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Solving linear programming problems under fuzziness and randomness environment using attainment values

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Abstract

In this paper, the author presents a model to measure attainment values of fuzzy numbers/fuzzy stochastic variables. These new measures are then used to convert the fuzzy linear programming problem or the fuzzy stochastic linear programming problem into the corresponding deterministic linear programming problem. Numerical comparisons are provided to illustrate the effectiveness of the proposed method.
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1. Introduction

Solving fuzzy linear programming problems has received a great deal of attention [10,6,4,2,9,3,8]. Recently, Inuiguchi and Ramik, [7] conducted a comprehensive review of fuzzy optimization. The most popular approach to handle the challenge of solving fuzzy linear programming problems is to convert the fuzzy linear program (FLP) into the corresponding deterministic linear program (LP). An example of this is the approach that compares fuzzy numbers by using some ordering measures of fuzzy numbers, such as the area compensation method [5], the expected mid-point approach [8], the grade of possibility and necessity measures [1], and the signed distance method [3].

In addition, a new line of research in the fuzzy optimization area has been extended to the case of fuzzy stochastic optimization, in which solving fuzzy stochastic linear programming problems is the main concern. Together, stochastic and fuzzy aspects are considered to provide an efficient tool to describe real-life problems where uncertainty and imprecise information are inherent. However, this kind of combination creates a great challenge for researchers to find efficient methods to deal with both fuzzy and stochastic terms. Like the approaches of fuzzy optimization, the general strategy to handle the fuzzy stochastic optimization problem

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is to de-fuzzify and/or de-randomize fuzzy random variables so as to convert the problem into the corresponding deterministic problem. There are two common methods to achieve this. The first is to perform the conversions (de-fuzzify/de-randomize) in a sequential manner [16,17,15]. Although the obtained results of Luhandjula's approaches are LPs, the discretizing process of fuzzy sets via α-levels creates quite a large number of constraints and variables. The second method is to perform both actions at the same time by calculating the expected value of fuzzy random variables [12,13,11,14]. Although the works of Liu proposed a promising measure in terms of expected value of fuzzy random variables, the computation process of these expected values is quite complex and time consuming.

In this paper, the author proposes a new method to solve linear programming problems under the fuzziness and randomness environment. In real-life, fuzzy stochastic linear programming arises in several situations. The parameters of linear programs such as the right-hand-sides (RHSs) and the coefficients of the objective and constraints could all be fuzzy random variables due to the fact that they depend upon many factors. Thus, it is difficult to exactly determine the values of these parameters. Moreover, the factors, which are fluctuating due to an uncertain environment, could cause these parameters to vary. These circumstances often happen in situations wherein the described conditions (objectives, constraints, coefficients) cannot be determined precisely and certainly, such as long-term planning, development strategies [16], engineering design [20], and financial modeling [21].

As an example of fuzzy stochastic linear programming, let's consider the case of a production planning problem where the objective is to minimize the total cost. This objective can be expressed as a fuzzy stochastic variable because the total cost includes the cost of inventory holding, materials, and operation. Production output depends on process parameters (cutting speeds and feed rates, for example) and machine running time. However, machine running time is fluctuating and hard to estimate precisely. In addition, available resources, demand, and constraints' coefficients can also be modeled as fuzzy random variables because of the variances in statistical data due to environmental conditions such as seasonal factors, market prices, and suppliers, which contribute to constraint parameters.

Another example is the case of preventive maintenance. Factory equipment breaks down from time to time, causing losses in production output. Preventive maintenance can be employed to reduce the frequency of such breakdowns, and could involve activities such as inspections, repairs, and replacing components when necessary. These activities are costly in terms of materials, wages, and the loss of production due to downtime while the activities are being conducted. The length of the downtime is also uncertain due to the complexity of the inspection, repair and replacement jobs, as well as the varying skills of maintenance staff. The problem is to determine the frequency of preventative maintenance such that the total downtime, which includes downtime due to breakdowns as well as preventive maintenance, is minimized while ensuring the associated costs do not exceed the available budget. Here, the running time of the machine is also uncertain. Therefore, we would rather consider these times and their associated costs as fuzzy random variables.

Examples such as these motivate the author to propose a new model for solving fuzzy stochastic linear programming problems. In this paper, the attainment measure of fuzzy numbers and/or fuzzy random variables is developed to convert the fuzzy stochastic linear programming problem into the corresponding deterministic LP. The obtained LP, with few additional constraints, is easily solved by standard optimization packages such as LINGO [18] or http://www.lindo.com.

This paper is organized as follows: Firstly, some important results of fuzzy random variables are summarized. Then, in Section 3, the attainment model is developed. Section 4 utilizes the attainment measures to convert the fuzzy linear programming problem into the corresponding deterministic LP. Section 5 presents the solution method for the fuzzy stochastic linear programming problem. Finally, Section 6 presents the conclusions.

2. Fuzzy random variable

In this section, we summarize some important concepts and results of the fuzzy random variable which is the basis for the development of the attainment measures. There are several alternative approaches to define a fuzzy random variable and its characteristics, and for this paper we utilize the results of Luhandjula [17] for the definition.

Definition 1 [17]. Consider a probability space $(\Omega, \Im, P)$, a fuzzy random variable on this space is a fuzzy set-valued mapping:

$$\begin{aligned} &\widetilde{\widetilde{X}} : \Omega \to F_0(\mathbb{R}) \\ &w \to \widetilde{X}_w \end{aligned}$$

such that for any Borel set B and for every $\alpha \in (0,1)$

$$\widetilde{\widetilde{X}}_{\alpha}^{-1}(B) = \{w \in \Omega | \widetilde{X}_w^{\alpha} \subset B\} \in \Im \tag{1}$$

where $F_0(\mathbb{R})$ and $\widetilde{X}_w^{\alpha}$ stand for the set of fuzzy numbers with compact supports and the α-level set of the fuzzy set $\widetilde{X}_w$, respectively.

Theorem 1 [17]. *$\widetilde{\widetilde{X}}$ is a fuzzy random variable if and only if given $w \in \Omega$, $\widetilde{X}_w^{\alpha}$ is a random interval $\forall \alpha \in (0,1]$.*

3. Attainment degree of fuzzy numbers and fuzzy random variables

Definition 2. Given two fuzzy numbers $\widetilde{P}, \widetilde{Q}$ and $\widetilde{P} \leqslant \widetilde{Q}$, the lower-side attainment index of $\widetilde{P}$ to $\widetilde{Q}$ is defined as

$$D(\widetilde{P}, \widetilde{Q}) = \int_0^1 \max\{0, \sup\{s \in \mathbb{R} : \widetilde{P}(s) \geqslant \alpha\} - \inf\{r \in \mathbb{R} : \widetilde{Q}(r) \geqslant \alpha\}\} \, d\alpha \tag{2}$$

Definition 3. The both-side attainment index of fuzzy number $\widetilde{P}$ to fuzzy number $\widetilde{Q}$ is defined as follows:

$$V(\widetilde{P}, \widetilde{Q}) = \max\{0, \min(D(\widetilde{P}, \widetilde{Q}), D(\widetilde{Q}, \widetilde{P}))\} \tag{3}$$

Let T be a collection of triangular fuzzy numbers (T-numbers) with the following membership function:

$$T = \{\tilde{t}; \tilde{t} = (t, l, r), l, r \geqslant 0\} \quad \text{and} \quad \mu_{\tilde{t}}(x) = \begin{cases} \max\left(0, 1 - \frac{t-x}{l}\right), & \text{if } x \leqslant t \\ 1, & \text{if } l = 0, r = 0, t = x \\ \max\left(0, 1 - \frac{x-t}{r}\right), & \text{if } x > t \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where the scalars $l, r \geqslant 0 (l, r \in \mathbb{R})$ are called the left and right spreads, respectively.

Proposition 1. *Consider two fuzzy numbers $\widetilde{P} = (u, a, b), \widetilde{Q} = (v, c, d) \in \mathrm{T}(u \leqslant v)$, the average lower-side attainment index of $\widetilde{P}$ to $\widetilde{Q}$ is*

$$\overline{D}(\widetilde{P}, \widetilde{Q}) = \frac{u - v + b + c}{2}. \tag{5}$$

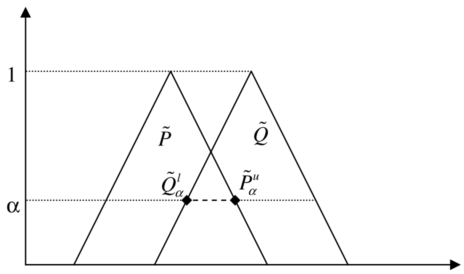

Fig. 1. Attainment degree of fuzzy number $\widetilde{P}$ to $\widetilde{Q}$ at α-level.

Proof. The α-cuts of $\widetilde{P}$ and $\widetilde{Q}$ are (see Fig. 1)

$$[P^l(\alpha), P^u(\alpha)] = [u - a(1-\alpha), u + b(1-\alpha)]$$
$$[Q^l(\alpha), Q^u(\alpha)] = [v - c(1-\alpha), v + d(1-\alpha)]$$

We also have the lower-side attainment index of fuzzy number $\widetilde{P}$ to $\widetilde{Q}$ at α-level:

$$D(\widetilde{P},\widetilde{Q})_\alpha = \max\{0, \widetilde{P}^u_\alpha - \widetilde{Q}^l_\alpha\}$$
$$D(\widetilde{P},\widetilde{Q})_\alpha = \max\{0, P^u(\alpha) - Q^l(\alpha)\} = \max\{0, (u-v) + (b+c)(1-\alpha)\}$$
$$\Rightarrow D(\widetilde{P},\widetilde{Q}) = \int_0^1 \max\{0, (u-v) + (b+c)(1-\alpha)\}\,\mathrm{d}\alpha$$
$$\iff D(\widetilde{P},\widetilde{Q}) = \int_0^{\lambda^*=\left(1-\frac{v-u}{b+c}\right)} [(u-v) + (b+c)(1-\alpha)]\,\mathrm{d}\alpha$$
$$\Rightarrow \overline{D}(\widetilde{P},\widetilde{Q}) = \frac{1}{\lambda^*}\int_0^{\lambda^*=\left(1-\frac{v-u}{b+c}\right)} [(u-v) + (b+c)(1-\alpha)]\,\mathrm{d}\alpha$$
$$\iff \overline{D}(\widetilde{P},\widetilde{Q}) = \frac{1}{\lambda^*}\left[(u-v+b+c)\alpha - \left((b+c)\frac{\alpha^2}{2}\right)\right]_0^{\lambda^*}$$
$$\iff \overline{D}(\widetilde{P},\widetilde{Q}) = \frac{1}{\lambda^*}\left[(u-v+b+c)\lambda^* - \left((b+c)\frac{\lambda^{*2}}{2}\right)\right] = (u-v+b+c) - \left((b+c)\frac{\lambda^*}{2}\right)$$
$$\iff \overline{D}(\widetilde{P},\widetilde{Q}) = (u-v+b+c) - \frac{(b+c)}{2}\left(1 - \frac{v-u}{(b+c)}\right) = u - v + b + c - \frac{(b+c)}{2} + \frac{(v-u)}{2}$$
$$\iff \overline{D}(\widetilde{P},\widetilde{Q}) = \frac{(u-v)+(b+c)}{2} \qquad \square$$

From (3) and (5), the average both-side attainment index of $\widetilde{P}$ to $\widetilde{Q}$ is defined as

$$\overline{V}(\widetilde{P},\widetilde{Q}) = \max\left\{0, \min\left\{\frac{u-v+b+c}{2}; \frac{v-u+a+d}{2}\right\}\right\}. \tag{6}$$

From (6) we derive the following result:

Proposition 2. *Let* $\widetilde{P}, \widetilde{Q}, 0 < \lambda$. *Then*

$$\overline{V}(\widetilde{P},\widetilde{Q}) \geqslant \lambda \tag{7}$$

iff

$$\overline{D}(\widetilde{P},\widetilde{Q}) = \frac{u-v+b+c}{2} \geqslant \lambda \tag{8}$$
$$\overline{D}(\widetilde{Q},\widetilde{P}) = \frac{v-u+a+d}{2} \geqslant \lambda \tag{9}$$

Definition 4. The lower-side attainment index of fuzzy random variable $\widetilde{\widetilde{P}}$ to fuzzy random variable $\widetilde{\widetilde{Q}}, \widetilde{\widetilde{P}} \leqslant \widetilde{\widetilde{Q}}$ is defined as

$$D(\widetilde{\widetilde{P}},\widetilde{\widetilde{Q}}) = \int_0^1 \max\{0, \sup\{s \in \mathbb{R} : \widetilde{P}_w(s) \geqslant \alpha\} - \inf\{r \in \mathbb{R} : \widetilde{Q}_w(r) \geqslant \alpha\}\}\,\mathrm{d}\alpha \tag{10}$$

Definition 5. The both-side attainment index of fuzzy random variable $\widetilde{\widetilde{P}}$ to fuzzy random variable $\widetilde{\widetilde{Q}}$ is defined as follows:

$$V(\widetilde{\widetilde{P}},\widetilde{\widetilde{Q}}) = \max\{0, \min(D(\widetilde{\widetilde{P}},\widetilde{\widetilde{Q}}), D(\widetilde{\widetilde{Q}},\widetilde{\widetilde{P}}))\} \tag{11}$$

For any $\alpha \in (0,1]$, consider the α-cut of the fuzzy sets $\widetilde{P}_w, \widetilde{Q}_w$:

$$\widetilde{P}_w^\alpha = [P_\alpha^l(w), P_\alpha^u(w)] \quad \widetilde{Q}_w^\alpha = [Q_\alpha^l(w), Q_\alpha^u(w)]$$

By Theorem 1, these intervals are random intervals. If $\widetilde{P}_w, \widetilde{Q}_w$ are T-numbers, the α-cut of $\widetilde{P}_w, \widetilde{Q}_w$ are: (see Fig. 2)

$$\widetilde{P}_w^\alpha = [P_\alpha^l(w), P_\alpha^u(w)] = [u(w) - a(w)(1-\alpha), u(w) + b(w)(1-\alpha)]$$
$$\widetilde{Q}_w^\alpha = [Q_\alpha^l(w), Q_\alpha^u(w)] = [v(w) - c(w)(1-\alpha), v(w) + d(w)(1-\alpha)]$$

Thus, the lower-side attainment index of $\widetilde{\widetilde{P}}$ to $\widetilde{\widetilde{Q}}$ at α-level is

$$D(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}})_\alpha = \max\{0, P_\alpha^u(w) - Q_\alpha^l(w)\}$$

From these definitions, we have the following results:

Proposition 3. *Considering two triangular fuzzy random variables $\widetilde{\widetilde{P}} \leqslant \widetilde{\widetilde{Q}}$, the average lower-side attainment index of $\widetilde{\widetilde{P}}$ to $\widetilde{\widetilde{Q}}$ is*

$$\overline{D}(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}}) = \frac{u(w) - v(w) + b(w) + c(w)}{2} \tag{12}$$

Proof

$$D(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}})_\alpha = \max\{0, P_\alpha^u(w) - Q_\alpha^l(w)\} = \max\{0, (u(w) - v(w)) + (b(w) + c(w))(1-\alpha)\}$$
$$\Rightarrow D(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}}) = \int_0^1 \max\{0, (u(w) - v(w)) + (b(w) + c(w))(1-\alpha)\}\,\mathrm{d}\alpha$$
$$\Longleftrightarrow D(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}}) = \int_0^{\lambda^* = \left(1 - \frac{v(w)-u(w)}{b(w)+c(w)}\right)} [(u(w) - v(w)) + (b(w) + c(w))(1-\alpha)]\,\mathrm{d}\alpha$$
$$\Rightarrow \overline{D}(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}}) = \frac{1}{\lambda^*} \int_0^{\lambda^*} [(u(w) - v(w)) + (b(w) + c(w))(1-\alpha)]\,\mathrm{d}\alpha$$
$$\Longleftrightarrow \overline{D}(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}}) = \frac{1}{\lambda^*} \left[(u(w) - v(w) + b(w) + c(w))\alpha - \left((b(w) + c(w)) \frac{\alpha^2}{2} \right) \right]_0^{\lambda^*}$$
$$\Longleftrightarrow \overline{D}(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}}) = \frac{1}{\lambda^*} \left[(u(w) - v(w) + b(w) + c(w))\lambda^* - \left((b(w) + c(w)) \frac{\lambda^{*2}}{2} \right) \right]$$
$$\Longleftrightarrow \overline{D}(\widetilde{\widetilde{P}}, \widetilde{\widetilde{Q}}) = u(w) - v(w) + b(w) + c(w) - \frac{(b(w) + c(w))}{2} + \frac{(v(w) - u(w))}{2} \qquad \square$$

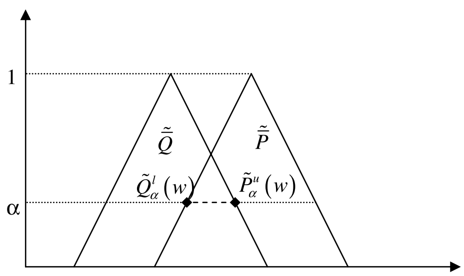

Fig. 2. Lower-side attainment degree of $\widetilde{\widetilde{P}}$ to $\widetilde{\widetilde{Q}}$ at α-level.

From (8) and (9), the average both-side attainment index of $\overline{\tilde{P}}$ to $\overline{\tilde{Q}}$ is defined as

$$\overline{V}(\overline{\tilde{P}},\overline{\tilde{Q}}) = \max\left\{0, \min\left\{\frac{u(w)-v(w)+b(w)+c(w)}{2};\frac{v(w)-u(w)+a(w)+d(w)}{2}\right\}\right\} \tag{13}$$

Similarly, from (10) we derive the following result:

Proposition 4. *Let* $\overline{\tilde{P}},\overline{\tilde{Q}}, 0<\lambda(w), w\in\Omega$. *Then*

$$\overline{V}(\overline{\tilde{P}},\overline{\tilde{Q}}) \geqslant \lambda(w) \tag{14}$$

iff

$$\frac{u(w)-v(w)+b(w)+c(w)}{2} \geqslant \lambda(w) \tag{15}$$

$$\frac{v(w)-u(w)+a(w)+d(w)}{2} \geqslant \lambda(w) \tag{16}$$

The main difference of the proposed approach with traditional ones such as the signed distance method [19], is the defuzzifying method used to obtain the attainment degree between fuzzy numbers and fuzzy stochastic variables. The signed distance method defuzzifies fuzzy numbers by using the absolute relationship between the converted points of fuzzy numbers, while the proposed approach uses the relative relationship between fuzzy numbers/fuzzy stochastic variables via attainment degrees. To illustrate the advantages of the proposed approach, in the next section the author will compare it with the signed distance method.

4. Fuzzy linear programming

Using the results obtained in Sections 2 and 3, the method for solving fuzzy linear programming problems is now developed in this section. Consider the following fuzzy linear program:

$$(P1):\begin{cases}\text{Max}\, cx\\ \sum_{j=1}^{n}(\tilde{a}_{ij})x_j \leqslant (\tilde{b}_i); \quad i=1,m\\ x_j \geqslant 0\end{cases}$$

where c is a $(1xn)$ matrix. The common way to solve this problem is to achieve the objective subject so as to avoid any violation of the constraints. Consequently, any violation of the left-hand-side (LHS) to the right-hand-side (RHS) of constraints is minimized. Thus, by applying the upper-side attainment index to constraints, the problem (P1) is reformulated as:

$$(P2):\begin{cases}\text{Max } cx - \sum_{i=1}^{m}\lambda_i\\ \text{s.t.}\\ \overline{D}_i\left((\tilde{b}_i),\sum_{j=1}^{n}(\tilde{a}_{ij})x_j\right) = \lambda_i, \quad i=1,m\\ x_j,\lambda_i \geqslant 0\end{cases}$$

According the result of (5), (P2) is a standard LP and can be solved by traditional optimization software, such as LINGO [18] or http://www.lindo.com.

To illustrate the advantages of the proposed approach, we consider the following production planning problem to compare the proposed method with the signed distance method (modified from [7]):

Example 1. A factory produces two products P and Q. Products P and Q are manufactured by two mixing processes M and N, respectively. It takes *about* 2 *h* at process M and *about* 6 *h* at process N to manufacture a batch of Product P. On the other hand, it takes *about* 3 *h* at process M and *about* 4 *h* at process N to manufacture a batch of product Q. Subject to many factors such as machine breakdown and time spent waiting for materials, the working time of process M (resp. N) is *substantially smaller than* 9 h (resp. 18). The profit rates ($1000/batch) of products P and Q are 7 and 9, respectively. What quantity of products P and Q should be manufactured under such circumstances to maximize total profit?

Here we have the following FLP problem

$$\begin{aligned} \text{Maximize} \quad & 7x_1 + 9x_2 \\ \text{Subject to} \quad & \tilde{2}x_1 + \tilde{3}x_2 \leqslant \tilde{9} \\ & \tilde{6}x_1 + \tilde{4}x_2 \leqslant \widetilde{18} \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

Assume all fuzzy numbers are in the form of $\tilde{t} = (t, \delta_1, \delta_2) \in T$ with $\delta_1 = \delta_2 = 1$ for all coefficients. Converting this example to the form of a linear program (P2) by using (5), we have:

$$\begin{aligned} \text{Maximize} \quad & 7x_1 + 9x_2 - \lambda_1 - \lambda_2 \\ \text{Subject to} \quad & \frac{1}{2}[(9 - 2x_1 - 3x_2) + (1 + x_1 + x_2)] = \lambda_1 \\ & \frac{1}{2}[(18 - 6x_1 - 4x_2) + (1 + x_1 + x_2)] = \lambda_2 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

The solution of this linear program is $x_1 = 0.875, x_2 = 4.875$ and objective value = 50.

If we use the signed distance method, the signed distance of fuzzy number $\tilde{\mu} = (\mu, \Delta_1, \Delta_2)$ to $\tilde{0}_1$ (y axis) is

$$d(\tilde{\mu}, \tilde{0}_1) = \frac{1}{4}(2\mu + \mu - \Delta_1 + \mu + \Delta_2) = \frac{1}{4}(4\mu - \Delta_1 + \Delta_2) \tag{17}$$

Then, the linear program (P2) is converted to

$$\begin{aligned} \text{Maximize} \quad & 7x_1 + 9x_2 \\ \text{Subject to} \quad & 2x_1 + 3x_2 \leqslant 9 \\ & 6x_1 + 4x_2 \leqslant 18 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

The solution of this linear program is $x_1 = 1.8, x_2 = 1.8$ and objective value = 28.8. Since our problem is maximized problem, the obtained result demonstrates that our approach gives a better solution −50 versus 28.8.

To confirm the advantages of the proposed approach over the signed distance method, we designed an experiment that utilizes real-life data collected from a detergent powder mixing process similar to the example mentioned above. All processing times and the available times of machines are approximately measured in the form of fuzzy numbers and given in Table 1. The profit rates per batch ($1000/batch) are also presented in Table 1. Fifteen tests are studied.

The obtained results of the 15 cases are presented in Table 2. Row 1 is our method's results. Row 2 is the signed distance method's results.

From the obtained results, it is seen that our method gives a better solution in terms of objective values. The reason for this is that our method compares fuzzy numbers based on their relative relationships. This comparison relaxes the constraints from the deterministic ones. Thus, the larger spreads of fuzzy numbers, the greater the relaxation amount. Therefore, overlapping areas between fuzzy numbers will be increased and the obtained results will be better. In contrast, the signed distance method uses the absolute relationship between fuzzy numbers. They convert the fuzzy numbers into the corresponding deterministic numbers. This method does not adjust the values of deterministic constraints, so it is similar to the deterministic case. In particular, when two spreads of fuzzy numbers are equal, the converted constraints are exactly the same as the deterministic constraints. Therefore, the obtained results will be close to the deterministic solution.

Table 1
Collected data

Case	c_j	a_{ij}	a^l_{ij}	a^r_{ij}	b_i	b^l_i	b^r_i
1	8.03	6.81	0.76	0.63	14.71	0.83	0.65
	9.07	2.24	0.11	0.52			
		6.82	0.87	0.41	5.89	0.90	0.80
		3.86	0.34	0.86			
2	1.73	0.78	0.52	0.07	5.84	1.17	1.23
	0.98	8.49	0.49	0.48			
		2.95	0.69	0.71	12.10	1.12	0.65
		7.02	0.61	0.75			
3	3.53	8.67	0.03	0.10	9.39	0.99	0.75
	4.69	0.72	0.94	0.84			
		5.46	0.85	0.23	14.08	0.94	1.14
		6.57	0.91	0.47			
4	9.86	3.49	0.42	0.45	13.80	0.77	0.93
	4.36	9.61	0.59	0.36			
		6.69	0.76	0.38	10.77	1.13	0.68
		5.23	0.10	0.52			
5	0.50	7.11	0.58	0.95	14.01	0.87	1.23
	7.28	2.26	0.23	0.36			
		8.26	0.06	0.71	14.11	1.04	0.72
		3.39	0.61	0.44			
6	5.42	6.65	0.08	0.02	14.89	0.56	1.30
	8.29	3.78	0.02	0.92			
		0.69	0.67	0.98	7.23	0.73	0.65
		4.15	0.93	0.22			
7	0.11	5.89	0.17	0.99	5.62	0.92	0.64
	6.68	7.78	0.07	0.56			
		3.41	0.96	0.31	12.95	0.97	0.89
		3.81	0.27	0.70			
8	8.36	9.03	0.75	0.12	14.81	1.12	1.13
	0.16	3.85	0.28	0.26			
		5.02	0.66	0.38	10.80	1.33	0.68
		2.85	0.57	0.78			
9	2.97	8.46	0.15	0.04	10.27	0.82	1.20
	1.52	0.94	0.39	0.41			
		4.63	0.22	0.94	12.42	0.56	1.48
		3.03	0.38	0.68			
10	7.79	6.85	0.82	0.28	9.01	1.36	0.65
	3.78	2.37	0.96	0.81			
		8.58	0.82	0.42	9.98	1.11	1.29
		0.86	0.70	0.16			
11	9.52	0.64	0.77	0.69	5.50	0.58	0.64
	1.68	3.57	0.35	0.76			
		5.41	0.79	0.46	7.31	1.23	0.56
		1.72	0.45	0.09			
12	3.90	4.02	0.46	0.24	12.55	0.71	1.24
	3.71	3.89	0.91	0.46			
		9.64	0.82	0.12	14.16	0.91	1.31
		6.20	0.51	0.72			
13	6.2	9.8	0.58	0.85	13.82	1.05	1.24
	8.34	2.26	0.7	0.64			
		4.74	0.68	0.68	6.36	0.55	0.91

Table 1 (*continued*)

Case	c_j	a_{ij}	a_{ij}^l	a_{ij}^r	b_i	b_i^l	b_i^r
	9.42	0.96	0.19				
14	5.16	1.96	0.57	0.45	12.46	0.81	0.81
	0.38	6.75	0.18	0.16			
		4.20	0.93	0.70	14.48	0.90	0.92
		4.05	0.48	0.17			
15	0.30	4.21	0.29	0.53	12.72	0.98	1.48
	6.63	4.10	0.66	1.00			
		6.84	0.76	0.38	13.11	0.58	0.76
		3.54	0.92	0.45			

5. Fuzzy stochastic linear programming problem

Consider the following fuzzy stochastic linear program:

$$(P3):\begin{cases}\text{Min } cx\\ \sum_{j=1}^{n}(\tilde{a}_{ij})_w x_j \leqslant (\tilde{b}_i)_w, \quad i=1,m\\ x_j \geqslant 0, \quad w\in\Omega, \quad k=1,l\end{cases}$$

where c is a $1xn$ matrix. A, b are mxn and $mx1$ matrices of fuzzy random variable constraint coefficients defined on a probability space (Ω, F, P). Applying the lower-side attainment index to minimize the achievement of the LHS to the RHS, problem (*P3*) is reformulated as follows:

$$(P4):\begin{cases}\text{Min } cx+\sum\limits_{i=1}^{m}\lambda_i(w)\\ \text{s.t.}\\ \overline{D}_i\left(\sum\limits_{j=1}^{n}(\tilde{a}_{ij})_w x_j, (\tilde{b}_i)_w\right)=\lambda_i(w), \quad i=1,m\\ x_j \geqslant 0, \quad w\in\Omega\end{cases}$$

(*P4*) is then de-randomized by using stochastic programming techniques. The corresponding deterministic program for this problem is:

$$(P5):\begin{cases}\text{Min } cx+E\left[\sum\limits_{i=1}^{m}\lambda_i(w)\right]\\ \text{s.t.}\\ \overline{D}_i\left((\tilde{b}_i)_w, \sum\limits_{j=1}^{n}(\tilde{a}_{ij})_w x_j\right)=\lambda_i(w), \quad i=1,m\\ x_j, \lambda_i(w) \geqslant 0, \quad w\in\Omega\end{cases}$$

where E denotes the mathematical expectation.

Example 2. We use this example to show that Luhandjula's model solution can be improved by the proposed method. Reconsider the example of [17]):

Table 2
Comparison between our method and signed method

Case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	17.24	9.75	12.27	19.05	38.84	26.02	5.42	16.1	8.48	25.92	14.39	10.08	10.73	24.3	26.39
2	13.33	7.02	10.26	15.93	30.51	20.13	4.69	13.96	6.77	14.31	12.76	8.46	8.44	18.04	20.35

$$\begin{aligned} &\text{Minimize} \quad 3x_1 + 2x_2 \\ &\text{Subject to} \quad \overline{\widetilde{A}}x \leqslant \overline{\tilde{b}} \\ &\qquad\qquad\quad x \geqslant 0, w \in \Omega \end{aligned}$$

where

$$(\overline{\widetilde{A}}, \overline{\tilde{b}}) = \begin{cases} (\widetilde{A}_{w1}, \tilde{b}_{w1}) = \left\{ \begin{pmatrix} \tilde{1} & \tilde{1} \\ \tilde{2} & \tilde{1} \end{pmatrix}, \begin{pmatrix} \tilde{3} \\ \tilde{4} \end{pmatrix} \right\} \\ (\widetilde{A}_{w2}, \tilde{b}_{w2}) = \left\{ \begin{pmatrix} \tilde{1} & \tilde{3} \\ \tilde{1} & \tilde{2} \end{pmatrix}, \begin{pmatrix} \tilde{5} \\ \tilde{4} \end{pmatrix} \right\} \end{cases}$$

$$p(w_1) = 0.25, \quad p(w_2) = 0.75, \quad \Omega = (w_1, w_2)$$

and $\tilde{t}$ denotes a triangular fuzzy number with the following membership function:

$$\mu_{\tilde{t}}(x) = \begin{cases} 0; & x \leqslant t - 0.5 \\ x - (t - 0.5); & (t - 0.5) < x \leqslant t \\ -x + (t + 0.5); & t \leqslant x < (t + 0.5) \\ 0; & (t + 0.5) \leqslant x \end{cases}$$

The program (*P5*) corresponding to this example is then

$$\begin{aligned} &\text{Minimize} \quad 3x_1 + 2x_2 + [(0.25\lambda_{11} + 0.25\lambda_{21} + 0.75\lambda_{12} + 0.75\lambda_{22})] \\ &\text{Subject to} \quad \overline{D}(\tilde{1}x_1 + \tilde{1}x_2, \tilde{3}) = \lambda_{11} \\ &\qquad\qquad\quad \overline{D}(\tilde{2}x_1 + \tilde{1}x_2, \tilde{4}) = \lambda_{21} \\ &\qquad\qquad\quad \overline{D}(\tilde{1}x_1 + \tilde{3}x_2, \tilde{5}) = \lambda_{12} \\ &\qquad\qquad\quad \overline{D}(\tilde{1}x_1 + \tilde{2}x_2, \tilde{4}) = \lambda_{22} \\ &\qquad\qquad\quad x_j, \lambda_{1h} = \lambda_1(w_h), \lambda_{2h} = \lambda_2(w_h) \geqslant 0 \\ &\qquad\qquad\quad i = 1,2; \quad j = 1,2; \quad k = 1,2; \quad h = 1,2 \end{aligned}$$

The corresponding LP is

$$\begin{aligned} &\text{Minimize} \quad 3x_1 + 2x_2 + [(0.25\lambda_{11} + 0.25\lambda_{21} + 0.75\lambda_{12} + 0.75\lambda_{22})] \\ &\text{Subject to} \quad \frac{1}{2}(x_1 + x_2 - 3 + 0.5x_1 + 0.5x_2 + 0.5) = \lambda_{11} \\ &\qquad\qquad\quad \frac{1}{2}(2x_1 + 1x_2 - 4 + 0.5x_1 + 0.5x_2 + 0.5) = \lambda_{21} \\ &\qquad\qquad\quad \frac{1}{2}(x_1 + 3x_2 - 5 + 0.5x_1 + 0.5x_2 + 0.5) = \lambda_{12} \\ &\qquad\qquad\quad \frac{1}{2}(x_1 + 2x_2 - 4 + 0.5x_1 + 0.5x_2 + 0.5) = \lambda_{22} \\ &\qquad\qquad\quad x_j, \lambda_{1h} = \lambda_1(w_h), \lambda_{2h} = \lambda_2(w_h) \geqslant 0 \\ &\qquad\qquad\quad i = 1,2; \quad j = 1,2; \quad k = 1,2; \quad h = 1,2 \end{aligned}$$

Using Lingo [18] or http://www.lindo.com, the obtained solution is $x_1 = 0.85, x_2 = 0.92$ and objective value = 4.9. The solution of Luhandjula is $x_1 = 1.34, x_2 = 1$ and objective value = 6.02, which is larger than our solution. Since this is a minimized problem, our solution is better. In addition, the number of constraints in our model is fewer than in the model of Luhandjula. Thus, for larger problems, the computation time of our model will be relatively shorter.

Table 3
Comparison between our method and Luhandjula's method

Case	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	15.2	25.27	17.85	35.35	11.61	0.60	14.30	32.94	25.54	3.77	3.80	18.79	8.16	27.65	6.72
2	23.34	88.2	21.17	57.38	26.91	0.72	16.33	118.42	33.19	6.89	8.56	25.28	9.83	32.81	8.43

Similar to the fuzzy linear programming case, we compare our approach with Luhandjula's method by an experiment in which all parameters are generated randomly in the form of uniform distributions. The parameters of Luhandjula's method do not change, i.e., $0 = \alpha_0 < 0.4 = \alpha_1 < 0.6 = \alpha_2 < 0.8 = \alpha_3 < \alpha_4 = 1$; $0 = \delta_4 < 0.01 = \delta_3 < 0.02 = \delta_2 < 0.03 = \delta_1$; and $g_i^{\alpha 1} = 2, g_i^{\alpha 2} = 3, g_i^{\alpha 3} = 4$. The generated data is given in Appendix A and the obtained results are presented in Table 3. In Table 3, row 1 contains our method's results, and row 2 are the results of Luhandjula's method.

From the obtained results, it is seen that our method gives a better solution in terms of objective values. This improvement is a consequence of the primary difference between our method and the Luhandjula's method – our approach uses overlapping areas to convert the fuzzy stochastic linear programming problem into the corresponding deterministic one, while Luhandjula's method discretizes fuzzy stochastic variables by α-levels. This discretizing process creates more constraints, causing the obtained deterministic LP to be closer to the deteriministic solution than ours. This also results in Luhandjula's method requiring more time to compute.

6. Conclusion

This paper has presented a simple method to convert the fuzzy and fuzzy stochastic linear programming problems into the conventional LPs through the use of attainment values of fuzzy numbers and/or fuzzy random variables. The proposed method provides simple deterministic LP models, which can then be solved easily using standard optimization packages. In addition, there is greater computation efficiency because the obtained models have fewer constraints.

Appendix A

See Table 4.

Table 4
Data for the fuzzy stochastic experiment

Case	c_j	p_1	p_2	a_{ij}	a_{ij}^l	a_{ij}^r	b_i	b_i^l	b_i^r
1	2.57	0.57	0.43	2.05	0.00	0.67	5.62	0.96	0.61
	2.46			1.13	0.74	0.87			
				1.46	0.34	0.96	14.99	1.33	1.32
				1.28	0.10	0.26			
				0.50	0.57	0.59	12.44	1.10	1.44
				6.32	0.04	0.43			
				5.52	0.69	0.04	10.26	0.81	1.20
				0.93	0.08	0.97			
2	8.5	0.23	0.77	2.25	0.12	0.29	10.68	0.66	1.11
	9.4			7.31	1	0.23			
				5.74	0.5	0.01	8.37	1.15	1.15
				2.71	0.66	0.99			
				5.86	0.49	0.69	6.24	0.88	0.69
				1.64	0.38	0.01			
				6.91	0.44	0.38	14.85	0.98	1.26
				1.39	0.64	0.13			

(continued on next page)

Table 4 (*continued*)

Case	c_j	p_1	p_2	a_{ij}	a_{ij}^l	a_{ij}^r	b_i	b_i^l	b_i^r
3	7.79	0.37	0.63	5.51	0.95	0.70	5.06	1.17	1.29
	1.05			9.34	0.96	0.32			
				1.63	0.76	0.67	12.84	1.35	1.47
				4.21	0.13	0.56			
				8.74	0.63	0.40	6.90	1.28	0.52
				1.18	0.13	0.31			
				3.25	0.22	0.06	9.00	0.88	1.22
				0.32	0.46	0.51			
4	6.13	0.15	0.85	2.05	0.38	0.06	13.41	1.24	0.79
	8.98			0.27	0.18	0.02			
				3.22	0.83	0.38	5.78	0.96	1.42
				2.17	0.89	0.67			
				9.23	0.25	0.46	8.77	1.24	0.96
				5.27	0.90	0.20			
				9.40	0.22	0.33	6.45	1.17	1.19
				3.82	0.63	0.54			
5	3.89	0.72	0.28	8.85	0.88	0.42	12.16	1.23	1.40
	5.51			1.44	0.42	0.48			
				5.08	0.40	0.10	13.75	0.53	0.50
				3.15	0.65	0.99			
				5.65	0.64	0.25	5.08	0.69	1.33
				0.81	0.96	0.42			
				2.38	0.45	0.69	8.79	1.49	1.32
				1.55	0.15	0.14			
6	6.57	0.64	0.36	6.04	0.94	0.56	8.69	0.65	0.56
	0.34			5.74	0.31	0.71			
				9.92	0.29	0.25	9.64	0.71	1.00
				8.50	0.01	0.37			
				4.07	0.23	0.44	9.66	0.53	1.18
				9.72	0.37	0.11			
				0.30	0.01	0.25	13.05	0.98	0.63
				5.77	0.94	0.99			
7	1.88	0.26	0.74	1.74	0.58	0.97	11.65	1.08	0.73
	8.49			8.90	0.20	0.02			
				4.11	0.37	0.06	13.74	0.54	0.65
				3.44	0.10	0.11			
				1.97	0.30	0.19	7.77	1.15	0.52
				3.63	0.17	0.82			
				0.98	0.37	0.14	10.34	0.55	0.95
				5.43	0.99	0.67			
8	2.25	0.10	0.90	5.15	0.47	0.49	6.90	0.71	0.64
	6.20			8.86	0.56	0.43			
				4.11	0.01	0.01	6.07	1.13	0.98
				5.92	0.55	0.13			
				2.52	0.61	0.58	11.12	1.06	0.65
				9.61	0.96	0.78			
				0.69	0.48	0.19	14.15	1.29	1.09
				0.31	0.45	0.94			
9	2.82	0.30	0.70	0.66	0.87	0.72	7.76	1.28	0.50
	7.92			5.54	0.40	0.39			
				5.83	0.30	0.37	6.44	0.85	0.82
				2.50	0.79	0.85			
				0.79	0.26	0.25	9.98	0.57	0.98
				0.43	0.60	0.31			

Table 4 (*continued*)

Case	c_j	p_1	p_2	a_{ij}	a^l_{ij}	a^r_{ij}	b_i	b^l_i	b^r_i
			9.18	0.04	0.37	5.16	1.05	0.75	
			4.43	0.16	0.43				
10	1.75	0.21	0.79	7.09	0.80	0.20	9.09	0.61	1.44
	3.54			8.70	0.82	0.95			
				5.77	0.18	0.78	14.79	0.65	0.90
				7.51	0.79	0.16			
				8.15	0.39	0.67	7.56	0.75	0.94
				3.83	0.57	0.76			
				5.92	0.50	0.42	5.93	1.27	1.12
				5.24	0.51	0.08			
11	7.69	0.67	0.33	9.91	0.35	0.01	13.14	1.48	0.59
	3.19			9.40	0.87	0.40			
				8.72	0.47	0.86	10.66	1.06	0.93
				7.22	0.90	0.96			
				3.93	0.10	0.29	5.75	1.16	0.75
				4.68	0.82	0.91			
				6.25	0.92	0.57	8.00	0.66	0.65
				9.98	0.88	0.05			
12	5.43	0.23	0.77	5.15	0.75	0.93	14.22	0.58	0.70
	5.74			8.94	0.61	0.68			
				0.04	0.05	0.59	7.92	1.22	0.67
				6.85	0.97	0.73			
				7.39	0.56	0.01	10.37	0.53	0.63
				2.32	0.10	0.97			
				2.25	0.50	0.52	9.36	0.92	1.37
				0.64	0.61	0.53			
13	2.91	0.51	0.49	3.69	0.69	0.66	12.82	1.44	1.04
	7.70			7.14	0.19	0.71			
				3.40	0.53	0.08	10.60	0.84	0.74
				0.44	0.04	0.46			
				5.37	0.24	0.97	11.00	1.39	1.02
				7.19	0.04	0.35			
				5.46	0.19	0.61	13.54	0.77	0.93
				8.54	0.56	0.15			
14	9.62	0.44	0.56	6.77	0.69	0.34	11.48	0.95	1.16
	8.71			4.22	0.43	0.21			
				7.36	0.36	0.25	13.11	1.03	0.76
				6.45	0.48	0.30			
				0.51	0.76	0.83	12.84	0.84	1.44
				5.94	0.69	0.36			
				4.29	0.82	0.44	10.08	1.16	1.38
				1.31	0.80	0.05			
15	5.65	0.70	0.30	6.45	0.42	0.19	8.74	0.98	1.42
	6.54			8.72	0.84	0.23			
				6.18	0.10	0.91	9.75	1.32	0.91
				0.86	0.91	0.89			
				5.95	0.79	0.16	6.13	0.87	1.27
				1.71	0.23	0.83			
				7.41	0.85	0.48	9.15	0.96	0.94
				1.58	0.50	0.80			

References

[1] L. Campos, J.L. Verdegay, Linear programming problems and ranking of fuzzy numbers, Fuzzy Sets and Systems 32 (1989) 1–11.

[2] S. Chanas, P. Zielinski, On the equivalence of two optimization methods for fuzzy linear programming problems, European Journal of Operational Research 121 (2000) 56–63.
[3] J. Chiang, Fuzzy linear programming based on statistical confidence interval and interval-valued fuzzy set, European Journal of Operational Research 129 (2001) 65–86.
[4] S.C. Fang, C.F. Hu, H.F. Wang, S.Y. Wu, Linear programming with fuzzy coefficients in constraints, Computers and Mathematics with Applications 37 (1999) 63–76.
[5] Ph. Fortemps, M. Roubens, Ranking and defuzzification methods based on area compensation, Fuzzy Sets and Systems 82 (1996) 319–330.
[6] S.M. Guu, Y.K. Wu, Two-phase approach for solving the fuzzy linear programming problems, Fuzzy Sets and Systems 107 (1999) 191–195.
[7] M. Inuiguchi, J. Ramik, Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem, Fuzzy Sets and Systems 111 (2000) 3–28.
[8] K.D. Jamison, W.A. Lodwick, Fuzzy linear programming using a penalty method, Fuzzy Sets and Systems 119 (2001) 97–110.
[9] H.R. Maleki, M. Tata, M. Mashinchi, Linear programming with fuzzy variables, Fuzzy Sets and Systems 109 (2000) 21–33.
[10] D. Wang, An inexact approach for linear programming problems with fuzzy objective and resources, Fuzzy Sets and Systems 89 (1997) 61–68.
[11] B. Liu, Y.K. Liu, Expected value of fuzzy variable and fuzzy expected value models, IEEE Transactions on Fuzzy Systems 10 (4) (2002) 445–450.
[12] B. Liu, Fuzzy random dependent-chance programming, IEEE Transactions on Fuzzy Systems 9 (5) (2001) 721–726.
[13] B. Liu, Fuzzy random chance-constrained programming, IEEE Transactions on Fuzzy Systems 9 (5) (2001) 713–720.
[14] Y.K. Liu, B. Liu, A class of fuzzy random optimization: expected value models, Information Sciences 155 (2003) 89–102.
[15] M.K. Luhandjula, Optimization under hybrid uncertainty, Fuzzy Sets and Systems 146 (2004) 187–203.
[16] M.K. Luhandjula, M.M. Gupta, On fuzzy stochastic optimization, Fuzzy Sets and Systems 81 (1996) 47–55.
[17] M.K. Luhandjula, Fuzziness and randomness in an optimization framework, Fuzzy Sets and Systems 77 (1996) 291–297.
[18] L.E. Schrage, Optimization Modeling with LINGO, LINDO Systems, Inc, Chicago, 1999.
[19] J.S. Yao, K. Wu, Ranking fuzzy numbers based on decomposition principle and signed distance, Fuzzy Sets and Systems (2001).
[20] C.J. Shih, R.A.S. Wangsawidjaja, Mixed fuzzy-probabilistic programming approach for multiobjective engineering optimization with random variables, Computers and Structures 59 (2) (1996) 283–290.
[21] Z. Zmeskal, Value at risk methodology under soft conditions approach (fuzzy-stochastic approach), European Journal of Operational Research 161 (2) (2005) 337–347.