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Normality Domains for Families of Holomorphic Maps

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In 1926 Julia [7] proved that the normality domain of a family of holomorphic functions is pseudoconvex. This paper generalizes Julia's theorem to families of holomorphic maps into complex manifolds. The main result states that if M and N are complex manifolds of dimensions m and n respectively $(0<n<m)$ and N has a Stein covering manifold, then the normality domain of a family of holomorphic maps from M into N is $(m-n)$-pseudoconvex. The machinery used to prove this theorem also yields a proof that a covering manifold is taut [12] if and only if the base manifold is taut.

1. Normal Families

For convenience we assume that all topological spaces (other than spaces of maps) are locally connected, locally compact, and metrizable. If M and N are two such spaces, $\mathscr{C}(M, N)$ denotes the set of continuous maps from M into N endowed with the compact-open topology [3, Definition 1.1, p. 257]. In this topology composition and evaluation are continuous maps [3, Theorems 2.2 and 2.4, pp. 259–260].

Suppose that N is a connected space, and let $\mathscr{F} \subset \mathscr{C}(M, N)$. For connected M, Grauert and Reckziegel [5, Definition 3, p. 114] and Wu [12, Definition 1.1, p. 197] call $\mathscr{F}$ *normal* iff every sequence in $\mathscr{F}$ has a subsequence which either converges in $\mathscr{C}(M, N)$ or diverges compactly. For not necessarily connected M, we say that $\mathscr{F}$ is *normal* if the restriction $\mathscr{F} \mid L \subset \mathscr{C}(L, N)$ is normal for every connectivity component L of M. The *normality domain* of $\mathscr{F}$ is the union $D(\mathscr{F})$ of the open subsets U of M such that $\mathscr{F} \mid U \subset \mathscr{C}(U, N)$ is normal. It need not be connected; but, since normality is a local property, $\mathscr{F} \mid D(\mathscr{F})$ is a normal family. The normality domain is the largest open subset of M having this property.

2. Covering Spaces

By a *covering* $p: L \to N$ we mean a covering space in the sense of [2, Definition 3, p. 40]. If $g \in \mathscr{C}(M, N)$, a *lifting* of g to L is a continuous map $f: M \to L$ such that $p \circ f = g$. Lifting is unique in the sense that if M is connected and two liftings agree at a point they are identical [2, Lemma 1, p. 51]. If M is simply connected, liftings always exist: for each $u \in M$ and $v \in p^{-1}(g(u))$ there is a lifting f of g to L such that $f(u) = v$ [2, Proposition 1, p. 50].

Lemma 1. *Let G be a topological space, and let $p: L \to N$ be a covering. Let u, $u_j \in G$ with $u_j \to u$ as $j \to \infty$. Let f, $f_j \in \mathscr{C}(G, L)$ with $p \circ f_j \to p \circ f$ and $f_j(u_j) \to f(u)$ as $j \to \infty$. Then*

(i) *There exists a neighborhood U of u in G such that $f_j | U \to f | U$ as $j \to \infty$.*

(ii) *If G is connected, then $f_j \to f$ as $j \to \infty$.*

Proof. (i) Let V be an open neighborhood of $f(u)$ such that $p|V: V \to p(V) = W$ is topological. Since $p \circ f(u) \in W$ and W is open, there is a compact connected neighborhood U of u such that $p \circ f(U) \subset W$. By the definition of the compact-open topology there exists jo such that $p \circ f_j(U) \subset W$ for $j > jo$. We can assume that $u_j \in U$ and $f_j(u_j) \in V$ for $j > jo$. Since lifting is unique, $f_j | U = (p|V)^{-1} \circ (p \circ f_j | U)$ for $j > jo$. Composition being continuous, $f_j | U \to (p|V)^{-1} \circ (p \circ f | U) = f | U$ as $j \to \infty$.

(ii) Suppose that $f_j \not\to f$ as $j \to \infty$. Since convergence in $\mathscr{C}(G, L)$ is a local property, (i) implies that there exists $t \in G$ such that $f_j(t) \not\to f(t)$ as $j \to \infty$. Thus there are a neighborhood V of $f(t)$ and a subsequence $\{g_k\}$ of $\{f_j\}$ such that $g_k(t) \notin V$ for all k. Let $E = \{s \in G | f(s)$ is an accumulation point of $\{g_k(s)\}\}$. By (i), E is open. To show that E is closed, take points $s_j \in E$ with $s_j \to s \in G$ as $j \to \infty$. Let d be a metric inducing the topology on L. An inductive construction yields integers $k_1 < k_2 < \cdots$ such that $d(g_{k_j}(s_j), f(s_j)) < 1/j$. Since $f(s_j) \to f(s)$, this implies that $g_{k_j}(s_j) \to f(s)$ as $j \to \infty$. By (i), $g_{k_j}(s) \to f(s)$ as $j \to \infty$, i.e., $s \in E$. Thus E is closed. Since $u \in E$ and $t \notin E$, G is not connected. □

3. Taut Manifolds

The theorem of this section will not be used in the sequel. We present it here because it follows easily from Lemma 1. Kobayashi has proved the analogous theorem for complete hyperbolic manifolds [8, Theorem 5.5, pp. 471–472].

By a *manifold* we mean a connected second countable complex manifold. For manifolds M and N, $\mathscr{A}(M, N)$ denotes the closed subspace of $\mathscr{C}(M, N)$ consisting of the holomorphic maps from M into N. According to Wu [12, Definition 1.2, p. 199], N is called *taut* iff $\mathscr{A}(M, N)$ is normal for all M. If E denotes the open unit disk in $\mathbb{C}$, N is taut if and only if $\mathscr{A}(E, N)$ is normal [1, Theorem 2, p. 430].

By a *complex covering* we mean a covering $p: L \to N$ in which the spaces are manifolds and the projection is locally biholomorphic. A lifting of a holomorphic map to a complex covering is again holomorphic.

Theorem 1. *Let $p: L \to N$ be a complex covering. Then L is taut if and only if N is taut.*

Proof. (i) Suppose that L is taut. Let $\{h_j\}$ be a sequence in $\mathscr{A}(E, N)$ which is not compactly divergent. Then there exist a subsequence $\{g_k\}$ of $\{h_j\}$ and compact sets $K \subset E$, $K' \subset N$ such that $g_k(K) \cap K' \neq \emptyset$ for all k. Since p is locally topological, there is a compact set $K'' \subset L$ with $p(K'') = K'$.

Existence of liftings gives $f_k \in \mathscr{A}(E, L)$ such that $p \circ f_k = g_k$ and $f_k(K) \cap K'' \neq \emptyset$ for all k. Since L is taut, $\{f_k\}$ has a convergent subsequence; composition being continuous $\{p \circ f_k\} = \{g_k\}$ has a convergent subsequence. Hence so does $\{h_j\}$. Therefore $\mathscr{A}(E, N)$ is normal.

(ii) Suppose that N is taut. Let $\{h_j\}$ be a sequence in $\mathscr{A}(E, L)$ which is not compactly divergent. Then there exist a subsequence $\{f_k\}$ of $\{h_j\}$, compact sets $K \subset E$, $K' \subset L$, and points $u_k \in K$ such that $f_k(u_k) \in K'$ for all k. By taking successive subsequences we can assume that $u_k \to u \in K$ and $f_k(u_k) \to t \in K'$ as $k \to \infty$. Now $p \circ f_k(u_k)$ belongs to the compact set $p(K')$, so $\{p \circ f_k\}$ does not have a compactly divergent subsequence. Since N is taut we can assume that $p \circ f_k \to g \in \mathscr{A}(E, N)$ as $k \to \infty$. By continuity of evaluation, $p(t) = g(u)$. Existence of liftings gives $f \in \mathscr{A}(E, L)$ with $p \circ f = g$ and $f(u) = t$. Then $p \circ f_k \to p \circ f$ and $f_k(u_k) \to f(u)$ as $k \to \infty$. By Lemma 1, $f_k \to f$ as $k \to \infty$. Therefore $\mathscr{A}(E, L)$ is normal. □

4. Pseudoconvexity

Let $\|z\| = \max |z_j|$ denote the maximum norm on $\mathbb{C}^n$ for all n. The following definitions are due to Tadokoro [11, p. 283] and Riemenschneider [9, pp. 314–315].

Definition 1. *Let $0 < k < m$ be integers. A pair (H, P) with*

$$H = \{(z, w) \mid z \in \mathbb{C}^k,\ \|z\| < r;\ w \in \mathbb{C}^{m-k},\ r_1' < \|w\| < r'\}$$
$$\cup \{(z, w) \mid z \in \mathbb{C}^k,\ \|z\| < r_1;\ w \in \mathbb{C}^{m-k},\ \|w\| < r'\},$$
$$P = \{(z, w) \mid z \in \mathbb{C}^k,\ \|z\| < r;\ w \in \mathbb{C}^{m-k},\ \|w\| < r'\}$$

where $0 < r_1 < r$, $0 < r_1' < r'$ is called a (k, m)-Hartogsfigure.

Definition 2. *Let G_0 be an open subset of the m-dimensional manifold G, $0 < k < m$. G_0 is said to be k-pseudoconvex in G iff $f(P) \subset G_0$ whenever (H, P) is a (k, m)-Hartogsfigure and $f: P \to f(P) \subset G$ is a biholomorphic map with $f(H) \subset G_0$.*

The k-pseudoconvexity of Definition 2 is equivalent to that of Rothstein [10] and to the $(m-k)$-convexity of Grauert [4]. Ordinary pseudoconvexity coincides with $(m-1)$-pseudoconvexity. If $0 < j < k < m$ and G_0 is k-pseudoconvex in G, then G_0 is j-pseudoconvex in G. If S is a pure k-dimensional analytic set in G, then $G - S$ is k-pseudoconvex in G. The following property of Hartogsfigures will be used in the next section.

Lemma 2. *Let (H, P) be a (k, m)-Hartogsfigure, let N be a Stein manifold, and let $\{f_j\}$ be a sequence in $\mathscr{A}(P, N)$. If $\{f_j \mid H\}$ converges in $\mathscr{A}(H, N)$, then $\{f_j\}$ converges in $\mathscr{A}(P, N)$.*

Proof. By the imbedding theorem for Stein manifolds [6, Theorem 13, p. 226], N can be considered as a closed complex submanifold of $\mathbb{C}^n$. The Cauchy integral formula shows that $\{f_j\}$ converges in $\mathscr{A}(P, \mathbb{C}^n)$, hence in $\mathscr{A}(P, N)$. □

5. Normality Domains

Theorem 2. *Let M and N be manifolds of dimensions m and n respectively $(0<n<m)$. Suppose that N has a complex covering by a Stein manifold. Let $\mathscr{F} \subset \mathscr{A}(M, N)$. Then the normality domain $D(\mathscr{F})$ is $(m-n)$-pseudoconvex in M.*

Proof. Following the economical tradition of erasing biholomorphic maps, we take an $(m-n, m)$-Hartogsfigure (H, P) with $P \subset M$, $H \subset D(\mathscr{F})$, and prove that $P \subset D(\mathscr{F})$.

Let $\{f_j\}$ be a sequence in $\mathscr{F}$. Since $H \subset D(\mathscr{F})$, there is a subsequence $\{g_k\}$ of $\{f_j\}$ which either converges or diverges compactly on H.

If $\{g_k | H\}$ converges, let g be the limit map. Fix $u \in H$. Then $g_k(u) \to g(u)$ as $k \to \infty$. Let $p: L \to N$ be a complex covering with L a Stein manifold. Since p is locally topological, there exist $t, t_k \in L$ such that $p(t) = g(u)$, $p(t_k) = g_k(u)$, and $t_k \to t$ as $k \to \infty$. Existence of liftings gives $h \in \mathscr{A}(H, L)$, $h_k \in \mathscr{A}(P, L)$ such that $h(u) = t$, $h_k(u) = t_k$, $p \circ h = g$, and $p \circ h_k = g_k | P$. By Lemma 1, $h_k | H \to h$ as $k \to \infty$. By Lemma 2, $\{h_k\}$ converges in $\mathscr{A}(P, L)$. Since composition is continuous, $\{p \circ h_k\} = \{g_k | P\}$ converges in $\mathscr{A}(P, N)$.

If $\{g_k | H\}$ diverges compactly, let $K \subset P$, $K' \subset N$ be compact. Let (H^*, P^*) be an $(m-n, m)$-Hartogsfigure with $H^* \subset\subset H$ and $P^* \supset K$. Since $\{g_k | H\}$ diverges compactly, there exists k_0 such that $g_k(H^*) \cap K' = \emptyset$ for $k > k_0$. For $v \in K'$, $k > k_0$, consider the analytic set $S = \{z \in P^* | g_k(z) = v\}$. Clearly $H^* \subset P^* - S$. Since the irreducible branches of S all have dimension at least $m-n$, $P^* - S$ is $(m-n)$-pseudoconvex in P^*. Thus $P^* \subset P^* - S$, i.e., $S = \emptyset$. This means that $g_k(P^*) \cap K' = \emptyset$ for $k > k_0$. Thus $\{g_k\}$ diverges compactly on P.

Therefore $P \subset D(\mathscr{F})$. □

6. An Example

We construct a family $\mathscr{F} \subset \mathscr{A}(\mathbb{C}^m, \mathbb{C}^n)$ whose normality domain is not $(m-n+1)$-pseudoconvex. This shows that the integer $m-n$ of Theorem 2 is in general best possible.

Let $2 \leqq n \leqq m$. For $z \in \mathbb{C}^{m-n+1}$, $w \in \mathbb{C}^{n-1}$, define $f_j(z, w) = j(z_1 - 2, w)$. Then $\mathscr{F} = \{f_j | j = 1, 2, \ldots\} \subset \mathscr{A}(\mathbb{C}^m, \mathbb{C}^n)$. Let (H, P) be the $(m-n+1, m)$-Hartogsfigure defined by $r = r' = 3$, $r_1 = r_1' = 1$. An easy computation shows that $\|f_j\| > j$ on H. Hence $\{f_j | H\}$ diverges compactly and $H \subset D(\mathscr{F})$. However $f_j(2, 0, \ldots, 0) = 0$, so that $\{f_j | P\}$ does not have a compactly divergent subsequence. Therefore $P \not\subset D(\mathscr{F})$.

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