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Conformally Homogeneous Model Universes - I (*).

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Summary. — Intrinsically inhomogeneous cosmological models are investigated that have the same « free gravitational fields » and the same conformal motions as the classical, spatially homogeneous, isotropic models. The present models are conformally equivalent to the classical ones and allow the physical interpretations of the classical models to be utilized. Exact equations are obtained which relate the salient physical parameters and which are equivalent to the Einstein field equations. The velocity vector (of energy transport) is shown to be the sum of a multiple of the velocity vector of the corresponding homogeneous model and a dispersion vector (relative to the homogeneous model). Exact solutions for the dispersion vector are obtained. If squares of the derivatives of the conformal coefficient, ψ, are neglected, explicit expressions for the density and pressure are obtained. The conformal factor is shown to satisfy a diffusion equation—$\nabla^2\psi = 3R^2(\dot{R}R^{-1}\partial_t\psi + \Lambda)$—when $'\varrho/\varrho = e^{-\psi}$ ($'\varrho$ denotes the density of the present models). For Friedman models with $k = +1$, unique spatial scales of inhomogeneity are obtained. Each scale of inhomogeneity decays with time with the relaxation time increasing with the scale, and the dispersion vector vanishes only at points where $'\varrho$ has a spatial maximum or minimum. Exact and approximate expressions are obtained for the expansion parameter.

1. – Introduction.

Complete satisfaction with (spatially) homogeneous isotropic modeling of the universe has waned in recent years. This is due in part to the inability to account for a time interval of sufficient duration for the formation of the observed aglomerations of matter when classical models are used as an underlying substratum. The problems of modeling an intrinsically inhomogeneous universe must consequently be removed from the domain of academic specula-

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tion and squarely faced. A number of analyses and methods have been aimed in this direction; to sight a few instances: BONNER (1), Newtonian model perturbations; IRVINE (2), Newtonian approximation in relativistic models: LIFSHITZ (3) and LIFSHITZ and KHALATNIKOV (4), metric perturbations in relativistic models: HAWKING (5), perturbations of the conformal tensor: CHANDRASEKHAR (6), post Newtonian approximations: EDELEN (7), metric and energy perturbations of conformally semistatic spaces.

To a greater or lesser extent the above analyses and methods ignore an aspect of the homogeneous isotropic models which underlies the success of these models in accounting for the properties of the universe « in the large »: the classical models are conformally flat (have vanishing free gravitational fields) and hence admit the maximal number of conformal motions, namely 15 (TAKENO (8)). The isotropy and homogeneity is then just the restriction that these conformal motions contain a subset which form a 6-parameter group of « metric » motions (isometries) isomorphic to the group of motions of a 3-dimensional Euclidean space. The approach taken here is to study model universes which are conformal transformations of homogeneous isotropic models. In this way we retain the same 15 independent conformal motions and the same free gravitational fields as exhibited by the homogeneous models, losing only the subgroup of isometries. We consider conformally homogeneous models in their full generality and obtain the exact equations in terms of the salient physical quantities. An approximation is then made in which products of derivatives of the conformal coefficient are neglected. This allows us to show that conformally homogeneous models equivalent to Freidman models with $k = +1$ are such that each scale of inhomogeneity is decaying during an expansion phase and the relaxation times for the various scales of inhomogeneity, which are intrinsically defined, increase with the scale size. We also show that a dispersion velocity always exists; it points toward the points of maximum density, and vanishes only where the density is either maximal or minimal.

2. – The classical models.

We collect here those results from the classical, isotropic, homogeneous cosmological models that are required for reference or for comparison. The

(1) W. B. BONNOR: *Month. Not.*, **117**, 104 (1957).
(2) W. IRVINE: *Ann. of Phys.*, **32**, 322 (1965).
(3) E. M. LIFSHITZ: *Journ. Phys. U.S.S.R.*, **10**, 116 (1946).
(4) E. M. LIFSHITZ and I. N. KHALATNIKOV: *Adv. in Phys.*, **12**, 185 (1963).
(5) S. W. HAWKING: *Ap. Journ.*, **145**, 544 (1966).
(6) S. CHANDRASEKHAR: *Ap. Journ.*, **142**, 1488 (1965).
(7) D. G. B. EDELEN: *Nuovo Cimento*, **43** B, 1095 (1966).
(8) H. TAKENO: *The Theory of Spherically Symmetric Space-Times* (Hiroshima, 1963).

reader is referred to ROBERTSON (9) or HECKMANN and SCHÜCKING (10) for details; in particular, to Robertson's comments concerning the purpose and scope of the homogeneous isotropic modeling of the universe.

Let $\mathscr{E}$ denote the class of Einstein-Riemann spaces admissible in homogeneous isotropic cosmologies and let $\boldsymbol{g}(x^\gamma)$ denote the metric tensor. We then have

$$\mathrm{d}s(\boldsymbol{g})^2 = g_{\alpha\beta}(x^\gamma)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \stackrel{*}{=} \mathrm{d}t^2 - R(t)^2\,\mathrm{d}u(\boldsymbol{L})^2\,, \qquad \alpha, \beta, \gamma = 0, 1, 2, 3\,, \tag{1}$$

where $\stackrel{*}{=}$ denotes evaluation in a co-ordinate system in which the matter in the model universe is at rest (comoving co-ordinates), the value of the speed of light in vacuo is unity, and

$$\mathrm{d}u(\boldsymbol{L})^2 = L_{ij}(x^k)\,\mathrm{d}x^i\mathrm{d}x^j \stackrel{*}{=} \left(1+\frac{k}{4}r^2\right)^{-2}\left((\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2\right)\,, \tag{2}$$

$$\left(r^2 = (x^1)^2+(x^2)^2+(x^3)^2\right)\,, \qquad i, j = 1, 2, 3\,,$$

is the metric tensor on a three-dimensional space $\mathscr{S}$ of constant curvature $k = (-1, 0, +1)$. The momentum-energy tensor $\boldsymbol{T}$ associated with any element of $\mathscr{E}$ is

$$T_{\alpha\beta} = (\varrho + p)\quad W_\alpha\, W_\beta - p g_{\alpha\beta}\,, \tag{3}$$

where

$$W^\alpha \stackrel{*}{=} \delta^\alpha_0 (W^\alpha\, W_\alpha = 1) \tag{4}$$

is the velocity vector and ϱ and p are the density and (Poincaré) pressure, respectively. If $\boldsymbol{G}(\boldsymbol{g})$ denotes the Einstein tensor formed from $\boldsymbol{g}$, the physics and the geometry are related by

$$G_{\alpha\beta}(\boldsymbol{g}) + \Lambda g_{\alpha\beta} = -\varkappa T_{\alpha\beta}\,, \tag{5}$$

where Λ is the cosmological constant. When a *–co-ordinate system is used, these equations yield

$$\left\{\begin{aligned} &\varkappa\varrho \stackrel{*}{=} \Lambda + 3(k+\dot{R}^2)R^{-2}\,,\\ &\varkappa p \stackrel{*}{=} \Lambda - 2\ddot{R}R^{-1} - (k+\dot{R}^2)R^{-2}\,,\\ &R\dot{\varrho} + 3\dot{R}(\varrho+p) \stackrel{*}{=} 0\,,\end{aligned}\right. \tag{6}$$

(9) H. P. ROBERTSON: *Rev. Mod. Phys.*, **5**, 62 (1933).

(10) O. HECKMANN and E. SCHÜCKING: *Gravitation: An Introduction to Current Research* (New York, 1962).

and the nonvanishing Christoffel symbols of the second kind are

$$(7)\qquad \begin{cases} \Gamma^i_{jk}(\boldsymbol{g}) \overset{*}{=} \Gamma^i_{jk}(\boldsymbol{L})\,, \\ \Gamma^0_{ij}(\boldsymbol{g}) \overset{*}{=} R\dot{R}L_{ij}\,, \qquad \Gamma^i_{0j}(\boldsymbol{g}) = \dot{R}R^{-1}\delta^i_j\,. \end{cases}$$

All elements of $\mathscr{E}$ are spherically symmetric and conformally flat, and hence admit the maximal number of conformal motions, namely 15 (TAKENO [8]). By a conformal motion, we mean a generating vector field $\xi^\alpha(x^\gamma)$ such that

$$\pounds(\boldsymbol{\xi})(g_{\alpha\beta}) \equiv \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = \varphi(x^\gamma) g_{\alpha\beta}$$

for some bounded function $\varphi(x^\gamma)$. Takeno's [8] detailed work gives a full and explicit account of the conformal motions as well as the groups of motions (Killing fields) admitted by the various elements of $\mathscr{E}$. In addition, TAKENO gives the following characterization of the classical cosmological line elements: *The elements of $\mathscr{E}$ are those hyperbolic-normal metric spaces which are conformally flat and such that there exists a vector field $\boldsymbol{v}(x^\gamma)$ for which*

$$\nabla_\alpha v_\beta = \tfrac{1}{4}(\nabla_\nu v^\nu) g_{\alpha\beta}\,, \qquad v^\alpha v_\alpha > 0\,,$$

i.e. the space admits a conformal motion v^α with

$$\varphi = \tfrac{1}{2}\nabla_\nu v^\nu \qquad \text{and} \qquad \nabla_\alpha v_\beta = \nabla_\beta v_\alpha\,.$$

With eqs. (4) and (7) and the definitions of the expansion $\theta = \nabla_\alpha W^\alpha$ and the rotation

$$\omega_{\alpha\beta} = \tfrac{1}{2}(\delta^\mu_\alpha - W_\alpha W^\mu)(\delta^\nu_\beta - W_\beta W^\nu)(\nabla_\mu W_\nu - \nabla_\nu W_\mu)\,,$$

we have

$$(8)\qquad \theta = 3\dot{R}R^{-1}\,, \qquad \omega_{\alpha\beta} = 0\,.$$

Hence, the Hubble parameter, H, is given by

$$(9)\qquad H = \dot{R}R^{-1} = \tfrac{1}{3}\theta\,.$$

3. – Similarly and inhomogeneity.

In view of the fundamental equivalence of geometry and physics stated by the Einstein field equations, physical inhomogeneities, such as the agglomeration of matter into galaxies and clusters of galaxies, imply spatial geometric inhomogeneities. The simplest manner of introducing such inhomogeneities, and one which is consistent with the idea that the inhomogeneous can be viewed as bumps, so to speak, on a homogeneous substratum, is that in which the spatial geometry of the inhomogeneous case is similar to that of the homogeneous case. The factor of proportionality would then describe the inhomogeneities through its variation from point to point. If there are spatial geometric inhomogeneities, however, the corresponding physical inhomogeneities, would result in inhomogeneities in the energy density, and these in turn would imply inhomogeneities in the proper-time rate. Since we must thus consider both spatial and temporal inhomogeneities, we are led to consider inhomogeneous cosmological models that are *conformally homogeneous*: the (4-dimensional) metric geometry of the inhomogeneous models is similar to the geometry of the corresponding homogeneous models. The factor of proportionality (conformal factor) describes all of the efforts of the inhomogeneities under consideration.

Above and beyond the arguments of both mathematical and physical simplicity, conformally homogeneous models possess a unique property which singles them out from all other possible inhomogeneous models. It is known that the complete curvature tensor of a metric space (a primitive observable in contrast to the metric tensor) can be decomposed uniquely into a sum of two tensors, one of which is the Weyl conformal curvature tensor and the other is a tensor that is uniquely determined by the metric tensor and the Ricci tensor (the momentum-energy tensor). Now, the Weyl tensor is that part of the curvature that is not determined locally by the matter (the momentum-energy tensor) and may thus be viewed as representing the «free gravitational field» (JORDAN, EHLERS, and KNUDT ([11])); further, the Weyl tensor is the unique curvature invariant under conformal changes in the metric tensor. It thus follows that conformally homogeneous models describe inhomogeneous distributions of matter whose free gravitational fields are identical with the free gravitational fields of the corresponding homogeneous models. As previously noted, the classical models are conformally flat and hence admit the full set of 15 conformal motions, and this fact, to a great extent, determines a majority of the observable properties of these models (HAWKING ([5])). In this respect, the conformally homogeneous models admit the same 15 conformal motions and are the *unique* models whose generating vectors of conformal motions are identical with those of the homogeneous models (TAKENO ([8]), p. 119). The conformally homogeneous models considered here may thus be expected to preserve the agreement between those predictions of the classical theory and the observables which do not depend on the inhomogeneities and yet include the freedom to model the actual agglomerations of matter in the universe.

([11]) P. JORDAN, J. EHLERS and W. KUNDT: *Abh. Akad. Wiss. Mainz*, no. 7 (1960).

Let $\mathscr{C}$ denote the class of Einstein-Riemann spaces that are conformally equivalent to elements of $\mathscr{E}$, and let $\boldsymbol{h}$ denote the metric tensor of such spaces. We then have

$$h_{\alpha\beta}(x^\gamma) = \exp[\psi(x^\gamma)]g_{\alpha\beta}(x^\gamma)\,, \tag{10}$$

where

$$\mathrm{d}s(\boldsymbol{h})^2 = h_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \exp[\psi]\,\mathrm{d}s(\boldsymbol{g})^2\,,$$

is the fundamental metric form and ψ is the conformal coefficient. The structure of such spaces is well known and will be used without further notice (see, for instance, SYNGE ([12]), SCHOUTEN ([13])). Let ∇ denote covariant differentiation based on $\Gamma^\alpha_{\beta\gamma}(\boldsymbol{g})$ (covariant differentiation in the corresponding homogeneous model) and let ∂ denote co-ordinate differentiation. We have

$$\begin{cases} \Gamma^\alpha_{\beta\gamma}(\boldsymbol{h}) = \Gamma^\alpha_{\beta\gamma}(\boldsymbol{g}) + \frac{1}{2}(\delta^\alpha_\beta\partial_\gamma\psi + \delta^\alpha_\gamma\partial_\beta\psi - g^{\alpha\lambda}g_{\beta\gamma}\partial_\lambda\psi)\,,\\ G_{\alpha\beta}(\boldsymbol{h}) = G_{\alpha\beta}(\boldsymbol{g}) + \nabla_\alpha\nabla_\beta\psi - \frac{1}{2}\partial_\alpha\psi\,\partial_\beta\,\psi - g_{\alpha\beta}\,g^{\gamma\lambda}(\nabla_\gamma\nabla_\lambda\psi + \frac{1}{4}\partial_\gamma\psi\,\partial_\lambda\psi)\,, \end{cases} \tag{11}$$

where $G_{\alpha\beta}(\boldsymbol{h})$ denotes the Einstein tensor formed from $h_{\alpha\beta}$. The field equations give

$$G_{\alpha\beta}(\boldsymbol{h}) = -\varkappa\,{}'T_{\alpha\beta} - {}'\Lambda h_{\alpha\beta}\,, \tag{12}$$

where ${}'T_{\alpha\beta}$ and ${}'\Lambda$ are the momentum-energy tensor and the cosmological constant of the conformally homogeneous model. It then follows from eq,. (3), (5), (10), (11) and (12) that

$$\begin{aligned} \varkappa\,{}'T_{\alpha\beta} = \varkappa(\varrho + p)\,W_\alpha\,W_\beta - \nabla_\alpha\nabla_\beta\psi + \tfrac{1}{2}\partial_\alpha\psi\,\partial_\beta\psi\, + \\ + \{\Lambda - {}'\Lambda e^\psi - \varkappa p + g^{\gamma\lambda}(\nabla_\gamma\nabla_\lambda\psi + \tfrac{1}{4}\partial_\gamma\psi\partial_\lambda\psi)\}g_{\alpha\beta}\,. \end{aligned} \tag{13}$$

4. – The governing equations.

We would like to assume that the momentum-energy tensors associated with conformally homogeneous models are representative of a perfect fluid, since this assumption is essential for physical interpretations in the homogeneous models. The perfect fluid momentum-energy tensor is, however, so structured that it has a simple eigenvalue yielding a timelike eigenvector and an eigenvalue of multiplicity three yielding an isotropic 3-space of spacelike

([12]) J. L. SYNGE: *Relativity: The General Theory* (Amsterdam, 1960).
([13]) J. A. SCHOUTEN: *Ricci-Calculus* (Berlin, 1954).

eigenvectors. For the conformally homogeneous models, the terms $\nabla_\alpha \nabla_\beta \psi$ and $\partial_\alpha \psi \partial_\beta \psi$ occurring in eq. (13) will in general obviate the possibility of such an eigenvalue and eigenvector structure for $'T_{\alpha\beta}$. We must therefore take

$$(14) \qquad 'T^{\alpha\beta} = ('\varrho + '\varrho)'W^{\alpha}\,'W^{\beta} - 'p h^{\alpha\beta} + Q^{\alpha\beta}, \qquad Q^{\alpha\beta} = Q^{\beta\alpha},$$

where $Q^{\alpha\beta}$ is to be determined. In order to insure that $'\varrho$ is still the eigenvalue corresponding to the timelike eigenvector $'W^\alpha$, we require

$$(15) \qquad Q^{\alpha\beta}\,'W^{\gamma} h_{\beta\gamma} = 0 .$$

Under these conditions, $'\varrho$ and $'\boldsymbol{W}$ can be identified with the total rest energy and the velocity vector of energy transport, respectively (EDELEN [14]). Conformally homogeneous models thus not only allow us to consider more general dynamical processes than those of a perfect fluid, they demand it. This is to the advantage, however, in view of the scope and energy of galactic magnetic fields (WOLTJER [15]) which are necessarily neglected in the homogeneous models. In this respect, it is important to note that $'\varrho$ is the density of total rest energy, and electromagnetic energy in the space as well as all other forms of energy other than free gravitational energy. Accordingly, $'\boldsymbol{W}$ is the velocity field of energy transport, not just of mass transport.

Since $\boldsymbol{W}$ and $'\boldsymbol{W}$ are both timelike with respect to their corresponding metric tensors and the conformal transformation relating these metric tensors maps null cones onto null cones, we may write

$$(16) \qquad 'W^\alpha = \lambda W^\alpha + V^\alpha, \qquad V^\alpha W^\beta h_{\alpha\beta} = 0 = V^\alpha W^\beta g_{\alpha\beta},$$

where λ and V^α are to be determined. With this decomposition we relate the velocity vectors of homogeneous and conformally homogeneous models and interpret V^α as the *dispersion vector* relative to the corresponding homogeneous model. In this way we can make physical sense of the conformally homogeneous models even though there are, in general, no isotropic homogeneous comoving co-ordinate systems on elements of $\mathscr{C}$. Because $'\boldsymbol{W}$ is the velocity vector of energy transport in $\mathscr{C}$, we must have $'W^\alpha\,'W^\beta h_{\alpha\beta} = 1$, and hence eqs. (4), (10), and (16) give

$$(17) \qquad \lambda = \sqrt{(e^{-\psi} + V^2)}, \qquad V^2 = -g_{\alpha\beta} V^\alpha V^\beta \geqslant 0 .$$

(The sign of the radical which evaluates λ is determined by the condition that

[14] D. G. B. EDELEN: *Nuovo Cimento*, **30**, 292 (1963).
[15] L. WOLTJER: *The Structure and Evolution of Galaxies* (London, 1965).

both $\boldsymbol{W}$ and $'\boldsymbol{W}$ point into the future null cone.) Combining our results, we have

$$(18)\qquad 'T_{\alpha\beta} = -'pe^{\psi}g_{\alpha\beta} + Q_{\alpha\beta}e^{2\psi} + \\ + ('\varrho + 'p)e^{2\psi}\{\lambda^2 W_\alpha W_\beta + V_\alpha V_\beta + \lambda(W_\alpha V_\beta + W_\beta V_\alpha)\},$$

$$(19)\qquad (\lambda W_\beta + V_\beta)Q^{\alpha\beta} = 0,$$

where

$$V_\alpha \equiv V^\beta g_{\beta\alpha}, \qquad Q_{\alpha\beta} \equiv Q^{\mu\nu} g_{\mu\alpha} g_{\nu\beta}.$$

We can now obtain the governing equations. Define the quantities $\Box\psi$ and χ by

$$(20)\qquad \Box\psi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi, \qquad \chi = -g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi,$$

then eqs. (13) and (18) give

$$(21)\qquad \varkappa('\varrho + 'p)e^{2\psi}\{\lambda^2 W_\alpha W_\beta + V_\alpha V_\beta + \lambda(W_\alpha V_\beta + W_\beta V_\alpha)\} - \\ -\varkappa' pe^{\psi}g_{\alpha\beta} + \varkappa e^{2\psi}Q_{\alpha\beta} = \varkappa(\varrho + p) W_\alpha W_\beta - \nabla_\alpha\nabla_\beta\psi + \tfrac{1}{2}\partial_\alpha\psi\partial_\beta\psi + \\ + (\Lambda - '\Lambda e^{\psi} - \varkappa p + \Box\psi - \tfrac{1}{4}\chi\} g_{\alpha\beta},$$

which, together with eqs. (3) and (5), are equivalent to the Einstein field equations. If we refer the model to a *–co-ordinate system of the corresponding homogeneous model and combine our previous results, we have

$$(26)\qquad V^0 \stackrel{*}{=} V_0 \stackrel{*}{=} 0, \qquad V_i \stackrel{*}{=} -R^2 L_{ij} V^j, \qquad V^2 = R^2 L_{ij} V^i V^j,$$

$$(27)\qquad 'W_0 \stackrel{*}{=} \lambda e^{\psi}, \qquad 'W_i = -R^2 e^{\psi} L_{ij} V^j, \qquad 'W^0 \stackrel{*}{=} \lambda, \qquad 'W^i \stackrel{*}{=} V^i,$$

$$(28)\qquad \begin{cases} Q^{00} \stackrel{*}{=} \lambda^{-2} Q^{ij} V_i V_j, \qquad Q^{0i} \stackrel{*}{=} -\lambda^{-1} Q^{ij} V_j \stackrel{*}{=} \lambda^{-1} R^2 Q^{ij} L_{jk} V^k, \\ Q_{00} \stackrel{*}{=} \lambda^{-2} Q^{ij} V_i V_j \stackrel{*}{=} \lambda^{-2} R^4 L_{ij} Q^{jk} L_{km} V^i V^m, \\ Q_{0i} \stackrel{*}{=} \lambda^{-1} R^2 L_{ij} Q^{jk} V_k \stackrel{*}{=} -\lambda^{-1} R^4 L_{ij} Q^{jk} L_{km} V^m, \end{cases}$$

$$(29)\qquad \begin{cases} \Box\psi \stackrel{*}{=} \partial_0^2\psi - R^{-2} L^{ij}(\partial_i\partial_j\psi - \Gamma^k_{ij}(\boldsymbol{L})\partial_k\psi - R\dot{R}L_{ij}\partial_0\psi), \\ \chi \stackrel{*}{=} R^{-2} L^{ij}\partial_i\psi\partial_j\psi - (\partial_0\psi)^2. \end{cases}$$

The system (21) then becomes

$$(30)\qquad \varkappa\{('\varrho + 'p)e^{2\psi}\lambda^2 - \varrho - e^{\psi}{}'p\} + '\Lambda e^{\psi} - \Lambda + \\ + \varkappa\lambda^{-2} Q^{ij} V_i V_j e^{2\psi} \stackrel{*}{=} -\partial_0^2\psi + \tfrac{1}{2}(\partial_0\psi)^2 + \Box\psi - \tfrac{1}{4}\chi,$$

for $\alpha = \beta = 0$,

$$(31)\qquad -\varkappa(`\varrho + `p)e^{2\psi}\lambda V^j L_{ji} R^2 - \varkappa\lambda^{-1} R^4 L_{ij} Q^{jk} L_{km} V^m e^{2\psi} \stackrel{*}{=} \stackrel{*}{=} -\partial_0\partial_i\psi + (\dot{R}R^{-1} + \tfrac{1}{2}\partial_0\psi)\partial_i\psi ,$$

for $\alpha = 0$, $\beta = i$, and

$$(32)\qquad \varkappa(`\varrho + `p)e^{2\psi}R^4 V^k V^m L_{ki} L_{mj} + \varkappa(`pe^{\psi} - p)R^2 L_{ij} + (\Lambda - `\Lambda e^{\psi})R^2 L_{ij} + \varkappa R^4 e^{2\psi} L_{jk} L_{jm} Q^{km} \stackrel{*}{=} \stackrel{*}{=} -\partial_i\partial_j\psi + \Gamma^k_{ij}(\boldsymbol{L})\partial_k\psi + R\dot{R}L_{ij}\partial_0\psi + \tfrac{1}{2}\partial_i\psi\partial_i\psi - R^2 L_{ij}(\Box\psi - \tfrac{1}{4}\chi) ,$$

for $\alpha = i$, $\beta = j$, where the quantities ϱ, p, R, and Λ must satisfy the system (6) and L_{ij} is given by (2). When the value $\lambda = \sqrt{(e^{-\psi} + V^2)}$ is substituted into (30) we obtain

$$(33)\qquad \varkappa\{`\varrho e^{\psi} - \varrho + V^2(`\varrho + `p)e^{2\psi}\} + `\Lambda e^{\psi} - \Lambda + \varkappa\lambda^{-2}e^{2\psi}Q^{ij}V_i V_j \stackrel{*}{=} \stackrel{*}{=} -\partial_0^2\psi + \tfrac{1}{2}(\partial_0\psi)^2 + \Box\psi - \tfrac{1}{4}\chi .$$

This system of equations, although equivalent to the field equations, gives direct relations between the salient physical quantities, the conformal coefficient, and the known quantities of the corresponding homogeneous model.

5. – Dispersion vectors.

The decomposition given by (16) allows us to represent the velocity vector $`\boldsymbol{W}$ of the conformally homogeneous model in terms of the velocity vector $\boldsymbol{W}$ of the corresponding homogeneous model and the *dispersion vector* $\boldsymbol{V}$. When (31) is used, we obtain

$$(34)\qquad \varkappa\left\{Q^{jk}L_{km} + \frac{\lambda^2}{R^2}(`\varrho + `p)\delta^j_m\right\}V^m \stackrel{*}{=} \frac{\lambda e^{-2\psi}}{R^4}\{\partial_0\partial_i\psi - (\dot{R}R^{-1} + \tfrac{1}{2}\partial_0\psi)\partial_i\psi\}L^{ij} ,$$

and hence V^m is uniquely determined provided $(-\lambda^2/R^2)(`\varrho + `p)$ is not an eigenvalue of $Q^{jk}L_{km}$. We henceforth confine our attention to *nondegenerate* models; namely those for which $(-\lambda^2/R^2)(`\varrho + `p)$ is not an eigenvalue of $Q^{ik}L_{km}$, and hence *the dispersion vector is uniquely determined.*

With Λ, ϱ, p, R and L_{ij} known from the corresponding homogeneous model, the system (31) through (33) gives 10 independent equations in the 13 unknowns $`\varrho$, $`p$, $`\Lambda$, ψ, V^i, Q^{ij}. We are thus at liberty to assign 3 more conditions if a deterministic system is to result. The simplest such system is

$$(35)\qquad Q^{ij}V_j \stackrel{*}{=} -R^2 Q^{ij} L_{jk} V^k \stackrel{*}{=} 0 ,$$

which implies, in view of $V^0 = V_0 \overset{*}{=} 0$, (15), and (16), that $Q^{\alpha\beta} V_\beta = 0$, $Q^{\alpha\beta} W_\beta = 0$. A model satisfying the conditions (35) will be said to be *normal.* The condition (35) results in the desirable situation in which the compensating field $Q^{\alpha\beta}$ of the conformal inhomogeneities has the velocity field of the corresponding homogeneous model in its null space—a situation which tends to minimize the physical deviations between the two kinds of models. When (35) is substituted into (34), we obtain

$$V^i \overset{*}{=} {}^{\backprime}W^i \overset{*}{=} \frac{\{\partial_0 \partial_i \psi - (\dot{R}R^{-1} + \frac{1}{2}\partial_0\psi)\partial_i\psi\} L^{ij}}{\varkappa({}^{\backprime}\varrho + {}^{\backprime}p)\lambda R^2 e^{2\psi}} \tag{36}$$

for normal models. Since the inhomogeneous models are assumed to be nonempty, ${}^{\backprime}\varrho + {}^{\backprime}p \neq 0$, and hence *the dispersion vector of a normal, nondegenerate, conformally homogeneous model vanishes only at those points for which*

$$(\dot{R}R^{-1} + \tfrac{1}{2}\partial_0\psi - \partial_0)\partial_i\psi = 0 \, .$$

The implications of the normality condition (35) are rather severe, however. When eq. (32) is contracted with V^j and eq. (33) is used, (35) yields

$$\begin{aligned} V^j\{\tfrac{1}{2}\partial_i\psi\partial_j\psi - \nabla_j\partial_i\psi\} \overset{*}{=} [-\varkappa\{({}^{\backprime}\varrho + {}^{\backprime}p)e^{\psi} - (\varrho + p) + 2V^2({}^{\backprime}\varrho + {}^{\backprime}p)e^{2\psi}\} + \\ + \dot{R}R^{-1}\partial_0\psi - \partial_0^2\psi + \tfrac{1}{2}(\partial_0\psi)^2]\, V_i \, , \end{aligned} \tag{37}$$

and hence only in this instance can a conformally homogeneous model be normal.

Now that V^i is determined we can calculate V^2. For a normal model, we have

$$V^2 \overset{*}{=} R^2 L_{ij} V^i V^j \overset{*}{=} L^{ij} f_i f_j / \mu^2 R^2 \lambda^2 \, ,$$

where

$$\begin{cases} f_i \overset{*}{=} \{\partial_0 - \dot{R}R^{-1} + \frac{1}{2}\partial_0\psi\}\partial_i\psi \, , \\ \mu = \varkappa({}^{\backprime}\varrho + {}^{\backprime}p)e^{2\psi} \, . \end{cases} \tag{38}$$

Hence, since $\lambda^2 = e^{-\psi} + V^2$, we obtain

$$V^2 \overset{*}{=} \frac{e^{-\psi}}{2}\left(\sqrt{1 + \frac{4L^{ij}f_i f_j e^{2\psi}}{\mu^2 R^2}} - 1\right) \tag{39}$$

and

$$\lambda \overset{*}{=} \frac{e^{-\psi/2}}{\sqrt{2}} \sqrt{1 + \sqrt{1 + \frac{4L^{ij}f_i f_j e^{2\psi}}{\mu^2 R^2}}} \, . \tag{40}$$

When these are substituted into (36), we finally obtain

$$V^i \overset{*}{=} {}'W^i = \frac{\sqrt{2}e^{\psi/2}L^{ij}f_j}{\mu R^2\sqrt{1+\sqrt{1+4L^{km}f_k f_m e^{2\psi}/\mu^2R^2}}}\,. \tag{41}$$

Equations (8) and (9) showed that the expansion, θ, of the velocity field of the homogeneous models is three times the Hubble expansion parameter $H(t)=\dot{R}R^{-1}$; hence θ is a primary cosmological quantity. For conformally homogeneous models, we have

$$\begin{aligned}{}'\theta &= \partial_\alpha {}'W^\alpha + \Gamma^\alpha_{\mu\alpha}(\boldsymbol{h})\,{}'W^\mu = \partial_\alpha {}'W^\alpha + \big(\Gamma^\alpha_{\mu\alpha}(\boldsymbol{g}) + 2\partial_\mu\psi\big)\,{}'W^n = \\ &= \nabla_\alpha {}'W^\alpha + 2\,{}'W^\alpha\partial_\alpha\psi \overset{*}{=} \lambda(\theta + 2\partial_0\psi + \lambda^{-1}\partial_0\lambda) + \partial_i V^i + V^i\big(\Gamma^j_{ij}(\boldsymbol{L}) + 2\partial_i\psi\big)\,.\end{aligned} \tag{42}$$

If the dispersion vector vanishes at a point P, we have ${}'W^\alpha(P) = W^\alpha(P)\cdot$ $\cdot\exp[-\psi/2]$, $f_i(P)=0$, and (40) gives $\lambda(P)=\exp[-\psi/2]$ and $\partial_0\lambda(P)\overset{*}{=}$ $\overset{*}{=}-\big(\exp[-\psi/2]/2\big)\partial_0\psi$. Equations (42) then yield

$$'\theta(P) = \nabla_\alpha {}'W^\alpha + 2\,{}'W^\alpha\partial_\alpha\psi \overset{*}{=} \exp[-\psi/2](\theta + \tfrac{3}{2}\partial_0\psi)\,,$$

and hence a pointwise coincidence of the velocity fields of a homogeneous model and a conformally homogeneous model does not imply a proportionality of the primary cosmological quantities θ and $'\theta$. There are basically two ways out of this difficulty. The first is where

$$0 = W^\alpha\partial_\alpha\psi \overset{*}{=} \partial_0\psi\,. \tag{43}$$

Although this situation leads to significant mathematical simplification, as a glance at eq. (36), shows, it is highly restrictive, in view of eqs. (32) and (33). Further, (43) states that ψ is constant on any trajectory of the energy transport field $\boldsymbol{W}$ of the corresponding homogeneous model—the geometric inhomogeneities move with energy distribution of the corresponding homogeneous model—a situation which is difficult to envision physically. We henceforth refer to models for which (43) holds as *hyper-regular* and consider them as restrictive special cases. The second is where

$$0 = {}'W^\alpha\partial_\alpha\psi \overset{*}{=} \lambda\partial_0\psi + V^i\partial_i\psi\,. \tag{44}$$

In this instance ψ is constant on any trajectory of the energy transport field $\boldsymbol{W}$ of the conformally homogeneous model under consideration—the geometric inhomogeneities move with the energy distribution that gives rise to these inhomogeneities—a situation which is physically reasonable and which may be interpreted as an equilibrium condition. We henceforth refers to models for which (44) holds as *regular*. (The subset of regular models for which $V^i \partial_i \psi \overset{*}{=} 0$ are *hyper-regular*.) For regular models we have

$$
\text{(45)} \qquad {}'\theta(P) = \nabla_\alpha {}'W^\alpha \overset{*}{=} \exp[-\psi/2]\theta(P)\,, \qquad V^i(P) = 0\,.
$$

For a hyper-regular model we have $f_i = -\dot{R}R^{-1}\partial_i\psi$ and hence

$$
\text{(46)} \qquad V^i \overset{*}{=} -\sqrt{2}e^{\psi/2}L^{ij}\dot{R}R^{-1}\partial_i\psi/\mu R^2 \sqrt{1+\sqrt{1+\frac{4e^{2\psi}\dot{R}^2L^{km}\partial_k\psi\partial_m\psi}{\mu^2R^4}}}\,,
$$

$$
V^2 \overset{*}{=} \frac{e^{-\psi}}{2}\left(\sqrt{1+\frac{4e^{2\psi}\dot{R}^2L^{km}\partial_k\psi\partial_m\psi}{R^2\mu^4}}-1\right),
$$

$$
\lambda \overset{*}{=} \frac{e^{-\psi/2}}{\sqrt{2}}\sqrt{1+\sqrt{1+\frac{4e^{2\psi}\dot{R}^2L^{km}\partial_k\psi\partial_m\psi}{\mu^2R^4}}}\,.
$$

In the case of regular models, (38), (41) and (44) give

$$
\lambda\partial_0\psi \overset{*}{=} \partial_i\psi L^{ij}(\dot{R}R^{-1} + \tfrac{1}{2}\partial_0\psi - \partial_0)\partial_i\psi/\mu R^2\lambda
$$

and hence

$$
\text{(47)} \qquad \partial_0\psi \overset{*}{=} \frac{2\partial_i\psi L^{ij}(\dot{R}R^{-1}-\partial_0)\partial_j\psi}{2\lambda^2\mu^2R^2 - L^{km}\partial_k\psi\partial_m\psi}\,.
$$

Since hyper-regular models are special cases of regular models under the restriction equivalent to $\partial_0\psi \overset{*}{=} 0$, eq. (47) gives the condition

$$
\partial_i\psi L^{ij}(\dot{R}R^{-1}-\partial_0)\partial_j\psi \overset{*}{=} 0\,.
$$

An integration then yields $\partial_i\psi L^{ij}\partial_j\psi \overset{*}{=} U(x^i)\sqrt{R(t)}$. Combining this with the condition $\partial_0\psi \overset{*}{=} 0$ and noting that $\partial_0 L^{ij} \overset{*}{=} 0$, we conclude that *a model can be hyper-regular if and only if* $R(t) = \text{constant}$, a highly uninteresting case.

6. – Approximations.

Up to this point the analysis has been exact. In order to demonstrate the intrinsic utility of conformally homogeneous models, we consider approximations in which particularly simple situations arise. The results presented here are meant primarily as a guide to the reader since we shall consider the exact equations and their implications in future papers.

We consider those models for which ψ is such that the product of any two or more of its derivatives may be neglected in comparison with terms that are either linear in ψ's derivatives or are independent of derivatives of ψ. If $\sim$ denotes equality in a *–co-ordinate system to within this accuracy, then

$$\Box\psi \sim \partial_0^2\psi - R^{-2}\nabla^2\psi + 3\dot{R}R^{-1}\partial_0\psi\,,$$

where $\nabla^2\psi = L^{ij}\big(\partial_i\partial_j\psi - \Gamma_{ij}^k(\boldsymbol{L})\partial_k\psi\big)$ is the Laplacian in the 3-dimensional Riemman space $\mathscr{S}$, and

$$\lambda \sim \exp[-\psi/2]\,,\qquad f_i \sim (\partial_0 - \dot{R}R^{-1})\partial_i\psi\,,\qquad V^i \sim \exp[\psi/2]L^{ij}f_i/\mu R^2\,.$$

When the field equations (6) of the corresponding homogeneous model are used, the system (32) and (33) becomes

$$(48)\qquad (\varkappa\text{`}\varrho + \text{`}\Lambda) \sim 3\{[k + \dot{R}^2 - \tfrac{1}{3}\nabla^2\psi)R^{-2} + \dot{R}R^{-1}\partial_0\psi\}e^{-\psi}\,,$$

$$(49)\qquad \begin{aligned}\varkappa R^4 e^{2\psi} L_{ik}L_{jm}Q^{km} \sim &-\partial_i\partial_j\psi + \Gamma_{ij}^k(\boldsymbol{L})\partial_k\psi + \\ &+ R\dot{R}L_{ij}\partial_0\psi - R^2L_{ij}(\partial_0^2\psi - R^{-2}\nabla^2\psi + 3\dot{R}R^{-1}\partial_0\psi) - \\ &- R^2L_{ij}(\varkappa\text{`}pe^{\psi} + 2\ddot{R}R^{-1} + (k + \dot{R}^2)R^{-2} - \text{`}\Lambda e^{\psi})\,,\end{aligned}$$

and the normality condition becomes

$$\varkappa\text{`}pe^{\psi} + 2\ddot{R}R^{-1} + (k + \dot{R}^2)R^{-2} - \text{`}\Lambda e^{\psi} \sim 0\,,$$

or, equivalently, by (2.6),

$$(50)\qquad \varkappa\text{`}p \sim (\varkappa p - \Lambda)e^{-\psi} + \text{`}\Lambda\,.$$

When (50) is substituted into (49) we then obtain

$$(51)\qquad \begin{aligned}\varkappa R^4 e^{2\psi} L_{ik}L_{jm}Q^{km} \sim &-\partial_i\partial_j\psi + L_{ij}^k(\boldsymbol{L})\partial_k\psi + R\dot{R}L_{ij}\partial_0\psi - \\ &- R^2L_{ij}(\partial_0^2\psi - R^{-2}\nabla^2\psi + 3\dot{R}R^{-1}\partial_0\psi)\,,\end{aligned}$$

while (42) gives

$$(52)\qquad \text{`}\theta \sim e^{-\psi/2}\left(\theta + \frac{3}{2}\partial_0\psi\right) + \frac{e^{\psi/2}}{\mu R^2}L^{ij}\big(\partial_i f_j + \Gamma_{jk}^k(\boldsymbol{L})f_k\big)\,.$$

It thus follows that for any *normal* model we obtain $\sim$-solutions for $\varkappa\text{`}\varrho + \text{`}\Lambda$, $\text{`}p$, $Q^{ij}$, and $\text{`}\theta$. On the other hand, (44) and the above results show that if the model is regular then $\partial_0 \psi \sim 0$ and hence it is hyper-regular to within $\sim$-equality. The regularity condition is thus too stringent within the approximation scheme considered here.

There is a particularly interesting subclass of solutions that obtains in the case $'\Lambda = 0$. If (6) is used to rewrite (48), the resulting equation is satisfied if

$$'\varrho/\varrho \sim e^{-\psi} \tag{53}$$

and ψ is such that

$$\nabla^2 \psi = 3R^2(\dot{R}R^{-1}\partial_0 \psi + \Lambda) = 3R^2(H\partial_0 \psi + \Lambda)\,, \tag{54}$$

i.e. ψ satisfies a time-dependent diffusion equation with source term $3\Lambda R^2$. There are two possibilities here. The first is to use observations to determine $'\varrho/\varrho$ at the present epoch and hence $e^{-\psi}$. From this we can then use (54) to evaluate $\partial_0\psi$ at the present epoch and test whether our galactic neighborhood has $\partial_0\psi > 0$ or $\partial_0\psi < 0$. In this context, it is worth noting the possibility of reducing the scatter in the red-shift-magnitude plot for our local neighborhood of galaxies by accounting for the ψ effects. We shall examine this and related questions in a forthcoming paper on the analysis of observables in conformally homogeneous models.

The second possibility is to solve (54) for ψ and then infer $'\varrho$ by (53). Separating variables in (54) yields $\psi = T(t)X(x^i)$, where

$$T(t) = \exp\left[-\frac{\eta^2}{3}\int_{t_0}^{t}\frac{d\tau}{R^2(\tau)H(\tau)}\right]\left\{T_0 + \Lambda\int_{t_0}^{t}\frac{d\tau}{H(\tau)}\exp\left[\frac{\eta^2}{3}\int_{t_0}^{\tau}\frac{dy}{R^2(y)H(y)}\right]\right\}, \tag{55}$$

$$\nabla^2 X + \eta^2 X = 0\,, \qquad X = X(x^i, \eta)\,. \tag{56}$$

Now, ∇^2 is the Laplacian on $\mathscr{S}$, and, if $\dot{k} = +1$, $\mathscr{S}$ is a compact, simply connected, complete Riemann space. Under this condition, (56) admits nontrivial, C^2 solutions if and only if the values of η are real and discrete; in which case, the solutions of (56) (eigenfunctions) are convex functions about the trivial solution $X(x^i) = 0$. For those conformally homogeneous models corresponding to Freidman models with $k = +1$, $\Lambda = 0$, and

$$T(t) = T_0 \exp\left[-\frac{\eta^2}{3}\int_{t_0}^{t}\frac{d\tau}{R^2(\tau)H(\tau)}\right], \qquad \eta^2 = n(n+2)\,. \tag{57}$$

The eigenfunctions which solve (56) on $\mathscr{S}$ will define scales of inhomogeneity, each scale being defined by the diameter of the cells into which 3-space is divided by the surfaces $X(x^i, \eta) = 0$. Now, the scale of an inhomogeneity will decrease monotonically with increasing η and (57) shows that the relaxation time of a scale decreases as η increases when the corresponding homogeneous model is in an *expansion phase.* The large-scale inhomogeneities will thus decay more slowly than the small-scale ones. The possibility is thus presented for describing large-scale inhomogeneities, such as clusters of galaxies, and second-order clusters which are such that they have not had time to dissipate during the current-expansion phase. An adequate accounting of galaxies, on the other hand, will probably require the full nonlinear equations without approximation since the conditions $|\partial_\alpha \psi \partial_\beta \psi| \ll 1$ will not be satisfied for « small-scale » agglomerations.

For the present approximate solutions,

$$f_i \sim (\partial_0 - \dot{R}R^{-1})\partial_i \psi = (\partial_0 - H)\partial_i \psi = -T\left(H + \frac{\eta^2}{3R^2 H}\right)\partial_i X\,,$$

and hence the points where the dispersion vanishes are those for which $X(x^i)$ possesses a local maximum or minimum. Since ${}'\varrho/\varrho \sim e^{-\psi}$, ${}'\varrho$ is consequently maximal when $X(x^i)$ is minimal. It then follows that $W \sim {}'W$ at a cluster center which, to a great extent, accounts for the agreement between the observed magnitude–red-shift law and that predicted by the homogeneous models (*i.e.* the process of averaging over a cluster effectively makes $\langle {}'\theta \rangle \sim \langle \theta \rangle$). Dispersion always exists for the inhomogeneous models, vanishing only at the points of spatial maximum or minimum of the density distribution (*i.e.* $\psi = -\log({}'\varrho/\varrho)$). Further, from the above $\sim$-evaluation of V^i, the dispersion vector points toward the cluster centers. Clusters may thus be viewed as dynamically bound entities whose binding decays with time as the universe expands. Since the inhomogeneities considered here ($k = +1$, $\Lambda = 0$) decay during an expansion phase, they must be present before or during the initiation of the expansion phase.

This Section has been included in order to show that conformally homogeneous models provide a means for a rational discussion of observed inhomogeneities within a cosmological context. The results are in no sense complete, being only indicative in nature. The next paper in this series examines the various observables and their interrelationships for conformally homogeneous models. Once this is at hand, precise statements can be made in all cases, $k = -1$, 0, $+1$, $\Lambda < 0$, $= 0$, > 0, etc.

* * *

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RIASSUNTO (*)

Si studiano modelli cosmologici intrinsecamente inomogenei che hanno gli stessi « campi gravitazionali liberi » e gli stessi moti conformi dei modelli isotropi classici spazialmente omogenei. I modelli trattati sono equivalenti a quelli classici e permettono l'interpretazione fisica dei modelli classici da utilizzare. Si ottengono equazioni esatte che collegano i parametri fisici salienti e sono equivalenti alle equazioni di campo di Einstein. Si mostra che il vettore velocità (di trasporto della energia) è la somma di un multiplo del vettore velocità del corrispondente modello omogeneo e di un vettore di dispersione (relativamente al modello omogeneo). Si ottengono soluzioni esatte per il vettore di dispersione. Se si trascurano i quadrati delle derivate del coefficiente conforme, ψ, si ottengono espressioni esplicite per la densità e la pressione. Si mostra che il fattore conforme soddisfa un'equazione di diffusione — $\nabla^2\psi = 3R^2(\dot{R}R^{-1}\partial_t\psi + \Lambda)$ — quando ${}'\varrho/\varrho = e^{-\psi}$ (${}'\varrho$ indica la densità dei modelli adottati). Per i modelli di Friedman con $k = +1$, si ottengono scale uniche spaziali di inomogeneità. Ogni scala di inomogeneità decresce col tempo mentre il tempo di rilassamento cresce con la scala, ed il vettore di dispersione si annulla solo nei punti dove ${}'\varrho$ ha un massimo o un minimo spaziale. Si ottengono espressioni esatte e approssimate per il parametro di sviluppo.

(*) *Traduzione a cura della Redazione.*

Конформно однородная модель вселенной. - I

Резюме (*). — Исследуются внутренне неоднородные космологические модели, чтобы иметь те же « свободные гравитационные поля » и те же конформные движения, как и в классических пространственно однородных, изотропных моделях. Настоящие модели являются конформно эквивалентными классическим и допускают использование физических интерпретаций для классических моделей. Получаются точные уравнения, которые связывают немые физические параметры и которые эквивалентны уравнениям поля Эйнштейна. Показывается, что вектор скорости (переноса энергии) является суммой кратного числа вектора скорости, соответствующей однородной модели, и вектора дисперсии (относительно этой однородной модели). Получаются точные решения уравнения для вектора дисперсии. Если пренебречь квадратами производных конформного коэффициента x, то получаются точные выражения для плотности и давления. Показывается, что конформный фактор удовлетворяет уравнению диффузии $\nabla^2\psi = 3R^2(\dot{R}R^{-1}\partial_t\psi + \Lambda)$ где ${}'\varrho/\varrho = e^{-\psi}$ (${}'\varrho$ обозначает плотность настоящих моделей). Для моделей Фридмана с $k=+1$, получаются уникальные пространственные масштабы неоднородностей. Каждый масштаб неоднородностей распадается со временем, увеличивающимся с масштабом, и вектор дисперсии обращается в нуль только в точках, где ${}'\varrho$ имеет пространственный максимум или минимум. Получаются точное и приближенное выражение для параметра разложения.

(*) *Переведено редакцией.*