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Differential equations with locally bounded delay

Hans-Otto Walther *

Mathematisches Institut, Universität Gießen, Arndtstr. 2, D 35392 Gießen, Germany

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ABSTRACT

For differential delay equations of the general form $x'(t) = g(x_t)$ which include equations with unbounded finite state-dependent delays we construct semiflows of continuously differentiable solution operators on suitable Banach manifolds and provide local stable and unstable manifolds at equilibria. Examples occur in feedback systems with delays caused by signal transmission.

1. Introduction

This paper deals with differential equations with state-dependent delays which at each state $\phi : (-\infty, 0] \to \mathbb{R}^n$ take into account only a restriction to a bounded interval. The union of these intervals may be unbounded. Examples are given by models for feedback systems which involve a transmission delay. Section 3 below discusses an equation of this kind.

In order to introduce the framework which will be used we need a bit of notation. We choose a norm on $\mathbb{R}^n$ and indicate derivatives of maps from an interval into $\mathbb{R}^n$ by a prime, as in ϕ'. Recall the familiar segment notation: If a map $x : I \to A$ and $t \in \mathbb{R}$ are given with $(-\infty, t] \subset I$ then the segment, or history of x at t, $x_t : [-\infty, 0] \to A$ is defined by $x_t(s) = x(t+s)$. Let $C = C((-\infty, 0], \mathbb{R}^n)$ and $C^1 = C^1((-\infty, 0], \mathbb{R}^n)$ denote the vector spaces of continuous and continuously differentiable maps $(-\infty, 0] \to \mathbb{R}^n$, respectively.

* Fax: +49 641 9932029.
E-mail address: Hans-Otto.Walther@math.uni-giessen.de.

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Let $a > 0$. Consider the Banach spaces

$$B = \{\phi \in C\colon\ e^{as}\phi(s) \to 0 \text{ as } t \to -\infty\}$$

with the norm given by $|\phi|_B = \sup_{s \leqslant 0} e^{as}|\phi(s)|$ and

$$B^1 = \{\phi \in B \cap C^1\colon\ e^{as}\phi'(s) \to 0 \text{ as } t \to -\infty\}$$

with the norm given by $|\phi|_{B^1} = |\phi|_B + |\phi'|_B$.

We study the initial value problem

$$x'(t) = g(x_t), \tag{1.1}$$

$$x_0 = \phi \tag{1.2}$$

for a continuously differentiable map $g : U \to \mathbb{R}^n$ which is defined on a non-empty open subset $U \subset B^1$ and has the following additional smoothness property:

(E) *Each derivative $Dg(\phi) : B^1 \to \mathbb{R}^n$ has a linear extension $D_e g(\phi) : B \to \mathbb{R}^n$, and the map*

$$U \times B \ni (\phi, \beta) \mapsto D_e g(\phi)\beta \in \mathbb{R}^n$$

is continuous.

These hypotheses on g are designed for applications to equations with state-dependent delay, like their analogues in [23,24,10] for equations with bounded delays.

Corollary 1.1. *For every $\phi \in U$ there exist a neighbourhood $N \subset U$ and $\lambda \geqslant 0$ with*

$$\big|g(\psi) - g(\chi)\big|_{B^1} \leqslant \lambda|\psi - \chi|_B \quad \textit{for all } \psi \in N,\ \chi \in N.$$

Proof. The previous statement about continuity yields that the map $D_e g : U \to L_c(B, \mathbb{R}^n)$ is locally bounded. It follows that there are a convex open neighbourhood N of ϕ in U and $\lambda \geqslant 0$ such that for all ψ, χ in N we have

$$\begin{aligned}\big|g(\psi) - g(\chi)\big|_{B^1} &= \left|\int_0^1 Dg\big(\chi + s(\psi - \chi)\big)(\psi - \chi)\,ds\right| \\ &= \left|\int_0^1 D_e g\big(\chi + s(\psi - \chi)\big)(\psi - \chi)\,ds\right| \\ &\leqslant \lambda|\psi - \chi|_B. \qquad \square\end{aligned}$$

The next hypothesis on g makes precise what we mean by locally bounded delay. We assume that

(LBD) *for each $\phi \in U$ there exist $h > 0$ and a neighbourhood $N \subset U$ so that for all ψ and ρ in N with*

$$\psi(s) = \rho(s) \quad \text{for all } s \in [-h, 0]$$

we have

$$g(\psi) = g(\rho).$$

Finally we assume that

(M) *the set*

$$M = \{\phi \in U: \phi'(0) = g(\phi)\}$$

is non-empty.

All of these hypotheses hold for the example with locally bounded and globally unbounded state-dependent delay in Section 3.

We define a solution of Eq. (1.1) on the interval $[t_0, t_e)$, $t_0 < t_e \leqslant \infty$, to be a continuously differentiable map $x : (-\infty, t_e) \to \mathbb{R}^n$ with $x_t \in U$ for $t_0 \leqslant t < t_e$ such that Eq. (1.1) holds for $t_0 \leqslant t < t_e$. Solutions on $\mathbb{R}$ are defined analogously. A solution of the initial value problem (1.1)–(1.2) is a solution on some interval $[0, t_e)$, $0 < t_e \leqslant \infty$, which satisfies $x_0 = \phi$. Observe that such solutions have all their segments x_t, $0 \leqslant t < t_e$, in M.

In Section 4 below we show that M is a continuously differentiable submanifold with codimension n in B^1 and that on M the initial value problem (1.1)–(1.2) defines a continuous semiflow of continuously differentiable solution operators (time-t-maps). Section 5 verifies that the derivatives of the solution operators are given by solutions v of linear variational equations with segments v_t tangent to M.

Most of these results are analogous to those obtained in [23,24,10] for differential equations with bounded delays. In fact, we shall exploit the latter, by means of relations between solutions of Eq. -(1.1) and solutions of associated equations

$$x'(t) = g_d(x_{d,t})$$

with functionals $g_d : U_d \to \mathbb{R}^n$ on open subsets U_d of the Banach spaces

$$C_d^1 = C^1([-d, 0], \mathbb{R}^n), \quad d > 0,$$

of continuously differentiable maps $\chi : [-d, 0] \to \mathbb{R}^n$, with the norm given by

$$|\chi|_{C_d^1} = \max_{-d \leqslant s \leqslant 0} |\chi(s)| + \max_{-d \leqslant s \leqslant 0} |\chi'(s)|.$$

The segment $x_{d,t} : [-d, 0] \to \mathbb{R}^n$ is given by $x_{d,t}(s) = x(t+s)$, for $x : I \to A$ with $[t-d, t] \subset I$.

There also are differences between the present case and bounded delay as in [23,24,10]. It is the choice of the exponential weight in the norm on $B^1 \subset C^1$ which yields that flowlines $t \mapsto x_t$ given by solutions x of Eq. (1.1) have range in B^1, and it is the limit at $-\infty$ in the definition of B^1 which gives their continuity, see Proposition 2.3. In contrast to the case of bounded delay, most flowlines are nowhere differentiable, see Remark 4.3.

In Sections 6 and 7 we find local stable and unstable manifolds at a stationary point ϕ of the semiflow on M. This is done in the following way. For suitable $d > 0$ a restriction $\phi_{d,0} = \phi|[-d, 0]$ is a stationary point of an associated equation with bounded delay. The latter defines a semiflow H_d on a submanifold $M_d \subset C_d^1$ of codimension n. We pick $t > 0$ and linearize the time-t-map from H_d at the fixed point $\phi_{d,0}$. Results from [10] yield a positively invariant decomposition of the tangent space $T_{\phi_{d,0}} M_d$ into the stable, center and unstable spaces, with the spectra of the induced endomorphisms inside, on and outside the unit circle, respectively. From this decomposition we construct its analogue for the linearized time-t-map from H on the tangent space $T_\phi M$. The dimensions of the subspaces in the decomposition are preserved due to our choice $a > 0$ for the definition of the ambient space B^1. Then we proceed as in Section 3.5 of [10] and get a local stable manifold for the time-t-map which finally yields a local stable manifold for the semiflow. The approach to the local unstable manifold is similar, with several details different. An advantage of this approach may be seen in the fact that apart from standard arguments it only requires a discussion of linear maps (see Propositions 6.1–6.4). Alternatively one may attempt to construct local stable and unstable manifolds for H directly from those for the semiflow H_d.

Let us mention an obvious fact: In contrast to the case of bounded delay for $t \to \infty$ the segments x_t of flowlines in the local stable manifold of Section 6 (and their derivatives) do not converge to the stationary point *uniformly* on their domain $(-\infty, 0]$.

Local center manifolds for the semiflow on M are not addressed here since in general they do not arise as local center manifolds for time-t-maps – see [16] for an example on the plane – while a direct construction would overload the present paper. Existence and smoothness of center and center-stable manifolds for differential equations which include cases of *bounded* state-dependent delay were established in [10,17] and in [21], respectively. For a class of equations of this kind local unstable manifolds were first constructed in [13,14]. A result on their smoothness which seems optimal was obtained in [15]. A result on Lipschitz-continuous local stable and unstable manifolds for certain equations with bounded state-dependent delay is contained in [2].

For results on functional differential equations with time-invariant infinite delay, see [11] and [4].

Further notation. $\mathbb{N}$ and $\mathbb{N}_0$ denote the positive and nonnegative integers, respectively. For a linear map $L : V \to W$, $ker(L) = L^{-1}(0)$. Let E, F be Banach spaces over $\mathbb{R}$. The Banach space of continuous linear maps $E \to F$ is denoted by $L_c(E, F)$. The Lipschitz constant of a map $f : U \to F$, $U \subset E$ is

$$lip(F) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leqslant \infty.$$

Let $I \subset \mathbb{R}$ be an interval, open or closed. $C(I, \mathbb{R}^n)$ and $C^1(I, \mathbb{R}^n)$ denote the vector spaces of maps $I \to \mathbb{R}^n$ which are continuous and continuously differentiable, respectively. For $t \in \mathbb{R}$ we consider the Banach spaces

$$B_t = \{\phi \in C((-\infty, t], \mathbb{R}^n) \colon e^{as}\phi(s) \to 0 \text{ as } s \to -\infty\}$$

with the norm given by

$$|\phi|_{B_t} = \sup_{s \leqslant t} e^{as}|\phi(s)|,$$

and

$$B_t^1 = \{\phi \in B_t \cap C^1((-\infty, t], \mathbb{R}^n) \colon e^{as}\phi'(s) \to 0 \text{ as } s \to -\infty\}$$

with the norm given by

$$|\phi|_{B_t^1} = |\phi|_{B_t} + |\phi'|_{B_t}.$$

Clearly $B = B_0$ and $B^1 = B_0^1$.

Differentiation $\partial : B^1 \ni \phi \mapsto \phi' \in B$ is linear and continuous, with

$$|\partial|_{L_c(B^1, B)} \leqslant 1.$$

For $d > 0$,

$$C_d = C\big([-d, 0], \mathbb{R}^n\big)$$

denotes the Banach space of continuous maps $\chi : [-d, 0] \to \mathbb{R}^n$ with the norm given by

$$|\chi|_{C_d} = \max_{-d \leqslant s \leqslant 0} \big|\chi(s)\big|.$$

Notice that for $\phi \in C_d^1$, $|\phi|_{C_d^1} = |\phi|_{C_d} + |\phi'|_{C_d}$. The restriction map

$$R_d : \big(\mathbb{R}^n\big)^{(-\infty,0]} \ni \phi \mapsto \phi|[-d, 0] \in \big(\mathbb{R}^n\big)^{[-d,0]}$$

induces linear, continuous and surjective maps

$$B \to C_d \quad \text{and} \quad B^1 \to C_d^1.$$

2. Preliminaries

We begin with the evaluation maps $ev : B \times (-\infty, 0] \ni (\phi, s) \mapsto \phi(s) \in \mathbb{R}^n$ and $ev_1 : B^1 \times (-\infty, 0] \ni (\phi, s) \mapsto \phi(s) \in \mathbb{R}^n$.

Proposition 2.1.

(i) *ev is continuous.*
(ii) *The restriction $ev_1|B^1 \times (-\infty, 0)$ is continuously differentiable with*

$$Dev_1(\phi, s)(\psi, u) = \psi(s) + u\phi'(s)$$

for all $\phi \in B^1$, $s < 0$, $\psi \in B^1$, $u \in \mathbb{R}$.

Proof. 1. Continuity of ev at (ϕ, s): Set $d = 1 - s > 0$. On $B \times [-d, 0]$ we have $ev(\phi, u) = ev_d(R_d\phi, u)$, with the evaluation map

$$ev_d : C_d \times [-d, 0] \ni (\chi, u) \mapsto \chi(u) \in \mathbb{R}^n$$

which is continuous (see [23]). Now the assertion becomes obvious.

2. Proof of (ii). It is sufficient to consider restrictions $ev_{1,d} = ev_1 \mid B^1 \times (-d, 0)$, for arbitrary $d > 0$. For $(\phi, s) \in B^1 \times (-d, 0)$ we have $ev_{1,d}(\phi, s) = ev_{1,dd}(R_d\phi, s)$ with the continuously differentiable map

$$ev_{1,dd} : C_d^1 \times (-d, 0) \ni (\chi, s) \mapsto \chi(s) \in \mathbb{R}^n \quad (\text{see [23]}).$$

Therefore the restriction is continuously differentiable, with

$$\begin{aligned} Dev_{1,d}(\phi, s)(\psi, u) &= Dev_{1,dd}(R_d\phi, s)(R_d\psi, u) \\ &= R_d\psi(s) + u(R_d\phi)'(s) \\ &= \psi(s) + u\phi'(s) \end{aligned}$$

for all $\phi \in B^1$, $s \in (-d, 0)$, $\psi \in B^1$, $u \in \mathbb{R}$. It follows that ev_1 is continuously differentiable with

$$Dev_1(\phi, s)(\psi, u) = \psi(s) + u\phi'(s)$$

for all $\phi \in B^1$, $s < 0$, $\psi \in B^1$, $u \in \mathbb{R}$. $\square$

Continuous forward extensions $x : (-\infty, t_e] \to \mathbb{R}^n$, $0 \leqslant t_e$, of elements $\phi \in B$ have segments $x_t : (-\infty, 0] \ni s \mapsto x(t+s) \in \mathbb{R}^n$, $t \leqslant t_e$, in B since for $t \leqslant t_e$ and $s \leqslant 0$ we have

$$e^{as}x_t(s) = e^{as}x(t+s) = e^{-at}e^{a(t+s)}x(t+s),$$

hence $e^{as}x_t(s) \to 0$ as $s \to -\infty$.

Proposition 2.2. *Suppose $x : (-\infty, t_e] \to \mathbb{R}^n$, $0 \leqslant t_e$, is continuous and $x_0 \in B$. Then the curve $(-\infty, t_e] \ni t \mapsto x_t \in B$ is continuous.*

Proof. Let $t_0 \leqslant t_e$. Suppose the sequence $(t_j)_1^\infty$ in $(-\infty, t_e]$ converges to t_0. There exists $r > 0$ with $r < e^{at_j}$ for all $j \in \mathbb{N}_0$. Let $\epsilon > 0$. As $x_0 \in B$ there exists $t_- < t_0$ with

$$e^{au}\left|x(u)\right| < \frac{\epsilon r}{4} \quad \text{for all } u \leqslant t_-.$$

As the restriction of x to $[t_- - 1, t_e]$ is uniformly continuous there exists $\delta \in (0, 1)$ such that for every $v \in [t_- - 1, t_e]$ and for all $w \in (-\delta, \delta)$ with $v + w \leqslant t_e$ have

$$\left|x(v+w) - x(v)\right| < \frac{\epsilon}{2}.$$

Now choose $J \in \mathbb{N}$ so large that for all integers $j \geqslant J$ we have

$$|t_j - t_0| < \delta.$$

Consider an integer $j \geqslant J$. Let $t \leqslant 0$. In case $t_- - 1 \leqslant t + t_j$ $(\leqslant t_e)$ we use $t_0 - t_j \in (-\delta, \delta)$ and get

$$e^{at}\left|x_{t_j}(t) - x_{t_0}(t)\right| \leqslant \left|x(t_0+t) - x(t_j+t)\right| = \left|x\big(t + t_j + (t_0 - t_j)\big) - x(t+t_j)\right| < \frac{\epsilon}{2}.$$

In case $t + t_j < t_- - 1$ $(< t_-)$ we get

$$t_0 + t = (t_0 - t_j) + t + t_j < \delta + (t + t_j) < \delta + t_- - 1 < t_-.$$

It follows that

$$\begin{aligned} e^{at}\left|x_{t_j}(t) - x_{t_0}(t)\right| &\leqslant e^{a(t_0+t)}\left|x(t_0+t)\right|e^{-at_0} + e^{a(t_j+t)}\left|x(t_j+t)\right|e^{-at_j} \\ &\leqslant \frac{\epsilon}{4}\frac{r}{e^{at_0}} + \frac{\epsilon}{4}\frac{r}{e^{at_j}} < \frac{\epsilon}{2}. \end{aligned}$$

Altogether, for every integer $j \geqslant J$,

$$|x_{t_j} - x_t|_B = \sup_{t \leqslant 0} e^{at}\left|x_{t_j}(t) - x_{t_0}(t)\right| \leqslant \frac{\epsilon}{2}. \qquad \square$$

Analogously continuously differentiable forward extensions $x:(-\infty,t_e]\to\mathbb{R}^n$, $0\leqslant t_e$, of elements $\phi\in B^1$ have segments x_t, $t\leqslant t_e$, in B^1.

Proposition 2.3. *Suppose $x:(-\infty,t_e]\to\mathbb{R}^n$, $0\leqslant t_e$, is continuously differentiable and $x_0\in B^1$. Then the curve $(-\infty,t_e]\ni t\mapsto x_t\in B^1$ is continuous.*

The proof is analogous to the previous one, or can be given using the previous result and the continuity of differentiation $\partial: B^1\to B$.

For reals $s\leqslant t$ and $\chi\in B_t$, $\phi\in B^1_t$ we have $\chi_s\in B$ and $\phi_s\in B^1$, due to

$$e^{au}\chi_s(u)=e^{-as}e^{a(s+u)}\chi(s+u)\quad\text{for all } u\leqslant 0$$

and to analogous equations for $\phi\in B^1_t$, and for reals $s\leqslant t$ the linear maps

$$B_t\ni\chi\mapsto\chi_s\in B,$$
$$B^1_t\ni\chi\mapsto\chi_s\in B^1$$

are continuous. For every $t<0$ the linear restriction map

$$B^1\ni\psi\mapsto\psi\mid(-\infty,t]\in B^1_t$$

is continuous, and for $0\leqslant t\leqslant d$ the linear map

$$B^1\ni\zeta\mapsto\zeta(t+\cdot)\mid(-\infty,-d]\in B^1_{-d}$$

is continuous. The last statement follows from the estimate

$$e^{au}\left|\zeta(t+u)\right|=e^{-at}e^{a(t+u)}\left|\zeta(t+u)\right|\leqslant|\zeta|_{B^1}\quad\text{for } u\leqslant -d$$

in combination with its analogue with ζ' in place of ζ.

Proposition 2.4. *For every $t\in\mathbb{R}$ the maps*

$$(-\infty,t]\times B_t\ni(s,\chi)\mapsto\chi_s\in B\quad\textit{and}\quad(-\infty,t]\times B^1_t\ni(s,\chi)\mapsto\chi_s\in B^1$$

are continuous.

Proof. Let reals $s\leqslant t$ and $\chi\in B_t$ be given. For all $u\in[s-1,t]$ and all $\rho\in B_t$ we have

$$\begin{aligned}
|\rho_u-\chi_s|_B&\leqslant|\rho_u-\chi_u|_B+|\chi_u-\chi_s|_B\\
&=\sup_{v\leqslant 0}e^{av}\left|\rho(u+v)-\chi(u+v)\right|+|\chi_u-\chi_s|_B\\
&=e^{-au}\sup_{v\leqslant 0}e^{a(u+v)}\left|\rho(u+v)-\chi(u+v)\right|+|\chi_u-\chi_s|_B\\
&\leqslant e^{-a(s-1)}|\rho-\chi|_{B_t}+|\chi_u-\chi_s|_B.
\end{aligned}$$

Observe that

$$\chi_u - \chi_s = (\chi_t)_{u-t} - (\chi_t)_{s-t},$$

with $\chi_t \in B$. Use this equation, the previous estimate, and Proposition 2.2 to show that the first map of the assertion is continuous at (s, χ). The proof for the second map in the assertion is similar. $\square$

For $d > 0$ we introduce maps $(\mathbb{R}^n)^{[-d,0]} \to (\mathbb{R}^n)^{(-\infty,0]}$. Let $h \in (0, d)$ and let a map $\phi : (-\infty, 0] \to \mathbb{R}^n$ be given. Choose $\epsilon > 0$ with $h + \epsilon < d - \epsilon$ and a continuously differentiable map $p : \mathbb{R} \to \mathbb{R}$ with $p(t) = 0$ for $-h - \epsilon \leqslant t \leqslant 0$ and $p(t) = 1$ for $t \leqslant -d + \epsilon$. Define the affine linear map $E_{h,d,\phi,\epsilon,p} : (\mathbb{R}^n)^{[-d,0]} \to (\mathbb{R}^n)^{(-\infty,0]}$ by

$$E_{h,d,\phi,\epsilon,p}(\chi)(s) = p(s)\phi(s) + \big(1 - p(s)\big)\chi^d(s),$$

where $\chi^d(s) = \chi(s)$ on $[-d, 0]$ and $\chi^d(s) = 0$ for $s < -d$. Then, with R_d from Section 1,

$$E_{h,d,\phi,\epsilon,p}(R_d\phi) = \phi,$$

and for all $\chi : [-d, 0] \to \mathbb{R}^n$,

$$E_{h,d,\phi,\epsilon,p}(\chi)(s) = \phi(s) \quad \text{on } (-\infty, -d], \qquad E_{h,d,\phi,\epsilon,p}(\chi)(s) = \chi(s) \quad \text{on } [-h, 0].$$

Proposition 2.5. *For $\phi \in B$ the map $E_{h,d,\phi,\epsilon,p}$ induces a continuous map $C_d \to B$, and for $\phi \in B^1$ a continuously differentiable map $C_d^1 \to B^1$.*

Proof. For $\phi \in B$ $E_{h,d,\phi,\epsilon,p}$ is the sum of a constant map with value in B and of a linear map, and the values of the linear map are functions which vanish for $t \leqslant -d$. Moreover, continuous maps are mapped to continuous maps. It is easy to complete the proof by estimates which show that the induced linear map $C_d \to B$ is continuous. The argument in case $\phi \in B^1$ is analogous. $\square$

Next we consider concatenation. It is easy to see that for every $t > 0$ the linear map

$$c_t : \big\{(\phi, \chi) \in B \times C\big([0, t], \mathbb{R}^n\big) \colon \phi(0) = \chi(0)\big\} \to B_t$$

given by $c_t(\phi, \chi)(s) = \phi(s)$ for $s \leqslant 0$ and $c_t(\phi, \chi)(s) = \chi(s)$ for $0 < s \leqslant t$ is continuous. Analogously we have that for every $t > 0$ the linear map

$$c_t^1 : \big\{(\phi, \chi) \in B^1 \times C^1\big([0, t], \mathbb{R}^n\big) \colon \phi(0) = \chi(0),\ \phi'(0) = \chi'(0)\big\} \to B_t^1$$

defined in the same way is continuous. The domains of these maps are closed subspaces of $B \times C([0, t], \mathbb{R}^n)$ and of $B^1 \times C^1([0, t], \mathbb{R}^n)$, respectively.

We now describe a reduction procedure which will frequently be used in the sequel. Consider $\phi \in M \subset U$ and $N \subset U$ and $h > 0$ as in the hypothesis (LBD). Choose $d > h$, and ϵ and p as above. The map $E = E_{h,d,\phi,\epsilon,p}$ sends an open neighbourhood $U_d \subset C_d^1$ of $R_d\phi$ into N. Define the continuously differentiable map $g_d : U_d \to \mathbb{R}^n$ by

$$g_d(\chi) = g\big(E(\chi)\big).$$

Choose an open neighbourhood $N_0 \subset N$ of ϕ with $R_d N_0 \subset U_d$.

Proposition 2.6. *For every $\rho \in N_0$ we have*

$$g_d(R_d\rho) = g(\rho),$$

and for every $\chi \in B^1$,

$$Dg(\rho)\chi = Dg_d(R_d\rho)R_d\chi.$$

Proof. For $\rho \in N_0 \subset N$ we have $E(R_d\rho) \in N$, and for $-h \leqslant s \leqslant 0$,

$$E(R_d\rho)(s) = (R_d\rho)(s) = \rho(s).$$

Using property (LBD) we infer

$$g_d(R_d\rho) = g\big(E(R_d\rho)\big) = g(\rho).$$

Differentiation yields the last part of the assertion. □

Each derivative $Dg_d(\chi): C_d^1 \to \mathbb{R}^n$, $\chi \in U_d$, has a linear extension $D_e g_d(\chi): C_d \to \mathbb{R}^n$ given by

$$D_e g_d(\chi)\zeta = D_e g\big(E(\chi)\big)E(\zeta),$$

and the map

$$U_d \times C_d \ni (\chi, \zeta) \mapsto D_e g_d(\chi)\zeta \in \mathbb{R}^n$$

is continuous as it is the composition of the maps

$$U_d \times C_d \ni (\chi, \zeta) \mapsto (E\chi, E\zeta) \in U \times B$$

and

$$U \times B \ni (\psi, \beta) \mapsto D_e g(\psi)\beta \in \mathbb{R}^n.$$

Set

$$M_d = \big\{\chi \in U_d\colon \chi'(0) = g_d(\chi)\big\}.$$

Corollary 2.7. $R_d(M \cap N_0) \subset M_d \neq \emptyset$.

Proof. For every $\psi \in M \cap N_0$, $R_d\psi \in U_d$, and Proposition 2.6 yields

$$(R_d\psi)'(0) = \psi'(0) = g(\psi) = g_d(R_d\psi).$$

In particular, $R_d\phi \in M_d$. □

Notice also that

$$E(M_d) \subset M$$

since for $\chi \in M_d \subset U_d$ the map $\psi = E(\chi) \in N \subset U$ satisfies

$$\psi'(0) = \chi'(0) = g_d(\chi) = g(E\chi) = g(\psi).$$

Using results from [23,24,10] we infer that the set M_d is a continuously differentiable submanifold of codimension n in the space C^1_d, with the tangent space at $\chi \in M_d$ given by

$$T_\chi M_d = \{\rho \in C^1_d\colon\ \rho'(0) = Dg_d(\chi)\rho\},$$

and that the maximal continuously differentiable solutions

$$x^{d,\chi} : [-d, t_{d,e}(\chi)) \to \mathbb{R}^n$$

of the initial value problem

$$x'(t) = g_d(x_{d,t}), \tag{2.1}$$

$$x_{d,0} = \chi \in M_d \tag{2.2}$$

define a continuous semiflow $H_d : \Omega_d \to M_d$ on M_d, by

$$\Omega_d = \{(t,\chi) \in [0,\infty) \times M_d\colon\ t < t_{d,e}(\chi)\},$$
$$H_d(t,\chi) = x^{d,\chi}_{d,t}.$$

All time-t-maps

$$H_{d,t} : \Omega_{d,t} \ni \chi \mapsto H_d(t,\chi) \in M_d, \quad t \geqslant 0,$$

with $\Omega_{d,t} = \{\chi \in M_d\colon\ t < t_{d,e}(\chi)\}$, are continuously differentiable, and we have

$$DH_{d,t}(\chi)\rho = v^{d,\chi,\rho}_{d,t}$$

with the (uniquely determined) continuously differentiable solution $v^{d,\chi,\rho} : [-d, t_{d,e}(\chi)) \to \mathbb{R}^n$ of the initial value problem

$$v'(t) = Dg_d(H_d(t,\chi))v_{d,t}, \tag{2.3}$$

$$v_{d,0} = \rho \in T_\chi M_d. \tag{2.4}$$

Moreover, arguments following the proof of Proposition 5 in [23] show immediately that for every $\chi \in M_d$ there exist $t_\chi > 0$ and a neighbourhood $N_{d,\chi} \subset U_d$ with $[0,t_\chi] \times (M_d \cap N_{d,\chi}) \subset \Omega_d$ so that the map

$$M_d \cap N_{d,\chi} \ni \rho \mapsto x^{d,\rho} \mid [0,t_\chi] \in C^1([0,t_\chi],\mathbb{R}^n)$$

is continuously differentiable.

3. Example

Consider an object on a line which regulates its position in the following way: A signal is emitted, travels with speed $c > 0$, reaches a reflector and returns. The object measures the signal travel time σ and converts it into a distance p from the reflector using the equation

$$c = \frac{2p}{\sigma}$$

(which gives the correct distance in case the object is in the same position when emitting and receiving the signal). Depending on the computed distance p the object accelerates or slows down.

For u denoting the (true) position, v the velocity, and the reflected signal arriving at time t we have the equations

$$\begin{aligned} c\sigma(t) &= u\big(t - \sigma(t)\big) + u(t), \\ u'(t) &= v(t), \\ v'(t) &= f\left(\frac{c\sigma(t)}{2}\right) \end{aligned}$$

with a feedback function $f : \mathbb{R} \to \mathbb{R}$.

State-dependent delays of a similar but more complicated type arise in the 2-body problem of electrodynamics [5–8,1,3]. A variant of the system above was proposed by Nussbaum [20]. The system itself and closely related ones were studied in [22,25,26,12] for solutions with $|u'| < c$ and the position u bounded by some constant. Let us give up the bound on positions. Let $c > 0$, consider a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ and B^1 with $n = 2$. Then

$$\begin{aligned} U = \Big\{\phi \in B^1 \colon &\text{ there exists } h > 0 \text{ so that } 0 < \phi_1(t) \text{ and } \left|\phi_1'(t)\right| < c \text{ on } [-h, 0], \\ &\text{and } \phi_1(-h) + \phi_1(0) < ch\Big\} \end{aligned}$$

is a non-empty open subset of B^1. Using continuity and the intermediate value theorem we infer that for every $\phi = (\phi_1, \phi_2) \in U$ there exists a smallest positive number $s = \sigma(\phi)$ so that

$$cs = \phi_1(-s) + \phi_1(0).$$

Moreover, there exists $h > \sigma(\phi)$ with $|\phi_1'(t)| < c$ for all $t \in [-h, 0]$. In particular,

$$\left|\phi_1'\big(-\sigma(\phi)\big)\right| < c.$$

Proposition 3.1. *The map $U \ni \phi \mapsto \sigma(\phi) \in \mathbb{R}$ is continuously differentiable, with*

$$D\sigma(\phi)(\psi) = \frac{\psi_1(-\sigma(\phi))}{\phi_1'(-\sigma(\phi)) + c}.$$

Proof. The properties of the evaluation map ev_1 in Proposition 2.1 yield that the map

$$F : U \times (0, \infty) \ni (\phi, s) \mapsto \phi_1(-s) + \phi_1(0) - cs \in \mathbb{R}$$

is continuously differentiable with $D_2F(\phi,s)1 = -\phi_1'(-s) - c$. Using the preceding inequality we infer $D_2F(\phi,\sigma(\phi))1 < 0$. Therefore the Implicit Function Theorem is applicable and yields an open interval $(u,v) \ni \sigma(\phi)$, an open ball $K \ni \phi$, and a continuously differentiable map $\hat{\sigma} : K \to (u,v)$ with

$$\hat{\sigma}(\phi) = \sigma(\phi),$$
$$F\big(\psi,\hat{\sigma}(\psi)\big) = 0 \quad \text{for all } \psi \in K,$$
$$s = \hat{\sigma}(\psi) \quad \text{for all } (\psi,s) \in K \times (u,v) \text{ with } F(\psi,s) = 0.$$

Using compactness of $[0,u]$ and $cs \neq \phi_1(-s) + \phi_1(0)$ on $[0,u]$ we find a smaller open ball $K_1 \subset K$ with $\phi \in K_1$ so that for all $\psi \in K_1$ and for all $s \in [0,u]$ we have

$$cs \neq \psi_1(-s) + \psi_1(0).$$

It follows that for each $\psi \in K_1$ the number $\hat{\sigma}(\psi) \in (u,v)$ is the smallest solution of the equation

$$cs = \psi_1(-s) + \psi_1(0)$$

which in turn gives $\hat{\sigma}(\psi) = \sigma(\psi)$ on K_1.

Differentiating the equation $F(\psi,\sigma(\psi)) = 0$ for $\psi \in K_1$ and using the formula for the derivative of ev_1 in Proposition 2.1 we obtain the last part of the assertion. □

Observe that the formula for $D\sigma(\phi)$ also defines a linear extension

$$D_e\sigma(\phi) : B \to \mathbb{R}.$$

The continuity of the evaluation map ev and of the differentiation operator $\partial : B^1 \to B$ combined yield that the map

$$U \times B \ni (\phi,\chi) \mapsto D_e\sigma(\phi)\chi \in \mathbb{R}$$

is continuous.

Now consider the map $g : U \to \mathbb{R}^2$ with components given by

$$g_1(\phi) = \phi_2(0)$$

and

$$g_2(\phi) = f\left(\frac{c\sigma(\phi)}{2}\right).$$

g_2 is continuously differentiable with

$$Dg_2(\phi)\psi = \frac{c}{2} f'\left(\frac{c\sigma(\phi)}{2}\right) D\sigma(\phi)\psi$$

for all $\phi \in U$, $\psi \in B^1$. Using the definition of g_1, the previous formula for $Dg_2(\phi)$, and the properties of σ we infer that the hypothesis (E) made in Section 1 is satisfied.

We proceed to the hypothesis (LBD). For $\phi \in U$ and $h > 0$ as in the definition of U we find a neighbourhood N_h of $R_h\phi$ in C_h^1 so that for all $\chi \in N_h$ we have

$$0 < \chi_1(t) \quad \text{and} \quad \left|\chi_1'(t)\right| < c \quad \text{on } [-h, 0],$$

and $\chi_1(-h) + \chi_1(0) < ch$. For each ψ in the neighbourhood $N = U \cap R_h^{-1}(N_h)$ of ϕ we get $\psi_1(0) + \psi_1(0) > 0 = c0$ and $\psi_1(-h) + \psi_1(0) < ch$. This yields $\sigma(\psi) < h$. Moreover, for ψ and ρ in N with $R_h\psi = R_h\rho$ we obtain $\sigma(\psi) = \sigma(\rho)$, and consequently

$$g_1(\psi) = \psi(0) = \rho(0) = g_1(\rho),$$
$$g_2(\psi) = f\left(\frac{c\sigma(\psi)}{2}\right) = f\left(\frac{c\sigma(\rho)}{2}\right) = g_2(\rho).$$

Finally, (M) holds as $\phi : (-\infty, 0] \to \mathbb{R}^2$ with $\phi(t) = 1$ for all $t \leqslant 0$ and ϕ_2 continuously differentiable with $\phi_2(t) = 0$ for $t \leqslant -1$, $\phi_2(0) = 0$, $(\phi_2)'(0) = f(1)$ belongs to U and satisfies $\phi'(0) = g(\phi)$.

4. The semiflow on the solution manifold

Analogously to Proposition 1 in [23] we have the following result.

Proposition 4.1. *The set $M \subset U \subset B^1$ is a continuously differentiable submanifold of codimension n in B^1.*

Proof. With the continuous linear map $p : B^1 \ni \phi \mapsto \phi'(0) \in \mathbb{R}^n$ we have

$$M = \{\phi \in U\colon\ (p - f)(\phi) = 0\},$$

and it is sufficient to show that each derivative $D(p - f)(\phi) = p - Df(\phi)$, $\phi \in M$, is surjective. (Then one can decompose B^1 into the kernel of $D(p - f)(\phi)$ and a complementary space on which $D(p - f)(\phi)$ defines an isomorphism, and close to ϕ the Implicit Function Theorem yields a local representation of M as the graph of a continuously differentiable map.) Choose a basis $(b_1, \ldots, b_n)$ of $\mathbb{R}^n$ and $\epsilon > 0$ so that every n-tuple $(x_1, \ldots, x_n)$ of vectors in $\mathbb{R}^n$ with $|x_j - b_j| < \epsilon$ for $j = 1, \ldots, n$ is a basis of $\mathbb{R}^n$. Let $\phi \in U$ be given. There exists $\delta > 0$ so that for all $\chi \in B$ with $|\chi|_B < \delta$ we have $|D_e f(\phi)\chi| < \epsilon$, as $D_e f(\phi)$ is continuous at $\chi = 0$. The ball $\{\chi \in B\colon\ |\chi|_B < \delta\}$ contains elements $\chi_1, \ldots, \chi_n$ in B^1 with $\chi_j'(0) = b_j$ for $j = 1, \ldots, n$. It follows that for $j = 1, \ldots, n$,

$$\left|\big(p - Df(\phi)\big)\chi_j - b_j\right| = \left|D_e f(\phi)\chi_j\right| < \epsilon.$$

Hence $D(p - f)(\phi)B^1$ contains a basis of $\mathbb{R}^n$. $\square$

The tangent spaces of the manifold M are given by

$$T_\phi M = \{\chi \in B^1\colon\ \chi'(0) = Dg(\phi)\chi\}.$$

We turn to existence of solutions to the initial value problem (1.1)–(1.2) on M.

Proposition 4.2. (i) *For every $\phi \in M$ there exist a neighbourhood $N_\phi \subset U$ of ϕ, $t_\phi > 0$ and a map*

$$M \cap N_\phi \ni \psi \mapsto x^{(\psi)} \in C^1\big((-\infty, t_\phi), \mathbb{R}^n\big)$$

so that for each $\psi \in M \cap N_\phi$ the map $x^{(\psi)}$ is a solution of the initial value problem

$$x'(t) = g(x_t),$$
$$x_0 = \psi.$$

(ii) *The map*

$$[0, t_\phi) \times (M \cap N_\phi) \ni (t, \psi) \mapsto x_t^{(\psi)} \in M$$

is continuous.

(iii) *For every $t \in [0, t_\phi)$ the map*

$$M \cap N_\phi \ni \psi \mapsto x_t^{(\psi)} \in M$$

is continuously differentiable.

Proof. 1. Let $\phi \in M$ be given. Choose N, h, d, ϵ, p and consider $E = E_{h,d,\phi,\epsilon,p}$, U_d, g_d, M_d as in Section 2. Choose a neighbourhood $N_d \subset U_d$ of $R_d\phi$ and $t_0 > 0$ so small that

$$[0, t_0] \times (M_d \cap N_d) \subset \Omega_d,$$

with the map

$$sol : M_d \cap N_d \ni \chi \mapsto x^{d,\chi} \mid [0, t_0] \in C^1\big([0, t_0], \mathbb{R}^n\big)$$

continuously differentiable.

Choose a neighbourhood $N_0 \subset N$ of ϕ so small that $R_d N_0 \subset N_d$. By Corollary 2.7, $R_d(M \cap N_0) \subset M_d$. Observe that for $\psi \in M \cap N_0$ we have $\psi(0) = (R_d\psi)(0) = x^{d,R_d\psi}(0) = sol(R_d\psi)(0)$ and $\psi'(0) = (R_d\psi)'(0) = (x^{d,R_d\psi})'(0) = sol(R_d\psi)'(0)$. Now we define

$$x^{(\psi)} = c^1_{t_0}\big(\psi, sol(R_d\psi)\big) \in B^1_{t_0}.$$

Recall that this implies $x_t^{(\psi)} \in B^1$ for $t \leqslant t_0$.

2. The map

$$[0, t_0] \times (M \cap N_0) \ni (t, \psi) \mapsto x_t^{(\psi)} \in B^1$$

is given by

$$\big(c_{t_0}\big(\psi, sol(R_d\psi)\big)\big)_t$$

and thereby continuous.

We conclude that there exist a neighbourhood $N_\phi \subset N_0$ of ϕ and $t_\phi \in (0, t_0]$ so that $x_t^{(\psi)} \in N$ for $\psi \in M \cap N_\phi$ and $0 \leqslant t \leqslant t_\phi$.

3. Let $0 < t \leqslant t_\phi$, $\psi \in M \cap N_\phi$, $\chi = R_d\psi$. The relation $x^{d,\chi}_{d,t} \in U_d$ yields $E(x^{d,\chi}_{d,t}) \in N$. By part 2, $x_t^{(\psi)} \in N \subset U$. For $-h \leqslant s \leqslant 0$ we have $-h < t + s \leqslant t_\phi$, hence

$$E\big(x^{d,\chi}_{d,t}\big)(s) = x^{d,\chi}(t+s) = x^{(\psi)}(t+s) = x_t^{(\psi)}(s).$$

Using this and hypothesis (LBD), and the choice of N and h, we infer

$$g\big(E\big(x_{d,t}^{d,\chi}\big)\big) = g\big(x_t^{(\psi)}\big)$$

which gives

$$\big(x^{(\psi)}\big)'(t) = \big(x^{d,\chi}\big)'(t) = g_d\big(x_{d,t}^{d,\chi}\big) = g\big(E\big(x_{d,t}^{d,\chi}\big)\big) = g\big(x_t^{(\psi)}\big).$$

Consequently, $x_t^{(\psi)} \in M$, and Eq. (1.1) holds. This completes the proof of assertion (i).

4. Proof of (ii). The map in question is given by the composition of the following continuous maps:

$$[0, t_\phi) \times (M \cap N_\phi) \ni (t, \psi) \mapsto (t, \psi, R_d\psi) \in \mathbb{R} \times B^1 \times C_d^1,$$

$$[0, t_0] \times B^1 \times (M_d \cap N_d) \ni (t, \psi, \chi) \mapsto \big(t, \psi, sol(\chi)\big) \in [0, t_0] \times B^1 \times C^1\big([0, t_0], \mathbb{R}^n\big),$$

$$[0, t_0] \times \big\{(\psi, y) \in B^1 \times C^1\big([0, t_0], \mathbb{R}^n\big)\colon\ \psi(0) = y(0),\ \psi'(0) = y'(0)\big\} \ni (t, \psi, y)$$
$$\mapsto \big(t, c_{t_0}(\psi, y)\big) \in (-\infty, t_0] \times B_{t_0}^1,$$

$$(-\infty, t_0] \times B_{t_0}^1 \ni (t, \rho) \mapsto \rho_t \in B^1.$$

5. Proof of (iii). Let $t \in [0, t_\phi)$ be given. The map in question is given by the composition of the following maps:

$$M \cap N_\phi \ni \psi \mapsto (\psi, R_d\psi) \in B^1 \times C_d^1,$$

$$B^1 \times (M_d \cap N_d) \ni (\psi, \chi) \mapsto \big(\psi, sol(\chi)\big) \in B^1 \times C^1\big([0, t_0], \mathbb{R}^n\big),$$

$$c_{t_0}, \quad \text{and}$$

$$B_{t_0}^1 \ni \rho \mapsto \rho_t \in B^1.$$

The first, third and fourth of these are linear and continuous, hence continuously differentiable as is the second one. $\square$

Notice that differentiability of a flowline $\xi : (-\infty, t_\phi] \ni t \mapsto x_t^{(\psi)} \in B^1$, $\psi \in M \cap N_\phi$, at some $t \in [0, t_\phi)$, with $D\xi(t)1 = \rho \in B^1$, implies that ψ has second order derivatives at each $u < 0$, as can be seen from the estimate

$$\begin{aligned}
&\left|\frac{1}{\Delta}\big(\psi'(u+\Delta) - \psi'(u)\big) - \rho'(u-t)\right| \\
&\quad = \left|\frac{1}{\Delta}\big(\xi(t+\Delta)'(u-t) - \xi(t)'(u-t)\big) - \rho'(u-t)\right| \\
&\quad = e^{a(t-u)}e^{a(u-t)}\left|\frac{1}{\Delta}\big(\xi(t+\Delta)'(u-t) - \xi(t)'(u-t)\big) - \rho'(u-t)\right| \\
&\quad \leqslant e^{a(t-u)} \sup_{v \leqslant 0} e^{av}\left|\frac{1}{\Delta}\big(\xi(t+\Delta)'(v) - \xi(t)'(v)\big) - \rho'(v)\right| \\
&\quad \leqslant e^{a(t-u)}\left|\frac{1}{\Delta}\big(\xi(t+\Delta) - \xi(t)\big) - \rho\right|_{B^1}
\end{aligned}$$

for $\Delta \neq 0$ with $u + \Delta \leqslant 0$ and $t + \Delta < t_\phi$. This justifies the next statement.

Remark 4.3. The flowlines $(-\infty, t_\phi] \ni t \mapsto x_t^{(\psi)} \in B^1$, $\psi \in M \cap N_\phi$, from Proposition 4.2 are in general not differentiable.

Proposition 4.4 *(Uniqueness). Suppose $x : (-\infty, t_x) \to \mathbb{R}^n$, $0 < t_x \leqslant \infty$, and $y : (-\infty, t_y) \to \mathbb{R}^n$, $0 < t_y \leqslant \infty$, are solutions of Eq.* (1.1) *on $[0, t_x)$ and on $[0, t_y)$, respectively, and $x_0 = y_0$. Then*

$$x(t) = y(t) \quad \textit{for all } t \in (-\infty, t_x) \cap (-\infty, t_y).$$

Proof. All segments x_t, $0 \leqslant t < t_x$, and y_t, $0 \leqslant t < t_y$, belong to M. Suppose $x(t) \neq y(t)$ for some $t \in (0, t_x) \cap (0, t_y)$. Then by continuity there exist $t_0 \in [0, t)$ and a sequence $(t_j)_1^\infty$ in (t_0, t) so that $\lim_{j \to \infty} t_j = t_0$, $x(s) = y(s)$ for all $s \leqslant t_0$, and $x(t_j) \neq y(t_j)$ for all $j \in \mathbb{N}$. Set $\phi = x_{t_0} \in M$.

Choose N, h, d, ϵ, p and consider $E = E_{h,d,\phi,\epsilon,p}$, U_d, g_d, M_d as in Section 2. Choose a neighbourhood $N_d \subset U_d$ of $R_d\phi$ and $t_d > 0$ so small that

$$[0, t_d] \times (M_d \cap N_d) \subset \Omega_d.$$

Choose a neighbourhood $N_0 \subset N$ of ϕ so small that $R_d N_0 \subset N_d$. By Corollary 2.7, $R_d(M \cap N_0) \subset M_d$.

Proposition 2.3 yields continuity of the curves

$$(\infty, t_x) \ni s \mapsto x_s \in B^1 \quad \text{and} \quad (\infty, t_y) \ni s \mapsto y_s \in B^1.$$

In particular there exists $\delta \in (0, \min\{t - t_0, t_d\})$ with $x_s \in N_0 \ni y_s$ for $t_0 \leqslant s < t_0 + \delta$. Consider the shifted restrictions

$$\hat{x} : [-d, \delta) \ni s \mapsto x(t_0 + s) \in \mathbb{R}^n \quad \text{and} \quad \hat{y} : [-d, \delta) \ni s \mapsto y(t_0 + s) \in \mathbb{R}^n.$$

For $0 \leqslant s < \delta$ we have $x_{t_0+s} \in N_0$. Using Proposition 2.6 we infer

$$\hat{x}'(s) = x'(t_0 + s) = g(x_{t_0+s}) = g_d(R_d x_{t_0+s}) = g_d(\hat{x}_{d,s}).$$

For $\hat{y}$ we have the same result. As $\hat{y}_{d,0} = R_d\phi = \hat{x}_{d,0}$ we infer from uniqueness for the initial value problem (2.1)–(2.2) that $\hat{x}(s) = \hat{y}(s)$ on $(0, \delta)$ which gives

$$x(s) = y(s) \quad \text{for all } s \in (t_0, t_0 + \delta),$$

in contradiction to the properties of the sequence $(t_j)_1^\infty$. $\square$

Now we obtain the unique maximal solution $x^\phi : (-\infty, t_e(\phi)) \to \mathbb{R}^n$, $0 < t_e(\phi) \leqslant \infty$, of the initial value problem (1.1)–(1.2) from the relations

$$t_e(\phi) = \sup\bigl\{t > 0\colon \text{there exists a solution } x : (-\infty, t) \to \mathbb{R}^n \text{ of (1.1)–(1.2)}\bigr\} \leqslant \infty,$$

and for $0 < s < t_e(\phi)$,

$$x^\phi(s) = x(s)$$

with any solution $x : (-\infty, t) \to \mathbb{R}^n$ of the initial value problem (1.1)–(1.2) which satisfies $s < t$. The semiflow

$$H : \Omega \to M$$

given by these maximal solutions is defined by

$$\Omega = \{(t, \phi) \in [0, \infty) \times M\colon\ t < t_e(\phi)\}$$

and

$$H(t, \phi) = x_t^{\phi}.$$

Then $(0, \phi) \in \Omega$ for all $\phi \in M$, and $H(0, \phi) = \phi$. It is easy to see that for all $(s, \phi) \in \Omega$ and all $t \geqslant 0$ with $(t, H(s, \phi)) \in \Omega$ we have $(t + s, \phi) \in \Omega$ and

$$H(t + s, \phi) = H\big(t, H(s, \phi)\big).$$

Proposition 2.3 yields the following result.

Corollary 4.5. *All flowlines $[0, t_e(\phi)) \ni t \mapsto H(t, \phi) \in M$, $\phi \in M$, are continuous.*

For $t \geqslant 0$ we set

$$\Omega_t = \{\phi \in M\colon\ t < t_e(\phi)\}.$$

In case $\Omega_t \neq \emptyset$ we define the time-t-map, or solution operator,

$$H_t : \Omega_t \to M$$

by $H_t(\phi) = H(t, \phi)$.

Corollary 4.5 and Proposition 4.2(iii) combined with standard arguments yield the next result, whose proof is included for convenience.

Proposition 4.6. *For every $t \geqslant 0$ with $\Omega_t \neq \emptyset$ this set is open in M, and the map H_t is continuously differentiable.*

Proof. For $t = 0$ we have $\Omega_t = M$ and $H_t = \mathrm{id}_M$. So let $t > 0$ and $\phi \in \Omega_t$ be given. Corollary 4.5 implies that the set $H([0, t] \times \{\phi\}) \subset B^1$ is compact. Proposition 4.2(iii) guarantees that for each $s \in [0, t]$ there exist $t_s > 0$ and an open neighbourhood $N_{H(s,\phi)}$ of $H(s, \phi)$ in U with $[0, t_s] \times (M \cap N_{H(s,\phi)}) \subset \Omega$ so that for each $u \in [0, t_s]$ the map

$$M \cap N_{H(s,\phi)} \ni \psi \mapsto H(u, \psi) \in B^1$$

is continuously differentiable. By compactness there are $J \in \mathbb{N}$ and $s_1, \ldots, s_J$ in $[0, t]$ with

$$\bigcup_1^J N_{H(s_j,\phi)} \supset H\big([0, t] \times \{\phi\}\big).$$

Choose $K \in \mathbb{N}$ so that

$$\frac{t}{K} \leqslant \min_j t_{s_j}$$

For $k = 0, \dots, K-1$ set $u_k = k\frac{t}{K}$. We show by induction that for every $k \in \{0, \dots, K-1\}$ there exists a neighbourhood $N_k \subset U$ of ϕ with $[0, u_{k+1}] \times (M \cap N_k) \subset \Omega$ and

$$M \cap N_k \ni \psi \mapsto H(u_{k+1}, \psi) \in B^1$$

continuously differentiable.

For $k = 0$, $\phi = H(0, \phi) \in N_{H(s_j,\phi)}$ for some $j \in \{1, \dots, J\}$. $N_0 = N_{H(s_j,\phi)}$ is a neighbourhood of ϕ in U. For all $\psi \in M \cap N_0 = M \cap N_{H(s_j,\phi)}$ we get

$$[0, u_1] \times (M \cap N_0) \subset [0, t_{s_j}] \times (M \cap N_{H(s_j,\phi)}) \subset \Omega,$$

and the map

$$M \cap N_0 = M \cap N_{H(s_j,\phi)} \ni \psi \mapsto H(u_1, \psi) \in B^1$$

is continuously differentiable.

Suppose now $K > 1$ and the assertion holds for some $k \in \{0, \dots, K-2\}$. Then $\phi_* = H(u_{k+1}, \phi) \in N_{H(s_j,\phi)}$ for some $j \in \{1, \dots, J\}$. $N_* = N_{H(s_j,\phi)}$ is a neighbourhood of ϕ_* in U, and we have

$$\left[0, \frac{t}{K}\right] \times (M \cap N_*) \subset [0, t_{s_j}] \times (M \cap N_{H(s_j,\phi)}) \subset \Omega,$$

and the map

$$M \cap N_* = M \cap N_{H(s_j,\phi)} \ni \psi \mapsto H\left(\frac{t}{K}, \psi\right) \in B^1$$

is continuously differentiable. By assumption also the map

$$M \cap N_k \ni \chi \mapsto H(u_{k+1}, \chi) \in B^1$$

is continuously differentiable. Continuity yields that there is a neighbourhood $N_{k+1} \subset N_k$ of ϕ in U with

$$H(u_{k+1}, \chi) \in N_* \quad \text{for all } \chi \in M \cap N_{k+1}.$$

For such χ we infer

$$(u_{k+2}, \chi) = \left(\frac{t}{K} + u_{k+1}, \chi\right) \in \Omega$$

and

$$H(u_{k+2}, \chi) = H\left(\frac{t}{K}, H(u_{k+1}, \chi)\right)$$

which shows that the map

$$M \cap N_{k+1} \ni \chi \mapsto H(u_{k+2}, \chi) \in B^1$$

is continuously differentiable. $\square$

Proposition 4.7. *Ω is an open subset of $[0,\infty)\times M$, and H is continuous.*

Proof. Let $(t,\phi)\in\Omega$ be given. Then $t<t_e(\phi)$. Choose $t_0\in(t,t_e(\phi))$. As Ω_{t_0} is an open subset of M there is a neighbourhood N_ϕ of ϕ in U with $M\cap N_\phi\subset\Omega_{t_0}$. It follows that

$$[0,t_0]\times(M\cap N_\phi)\subset\Omega.$$

This proves the first part of the assertion.

For $(s,\psi)\in[0,t_0]\times(M\cap N_\phi)\subset\Omega$ we obtain

$$\big|H(s,\psi)-H(t,\phi)\big|_{B^1}\leqslant\big|H(s,\psi)-H(s,\phi)\big|_{B^1}+\big|H(s,\phi)-H(t,\phi)\big|_{B^1},$$

with

$$\begin{aligned}
&\big|H(s,\psi)-H(s,\phi)\big|_{B^1}\\
&\quad=\sup_{u\leqslant0}e^{au}\big|x^\psi(s+u)-x^\phi(s+u)\big|+\sup_{u\leqslant0}e^{au}\big|\big(x^\psi\big)'(s+u)-\big(x^\phi\big)'(s+u)\big|\\
&\quad=e^{-as}\sup_{u\leqslant0}e^{a(s+u)}\big|x^\psi(s+u)-x^\phi(s+u)\big|\\
&\qquad+e^{-as}\sup_{u\leqslant0}e^{a(s+u)}\big|\big(x^\psi\big)'(s+u)-\big(x^\phi\big)'(s+u)\big|\\
&\quad=e^{-as}\Big(\sup_{v\leqslant s}e^{av}\big|x^\psi(v)-x^\phi(v)\big|+\sup_{v\leqslant s}e^{av}\big|\big(x^\psi\big)'(v)-\big(x^\phi\big)'(v)\big|\Big)\\
&\quad\leqslant e^{-as}\Big(\sup_{v\leqslant t_0}e^{av}\big|x^\psi(v)-x^\phi(v)\big|+\sup_{v\leqslant t_0}e^{av}\big|\big(x^\psi\big)'(v)-\big(x^\phi\big)'(v)\big|\Big)\\
&\quad=e^{-as}\sup_{w\leqslant0}e^{a(t_0+w)}\big|x^\psi(t_0+w)-x^\phi(t_0+w)\big|\\
&\qquad+e^{-as}\sup_{w\leqslant0}e^{a(t_0+w)}\big|\big(x^\psi\big)'(t_0+w)-\big(x^\phi\big)'(t_0+w)\big|\\
&\quad=e^{a(t_0-s)}\big|H(t_0,\psi)-H(t_0,\phi)\big|_{B^1}\\
&\quad\leqslant e^{at_0}\big|H(t_0,\psi)-H(t_0,\phi)\big|_{B^1}.
\end{aligned}$$

Using the previous estimates, the continuity of H_{t_0}, and the continuity of the flowline $[0,t_e(\phi))\ni u\mapsto H(u,\phi)\in M$ the proof is easily completed. □

Proposition 4.8. *For $\phi\in M$ consider N,h,d,ϵ,p and $U_d,g_d,N_0,M_d,\Omega_d,H_d$ as in Section 2.*

(i) *There exist a neighbourhood $N_\phi\subset N_0\subset N\subset U$ and $t_\phi>0$ so that for all $t\in[0,t_\phi)$ and for all $\psi\in M\cap N_\phi$ we have $\psi\in\Omega_t$, $R_d\psi\in\Omega_{d,t}$, and*

$$R_dH_t(\psi)=H_{d,t}(R_d\psi).$$

(ii) *R_dN_0 is an open neighbourhood of $R_d\phi$, and*

$$R_d(M\cap N_0)=M_d\cap R_dN_0.$$

(iii) *For every* $\psi \in M \cap N_0$,

$$R_d T_\psi M = T_{R_d \psi} M_d.$$

Proof. 1. In order to prove assertion (i) recall from the proof of Proposition 4.2 that for $\psi \in M \cap N_\phi$ we have

$$x^{(\psi)}(t) = x^{d, R_d \psi}(t) \quad \text{for all } t \in [-d, t_0],$$

and that the restriction $x^{(\psi)} \mid (-\infty, t_\phi)$ is a solution of the initial value problem $x'(s) = g(x_s)$, $x_0 = \psi$. Hence $R_d H(t, \psi) = R_d x_t^\psi = R_d x_t^{(\psi)} = x_{d,t}^{d, R_d \psi} = H_d(t, R_d \psi)$ for $0 \leqslant t < t_\phi$.

2. Proof of assertion (ii). The Open Mapping Theorem guarantees that $R_d N_0 \ni R_d \phi$ is an open subset of C_d^1. For $\chi \in M_d \cap R_d N_0$ we have $M_d \ni \chi = R_d \psi$ with $\psi \in N_0 \subset U$. Using Proposition 2.6 we infer

$$\psi'(0) = \chi'(0) = g_d(\chi) = g_d(R_d \psi) = g(\psi)$$

which gives $\psi \in M$. Consequently,

$$M_d \cap R_d N_0 \subset R_d(M \cap N_0).$$

The other inclusion is obvious from $R_d(M \cap N_0) \subset M_d$ in Corollary 2.7.

3. Proof of assertion (iii). Let $\psi \in M \cap N_0$ be given. The previous inclusion yields $R_d T_\psi M \subset T_{R_d \psi} M_d$. Let $\chi \in T_{R_d \psi} M_d \subset C_d^1$ be given. Choose a backward continuation $\chi^c \in B^1$ of χ. Then $\chi = R_d \chi^c$. Using Proposition 2.6 we infer

$$(\chi^c)'(0) = \chi'(0) = Dg_d(R_d \psi)\chi = Dg_d(R_d \psi) R_d \chi^c = Dg(\psi)\chi^c,$$

or, $\chi^c \in T_\psi M$. $\square$

5. Linearization and variational equations

Proposition 5.1. *For $\phi \in M$ consider N, h, d, ϵ, p and g_d, M_d, H_d as in Section 2. There exist a neighbourhood $N_\phi \subset N \subset U$ and $t_\phi \in (0, d]$ with $[0, t_\phi) \times (M \cap N_\phi) \subset \Omega$ so that for every $\psi \in M \cap N_\phi$ and for every $\chi \in T_\psi M$ there is a solution $v : (-\infty, t_\phi) \to \mathbb{R}^n$ of the initial value problem*

$$v'(t) = Dg\big(H(t, \psi)\big) v_t, \tag{5.1}$$

$$v_0 = \chi \tag{5.2}$$

with

$$DH_t(\psi)\chi = v_t \quad \text{for all } t \in [0, t_\phi).$$

Here a solution $v : I \to \mathbb{R}^n$, $I = (-\infty, t_*]$ with $0 < t_* < t_e(\psi)$ or $I = (-\infty, t_*)$ with $0 < t_* \leqslant t_e(\psi) \leqslant \infty$, of the initial value problem (5.1)–(5.2) is defined in analogy with the corresponding notion for the autonomous problem in Section 1: We require that v is continuously differentiable with $v_0 = \chi \in T_\psi M$ (which implies $v_t \in B^1$ for $0 \leqslant t \in I$) and that Eq. (5.1) holds for $0 \leqslant t \in I$, which is equivalent to the relation $v_t \in T_{H(t,\psi)}M$ for these t.

Proof of Proposition 5.1. 1. Let $\phi \in M$ be given. Consider N, h, d, ϵ, p and g_d, M_d, H_d, and $N_\phi \subset N_0 \subset N \subset U$ and $t_\phi \in (0, d]$ as in Proposition 4.8(i). Let $\psi \in M \cap N_\phi$, $\chi \in T_\psi M$ be given. Then $R_d\psi \in M_d$ and $R_d\chi = DR_d(\psi)\chi \in T_{R_d\psi}M_d$, and differentiation of the formula from Proposition 4.8(i) yields

$$R_d DH_t(\psi)\chi = DH_{d,t}(R_d\psi)R_d\chi = v_{d,t}^{d,R_d\psi,R_d\chi}$$

for every $t \in [0, t_\phi)$.

2. The restriction operator

$$R_{dd} : B^1 \ni \rho \mapsto \rho \mid (-\infty, -d] \in B^1_{-d}$$

is linear and continuous. Let $0 \leqslant t < t_\phi (\leqslant d)$. For every $\zeta \in M \cap N_\phi$ and for all $s \leqslant -d$ we have

$$\big(R_{dd} H_t(\zeta)\big)(s) = \big(R_{dd} x_t^\zeta\big)(s) = x^\zeta(t+s) = \zeta(t+s).$$

The linear map

$$B^1 \ni \zeta \mapsto \zeta(t+\cdot) \mid (-\infty, -d] \in B^1_{-d}$$

is continuous. Therefore differentiation of $R_{dd} \circ H_t$ at ψ yields

$$R_{dd} DH_t(\psi)\chi = D(R_{dd} \circ H_t)(\psi)\chi = \chi(t+\cdot) \mid (-\infty, -d].$$

3. The map $v : (-\infty, t_\phi) \to \mathbb{R}^n$ given by

$$v(s) = v^{d,R_d\psi,R_d\chi}(s) \quad \text{for } 0 \leqslant s < t_\phi,$$
$$v(s) = \chi(s) \quad \text{for } s < 0$$

is continuously differentiable, with all segments v_t, $0 \leqslant t < t_\phi$, in B^1. Let $0 \leqslant t < t_\phi$. Then

$$R_d v_t = v_{d,t}^{d,R_d\psi,R_d\chi} = R_d DH_t(\psi)\chi.$$

Also,

$$R_{dd} v_t = v(t+\cdot) \mid (-\infty, -d] = \chi(t+\cdot) \mid (-\infty, -d] = R_{dd} DH_t(\psi)\chi.$$

The last equations combined yield

$$DH_t(\psi)\chi = v_t.$$

Using $N_\phi \subset N_0$ and Proposition 2.6 we also get

$$v'(t) = \left(v^{d,R_d\psi,R_d\chi}\right)'(t) = Dg_d\left(H_{d,t}(R_d\psi)\right)v_{d,t}^{d,R_d\psi,R_d\chi}$$
$$= Dg_d\left(R_d H_t(\psi)\right)R_d v_t = Dg\left(H_t(\psi)\right)v_t. \qquad \square$$

Corollary 5.2. *For every $\phi \in M$ and for every $\chi \in T_\phi M$ there is a solution $v : (-\infty, t_e(\phi)) \to \mathbb{R}^n$ of the initial value problem* (5.1)–(5.2) *with*

$$DH_t(\phi)\chi = v_t \quad \textit{for all } t \in \left[0, t_e(\phi)\right).$$

Proof. 1. Let $\phi \in M$ and $\chi \in T_\phi M$ be given. Let $t \in [0, t_e(\phi))$. For each point $H(s,\phi)$, $0 \leqslant s \leqslant t$, there exist an open neighbourhood $N_{H(s,\phi)}$ and $t_{H(s,\phi)} > 0$ as in Proposition 5.1. Compactness yields $J \in \mathbb{N}$ and $s_1, \ldots, s_J$ in $[0,t]$ with

$$\bigcup_{j=1}^{J} N_{H(s_j,\phi)} \supset H\left([0,t] \times \{\phi\}\right).$$

Choose an integer $K \geqslant 2$ with $\frac{t}{K} < \min_{j=1,\ldots,J} t_{H(s_j,\phi)}$. For $k = 0, \ldots, K$ define $\phi_k \in M$ and $\chi_k \in T_{\phi_k} M$ by

$$\phi_0 = \phi \quad \text{and} \quad \phi_k = H_{t/K}(\phi_{k-1}) \quad \text{for } k = 1, \ldots, K,$$
$$\chi_0 = \chi \quad \text{and} \quad \chi_k = DH_{t/K}(\phi_{k-1})\chi_{k-1} \quad \text{for } k = 1, \ldots, K.$$

For each point ϕ_k, $k = 0, \ldots, K-1$, there exists $j \in \{1, \ldots, J\}$ with $\phi_k \in N_{H(s_j,\phi)}$. We apply Proposition 5.1 to ϕ_k in this neighbourhood and to $\chi_k \in T_{\phi_k} M$. As $\frac{t}{K} < t_{H(s_j,\phi)}$ we obtain solutions $v^{((k))} : (-\infty, \frac{t}{K}] \to \mathbb{R}^n$, $k = 1, \ldots, K$, of the initial value problems

$$v'(s) = Dg\left(H(s, \phi_{k-1})\right)v_s,$$
$$v_0 = \chi_{k-1},$$

with

$$DH_s(\phi_{k-1})\chi_{k-1} = v_s^{((k))} \quad \text{for } 0 \leqslant s \leqslant \frac{t}{K}. \tag{5.3}$$

In particular,

$$\chi_k = DH_{t/K}(\phi_{k-1})\chi_{k-1} = v_{\frac{t}{K}}^{((k))} \quad \text{for } k = 1, \ldots, K. \tag{5.4}$$

Define $v^{(t)} : (-\infty, t] \to \mathbb{R}^n$ by $v_0^{(t)} = \chi$ and

$$v^{(t)}(s) = v^{((k))}\left(s - \frac{(k-1)t}{K}\right) \quad \text{for } \frac{(k-1)t}{K} < s \leqslant \frac{kt}{K},\ k = 1, \ldots, K.$$

2. *Claim*: For every $k \in \{1, \ldots, K\}$ and for $\frac{k-1}{K}t \leqslant s \leqslant \frac{k}{K}t$ we have

$$v^{((k))}_{s-\frac{k-1}{K}t} = v^{(t)}_s.$$

Proof by induction: For $k = 1$ we consider $0 \leqslant s \leqslant \frac{t}{K}$ and $u \leqslant 0$. In case $s + u \leqslant 0$ we have

$$v^{(t)}_s(u) = v^{(t)}(s+u) = \chi(s+u) = v^{((1))}(s+u) = v^{((1))}_s(u)$$

while in case $0 < s + u$,

$$v^{(t)}_s(u) = v^{(t)}(s+u) = v^{((1))}(s+u) = v^{((1))}_s(u)$$

by definition.

Now assume the assertion holds for some $k \in \{1, \ldots, K-1\}$. Let $\frac{(k+1)-1}{K}t \leqslant s \leqslant \frac{k+1}{K}t$ and $u \leqslant 0$. In case

$$s + u \leqslant \frac{(k+1)-1}{K}t = \frac{k}{K}t$$

we have

$$s + u - \frac{k}{K}t \leqslant 0$$

and get

$$\begin{aligned}
v^{(t)}_s(u) &= v^{(t)}(s+u) = v^{(t)}\left(\frac{k}{K}t + \left(s + u - \frac{k}{K}t\right)\right) \\
&= v^{(t)}_{\frac{k}{K}t}\left(s + u - \frac{k}{K}t\right) = v^{((k))}_{\frac{k}{K}t - \frac{k-1}{K}t}\left(s + u - \frac{k}{K}t\right) \quad \text{(by assumption)} \\
&= v^{((k))}_{\frac{t}{K}}\left(s + u - \frac{k}{K}t\right) = \chi_k\left(s + u - \frac{k}{K}t\right) \quad \big(\text{by (5.4)}\big) \\
&= v^{((k+1))}\left(s + u - \frac{k}{K}t\right) = v^{((k+1))}_{s-\frac{(k+1)-1}{K}t}(u).
\end{aligned}$$

In case

$$\left(\frac{k}{K}t =\right) \quad \frac{(k+1)-1}{K}t < s + u \leqslant \frac{k+1}{K}t$$

the definition of $v^{(t)}$ yields

$$v^{(t)}_s(u) = v^{(t)}(s+u) = v^{((k+1))}\left(s + u - \frac{(k+1)-1}{K}t\right) = v^{((k+1))}_{s-\frac{(k+1)-1}{K}t}(u).$$

3. It follows that $v^{(t)}$ is continuously differentiable with initial value χ, and all segments $v^{(t)}_s$, $0 \leqslant s \leqslant t$, belong to B^1, due to Proposition 2.3. Moreover, for $k \in \{1, \ldots, K\}$ and

$$\frac{k-1}{K}t < s \leqslant \frac{k}{K}t$$

we infer

$$\begin{aligned}
\big(v^{(t)}\big)'(s) &= \big(v^{((k))}\big)'\left(s - \frac{k-1}{K}t\right) \\
&= Dg\left(H\left(s - \frac{k-1}{K}t, \phi_{k-1}\right)\right)v^{((k))}_{s-\frac{k-1}{K}t} \\
&= Dg\left(H\left(s - \frac{k-1}{K}t, H\left(\frac{k-1}{K}t, \phi\right)\right)\right)v^{(t)}_s \\
&= Dg\big(H(s,\phi)\big)v^{(t)}_s.
\end{aligned}$$

Also,

$$\big(v^{(t)}\big)'(0) = \chi'(0) = Dg(\phi)\chi = Dg\big(H(0,\phi)\big)v^{(t)}_0$$

as $\chi \in T_\phi M$.

4. Let $s \in [0,t]$ be given and choose $k \in \{1,\dots,K\}$ with

$$\frac{k-1}{K}t \leqslant s \leqslant \frac{k}{K}t.$$

Then

$$0 \leqslant s - \frac{k-1}{K}t \leqslant \frac{t}{K},$$

and using this and the result of part 2 once more we obtain

$$\begin{aligned}
v^{(t)}_s &= v^{((k))}_{s-\frac{k-1}{K}t} = DH_{s-\frac{k-1}{K}t}(\phi_{k-1})\chi_{k-1} \quad \big(\text{see (5.3)}\big) \\
&= DH_{s-\frac{k-1}{K}t}\big(H_{\frac{k-1}{K}t}(\phi)\big)DH_{\frac{k-1}{K}t}(\phi)\chi \\
&= DH_s(\phi)\chi.
\end{aligned}$$

5. Finally, observe that for $0 \leqslant t < u < t_e(\phi)$ we have

$$v^{(u)}(s) = v^{(t)}(s) \quad \text{for all } s \leqslant t$$

since $v^{(u)}_t = DH_t(\phi)\chi = v^{(t)}_t$. Now define $v : (-\infty, t_e(\phi)) \to \mathbb{R}^n$ by $v(s) = v^{(t)}(s)$ for $s < t_e(\phi)$, with some $t \in [s, t_e(\phi))$. $\square$

6. Local stable manifolds at equilibria

Let a constant map $\phi \in U \subset B^1$ be given with value ξ and $g(\phi) = 0$. Then $\phi \in M$ and $\mathbb{R} \ni t \mapsto \xi \in \mathbb{R}^n$ is a solution on $\mathbb{R}$ for Eq. (1.1). Consider N, h, d, ϵ, p and $U_d, g_d, N_0, M_d, \Omega_d, H_d$ as in Section 2. Choose a neighbourhood N_ϕ of ϕ in $N_0 \subset N \subset U \subset B^1$ and $t_\phi > 0$ according to Proposition 4.8(i).

Notice that our previous theory applies to $Dg(\phi) : B^1 \to \mathbb{R}^n$ in place of g and to the solution manifold $T = T_\phi M$ in place of M. In particular each $\chi \in T$ uniquely defines a (continuously differentiable) solution $v^{\phi,\chi} : \mathbb{R} \to \mathbb{R}^n$ of the initial value problem

$$v'(t) = Dg(\phi)v_t,$$

$$v_0 = \chi,$$

and the maps $DH_t(\phi) : T \ni \chi \mapsto v_t^{\phi,\chi} \in T$, $t \geqslant 0$, form a strongly continuous semigroup.

Fix $t \in (0, t_\phi) \cap (0, d)$ and set $L = DH_t(\phi)$, $L_d = DH_{d,t}(R_d\phi)$. Also define $T_d = T_{R_d\phi}M_d$. From Proposition 4.8 we infer

$$R_d L\chi = L_d R_d \chi \quad \text{for all } \chi \in T,$$

$$R_d T = T_d, \quad \text{and}$$

$$R_d(M \cap N_0) = M_d \cap (R_d N_0).$$

It seems worth mentioning that we have

$$B^1 \cap ker(R_d) \subset T$$

although we shall not make use of this in the sequel. Proof of the preceding inclusion: Let $\chi \in B^1 \cap ker(R_d)$ be given. Using Proposition 2.6 we obtain

$$Dg(\phi)\chi = Dg_d(R_d\phi)R_d\chi = 0 = \chi'(0)$$

which means $\chi \in T$.

In order to obtain local stable and unstable manifolds of the map H_t at its fixed point ϕ we need a direct sum decomposition of T into 3 closed subspaces which are positively invariant under L, with the spectrum of one of the induced endomorphisms contained in the open unit disk Δ and the spectra of the other induced endomorphisms on and outside the unit circle, respectively.

We begin with spectral properties of the map $L_d = DH_{d,t}(R_d\phi)$. From results in Chapter 3 of [10] about the spectrum of the generator of the semigroup $(DH_{d,s}(\mathbb{R}_d\phi))_{s\geqslant 0}$ on $T_d = T_{R_d\phi}M_d$ we obtain that the spectrum of L_d decomposes into a non-empty compact subset $\sigma_{d,s} \subset \Delta$ and finite sets $\sigma_{d,c}$ and $\sigma_{d,u}$ of eigenvalues on and outside the unit circle, respectively. The realified generalized eigenspaces $T_{d,c}$ of $\sigma_{d,c}$ and $T_{d,u}$ of $\sigma_{d,u}$ are finite-dimensional. We assume that also $\sigma_{d,u}$ is non-empty. Choose a positive real number $\gamma_s < 1$ with

$$|z| < \gamma_s \quad \text{for all } z \in \sigma_{d,s}.$$

As $a > 0$ we may assume

$$-\log(\gamma_s) < at \tag{6.1}$$

and find $\gamma_c \in (0,1)$ with

$$-\log(\gamma_c) < at. \tag{6.2}$$

Choose $\gamma_u > 1$ with $\gamma_u < |z|$ for all $z \in \sigma_{d,u}$. With $T_{d,s}$ denoting the realified generalized eigenspace of the spectral set $\sigma_{d,s}$ we have the decomposition

$$T_d = T_{d,s} \oplus T_{d,c} \oplus T_{c,u}$$

into closed subspaces with $L_d T_{d,s} \subset T_{d,s}$, $\dim T_{d,c} < \infty$, $L_d T_{d,c} = T_{d,c}$, $0 < \dim T_{d,u} < \infty$, $L_d T_{d,u} = T_{d,u}$, and there are constants $k_s \geqslant 1$, $k_c > 0$, $k_u > 0$ such that for all $j \in \mathbb{N}$ we have

$$\begin{aligned} \left|L_d^j \chi\right|_{C_d^1} &\leqslant k_s \gamma_s^j |\chi|_{C_d^1} \quad \text{for all } \chi \in T_{d,s}, \\ \left|L_d^j \chi\right|_{C_d^1} &\geqslant k_c \gamma_c^j |\chi|_{C_d^1} \quad \text{for all } \chi \in T_{d,c}, \\ \left|L_d^j \chi\right|_{C_d^1} &\geqslant k_u \gamma_u^j |\chi|_{C_d^1} \quad \text{for all } \chi \in T_{d,u}. \end{aligned}$$

$T_{d,s}$ and $T_{d,u}$ are the *stable space* and the *unstable space*, respectively.

For the closed subspace

$$T_s = B^1 \cap (R_d)^{-1} T_{d,s}$$

we have $T_s \subset T$ since $\chi \in B^1$, $R_d \chi \in T_{d,s}$ and Proposition 2.6 combined yield

$$\chi'(0) = (R_d \chi)'(0) = Dg_d(R_d \phi) R_d \chi = Dg(\phi)\chi.$$

Also, $LT_s \subset T_s$ because for every $\chi \in T_s \subset T \subset B^1$ we have $L\chi \in T \subset B^1$, $R_d \chi \in T_{d,s}$, and

$$R_d L \chi = L_d R_d \chi \in L_d T_{d,s} \subset T_{d,s}.$$

Next we construct a space T_c. The previous estimate shows that the endomorphism $L_{d,c} : T_{d,c} \to T_{d,c}$ given by L_d is injective. As $T_{d,c}$ has finite dimension $L_{d,c}$ is an isomorphism. Therefore each element $\chi \in T_{d,c}$ defines a sequence of backward iterates $\chi_j \in T_{d,c}$,

$$\chi_0 = \chi \quad \text{and} \quad \chi_j = L_{d,c} \chi_{j-1} \quad \text{for all } j \in -\mathbb{N}_0.$$

Proposition 6.1. *For every $\chi \in T_{d,c}$ there exists $\hat{\chi} \in B^1$ with*

$$\hat{\chi}_{d,jt} = \chi_j \quad \textit{for all } j \in -\mathbb{N}_0.$$

Proof. 1. Recall $0 < t < d$. For each $j \in -\mathbb{N}_0$ we have $\chi_j = v_t^{d, R_d \phi, \chi_{j-1}}$. For $-d \leqslant s \leqslant -t$ this yields

$$\chi_j(s) = v^{d, R_d \phi, \chi_{j-1}}(t+s) = \chi_{j-1}(s+t).$$

Claim: For every $j \in -\mathbb{N}_0$ and for all $k \in \mathbb{N}$ with $-d \leqslant -kt$ we have

$$\chi_j(s) = \chi_{j-k}(s+kt) \quad \text{for all } s \in [-d, -kt].$$

Proof by induction: For $k = 1$, see the previous statement. Suppose the assertion holds for $k \in \mathbb{N}$, and $-d \leqslant s \leqslant -(k+1)t$. Then $s \leqslant -kt$, hence

$$\chi_j(s) = \chi_{j-k}(s+kt) = \chi_{(j-k)-1}\big((s+kt)+t\big)$$

by the assumption for k in combination with the previous statement.

2. *Claim*: For $m < j$ in $-\mathbb{N}_0$ and $s \leqslant 0$ with

$$mt - d \leqslant s \leqslant mt \quad \text{and} \quad jt - d \leqslant s \leqslant jt$$

we have

$$\chi_j(s - jt) = \chi_m(s - mt).$$

Proof: Apply the preceding claim with $k = j - m$ and $s - jt \in [-d, -kt]$ in place of s.

3. It follows that the equations

$$\hat{\chi}(s) = \chi_j(s - jt) \quad \text{for } j \in -\mathbb{N}_0 \quad \text{and} \quad jt - d \leqslant s \leqslant jt$$

define a continuously differentiable map $\hat{\chi} : (-\infty, 0] \to \mathbb{R}^n$. For every $j \in -\mathbb{N}_0$ and for all $u \in [-d, 0]$ we have

$$\hat{\chi}_{d,jt}(u) = \hat{\chi}(jt + u) = \chi_j(u),$$

which gives $\hat{\chi}_{d,jt} = \chi_j$ for $j \in -\mathbb{N}_0$.

4. Proof of $\hat{\chi} \in B^1$. Let $s \leqslant 0$. Choose $j \in -\mathbb{N}_0$ with $jt - t \leqslant s \leqslant jt$. Then $jt - d \leqslant s \leqslant jt$. Hence

$$\begin{aligned}
e^{as}\big|\hat{\chi}(s)\big| &= e^{as}\big|\chi_j(s - jt)\big| \leqslant e^{as}|\chi_j|_{C^1_d} \\
&= e^{as}\big|L^j_{d,c}\chi_0\big|_{C^1_d} \leqslant \frac{1}{k_c}e^{as}\gamma_c^j|\chi|_{C^1_d} \\
&= \frac{1}{k_c}e^{as + j\log(\gamma_c)}|\chi|_{C^1_d} \\
&\leqslant \frac{1}{k_c}e^{ajt + j\log(\gamma_c)}|\chi|_{C^1_d}.
\end{aligned}$$

From this estimate in combination with inequality (6.2) we infer $e^{as}\hat{\chi}(s) \to 0$ as $s \to -\infty$. Arguing in the same way we find $e^{as}\hat{\chi}'(s) \to 0$ as $s \to -\infty$. $\square$

The map $S_{d,c} : T_{d,c} \ni \chi \mapsto \hat{\chi} \in B^1$ is linear and injective, and

$$R_d S_{d,c}\chi = R_d\hat{\chi} = \hat{\chi}_{d,0} = \chi_0 = \chi \quad \text{for all } \chi \in T_{d,c}.$$

It follows that the finite-dimensional subspace

$$T_c = S_{d,c}T_{d,c} \subset B^1$$

is isomorphic to $T_{d,c}$ and that the map $R_{d,c} : T_c \to T_{d,c}$ given by R_d is an isomorphism.

We have $T_c \subset T$ since for every $\hat{\chi} \in T_c$, $\hat{\chi} = S_{d,c}\chi$ with $\chi \in T_{d,c}$, and using Proposition 2.6 we get

$$\hat{\chi}'(0) = \chi'(0) = Dg_d(R_d\phi)\chi = Dg_d(R_d\phi)R_d\hat{\chi} = Dg(\phi)\hat{\chi}.$$

Proposition 6.2. $LT_c \subset T_c$.

Proof. Let $\hat{\chi} \in T_c$ be given. Then $\hat{\chi} = S_{d,c}\chi$ with $\chi \in T_{d,c}$. Consider the sequence of the points $\chi_j = L_{d,c}^j \chi \in T_{d,c}$, $j \in -\mathbb{N}_0$. Set

$$\zeta = L_d\chi \in T_{d,c} \quad \text{and} \quad \zeta_j = L_{d,c}^j \zeta \quad \text{for } j \in -\mathbb{N}_0.$$

Then $\zeta_j = \chi_{j+1}$ for all $j \in -\mathbb{N}$. We get

$$\zeta_0 = \zeta = L_d R_d \hat{\chi} = R_d L \hat{\chi} = (L\hat{\chi})_{d,0}.$$

The equations

$$L\hat{\chi}(u) = v^{\phi,\hat{\chi}}(t+u) = \hat{\chi}(t+u) \quad \text{for all } u \leqslant -t$$

in combination with Proposition 6.1 imply that for every $j \in -\mathbb{N}$ we have

$$(L\hat{\chi})_{d,jt} = \hat{\chi}_{d,(j+1)t} = \chi_{j+1} = \zeta_j.$$

Using Proposition 6.1 again we infer

$$\hat{\zeta}_{d,jt} = \zeta_j = (L\hat{\chi})_{d,jt} \quad \text{for all } j \in -\mathbb{N}_0$$

which yields

$$L\hat{\chi} = \hat{\zeta} \in T_c. \qquad \Box$$

In the same way as before we construct a linear injective map $S_{d,u} : T_{d,u} \to B^1$ the only difference being that in the proof of the analogue of Proposition 6.1 we use the obvious estimate $0 < at + \log(\gamma_u)$ instead of (6.2). We have

$$R_d S_{d,u}\chi = \chi \quad \text{for all } \chi \in T_{d,u},$$

R_d defines an isomorphism $R_{d,u}$ from

$$T_u = S_{d,u}T_{d,u}$$

onto $T_{d,u}$, and $T_u \subset T$, $LT_u \subset T_u$.

Notice that the isomorphisms $L_c : T_c \to T_c$ and $L_u : T_u \to T_u$ given by L are conjugate to $L_{d,c}$ and $L_{d,u}$, respectively. It follows that their spectra coincide with $\sigma_{d,c}$ and $\sigma_{d,u}$, respectively.

Proposition 6.3.

$$T = T_s \oplus T_c \oplus T_u.$$

Proof. Let $\psi \in T$. Then $R_d\psi = \chi + \rho + \eta$ with $\chi \in T_{d,s}$ and $\rho \in T_{d,c}$ and $\eta \in T_{d,u}$. For $\psi - S_{d,c}\rho - S_{d,u}\eta \in T$ we obtain $R_d(\psi - S_{d,c}\rho - S_{d,u}\eta) = R_d\psi - \rho - \eta = \chi \in T_{d,s}$, and it follows that $\psi - S_{d,c}\rho - S_{d,u}\eta \in B^1 \cap R_d^{-1}(T_{d,s}) = T_s$. This shows $T \subset T_s + T_c + T_u$.

For $\psi \in T_s \cap T_c \cap T_u$ we get $R_d\psi \in T_{d,s} \cap T_{d,c} \cap T_{d,u} = \{0\}$ and thereby $\psi = S_{d,c}R_d\psi = 0$. □

Proposition 6.4. *There exists $\kappa_s \geqslant 1$ with*

$$\left|L^j\psi\right|_{B^1} \leqslant \kappa_s\gamma_s^j|\psi|_{B^1} \quad \textit{for all } j \in \mathbb{N}, \psi \in T_s.$$

Proof. Let $\psi \in T_s$ and $j \in \mathbb{N}$ be given. Set $v = v^{\phi,\psi}$ and observe that due to Proposition 2.6 we have

$$v'(s) = Dg(\phi)v_s = Dg_d(R_d\phi)R_dv_s = Dg_d(R_d\phi)v_{d,s} \quad \text{for all } s \geqslant 0.$$

This shows that $v|[-d,\infty) = v^{d,R_d\phi,R_d\psi}$. For every $u \leqslant 0$ with $0 \leqslant jt + u$ we obtain the equation

$$\left(L^j\psi\right)(u) = v_{jt}(u) = v(jt+u) = v^{d,R_d\phi,R_d\psi}(jt+u)$$

while in case $u \leqslant 0$ and $jt + u \leqslant 0$ we have

$$\left(L^j\psi\right)(u) = v_{jt}(u) = v(jt+u) = \psi(jt+u).$$

In this second case we use the inequality (6.1) and obtain the estimate

$$e^{au}\left|\left(L^j\psi\right)(u)\right| = e^{a(jt+u)}\left(e^{-at}\right)^j\left|\psi(jt+u)\right| \leqslant \gamma_s^j|\psi|_{B^1}.$$

Analogously we find

$$e^{au}\left|\left(L^j\psi\right)'(u)\right| \leqslant \gamma_s^j|\psi|_{B^1}.$$

In the other case, namely $u \leqslant 0$ with $0 \leqslant jt + u$, there exists $\iota \in \{1,\dots,j\}$ with

$$(\iota t - d \leqslant) \quad (\iota - 1)t \leqslant jt + u \leqslant \iota t.$$

We obtain

$$\begin{aligned} e^{au}\left|\left(L^j\psi\right)(u)\right| &= e^{au}\left|v^{d,R_d\phi,R_d\psi}(jt+u)\right| \leqslant e^{au}\left|v_{d,\iota t}^{d,R_d\phi,R_d\psi}\right|_{C_d^1} \\ &= e^{au}\left|(L_d)^\iota R_d\psi\right|_{C_d^1} \leqslant e^{au}k_s\gamma_s^\iota|R_d\psi|_{C_d^1} \quad (\text{since} R_d\psi \in T_{d,s}) \\ &\leqslant e^{au+\iota\log(\gamma_s)}k_sr_d|\psi|_{B^1}, \end{aligned}$$

where r_d denotes the norm of the map $B^1 \to C_d^1$ induced by R_d. From (6.1) we have

$$(j-\iota)\log(\gamma_s) \geqslant (\iota - j)at \geqslant au,$$

hence

$$au + \iota\log(\gamma_s) \leqslant j\log(\gamma_s).$$

Using this we infer

$$e^{au}\left|\left(L^j\psi\right)(u)\right| \leqslant \gamma_s^j k_s r_d |\psi|_{B^1}.$$

Analogously we find

$$e^{au}\left|\left(L^j\psi\right)'(u)\right| \leqslant \gamma_s^j k_s r_d |\psi|_{B^1}$$

in the case considered. Combining both cases we deduce the estimate

$$\left|L^j\psi\right|_{B^1} \leqslant 2(1+k_s r_d)\gamma_s^j |\psi|_{B^1}. \qquad \square$$

Using the spectral radius formula we infer from Proposition 6.4 that the spectrum of the map $L_s : T_s \ni \psi \mapsto L\psi \in T_s$ is contained in the closed disk of radius $\gamma_s < 1$. Choose $\gamma \in (0,1)$ with

$$\gamma_s < \gamma \quad \text{and} \quad \frac{1}{\gamma_u} < \gamma.$$

From here on we proceed as in Section 3.5 of [10] in order to obtain a local stable manifold of the semiflow H at the stationary point ϕ. We begin with a manifold chart of M at ϕ. There is a subspace $E \subset B^1$ of dimension n which is a complement of T in B^1. Let $P : B^1 \to B^1$ denote the projection along E onto T. Then the equation $K(\psi) = P(\psi - \phi)$ defines a manifold chart on an open neighbourhood V of ϕ in $\Omega_t \subset M$, with $T_0 = K(V)$ an open neighbourhood of $0 = K(\phi)$ in the Banach space T (with the norm given by $|\cdot|_{B^1}$). The inverse of K is given by a continuously differentiable map $R : T_0 \to B^1$. Both derivatives $DK(\phi)$ and $DR(0)$ are given by the identity on T. We may assume $lip(R) < \infty$. Choose an open neighbourhood $T_1 \subset T_0$ of 0 in T with $H_t(R(T_1)) \subset V$. In local coordinates the map H_t is represented by the continuously differentiable map

$$F : T_1 \ni \chi \mapsto K\big(H_t\big(R(\chi)\big)\big) \in T.$$

Obviously, $F(0) = 0$, $DF(0) = DH_t(\phi) = L$, and $F(T_1) \subset T_0$. Proposition 6.3, the location of the spectra of the maps L_s, L_c, L_u, and the choice of γ show that Theorem I.2 from [18] about a local stable manifold of the continuously differentiable map F at its fixed point $0 \in T_1$ can be applied; Theorem I.2 in [18] goes back to a stable manifold theorem in [9] and a version of the latter in [19]. With $T_{cu} = T_c \oplus T_u$ we obtain the following result.

Proposition 6.5. *There exist $\alpha \in (0,\gamma)$, convex open neighbourhoods $T_{s,2}$ of 0 in T_s and $T_{cu,2}$ of 0 in T_{cu} with $T_2 = T_{s,2} + T_{cu,2} \subset T_1$, a continuously differentiable map $w : T_{s,2} \to T_{cu,2}$ with $w(0) = 0$ and $Dw(0) = 0$, and an equivalent norm $|\cdot|_F$ on T such that the following holds.*

(i) *The graph $W = \{\chi + w(\chi):\ \chi \in T_{s,2}\}$ is equal to the set of all initial points $\psi = \psi_0$ of trajectories $(\psi_j)_0^\infty$ of F which satisfy $\gamma^{-j}\psi_j \in T_2$ for all $j \in \mathbb{N}_0$ and $\gamma^{-j}\psi_j \to 0$ as $j \to \infty$.*
(ii) *$F(W) \subset W$.*
(iii) *$|F(\chi) - F(\psi)|_F \leqslant \alpha|\chi - \psi|_F$ for all $\psi \in W$, $\chi \in W$.*
(iv) *For every trajectory $(\psi_j)_0^\infty$ of F with $\gamma^{-j}\psi_j \in T_2$ for all $j \in \mathbb{N}_0$,*

$$\psi_0 \in W.$$

Here, trajectories are defined by the equations $\psi_{j+1} = F(\psi_j)$ for all integers $j \geqslant 0$.

As a *local stable manifold* of H_t at ϕ we take the continuously differentiable submanifold

$$W_s = R(W)$$

of M. Obviously, $W_s \subset V$, $\phi \in W_s$, and $T_\phi W_s = T_s$.

Corollary 6.6. (i) *$H_t(W_s) \subset W_s$, and each neighbourhood of ϕ in W_s contains a neighbourhood $W_{s,1}$ of ϕ in W_s with $H_t(W_{s,1}) \subset W_{s,1}$.*

(ii) *There exists $c_s \geqslant 0$ so that for every trajectory $(\psi_j)_0^\infty$ of H_t in W_s and for all integers $j \geqslant 0$,*

$$|\psi_j - \phi|_{B^1} \leqslant c_s \alpha^j |\psi_0 - \phi|_{B^1}.$$

Proof. 1. The first inclusion in assertion (i) follows from

$$K\big(H_t(W_s)\big) = K\big(H_t\big(R(W)\big)\big) = F(W) \subset W = K\big(R(W)\big)$$

by application of R. Proof of the second part of (i): For $\epsilon > 0$, set

$$T_{F,\epsilon} = \big\{\psi \in T\colon |\psi|_F < \epsilon\big\}.$$

Any given neighbourhood of ϕ in $V \subset M$ contains $V_\epsilon = R(T_{F,\epsilon})$ for some $\epsilon > 0$, and $R(W \cap T_{F,\epsilon}) = R(W) \cap R(T_{F,\epsilon}) = W_s \cap V_\epsilon$. Parts (ii) and (iii) of Proposition 6.5 yield $H_t(W_s \cap V_\epsilon) = R(K(H_t(R(W \cap T_{F,\epsilon})))) = R(F(W \cap T_{F,\epsilon})) \subset R(W \cap T_{F,\epsilon}) = W_s \cap V_\epsilon$.

2. Proof of assertion (ii). There are positive constants $c_1 \leqslant c_2$ with

$$c_1|\chi|_{B^1} \leqslant |\chi|_F \leqslant c_2|\chi|_{B^1} \quad \text{for all } \chi \in T.$$

Let a trajectory $(\psi_j)_0^\infty$ of H_t in W_s be given. The points $\chi_j = K(\psi_j) \in W$ form a trajectory of F since

$$\chi_{j+1} = K(\psi_{j+1}) = K\big(H_t(\psi_j)\big) = K\big(H_t\big(R(\chi_j)\big)\big) = F(\chi_j)$$

for each integer $j \geqslant 0$. Hence

$$\begin{aligned} |\psi_j - \phi|_{B^1} &= \big|R(\chi_j) - R(0)\big|_{B^1} \leqslant lip(R)|\chi_j|_{B^1} \\ &\leqslant lip(R)c_1^{-1}|\chi_j|_F \leqslant lip(R)c_1^{-1}\alpha^j|\chi_0|_F \\ &\leqslant lip(R)\frac{c_2}{c_1}\alpha^j|\chi_0|_{B^1} \leqslant lip(R)\frac{c_2}{c_1}\alpha^j|P|_{L_c(B^1,B^1)}|\psi_0 - \phi|_{B^1}. \qquad \square \end{aligned}$$

Before we can show that flowlines $H(\cdot,\psi)$ starting in W_s close to ϕ remain in W^s we need a simple growth estimate for solutions of the initial value problem (1.1)–(1.2) which corresponds to Proposition 3.5.3 in [10].

Proposition 6.7. *There are a neighbourhood N_t of ϕ in U and $c \geqslant 0$ with $[0,t] \times (M \cap N_t) \subset \Omega$ and*

$$\big|H(s,\psi) - \phi\big|_{B^1} \leqslant c|\psi - \phi|_{B^1} \quad \textit{for all } s \in [0,t],\ \psi \in M \cap N_t.$$

Proof. 1. Choose a neighbourhood N_λ of ϕ in U and $\lambda \geqslant 0$ according to Corollary 1.1. $\Omega \subset [0,\infty) \times M$ is open, $[0,t] \times \{\phi\} \subset \Omega$ is compact, and H is continuous. Using these facts we find a neighbourhood N_t of ϕ in U such that $[0,t] \times N_t \subset \Omega$ and $H([0,t] \times N_t) \subset N_\lambda$. Let $\psi \in N_t$ be given and set $x = x^\psi$.

2. In case $s \in [0,t]$ and $|H(s,\psi) - \phi|_B = e^{au}|(H(s,\psi)-\phi)(u)|$ for some $u \in [-s,0]$,

$$\begin{aligned}
\left|H(s,\psi) - \phi\right|_B &= e^{au}\left|x(s+u) - \xi\right| \\
&\leqslant \left|\psi(0) - \xi\right| + \left|\int_0^{s+u} x'(w)\,dw\right| \\
&\leqslant |\psi - \phi|_B + \left|\int_0^{s+u} \bigl(g(x_w) - g(\phi)\bigr)\,dw\right| \\
&\leqslant |\psi - \phi|_B + \lambda \int_0^s |x_w - \phi|_B\,dw \\
&= |\psi - \phi|_B + \lambda \int_0^s \left|H(w,\psi) - \phi\right|_B\,dw.
\end{aligned}$$

In case $s \in [0,t]$ and $|H(s,\psi) - \phi|_B > \max_{-s \leqslant u \leqslant 0} e^{au}|(H(s,\psi)-\phi)(u)|$ we have

$$\left|H(s,\psi) - \phi\right|_B = \sup_{u \leqslant -s} e^{au}\left|x(s+u) - \xi\right| \leqslant \sup_{u \leqslant -s} e^{a(s+u)}\left|x(s+u) - \xi\right| = |\psi - \phi|_B,$$

and the previous estimate holds as well. Gronwall's lemma yields

$$\left|H(s,\psi) - \phi\right|_B \leqslant |\psi - \phi|_B e^{\lambda s} \quad \text{for all } s \in [0,t].$$

3. For all $s \in [0,t]$ and $u \in [-s,0]$ we obtain

$$\begin{aligned}
e^{au}\left|\bigl(H(s,\psi) - \phi\bigr)'(u)\right| &= e^{au}\left|x'(s+u)\right| = e^{au}\left|g(x_{s+u}) - g(\phi)\right| \\
&\leqslant \lambda |x_{s+u} - \phi|_B \leqslant \lambda |\psi - \phi|_B e^{\lambda(s+u)}
\end{aligned}$$

while for $s \in [0,t]$ and $u \leqslant -s$ we have

$$\begin{aligned}
e^{au}\left|\bigl(H(s,\psi) - \phi\bigr)'(u)\right| &= e^{au}\left|x'(s+u)\right| = e^{au}\left|\psi'(s+u) - \phi'(s+u)\right| \\
&\leqslant e^{a(s+u)}\left|\psi'(s+u) - \phi'(s+u)\right| \leqslant \left|\psi' - \phi'\right|_B.
\end{aligned}$$

Let $s \in [0,t]$. Combining all estimates we arrive at

$$\left|H(s,\psi) - \phi\right|_{B^1} \leqslant |\psi - \phi|_{B^1}(2+\lambda)e^{\lambda t}. \qquad \square$$

The next result corresponds to Proposition 3.5.4 in [10].

Proposition 6.8. *There exist an open neighbourhood W^s of ϕ in W_s and $c_{H,s} \geqslant 0$ such that $[0,\infty) \times W^s \subset \Omega$, $H([0,\infty) \times W^s) \subset W_s$, and*

$$\big|H(u,\psi) - \phi\big|_{B^1} \leqslant c_{H,s} e^{u \frac{\log(\alpha)}{t}} |\psi - \phi|_{B^1} \quad \textit{for all } u \geqslant 0, \psi \in W^s.$$

Proof. 1. Set $V_2 = R(T_2)$. Choose c_s according to Corollary 6.6 (ii) and a neighbourhood N_t of ϕ in U and c according to Proposition 6.7. It follows that there is an open neighbourhood W^s of ϕ in $W_s \cap N_t \subset V_2 \cap N_t$ so that

$$H_t\big(W^s\big) \subset W^s, \tag{6.3}$$

$$H\big([0,t] \times W^s\big) \subset V_2, \tag{6.4}$$

and

$$\Big\{\chi \in T\colon |\chi|_{B^1} \leqslant |P|_{L_c(B^1,B^1)} c c_s \sup_{\eta \in W^s} |\eta - \phi|_{B^1}\Big\} \subset T_2. \tag{6.5}$$

Using (6.3) and (6.4) and properties of the semiflow we get $[0,\infty) \times W^s \subset \Omega$ and $H([0,\infty) \times W^s) \subset V_2$.

2. Proof of $H([0,\infty) \times W^s) \subset W_s$. Let $u \geqslant 0$ and $\psi \in W^s$ be given. The assertion $\rho = H(u,\psi) \in W_s$ is equivalent to

$$K(\rho) \in K(W_s) = W. \tag{6.6}$$

By the remarks in part 1 the point ρ defines a trajectory $(\rho_j)_0^\infty$ of H_t in V_2, with $\rho_0 = \rho$, and the point $\psi \in W^s$ defines a trajectory $(\psi_j)_0^\infty$ of H_t in $W^s \subset V_2$, with $\psi_0 = \psi$. The points $\chi_j = K(\rho_j)$ form a trajectory of F in T_2 since

$$\chi_{j+1} = K(\rho_{j+1}) = K\big(H_t(\rho_j)\big) = K\big(H_t\big(R(\chi_j)\big)\big) = F(\chi_j)$$

for all integers $j \geqslant 0$. Proposition 6.5 (iv) shows that the relation (6.6) follows from

$$\gamma^{-j} \chi_j \in T_2 \quad \text{for all integers } j \geqslant 0. \tag{6.7}$$

Proof of (6.7): Let $j \in \mathbb{N}_0$, $k \in \mathbb{N}_0$, $kt \leqslant u < (k+1)t$. Then

$$\begin{aligned}
|\chi_j|_{B^1} &= \big|K(\rho_j)\big|_{B^1} = \big|P(\rho_j - \phi)\big|_{B^1} \leqslant |P|_{L_c(B^1,B^1)} |\rho_j - \phi|_{B^1} \\
&= |P|_{L_c(B^1,B^1)} \big|H(jt,\rho) - \phi\big|_{B^1} \\
&= |P|_{L_c(B^1,B^1)} \big|H\big(u - kt, H\big((j+k)t,\psi\big)\big) - \phi\big|_{B^1} \\
&\leqslant |P|_{L_c(B^1,B^1)} c |\psi_{j+k} - \phi|_{B^1} \quad \text{(by Proposition 6.7)} \\
&\leqslant |P|_{L_c(B^1,B^1)} c c_s \alpha^{j+k} |\psi - \phi|_{B^1} \quad \big(\text{by Corollary 6.6(ii)}\big),
\end{aligned}$$

hence

$$\big|\gamma^{-j} \chi_j\big|_{B^1} \leqslant |P|_{L_c(B^1,B^1)} c c_s \cdot 1 \cdot \sup_{\eta \in W^s} |\eta - \phi|_{B^1},$$

which yields

$$\gamma^{-j}\chi_j \in T_2,$$

according to (6.5).

3. Proof of the estimate: Let $\psi \in W^s$, $u \geqslant 0$, $j \in \mathbb{N}_0$, $jt \leqslant u < (j+1)t$. Then

$$\begin{aligned}
\bigl|H(u,\psi)-\phi\bigr|_{B^1} &= \bigl|H\bigl(u-jt, H(jt,\psi)\bigr)-\phi\bigr|_{B^1}\\
&\leqslant c\bigl|H(jt,\psi)-\phi\bigr|_{B^1} \quad \text{(by Proposition 6.7)}\\
&\leqslant cc_s\alpha^j|\psi-\phi|_{B^1} \quad \bigl(\text{by Corollary 6.6(ii)}\bigr)\\
&= cc_s e^{j\log(\alpha)}|\psi-\phi|_{B^1}\\
&= cc_s e^{u\frac{\log(\alpha)}{t}} e^{(j-\frac{u}{t})\log(\alpha)}|\psi-\phi|_{B^1}\\
&\leqslant cc_s e^{u\frac{\log(\alpha)}{t}}\alpha^{-1}|\psi-\phi|_{B^1}. \qquad \square
\end{aligned}$$

The set W^s is the desired local stable manifold of the semiflow H. It absorbs all flowlines whose segments converge to ϕ at a certain rate. The precise statement is as follows.

Proposition 6.9. *For every $\psi \in M$ with $t(\psi) = \infty$ and*

$$\sup_{u\geqslant 0} e^{-u\frac{\log(\gamma)}{t}}\bigl|H(u,\psi)-\phi\bigr|_{B^1} < \infty$$

there exists $t_\psi \geqslant 0$ with $H(u,\psi) \in W^s$ for all $u \geqslant t_\psi$.

Proof. 1. There exists $c \geqslant 0$ with

$$\bigl|H(u,\psi)-\phi\bigr|_{B^1} \leqslant ce^{u\frac{\log(\gamma)}{t}} \quad \text{for all } u \geqslant 0.$$

Choose an open neighbourhood $N_* \subset U$ of ϕ with $W^s = W_s \cap N_*$. It follows that there exists $t_c \geqslant 0$ such that for all $u \geqslant t_c$,

$$H(u,\psi) \in N_* \cap R(T_2).$$

So it remains to find $t_\psi \geqslant t_c$ such that for all $u \geqslant t_c$ we have $H(u,\psi) \in W_s$, or equivalently, $K(H(u,\psi)) \in W$.

2. Choose $t_\psi \geqslant t_c$ so large that

$$\bigl\{\chi \in T\colon\, |\chi|_{B^1} \leqslant c|P|_{L_c(B^1,B^1)}e^{t_\psi\frac{\log(\gamma)}{t}}\bigr\} \subset T_2.$$

Let $u \geqslant t_\psi$. For every $j \in \mathbb{N}_0$ we have

$$H_t^j\bigl(H(u,\psi)\bigr) = H(jt+u,\psi) \in R(T_2).$$

Using induction we infer that for each $j \in \mathbb{N}_0$ $K(H(u,\psi))$ is in the domain of F^j, and

$$F^j\big(K\big(H(u,\psi)\big)\big) = K\big(H_t^j\big(H(u,\psi)\big)\big),$$

hence

$$\begin{aligned}
c|P|_{L_c(B^1,B^1)} e^{t_\psi \frac{\log(\gamma)}{t}} &\geqslant c|P|_{L_c(B^1,B^1)} e^{u \frac{\log(\gamma)}{t}} \\
&= \gamma^{-j} c|P|_{L_c(B^1,B^1)} e^{(jt+u)\frac{\log(\gamma)}{t}} \\
&\geqslant \big|\gamma^{-j} P\big(H(jt+u,\psi) - \phi\big)\big|_{B^1} \\
&= \big|\gamma^{-j} K\big(H(jt+u,\psi)\big)\big|_{B^1} \\
&= \big|\gamma^{-j} K\big(H_t^j\big(H(u,\psi)\big)\big)\big|_{B^1} \\
&= \big|\gamma^{-j} F^j\big(K\big(H(u,\psi)\big)\big)\big|_{B^1}.
\end{aligned}$$

By the choice of t_ψ, $\gamma^{-j} F^j(K(H(u,\psi))) \in T_2$. Proposition 6.5(iv) gives $K(H(u,\psi)) \in W$. $\square$

7. Local unstable manifolds

The construction of a local unstable manifold of the semiflow H at the stationary point ϕ is not entirely parallel to the construction of W^s. In the sequel we use what is prepared in Section 6. Recall the choice of $\gamma \in (0,1)$ prior to Proposition 6.5.

Below it is convenient to use upper indices in the notation of trajectories of maps, in order to avoid confusion with solution segments in B^1.

Theorem I.3 from [18] about local unstable manifolds for continuously differentiable maps in Banach spaces, which goes back to [9,19], yields the following result, with $T_{cs} = T_c \oplus T_s$.

Proposition 7.1. *There exist $\beta \in (0,\gamma)$, convex open neighbourhoods $T_{u,2}$ of 0 in T_u and $T_{cs,2}$ of 0 in T_{cs} with $T_2 = T_{u,2} + T_{cs,2} \subset T_1$, a continuously differentiable map $w : T_{u,2} \to T_{cs,2}$ with $w(0) = 0$ and $Dw(0) = 0$, and an equivalent norm $|\cdot|_F$ on T such that the following holds.*

(i) *The graph $W = \{\chi + w(\chi):\ \chi \in T_{u,2}\}$ is equal to the set of all initial points $\psi = \psi^0$ of trajectories $(\psi^j)_{-\infty}^0$ of F which satisfy $\gamma^j \psi^j \in T_2$ for all $j \in -\mathbb{N}_0$ and $\gamma^j \psi^j \to 0$ as $j \to -\infty$.*

(ii) *There is an open neighbourhood $\tilde{T_2}$ of 0 in T_2 such that $F|W \cap \tilde{T_2}$ defines a C^1-diffeomorphism F_u onto W whose inverse satisfies*

$$\big|F_u^{-1}(\psi) - F_u^{-1}(\chi)\big|_F \leqslant \beta|\psi - \chi|_F \quad \textit{for all } \psi \in W,\ \chi \in W.$$

(iii) *For every trajectory $(\psi^j)_{-\infty}^0$ of F with $\gamma^j \psi^j \in T_2$ for all $j \in -\mathbb{N}_0$,*

$$\psi^0 \in W.$$

As a *local unstable manifold* of H_t at ϕ we take the continuously differentiable submanifold

$$W_u = R(W)$$

of M. Obviously, $W_u \subset V$, $\phi \in W_u$, and $T_\phi W_u = T_u$. Also,

$$\tilde{W_u} = R(W \cap \tilde{T_2}) = W_u \cap R(\tilde{T_2})$$

is an open neighbourhood of ϕ in W_u.

Corollary 7.2. (i) *$H_t(\tilde{W_u}) = W_u$, and $H_t|\tilde{W_u}$ is injective.*

(ii) *For every $\psi \in W_u$ there exists a unique trajectory $(\psi^j)_{-\infty}^0$ of H_t in W_u with $\psi = \psi^0$, and there exists $c_u \geqslant 0$ with*

$$\left|\psi^j - \phi\right|_{B^1} \leqslant c_u \beta^{-j} |\psi - \phi|_{B^1} \quad \text{for all } j \in -\mathbb{N}.$$

Proof. 1. The equation in assertion (i) follows from

$$K\big(H_t(\tilde{W_u})\big) = K\big(H_t\big(R(W \cap \tilde{T_2})\big)\big) = F(W \cap \tilde{T_2}) = W = K\big(R(W)\big) = K(W_u)$$

by application of R. Injectivity of $H_t|\tilde{W_u}$ is obvious from

$$K(\psi) \in W \cap \tilde{T_2} \quad \text{and} \quad H_t(\psi) = R\big(F\big(K(\psi)\big)\big) \quad \text{for all } \psi \in \tilde{W_u}.$$

2. Assertion (i) yields the first statement of assertion (ii). There are positive constants $c_1 \leqslant c_2$ with

$$c_1 |\chi|_{B^1} \leqslant |\chi|_F \leqslant c_2 |\chi|_{B^1} \quad \text{for all } \chi \in T.$$

Let a trajectory $(\psi^j)_{-\infty}^0$ of H_t in W_u be given. The points $\chi^j = K(\psi^j) \in W$ form a trajectory of F in W since

$$\chi^{j+1} = K\big(\psi^{j+1}\big) = K\big(H_t\big(\psi^j\big)\big) = K\big(H_t\big(R\big(\chi^j\big)\big)\big) = F\big(\chi^j\big)$$

for each integer $j \in -\mathbb{N}$. For $j \in -\mathbb{N}$ we get

$$\begin{aligned}
\left|\psi^j - \phi\right|_{B^1} &= \left|R\big(\chi^j\big) - R(0)\right|_{B^1} \leqslant lip(R)\left|\chi^j\right|_{B^1} \\
&\leqslant lip(R) c_1^{-1} \left|\chi^j\right|_F \\
&\leqslant lip(R) c_1^{-1} \beta^{-j} \left|\chi^0\right|_F \\
&\leqslant lip(R) \frac{c_2}{c_1} \beta^{-j} \left|\chi^0\right|_{B^1} \\
&\leqslant lip(R) \frac{c_2}{c_1} \beta^{-j} |P|_{L_c(B^1, B^1)} |\psi - \phi|_{B^1}. \qquad \square
\end{aligned}$$

Proposition 7.3. *There exist an open neighbourhood W^u of ϕ in W_u and $c_{H,u} \geqslant 0$ such that for every $\psi \in W^u$ and for all $r \leqslant 0$ we have $\psi_r \in W_u$ and*

$$|\psi_r - \phi|_{B^1} \leqslant c_{H,u} e^{-r \frac{\log(\beta)}{t}} |\psi - \phi|_{B^1}.$$

Notice that the relations $\psi_r \in W_u \subset M$ for $r \leqslant 0$ imply that ψ satisfies Eq. (1.1) for all $r \leqslant 0$. The curve $(-\infty, 0] \ni r \mapsto \psi_r \in M$ is a flowline of H, which means

$$\psi_{r+s} = H(s, \psi_r) \quad \text{for all } s \geqslant 0 \quad \text{and} \quad r \leqslant -s.$$

Proof of Proposition 7.3. 1. Set $V_2 = R(T_2)$. Choose c_u according to Corollary 7.2(ii) and a neighbourhood N_t of ϕ in U and c according to Proposition 6.7. Using this and Corollary 7.2(ii) we infer that there is an open neighbourhood W_u^* of ϕ in $\tilde{W}_u \cap N_t \subset V_2 \cap N_t$ so that

$$H\big([0,t] \times W_u^*\big) \subset V_2$$

and

$$\Big\{\chi \in T\colon\ |\chi|_{B^1} \leqslant |P|_{L_c(B^1,B^1)} c c_u \sup_{\eta \in W_u^*} |\eta - \phi|_{B^1}\Big\} \subset T_2, \tag{7.1}$$

and that there is a further open neighbourhood W^u of ϕ in W_u^* such that for each $\psi \in W^u$ the trajectory $(\psi^j)_{-\infty}^0$ given by Corollary 7.2(ii) has all points in W_u^*.

2. Let $\psi \in W^u$ be given and consider the trajectory $(\psi^j)_{-\infty}^0$ in W_u^* as in part 1. Let $j \in -\mathbb{N}_0$ and $s \in [0,t]$ be given. Proof of $H(s,\psi^j) \in W_u$: The sequence $(\psi^{j+k})_{k=-\infty}^0$ is a trajectory of H_t in W_u^*. For $k \in -\mathbb{N}_0$ define $\rho^k = H(s,\psi^{j+k}) \in V_2 = R(T_2)$, and $\chi^k = K(\rho^k) \in T_2$. The points $\chi^k \in T_2$ form a trajectory of F since for every $k \in -\mathbb{N}$ we have

$$\begin{aligned}
\chi^{k+1} &= K\big(\rho^{k+1}\big) = K\big(H\big(s,\psi^{j+k+1}\big)\big) = K\big(H\big(s+t,\psi^{j+k}\big)\big) \\
&= K\big(H\big(t,H\big(s,\psi^{j+k}\big)\big)\big) = K\big(H_t\big(\rho^k\big)\big) \\
&= K\big(H_t\big(R\big(\chi^k\big)\big)\big) = F\big(\chi^k\big).
\end{aligned}$$

The assertion $H(s,\psi^j) \in W_u$ is equivalent to $\chi^0 = K(\rho^0) = K(H(s,\psi^j)) \in W$. According to Proposition 7.1(iii) the preceding relation follows from

$$\gamma^k \chi^k \in T_2 \quad \text{for all } k \in -\mathbb{N}_0.$$

Proof of this: We have $\gamma^0\chi^0 \in T_2$. Let $k \in -\mathbb{N}$. Then

$$\begin{aligned}
\big|\chi^k\big|_{B^1} &= \big|K\big(\rho^k\big)\big|_{B^1} = \big|P\big(\rho^k - \phi\big)\big|_{B^1} \\
&\leqslant |P|_{L_c(B^1,B^1)}\big|\rho^k - \phi\big|_{B^1} = |P|_{L_c(B^1,B^1)}\big|H\big(s,\psi^{j+k}\big) - \phi\big|_{B^1} \\
&\leqslant |P|_{L_c(B^1,B^1)} c \big|\psi^{j+k} - \phi\big|_{B^1} \quad \text{(by Proposition 6.7)} \\
&\leqslant |P|_{L_c(B^1,B^1)} c c_u \beta^{-j-k} |\psi - \phi|_{B^1} \quad \big(\text{by Corollary 7.2(ii)}\big) \\
&\leqslant |P|_{L_c(B^1,B^1)} c c_u \gamma^{-j-k} |\psi - \phi|_{B^1} \\
&\leqslant |P|_{L_c(B^1,B^1)} c c_u \gamma^{-k} |\psi - \phi|_{B^1},
\end{aligned}$$

and using (7.1) we get $\gamma^k\chi^k \in T_2$.

3. Let $\psi \in W^u$ be given and consider the trajectory $(\psi^j)_{-\infty}^0$ in W_u^* as in part 1. Let $r < 0$. Proof of $\psi_r = H(r - jt, \psi^j)$ for $j \in -N$ with $jt \leqslant r < (j+1)t$: For each $s \leqslant 0$ we have

$$\begin{aligned}\psi_r(s) &= \psi(r+s) = \psi^0(r+s) = (H_t)^{-j}\big(\psi^j\big)(r+s)\\ &= H\big(-jt, \psi^j\big)(r+s) = x^{\psi^j}_{-jt}(r+s)\\ &= x^{\psi^j}(-jt+r+s) = x^{\psi^j}_{r-jt}(s)\\ &= H\big(r-jt, \psi^j\big)(s).\end{aligned}$$

4. Parts 2 and 3 combined yield $\psi_r \in W_u$ for all $\psi \in W^u$ and $r \leqslant 0$. Proof of the estimate: Let $\psi \in W^u$, $r \leqslant 0$, $j \in -\mathbb{N}$, $jt \leqslant r < (j+1)t$. Then

$$\begin{aligned}|\psi_r - \phi|_{B^1} &= \big|H\big(r-jt, \psi^j\big) - \phi\big|_{B^1}\\ &\leqslant c\big|\psi^j - \phi\big|_{B^1} \quad \text{(by Proposition 6.7)}\\ &\leqslant cc_u\beta^{-j}|\psi - \phi|_{B^1} \quad \big(\text{by Corollary 7.2(ii)}\big)\\ &= cc_u e^{-j\log(\beta)}|\psi - \phi|_{B^1}\\ &= cc_u e^{-r\frac{\log(\beta)}{t}} e^{(r-jt)\frac{\log(\beta)}{t}}|\psi - \phi|_{B^1}\\ &\leqslant cc_u e^{-r\frac{\log(\beta)}{t}}|\psi - \phi|_{B^1}. \qquad \square\end{aligned}$$

The set W^u is the local unstable manifold of the semiflow H which we were looking for. Finally we state the analogue of Proposition 6.9 for the local unstable manifold.

Proposition 7.4. *For every $\psi \in M$ with $\psi_s \in M$ for all $s \leqslant 0$ and*

$$\sup_{s\leqslant 0} e^{s\frac{\log(\gamma)}{t}}|\psi_s - \phi|_{B^1} < \infty$$

there exists $t_\psi \leqslant 0$ with $\psi_s \in W^u$ for all $s \leqslant t_\psi$.

The proof is similar to the proof of Proposition 6.9 and is left to the reader.

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