABz^2),$$

hence,

$$\det(1 + (1 + Az)^{-1} Bz) =$$
$$= \det(1 + (1 + 4j^2 z^2)^{-1} (Bz - ABz^2)),$$

where

$$(Bz - ABz^2)_i^{\ h} = (2j_i j^h + 4jj_i \delta^h z) z.$$

As we have, in view of (2.3),

$$(2j_t j^h + 4jj_t \delta^h z)(2j_i j^t + 4jj_i \delta^t z)$$
$$= -4j^2 (2j_i j^h + 4jj_i \delta^h z) z,$$

we get

$$(Bz - ABz^2)^{k+1} = (-4j^2 z^2)^k (Bz - ABz^2),$$

hence,

$$\operatorname{trace}(Bz - ABz^2)^{k+1} = (-4j^2 z^2)^k \operatorname{trace}(Bz - ABz^2)$$
$$= -(-4j^2 z^2)^{k+1}.$$

In view of Lemma 3.1 we get

$$\log\det(1 + (1 + Az)^{-1} Bz)$$
$$= \log\det(1 + (1 + 4j^2 z^2)^{-1} (Bz - ABz^2))$$
$$= \sum_{k=1}^{\infty} \frac{1}{k} (1 + 4j^2 z^2)^{-k} (4j^2 z^2)^k$$
$$= -\log(1 - (1 + 4j^2 z^2)^{-1} 4j^2 z^2)$$
$$= \log(1 + 4j^2 z^2).$$

This and (4.1) prove

$$\det(1 + K^* z) = (1 + 4j^2 z^2) \det(1 + (K + 2jJ) z).$$

In view of (2.4) we get the following theorem.

THEOREM 4.1. *Let $\pi:(\tilde{M}, \tilde{g}, \tilde{\xi}) \to (M, g, J)$ be a Sasakian submersion. Then the matrix $\tilde{K}$ of the curvature form on $\tilde{M}$, the matrix K of the curvature form on M, the complex structure J and the Kaehler form j satisfy*

$$\det(1 + \tilde{K}) = \pi^*((1 + 4j^2)\det(1 + K + 2jJ)). \tag{4.2}$$

5. THE RELATION BETWEEN THE PONTRJAGIN CLASS OF $\tilde{M}$ AND THE CHERN CLASS OF M

Given any point p in the Kaehler manifold (M, g, J) we can take a neighborhood U of p in M such that there exists an orthonormal frame field consisting of $2m$ vector fields $X_1, \dots, X_m, Y_1, \dots, Y_m$ satisfying

$$JX_\alpha = Y_\alpha, \qquad JY_\alpha = -X_\alpha \qquad (\alpha, \beta = 1, \dots, m); \tag{5.1}$$

namely,

$$J_t{}^h X_\alpha{}^t = Y_\alpha{}^h, \qquad J_t{}^h Y_\alpha{}^t = -X_\alpha{}^h,$$

where $X_\alpha{}^h$ and $Y_\alpha{}^h$ are the local components of the vectors X_α and Y_α, respectively. From the curvature form $K = [K_i{}^h]$ of M we define

$$L_{\beta\alpha} = K_{ih} X_\beta{}^i X_\alpha{}^h, \qquad M_{\beta\alpha} = K_{ih} X_\beta{}^i Y_\alpha{}^h, \tag{5.2}$$

where $K_{ih} = K_i{}^t g_{th}$. Then we have, in view of (1.6),

$$L_{\beta\alpha} = K_{ih} Y_\beta{}^i Y_\alpha{}^h, \qquad M_{\beta\alpha} = -K_{ih} Y_\beta{}^i X_\alpha{}^h. \tag{5.3}$$

We have $L_{\alpha\beta} = -L_{\beta\alpha}$, $M_{\alpha\beta} = M_{\beta\alpha}$. The m-rowed square matrix

$$\Omega = [\Omega_{\beta\alpha}], \qquad \Omega_{\beta\alpha} = \tfrac{1}{2}(-L_{\beta\alpha} + iM_{\beta\alpha}) \tag{5.4}$$

is the curvature form of the complex manifold M. We denote by $\bar{\Omega}_{\beta\alpha}$ the complex conjugate of $\Omega_{\beta\alpha}$.

Now let σ_k and $\bar{\sigma}_k$ be defined by the expansion formulas

$$\det(1 - \Omega) = 1 + \sigma_1 + \sigma_2 + \cdots, \tag{5.5}$$

$$\det(1 - \bar{\Omega}) = 1 + \bar{\sigma}_1 + \bar{\sigma}_2 + \cdots, \tag{5.5$'$}$$

then we have

$$\bar{\sigma}_k = (-1)^k \sigma_k$$

and the Chern class $c_k = [c_k]$ is represented by [3]

$$c_k = (2\pi i)^{-k} \sigma_k. \tag{5.6}$$

As we have

$$K_{ih} + 2jJ_{ih} = (K_i{}^t + 2jJ_i{}^t) g_{th},$$

we get

$$(K_{ih} + jJ_{ih})X_\beta{}^i X_\alpha{}^h = L_{\beta\alpha},$$
$$(K_{ih} + 2jJ_{ih})X_\beta{}^i Y_\alpha{}^h = M_{\beta\alpha} + 2j\delta_{\beta\alpha},$$
$$(K_{ih} + 2jJ_{ih})Y_\beta{}^i X_\alpha{}^h = -M_{\beta\alpha} - 2j\delta_{\beta\alpha},$$
$$(K_{ih} + 2jJ_{ih})Y_\beta{}^i Y_\alpha{}^h = L_{\beta\alpha}.$$

Let us define n-rowed square matrices

$$S = \begin{bmatrix} X_1{}^1 & \dots & X_m{}^1 & Y_1{}^1 & \dots & Y_m{}^1 \\ \vdots & & & & & \\ X_1{}^n & \dots & X_m{}^n & Y_1{}^n & \dots & Y_m{}^n \end{bmatrix}$$

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} E & E \\ -iE & iE \end{bmatrix},$$

where E is the m-rowed unit matrix. Then we get

$$T^{-1}S^{-1}(K + 2jJ)ST = \begin{bmatrix} L + i(M + 2jE) & 0 \\ 0 & L - i(M + 2jE) \end{bmatrix},$$

where $L = [L_{\beta\alpha}]$, $M = [M_{\beta\alpha}]$. As this is equivalent to

$$T^{-1}S^{-1}(K + 2jJ)ST = \begin{bmatrix} -2\bar{\Omega} + 2ijE & 0 \\ 0 & -2\Omega - 2ijE \end{bmatrix},$$

we get

$$\text{(5.7)} \qquad \det(1 + (K + 2jJ)z) = \det(E + 2(-\Omega - ijE)z)\det(E + 2(-\bar{\Omega} + ijE)z).$$

Now our algebra here is the graded algebra generated by 1- and 2-forms on M. Hence, we can consider

$$(1 - 2izj)^{-1} = \sum_{k=0}^{\infty} (2izj)^k$$

where the second member is essentially a finite series and get

$$\det(E + 2(-\Omega - ijE)z) = \det((1 - 2izj)E - 2\Omega z) = (1 - 2izj)^m \det(E - \Omega\mathfrak{m}(z)),$$

where

$$\text{(5.8)} \qquad \mathfrak{m}(z) = 2z/(1 - 2izj)$$

and, similarly,

$$\det(E + 2(-\bar{\Omega} + ijE)z) = (1 + 2izj)^m \det(E - \bar{\Omega}\bar{\mathfrak{m}}(z)).$$

In view of (5.5), (5.5)′, (5.7) and Theorem 4.1 we get

$$(5.9) \qquad \det(1 + \tilde{K}z) = \pi^*((1 + 4j^2z^2)^{m+1}|1 + \sigma_1\mathfrak{m}(z) + \sigma_2(\mathfrak{m}(z))^2 + \cdots|^2).$$

As we have

$$\det(1 + \tilde{K}z) = \sum_{k=0}^{\infty} (4\pi^2)^k \tilde{\mathfrak{p}}_k z^{2k}$$

and (5.6) we get the following lemma.

LEMMA 5.1. *Let $\pi:(\tilde{M}, \tilde{g}, \tilde{\xi}) \to (M, g, J)$ be a Sasakian submersion, where* $\dim \tilde{M} = 2m + 1$. *Then we have*

$$(5.10) \qquad \sum_{k=0}^{\infty} (4\pi^2)^k \tilde{\mathfrak{p}}_k = \pi^*\left((1 + 4j^2)^{m+1}\left|\sum_{k=0}^{\infty} \mathfrak{c}_k (2\pi i\mathfrak{m})^k\right|^2\right),$$

$$(5.11) \qquad \mathfrak{m} = 2/(1 - 2ij),$$

where j is the Kaehler form, $\tilde{\mathfrak{p}}_k$ are defined by (1.7) and (1.8) and $\mathfrak{c}_k$ by (1.10).

Thus we can express $\tilde{\mathfrak{p}}_k$ as a sum such as

$$(5.12) \qquad \tilde{\mathfrak{p}}_k = \pi^* \sum_{p,q,r} \lambda_{k,p,q,r} j^p \mathfrak{c}_q \mathfrak{c}_r.$$

Applying π^* to cohomology classes of M in the sense that if α is a closed form of M then $\pi^*[\alpha]$ is the cohomology class $[\pi^*\alpha]$ of $\tilde{M}$, we get

$$\tilde{p}_k = \pi^* \sum_{p,q,r} \lambda_{k,p,q,r} [j]^p c_q c_r$$

from (5.12); hence,

$$(5.13) \qquad \sum_{k=0}^{\infty} (4\pi^2)^k \tilde{p}_k = \pi^*\left((1 + 4[j]^2)^{m+1}\left|\sum_{k=0}^{\infty} c_k(2\pi i\mu)^k\right|^2\right),$$

where

$$\mu = 2/(1 - 2i[j]).$$

Thus we have obtained the following theorem.

THEOREM 5.2. *Let $\pi: (\tilde{M}, \tilde{g}, \tilde{\xi}) \to (M, g, J)$ be a Sasakian submersion where* $\dim \tilde{M} = 2m + 1$. *Then the Pontrjagin classes $\tilde{p}_k$ of $\tilde{M}$ and the Chern classes c_k of M satisfy the relation* (5.13).

COROLLARY 5.3. *If in a Sasakian submersion*

$$(1 + 4j^2)^{m+1} \left| \sum_{k=0}^{\infty} c_k (2\pi i m)^k \right|^2 \sim 0$$

holds, then the Pontrjagin classes of M vanish.

A trivial example is the case $\pi: S^{2m+1} \to P^m(C)$.

As a non-trivial example we have the following corollary [6].

COROLLARY 5.4. *If in a Sasakian manifold $(\tilde{M}, \tilde{g}, \tilde{\xi})$ with Sasakian submersion the base manifold is the product of three 2-dimensional spheres, then the Pontrjagin classes of $\tilde{M}$ vanish.*

REFERENCES

1. Hatakeyama, Y., Ogawa, Y. and Tanno, S.: 'Some Properties of Manifolds with Contact Metric Structure'. *Tôhoku Math. J.* **15** (1963), 42–48.
2. Ishihara, S. and Konishi, M.: 'Differential Geometry of Fibred Spaces'. *Publications of the Study Group of Geometry*, **8**, Tokyo, 1973.
3. Kobayashi, S. and Nomizu, K.: *Foundations of Differential Geometry*, Vol. II. Wiley, Interscience, New York, 1969.
4. Mutō, Y.: 'Riemannian Submersions and Critical Riemannian Metrics'. *J. Math. Soc. Japan* **29** (1977), 493–511.
5. Mutō, Y.: 'Riemannian Submersion and the Laplace–Beltrami Operator'. *Kōdai Math. J.* **1** (1978), 329–338.
6. Mutō, Y.: 'A Sasakian Submersion where the Base Manifold is a Product Manifold'. *Tensor*, N.S. **37** (1982), 273–278.
7. Sasaki, S. and Hatakeyama, Y.: 'On Differentiable Manifolds with Contact Metric Structures'. *J. Math. Soc. Japan* **14** (1962), 249–271.
8. Yano, K. and Ishihara, S.: 'Fibred Spaces with Invariant Riemannian Metric'. *Kōdai Math. Sem. Rep.* **19** (1967), 317–360.

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