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On the Mass-Independent Renormalization of the Massive Thirring Model.

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Summary. — Weinberg's renormalization scheme breaks down in the 2-dimensional massive Thirring model, due to the same difficulty as in the 4-dimensional φ^4-theory, *i.e.* the mass renormalization factor is infra-red divergent. It is shown that the dimensional regularization copes with the difficulty.

Wilson's conjecture (1) had stimulated a great interest in the behaviour of quantum field theories in the zero-mass limit, in particular, of the massive Thirring model, whose associated zero-mass theory is exactly solvable (2-4). MUELLER and TRUEMAN (5) and GOMES and LOWENSTEIN (6) have shown in the perturbation theory that the Green's functions of the massive Thirring model pass smoothly into those of the massless model, as far as the relevant momenta are nonexceptional.

The renormalization employed by them is mass dependent and the crucial point of the problem lay in the coexistence of infra-red and ultraviolet divergences in the zero-mass limit.

(1) K. G. WILSON: *Phys. Rev.*, **179**, 1499 (1969).
(2) W. THIRRING: *Ann. Phys.* (*N. Y.*), **3**, 91 (1958).
(3) K. JOHNSON: *Nuovo Cimento*, **20**, 773 (1961).
(4) B. KLAIBER: *Lectures in Theoretical Physics* (New York, N. Y., 1968), p. 141.
(5) A. H. MUELLER and T. L. TRUEMAN: *Phys. Rev. D*, **4**, 1635 (1971).
(6) M. GOMES and J. H. LOWENSTEIN: *Nucl. Phys. B*, **45**, 252 (1972).

Now we know the mass-independent renormalization which has been established by WEINBERG [7] and by 'T HOOFT and VELTMAN [8]. When examined in this scheme, the problem of the zero-mass limit becomes much simplified. According to Weinberg's argument [7], there does not exist any infra-red divergence in the Green's functions for nonexceptional momenta. (But the derivative with respect to the renormalized mass diverges logarithmically at $m_R = 0$, in the massive Thirring model.) Thus the existence of the zero-mass limit is guaranteed.

Then the only question that remains to us is whether or not the mass-independent renormalization could actually work.

It is well known that Weinberg's renormalization procedure suffers from the difficulty of the quadratic divergence in the φ^4-theory. To make matters worse, it still breaks down when dealing with the mass renormalization factor in the same theory, as pointed out by COLLINS [9]. The second trouble spoils Weinberg's procedure in the massive Thirring model as well: The mass renormalization factor defined as

$$Z_m = 1 - \left.\frac{\partial \Sigma}{\partial m_B}\right|_{m_B=0} \tag{1}$$

diverges, since $\sum$ contains terms like $m_B \ln m_B$.

However, this infra-red divergence must be dismissed away in the scheme [8] based on the dimensional regularization, as well known on general grounds [8,10]. For the φ^4-theory, it has been verified by COLLINS [9]. The aim of this paper is to show that, also in the massive Thirring model, the dimensional regularization is released from the above difficulty and actually works.

The massive Thirring model in the 2-dimensional space-time is described by

$$L_F = \bar{\psi}(i\not{\partial} - m_R)\psi \tag{2}$$

and

$$L_I = -(m_B - m_R)\bar{\psi}\psi - \frac{g_B}{2}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi) \tag{3}$$

with the renormalization constants

$$m_B = m_R\left[1 + \sum_{\nu=1}^{\infty}\frac{b_\nu(g_R)}{(n-2)^\nu}\right], \tag{4}$$

$$g_B = \mu^{2-n}\left[g_R + \sum_{\nu=1}^{\infty}\frac{a_\nu(g_R)}{(n-2)^\nu}\right], \tag{5}$$

(7) S. WEINBERG: *Phys. Rev. D*, **8**, 3497 (1973).
(8) G. 'T HOOFT and M. VELTMAN: *Nucl. Phys. B*, **44**, 189 (1972).
(9) J. C. COLLINS: *Phys. Rev. D*, **10**, 1213 (1974).
(10) G. 'T HOOFT: *Nucl. Phys. B*, **61**, 455 (1973).

and

$$Z = 1 + \sum_{\nu=1}^{\infty} \frac{C_\nu(g_\mathrm{R})}{(n-2)^\nu}\,. \tag{6}$$

Let us begin with the renormalization of the self-energy of order g_R^2 (fig. 1).

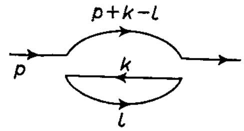

Fig. 1. – Self-energy diagram of order g_R^2.

It is

$$\begin{aligned}
(7)\qquad -i\sum &= -\frac{(-ig_\mathrm{B})^2}{(2\pi)^{2n}}\int \mathrm{d}^n k\,\mathrm{d}^n l\,\gamma_\mu \frac{i}{\not p + \not k - \not l - m_\mathrm{R}}\gamma_\nu\, T_r\left(\gamma^\mu \frac{i}{\not l - m_\mathrm{R}}\gamma^\nu \frac{i}{\not k - m_\mathrm{R}}\right) = \\
&= \frac{-ig_\mathrm{B}^2}{(2\pi)^{2n}}\, 2^{n/2}\int \mathrm{d}^n k\,\mathrm{d}^n l\, \frac{1}{(k^2-m_\mathrm{R}^2)(l^2-m_\mathrm{R}^2)\{(p+k+l)^2-m_\mathrm{R}^2\}}\,\cdot \\
&\cdot[-2(k^2\not l + l^2\not k) + (2-n)kl(\not k + \not l) - (\not k\not p\not l + \not l\not p\not k) + \\
&\qquad + (2-n)kl\not p + (n-2)m_\mathrm{R}kl + m_\mathrm{R}^2\gamma_\mu(\not p + \not k + \not l + m_\mathrm{R})\gamma^\mu]\,.
\end{aligned}$$

For convenience, taking off the square bracket, we denote each integral $I_1, \dots, I_6$, in that order, *i.e.* $-i\sum = \sum I_i$. The following formulae are needed to evaluate I_i:

$$\begin{aligned}
(8)\qquad &\int \mathrm{d}^n k\,\mathrm{d}^n l\, \frac{l_\mu k_\nu}{(k^2-m_\mathrm{R}^2)(l^2-m_\mathrm{R}^2)\{(p+k+l)^2-m_\mathrm{R}^2\}} = \\
&= \pi^n\,\Gamma(3-n)\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\delta(\alpha+\beta+\gamma-1)\,\frac{\alpha\beta\gamma^2}{C^{n/2+2}}\left(\frac{D}{C}\right)^{n-3} p_\mu p_\nu + \\
&\qquad + \pi^n\,\Gamma(2-n)\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\delta(\alpha+\beta+\gamma-1)\,\frac{\gamma}{C^{n/2+1}}\left(\frac{D}{C}\right)^{n-2} \frac{1}{2} g_{\mu\nu}
\end{aligned}$$

and

$$\begin{aligned}
(9)\qquad &\int \mathrm{d}^n k\,\mathrm{d}^n l\, \frac{l_\mu l_\nu k_\varrho}{(k^2-m_\mathrm{R}^2)(l^2-m_\mathrm{R}^2)\{(p+k+l)^2-m_\mathrm{R}^2\}} = \\
&= -\pi^n\,\Gamma(3-n)\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\delta(\alpha+\beta+\gamma-1)\,\frac{(\gamma\alpha)^2\beta\gamma}{C^{n/2+3}}\left(\frac{D}{C}\right)^{n-3} p_\mu p_\nu p_\varrho +
\end{aligned}$$

$$+ \pi^n \Gamma(2-n)\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\delta(\alpha+\beta+\gamma-1)\frac{\gamma}{C^{n/2+1}}\left(\frac{D}{C}\right)^{n-2}\cdot$$

$$\cdot\left\{\frac{1}{2}g_{\mu\nu}p_\varrho - \frac{1}{2}(g_{\mu\nu}p_\varrho + g_{\mu\varrho}p_\nu + g_{\nu\varrho}p_\mu)\frac{\gamma\alpha}{C}\right\},$$

where

$$C = \alpha\beta + \beta\gamma + \gamma\alpha$$

and

$$D = (\alpha\beta + \beta\gamma + \gamma\alpha)m_{\mathrm{R}}^2 - \alpha\beta\gamma p^2 .$$

Note that the integrals in the second terms in (8) and (9) diverge at $\alpha + \beta = 0$ when $n \geqslant 2$, so that they give rise to the pole at $n = 2$ which contributes to ultraviolet divergence. On the other hand, those in the first terms are finite at $n = 2$.

(For $m_{\mathrm{R}} = 0$, they also diverge at $n = 2$. In this regard, see the comment at the end of the paper.)

For the divergent part of I_1, we get

$$I_1^{\mathrm{div}} = \int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\lambda\, H\left[\left(1-\frac{2\gamma\alpha}{C}\right) + \left(1-\frac{\gamma\alpha}{C}\right)\frac{n-2}{2}\right](-4\not{p}) \tag{10}$$

with

$$H = \frac{-ig_{\mathrm{B}}^2}{(2\pi)^{2n}}\,2^{n/2}\,\pi^n\,\Gamma(2-n)\,\delta(\alpha+\beta+\gamma-1)\frac{\gamma}{C^{n/2+1}}\left(\frac{D}{C}\right)^{n-2}.$$

Now it is suitable to transform the integral variable such that

$$\alpha = \varrho x\,, \qquad \beta = \varrho(1-x)\,, \qquad \gamma = 1-\varrho\,. \tag{11}$$

Then the integral coming from the first term in the bracket happens to be finite at $n = 2$:

$$\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\delta(\alpha+\beta+\gamma-1)\frac{\gamma}{C^2}\left(1-\frac{2\gamma\alpha}{C}\right) = \frac{1}{2}\,, \tag{12}$$

while the remaining integral diverges at the origin in the ϱ-integration. By using

$$C \sim \varrho \qquad \text{as } \varrho \to 0$$

and

$$\frac{D}{C} \sim m_{\mathrm{R}}^2 \qquad \text{as } \varrho \to 0\,,$$

the singular part can be calculated:

$$\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\delta(\alpha+\beta+\gamma-1)\,\frac{\gamma}{C^{n/2+1}}\left(\frac{D}{C}\right)^{n/2}\left(1-\frac{\gamma\alpha}{C}\right)= \tag{13}$$

$$=\int_0^1 \varrho\,\mathrm{d}\varrho\int_0^1 \mathrm{d}x\,\frac{1}{\varrho^{n/2+1}}\,(m_{\mathrm{R}}^2)^{n-2}(1-x)+O\big((n-2)^0\big)=\frac{1}{2-n}\,(m_{\mathrm{R}}^2)^{n-2}+O\big((n-2)^0\big)\,.$$

Substituting (12) and (13) into (10), we obtain

$$I_1^{\mathrm{div}}=\frac{-ig_{\mathrm{R}}^2}{4\pi^2}\left(\frac{\sqrt{2}}{4\pi\mu}\right)^{n-2}\frac{1}{n-2}\,\{1-(m_{\mathrm{R}}^2)^{n-2}\}\,\not{p}+O\big((n-2)^0\big)\,. \tag{14}$$

Similar calculations can be undertaken also for $I_2, \dots, I_6$. The result is

$$I_2^{\mathrm{div}}=\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,H\left[\left(1-\frac{4\gamma\alpha}{C}\right)(2-n)+\frac{\gamma\alpha}{C}\,(2-n)^2\right]\not{p}= \tag{15}$$

$$=\frac{-ig_{\mathrm{R}}^2}{4\pi^2}\left(\frac{\sqrt{2}}{4\pi\mu}\right)^{n-2}\frac{1}{n-2}\,(m_{\mathrm{R}}^2)^{n-2}\,\not{p}+O\big((n-2)^0\big)\,,$$

$$I_3^{\mathrm{div}}+I_4^{\mathrm{div}}=\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,H(n-2)^2(-\tfrac{1}{2}\not{p})=O\big((n-2)^0\big)\,, \tag{16}$$

$$I_5^{\mathrm{div}}=\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,H\,\frac{n(n-2)}{2}\,m_{\mathrm{R}}= \tag{17}$$

$$=\frac{-ig_{\mathrm{R}}^2}{4\pi^2}\left(\frac{\sqrt{2}}{4\pi\mu}\right)^{n-2}\frac{1}{n-2}\,(m_{\mathrm{R}}^2)^{n-2}m_{\mathrm{R}}+O\big((n-2)^0\big)$$

and I_6 is finite. In summary we get the singular part for $\sum$

$$\sum\nolimits^{\mathrm{div}}=\frac{g_{\mathrm{R}}^2}{4\pi^2}\left(\frac{\sqrt{2}}{4\pi\mu}\right)^{n-2}\frac{1}{n-2}\,\not{p}+\frac{g_{\mathrm{R}}^2}{4\pi^2}\left(\frac{\sqrt{2}}{4\pi\mu}\right)^{n-2}\frac{1}{n-2}\,m_{\mathrm{R}}+O\big((n-2)^0\big)\,, \tag{18}$$

from (14)-(17).

No double pole exists in (18), although $\sum$ contains the two-loop integral corresponding to fig. 1. It implies that the overlapping divergence disappears. This fact can be shown also according to the argument ([8]). There are three one-loop diagrams in $\sum$, as shown in fig. 2. Figure 2a) yields

$$T_r\int\frac{\mathrm{d}^n k}{(2\pi)^n}\,\gamma_\mu\,\frac{i}{\not{k}-m_{\mathrm{R}}}\,\gamma_\nu\,\frac{i}{\not{k}+\not{q}-m_{\mathrm{R}}}\,,$$

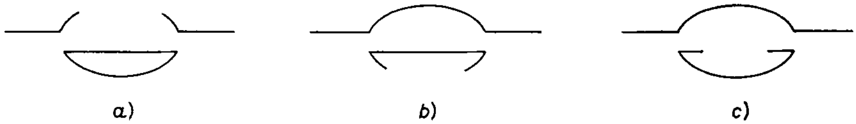

Fig. 2.

which is regular at $n=2$. For fig. 2*b*) see the k-integral in (7). Then only I_1 is divergent at $n=2$ in the k-integral. We have

$$(19)\qquad I_1 \sim \int \mathrm{d}^n l\, \frac{\not{l}}{l^2-m_R^2}\int \mathrm{d}^n k\, \frac{k^2}{(k^2-m_R^2)\{(p+k+l)^2-m_R^2\}} =$$

$$= \frac{-i\pi^{n/2}n}{2}\,\frac{\Gamma(1-n/2)}{\Gamma(2)}\int \mathrm{d}^n l\, \frac{\not{l}}{l^2-m_R^2}\,\frac{1}{[m_R^2-(p+l)^2x(1-x)]^{1-n/2}} + O((n-2)^0)\,,$$

keeping the pole term in the k-integral only.

The remaining l-integral in (19) does not give rise to any more pole at $n=2$. Similar discussions can be repeated also for fig. 2*c*). Thus $\sum$ is free from overlapping divergence. As a consequence we do not have to take account of the extra diagrams that are needed in the φ^4-theory ([9]).

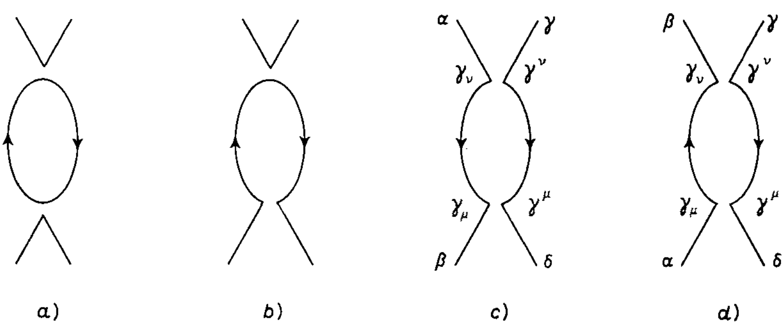

Fig. 3. – Vertex diagrams of order g_R^2.

Next let us go on to the renormalization of the vertex. The diagrams of order g_R^2 are shown in fig. 3. It is easy to see that fig. 3*a*) and *b*) do not give ultraviolet divergence. Figures 3*c*) and *d*) yield, respectively,

$$\int \mathrm{d}^n k \left[\gamma_\mu \frac{i}{-\not{k}-m_R}\gamma_\nu\right]_{\alpha\beta} \otimes \left[\gamma^\mu \frac{i}{\not{k}+\not{p}-m_R}\gamma^\nu\right]_{\gamma\delta} =$$

$$= -\frac{i\pi}{2}\,\frac{1}{n-2}\,(\gamma_\mu\gamma_\varrho\gamma_\nu)_{\alpha\beta}\otimes(\gamma^\mu\gamma^\varrho\gamma^\nu)_{\gamma\delta} + O((n-2)^0)$$

and

$$\int \mathrm{d}^n k \left[\gamma_\nu \frac{i}{k - m_\mathrm{R}} \gamma_\mu\right]_{\alpha\beta} \otimes \left[\gamma^\mu \frac{i}{k + p - m_\mathrm{R}} \gamma^\nu\right]_{\gamma\delta} = \\ = \frac{i\pi}{2} \frac{1}{n-2} (\gamma_\nu \gamma_\varrho \gamma_\mu)_{\alpha\beta} \otimes (\gamma^\mu \gamma^\varrho \gamma^\nu)_{\gamma\delta} + O((n-2)^0) .$$

In the limit $n = 2$ the singular parts are cancelled between each other in combining them. In this way the coupling constant is not renormalized to order g_R^2.

Now a_ν, b_ν and c_ν in (4)-(6) can be evaluated to order g_R^2. From (18), taking into account the mass counterterm $m_\mathrm{B} - m_\mathrm{R}$, we have the total propagator (to order g_R^2)

$$S_\mathrm{F}'^{-1} = p - m_\mathrm{B} - \sum = p \left[1 - \frac{g_\mathrm{R}^2}{4\pi^2} \frac{1}{n-2}\right] - m_\mathrm{R}\left[1 + \frac{g_\mathrm{R}^2}{n-2}\left(b_1 + \frac{1}{4\pi}\right)\right] . \tag{20}$$

Define Z and b to be such that the term of order g_R^2 in $ZS_\mathrm{F}'^{-1}$ is regular at $n = 2$. Then

$$m_\mathrm{B} = m_\mathrm{R}\left[1 - \frac{g_\mathrm{R}^2}{2\pi^2} \frac{1}{n-2} + O(g_\mathrm{R}^3)\right], \tag{21}$$

$$Z = 1 + \frac{g_\mathrm{R}^2}{4\pi^2} \frac{1}{n-2} + O(g_\mathrm{R}^3) \tag{22}$$

and

$$g_\mathrm{B} = O(g_\mathrm{R}^3) . \tag{23}$$

We have demonstrated to order g_R^2 that, with resort to the dimensional. regularization, the mass-independent renormalization actually works in the massive Thirring model as well. Some comments are in order. The reader might worry about infra-red divergences in the regular part of I_i. They should be, of course, cancelled in the final form of $\sum$, as has been discussed in the beginning of this paper.

We can check this fact as follows. At first, the regular part obtained from $\sum^\mathrm{div}$, eq. (18), is not divergent at $m_\mathrm{R} = 0$. Note here a cancellation of infra-red divergent terms like $\ln m_\mathrm{R}^2$, which come from each regular part (14) and (15) by using $(m_\mathrm{R}^2)^{n-2} = 1 + (n-2) \ln m_\mathrm{R}^2$.

Next the regular part $\sum_i^\mathrm{fin}$ must be examined, which comes out through the first terms in formulae (8) and (9). Each of them is infra-red divergent. However, when the combinations

$$I_1^\mathrm{fin} + I_3^\mathrm{fin} = \int_0^1 \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\, F \left(1 - \frac{2\gamma\alpha}{C}\right)(-2p^2 p)$$

and

$$I_2^{\text{fin}} + I_4^{\text{fin}} = \int_0^1 \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\, F \left(1 - \frac{2\gamma\alpha}{C}\right)(2-n)p^2 \not{p}$$

with

$$F = \frac{-ig_{\mathrm{B}}^2}{(2\pi)^{2n}}\, 2^{n/2}\, \pi^n\, \Gamma(3-n)\, \delta(\alpha+\beta+\gamma-1)\, \frac{\alpha\beta\gamma^2}{C^{n/2+2}} \left(\frac{D}{C}\right)^{n-3}$$

are evaluated at $m_{\mathrm{R}} = 0$ by the transformation (11), then they become finite at $n = 2$ due to the factor $1 - 2\gamma\alpha/C$, *i.e.* in the same way as (12) did. Thus the regular part of $\sum$ is free from infra-red divergence.

Finally, we mention the other appealing issue which has been discussed with great energy in ref. [5,6], besides the problem of the zero-mass limit. That is the absence of the coupling constant renormalization. We have shown it only to order g_{R}^2 in this paper. It is hoped that this fact will be proved to all orders in the scheme of the dimensional regularization.

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● RIASSUNTO (*)

Lo schema di rinormalizzazione di Weinberg è violato nel modello di Thirring massivo a 2 dimensioni, a causa della stessa difficoltà del modello quadridimensionale della teoria φ^4, cioè il fattore di rinormalizzazione di massa è divergente nell'infrarosso. Si mostra che la regolarizzazione dimensionale fa fronte alla difficoltà.

(*) *Traduzione a cura della Redazione.*

Не зависящая от массы перенормировка массивной модели Тирринга.

Резюме (*). — Схема перенормировки Вейнберга нарушается в двумерной массивной модели Тирринга, из-за аналогичной трудности, которая существует в четырехмерной φ^4-теории, т.е. из-за того, что перенормировка массы содержит инфракрасную расходимость. Показывается, что размерная регуляризаия разрешает указанную трудность.

(*) *Переведено редакцией.*