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A geometry for groups of J_3-type

Barbara Baumeister

Abstract. The proof of the existence and of the uniqueness of groups of J_3-type by G. Higman and J. McKay is based on the fact that a group of J_3-type is a faithful completion of an amalgam of J_3-type, see [11]. In this paper here, we provide a direct reference for that fact. The proofs in this paper are elementary and we do not use any character theory.

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1. Introduction. A finite simple group G is said to be of J_3-type provided that all involutions of G are conjugate and the centralizer of an involution is a split extension of an extraspecial group of order 32 by Alt(5). Janko presented the inital evidence of a group of J_3-type [13], G. Higman and J. McKay showed the existence and the uniqueness of groups of J_3-type [11]. Their proof is computer-based and uses, moreover, the fact that a group of J_3-type is a faithful completion of an amalgam of J_3-type.

An *amalgam of rank* n is a family

$$\mathcal{A} = (\alpha_{J,K} : P_J \to P_K \mid \emptyset \neq K \subset J \subseteq I),$$

where $I = \{1, \dots, n\}$, of group homomorphisms such that for all $L \subset K \subset J \subseteq I$

$$\alpha_{J,K}\alpha_{K,L} = \alpha_{J,L}.$$

To shorten notation we will simply write $\mathcal{A} = (P_J \mid \emptyset \neq J \subseteq I)$. A *completion* $\beta : \mathcal{A} \to G$ for $\mathcal{A}$ is a family $\beta = (\beta_J : P_J \to G)$ of group homomorphisms such that $G = \langle P_J^{\beta_J} \mid J \subset I\rangle$ and for all $K \subset J \subset I$ it holds $\alpha_{J,K}\beta_K = \beta_J$. A completion is said to be *faithful* if each β_J is an injection and a faithful completion $\gamma : \mathcal{A} \to G(\mathcal{A})$ is *universal* if for every completion $\beta : \mathcal{A} \to G$ there is a group homomorphism

φ of $G(\mathcal{A})$ onto G such that $\gamma_J\varphi = \beta_J$ for all $J \subseteq I$. These definitions are taken from [3]. In the following we omit brackets in $G_{\{i,j\}}$ by writing G_{ij}.

An amalgam of J_3-*type* is an amalgam $\mathcal{A} = \{G_1, G_2, G_3, G_{12}, G_{13}, G_{23}, G_{123}\}$ of rank 3 satisfying the following conditions, where $B := G_{123}$.

(i) $G_1 \cong L_2(16) : 2,\ G_2 \cong 2^4 : GL_2(4),\ G_3 \cong 3 : PGL_2(9)$;
(ii) $G_{12} \cong 2^4 : (3 \times D_{10}),\ G_{23} \cong GL_2(4) \cong 3 \times \mathrm{Alt}(5),\ G_{13} \cong \mathrm{Sym}(3) \times D_{10}$;
(iii) $B \cong 3 \times D_{10}$,

It was shown by J.G. Thompson that a group of J_3-type has a subgroup isomorphic to $L_2(16) : 2$, but this result was never published.

Meanwhile, there are also existence proofs of a group of J_3-type which are not computer dependent [15, 2, 4] and there is a computer-free uniqueness proof due to D. Frohardt [9].

In this paper we provide a direct reference for the fact that a group of J_3-type is a faithful completion of an amalgam of J_3-type. We show

Theorem 1. *Let G be a group of J_3-type. Then G is a completion of an amalgam of J_3-type.*

The hope is that we can use Theorem 1 to give a more simple uniqueness proof for groups of J_3-type because of the following facts. A completion of an amalgam of J_3-type acts flag-transitively on a Buekenhout geometry, namely on a dual extended quadrangle DEQ (see [5]) which is a geometry consisting of points, lines and quads such that

(res(p)). For a point p the lines and the quads which are incident with p form a complete graph whose vertices are the lines and whose edges are the quads;
(res(l)). Any point on a line l is incident to any quad which is incident with l;
(res(q)). For a quad q the points and the lines which are incident to q form a generalized quadrangle.

See [8] or [14] for an introduction to diagram geometries.

Let $\mathcal{A}$ be an amalgam of J_3-type and let G be a faithful completion of $\mathcal{A}$. Then the coset geometry $\Gamma = \Gamma(G, (G_1, G_2, G_3))$, a rank three geometry consisting of points, lines and quads, which are the cosets of G_i for $i = 1, 2, 3$ in G, respectively, such that two elements of the geometry are incident if and only if the respective cosets intersect non-trivially, is a DEQ and G acts flag-transitively on Γ. In [4] it was shown that there is up to isomorphism only one amalgam of J_3-type. This shows that there is at most one universal completion of an amalgam of J_3-type. By [4, Lemma 2.2] the latter group is finite. Moreover, in the same paper a DEQ, $\hat{\Gamma}$ has been constructed which admits a group of J_3-type as flag-transitive group of automorphisms.

The two latter facts and Theorem 1 imply the following.

Corollary 1.1. *Let G be a group of J_3-type. Then G acts flag-transitively on a DEQ which is a quotient of the universal cover of $\hat{\Gamma}$. In particular, G is isomorphic to a quotient of the universal completion of $\mathcal{A}$.*

To show that there is only one group of J_3-type up to isomorphism it remains to determine the universal cover of the geometry $\hat{\Gamma}$ and to study their quotients. Until now, this has been done only with the aid of a computer, see for instance [5].

Theorem 2. [5] *The universal cover of $\hat{\Gamma}$ is a triple cover of $\hat{\Gamma}$.*

The previous theorem implies that the completion of an amalgam of J_3-type is either a group of J_3-type or a triple cover of a groups of J_3-type and that there is exactly one group of J_3-type up to isomorphism.

The proof of Theorem 1 is almost self-contained. We only quote some standard group theory and the result of Bender which states that a group whose involution centralizers are dihedral groups of order 8 is of order either $8 \cdot 3 \cdot 7$ or $8 \cdot 9 \cdot 5$, see [6]. His proof is very short and elementary. We cite his result to construct the third parabolic subgroup G_3. The first parabolic subgroup G_1 is constructed using the amalgam method while we choose G_2 and G_3 as normalizers of an elementary abelian subgroup and a cyclic subgroup of order 16 and 3, respectively.

Contrary to Janko [13] we do not use any character theory.

2. Proof of Theorem 1. Let G be a group of J_3-type. Then all the involutions in G are conjugate and the centralizer of an involution is a split extension of an extraspecial group of order 32 by $\mathrm{Alt}(5)$.

Notation. For $g \in G$ let $C_g = C_G(g)$ and $N_g = N_G(\langle g \rangle)$. For $i \in G$ an involution, set $Q_i = O_2(C_i)$ and let T_i be a complement to Q_i in C_i.

So $|Q_i| = 32$ and $T_i \cong \mathrm{Alt}(5)$, for every involution i in G.

Lemma 2.1. *Assume that $i \in G$ is an involution. The following holds.*

(i) $C_{C_i}(Q_i) \leq Q_i$ *and* $Q_i \cong D_8 * Q_8$.
(ii) $Q_i/\langle i \rangle$ *is the even part of the permutation module for* $T_i \cong \mathrm{Alt}(5)$.
(iii) $Q_i/\langle i \rangle$ *is the* $\mathrm{O}_4^-(2)$*-module for* $T_i \cong \mathrm{O}_4^-(2)$ *and* T_i *is transitive on the singular subspaces of* $Q_i/\langle i \rangle$.
(iv) $Q_i/\langle i \rangle$ *is a projective module for* T_i.
(v) *Let s be an element of order 3 in T_i. Then* $C_s \cap Q_i \cong D_8$.

Proof. Assume $C_{C_i}(Q_i) \not\leq Q_i$. Then, as $C_{C_i}(Q_i)$ is normal in C_i, we have $C_{C_i}(Q_i)Q_i = C_i$. Therefore, there is a complement T to $C_{C_i}(Q_i) \cap Q_i = \langle i \rangle$ in $C_{C_i}(Q_i)$ which is isomorphic to $\mathrm{Alt}(5)$. Let j be an involution in T. Then C_i and C_j intersect in a Sylow 2-subgroup S. This is not possible, since i is a commutator in S, but j is not, which contradicts the fact that all involutions are conjugate in G. Hence $C_{C_i}(Q_i) \le Q_i$.

As Q_i is an extraspecial group, there is a non-degenerate quadratic form on Q_i which is left invariant by T_i. The fact that $T_i \cong O_4^-(2)$ implies that Q_i is of -type, that is $Q_i \cong D_8 * Q_8$. This also shows the first part of (iii) and application of the Lemma of Witt yields the second part of (iii).

It follows from (iii) that T_i has two orbits of size 5 and 10 on the set of involutions in $Q_i/\langle i\rangle$. Hence, $Q_i/\langle i\rangle$ is not a $GF(4)$-module for T_i. It is an easy exercise that there is exactly one module of order 2^4 which is not a $GF(4)$-module for T_i such that T_i has two orbits of size 5 and 10 on the set of involutions of the module. As the even part of the permutation module for $T_i \cong \mathrm{Alt}(5)$ satisfies these conditions, (ii) holds.

According to [10, Theorem 2.8.7] $Q_i/\langle i\rangle$ is a projective module for T_i as stated in (iv).

Let s be an element of order 3 in T_i. Then s centralizes a subgroup U of order 4 in the even part of the permutation module $Q_i/\langle i\rangle$. The preimages of two elements of U are of order 4 in Q_i, which implies $C_s \cap Q_i \cong D_8$, statement (v). □

Lemma 2.2. *G acts transitively on the set*

$$P := \{(j, W) \mid j \in G \textit{ an involution}, \ j \in W, \ W \textit{ elementary abelian of order } 2^4\}.$$

Proof. Let $U \le Q_i$, with i an involution, be an elementary abelian subgroup of maximal rank. By Lemma 2.1 U is of order 4, the involution i is in U and $N_{T_i}(U) \cong \mathrm{Alt}\,(4)$, as $U/Z(Q_i)$ is a singular point in $Q_i/Z(Q_i)$. Set $V = UO_2(N_{T_i}(U))$.

We claim that V is elementary abelian. As every element of order 3 of $N_{T_i}(U)$ acts trivially on U, also $O_2(N_{T_i}(U))$ acts trivially on U. Thus V is elementary abelian of order 16.

Now let W be some elementary abelian subgroup of order 16 in C_i. Then $Q_i \cap W \cong 2^2$ and $(Q_i \cap W)/\langle i\rangle$ is a singular point.

We claim that all the complements to $Q_i \cap W$ in W are conjugate under $C = C_{C_i}(Q_i \cap W)$. We have $C_{Q_i}(Q_i \cap W) \cong Q_8 \times 2$ and $C \cong (Q_8 \times 2) : \mathrm{Alt}(4)$. We count the elementary abelian subgroups of $Z \setminus Q_i$ of order 4 where $Z = O_2(C)$. Let f be an involution in $Z \cap T_i$. Then we see in the permutation module $Q_i/\langle i\rangle$ for T_i that f inverts two subgroups $\langle c_1\rangle$, $\langle c_2\rangle$ of order 4 in $C \cap Q_i$ and that $c_1c_2 \in Q_i \cap W$. Hence there are two different elementary abelian subgroups of order 8 in $C \cap Q_i : \langle f\rangle$ and therefore there are precisely four complements to $C \cap Q_i$ in Z. It is $|Z \cap T_i| = 4$ and $N_Z(Z \cap T_i)$ is of order 2^4, which implies, as $|Z| = 2^6$, that $|(Z \cap T_i)^Z| = 4$. Thus all the complements to $C \cap Q_i$ in Z are conjugate. This yields that all the complements to $Q_i \cap W$ in W are conjugate in C_i as asserted.

Therefore we may assume $W = (Q_i \cap W)O_2(N_{T_i}(Q_i \cap W))$. Thus, since T_i acts transitively on the singular points in $Q_i/\langle i\rangle$, the centralizer C_i acts transitively on the elementary abelian subgroups of C_i of maximal rank. As in G there is only one class of involutions, G acts transitively on P, as claimed. □

In the following let V be a maximal elementary abelian subgroup of order 2^4.

Lemma 2.3. $N_G(V) \cong 2^4 : GL_2(4)$.

Proof. By Lemma 2.2 $N_G(V)$ acts transitively on $V^{\#}$. Let i be an involution in V, then $N_{C_i}(V) \cong 2^4 : \mathrm{Alt}(4)$. It can be observed in C_i that $C_G(V) = V$, so we obtain that $N_{C_i}(V)$ induces on V a group of order 12 which is in fact the stabilizer of an element of $V^{\#}$ in $N_G(V)$. Thus $N_G(V)$ induces on V a group of order $12 \cdot 15$ which is transitive on $V^{\#}$. This yields, as $N_G(V)$ is a subgroup of $2^4 : SL_4(2)$, that $N_G(V)/V \cong GL_2(4)$, see [12, II (8.27)]. Let S be a Sylow 3-subgroup of $O_{2,3}(N_G(V))$. Then, as S acts fixed point freely on V, the Frattini argument implies that the normalizer of S in $N_G(V)$ is a complement to V in $N_G(V)$. Thus $N_G(V)$ splits over V, which proves the assertion. □

Set
$$G_2 := N_G(V).$$

Let L_1 be a subgroup of $N_G(V)$ isomorphic to $2^4 : (3 \times D_{10})$ and let L_{12} be a subgroup of L_1 isomorphic to $3 \times D_{10}$. Let $\langle s\rangle = O_3(L_{12})$. Next, we construct G_3.

Lemma 2.4. $N_s \cong (3 \times \mathrm{Alt}(6)) : 2 \cong 3 : PGL_2(9)$.

Proof. The element s is centralized by an involution i. By 2.1 (v) we have $C_s \cap C_i \cong D_8 \times 3$. As all the involutions of G are conjugate and as $\langle s\rangle$ is a Sylow-3-subgroup of C_i, all the involutions in C_s are conjugate and the centralizer of every involution in $\overline{C}_s = C_s/\langle s\rangle$ is a dihedral group of order 8. By the result of Bender [6] we have $|\overline{C}_s| = 3 \cdot 8 \cdot 7$ or $8 \cdot 9 \cdot 5$. As 5 divides $|\overline{C}_s|$, the latter holds. Let $R := C_{G_2}(s) \cong 3 \times \mathrm{Alt}(5)$, then $\overline{R}$ is a subgroup of index 6 of $\overline{C}_s$ and it follows that $\overline{C}_s \cong \mathrm{Alt}(6)$. As there is an involution in C_i which inverts s, it follows $N_s/\langle s\rangle \cong PGL_2(9)$ or $\mathrm{Sym}(6)$. Assume the latter. Then every Sylow 2-subgroup U of N_s is isomorphic to $D_8 \times 2$. Let $C_i = Q_i : T_i$. As $Q_i/\langle i\rangle$ is the even part of the permutation module for $T_i \cong \mathrm{Alt}(5)$, see 2.1, we see easily that $U \ncong D_8 \times 2$. Thus N_s is an extension of $\langle s\rangle$ by a group isomorphic to $PGL_2(9)$.

It remains to show that this extension splits. Let σ be an element of order 3 in $R' \cong \mathrm{Alt}(5)$. Then $N_R(\langle\sigma\rangle) = \langle s\rangle \times A$ with $A \leq R'$ and $A \cong \mathrm{Sym}(3)$ and there is an involution which inverts σ and centralizes s. As there is no involution in N_s which inverts s and centralizes an element of order 3 in N_s, the subgroups $\langle s\rangle$ and $\langle\sigma\rangle$ are not conjugate in G. As $s \cdot \sigma$ centralizes an involution in $O_2(G_2)$, this element is conjugate to s. If N_s were a non-split extension, then a Sylow 3-subgroup of N_s would be an extraspecial group of order 27 and the elements σ and $s \cdot \sigma$ would be conjugate in C_s. Since this is not the case, we have proven the assertion of the lemma. $\square$

Recall the definition of L_1 and L_{12} just before Lemma 2.4. Set

$$G_3 := N_s.$$

The next result follows from Lemma 2.4.

Lemma 2.5. $N_G(L_{12}) \cong \mathrm{Sym}(3) \times D_{10}$.

Set

$$L_2 := N_G(L_{12}).$$

Notice, that $L_{12} = L_1 \cap L_2$.

Let f be an involution and w an element of order 5 in L_{12}. Then f inverts $\langle w \rangle$ and centralizes $\langle s \rangle$ and $C_{L_1}(f) \cong 2^2 : 3 \times 2 \cong \mathrm{Alt}(4) \times 2$ and $C_{L_2}(f) \cong \mathrm{Sym}(3) \times 2$.

Set

$$L_3 := \langle C_{L_1}(f), C_{L_2}(f) \rangle.$$

Lemma 2.6. $L_3 \cong \mathrm{Alt}(5) \times 2$.

Proof. We have $N_{C_f}(\langle s \rangle) \cong 3 : D_{16}$ and all the subgroups isomorphic to

$$C_{L_2}(f) \cong \mathrm{Sym}(3) \times 2$$

are conjugate in $N_{C_f}(\langle s \rangle)$. Therefore, we may choose a complement $T_f \cong \mathrm{Alt}(5)$ to Q_f in C_f such that $T_f \cap C_{L_2}(f) \cong \mathrm{Sym}(3)$.

It remains to show that $C_{L_1}(f)$ is contained in a conjugate of $\langle f \rangle \times T_f$ under the action of the normalizer of $C_{L_2}(f)$ in C_f.

Assume $C_{L_1}(f) \cap Q_f > \langle f \rangle$. Then $C_{L_1}(f) \cap Q_f$ is elementary abelian of order 8, which contradicts the fact that $Q_f \cong D_8 * Q_8$ is of minus-type, see Lemma 2.1. Therefore, we have $C_{L_1}(f) \cap Q_f = \langle f \rangle$.

We claim that all the subgroups isomorphic to $2 \times \mathrm{Alt}(4)$ which intersect Q_f precisely in $\langle f \rangle$ and which contain s are conjugate in $C_f \cap C_s$. Let X be such a subgroup. Let U be the projection of XQ_f/Q_f onto T_f and let u be an involution in U. Then $\tilde{C} = C_{Q_f/\langle f \rangle}(u) = 2^2$ with preimage $K \cong 4 \times 2$ and u inverts every element of order 4 of K. Let $C_K(s) = \langle f, b \rangle$. Then b is an involution and notice, if $\langle qu, (qu)^s \rangle \cong 2^2$ for some $q \in K$, then $\langle bqu, (bqu)^s \rangle \not\cong 2^2$. This shows that there are precisely two subgroups $\langle qu, (qu)^s \rangle$ with $q \in K$ which are elementary abelian of order 4. We have $C_{Q_f}(s) \cong D_8$ and $C_{Q_f}(\langle s, u \rangle) \cong 2^2$ which implies that the two subgroups are conjugate under $C_{Q_f}(s)$. This proves the claim.

Hence, $C_{L_1}(f)$ is conjugate to a subgroup of $\langle f \rangle \times T_f$ under the action of the normalizer of $C_{L_2}(f)$ in C_f. So, we may assume that T_f is chosen such that $C_{L_1}(f) \leq \langle f \rangle \times T_f$. This yields the assertion. $\square$

Set

$$L = \langle L_1, L_2 \rangle.$$

Then $L_3 \leq L$. Recall that

$$L_1 \cong 2^4 : (3 \times D_{10}), L_2 \cong \mathrm{Sym}(3) \times D_{10} \text{ and } L_3 \cong 2 \times \mathrm{Alt}(5).$$

To prove that $L \cong L_2(16) : 2$, we show the following.

Lemma 2.7. *Let H be a group and H_1, H_2, H_3 subgroups of H such that*

(i) $H = \langle H_1, H_2 \rangle$*;*
(ii) $H_1 \cong 2^4 : (3 \times D_{10}), C_{H_1}(O_2(H_1)) = O_2(H_1)$*;* $H_2 \cong \mathrm{Sym}(3) \times D_{10}$*;* $H_3 \cong \mathrm{Alt}(5) \times 2$*; and*
(iii) $H_1 \cap H_2 \cong 3 \times D_{10}$*;* $H_1 \cap H_3 \cong \mathrm{Alt}(4) \times 2$*;* $H_2 \cap H_3 \cong \mathrm{Sym}(3) \times 2$.

Then H is a triply transitive permutation group of degree 17; in this action H_1 is the stabilizer of a point and $|H| = 2 \cdot 15 \cdot 16 \cdot 17$.

Proof. Let $\langle s \rangle = O_3(H_2)$, $\langle w \rangle = O_5(H_2)$ and let b, i be involutions in $H_2 \cap H_3$ with

$$s^b = s^{-1},\ w^b = w \text{ and } s^i = s, w^i = w^{-1}.$$

Let Θ be a graph whose vertices are the cosets of H_1 in H and whose edges are the sets $\{H_1 x, H_1 bx\}$ with $x \in H$. As by (i) $H = \langle H_1, H_2 \rangle = \langle H_1, b \rangle$, this graph is connected.

We claim that Θ is a graph of valency 16. Clearly, b normalizes $H_1 \cap H_2$. If b would also normalize H_1, then $H = H_1 \langle b \rangle$ in contradiction to $2 \times \mathrm{Alt}(5) \cong H_3 \leq H$. Since $C_{H_1}(O_2(H_1)) = O_2(H_1)$, the intersection $H_1 \cap H_2$ is maximal in H_1 which implies $H_1 \cap H_1^b = H_1 \cap H_2 \cong 3 \times D_{10}$ is the stabilizer of the two neighbours H_1 and $H_1 b$ in H. Thus Θ is of valency $|H_1 : H_1 \cap H_1^b| = 16$, as claimed.

Therefore, it follows that $O_2(H_1)$ acts regularly on $\Theta(H_1)$. Moreover, as $H_1 \cap H_2$ is transitive on $O_2(H_1)^\#$, it follows that H_1 acts doubly transitively on its neighbours $\Theta(H_1)$.

Next, we show that Θ is a complete graph. Notice, that the facts $H_3 \cong 2 \times \mathrm{Alt}(5)$, $b \in (H_2 \cap H_3) \setminus H_1$ and $H_1 \cap H_3 \cong 2 \times \mathrm{Alt}(4)$ yield that there is an $h \in H_1 \cap H_3$ such that $(bh)^3 \in \langle i \rangle$. Hence

$$H_1 bhb = H_1 hbh = H_1 bh$$

is a common neighbour of H_1 and $H_1 b$. This shows that there is a triangle in Θ. Now, the fact that H_1 acts doubly transitively on $\Theta(H_1)$ implies that every vertex in $\Theta(H_1)$ is a neighbour of $H_1 b$, so Θ is a complete graph.

Thus Θ consists of 17 vertices and $|H : H_1| = 17$ which implies $|H| = |H_1| \cdot 17 = 2 \cdot 15 \cdot 16 \cdot 17$ and H acts triply transitively on the cosets of H_1 in H. □

Corollary 2.8. *Let L be a faithful completion of an amalgam*

$$\mathcal{B} = \{H_1, H_2, H_3, H_{12}, H_{13}, H_{23}\},$$

where the groups H_1, H_2 and H_3, $H_{ij} := H_i \cap H_j$ $(1 \leq i < j \leq 3)$ are as described in Lemma 2.7. Then $|L| = 2 \cdot 15 \cdot 16 \cdot 17$. In particular, every faithful completion of such an amalgam is already universal.

Notice that $H = L_2(16) : 2$ possesses such an amalgam $\mathcal{B}$: Let H_1 be a point stabilizer in H in its action of degree 17. Then $H_1 \cong 2^4 : (3 \times D_{10})$. Let H_2 be the setwise stabilizer of two points such that $H_1 \cap H_2 \cong 3 \times D_{10}$. Finally, let f be an involution in $H_1 \cap H_2$ and set $H_3 = C_H(f)$. Then H_1, H_2, H_3, $H_{ij} := H_i \cap H_j$ $(1 \leq i < j \leq 3)$ form an amalgam as described in Lemma 2.7. By Lemma 2.8 a completion of an amalgam of type $\mathcal{B}$ is a triply transitive permutation group of degree 17.

Lemma 2.9. *The embeddings of $H_1 \cong 2^4 : (3 \times D_{10})$ and of $H_2 \cong \mathrm{Sym}(3) \times D_{10}$ in $\mathrm{Sym}(17)$ as the stabilizer of a point and of a 2-set containing that point, respectively, are unique up to conjugation in $\mathrm{Sym}(17)$.*

Proof. Let H_1 be the stabilizer of 1. Then $O_2(H_1)$ acts regularly on $\{2, \ldots, 17\} =: \Omega$. Let $K = \mathrm{Sym}(\Omega)$. Then $N_K(O_2(H_1)) \cong 2^4 : L_4(2)$. We may assume that $O_3(H_1 \cap H_2)$ fixes $2 \in \Omega$. As $O_3(H_1 \cap H_2)$ acts fixed point freely on $O_2(H_1)^{\#}$, it follows that $H_1 \cap H_2 = C_{H_1}(O_3(H_1 \cap H_2))$ is a subgroup of the stabilizer of 2 in H_1 and therefore $H_1 \cap H_2$ is the stabilizer of 2 in H_1. Moreover, the action of $H_1 \cap H_2$ on Ω is uniqely determined up to conjugation in $N_K(O_2(H_1))$. Let a be an involution in $H_2 \setminus H_1 \cap H_2$ which centralizes $O_5(H_1 \cap H_2)$. Then a interchanges 1 and 2 and it fixes all 5 3-cycles of $O_3(H_1 \cap H_2)$ on the set $\Omega \setminus \{2\}$. We may assume the action of a on one of the 3-cycles which then determines uniquely the action of a on Ω. □

The previous lemma yields that the amalgam $\mathcal{B}$ is uniquely determined. This shows the following.

Corollary 2.10. *The universal completion of $\mathcal{B}$ is isomorphic to $L_2(16) : 2$. In particular, L is isomorphic to $L_2(16) : 2$.*

Set

$$G_1 := L.$$

Lemma 2.11. *$\mathcal{A} = \{G_1, G_2, G_3, G_{12}, G_{13}, G_{23}, B\}$ is an amalgam of type J_3.*

Proof. By construction $\mathcal{A}$ is of type J_3. □

Lemma 2.11 completes the proof of Theorem 1.

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Barbara Baumeister, Institut für Mathematik II, Fachbereich Mathematik und Informatik, FU Berlin, D-14195 Berlin, Germany
e-mail: baumeist@math.fu-berlin.de

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