PARTIAL DERIVATIVES OF ECLIPSE FUNCTIONS α^{oc} AND α^{tr}

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Abstract. Expressions are given for partial derivatives of eclipse functions with respect to geometrical depth, p, and the ratio of radii, k. The derivatives are evaluated for critical combinations of p and k at which indeterminacies occur and the resulting expressions are listed. All expressions are given in a form suitable for numerical evaluation. Notation employed is that of Merrill.

1. Introduction

Improvement of the elements of eclipsing binary systems by the method of least squares requires knowledge of the sensitivity coefficients of the fractional loss of light with respect to the elements being adjusted. This sensitivity is measured by the partial derivatives of the fractional loss of light with respect to the geometrical and photometric elements of the eclipse. The differential coefficients with respect to the former elements ultimately depend on the derivatives $\partial\alpha(k, p)/\partial p$ and $\partial\alpha(k, p)/\partial k$ where the notation is that of Merrill (1950). Kopal (1947), Irwin (1947), and Tsessevich (1947) suggested that these values be obtained by numerical differentiation of the appropriate tables of α-functions. It is claimed that a four-place precision is obtainable with the available tabulations. However, such a procedure is extremely tedious if done manually, and totally wasteful if employed with electronic computers since it requires storage of tables of α-functions. The latter approach violates good computer practice. Consequently for automatic computation analytical expressions are much preferred.

In the following paragraphs analytical expressions are given for $\partial\alpha/\partial p$ and $\partial\alpha/\partial k$ for both the transit and occultation eclipses. In addition expressions are given resulting from the evaluation of indeterminacies arising in the critical cases. A separate treatment of such cases is necessary because the general equations break down for certain values of k and p.

The derivation of the explicit expressions employs the two lower order associated alpha functions of Kopal (1943, 1944, 1959) as well as his integrals $I^m_{\beta,\gamma}$ or $J^m_{\beta,\gamma}$.

2. Partial Derivatives $\partial\alpha(p, k)/\partial p$

A. PRELIMINARY EXPRESSIONS

Partial derivatives of $\alpha(p, k)$ with respect to the geometrical depth p are given by $\partial\alpha/\partial p = k\partial\alpha/\partial\delta$. In terms of Kopal's I-integrals partial derivatives for the uniform light case can be written as

$$\frac{\partial^0\alpha}{\partial\delta_K} = -\frac{2}{r_2}\left(\frac{r_2}{r_1}\right)^2 I^1_{-1,0} \tag{1}$$

and those for stars totally darkened at the limb as

$$\frac{\partial^1 \alpha^x}{\partial \delta_K} = -\frac{3}{2}\frac{2}{r_2}\frac{1}{{}^1\tau(k)}\left(\frac{r_2}{r_1}\right)^3 I^1_{-1,1}\,. \tag{2}$$

The latter expression applies directly to the transit case which in the usual notation of Merrill is denoted by $\partial^1\alpha^{tr}/\partial\delta$. For the occultation case x is replaced by oc and ${}^1\tau(k)$ is omitted.

In the above expressions the ratio of radii, k, and the apparent separation, δ, are defined by

$$k = r_1/r_2 \quad \text{and} \quad \delta = \delta_K/r_2 = 1 + kp$$

where r_1 and r_2 are the radii of the smaller and larger components of the binary system and $\delta_K = r_2 + r_1 p$. Quantities subscripted with 'K' refer to Kopal's definitions whereas those not so subscripted to definitions of Merrill.

B. OCCULTATION OF UNIFORMLY BRIGHT STARS

In this case Kopal's α_0^0 function is equivalent to ${}^0\alpha$ function of Merrill. Kopal's $I^1_{-1,0}$ integral can be written as

$$I^1_{-1,0} = \frac{1}{\pi}\sqrt{1-\mu^2}\,,$$

where

$$\mu_K = \frac{r_2^2 - r_1^2 + \delta_K^2}{2\delta_K r_2}\,.$$

By the use of previous definitions the quantity $\partial^0\alpha/\partial p$ can be written as

$$\frac{\partial^0\alpha}{\partial p} = -\frac{2}{\pi}\sqrt{1 - \left(\frac{\delta^2 - 1 + k^2}{2k\delta}\right)^2}\,.$$

For the limiting case $k=0$ the second term under the radical becomes indeterminate. The removal of the indeterminacy yields

$$\frac{\partial^0\alpha(p,0)}{\partial p} = -\frac{2}{\pi}\sqrt{1-p^2}$$

Another indeterminacy arises for the case $k=1, p=-1$.

For $k=1$ we have

$$\frac{\partial^0\alpha(p,1)}{\partial p} = -\frac{1}{\pi}\sqrt{(3+p)(1-p)}$$

so that

$$\frac{\partial^0\alpha(-1,1)}{\partial p} = -\frac{2}{\pi}\,.$$

C. OCCULTATION OF TOTALLY LIMB-DARKENED STARS

For this case $I^1_{-1,1}$ in Expression (2) is given by

$$I^1_{-1,1} = \frac{2}{3\pi}\sqrt{\frac{\delta_K}{r_2}}\left\{(1+\mu_K)\,F(\kappa_K) - 2\mu_K E(\kappa_K)\right\},$$

where

$$\mu_K = \frac{r_2^2 - r_1^2 + \delta_K^2}{2\delta_K r_2} \quad \text{and} \quad \kappa_K = \frac{1-\mu_K}{2}.$$

Substitution in Equation (2) yields

$$\frac{\partial^1\alpha^{oc}}{\partial p} = -\frac{2}{\pi}\left\{\frac{(1+\delta)^2 - k^2}{2k^2\sqrt{\delta}}\,F(\kappa) - \frac{1-k^2+\delta^2}{k^2\sqrt{\delta}}\,E(\kappa)\right\}, \tag{3}$$

where

$$\kappa = \frac{1}{2}\sqrt{\frac{k^2-(1-\delta)^2}{\delta}}$$

and F and E are the complete elliptic integrals of the first and second kind, respectively. The evaluation of the indeterminate form which arises for $k=0$ yields

$$\frac{\partial^1\alpha^{oc}(p,0)}{\partial p} = -\tfrac{3}{4}(1-p^2).$$

The case $k=1, p=-1$ leads to indeterminacy of the coefficients of F and E. Evaluation of this indeterminacy leads to

$$\frac{\partial^1\alpha^{oc}(p,1)}{\partial p} = -\frac{2}{\pi}\left\{\frac{(3+p)\sqrt{1+p}}{2}\,F - (1+p)^{3/2}\,E\right\}$$

from which $\partial^1\alpha^{oc}(-1,1)/\partial p = 0$. At the time of external tangency we have $\partial\alpha(1,1)\;\partial p = 0$.

D. PARTIAL TRANSIT PHASES OF TOTALLY LIMB DARKENED STARS

Partial derivatives are described by Equation (2). The form of the integral $I^1_{-1,1}$ remains the same as in the case of occultation eclipses. Hence we can write

$$\frac{\partial^1\alpha^{tr}}{\partial p} = -\frac{2k^2}{\pi^1\tau(k)}\left\{\frac{(k+\delta)^2-1}{2\sqrt{k\delta}}\,F(\kappa) - \frac{k^2-1+\delta^2}{\sqrt{k\delta}}\,E(\kappa)\right\}, \tag{4}$$

where

$$\mu = \frac{k^2-1+\delta^2}{2\delta k} \quad \text{and} \quad \kappa = \frac{1}{2}\sqrt{\frac{1-(k-\delta)^2}{k\delta}}$$

and r_1 is taken as the radius of the larger component of the binary system.

The evaluation of Equation (4) for the case $k=0$ involves cumbersome algebra which eventually yields

$$\frac{\partial^1\alpha^{tr}(p,0)}{\partial p} = \frac{5}{8}\left[-\left(\frac{1+p}{2}\right)F + 5pE\right],$$

where the modulus of the elliptic integrals is given by

$$\kappa = \sqrt{(1-p)/2}\,.$$

For the case $k=1$ the form of $\partial^1\alpha^{tr}/\partial p\ (p,\ 1)$ is the same as that of $\partial^1\alpha^{oc}(p,\ 1)/\partial p$. Consequently for the case $k=1$ and $|p|=1$ we have

$$\partial^1\alpha^{tr}(-1,1)/\partial p = 0 \quad \text{and} \quad \partial^1\alpha^{tr}(1,1)/\partial p = 0\,.$$

E. ANNULAR PHASES OF TRANSIT ECLIPSES

Equation (2) continues to serve as the starting point. However, during the annular phases of the transit eclipse the quantity $I^1_{-1,1}$ is given by

$$I^1_{-1,1} = \sqrt{\frac{\delta_K}{r_2}}\,\frac{4}{3\pi\kappa}\{(1+\mu)\,F(\kappa) - \mu E(\kappa)\}$$

where

$$\mu = \frac{k^2 - 1 + \delta^2}{2\delta k} \quad \text{and} \quad \kappa = 2\sqrt{\frac{k\delta}{1-(k-\delta)^2}}\,.$$

Inserting this value of $I^1_{-1,1}$ in Equation (2) we obtain

$$\frac{\partial^1\alpha^{ann}}{\partial p} = -\frac{2}{\pi\,{}^1\tau(k)}\,k\sqrt{1-(k-\delta)^2} \times\left\{\frac{(k+\delta)^2-1}{2\delta}\,F(\kappa) - \frac{k^2-1+\delta^2}{2\delta}\,E(\kappa)\right\}. \tag{5}$$

Evaluation of Equation (5) for $k=0$ involves lengthy algebraic development which eventually yields

$$\frac{\partial^1\alpha^{ann}(p,0)}{\partial p^{ann}} = \frac{5}{8}\left(\frac{2}{1-p^{tr}}\right)^{3/2}\left\{(1+p^{tr})\,F(\kappa) - \left(\frac{3+p^{tr}}{2}\right)E(\kappa)\right\},$$

where

$$\kappa = \sqrt{\frac{1-p^{tr}}{2}}$$

and the geometrical depths are related by

$$p^{tr} = -\frac{3+p^{ann}}{1+p^{ann}}, \qquad -\frac{1}{k} \leqslant p^{ann} < -1\,.$$

When $k=1$ Equation (5) can be written as

$$\frac{\partial^1\alpha^{ann}(p,1)}{\partial p} = -\frac{2}{\pi}\sqrt{1-p^2}\,\{(3+p)\,F - (1-p)\,E\}\,.$$

where superscript tr on p has been omitted. The latter expression indicates that

$$\frac{\partial^1\alpha^{\mathrm{ann}}(-1,1)}{\partial p}=0.$$

Note that $\partial^1\alpha^{\mathrm{ann}}(p,k)/\partial p$ is undefined for $p>-1$. The limiting case for $p=-1/k$ leads to an indeterminate expression of the form $(\infty-\infty)$. Lengthy algebraic development is again involved in removing this indeterminacy. It is found that

$$\lim_{p\to-1/k}\frac{\partial^1\alpha^{\mathrm{ann}}}{\partial p}=0.$$

Numerical values of various derivatives evaluated at critical points are summarized in Table I.

TABLE I

k	$p=1$ $^0\dot{\alpha}_p$	$^1\dot{\alpha}_p{}^{\mathrm{oc}}$	$^1\dot{\alpha}_p{}^{\mathrm{tr}}$	$^1\dot{\alpha}_p{}^{\mathrm{ann}}$	$p=-1$ $^0\dot{\alpha}_p$	$^1\dot{\alpha}_p{}^{\mathrm{oc}}$	$^1\dot{\alpha}_p{}^{\mathrm{tr}}$	$^1\dot{\alpha}_p{}^{\mathrm{ann}}$
0	0	0	0		0	0	$-\frac{5}{8}$	$-\frac{5}{8}$
1	0	0	0		$-2/\pi$	0	0	0

Vanishing of all derivatives at the time of external tangency is, of course, expected on physical grounds.

Note that $\partial^0\alpha/\partial p=0$ for any k and $p<-1$ and that $\partial^1\alpha^{\mathrm{ann}}/\partial p$ for any k and $p>-1$ does not exist. The behavior of $\partial^1\alpha^{\mathrm{tr}}(-1,k)/\partial p$ is described by

$$\frac{\partial^1\alpha^{\mathrm{tr}}(-1,k)}{\partial p}=-\frac{4}{\pi\,{}^1\tau(k)}k^2\sqrt{k(1-k)}.$$

Finally, recall that $\partial^1\alpha^{\mathrm{ann}}/\partial p$ vanishes when p approaches its limiting value of $-1/k$.

The values of $\partial\alpha(k,p)/\partial p$ for arbitrary values of limb darkening are obtained by the straightforward use of the well-known interpolation formulae (e.g. Merrill, 1950).

This completes the discussion of the partial derivatives with respect to the geometrical depth.

3. Partial Derivatives $\partial\alpha(k,p)/\partial k$

A. OCCULTATION OF UNIFORMLY BRIGHT STARS

Differentiating Kopal's function $\alpha_0^0(k,p)$ with respect to r_2 and solving the resulting expression for $\partial\alpha_0^0/\partial k$ we obtain

$$\frac{\partial\alpha_0^0}{\partial k}=\frac{2}{k^3}\{I_{-1,0}^{-1}-I_{-1,0}^{0}\}.$$

Substitution of explicit expressions for the I integrals yields

$$\frac{\partial^0\alpha(p,k)}{\partial k}=\frac{2}{\pi k^3}\{\sqrt{1-\mu^2}-\cos^{-1}\mu\},$$

where

$$\mu = \frac{1 - k^2 + \delta^2}{2\delta}.$$

As k approaches very small values the derivative becomes indeterminate. Examination of the indeterminacy yields

$$\frac{\partial^0 \alpha(p, 0)}{\partial k} = -\frac{1}{3\pi}(1 - p^2)^{3/2}.$$

The indeterminacy at $p=1, k=-1$ leads to $\partial^0\alpha\,(-1,\ 1)/\partial k=0$. In fact this derivative vanishes for any values of k and $p=-1$.

B. OCCULTATION OF TOTALLY LIMB DARKENED STARS

Differentiating Kopal's function ${}^1f^{oc}=\frac{3}{2}\alpha_1^0$ with respect to r_2, solving the resulting expression for $\partial^1 f^{oc}/\partial k$, and employing the appropriate I integrals we obtain

$$\frac{\partial^1 \alpha^{oc}(p, k)}{\partial k} = \frac{3}{k^4}(I^1_{-1,1} - I^0_{-1,1}).$$

Substitution of explicit expressions for the I integrals yields

$$\frac{\partial^1 \alpha^{oc}}{\partial k} = \frac{4\sqrt{\delta}}{\pi k^4}\{2(1 + \mu)\,F(\kappa) - 2(3 + \mu)\,E(\kappa)\} \tag{6}$$

where

$$\mu = \frac{1 - k^2 + \delta^2}{2\delta} \quad \text{and} \quad \kappa = \frac{k}{2}\sqrt{\frac{1 - p^2}{1 + kp}}.$$

As $k\to 0$ the derivative becomes indeterminate. The indeterminacy can be removed either by the application of l'Hospital's rule or by expansion of $\partial^1\alpha^{oc}/\partial k$ into a series in powers of k. The latter approach shows that in the neighborhood of $k=0$ $\partial^1\alpha^{oc}/\partial k$ behaves as

$$\frac{\partial^1 \alpha^{oc}}{\partial k} \sim -\tfrac{3}{32}(1 - p^2)^2 - \tfrac{3}{64}kp(1 - p^2)^2 + \cdots.$$

Consequently

$$\frac{\partial^1 \alpha^{oc}(p, 0)}{\partial k} = -\tfrac{3}{32}(1 - p^2)^2.$$

For cases $k=1$, $p=-1$ and $k=1$, $p=1$ the derivative vanishes. In fact for $p=|1|$ the derivative vanishes regardless of the value of k.

C. PARTIAL TRANSIT PHASES OF TOTALLY LIMB DARKENED STARS

To obtain the required expression we differentiate Kopal's function $\alpha_1^0 = \frac{2}{3}\,{}^1\tau(k)^1\alpha^{tr}$ with respect to r_1. Solving the resulting expression for $\partial^1\alpha^{tr}/\partial k$ and replacing in it various derivatives by their equivalents in terms of the I-integrals we obtain

$$\frac{\partial^1\alpha^{tr}(p,k)}{\partial k} = -\frac{2k\sqrt{k\delta}}{\pi^1\tau(k)} \times \left\{(1+\mu)(3+p)F(\kappa) - 2(3+\mu p)E(\kappa) + 8\sqrt{\frac{1-k}{1+kp}}\,{}^1\alpha^{tr}(p,k)\right\}, \tag{7}$$

where

$$\mu = \frac{k^2-1+\delta^2}{2\delta k} \quad \text{and} \quad \kappa = \frac{1}{2}\sqrt{\frac{1-(k-\delta)^2}{k\delta}}.$$

The derivative $\partial^1\alpha^{tr}(p,k)/\partial k$ becomes indeterminate when k approaches very small values. Evaluation of this indeterminacy, although straightforward, is quite involved. It yields the following result

$$\frac{\partial^1\alpha^{tr}(p,0)}{\partial k} = \tfrac{5}{224}(1+p)\left\{(p^2-8p-9)F - (2p^2-6p-12)E\right\}.$$

As for the case of occultation eclipses the derivative vanishes for all $k \neq 0$ and $p = |1|$.

D. ANNULAR PHASES OF TOTALLY LIMB DARKENED STARS

The procedure for arriving at $\partial^1\alpha^{ann}/\partial k$ is the same as that for the partial phases of a transit eclipse with the exception that $I^1_{-1,1}$ and $I^0_{-1,1}$ are different. Taking into account this distinction we arrive at

$$\frac{\partial^1\alpha^{ann}(p,k)}{\partial k} = -\frac{2k\sqrt{1-(k-\delta)^2}}{\pi^1\tau(k)} \times \left\{p(1+\mu)F(\kappa) - (3+\mu p)E(\kappa) + 4^1\alpha^{ann}\kappa\sqrt{\frac{1-k}{\delta}}\right\}, \tag{8}$$

where

$$\mu = \frac{k^2-1+\delta^2}{2k\delta} \quad \text{and} \quad \kappa = 2\sqrt{\frac{k\delta}{1-(k-\delta)^2}}.$$

By arguments similar to those employed previously we can show that for vanishingly small values of k the derivative is described by

$$\frac{\partial^1\alpha^{ann}(p,0)}{\partial k} = -\frac{5\sqrt{2}}{224}\sqrt{1-p}(1+p) \times \left\{(p^2-2p-3)F - (p^2-3p-6)E\right\}, \tag{9}$$

where $p = p^{ann} \leqslant -1$.

Equation (9) expressed in terms of p^{tr} becomes

$$\frac{\partial^1\alpha^{ann}(p,0)}{\partial k} = \frac{5}{224}\left(\frac{2}{1-p^{tr}}\right)^{7/2} \times (1+p^{tr})\left\{(p^{tr^2}-2p^{tr}-3)F - (2p^{tr^2}-3p^{tr}-3)E\right\}.$$

At the instant of internal tangency $p^{ann} = -1$, $\mu = -1$, and $\kappa = 1$. Consequently for any given $k \neq 0$ and $p^{ann} = -1$ the last two terms of Equation (8) cancel each other and the first, which is of the form $0 \cdot \infty$, can be shown to vanish.

The indeterminate expression which results at $p = -1/k$, describing the central phase, can be shown to reduce to

$$\frac{\partial^1 \alpha^{ann}(-1/k, k)}{\partial k} = -\frac{2k\sqrt{1-k^2}}{\pi\, {}^1\tau(k)} \times \left\{ \frac{\pi}{2(1-k^2)} - \frac{\pi(3k+1)}{2k} + 8\sqrt{\frac{k}{1+k}}\, {}^1\alpha^{ann}(-1/k, k) \right\},$$

valid for $0 < k < 1$.

E. TRANSIT ECLIPSES FOR STARS OF ARBITRARY LIMB DARKENING

For occultation eclipses partial derivatives with respect to the ratio of radii are obtained by the use of interpolation formulae mentioned previously. No difficulties arise because the coefficients of $\partial^0\alpha/\partial k$ and $\partial^1\alpha^{oc}/\partial k$ occurring in these formulae are independent of k. However, this is not the case for transit eclipses. For these we have

$$ {}^x\alpha^{tr} = \frac{\kappa(1-x)}{\kappa - x}\, {}^0\alpha + \frac{(\kappa-1)\,x}{\kappa - x}\, {}^1\alpha^{tr}$$

$$ {}^x\alpha^{ann} = \frac{(\kappa-1)\,x}{(\kappa - x)}\, {}^1\alpha^{ann}$$

where

$$\kappa = \frac{3k^2}{3k^2 - 2\,{}^1\tau(k)}.$$

Hence

$$\frac{\partial^x \alpha^{tr}}{\partial k} = \frac{4}{3\pi}\kappa^2 \left[\frac{8k^{5/2}\sqrt{1-k} - \pi\,{}^1\tau(k)}{k^3} \right] \frac{x(1-x)}{(\kappa-x)^2} \times ({}^1\alpha^{tr} - {}^0\alpha) + \frac{\kappa(1-x)}{(\kappa-x)} \frac{\partial k}{\partial^0 \alpha} + \frac{(\kappa-1)\,x}{(\kappa-x)} \frac{\partial^1\alpha^{tr}}{\partial k} \tag{10}$$

and

$$\frac{\partial^x \alpha^{ann}}{\partial k} = \frac{4}{3\pi}\kappa^2 \left[\frac{8k^{5/2}\sqrt{1-k} - \pi\,{}^1\tau(k)}{k^3} \right] \frac{x(1-x)}{(\kappa-x)^2} \times ({}^1\alpha^{ann} - 1) + \frac{(\kappa-1)\,x}{(\kappa-x)} \frac{\partial^1\alpha^{ann}}{\partial k}. \tag{11}$$

For small values of k the quantity κ varies as

$$\kappa \sim \frac{32}{15\pi} k^{-1/2}$$

and hence grows without limit as $k \to 0$. As long as $p \neq |1|$ other terms and factors in (10) remain finite and $\partial\kappa/\partial k$ dominates the behavior of $\partial^x\alpha^{tr}/\partial k$. However, for $p = |1|$

the difference $({}^{1}\alpha^{\mathrm{tr}} - {}^{0}\alpha)$ tends to zero as k, hence more strongly than $\partial\kappa/\partial k$ and, therefore, $\partial^{x}\alpha^{\mathrm{tr}}(\pm 1, 0)/\partial k \to 0$. Similar arguments show that $\partial\kappa/\partial k$ dominates the variation of $\partial^{x}\alpha^{\mathrm{ann}}/\partial k$ for $k=0$ and for values of $p \neq -1$, and that for $k=0$ and $p=-1$ the derivative $\partial^{1}\alpha^{\mathrm{ann}}/\partial k$ vanishes.

Note that all basic derivatives with respect to k vanish at critical points where $k=0$ or 1 and $p=|1|$.

4. Remarks

Forms of $\partial\alpha/\partial p$ and $\partial\alpha/\partial k$ as given are convenient for numerical computation since they involve quantities which are needed in any event to compute $\alpha^{\mathrm{oc}}(k, p)$, $\alpha^{\mathrm{tr}}(k, p)$, and $\alpha^{\mathrm{ann}}(k, p)$. The computation of all $\partial\alpha/\partial p$ has been incorporated by the author in an automatic scheme to solve the light curves of eclipsing binaries by the second method of Kopal. The derivatives $\partial\alpha/\partial k$ have been tabulated and are available. Computational details as well as detailed derivations of the expressions given in this paper are available from the author.

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