Mathematical Programming 16 (1979) 21–36.
North-Holland Publishing Company

A CONSTRUCTION METHOD IN PARAMETRIC PROGRAMMING

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Received 9 June 1975
Revised manuscript received 24 July 1978

We consider a linear programming problem, with two parameters in the objective function, and present an algorithm for finding the decomposition of the parameter space into maximal polyhedral areas in which particular basic solutions are optimal. Special attention is paid to fill up areas of degenerate solutions.

Key words: Parametric Linear Programming, Construction Method, Degeneracy, Primal Lexicographic Method.

1. Basic concepts

We consider the LP-problem

$$\text{(P)} \qquad h(\lambda, \mu) = \max_{x \in F} f(\lambda, \mu; x)$$

with

$$f(\lambda, \mu; x) = (c_0 + \lambda c_1 + \mu c_2)'x, \quad F = \{x \in \mathbf{R}^n \mid Ax \leqq b, x \geqq 0$$

where $c_i \in \mathbf{R}^n$, $i = 0, 1, 2$, the $m \times n$ matrix A, $b \in \mathbf{R}^m$ are given and λ and μ are two scalar parameters.

We assume that the following regularity conditions hold:

(1) F has a nonempty interior.

(2) F is bounded.

(3) c_1 and c_2 are linearly independent.

Assumptions (1) and (2) imply that F is the convex hull of its extreme points $x_1^\dagger, \dots, x_L^\dagger$; $n + 1 \leqq L < \infty$.

We denote the closure, the interior and the boundary of a set S by $\operatorname{cl} S$, $\operatorname{int} S$ and $\operatorname{bd} S$, respectively; $\operatorname{bd} S = \operatorname{cl} S \backslash \operatorname{int} S$. The set of extreme points of a compact convex set S is denoted by $\operatorname{extr} S$, and $[e, f]$ is the convex hull of $e \in \mathbf{R}^2$ and $f \in \mathbf{R}^2$.

As λ and μ range over $\mathbf{R}^2$, the optimal extreme point changes and this gives rise to a decomposition of $\mathbf{R}^2$ into certain polyhedral areas. To our knowledge no systematic method for constructing all areas has been proposed up to now. Gal [1] finds all areas and their corresponding optimal solutions by simple enumeration. His approach only works satisfactorily with small examples. In this article we describe a method to constructing all areas and the corresponding optimal solutions in a systematic way. The method gives explicit representations for the boundaries of the areas, which together will be called the *construction figure* $\mathscr{V}$.

The relevant areas are

$$\bar{V}_i = \{(\lambda, \mu) \in \mathbf{R}^2 \mid (c_0 + \lambda c_1 + \mu c_2)'(x_i^* - x_j^*) \geqq 0, \text{ all } j\}, \quad i = 1, \dots, L.$$

which are intersections of the plane

$$\{c \in \mathbf{R}^n \mid c = c_0 + \lambda c_1 + \mu c_2; (\lambda, \mu) \in \mathbf{R}^2\}$$

with the normal cones

$$W_i = \{c \in \mathbf{R}^n \mid c' x_i^* \geqq c' x_j^*, \text{ all } j\}, \quad i = 1, \dots, L.$$

at the extreme points x_i^*, $i = 1, \dots, L$. It is possible of course that $\bar{V}_i = \emptyset$ for some i, or that int $\bar{V}_i = \emptyset$ for some i; in such cases $\bar{V}_i$ is a *false area*. Renumber the x_i^* so that int $\bar{V}_i \neq \emptyset$ if $i \leqq K$ and int $\bar{V} = \emptyset$ if $i > K$, and let $V_i = \bar{V}_i$ for $i \leqq K$. The sets V_i, $i = 1, \dots, K$ are called *areas*.

Some properties of the decomposition of $\mathbf{R}^n$ into the cones W_i can be clarified with aid of the following concept [4, p. 39, 41]. A finite family $\mathscr{C}$ of polyhedral sets in $\mathbf{R}^d$ is called a polyhedral complex provided that:

(i) Every face of a member of $\mathscr{C}$ is itself a member of $\mathscr{C}$.

(ii) The intersection of any two members of $\mathscr{C}$ is a face of each of them. Any nonempty face of W_i can be expressed as

$$\{c \in \mathbf{R}^n \mid c' x_i^* \geqq c' x_j^* \text{ all } j, \text{ with equality for } j \in J\} = \bigcap_{j \in J} W_j$$

for some index set J, $\{i\} \subset J \subset \{1, \dots, L\}$. Conversely, this set is actually a face for each W_j, $j \in J$, or more generally, $\bigcap_{j \in J} W_j$ is a face of $\bigcap_{j \in I} W_j$ if $\emptyset \neq I \subset J$.

As immediate consequence we have that

$$\mathscr{W} = \{\emptyset\} \cup \left\{ \bigcap_{i \in I} W_i \mid \emptyset \neq I \subset \{1, \dots, L\} \right\}$$

is a polyhedral complex in $\mathbf{R}^n$.

It is easily verified that the intersection of the members of $\mathscr{W}$ with the plane $\{c \in \mathbf{R}^n \mid c = c_0 + \lambda c_1 + \mu c_2; (\lambda, \mu) \in \mathbf{R}^2\}$ constitute a polyhedral complex in $\mathbf{R}^2$. So we have the following characterization of intersections of areas.

Theorem 1. (a) *V_i is convex, closed, polyhedral, has a nonempty interior and does not contain any two-sided finite line*; $i = 1, \dots, K$.

(b) *For each V_i, x_i^* is an optimal solution for* (P) *for all* $(\lambda, \mu) \in V_i$; *this optimal solution is unique on* int V_i.

(c) $\bigcup_{i=1}^K V_i = \mathbf{R}^2$, int $V_i \cap$ int $V_j = \emptyset$, $i \neq j$, $i, j = 1, \dots, K$.

Let $I \subset \{1, \dots, K\}$ such that $|I| \geqq 2$ and $V_I = \bigcap_{i \in I} V_i \neq \emptyset$. Then V_i is either an edge of V_i for each $i \in I$ or an extreme point of V_i for each $i \in I$.

Of course it is possible to characterize the areas V_i algebraically. Let any basis matrix B (a nonsingular $m \times m$-submatrix of $\tilde{A} = (A\, I)$) be given. Consider the partitions $\tilde{A} = (B\, N)$, $\tilde{c}' = (c'_B\, c'_N)$, $\tilde{x}' = (x'_B\, x'_N)$, $\tilde{x}^{*\prime} = (\tilde{x}^{*\prime}_{iB}\, \tilde{x}^{*\prime}_{iN})$ where $\tilde{c}, \tilde{x}, \tilde{x}^*_i \in \mathbf{R}^{m+n}$ arise from $c, x, \tilde{x}_i \in \mathbf{R}^n$ after adding the components of the slack variables. As usual, x^*_i (or $\tilde{x}^*_i$) is called degenerate if more than n co-ordinates of $\tilde{x}^*_i$ are equal to zero. Applying well-known theorems of linear programming we have:

Theorem 2 (algebraic characterizations of V_i). *Corresponding to every non-degenerate x^*_i, $i \in \{1, \dots, K\}$, there is a unique basis matrix B_i such that*

$$V_i = \{(\lambda, \mu) \in \mathbf{R}^2 \mid d_{0i} + \lambda d_{1i} + \mu d_{2i} \geqq 0\},$$

$$\operatorname{int} V_i = \{(\lambda, \mu) \in \mathbf{R}^2 \mid d_{0i} + \lambda d_{1i} + \mu d_{2i} > 0\}$$

where $d_{ji} \in \mathbf{R}^n$ are the vectors of reduced coefficients defined by

$$d'_{ji} = (\tilde{c}_j)'_{B_i} B_i^{-1} N_i - (\tilde{c}_j)'_{N_i}, \quad j = 0, 1, 2.$$

Corresponding to every degenerate x^*_i, $i \in \{1, \dots, K\}$, in general several basis matrices B_{ih}, $h = 1, \dots,$ exist, such that the following holds: Let d_{jih} be defined by $d_{jih} = (\tilde{c}_j)'_{B_{ih}} B_{ih}^{-1} - (\tilde{c}_j)'_{N_{ij}}$, $j = 0, 1, 2$, and let

$$V_{ih} = \{(\lambda, \mu) \in \mathbf{R}^2 \mid d_{0ih} + \lambda d_{1ih} + \mu d_{2ih} \geqq 0\},$$

so that

$$\operatorname{int} V_{ih} = \{(\lambda, \mu) \in \mathbf{R}^2 \mid d_{0ih} + \lambda d_{1ih} + \mu d_{2ih} > 0\}.$$

Then $V_i = \bigcup_h V_{ih}$, where the union can be restricted to $h = 1, \dots, H_i$ corresponding to those h for which $\operatorname{int} V_{ih} \neq \emptyset$.

The sets V_{ih}, $h = 1, \dots, H_i$ are called *subareas*. An area corresponding with a nondegenerate x^*_i is called a subarea too. Note that by definition each subarea has a nonempty interior. Just like V_i, each V_{ih} is convex, closed, polyhedral and does not contain two-sided infinite lines. However, it should be noted that it is quite possible that $\operatorname{int} V_{ip} \cap \operatorname{int} V_{iq} \neq \emptyset$ for different p and q. Furthermore, it is possible that $V_{ih} = V_{ik}$ whereas B_{ih} and B_{ik} belong to different sets of basic variables.

The construction method to be described in Section 2 is based on the simplex method, so it works with subareas instead of with areas. Therefore, an unfavourable sequence of different basic feasible solutions may give rise to subareas which overlap each other partially or completely. In order to circumvent this complication, we need a refinement of Theorem 2 guaranteeing an unique decomposition of the parameter space into subareas with disjoint interior. This can be ascertained by applying the standard primal lexicographic method.

Let e be a column vector of which the i-th component consists of the i-th power of a small positive scalar ϵ. Define $\tilde{b}(\epsilon) = b + Ie$, where I denotes the identity matrix. Any basis matrix B is called lexicographically feasible if $B^{-1}\tilde{b}(\epsilon) > 0$ for small positive values of ϵ.

Corollary 3. *Consider an arbitrary, but fixed, primal perturbation of* (P). *If in Theorem* 2 *only lexicographically feasible basis matrices are considered, then the corresponding subareas cover the parametric space and have disjoint interiors.*

Theorem 4. *Let V be any subarea and let T be the simplex tableau of the corresponding optimal solution, and let L be any boundary edge of V. Then there is at least one L-neighbouring subarea V', such that $V \cap V' = L$. Such a V' can be found by the standard parametric programming method* (*one parameter in the objective function*), *starting with the simplex tableau T, and using the standard primal lexicographic method.*

Proof. According to Theorem 2, $V = \{(\lambda, \mu) \in \mathbf{R}^2 \mid d_0 + \lambda d_1 + \mu d_2 \geqq 0\}$ for certain $d_j \in \mathbf{R}^n$ given by T. Let S_L be the (nonempty) set of indices corresponding to those nonbasic variables of T which determine L, i.e. $L = \{(\lambda, \mu) \in V \mid (d_0 + \lambda d_1 + \mu d_2)_p = 0\}$ for an arbitrary $p \in S_L$. Since $\dim L = 1$ and $\dim V = 2$ we must have $(d_0, d_1, d_2)_p = \sigma_p(\rho_0, \rho_1, \rho_2)$ for some $\sigma_p > 0$ and such that $\rho_1^2 + \rho_2^2 = 1$ for each $p \in S_L$.

Let $(\bar{\lambda}, \bar{\mu})$ be a relatively interior point of L and restrict (λ, μ) for the moment to be on the line through $(\bar{\lambda}, \bar{\mu})$ perpendicular to L, i.e. consider the LP program with one parameter τ in the objective function

$$(\mathrm{P}') \qquad \max_{x \in F} \{f(\lambda(\tau), \mu(\tau); x)\} \quad \text{where } (\lambda(\tau), \mu(\tau)) = (\bar{\lambda}, \bar{\mu}) + \tau(\rho_1, \rho_2).$$

Since V is an area, int $V \neq \emptyset$, hence the basic feasible solution of T is optimal for P' if $\tau \in [0, \delta_1]$, $\delta_1 > 0$, but is not optimal if $\tau < 0$. In more detail, the representation of the objective function of P' in the tableau T is given by the coefficients

$$\begin{aligned}(d_0 + \lambda d_1 + \mu d_2)_p &= (d_0 + \bar{\lambda} d_1 + \bar{\mu} d_2)_p + \tau(\rho_1 d_1 + \rho_2 d_2)_p \\ &= \alpha_p + \beta_p \tau, \quad p = 1, \ldots, n.\end{aligned}$$

For $p \notin S_L$ we have either $p \in S_+ = \{p \mid \alpha_p > 0\}$ or $p \in S_0 = \{p \mid \alpha_p = \beta_p = 0\}$, and for $p \in S_L$ it holds that $\alpha_p = 0$ and $\beta_p = \sigma_p > 0$. The standard parametric programming technique yields a simplex tableau T' with a basic feasible solution which is not optimal for P' if $\tau > 0$ and which is optimal if $\tau \in [-\delta_2, 0]$ for some $\delta_2 > 0$. In fact, an optimal solution of P' for small negative τ is found by applying the simplex method to the problem $\max_{x \in F}\{\sum_{p \notin S_+} \beta_p x_p\}$ with initial tableau T. If $|S_L \cup S_0| = 1$ this problem is solved in one step. If $|S_L \cup S_0| \geqq 2$, more than one iteration step might be required, because of τ-intervals of length zero. Since an anticycling technique is applied it is assured that after a finite number of steps an optimal solution for $\tau \in [-\delta_2, 0]$ is found; the transformed simplex tableau is called T'. We shall show that its basic feasible solution determines the subarea V' which is an L-neighbouring subarea of V.

Let A_{21} be the square submatrix of T, such that its rows correspond to the basic variables of T which are nonbasic in T', and such that its columns correspond to the nonbasic variables of T which are basic in T'. Partition T as follows:

$$T \quad \begin{array}{|cccc|c|} \hline 0 & 0 & d'_{j1} & d'_{j2} & g_j \\ \hline I & 0 & A_{11} & A_{12} & e_1 \\ 0 & I & A_{21} & A_{22} & e_2 \\ \hline \end{array}$$

The tableau T' has the representation

$$T' \quad \begin{array}{|cccc|c|} \hline 0 & -d'_{j1}A_{21}^{-1} & 0 & d'_{j2} - d'_{j1}A_{22}^{-1}A_{22} & g_j - d'_{j1}A_{21}^{-1}e_2 \\ \hline I & -A_{11}A_{21}^{-1} & 0 & A_{12} - A_{11}A_{21}^{-1}A_{22} & e_1 - A_{11}A_{21}^{-1}e_2 \\ 0 & A_{21}^{-1} & I & A_{21}^{-1}A_{22} & A_{21}^{-1}e_2 \\ \hline \end{array}$$

Clearly, we must have

$$V = \{(\lambda, \mu) \in \mathbf{R}^2 \mid (d_{01} + \lambda d_{11} + \mu d_{21})' \geqq 0, (d_{02} + \lambda d_{12} + \mu d_{22})' \geqq 0\},$$

$$V' = \{(\lambda, \mu) \in \mathbf{R}^2 \mid (d_{01} + \lambda d_{11} + \mu d_{21})'(-A_{21}^{-1}) \geqq 0,$$
$$(d_{02} + \lambda d_{12} + \mu d_{22})' + (d_{01} + \lambda d_{11} + \mu d_{21})'(-A_{21}^{-1}A_{22}) \geqq 0\}.$$

Define the affine set $M = \{(\lambda, \mu) \in \mathbf{R}^2 \mid d_{01} + \lambda d_{11} + \mu d_{21} = 0\}$. It follows immediately that $V' \cap M = V \cap M$ and $\operatorname{int} V \cap M = \emptyset$, so $\dim M \leqq 1$. From the analysis of P' we know that going from T to T' only incoming variables are chosen such that the corresponding coordinates of $d_0 + \bar{\lambda} d_1 + \bar{\mu} d_2$ (in T) equal zero. Hence $d_{01} + \bar{\lambda} d_{11} + \bar{\mu} d_{21} = 0$ so $(\bar{\lambda}, \bar{\mu}) \in M$. It is even true that $L \subset M$. Assume to the contrary that for some i, $(\rho_2 d_{11} - \rho_1 d_{21})_i \neq 0$, say >0. Then $(d_{01} + \lambda d_{11} + \mu d_{21})_i < 0$ if $(\lambda, \mu) = (\bar{\lambda}, \bar{\mu}) - \epsilon(\rho_2, -\rho_1)$ for all $\epsilon > 0$ so that all these (λ, μ) would not belong to V. However, for ϵ sufficiently small they must belong to $L \subset V$ since $(\bar{\lambda}, \bar{\mu})$ is a relatively interior point of L, contradiction. Hence $L \subset M$ so that M is the two-sided infinite line containing L. So $V' \cap M = V \cap M = L$.

Since $L \subset V'$ and also $(\lambda, \mu) - \delta_2(\rho_1, \rho_2) \in V' \setminus M$ it follows that int $V' \neq \emptyset$; so V' is a subarea. Furthermore, if there were a $(\lambda_0, \mu_0) \in V' \cap V$ but $(\lambda_0, \mu_0) \notin M$, then we would have $(d_{01} + \lambda_0 d_{11} + \mu_0 d_{21})_j > 0$ for some j. Since $(\bar{\lambda}, \bar{\mu})$ is a relatively interior point of $L, L \subset V'$, V' convex, a $\tau_0 > 0$ would exist such that $(\lambda, \mu) + \tau_0(\rho_1, \rho_2) \in V'$. This is a contradiction with the fact that the solution of T' is not optimal if $\tau > 0$. Therefore $V' \cap V = V' \cap V \cap M = L$ and int $V \cap$ int $V' = \emptyset$. This means that V' is an L-neighbouring area of V.

The construction method to be described in Section 2 restricts itself to the bounded rectangle

$$\mathscr{X} = \{(\lambda, \mu) \in \mathbf{R}^2 \mid 0 \leqq \lambda \leqq \lambda_{\max}, 0 \leqq \mu \leqq \mu_{\max}\}.$$

Of course, this is no serious limitation, since by making $\mathscr{X}$ large we can get all extreme points and extreme directions of all V_i from the extreme points of $Z_i = \mathscr{X} \cap V_i$, $i = 1, \dots, K$. However, from a practical point of view one may only be interested in a bounded part of the parameter space. In the sequel we shall restrict the attention to that part of the construction figure that is contained in $\mathscr{X}$. Of course it may happen that int $Z_i = \emptyset$ for some $i \in \{1, \dots, K\}$. Then renumber the V_i so that int $Z_i \neq \emptyset$ if $i \leqq N$ and int $Z_i = \emptyset$ if $i > N$. Since confusion is out of the question the Z_i, $i = 1, \dots, N$ will be referred to as areas. The neighbouring relations between areas together with the definition of the common boundary edge, is adjusted in an analogous way.

In the development the construction algorithm we use of the concept (simply connected) region, called *ring*: Suppose that the areas $z_1, z_2, \dots, z_n$ have been added to the construction figure. Define $Q = \bigcup_{i=1}^N \text{bd } Z_i$ and $Q_1 = \bigcup_{i=1}^N \text{extr } Z_i$. So the elements of Q are line segments and the elements of Q_1 are points. A ring R is defined as a sequence of elements of Q_1, say $(e_1, \dots, e_t)$ such that

(i) $e_i \neq e_j$ if $j \neq i$,

(ii) all $[e_s, e_{s+1}]$, $s = 1, \dots, t$ are elements of Q.

In this definition we set $e_{t+1} = e_1$.

Note, that a ring is defined as a sequence of elements ordered in a special way. Now define $Y(R) = \bigcup_{s=1}^t [e_s, e_{s+1}]$ as the *ringset* of R. Clearly $Y(R)$ forms a closed curve $Y(R)$ in $\mathscr{X}$. Moreover, it can easily be seen that the elements of $Y(R)$, if ordered appropriately, do constitute a simple circuit from the vertices (extreme points) e_1 to e_{t+1}. The terminology "simple circuit" is borrowed from the theory of graphs. For an introduction to this concept and other concepts that are borrowed from this theory the reader should consult the text from Busacker and Saaty [3]. Each ring that is obtained via the construction algorithm, to be described in Section 2, has at least the point $(0, 0)$ with bd $\mathscr{X}$ in common. Thus, if $R \neq \emptyset$ there exists an element $e_q \in R$ such that $e_q = (0, 0)$. Moreover if $R \neq \text{bd } \mathscr{X}$, the index

$$w = \min\{j \mid j \geqq q, [e_j, e_{j+1}] \not\subset \text{bd } \mathscr{X}\}$$

and the index

$$z = \min\{j \mid j \leqq q, [e_j, e_{j+1}] \not\subset \text{bd } \mathscr{X}\}$$

can be obtained. If necessary set $e_j = e_{j+t}$ for all $j = 1, \ldots, t$. Then e_w is called the *first ringpoint* and e_z is called the *last ringpoint* of R. Note, that $w = z$ is possible, and that e_w becomes the last ringpoint and e_z the first ringpoint if the order of the elements of R is reversed.

Let $I(R)$ denote the indexset of all areas that are enclosed by the ringset $Y(R)$ of R. These areas complete the inner region of $Y(R)$ which can be described as $\text{int} \cup \{Z_i \mid i \in I(R)\}$. By shifting the elements of R clockwise or reversing the order of the elements a different ring is obtained. So different rings may have the same ringset (and thus the same indexset). Clearly not every set $I \subset \{1, \ldots, N\}$ is an indexset of a ring, as generally it is not possible to order the elements of $\text{extr} \bigcup_{i \in I} Z_i$ such that a sequence $\bar{e}_1, \ldots, \bar{e}_v$ is obtained that satisfies condition (ii). Then $\bar{e}_i, \ldots, \bar{e}_{v+1}$ cannot be connected by a simple circuit or, in other words, $\text{int} \bigcup_{i \in I} Z_i$ is not a simply connected set, containing at least one "inner hole". A set I which is the indexset of a ring is called *regular*. For example $I = \emptyset$, $I = \{i\}$, and $I = \{i, \ldots, N\}$ are regular indexsets, and the corresponding ringsets are $\emptyset$, $\text{bd } Z_i$, and $\text{bd } \mathscr{X}$ respectively.

Now consider two rings $R = (e_1, \ldots, e_t)$ and $S = (f_1, \ldots, f_u)$, and let $Y(R) \neq Y(S)$. Define $e_{t+i} = e_i$ and $f_{u+i} = f_i$ for all integers i. If there exists an index y, and index z, and an integer $p(y)$ such that $e_{y+i} = f_{u-i}$ for $i = 0, \ldots, p(y)$, then R and S are said to *intersect* each other. If $p(y) \geqq 1$ the simple path from e_1 to e_t and the simple path from f_i to f_u have one or more edges in common; or, in other words, the two rings R and S share one or more common edges. With respect to R and e_y, the point f_u is called the *meeting point* of S and $[f_u, f_{u+1}]$ is called the *the meeting edge* of S. Finally, $f_{u-p(y)}$ is called the *leaving point* of S with respect to R. If the triple $(y, z, p(y))$ is unique and $p(y) \geqq 1$ then the two rings R and S wander off in any different directions, inside or outside each other, after sharing a few common edges. In that case, the *combination* of R and S, to denote by $R \,\&\, S$ and defined by the sequence

$$(e_1, e_2, \ldots, e_y = f_u, f_1, f_2, \ldots, f_{u-p(y)}, e_{y+p(y)+1}, \ldots, e_t)$$

clearly is a ring too, whereas $I(R \,\&\, S) = I(R) \cup I(S)$, holds. On the other hand, if the triple $(y, z, p(y))$ is not unique, $R \,\&\, S$ is a self-intersecting curve and not a ring as a consequence. If $R \,\&\, S$ is a ring, then for each $s = y, y+1, \ldots, y+p(y)-1$ there exists an index $i \in I(R)$ and an index $j \in I(S)$ such that Z_i and Z_j are neighbouring areas with $[e_s, e_{s+1}]$ as common boundary line. Then Z_j is called a neighbouring area of $\cup \{Z_i \mid i \in I(R)\}$.

An essential aspect of the construction algorithm is that a ring can be extended step by step. We state the following basic construction theorem.

Theorem 5 (Basic Construction Theorem). *Let* $R = (e_1, \ldots, e_t)$ *be a ring with*

$e_1 = (0, 0)$ and $I(R) \neq \{1, \dots, N\}$. Then there exists at least one ring S with $I(S) = \{j\}$ such that $R \,\&\, S$ is a ring.

Proof. Let M be the indexset of all neighbouring areas of $\bigcup_{i \in I(R)} Z_I$. Then $\emptyset \neq M \subset I^c = \{1, \dots, N\} \setminus I(R)$. It should be proved that

$$\bar{M} = \{j \in M \mid \operatorname{int} \cup \{Z_i \mid i \in I(R) \cup j\} \text{ is simply connected}\} \neq \emptyset.$$

Take an arbitrary $j_0 \in M$. If $j_0 \notin \bar{M}$, then there exists a nonempty indexset $J_1 \subset I^c \setminus \{j_0\}$ such that $\cup\{Z_i \mid i \in J_1\}$ is completely embraced by $\operatorname{int} \cup \{Z_i \mid i \in I(R) \cup \{j_0\}\}$. Since j_0 is a regular indexset, $M_1 = M \cap J_1 \neq \emptyset$ must hold. Now take $j_1 \in M_1$. If $j_1 \notin \bar{M}$, then there exists a nonempty indexset $J_2 \subset J_1 \setminus \{j_1\}$ such that $\cup\{Z_i \mid i \in J_2\}$ is completely embraced by $\operatorname{int} \cup \{Z_i \mid i \in I(R) \cup \{j_1\}\}$. Since $\{j_1\}$ is a regular indexset $M_2 = M \cap J_2 \neq \emptyset$ must hold, and so on. Since $|M| < N$ and M_{k+1} is a proper subset of M_k as long as $j_{k+1} \notin \bar{M}$, finally a $j_r \in M_r$ will be found such that $j_r \in \bar{M}$.

In fact Theorem 5 states that given any simple (bounded by a circuit) subregion of a decomposition of a rectangle into convex polyhedra, there is at least one adjacent polyhedron that can be added to the region to form another simple region. The algorithm to be described in Section 2 can be used to find one.

2. The construction method

In Section 1 we developed the logic of an algorithm to construct the extreme points of all areas Z_i, $i = 1, \dots, N$ together with the corresponding optimal basic solutions x_i^* of problem (P). We shall now describe the algorithm explicitly.

The following three subroutines are basic in the algorithm: PARALP, CROSS, and VERTEX. The subroutine PARALP is a linear programming code to deal with the case of one parameter in the objective function and is supplied with the standard anticycling technique based on a fixed perturbation which is maintained throughout the computation. The subroutine CROSS works exactly along the lines of Theorem 4. Given a simplex tableau T with corresponding area Z and boundary line $\bar{L}$ of $Z(\bar{L} \not\subset \operatorname{bd} \mathscr{X})$, CROSS generates a simplex tableau T' that determines the area Z' to which applies $Z' \cap Z = \bar{L}$. The line through $\bar{L}$ is called the *crossing edge*. The subroutine VERTEX produces the ring $\bar{S}$ of Z, where Z is an area known from a simplex tableau T as $\{(\lambda, \mu) \in \mathscr{X} \mid d_0 + \lambda d_1 + \mu d_2 \geqq 0\}$. In fact VERTEX produces a specific ring by prescribing the *starting point* f_1 together with the *starting edge* (the direction $f_2 - f_1$ determined by $[f_1, f_2]$).

Basically, given VERTEX the construction problem reduces itself to an organization problem with respect to the basic feasible solutions; if one is willing to generate all basic feasible solutions of (P) the subroutine yields all areas.

However, efficient computation of all basic feasible solutions (that is without computing the same basic feasible solution many times and without using a big computer memory space) is not easy. Moreover, one does not need all basic solutions, since generally only a small fraction gives rise to an area. Therefore, it is quite obvious that the already constructed areas should give information concerning the basic feasible solution to be considered next, likewise the one-parameter case. However, unlike the case of one-parametric programming, careful organization is needed to assure that no area is forgotten. In principle this may be realized by constructing all the neighbouring areas of one specific area; next all the neighbours of the neighbours and so on. However, the disadvantage of that method is that the relevant data of all simplex tableaux corresponding to those areas of which not all the neighbours have been constructed already, have to be stored. Moreover, it has to be checked whether one of the neighbours of some area has been found in an earlier stage as the neighbour of some other area. The time needed for this increases with the number of constructed areas and therefore this will be prohibitive for problems with many areas. We conclude that such a method is poor from both a computational and "book-keeping" point of view.

Utilizing the concept 'ring' as discussed in Section 1 it is guaranteed that no area is forgotten, whereas only a small number of simplex tableaux has to be stored. Moreover, it is ascertained that no areas are considered that have been added to the construction figure in an earlier stage. The number of multiple calculations of the same area, which cannot be avoided completely if one is not willing to store many simplex tableaux, is limited. In the following the basic steps of the algorithm are described. A flowchart is given in Fig. 1. Capitalized names used in the steps refer to names of subroutines. After explanation of the steps further comments are given.

Step 1. Initialize the set-up tableau of problem (P) and the bounded rectangle $\mathscr{X}$. If degeneracy is present then apply the standard (primal) lexicographic method. Set $R := \emptyset$, and CLOCK := true.

Step 2. Call subroutine START to find an optimal tableau $\bar{T}$ for $\lambda = 0$ and $0 \leqq \mu \leqq \bar{\mu}$, where $\bar{\mu} > 0$. $\bar{T}$ is chosen so that it determines an area $\bar{Z}$; if necessary, a small perturbation of $\lambda = 0$ is considered. The solution of $\bar{T}$ is denoted by x^*.

Step 3. Call subroutine VERTEX to find the ring $\bar{S} = (f_1, \dots, f_u)$ of the area $\bar{Z}$ determined by $\bar{T}$. If CLOCK = true, then the elements of $\bar{S}$ are found in the clockwise direction; otherwise the elements of $\bar{S}$ are found in the anti-clockwise direction. If $R = \emptyset$, then $(0, 0)$ is the starting point and $\{(\lambda, \mu) \mid \lambda = 0, \mu \geqq 0\}$ is the starting edge; otherwise the starting point is the current meeting point MP and the starting line is the current meeting edge ML.

Step 4. Call subroutine MEET to obtain the meeting point $\overline{\text{MP}}$ and the meeting edge $\overline{\mathrm{ML}}$ of $\bar{S}$ with respect to R and MP. If $R=\emptyset$, then $\overline{\mathrm{MP}}:=f_u$ and $\overline{\mathrm{ML}}:=[f_u, f_1]$.

Step 5. If $R=\emptyset$, then go to 6. Otherwise, determine f_a = leaving point of $\bar{S}$ with respect to R and f_1, and determine y such that $e_y=f_{a-1}$. Check whether $\bar{S}\,\&\,R$ is a ring by setting $p(y)=\max\{i \mid e_{y+i}=f_{u-i}\}$ and verifying the conditions $p(y)\geqq 1$, $f_j\neq e_i$ for all $i=1,\dots,t$ and all $j=1,\dots,u-p(y)-1$. If $\bar{S}\,\&\,R$ is not a ring, then go to 11.

Step 6. Call subroutine CONSTRUCT to add $\bar{Z}$ to the construction figure and print the corresponding solution x^*.

Step 7. If $R=\emptyset$, then $R:=\bar{S}$ else $R:=R\,\&\,\bar{S}$.

Step 8. Check whether $Y(R)=\mathrm{bd}\,\mathscr{X}$. If not, then go to 10.

Step 9. Stop, the construction figure has been filled up completely.

Step 10. Given ring $R=(e_1,\dots,e_t)$ determine e_q such that $e_q=(0,0)$. Next, determine $w=\min\{j \mid j\geqq q,\ [e_j,e_{j+1}]\not\subset \mathrm{bd}\,\mathscr{X}\}$ and $z=\min\{j \mid j\leqq q,\ [e_j,e_{j+1}]\not\subset \mathrm{bd}\,\mathscr{X}\}$. Set FRP $:=e_w$ and LRP $:=e_z$.

Step 11. Check whether $\overline{\mathrm{ML}}\subset \mathrm{bd}\,\mathscr{X}$. If not, then go to 16.

Step 12. Check whether $\overline{\mathrm{MP}}=\mathrm{LRP}$. If not, then go to 14.

Step 13. A construction round has been completed. Set CLOCK := not CLOCK, and reverse the order of the elements of R. Set LRP$'$:= FRP. Determine $q=\min\{j \mid j>q,\ [e_j,e_{j+1}]\not\subset \mathrm{bd}\,\mathscr{X}\}$. Set FRP$'$ $:=e_{j+1}$.

Step 14. Given $R=(e_1,\dots,e_t)$ determine q such that $e_q=\mathrm{MP}$. Let $W=\min\{j \mid j>q,\ [e_j,e_{j+1}]\not\subset \mathrm{bd}\,\mathscr{X}\}$. Set MP$'$ $:=e_{w+1}$ and ML$'$ $:=[e_w,e_{w+1}]$. Call subroutine PARALP to obtain tableau T' that is optimal for $(\lambda,\mu)=\mathrm{MP}'$, starting from the tableau $\bar{T}$ at hand. Note, that T' may be equal to $\bar{T}$.

Step 15. Set MP := MP$'$; ML := ML$'$; FRP := FRP$'$, and LRP := LRP$'$. Note that MP and ML are input data for the subroutine VERTEX that is called in Step 3.

Step 16. Apply subroutine CROSS on tableau $\bar{T}$ with as crossing edge $\overline{\mathrm{ML}}$ containing $\overline{\mathrm{MP}}$ to obtain tableau T' that determines the $\overline{\mathrm{ML}}$-neighbouring area Z' of $\bar{Z}$. The solution of T' is denoted by x'.

Step 17. Set MP := MP$'$ and ML := ML$'$. Note that MP and ML are input data for the subroutine VERTEX that is called in step 3.

Step 18. Set $\bar{T}:=T'$; $\bar{Z}:=Z'$, and $x^*:=x'$. Go to 3.

Comments

The direction of construction is reversed as soon as one construction round has been completed. This implies that the order of the elements of the ring at hand is reversed, and the order in which the elements $\bar{S}$ are obtained via VERTEX is changed from clockwise into anti-clockwise or vice-versa (step 13). FRP and LRP are defined as respectively the first ringpoint and the last ringpoint and are connected by segments of bd $\mathscr{X}$. When the direction of construction is reversed LRP and FRP are updated (step 13). Tableau $\bar{T}$, meeting edge $\overline{\mathrm{ML}}$, and meeting point $\overline{\mathrm{MP}}$ are used to enter the appropriate $\overline{\mathrm{ML}}$-neighbouring area of $\bar{Z}$ via

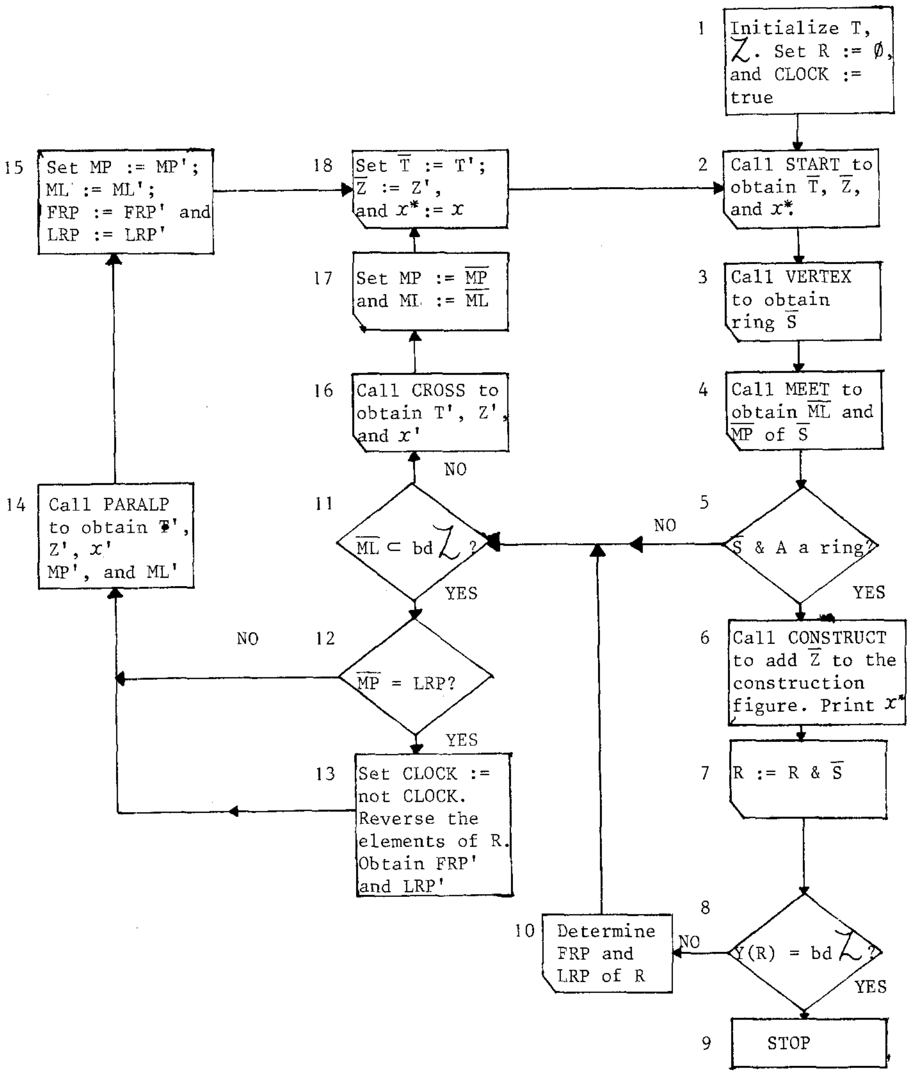

Fig. 1. Flowchart of the algorithm.

$\mathcal{Z} = \{0 \leqq \lambda \leqq \lambda_{MAX}, 0 \leqq \mu \leqq \mu_{MAX}\}$
R = ring of formerly constructed areas.
$Y(R)$ = ringset of R.
FRP = first ringpoint.
LRP = last ringpoint.
$\bar{Z}$ = just found area.
$\bar{T}$ = tableau of $\bar{Z}$.
x^* = solution of tableau $\bar{T}$.
$\bar{S}$ = ring of S.
$\overline{\text{MP}}$ = meeting point of $\bar{S}$.
$\overline{\text{ML}}$ = meeting edge of $\bar{S}$.

CLOCK = boolean, specifying the direction of search for the elements of $\bar{S}$.
Z' = area found via CROSS or PARALP.
T' = tableau of Z'.
x' = solution of tableau T'.
MP′ = meeting point found via PARALP.
ML′ = meeting edge found via PARALP.
MP = starting point for VERTEX.
ML = starting edge for VERTEX.
FRP′ = updated first ringpoint.
LRP′ = updated last ringpoint.

CROSS (step 16). However, if $\overline{\mathrm{ML}} \subset \mathrm{bd}\,\mathscr{X}$, CROSS cannot be applied in this way. By using PARALP and the knowledge of R it is then possible to continue with the next hole between R and $\mathrm{bd}\,\mathscr{X}$ (step 14). Another possibility to continue the search would have been saving a restart tableau for each separate hole between R and $\mathrm{bd}\,\mathscr{X}$. Finally, note that MP and ML are used in VERTEX (step 3) as starting point and starting direction, if the ring at hand is not empty.

Proof of the algorithm. With reference to the last part of Section 1 it is enough to state the central part of the proof. Suppose that the areas $Z_1, \dots, Z_n$ have been constructed already, such that a ring R_n with $I(R_n) = \{1, \dots, n\}$ is known. Theorem 5 states that there exists a neighbouring area Z_{n+1} of $\bigcup_{i=1}^{n} Z_i$ such that $\{1, \dots, n+1\}$ is regular too. The construction method considers the neighbours of $\bigcup_{i=1}^{n} Z_i$ in the order determined by R_n, and therefore an area must finally be found (if $R_n \neq \mathrm{bd}\,\mathscr{X}$). Since the partition of $\mathscr{X}$ into areas is unique (Theorem 1), and all areas are the convex hulls of a finite number of extreme points, and each extreme point of some area is an extreme point of every area to which it belongs, the verification of $Z \overset{?}{=}$ neighbour of $\bigcup_{i=1}^{n} Z_i$ can be performed unambiguously by working with R_n and a ring S with ringset $\mathrm{bd}\,Z$, or, in other words, by working with $\mathrm{extr} \bigcup_{i=1}^{n} Z_i$ and $\mathrm{extr}\,Z$. Since after a finite number of steps a new area is added to the construction figure, as long as not all areas have been constructed already, the algorithm terminates after a finite number of steps.

3. Numerical example

To illustrate how the construction method works, consider the following example.

$$
\begin{aligned}
\max \quad & f = -3x_1 + 2x_2 - 0.5x_3 - 0.5x_4 + \lambda(x_1 - x_2 + x_3 + x_4) \\
& \qquad + \mu(x_1 - x_2 + x_3 - x_4),
\end{aligned}
$$

$$
(\mathrm{P}^*) \qquad
\begin{aligned}
\text{s.t.} \quad x_1 - x_2 - x_3 - x_4 &\leqq 0, \\
-x_1 + x_2 - x_3 - x_4 &\leqq 0, \\
-x_1 - x_2 + x_3 - x_4 &\leqq 0, \\
-x_1 - x_2 - x_3 + x_4 &\leqq 0, \\
x_1 + x_2 + x_3 + x_4 &\leqq 1,
\end{aligned}
\qquad x_1, x_2, x_3, x_4 \geqq 0.
$$

Restrict the search to

$$\mathscr{X} = \{(\lambda, \mu) \in \mathbf{R}^2 \mid 0 \leqq \lambda \leqq 2.7, 0 \leqq \mu \leqq 4.8\}.$$

Denote the slack variables by $x_5, x_6, \dots, x_9$.

It can be shown that P* satisfies the regularity conditions (1), (2) and (3). As degeneracy is present the standard primal lexicographic method is applied, based on a fixed perturbation that is maintained throughout the computation. The

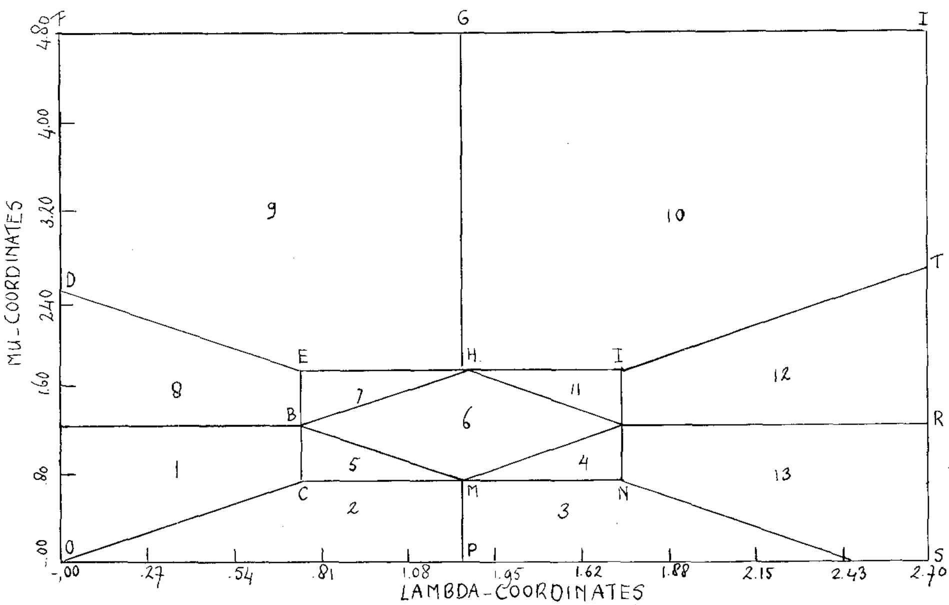

Fig. 2. Construction of P^*.

resulting construction figure of P* is shown in Fig. 2. The coordinates of the points A, ... , T are:

A = (0.00, 1.25);	B = (0.75, 1.25);	C = (0.75, 0.75);
D = (0.00, 2.50);	E = (0.75, 1.75);	F = (0.00, 4.80);
G = (1.25, 4.80);	H = (1.25, 1.75);	I = (2.70, 4.80);
J = (1.75, 1.75);	K = (1.25, 1.25);	L = (1.75, 1.25);
M = (1.25, 0.75);	N = (1.75, 0.75);	O = (0.00, 0.00);
P = (1.25, 0.00);	Q = (2.50; 0.00),	R = (2.70, 1.25);
S = (2.70, 0.00);	T = (2.70, 3.00).	

The construction figure of P* consists of 13 areas Z_i (with ring S_i) with corresponding solutions x_i^* as shown in Table 1.

Table 2 shows the order in which the areas are found by the construction method. After step 2 the first construction round has been completed and the order of the elements of R is reversed, whereas the elements of ring S are generated in the anti-clockwise direction. After step 9 the second construction round has been completed and again the order of the elements of R is reversed, whereas the elements of S are now generated in the clockwise direction, as was the case in the start. Note, that at step 5 an area is found (Z_6) that cannot be added to the construction figure because the combination of its rings (S_6) and the ring at hand is not a ring again. In Fig. 2 the areas Z_1 and Z_8 complete the set of parameter values for which the degenerate extreme point $(0, \frac{1}{2}, \frac{1}{2}, 0, 1, 1, 0, 0, 0)$ is

Table 1
Areas of P*

Area Z_i	Ring S_i	Nonbasic variables
Z_1	O, A, B, C	1, 6, 9, 4
Z_2	O, C, M, P	1, 6, 9, 8
Z_3	P, M, N, Q	3, 6, 9, 8
Z_4	M, N, L	5, 6, 3, 8
Z_5	M, B, C	1, 3, 6, 4
Z_6	B, M, L, H	1, 2, 3, 4
Z_7	B, H, E	5, 6, 4, 7
Z_8	B, E, D, A	1, 7, 9, 4
Z_9	E, D, F, G, H	5, 7, 9, 4
Z_{10}	H, G, I, T, J	5, 7, 9, 2
Z_{11}	J, H, L	5, 7, 2, 8
Z_{12}	L, J, T, R	5, 8, 9, 2
Z_{13}	L, R, S, Q, N	5, 8, 9, 3

optimal; Z_4, Z_5, Z_6, Z_7, and Z_{11} complete the set for which the degenerate extreme point $(0,0,0,0,0,0,0,0,1)$ is optimal; Z_9 and Z_{10} complete the set for which the degenerate extreme point $(\frac{1}{2},0,\frac{1}{2},0,0,1,1,0,0)$ is optimal; Z_{12} and Z_{13} complete the set for which the degenerate extreme point $(\frac{1}{2},0,0,\frac{1}{2},0,0,1,1,0)$ is optimal, and finally Z_2 and Z_3 complete the set for which the degenerate extreme point $(0,\frac{1}{2},0,\frac{1}{2},0,0,1,1,0)$ is optimal. One has to inspect the corresponding basic solutions to figure out what areas complete the set of parameter values for which one degenerate extreme point is optimal.

4. Remarks

(A) The algorithm and its proof are based on the regularity conditions. We shall investigate how restrictive they are.

(1) If $F=\emptyset$, then the problem is meaningless. If $F\neq\emptyset$ but $\operatorname{int} F=\emptyset$, then there is no serious complication, since the relative interior of F must be nonempty. For example, when $F=\{x\in\mathbf{R}^n \mid Cx=d,\ x\geqq 0\}$ one should eliminate some variables to make sure that $\operatorname{int} F\neq\emptyset$.

(2) If F is unbounded, no complications occur unless for some values of (λ,μ) the problem (P) is unbounded. In that case it is possible to define the construction figure as the (proper) subset of $\mathscr{X}_1$ of $\mathscr{X}$ corresponding with values of (λ,μ) for which (P) has a finite solution. It is possible to adjust the algorithm to meet the boundary of $\mathscr{X}_1$. However, one should note that no unbounded solutions occur in well-formulated practical problems.

(3) If c_1 and c_2 are dependent the number of parameters should be decreased.

(B) A numerical difficulty of the algorithm, as well as for many other iterative procedures, is the possibility of rounding errors. In this algorithm such errors

Table 2
Working of the construction method

Step	Ring R at hand	FRP	LRP	Area found	Ring S	New ring R
1	–	–	–	Z_1	O, A, B, C, O	O, A, B, C, O
2	O, A, B, C, O	A	O	Z_2	O, C, M, P, O	O, A, B, C, M, P, O
3	P, M, C, B, A, O, P	P	A	Z_3	M, P, Q, N, M	Q, N, M, C, B, A, O, P, Q
4	Q, N, M, C, B, A, O, P, Q,	Q	A	Z_4	M, N, L, M	Q, N, L, M, C, B, A, O, P, Q
5	Q, N, L, M, C, B, A, O, P, Q	Q	A	Z_6	M, L, H, B, M	Q, N, L, M, C, B, A, O, P, Q
6	Q, N, L, M, C, B, A, O, P, Q	Q	A	Z_5	B, M, C, B	Q, N, L, M, B, A, O, P, Q
7	Q, N, L, M, B, A, O, P, Q	Q	A	Z_6	B, M, L, H, B	Q, N, L, H, B, A, O, P, Q
8	Q, N, L, H, B, A, O, P, Q	Q	A	Z_7	B, H, E, B	Q, N, L, H, E, B, A, O, P, Q
9	Q, N, L, H, E, B, A, O, P, Q	Q	A	Z_8	B, E, D, A	Q, N, L, H, E, D, A, O, P, Q
10	D, E, H, L, N, Q, P, O, A, D	D	Q	Z_9	E, D, F, G, H, E	G, H, L, N, Q, P, O, A, D, F, G
11	G, H, L, N, Q, P, O, A, D, F, G,	G	Q	Z_{10}	H, G, I, T, J, H	T, J, H, L, N, Q, P, O, A, D, F, G, I, T
12	T, J, H, L, N, Q, P, O, A, D, F, G, I, T	T	Q	Z_{11}	H, J, L, H	T, J, L, N, Q, P, O, A, D, F, G, I, T
13	T, J, L, N, Q, P, O, A, D, F, G, I, T	T	Q	Z_{12}	J, L, R, T, J	R, L, N, Q, P, O, A, D, F, G, I, T, R
14	R, L, N, Q, P, O, A, D, F, G, I, T, R	R	Q	Z_{13}	L, R, S, Q, N, L	S, Q, P, O, A, D, F, G, I, T, R, S

may become fatal because ring points serving as orientation points cannot be recognized. Two possibilities may arise:

(1) Two points can be found to be equal while they are not.

(2) Two points can be found to be unequal while they are equal.

In our practical experience only once in a great number of calculations did problem 1 arise; problem 2 never did.

Acknowledgment

We are grateful to a referee for drawing our attention to the degenerate case and to J. Ponstein for helping to find the example of Section 3.

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