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Invariant Characteristic Representations for Classical and Micropolar Anisotropic Elasticity Tensors

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Abstract. By extending and developing the characteristic notion of the classical linear elasticity initiated by Lord Kelvin, a new type of representation for classical and micropolar anisotropic elasticity tensors is introduced. The new representation provides general expressions for characteristic forms of the two kinds of elasticity tensors under the material symmetry restriction and has many properties of physical and mathematical significance. For all types of material symmetries of interest, such new representations are constructed explicitly in terms of certain invariant constants and unit vectors in directions of material symmetry axes and hence they furnish invariants which can completely characterize the classical and micropolar linear elasticities. The results given are shown to be useful. In the case of classical elasticity, the spectral properties disclosed by our results are consistent with those given by similar established results.

1. Introduction

Properties of the elasticity tensor and the related linear stress-strain relation have been receiving much attention. Some important decompositions of the classical elasticity tensor were suggested in [1–9]; Ting [11], Hahn [10], and Srinivasan and Nigam [2] studied invariants of the classical elasticity tensor; Cowin and Mehrabadi [12], Cowin [13], and Norris [14–15] discussed identification of elastic symmetry; and Huo and Del Piero [16] proved the completeness of the crystallographic symmetries in describing the symmetries of the classical elasticity tensor. Moreover, Gurtin [17–18], Martins and Podio-Guidugli [19], Guo [20–21], and De Boor [22] offered new, concise proofs for isotropic, linear symmetric and asymmetric stress-strain relations respectively.

Recently, by extending and developing the characteristic notion of the classical linear elasticity initiated by Lord Kelvin [23–24] (cf. Todhunter and Pearson [25]) and later proposed again by Pipkin [26] and Rychlewski [27], Mehrabadi and Cowin [28] and Sutcliffe [29] systematically studied the spectral properties of various types of anisotropic elasticity tensors, and this author [30] introduced and determined invariant characteristic representations for symmetric material tensors including classical and micropolar elasticity tensors.

In this paper, we shall report the main results for the latter. We shall present general expressions for characteristic forms of various types of classical and micropolar anisotropic elasticity tensors, which are constructed explicitly in terms of certain invariant constants and unit vectors in directions of material symmetry axes. These results are shown to have many properties of physical and mathematical significance and to be useful. In the case of the classical elasticity, the spectral properties disclosed by our results are consistent with those given by Mehrabadi and Cowin's and Sutcliffe's results (cf. [28–29]).

Throughout, the results involving vectors and tensors are given in direct notation. Such results are more compact and clear than component results and from them component results under any given coordinate system can be obtained directly, without resorting to the commonly used tedious component transformation formulae.

2. Classical and Micropolar Elasticity Tensors

Let V be a 3-dimensional Euclidean space and Lin the space of second order tensors over V. Denote the symmetric subspace and the skewsymmetric subspace of Lin by Sym and Skw, respectively. It is easily understood that Lin is an inner product space endowed with the inner product

$$(\mathbf{X}, \mathbf{Y}) = \mathbf{X}\colon \mathbf{Y} = \operatorname{tr}(\mathbf{X}\mathbf{Y}^T) \quad (\forall \mathbf{X}, \mathbf{Y} \in \mathrm{Lin}). \tag{2.1}$$

Here and henceforth the sign : is used to denote the double dot product.

(a) Classical elasticity tensors. The generalized Hooke's law for the classical linear elastic material is of the form

$$\boldsymbol{\sigma} = \mathbf{C}\colon \boldsymbol{\varepsilon}, \quad \text{i.e.} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{2.2}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress tensor and the strain tensor, both symmetric, and the fourth order tensor $\mathbf{C}$ over V is known as the stiffness tensor, with the properties

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk}, \quad C_{ijkl} = C_{klij}. \tag{2.3}$$

The inverse of (2.2) is as follows:

$$\boldsymbol{\varepsilon} = \mathbf{S}\colon \boldsymbol{\sigma}, \quad \text{i.e.} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl}, \tag{2.4}$$

where the fourth order tensor $\mathbf{S}$ is called the compliance tensor, with the symmetry properties as shown by (2.3) and the property

$$\mathbf{S}\colon \mathbf{C} = \mathbf{C}\colon \mathbf{S} = \mathbf{II}^s, \quad \text{i.e.}\ S_{ijpq}C_{pqkl} = C_{ijpq}S_{pqkl} = II^s_{ijkl}, \tag{2.5}$$

where

$$II^s_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.6}$$

and δ_{ij} is the Kronecker delta.

C and **S** are called the classical elasticity tensors.

The elastic energy function U_c of a classical linear elastic material is given by

$$2U_c = \boldsymbol{\sigma}\colon\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\colon\mathbf{C}\colon\boldsymbol{\varepsilon} = \boldsymbol{\sigma}\colon\mathbf{S}\colon\boldsymbol{\sigma}. \tag{2.7}$$

It is required that U_c be a positive definite bilinear form, i.e.

$$U_c > 0 \text{ for all } \boldsymbol{\varepsilon} \neq \mathbf{0} \text{ and } \boldsymbol{\sigma} \neq \mathbf{0}. \tag{2.8}$$

Thus, the elasticity tensors **C** and **S** must be positive definite.

(b) Micropolar elasticity tensors. The constitutive equations of a linear micropolar elastic material are of the form

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{A}\colon\boldsymbol{\varepsilon} + \mathbf{B}\colon\boldsymbol{\tau}, \\ \mathbf{m} = \mathbf{B}^T\colon\boldsymbol{\varepsilon} + \mathbf{D}\colon\boldsymbol{\tau}, \end{cases} \text{i.e.} \begin{cases} \sigma_{ij} = A_{ijkl}\varepsilon_{kl} + B_{ijkl}\tau_{kl}, \\ m_{ij} = B_{klij}\varepsilon_{kl} + D_{ijkl}\tau_{kl}, \end{cases} \tag{2.9}$$

where $\boldsymbol{\sigma}$ and $\mathbf{m}, \boldsymbol{\varepsilon}$ and $\boldsymbol{\tau}$ are the stress tensor and the couple-stress tensor, the strain tensor and the torsion tensor, respectively, and each of them, in general, is asymmetric; the fourth order tensors $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ are called the micropolar stiffness tensors. Of them, **B** is a pseudo-tensor and **A** and **D** are absolute tensors, with the properties

$$A_{ijkl} = A_{klij}, \quad D_{ijkl} = D_{klij}. \tag{2.10}$$

The inverse of (2.9) is as follows:

$$\begin{cases} \boldsymbol{\varepsilon} = \bar{\mathbf{A}}\colon\boldsymbol{\sigma} + \bar{\mathbf{B}}\colon\mathbf{m}, \\ \boldsymbol{\tau} = \bar{\mathbf{B}}^T\colon\boldsymbol{\sigma} + \bar{\mathbf{D}}\colon\mathbf{m}, \end{cases} \text{i.e.} \begin{cases} \varepsilon_{ij} = \bar{A}_{ijkl}\sigma_{kl} + \bar{B}_{ijkl}m_{kl}, \\ \tau_{ij} = \bar{B}_{klij}\sigma_{kl} + \bar{D}_{ijkl}m_{kl}, \end{cases} \tag{2.11}$$

where $\bar{\mathbf{A}}, \bar{\mathbf{B}}$ and $\bar{\mathbf{D}}$ are called the micropolar compliance tensors. Of them, $\bar{\mathbf{B}}$ is a pseudo-tensor and $\bar{\mathbf{A}}$ and $\bar{\mathbf{D}}$ are absolute tensors and have the symmetry properties as shown by (2.10). Moreover,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix} \colon \begin{bmatrix} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \bar{\mathbf{B}}^T & \bar{\mathbf{D}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \bar{\mathbf{B}}^T & \bar{\mathbf{D}} \end{bmatrix} \colon \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{II} & \mathbf{O} \\ \mathbf{O} & \mathbf{II} \end{bmatrix}, \tag{2.12}$$

where

$$II_{ijkl} = \delta_{ik}\delta_{jl}. \tag{2.13}$$

The elastic energy function U_m of a linear micropolar elastic material is given by

$$\begin{aligned} 2U_m = \boldsymbol{\sigma}{:}\,\boldsymbol{\varepsilon} + \mathbf{m}{:}\,\boldsymbol{\tau} &= \boldsymbol{\varepsilon}{:}\,\mathbf{A}{:}\,\boldsymbol{\varepsilon} + \boldsymbol{\tau}{:}\,\mathbf{D}{:}\,\boldsymbol{\tau} + 2\boldsymbol{\varepsilon}{:}\,\mathbf{B}{:}\,\boldsymbol{\tau} \\ &= \boldsymbol{\sigma}{:}\,\bar{\mathbf{A}}{:}\,\boldsymbol{\sigma} + \mathbf{m}{:}\,\bar{\mathbf{D}}{:}\,\mathbf{m} + 2\boldsymbol{\sigma}{:}\,\bar{\mathbf{B}}{:}\,\mathbf{m}. \end{aligned} \tag{2.14}$$

The fourth order tensors **A**, **B** and **D**; $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{D}}$ are called the micropolar elasticity tensors. For centrosymmetric materials, the pseudo-tensors **B** and $\bar{\mathbf{B}}$ vanish and hence (2.9, 11, 12, 14) are reduced to

$$\boldsymbol{\sigma} = \mathbf{A}{:}\,\boldsymbol{\varepsilon},\ \ \mathbf{m} = \mathbf{D}{:}\,\boldsymbol{\tau}; \tag{2.15}$$

$$\boldsymbol{\varepsilon} = \bar{\mathbf{A}}{:}\,\boldsymbol{\sigma},\ \ \boldsymbol{\tau} = \bar{\mathbf{D}}{:}\,\mathbf{m}; \tag{2.16}$$

$$\mathbf{A}{:}\,\bar{\mathbf{A}} = \bar{\mathbf{A}}{:}\,\mathbf{A} = \mathbf{D}{:}\,\bar{\mathbf{D}} = \bar{\mathbf{D}}{:}\,\mathbf{D} = \mathbf{II}; \tag{2.17}$$

$$2U_m = \boldsymbol{\varepsilon}{:}\,\mathbf{A}{:}\,\boldsymbol{\varepsilon} + \boldsymbol{\tau}{:}\,\mathbf{D}{:}\,\boldsymbol{\tau} = \boldsymbol{\sigma}{:}\,\bar{\mathbf{A}}{:}\,\boldsymbol{\sigma} + \mathbf{m}{:}\,\bar{\mathbf{D}}{:}\,\mathbf{m}. \tag{2.18}$$

It is required that the elastic energy function U_m be a positive definite bilinear form, i.e.

$$\begin{aligned} &U_m > 0 \quad \text{for all } \boldsymbol{\varepsilon} \neq \mathbf{O} \quad \text{and} \quad \boldsymbol{\tau} \neq \mathbf{O} \quad \text{or for all } \boldsymbol{\sigma} \neq \mathbf{O} \\ &\text{and} \quad \mathbf{m} \neq \mathbf{O}. \end{aligned} \tag{2.19}$$

Thus, the absolute micropolar elasticity tensors **A**, **D**, $\bar{\mathbf{A}}$ and $\bar{\mathbf{D}}$ must be positive definite. From (2.18) it can be seen that the latter is the necessary and sufficient condition of (2.19) for centrosymmetric materials.

Henceforth we denote any of the classical elasticity tensors and any of the absolute micropolar elasticity tensors by **E** and **M**, respectively. The 6×6 and 9×9 elastic matrices for various types of **E** and **M** are available, refer to Gurtin [17] and Ilcewicz et al. [37].

3. Characteristic Forms of E and M

Let $Z \subset \mathrm{Lin}$ be a nonzero subspace. The orthogonal projection of Z is a fourth order tensor over V and also a symmetric second order tensor over Lin, defined by

$$\mathbf{P}(Z){:}\,\mathbf{X} = \begin{cases} \mathbf{X}, & \mathbf{X} \in Z, \\ \mathbf{O}, & \mathbf{X} \in \bar{Z}, \end{cases} \tag{3.1}$$

where $\bar{Z}$ is the orthocomplement of the subspace $Z \subset \mathrm{Lin}$ with respect to Lin. Let $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_c$ be an orthonormal basis of Z, i.e.

$$\boldsymbol{\omega}_i{:}\,\boldsymbol{\omega}_j = \delta_{ij}, i, j = 1, 2, \ldots, c; c = \dim Z. \tag{3.2}$$

Then

$$\mathbf{P}(Z) = \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 + \cdots + \boldsymbol{\omega}_c \otimes \boldsymbol{\omega}_c. \tag{3.3}$$

The classical and micropolar elasticity tensors $\mathbf{E}$ and $\mathbf{M}$ are fourth order tensors over V and also are symmetric second order tensors over Sym and Lin, respectively. Let $\lambda_1, \ldots, \lambda_r$ be all the distinct eigenvalues of $\mathbf{T} \in \{\mathbf{E}, \mathbf{M}\}$ and $Z_1, \ldots, Z_r$ be the subordinate eigenspaces. Then $\mathbf{T}$ has the following unique characteristic form:

$$\mathbf{T} = \sum_{\sigma=1}^{r} \lambda_\sigma \mathbf{P}(Z_\sigma), \tag{3.4}$$

where the subordinate eigenspaces constitute the orthogonal decomposition of Sym ($\mathbf{T} = \mathbf{E}$) or Lin ($\mathbf{T} = \mathbf{M}$), i.e.

$$\mathbf{P}(Z_\sigma) : \mathbf{P}(Z_\tau) = \delta_{\sigma\tau} \mathbf{P}(Z_\tau), \sigma, \tau = 1, \ldots, r, \tag{3.5}$$

$$\mathbf{P}(Z_1) + \cdots + \mathbf{P}(Z_r) = \mathbf{U}, \tag{3.6}$$

where

$$\mathbf{U} = \begin{cases} \mathbf{P}(\mathrm{Sym}) &= \mathbf{\Pi}^s, \ \mathbf{T} = \mathbf{E}, \\ \mathbf{P}(\mathrm{Lin}) &= \mathbf{\Pi}, \ \ \mathbf{T} = \mathbf{M}. \end{cases} \tag{3.7}$$

The multiplicity n_σ of the eigenvalue λ_σ equals the dimension of the subordinate eigenspace Z_σ, i.e.

$$n_\sigma = \dim Z_\sigma = \mathbf{P}(Z_\sigma)_{ijij}. \tag{3.8}$$

Substituting

$$\mathbf{T} - \lambda_\beta \mathbf{U} = \sum_{\sigma=1}^{r} (\lambda_\sigma - \lambda_\beta) \mathbf{P}(Z_\sigma)$$

into the following continued product, we obtain sequentially

$$\begin{aligned} \prod_{\beta \neq \alpha} (\mathbf{T} - \lambda_\beta \mathbf{U}) &= \prod_{\beta \neq \alpha} \left(\sum_{\sigma=1}^{r} (\lambda_\sigma - \lambda_\beta) \mathbf{P}(Z_\sigma) \right) \\ &= \sum_{\sigma=1}^{r} \left(\prod_{\beta \neq \alpha} (\lambda_\sigma - \lambda_\beta) \right) \mathbf{P}(Z_\sigma) \\ &= \left(\prod_{\beta \neq \alpha} (\lambda_\alpha - \lambda_\beta) \right) \mathbf{P}(Z_\alpha), \end{aligned}$$

where $\Pi_{\beta\neq\alpha}$ means the continued product for $\beta = 1, 2, \ldots, r$ except α and the condition (3.5) is used. Thus

$$\mathbf{P}(Z_\alpha) = p_\alpha^{-1} \prod_{\beta\neq\alpha} (\mathbf{T} - \lambda_\beta \mathbf{U}), \tag{3.9}$$

where

$$p_\alpha = \prod_{\beta\neq\alpha} (\lambda_\alpha - \lambda_\beta) \neq 0. \tag{3.10}$$

Since the eigenprojections of $\mathbf{T}$ are isotropic functions of $\mathbf{T}$ and the eigenvalues of $\mathbf{T}$ are isotropic invariants of $\mathbf{T}$, the characteristic form (3.4), i.e. the spectral decomposition of $\mathbf{T}$, is a coordinate-free invariant form. On the other hand, (3.3) and (3.4) yield another characteristic form of $\mathbf{T}$,

$$\mathbf{T} = \lambda_1 \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 + \cdots + \lambda_s \boldsymbol{\omega}_s \otimes \boldsymbol{\omega}_s, \tag{3.11}$$

where $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_s$ are eigentensors of $\mathbf{T}$ and orthonormal, and

$$s = U_{ijij} = \begin{cases} 6, & \mathbf{T} = \mathbf{E}, \\ 9, & \mathbf{T} = \mathbf{M}. \end{cases} \tag{3.12}$$

The form (3.11), in general, is not unique, except for the case when every eigenvalue of $\mathbf{T}$ is simple.

From the characteristic form (3.4) of the elasticity tensor $\mathbf{T}$ it follows that the inverse of $\mathbf{T}$ is given by

$$\mathbf{T}^{-1} = \sum_{\sigma=1}^{r} \lambda_\sigma^{-1} \mathbf{P}(Z_\sigma) \tag{3.13}$$

and that the elastic energy functions are of the forms

$$2U_c = c_1\varepsilon_1^2 + \cdots + c_r\varepsilon_r^2 = \sigma_1^2/c_1 + \cdots + \sigma_r^2/c_r, \tag{3.14}$$

$$2U_m = a_1\varepsilon_1^2 + \cdots + a_s\varepsilon_s^2 + d_1\tau_1^2 + \cdots + d_t\tau_t^2 \tag{3.15}$$

$$= \sigma_1^2/a_1 + \cdots + \sigma_s^2/a_s + m_1^2/d_1 + \cdots + m_t^2/d_t, \tag{3.16}$$

where $c_1, \ldots, c_r$ are all the distinct eigenvalues of the stiffness tensor $\mathbf{C}$, $a_1, \ldots, a_s$ and $d_1, \ldots, d_t$ are all the distinct eigenvalues of the absolute micropolar stiffness tensors $\mathbf{A}$ and $\mathbf{D}$, respectively, and moreover,

$$x_\alpha = (\mathbf{x}_\alpha : \mathbf{x}_\alpha)^{1/2}, \quad \mathbf{x}_\alpha = \mathbf{P}(Z_\alpha) : \mathbf{x}, \tag{3.17}$$

where $\mathbf{x} \in \{\boldsymbol{\sigma}, \mathbf{m}, \boldsymbol{\varepsilon}, \boldsymbol{\tau}\}$ and Z_α is an eigenspace of the related elasticity tensor $\mathbf{T} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$. We mention that (3.15) and (3.16) hold only for centrosymmetric materials. From (3.14)–(3.15) it follows that (2.8) and (2.19) hold iff

$$c_1 > 0, \ldots, c_r > 0, \tag{3.18}$$

$$a_1 > 0, \ldots, a_s > 0 \quad \text{and} \quad d_1 > 0, \ldots, d_t > 0. \tag{3.19}$$

4. Invariant Characteristic Representation

(a) Material symmetry restriction. Consider a classical or micropolar linear elastic material with the material symmetry group G. The elasticity tensor $\mathbf{T} \in \{\mathbf{E}, \mathbf{M}\}$ of this material should obey the following invariance conditions:

$$\mathbf{Q} * \mathbf{T} = \mathbf{T}, \quad \text{i.e. } Q_{ip}Q_{jq}Q_{kr}Q_{ls}T_{pqrs} = T_{ijkl} \quad (\forall \mathbf{Q} \in G). \tag{4.1}$$

The above invariance conditions restrict the form of the elasticity tensor $\mathbf{T} \in \{\mathbf{E}, \mathbf{M}\}$ and hence restrict the characteristic form of $\mathbf{T}$. In reality, from (4.1) and the uniqueness of the characteristic form (3.4), we can infer

$$\mathbf{Q} * \mathbf{P}(Z_\sigma) = \mathbf{P}(Z_\sigma), \sigma = 1, 2, \ldots, r, \quad (\forall \mathbf{Q} \in G), \tag{4.2}$$

i.e.

$$\mathbf{QSQ}^T \in Z_\sigma \quad \text{for every } \mathbf{S} \in Z_\sigma \quad \text{and} \quad \mathbf{Q} \in G, \quad \sigma = 1, 2, \ldots, r. \tag{4.3}$$

(4.2) and (4.3) indicate that every eigenprojection and eigenspace of $\mathbf{T}$ are invariant under the group G. As a result, the form of every eigenprojection, the dimension of every eigenspace and the multiplicity of every eigenvalue are limited due to the invariance conditions (4.1). A general expression for the characteristic form of an elasticity tensor that is invariant under a given material symmetry group G is called a characteristic representation of the elasticity tensor under the group G.

To determine the characteristic representation of the elasticity tensor $\mathbf{E}$ or $\mathbf{M}$ under a material symmetry group G, it is required to determine general forms of the G-invariant subspaces $Z_1, \ldots, Z_r$ which constitute orthogonal decomposition of the tensor space Sym (for $\mathbf{E}$) or Lin (for $\mathbf{M}$). The latter seems an involved problem not easily treated. In order to clarify it and make it tractable, in what follows we make a further investigation.

(b) Irreducible invariant subspaces. Let G be an orthogonal subgroup. A subspace $Z \subset$ Lin is an irreducible G-invariant subspace if it is a G-invariant subspace and contains no G-invariant subspace but itself and the null space. It is obvious that every one-dimensional G-invariant subspace of Lin is an irreducible G-invariant subspace.

Since the action of any orthogonal subgroup G on the tensor space Lin,

$$(\mathbf{Q},\mathbf{S})\in G\times \mathrm{Lin}\mapsto \mathbf{QSQ}^T\in \mathrm{Lin}, \tag{4.4}$$

is inner product preserving, it can be readily proved that any G-invariant subspace $Z\subset \mathrm{Lin}$ can be decomposed into an orthogonal direct sum of a certain number of irreducible G-invariant subspaces. Thus, from this fact and (4.3) we conclude that the following holds.

THEOREM 1. *Let G be an orthogonal subgroup. Then every elasticity tensor $\mathbf{T}\in\{\mathbf{E},\mathbf{M}\}$ that is invariant under the group G is expressible as the following form*:

$$\mathbf{T}=\sum_{\sigma=1}^{g}\lambda_\sigma \mathbf{P}(Z_\sigma^0), \tag{4.5}$$

where $\lambda_1,\ldots,\lambda_g$ are the eigenvalues of $\mathbf{T}$ and some of them may coincide; Z_σ^0 is an irreducible G-invariant subspace of the λ_σ-eigenspace of $\mathbf{T}$; and $Z_1^0,\ldots,Z_g^0$ constitute orthogonal decomposition of the tensor space Sym *(for $\mathbf{T}=\mathbf{E}$) or* Lin *(for $\mathbf{T}=\mathbf{M}$).*

The above theorem can be derived by decomposing every eigenspace of $\mathbf{T}$, which is invariant under the group G, into an orthogonal sum of irreducible G-invariant subspaces. According to this theorem, determination of the characteristic representation of the elasticity tensor $\mathbf{T}\in\{\mathbf{E},\mathbf{M}\}$ under the group G can be reduced to determination of general forms of the irreducible G-invariant subspaces which constitute orthogonal decomposition of the tensor space Sym (for $\mathbf{T}=\mathbf{E}$) or Lin (for $\mathbf{T}=\mathbf{M}$). A considerable reduction can be expected due to the following properties of irreducible invariant subspaces.

THEOREM 2. *A subspace $Z\subset$* Lin *is an irreducible G-invariant subspace iff*

$$\mathrm{span}\{\mathbf{QSQ}^T\mid \mathbf{Q}\in G\}=Z\quad \textit{for every}\ \mathbf{O}\neq\mathbf{S}\in Z, \tag{4.6}$$

i.e.

$$\mathrm{rank}\{\mathbf{QSQ}^T\mid \mathbf{Q}\in G\}=\dim Z\quad \textit{for every}\ \mathbf{O}\neq\mathbf{S}\in Z. \tag{4.7}$$

THEOREM 3. *Any two irreducible G-invariant subspaces of* Lin *with different dimensions are orthogonal.*

The proofs for the above two theorems will be given in the Appendix.

(c) Determination of characteristic representations. Simple procedures for constructing characteristic representations of the elasticity tensor $\mathbf{E}$ and $\mathbf{M}$ under any given material symmetry group G can be designed by applying Theorems 1–3.

Step 1. Determine all one-dimensional G-invariant subspaces of Sym or Lin by the condition

$$\mathrm{rank}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\} = 1 \quad \text{for } \mathbf{S} \in \mathrm{Sym} \text{ or Lin}. \tag{4.8}$$

Step 2. Let Z^{I} be the sum of all one-dimensional G-invariant subspaces of Sym or Lin. Then the orthocomplement of Z^{I} with respect to Sym or Lin is available, denoted by $\bar{Z}^{\mathrm{I}}$. Then, determine all two-dimensional irreducible G-invariant subspaces of Sym or Lin by the condition

$$\mathrm{rank}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\} = 2 \quad \text{for } \mathbf{S} \in \bar{Z}^{\mathrm{I}}. \tag{4.9}$$

Step 3. Let Z^{II} be the sum of all two-dimensional irreducible G-invariant subspaces of Sym or Lin. Then the orthocomplement of Z^{II} with respect to $\bar{Z}^{\mathrm{I}}$ is available, denoted by $\bar{Z}^{\mathrm{II}}$. Then, determine all three-dimensional irreducible G-invariant subspaces of Sym or Lin by the condition

$$\mathrm{rank}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\} = 3 \quad \text{for } \mathbf{S} \in \bar{Z}^{\mathrm{II}}. \tag{4.10}$$

The irreducible G-invariant subspaces of Sym or Lin with higher dimensions, if any, can be determined in a similar way.

Step 4. The results given by the above steps thus provide general forms of the irreducible G-invariant subspaces $Z_1^0, \ldots, Z_g^0$ which constitute orthogonal decomposition of Sym or Lin. Then, the characteristic representation under the group G is available, given by (4.5).

For every orthogonal subgroup G, the above procedures are easily fulfilled. In reality, for any subgroup $G \subset D_{\infty h}$, Lin contains no irreducible G-invariant subspace whose dimension is greater than 2. For any group G belonging to the cubic crystal system, Lin contains no irreducible G-invariant subspace whose dimension is greater than 3. Moreover, for the case of isotropy, including the icosahedral classes, Lin contains only three irreducible invariant subspaces, i.e. the spherical subspace $S = \mathrm{span}\{\mathbf{I}\}$, the skewsymmetric subspace Skw of Lin and the deviatoric subspace D of Sym.

Characteristic representations of the classical and micropolar elasticity tensors $\mathbf{E}$ and $\mathbf{M}$ under every subgroup can be constructed by fulfilling the above procedures. In the next two sections, we merely present the final results, omitting the details in deriving them. As an illustration, in the Appendix an example will be given to show how to construct the given characteristic representations by following the above procedures.

In the next two sections, $\mathbf{n}$ is a unit vector in the direction of the principal axis of the material symmetry group in question, if any, unless stated otherwise. The notations $\lambda_\sigma^{\mathrm{I}}$, $\lambda_\sigma^{\mathrm{II}}$, $\lambda_\sigma^{\mathrm{III}}$, and λ^V etc. are used to denote eigenvalues and the notations J_σ^{II}, J_σ^{III}, and Z_σ^{II}, Z_σ^{III}, etc. to denote eigenspaces, in which the capital

Roman numerals I, II, III and V indicate the multiplicities and the dimensions of the eigenvalues and the subordinate eigenspaces, respectively. For vectors $\mathbf{p}$ and $\mathbf{q}$, we denote

$$\mathbf{p} \vee \mathbf{q} = \mathbf{p} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{p}; \mathbf{p} \wedge \mathbf{q} = \mathbf{p} \otimes \mathbf{q} - \mathbf{q} \otimes \mathbf{p}. \tag{4.11}$$

5. Invariant Characteristic Representations of the Classical Elasticity Tensor E

(a) Triclinic crystal classes C_1 and S_2.

$$\mathbf{E} = \sum_{\sigma=1}^{6} \lambda_\sigma^I \omega_\sigma \otimes \omega_\sigma, \tag{5.1}$$

where the eigentensors $\omega_1, \ldots, \omega_6 \in \mathrm{Sym}$ are orthonormal, i.e.

$$\omega_\sigma : \omega_\tau = \delta_{\sigma\tau}, \sigma, \tau = 1, \ldots, 6, \tag{5.2}$$

and no further restriction is imposed.

(b) Monoclinic crystal classes C_2, C_{1h} and C_{2h}. The ICR (here and henceforth the abbreviation ICR is used to represent the term, invariant characteristic representation.) of $\mathbf{E}$ under C_{2h} is also of the form (5.1), where the eigentensors ω_5 and ω_6 are the pure shears

$$\begin{cases} \omega_5 = 1/\sqrt{2}\mathbf{n} \vee \mathbf{p}; \omega_6 = 1/\sqrt{2}\mathbf{n} \vee \mathbf{q}, \\ \mathbf{p} \cdot \mathbf{p} = \mathbf{q} \cdot \mathbf{q} = 1, \mathbf{p} \cdot \mathbf{q} = \mathbf{p} \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n} = 0, \end{cases} \tag{5.3}$$

and the eigentensors ω_σ, $\sigma = 1, 2, 3, 4$, are given by

$$\begin{cases} \omega_\sigma = r_{\sigma 1}\mathbf{p} \otimes \mathbf{p} + r_{\sigma 2}\mathbf{q} \otimes \mathbf{q} + r_{\sigma 3}\mathbf{n} \otimes \mathbf{n} + r_{\sigma 4}\mathbf{p} \vee \mathbf{q}, \\ r_{\sigma 1}r_{\tau 1} + r_{\sigma 2}r_{\tau 2} + r_{\sigma 3}r_{\tau 3} + 2r_{\sigma 4}r_{\tau 4} = \delta_{\sigma\tau}, \sigma, \tau = 1, 2, 3, 4. \end{cases} \tag{5.4}$$

Only six of the sixteen dimensionless elastic constants $r_{\sigma\tau}$ are independent. The orthonormal vectors $\mathbf{p}$ and $\mathbf{q}$ in the $\mathbf{n}$-plane depend on the elasticity of the material in question, which can be determined by one-dimensionless constant. In reality, let $(\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{n})$ be a chosen orthonormal system, where $\mathbf{e}_1^0$ and $\mathbf{e}_2^0$ are two chosen orthonormal vectors in the $\mathbf{n}$-plane. Then $\mathbf{p} = \mathbf{e}_1$, $\mathbf{q} = \mathbf{e}_2$,

$$\begin{cases} \mathbf{e}_1 = \mathbf{e}_1^0 \cos x + \mathbf{e}_2^0 \sin x, \\ \mathbf{e}_2 = -\mathbf{e}_1^0 \sin x + \mathbf{e}_2^0 \cos x, \quad 0 \leqslant x < \pi/2, \end{cases} \tag{5.5}$$

and under this system the eigentensors ω_5 and ω_6 are of the forms

$$\omega_5 = a\mathbf{n} \vee \mathbf{e}_1^0 + b\mathbf{n} \vee \mathbf{e}_2^0, \quad \omega_6 = -b\mathbf{n} \vee \mathbf{e}_1^0 + a\mathbf{n} \vee \mathbf{e}_2^0. \tag{5.6}$$

Then the angle x is determined by

$$tgx = b/a. \tag{5.7}$$

The invariant constants characterizing the classical linear elasticities of the materials in question are as follows:

$$\{\lambda_\sigma^{\mathrm{I}}, r_{\sigma\tau}: \sigma, \tau = 1,2,3,4; \lambda_5^{\mathrm{I}}, \lambda_6^{\mathrm{I}}, \mathbf{p}, \mathbf{q}\}. \tag{5.8}$$

Under the orthonormal system $(\mathbf{p}, \mathbf{q}, \mathbf{n})$ there are twelve independent elastic constants only.

(c) Rhombic crystal classes D_2, C_{2v} and D_{2h}. The ICR of $\mathbf{E}$ under D_{2h} is also of the form (5.1), where the eigentensors ω_α, $\alpha = 1,2,3$, are given by

$$\begin{cases} \omega_\alpha = r_{\alpha 1}\mathbf{n}_1 \otimes \mathbf{n}_1 + r_{\alpha 2}\mathbf{n}_2 \otimes \mathbf{n}_2 + r_{\alpha 3}\mathbf{n}_3 \otimes \mathbf{n}_3, \\ r_{\alpha 1}r_{\beta 1} + r_{\alpha 2}r_{\beta 2} + r_{\alpha 3}r_{\beta 3} = \delta_{\alpha\beta}, \quad \alpha, \beta = 1,2,3, \end{cases} \tag{5.9}$$

and the eigentensors $\omega_\sigma, \sigma = 4,5,6$, are the following pure shears:

$$\omega_4 = 1/\sqrt{2}\,\mathbf{n}_1 \vee \mathbf{n}_2,\ \omega_5 = 1/\sqrt{2}\,\mathbf{n}_2 \vee \mathbf{n}_3, \omega_6 = 1/\sqrt{2}\,\mathbf{n}_3 \vee \mathbf{n}_1. \tag{5.10}$$

Here $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ are three unit vectors in the directions of the three 2-fold axes of D_{2h} and they are orthonormal. Only three of the nine dimensionless elastic constants $r_{\alpha\beta}$ are independent. The invariant elastic constants characterizing the linear elasticities of the materials in question are as follows:

$$\{\lambda_\alpha^{\mathrm{I}}, r_{\alpha\beta}: \alpha, \beta = 1,2,3; \lambda_4^{\mathrm{I}}, \lambda_5^{\mathrm{I}}, \lambda_6^{\mathrm{I}}\} \tag{5.11}$$

and there are nine independent elastic constants.

(d) Trigonal crystal classes C_3 and S_6. The ICR of $\mathbf{E}$ under S_6 is given by

$$\mathbf{E} = \sum_{\sigma=1}^{2}(\lambda_\sigma^{\mathrm{I}}\omega_\sigma \otimes \omega_\sigma + \lambda_\sigma^{\mathrm{II}}\mathbf{P}(Z_\sigma^{\mathrm{II}})), \tag{5.12}$$

where the eigentensors ω_1 and ω_2 are of the forms

$$\begin{cases} \omega_1 = \mathbf{n} \otimes \mathbf{n} \sin\theta_0 + 1/\sqrt{2}(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\cos\theta_0, \\ \omega_2 = \mathbf{n} \otimes \mathbf{n} \cos\theta_0 - 1/\sqrt{2}(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\sin\theta_0, \end{cases} \tag{5.13}$$

$$0 \leqslant \theta_0 < \pi/2; \quad I_{ij} = \delta_{ij},$$

and the eigenprojections $\mathbf{P}(Z_\sigma^{\mathrm{II}})$ are given by

$$\begin{cases} \mathbf{P}(Z_1^{\mathrm{II}}) = \omega_3 \otimes \omega_3 + \omega_4 \otimes \omega_4, \\ \omega_3 = 1/\sqrt{2}((\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2)\sin\theta_1 + \mathbf{n} \vee \mathbf{e}_2 \cos\theta_1), \\ \omega_4 = 1/\sqrt{2}(\mathbf{e}_1 \vee \mathbf{e}_2 \sin\theta_1 + \mathbf{n} \vee \mathbf{e}_1 \cos\theta_1) \end{cases} \tag{5.14}$$

and

$$\begin{cases} \mathbf{P}(Z_2^{\mathrm{II}}) = \omega_5 \otimes \omega_5 + \omega_6 \otimes \omega_6, \\ \omega_5 = 1/\sqrt{2}((\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2)\cos\theta_1 - \mathbf{n} \vee \mathbf{e}_2 \sin\theta_1), \\ \omega_6 = 1/\sqrt{2}(\mathbf{e}_1 \vee \mathbf{e}_2 \cos\theta_1 - \mathbf{n} \vee \mathbf{e}_1 \sin\theta_1), \end{cases} \tag{5.15}$$

where the dimensionless elastic constant θ_1 can be taken as

$$0 \leqslant \theta_1 < \pi/2. \tag{5.16}$$

$\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors in the $\mathbf{n}$-plane, which depend on the elasticity of the material in question. The above two eigenprojections are expressible as the following forms:

$$\begin{aligned} \mathbf{P}(Z_\sigma^{\mathrm{II}}) \;=\; & \tfrac{1}{2}(\chi_1 + \chi_2 - \chi_0)((\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})) \sin^2 x_\sigma \\ & + \tfrac{1}{2}(\chi_1 + \chi_2)(\mathbf{n} \otimes \mathbf{n} \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \\ & + (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes \mathbf{n} \otimes \mathbf{n}) \cos^2 x_\sigma \\ & + \frac{1}{3} \sum_{\alpha=1}^{3} ((\mathbf{n} \vee \mathbf{p}_\alpha) \otimes \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha \\ & + \mathbf{p}_\alpha \otimes \mathbf{p}_\alpha \otimes (\mathbf{p}_\alpha \vee \mathbf{n})) \sin 2x_\sigma, \end{aligned} \tag{5.17}$$

where $\sigma = 1, 2$, and

$$x_1 = \theta_1, \quad x_2 = \theta_1 + \pi/2. \tag{5.18}$$

Moreover, $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ are three unit vectors in the $\mathbf{n}$-plane that make the angle $2\pi/3$ with one another. The following relations hold:

$$\mathbf{p}_1 = \mathbf{e}_2, \quad \mathbf{p}_2 = \sqrt{3}/2\, \mathbf{e}_1 - \tfrac{1}{2}\mathbf{e}_2, \quad \mathbf{p}_3 = -\sqrt{3}/2\, \mathbf{e}_1 - \tfrac{1}{2}\mathbf{e}_2; \tag{5.19}$$

$$\mathbf{e}_1 = (\mathbf{p}_2 - \mathbf{p}_3)/\sqrt{3}, \quad \mathbf{e}_2 = \mathbf{p}_1. \tag{5.20}$$

Finally, χ_0 is the identity transformation on the space of fourth order tensors over V, and χ_1 and χ_2 are two isotropic linear transformations on the space of fourth order tensors over V, given by

$$\chi_1(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_4) = \mathbf{v}_1 \otimes \mathbf{v}_3 \otimes \mathbf{v}_2 \otimes \mathbf{v}_4, \tag{5.21}$$

$$\chi_2(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_4) = \mathbf{v}_3 \otimes \mathbf{v}_2 \otimes \mathbf{v}_1 \otimes \mathbf{v}_4 \quad (\forall \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in V). \tag{5.22}$$

The classical linear elasticities of the materials in question are characterized by

$$\{\lambda_1^{\mathrm{I}}, \lambda_2^{\mathrm{I}}, \theta_0; \lambda_1^{\mathrm{II}}, \lambda_2^{\mathrm{II}}, \theta_1, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}. \tag{5.23}$$

Under the orthonormal system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ there are six independent elastic constants only. $(\mathbf{e}_1, \mathbf{e}_2)$ or $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ can be determined by one dimensionless constant. In reality, let $(\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{n})$ be a chosen orthonormal system. Then (5.5) holds and under this chosen system every eigentensor $\omega \in Z_1^{\mathrm{II}} \cup Z_2^{\mathrm{II}}$ is of the form

$$\omega = u(\mathbf{e}_1^0 \otimes \mathbf{e}_1^0 - \mathbf{e}_2^0 \otimes \mathbf{e}_2^0) + v\mathbf{e}_1^0 \vee \mathbf{e}_2^0 + p\mathbf{n} \vee \mathbf{e}_1^0 + q\mathbf{n} \vee \mathbf{e}_2^0. \tag{5.24}$$

Given u, v, p and q, the angle x is determined by (cf. Appendix)

$$\operatorname{tg} 3x = (vq - up)/(uq + vp). \tag{5.25}$$

From the expressions (5.17), the physical meaning of the unit vectors $(\mathbf{e}_1, \mathbf{e}_2)$ or $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ can be seen. In reality, let $\mathbf{R}_{\mathbf{a}}^{\theta}$ represent the right-handed rotation through the angle θ about an axis in the direction of the unit vector $\mathbf{a}$. Then

$$\mathbf{R}_{\mathbf{l}_i}^{\pi} * \mathbf{E} = \mathbf{E}, \mathbf{l}_i = \mathbf{n} \times \mathbf{p}_i, \quad i = 1, 2, 3. \tag{5.26}$$

From these it follows that the planes $\pi_{\mathbf{l}_i} = \operatorname{span}\{\mathbf{n}, \mathbf{p}_i\}$, $i = 1, 2, 3$, turn out to be three planes of elastic symmetry. It should be pointed out that the planes $\pi_{\mathbf{l}_i}$, $i = 1, 2, 3$, have to be determined by the elasticity of the material in question, since the crystal classes C_3 and S_6 have no crystallographic symmetry plane and symmetry axis in the $\mathbf{n}$-plane.

(e) Trigonal cyrstal classes D_3, C_{3v} and D_{3d}. The ICR of $\mathbf{E}$ under D_{3d} is also given by (5.12)–(5.20), except for the fact that the unit vectors $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ are determined by the three 2-fold axes of D_{3d}, i.e.

$$\begin{aligned} &\mathbf{p}_1 = (\mathbf{l}_2 - \mathbf{l}_3)/\sqrt{3}, \quad \mathbf{p}_2 = (\mathbf{l}_3 - \mathbf{l}_1)/\sqrt{3}, \\ &\mathbf{p}_3 = (\mathbf{l}_1 - \mathbf{l}_2)/\sqrt{3}; \;\; \mathbf{p}_k = \mathbf{n} \times \mathbf{l}_k, \quad k = 1, 2, 3, \end{aligned} \tag{5.27}$$

where $\mathbf{l}_1$, $\mathbf{l}_2$ and $\mathbf{l}_3$ are three unit vectors in the directions of the three 2-fold axes of D_{3d}, which make the angle $2\pi/3$ with one another. The classical linear elasticities of the materials in question are characterized by

$$\{\lambda_1^{\mathrm{I}}, \lambda_2^{\mathrm{I}}, \theta_0; \lambda_1^{\mathrm{II}}, \lambda_2^{\mathrm{II}}, \theta_1\}. \tag{5.28}$$

(f) Tetragonal crystal classes C_4, S_4 and C_{4h}. The ICR of $\mathbf{E}$ under C_{4h} is given by

$$\mathbf{E} = \sum_{\sigma=1}^{4} \lambda_\sigma^{\mathrm{I}} \omega_\sigma \otimes \omega_\sigma + \lambda^{\mathrm{II}} \mathbf{P}(J^{\mathrm{II}}), \tag{5.29}$$

where the eigentensors ω_1 and ω_2 are given by (5.13); the eigentensors ω_3 and ω_4 are the following pure shears:

$$\omega_3 = 1/\sqrt{2}\, \mathbf{e}_1 \vee \mathbf{e}_2, \quad \omega_4 = 1/\sqrt{2}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \tag{5.30}$$

where the orthonormal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathbf{n}$-plane depend on the elasticity of the material in question. The eigenprojection $\mathbf{P}(J^{\mathrm{II}})$ is of the form

$$\mathbf{P}(J^{\mathrm{II}}) = \omega_5 \otimes \omega_5 + \omega_6 \otimes \omega_6 \tag{5.31}$$

$$= \tfrac{1}{2}(\chi_1 + \chi_2)(\mathbf{n} \otimes \mathbf{n} \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) + (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes \mathbf{n} \otimes \mathbf{n}), \tag{5.32}$$

$$\omega_5 = 1/\sqrt{2}\,\mathbf{n} \vee \mathbf{e}_1, \quad \omega_6 = 1/\sqrt{2}\,\mathbf{n} \vee \mathbf{e}_2, \tag{5.33}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ can be taken as any two orthonormal vectors in the $\mathbf{n}$-plane.

The classical linear elasticities of the materials in question are characterized by

$$\{\lambda_1^{\mathrm{I}}, \lambda_2^{\mathrm{I}}, \theta_0; \lambda^{\mathrm{II}}; \lambda_3^{\mathrm{I}}, \lambda_4^{\mathrm{I}}, \mathbf{e}_1, \mathbf{e}_2\}, \tag{5.34}$$

and under the orthonormal system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ there are six independent elastic constants only. Moreover, the orthonormal vectors $(\mathbf{e}_1, \mathbf{e}_2)$ in the $\mathbf{n}$-plane are determined by one constant. In reality, let $(\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{n})$ be a chosen orthonormal system. Then (5.5) holds and under this chosen system the eigentensors ω_3 and ω_4 are of the forms

$$\begin{cases} \omega_3 = a(\mathbf{e}_1^0 \otimes \mathbf{e}_1^0 - \mathbf{e}_2^0 \otimes \mathbf{e}_2^0) + b\mathbf{e}_1^0 \vee \mathbf{e}_2^0, \\ \omega_4 = b(\mathbf{e}_1^0 \otimes \mathbf{e}_1^0 - \mathbf{e}_2^0 \otimes \mathbf{e}_2^0) - a\mathbf{e}_1^0 \vee \mathbf{e}_2^0. \end{cases} \tag{5.35}$$

Given the constants a and b, the angle x is determined by

$$\mathrm{tg}\, 2x = b/a. \tag{5.36}$$

It can be readily seen that the $\mathbf{e}_i$-planes, $i = 1, 2$, are two orthogonal planes of elastic symmetry. The two planes have to be determined by the elasticity of the material in question, since the classes C_4, S_4 and C_{4h} have no crystallographic symmetry plane and symmetry axis in the $\mathbf{n}$-plane.

(g) Tetragonal crystal classes D_4, C_{4v}, D_{2d} and D_{4h}. The ICR of $\mathbf{E}$ under the group $G \in \{D_4, C_{4v}, D_{2d}, D_{4h}\}$ is also of the form (5.29–33), except for the fact that the orthonormal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in the $\mathbf{n}$-plane are in the directions of two mutually orthogonal 2-fold axes of G. The classical linear elasticities of the materials in question are characterized by

$$\{\lambda_1^{\mathrm{I}}, \lambda_2^{\mathrm{I}}, \theta_0; \lambda_3^{\mathrm{I}}, \lambda_4^{\mathrm{I}}, \lambda^{\mathrm{II}}\}, \tag{5.37}$$

and there are six independent elastic constants.

(h) Hexagonal crystal classes $C_6, C_{3h}, C_{6h}, D_6, C_{6v}, D_{3h}$ and D_{6h} and more generally each subgroup $G \subset D_{\infty h}$ including a rotation or a rotatory-inversion through the angle $2\pi/m$, $m \geqslant 5$.

The ICR of $\mathbf{E}$ under each group G in question is given by

$$\mathbf{E} = \sum_{\sigma=1}^{2}(\lambda_\sigma^{\mathrm{I}}\boldsymbol{\omega}_\sigma \otimes \boldsymbol{\omega}_\sigma + \lambda_\sigma^{\mathrm{II}}\mathbf{P}(J_\sigma^{\mathrm{II}})), \tag{5.38}$$

where the eigentensors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are given by (5.13); the eigenprojection $\mathbf{P}(J_1^{\mathrm{II}})$ is given by

$$\begin{aligned}\mathbf{P}(J_1^{\mathrm{II}}) &= \boldsymbol{\omega}_3 \otimes \boldsymbol{\omega}_3 + \boldsymbol{\omega}_4 \otimes \boldsymbol{\omega}_4 \\ &= \tfrac{1}{2}(\chi_1 + \chi_2 - \chi_0)((\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})),\end{aligned} \tag{5.39}$$

$$\boldsymbol{\omega}_3 = 1/\sqrt{2}\, \mathbf{e}_1 \vee \mathbf{e}_2, \boldsymbol{\omega}_4 = 1/\sqrt{2}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \tag{5.40}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ can be taken as any two orthonormal vectors in the $\mathbf{n}$-plane; and moreover the eigenprojection $\mathbf{P}(J_2^{\mathrm{II}})$ is given by (5.31)–(5.33). It can be seen that every eigenprojection merely depends on the unit vector $\mathbf{n}$ in the direction of the principal axis and that every $\mathbf{e}$-plane, $\mathbf{e}$ being any vector in the $\mathbf{n}$-plane, is a plane of elastic symmetry.

The classical linear elasticities of the materials in question are characterized by

$$\{\lambda_1^{\mathrm{I}}, \lambda_2^{\mathrm{I}}, \theta_0; \lambda_1^{\mathrm{II}}, \lambda_2^{\mathrm{II}}\}, \tag{5.41}$$

and there are five independent constants.

(i) Cubic crystal classes T, T_h, O, T_d and O_h. The ICR of $\mathbf{E}$ under $G \in \{T, T_h, O, T_d, O_h\}$ is given by

$$\mathbf{E} = \lambda^{\mathrm{I}}\mathbf{P}(S) + \lambda^{\mathrm{II}}\mathbf{P}(K^{\mathrm{II}}) + \lambda^{\mathrm{III}}\mathbf{P}(K^{\mathrm{III}}), \tag{5.42}$$

$$\mathbf{P}(S) = 1/3\, \mathbf{I} \otimes \mathbf{I}, \tag{5.43}$$

$$\mathbf{P}(K^{\mathrm{II}}) = \sum_{\sigma=1}^{3} \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma - 1/3\, \mathbf{I} \otimes \mathbf{I}, \tag{5.44}$$

$$\mathbf{P}(K^{\mathrm{III}}) = \mathbf{II}^s - \sum_{\sigma=1}^{3} \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma \otimes \mathbf{n}_\sigma, \tag{5.45}$$

where $S = \mathrm{span}\{\mathbf{I}\}$ is the spherical subspace of Lin and its orthocomplement $\bar{S} = K^{\mathrm{II}} \oplus K^{\mathrm{III}}$, the deviatoric subspace of Sym, and $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ are three orthonormal vectors in the directions of the three 2-fold (for $G \in \{T, T_h\}$) or 4-fold (for $G \in \{O, T_d, O_h\}$) axes.

The classical linear elasticities of the materials in question are characterized by

$$\{\lambda^{\mathrm{I}}, \lambda^{\mathrm{II}}, \lambda^{\mathrm{III}}\}, \tag{5.46}$$

and there are three independent elastic constants.

(j) Isotropy and the icosahedral classes I and I_h. The ICR of $\mathbf{E}$ under the group $G \in \{\mathrm{Orth}, \mathrm{Orth}^+, \mathrm{I}, \mathrm{I}_h\}$ is given by

$$\mathbf{E} = \lambda^{\mathrm{I}}\mathbf{P}(S) + \lambda^{V}\mathbf{P}(D), \tag{5.47}$$

$$\begin{aligned} \mathbf{P}(S) &= 1/3\,\mathbf{I} \otimes \mathbf{I}, \\ \mathbf{P}(D) &= \mathbf{II}^s - 1/3\,\mathbf{I} \otimes \mathbf{I}, D = \bar{S}. \end{aligned} \tag{5.48}$$

The classical linear elasticities of the materials in question are characterized by

$$\{\lambda^{\mathrm{I}}, \lambda^{V}\}, \tag{5.49}$$

and there are two independent elastic constants.

(k) Remark. From the ICRs given in this section it follows that the types of the classical elasticity tensors under all possible orthogonal subgroups can be reduced to eight types. As a corollary, one can infer that the crystallographic symmetries, together with the isotropy, can completely describe the symmetries of the classical elasticity tensor (cf. Huo and Del Piero [16]).

6. Invariant Characteristic Representations of the Micropolar Elasticity Tensor M

(a) Triclinic crystal classes C_1 and S_2. The ICR of $\mathbf{M}$ is given by

$$\mathbf{M} = \sum_{\sigma=1}^{9} \lambda^{\mathrm{I}}_{\sigma}\mathbf{W}_\sigma \otimes \mathbf{W}_\sigma, \tag{6.1}$$

$$\mathbf{W}_\sigma : \mathbf{W}_\tau = \delta_{\sigma\tau}, \sigma, \tau = 1, \ldots, 9. \tag{6.2}$$

(b) Monoclinic crystal classes C_2, C_{1h} and C_{2h}. The ICR of $\mathbf{M}$ is given by (6.1), where the eigentensors $\mathbf{W}_1, \ldots, \mathbf{W}_4$ are an orthonormal basis of the subspace

$$J_1 = \mathrm{span}\{\mathbf{n} \otimes \mathbf{e}_1^0, \mathbf{n} \otimes \mathbf{e}_2^0, \mathbf{e}_1^0 \otimes \mathbf{n}, \mathbf{e}_2^0 \otimes \mathbf{n}\}, \tag{6.3}$$

and the eigentensors $\mathbf{W}_5, \ldots, \mathbf{W}_9$ are an orthonormal basis of the subspace

$$J_2 = \mathrm{span}\{\mathbf{e}_1^0 \otimes \mathbf{e}_1^0, \mathbf{e}_2^0 \otimes \mathbf{e}_2^0, \mathbf{e}_1^0 \otimes \mathbf{e}_2^0, \mathbf{e}_2^0 \otimes \mathbf{e}_1^0, \mathbf{n} \otimes \mathbf{n}\}. \tag{6.4}$$

Here $\mathbf{e}_1^0$ and $\mathbf{e}_2^0$ are two chosen orthonormal vectors in $\mathbf{n}$-plane.

The first four eigentensors include six independent dimensionless elastic constants and the other five eigentensors include ten independent dimensionless elastic constants. Thus, there are 25 independent elastic constants in all.

(c) Rhombic crystal classes D_2, C_{2v} and D_{2h}. The ICR of $\mathbf{M}$ is given by (6.1), where the eigentensors $\mathbf{W}_1, \mathbf{W}_2$ and $\mathbf{W}_3$ are given by (5.9), and the eigentensors $\mathbf{W}_4, \ldots, \mathbf{W}_9$ are of the forms

$$\begin{cases} \mathbf{W}_4 = \mathbf{n}_1 \otimes \mathbf{n}_2 \sin\theta_1 + \mathbf{n}_2 \otimes \mathbf{n}_1 \cos\theta_1, \\ \mathbf{W}_5 = \mathbf{n}_1 \otimes \mathbf{n}_2 \cos\theta_1 - \mathbf{n}_2 \otimes \mathbf{n}_1 \sin\theta_1, \end{cases} \tag{6.5}$$

$$\begin{cases} \mathbf{W}_6 = \mathbf{n}_2 \otimes \mathbf{n}_3 \sin\theta_2 + \mathbf{n}_3 \otimes \mathbf{n}_2 \cos\theta_2, \\ \mathbf{W}_7 = \mathbf{n}_2 \otimes \mathbf{n}_3 \cos\theta_2 - \mathbf{n}_3 \otimes \mathbf{n}_2 \sin\theta_2, \end{cases} \tag{6.6}$$

$$\begin{cases} \mathbf{W}_8 = \mathbf{n}_3 \otimes \mathbf{n}_1 \sin\theta_3 + \mathbf{n}_1 \otimes \mathbf{n}_3 \cos\theta_3, \\ \mathbf{W}_9 = \mathbf{n}_3 \otimes \mathbf{n}_1 \cos\theta_3 - \mathbf{n}_1 \otimes \mathbf{n}_3 \sin\theta_3, \end{cases} \tag{6.7}$$

$$0 \leqslant \theta_1, \theta_2, \theta_3 < \pi/2. \tag{6.8}$$

There are 15 independent elastic constants in all.

(d) Triginal crystal classes C_3 and S_6. The ICR of $\mathbf{M}$ is given by

$$\mathbf{E} = \sum_{\sigma=1}^{3} (\lambda_\sigma^{\mathrm{I}} \mathbf{W}_\sigma \otimes \mathbf{W}_\sigma + \lambda_\sigma^{\mathrm{II}} \mathbf{P}(Z_\sigma^{\mathrm{II}})), \tag{6.9}$$

where the eigentensors $\mathbf{W}_1, \mathbf{W}_2$ and $\mathbf{W}_3$ are of the forms

$$\begin{cases} \mathbf{W}_\sigma = r_{\sigma 1} \mathbf{n} \otimes \mathbf{n} + r_{\sigma 2}(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) + r_{\sigma 3} \mathbf{e}_1 \wedge \mathbf{e}_2, \\ r_{\sigma 1} r_{\tau 1} + 2 r_{\sigma 2} r_{\tau 2} + 2 r_{\sigma 3} r_{\tau 3} = \delta_{\sigma\tau}, \sigma, \tau = 1, 2, 3, \end{cases} \tag{6.10}$$

and the eigenprojections $\mathbf{P}(Z_\sigma^{\mathrm{II}}), \sigma = 1, 2, 3$, are determined by three subspaces Z_σ^{II} that constitute orthogonal decomposition of the subspace $J = \mathrm{span}\{\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{n} \otimes \mathbf{e}_1, \mathbf{n} \otimes \mathbf{e}_2, \mathbf{e}_1 \otimes \mathbf{n}, \mathbf{e}_2 \otimes \mathbf{n}\}$ and are invariant under S_6. $\mathbf{P}(Z_\sigma^{\mathrm{II}}), \sigma = 1, 2, 3$, are given by

$$\mathbf{P}(Z_\sigma^{\mathrm{II}}) = \mathbf{W}_{2\sigma+2} \otimes \mathbf{W}_{2\sigma+2} + \mathbf{W}_{2\sigma+3} \otimes \mathbf{W}_{2\sigma+3}, \sigma = 1, 2, 3, \tag{6.11}$$

$$\begin{cases} \mathbf{W}_{2\sigma+2} = u_\sigma(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) + v_\sigma \mathbf{e}_1 \vee \mathbf{e}_2 + p_\sigma \mathbf{n} \otimes \mathbf{e}_1 \\ \qquad + q_\sigma \mathbf{n} \otimes \mathbf{e}_2 + r_\sigma \mathbf{e}_1 \otimes \mathbf{n} + s_\sigma \mathbf{e}_2 \otimes \mathbf{n}, \\ \mathbf{W}_{2\sigma+3} = v_\sigma(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) + u_\sigma \mathbf{e}_1 \vee \mathbf{e}_2 - q_\sigma \mathbf{n} \otimes \mathbf{e}_1 \\ \qquad + p_\sigma \mathbf{n} \otimes \mathbf{e}_2 - s_\sigma \mathbf{e}_1 \otimes \mathbf{n} + r_\sigma \mathbf{e}_2 \otimes \mathbf{n}, \end{cases} \tag{6.12}$$

$$\begin{cases} 2u_\sigma^2 + 2v_\sigma^2 + p_\sigma^2 + q_\sigma^2 + r_\sigma^2 + s_\sigma^2 = 1, \\ \mathbf{P}(Z_1^{\mathrm{II}}) + \mathbf{P}(Z_2^{\mathrm{II}}) + \mathbf{P}(Z_3^{\mathrm{II}}) = \mathbf{P}(J). \end{cases} \tag{6.13}$$

In the above, $\mathbf{e}_1$ and $\mathbf{e}_2$ are two chosen orthonormal vectors in the $\mathbf{n}$-plane. By using the conditions (6.13), it can be proved that only six of the 18 dimensionless elastic constants $u_\sigma, \ldots, s_\sigma$ are independent. Thus, there are 15 independent elastic constants in all.

(e) Trigonal crystal classes D_3, C_{3v} and D_{3d}. The ICR of $\mathbf{M}$ is given by (6.9), where the eigentensors $\mathbf{W}_1$ and $\mathbf{W}_2$ are given by (5.13), and the eigentensor $\mathbf{W}_3$ is of the form

$$\mathbf{W}_3 = 1/\sqrt{2}\,\mathbf{e}_1 \wedge \mathbf{e}_2 = 1/\sqrt{2}\,\varepsilon\mathbf{n}, \tag{6.14}$$

where ε is the third order alternating tensor, and the eigenprojections $\mathbf{P}(Z_\sigma^{\mathrm{II}})$ are given by (6.11) and

$$\begin{cases} \mathbf{W}_{2\sigma+2} = r_{\sigma 1}\mathbf{e}_2 \otimes \mathbf{n} + r_{\sigma 2}\mathbf{n} \otimes \mathbf{e}_2 + r_{\sigma 3}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \\ \mathbf{W}_{2\sigma+3} = r_{\sigma 1}\mathbf{e}_1 \otimes \mathbf{n} + r_{\sigma 2}\mathbf{n} \otimes \mathbf{e}_1 + r_{\sigma 3}\mathbf{e}_1 \vee \mathbf{e}_2, \\ r_{\sigma 1}r_{\tau 1} + r_{\sigma 2}r_{\tau 2} + 2r_{\sigma 3}r_{\tau 3} = \delta_{\sigma\tau}, \sigma, \tau = 1, 2, 3. \end{cases} \tag{6.15}$$

In the above, $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors in the $\mathbf{n}$-plane, given by

$$\mathbf{e}_1 = \mathbf{l}_1, \mathbf{e}_2 = (\mathbf{l}_2 - \mathbf{l}_3)/\sqrt{3}, \tag{6.16}$$

where $\mathbf{l}_1, \mathbf{l}_2$ and $\mathbf{l}_3$ are three unit vectors in directions of the three 2-fold axes of D_{3d}. The explicit expressions of $\mathbf{P}(Z_\sigma^{\mathrm{II}})$ in terms of the symmetry axes of D_{3d} are as follows:

$$\begin{aligned} \mathbf{P}(Z_\sigma^{\mathrm{II}}) = {} & r_{\sigma 3}^2(\chi_1 + \chi_2 - \chi_0)((\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})) \\ & + r_{\sigma 1}r_{\sigma 2}(\chi_1 + \chi_2)(\mathbf{n} \otimes \mathbf{n} \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \\ & + (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes \mathbf{n} \otimes \mathbf{n}) + (r_{\sigma 1} - r_{\sigma 2})\chi_1(r_{\sigma 1}(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes \mathbf{n} \otimes \mathbf{n} \\ & - r_{\sigma 2}\mathbf{n} \otimes \mathbf{n} \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})) + \frac{2}{3} r_{\sigma 3} \sum_{k=1}^{3} ((r_{\sigma 1} + r_{\sigma_2})((\mathbf{n} \vee \mathbf{p}_k) \otimes \mathbf{p}_k \otimes \mathbf{p}_k \\ & + \mathbf{p}_k \otimes \mathbf{p}_k \otimes (\mathbf{p}_k \vee \mathbf{n})) - (r_{\sigma 1} - r_{\sigma 2})((\mathbf{n} \wedge \mathbf{p}_k) \otimes \mathbf{p}_k \otimes \mathbf{p}_k \\ & - \mathbf{p}_k \otimes \mathbf{p}_k \otimes (\mathbf{p}_k \wedge \mathbf{n}))), \end{aligned} \tag{6.17}$$

where $\mathbf{p}_k$ is a unit vector perpendicular to $\mathbf{l}_k$ and $\mathbf{n}$, given by (5.27).

Only three of the nine dimensionless elastic constants $r_{\sigma\tau}$ are independent due to $(6.15)_3$. Thus, there are ten independent elastic constants in all.

(f) Tetragonal crystal classes C_4, S_4 and C_{4h}. The ICR of $\mathbf{M}$ is given by

$$\mathbf{E} = \sum_{\sigma=1}^{5} \lambda_\sigma^{\mathrm{I}} \mathbf{W}_\sigma \otimes \mathbf{W}_\sigma + \sum_{\alpha=1}^{2} \lambda_\alpha^{\mathrm{II}} \mathbf{P}(Z_\alpha^{\mathrm{II}}), \tag{6.18}$$

where the eigentensors $\mathbf{W}_\sigma, \sigma = 1,2,3$, are given by (6.10) and the eigentensors $\mathbf{W}_4$ and $\mathbf{W}_5$ are the pure shears given by

$$\mathbf{W}_4 = 1/\sqrt{2}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \mathbf{W}_5 = 1/\sqrt{2}\, \mathbf{e}_1 \vee \mathbf{e}_2, \tag{6.19}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors in the $\mathbf{n}$-plane determined by one elastic constant (cf. (5.5, 35, 36)). Moreover, the eigenprojections $\mathbf{P}(Z_\alpha^{\mathrm{II}})$ are given by

$$\mathbf{P}(Z_\alpha^{\mathrm{II}}) = \mathbf{W}_{2\alpha+4} \otimes \mathbf{W}_{2\alpha+4} + \mathbf{W}_{2\alpha+5} \otimes \mathbf{W}_{2\alpha+5}, \alpha = 1,2, \tag{6.20}$$

$$\begin{cases} \mathbf{W}_6 = (\mathbf{e}_1 \otimes \mathbf{n} \cos\theta_2 + \mathbf{e}_2 \otimes \mathbf{n} \sin\theta_2) \sin\theta_1 \\ \qquad +(\mathbf{n} \otimes \mathbf{e}_1 \sin\theta_2 + \mathbf{n} \otimes \mathbf{e}_2 \cos\theta_2) \cos\theta_1, \\ \mathbf{W}_7 = (\mathbf{e}_1 \otimes \mathbf{n} \sin\theta_2 - \mathbf{e}_2 \otimes \mathbf{n} \cos\theta_2) \sin\theta_1 \\ \qquad +(\mathbf{n} \otimes \mathbf{e}_1 \cos\theta_2 - \mathbf{n} \otimes \mathbf{e}_2 \sin\theta_2) \cos\theta_1, \end{cases} \tag{6.21}$$

$$\begin{cases} \mathbf{W}_8 = (\mathbf{e}_1 \otimes \mathbf{n} \cos\theta_2 + \mathbf{e}_2 \otimes \mathbf{n} \sin\theta_2) \cos\theta_1 \\ \qquad -(\mathbf{n} \otimes \mathbf{e}_1 \sin\theta_2 + \mathbf{n} \otimes \mathbf{e}_2 \cos\theta_2) \sin\theta_1, \\ \mathbf{W}_9 = (\mathbf{e}_1 \otimes \mathbf{n} \sin\theta_2 - \mathbf{e}_2 \otimes \mathbf{n} \cos\theta_2) \cos\theta_1 \\ \qquad -(\mathbf{n} \otimes \mathbf{e}_1 \cos\theta_2 - \mathbf{n} \otimes \mathbf{e}_2 \sin\theta_2) \sin\theta_1, \end{cases} \tag{6.22}$$

$$0 \leqslant \theta_1 < \pi/2, 0 \leqslant \theta_2 < \pi. \tag{6.23}$$

The explicit expressions for $\mathbf{P}(Z_\alpha^{\mathrm{II}})$ are as follows:

$$\begin{aligned} \mathbf{P}(Z_\alpha^{\mathrm{II}}) = {} & \chi_1(\mathbf{n} \otimes \mathbf{n} \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \sin^2 x_\alpha + (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes \mathbf{n} \otimes \mathbf{n} \cos^2 x_\alpha) \\ & +\chi_2((\mathbf{n} \otimes \mathbf{n} \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) + (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes \mathbf{n} \otimes \mathbf{n}) \sin 2y_\alpha \\ & +(\mathbf{n} \otimes \mathbf{n} \otimes \mathbf{N} + \mathbf{N} \otimes \mathbf{n} \otimes \mathbf{n}) \cos 2y_\alpha) \sin x_\alpha \cos x_\alpha, \end{aligned} \tag{6.24}$$

where $\alpha = 1,2$, and

$$\mathbf{N} = \epsilon\mathbf{n}; x_1 = \theta_1, y_1 = \theta_2; x_2 = \pi/2 - \theta_1, y_2 = \theta_2 + \pi/2, \tag{6.25}$$

ϵ being the third order alternating tensor.

Under the orthonormal system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ there are 12 independent elastic constants in all.

(g) Tetragonal crystal classes D_4, C_{4v}, D_{2d} and D_{4h}. The ICR of $\mathbf{M}$ is given by (6.18), where the eigentensors $\mathbf{W}_1, \mathbf{W}_2$ and $\mathbf{W}_3$ are given by (5.13) and (6.15), respectively, the eigentensors $\mathbf{W}_4$ and $\mathbf{W}_5$ are given by (6.19), and the eigenprojections $\mathbf{P}(Z_\alpha^{\mathrm{II}})$ are given by (6.20) and

$$\begin{cases} \mathbf{W}_{2\alpha+4} = \mathbf{e}_1 \otimes \mathbf{n} \sin x_\alpha + \mathbf{n} \otimes \mathbf{e}_1 \cos x_\alpha, \\ \mathbf{W}_{2\alpha+5} = \mathbf{e}_2 \otimes \mathbf{n} \sin x_\alpha + \mathbf{n} \otimes \mathbf{e}_2 \cos x_\alpha, \end{cases} \tag{6.26}$$

where $\alpha = 1, 2$, and

$$x_1 = \theta_1, x_2 = \theta_1 + \pi/2, 0 \leqslant \theta_1 < \pi/2. \tag{6.27}$$

Moreover,

$$\begin{aligned} \mathbf{P}(Z_\alpha^{\mathrm{II}}) &= \chi_1(\mathbf{n} \otimes \mathbf{n} \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \sin^2 x_\alpha + (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \\ &\quad \otimes \mathbf{n} \otimes \mathbf{n} \cos^2 x_\alpha) + \chi_2(\mathbf{n} \otimes \mathbf{n} \otimes (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \\ &\quad + (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \otimes \mathbf{n} \otimes \mathbf{n}) \sin x_\alpha \cos x_\alpha. \end{aligned} \tag{6.28}$$

In the above, $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors in the directions of two mutually orthogonal 2-fold axes of D_{4h}. There are nine independent elastic constants in all.

(h) Hexagonal crystal classes C_6, C_{3h} and C_{6h} and more generally each group $G \subset C_{\infty h}$ including a rotation or a rotatory-inversion through the angle $2\pi/m, m \geqslant 5$. The ICR of $\mathbf{M}$ under each group in question is given by

$$\mathbf{M} = \sum_{\sigma=1}^{3} \lambda_\sigma^{\mathrm{I}} \mathbf{W}_\sigma \otimes \mathbf{W}_\sigma + \sum_{\alpha=1}^{2} \lambda_\alpha^{\mathrm{II}} \mathbf{P}(Z_\alpha^{\mathrm{II}}) + \lambda^{\mathrm{II}} \mathbf{P}(J_1^{\mathrm{II}}), \tag{6.29}$$

where the eigentensors $\mathbf{W}_\sigma, \sigma = 1, 2, 3$, are given by (6.10); the eigenprojections $\mathbf{P}(Z_\alpha^{\mathrm{II}}), \alpha = 1, 2$, and $\mathbf{P}(J_1^{\mathrm{II}})$ are given by (6.20)–(6.25) and (5.39, 40), respectively. In these expressions, $\mathbf{e}_1$ and $\mathbf{e}_2$ can be taken as any two orthonormal vectors in the $\mathbf{n}$-plane.

There are eleven independent elastic constants in all.

(i) Hexagonal crystal classes D_6, C_{6v}, D_{3h} and D_{6h} and more generally each subgroup $G \subset D_{\infty h}$ which includes a rotation or a rotatory-inversion through the angle $2\pi/m, m \geqslant 5$, and has a 2-fold axis. The ICR of $\mathbf{M}$ under each group in question is given by (6.29), where the eigenvectors $\mathbf{W}_1, \mathbf{W}_2$ and $\mathbf{W}_3$ are given by (5.13) and (6.15), respectively, and the eigenprojections $\mathbf{P}(Z_\alpha^{\mathrm{II}}), \alpha = 1, 2$, are given by (6.20) and (6.26)–(6.28). Moreover, the projection $\mathbf{P}(J_1^{\mathrm{II}})$ is given by (5.39, 40). In these expressions, $\mathbf{e}_1$ and $\mathbf{e}_2$ can be taken as any two orthonormal vectors in the $\mathbf{n}$-plane.

There are eight independent elastic constants in all.

(j) Cubic crystal classes T and T_h. The ICR of **M** is given by

$$\mathbf{M} = \lambda^{\mathrm{I}}\mathbf{P}(S) + \lambda^{\mathrm{II}}\mathbf{P}(K^{\mathrm{II}}) + \sum_{\sigma=1}^{2} \lambda_{\sigma}^{\mathrm{III}}\mathbf{P}(Z_{\sigma}^{\mathrm{III}}), \tag{6.30}$$

where the eigenprojections $\mathbf{P}(S)$ and $\mathbf{P}(K^{\mathrm{II}})$ are given by (5.43) and (5.44), respectively, and the projections $\mathbf{P}(Z_{\sigma}^{\mathrm{III}})$ are given by

$$\begin{aligned} \mathbf{P}(Z_{\sigma}^{\mathrm{III}}) &= \mathbf{W}_{3\sigma+1} \otimes \mathbf{W}_{3\sigma+1} + \mathbf{W}_{3\sigma+2} \otimes \mathbf{W}_{3\sigma+2} \\ &\quad + \mathbf{W}_{3\sigma+3} \otimes \mathbf{W}_{3\sigma+3}, \sigma = 1, 2, \end{aligned} \tag{6.31}$$

$$\begin{cases} \mathbf{W}_{3\sigma+1} = \mathbf{n}_1 \otimes \mathbf{n}_2 \sin x_{\sigma} + \mathbf{n}_2 \otimes \mathbf{n}_1 \cos x_{\sigma}, \\ \mathbf{W}_{3\sigma+2} = \mathbf{n}_2 \otimes \mathbf{n}_3 \sin x_{\sigma} + \mathbf{n}_3 \otimes \mathbf{n}_2 \cos x_{\sigma}, \\ \mathbf{W}_{3\sigma+3} = \mathbf{n}_3 \otimes \mathbf{n}_1 \sin x_{\sigma} + \mathbf{n}_1 \otimes \mathbf{n}_3 \cos x_{\sigma}, \end{cases} \tag{6.32}$$

where

$$x_1 = \theta, x_2 = \theta + \pi/2, 0 \leqslant \theta < \pi/2. \tag{6.33}$$

Moreover, $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ are three orthonormal vectors in the directions of the three 2-fold axes of T_h. There are five independent elastic constants.

(k) Cubic crystal classes O, T_d and O_h. The ICR of **M** is given by

$$\mathbf{M} = \lambda^{\mathrm{I}}\mathbf{P}(S) + \lambda^{\mathrm{II}}\mathbf{P}(K^{\mathrm{II}}) + \lambda_1^{\mathrm{III}}\mathbf{P}(K^{\mathrm{III}}) + \lambda_2^{\mathrm{III}}\mathbf{P}(\mathrm{Skw}), \tag{6.34}$$

where the eigenprojections $\mathbf{P}(S), \mathbf{P}(K^{\mathrm{II}})$ and $\mathbf{P}(K^{\mathrm{III}})$ are given by (5.43), (5.44) and (5.45), respectively, and the eigenprojection $\mathbf{P}(\mathrm{Skw})$ is given by

$$\begin{aligned} &\mathbf{P}(\mathrm{Skw}) = \mathbf{II} - \mathbf{II}^s, \\ &\mathbf{P}(\mathrm{Skw})_{ijkl} = 1/2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{aligned} \tag{6.35}$$

There are four independent elastic constants.

(l) Isotropy and the icosahedral classes I and I_h. The ICR of **M** is given by

$$\mathbf{M} = \lambda^{\mathrm{I}}\mathbf{P}(S) + \lambda^{\mathrm{III}}\mathbf{P}(\mathrm{Skw}) + \lambda^{V}\mathbf{P}(D), \tag{6.36}$$

where the eigenprojections $\mathbf{P}(S), \mathbf{P}(\mathrm{Skw})$ and $\mathbf{P}(D)$ are given by (5.43), (6.35) and (5.48), respectively.

There are three independent elastic constants.

(m) Remark. From the ICRs given in this section it follows that the types of the absolute micropolar elasticity tensor under all possible orthogonal subgroups can be reduced to twelve types. As a corollary, one can infer that the crystallographic symmetries, together with the isotropy, can completely describe the symmetries of the absolute micropolar elasticity tensor.

7. Discussion

From 6×6 elastic matrices, Mehrabadi and Cowin [28] derived eigenvalues and eigentensors of various types of classical anisotropic elasticity tensors (see also Sutcliffe [29]). It can be seen that the spectral properties disclosed by the results given in Section 5 are consistent with those given by these established similar results.

Since all eigenprojections, except for some cases of low symmetry, are constructed explicitly in terms of unit vectors in directions of material symmetry axes and the transformation property of the latter under the material symmetry group is obvious, they directly indicate the material symmetries involved. The eigenprojections of distinct form in various kinds of characteristic representations identify the types of elastic symmetries, while the eigenprojections of identical form reveal the common properties of different types of anisotropic elastic materials.

From the results given in direct notion, the components of the elasticity tensors **E** and **M** under any chosen coordinate system can be obtained in terms of the invariant elastic constants given and the components of the unit vectors in directions of material symmetry axes, without resorting to the commonly used tedious component transformation formulae. For example, from the ICR for the cubic crystals (cf. Section 5(i)) we obtain

$$\begin{cases} E_{ijkl} = r_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + r_2\delta_{ij}\delta_{kl} + r_3 \sum_{\sigma=1}^{3} n_{\sigma i}n_{\sigma j}n_{\sigma k}n_{\sigma l}, \\ r_1 = \lambda^{\mathrm{III}}/2, r_2 = \lambda^{\mathrm{II}} - \lambda^{\mathrm{III}}, r_3 = (\lambda^{\mathrm{I}} - \lambda^{\mathrm{II}})/3. \end{cases} \tag{7.1}$$

The eigenvalues and eigentensors of various types of anisotropic elasticity tensors can be expressed in terms of the elements of the elastic matrices. For the case of the classical elasticity, refer to Mehrabadi and Cowin [28].

Moreover, from the ICR given, the inverse of each type of anisotropic elasticity tensor can be obtained directly and the inequality restrictions on the elastic constants due to the positive definiteness of the elastic energy function can be derived easily, refer to Section 3.

For centrosymmetric micropolar elastic materials, only the absolute micropolar elasticity tensors **A**, **D** and $\bar{\mathbf{A}}, \bar{\mathbf{D}}$ are involved and hence similar remarks can be made. For noncentrosymmetric micropolar elastic materials, however, the asymmetric micropolar elasticity pseudo-tensors **B** and $\bar{\mathbf{B}}$ have to be considered. As a result, it is not easy to deal with the above problems. It has been shown that the ICR of **M** given in Section 6 is useful in discussing the positive definiteness of the elastic energy function, etc, (cf. Xiao [31]).

From the ICRs for $\mathbf{E}$ and $\mathbf{M}$ given, one can infer that the crystallographic symmetries can completely describe the symmerties of the classical elasticity tensor and the absolute micropolar elasticity tensor, refer to Section 5(k) and Section 6(m).

The physical and mathematical significance and some applications of the spectral properties of the elasticity tensors were discussed in [28–29]. Other applications in the formulation of the limit criteria for anisotropic plasticity and in the mechanical behaviours of composite materials can be found in [32–36].

Appendix

In this Appendix we first prove Theorems 2 and 3 given in Section 4 and then show how to construct the given ICRs through an example.

Proof for Theorem 2. First, for every $\mathbf{0} \neq \mathbf{S} \in \mathrm{Lin}$, the subspace span $\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\}$ is a G-invariant subspace. In reality, let $\mathbf{Q}_k\mathbf{S}\mathbf{Q}_k^T$, $k = 1, \ldots, r$, be a basis of this subspace, where $\mathbf{Q}_k \in G$. Then any $\mathbf{H} \in \mathrm{span}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\}$ is expressible as

$$\mathbf{H} = \sum_{k=1}^{r} c_k \mathbf{Q}_k \mathbf{S} \mathbf{Q}_k^T.$$

Hence, by using this and the fact

$$(\mathbf{RQ}_k)\mathbf{S}(\mathbf{RQ}_k)^T \in \mathrm{span}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\}, \text{for any } \mathbf{R} \in G,$$

we infer

$$\mathbf{RHR}^T = \sum_{k=1}^{r} c_k (\mathbf{RQ}_k)\mathbf{S}(\mathbf{RQ}_k)^T \in \mathrm{span}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\}$$

for any $\mathbf{R} \in G$ and $\mathbf{H} \in \mathrm{span}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\}$.

Then, if $Z \subset \mathrm{Lin}$ is an irreducible G-invariant subspace, by using the result just shown we infer that (4.6) holds. Next, if $Z \subset \mathrm{Lin}$ is a subspace satisfying (4.6), then Z is invariant under G. Now suppose that Z is not an irreducible G-invariant subspace. Then there exists a proper G-invariant subspace $Z^0 \subset Z$ such that $\dim Z^0 < \dim Z$. Since Z^0 is invariant under G, we infer

$$\mathrm{span}\{\mathbf{QS}_0\mathbf{Q}^T \mid \mathbf{Q} \in G\} \subset Z^0 \text{ for every } \mathbf{0} \neq \mathbf{S}_0 \in Z^0.$$

Hence,

$$\mathrm{rank}\{\mathbf{QS}_0\mathbf{Q}^T \mid \mathbf{Q} \in G\} \leqslant \dim Z_0 < \dim Z,$$

which violates the condition (4.6).

Proof for Theorem 3. First, we have $Z_1 \cap Z_2 = \{\mathbf{0}\}$, where Z_1 and Z_2 are two irreducible G-invariant subspaces of Lin with different dimensions. In reality, suppose that there exists $\mathbf{0} \neq \mathbf{S} \in Z_1 \cap Z_2$. Then from Theorem 2 we derive

$$Z_1 = \operatorname{span}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\} = Z_2,$$

which contradicts the fact $Z_1 \neq Z_2$.

Next, without loss of generality, let $\dim Z_1 < \dim Z_2$. Let $\bar{Z}_1$ be the orthocomplement of Z_1 with respect to the subspace $Z_1 + Z_2$. Then by using $Z_1 \cap Z_2 = \{\mathbf{0}\}$ we infer $\dim \bar{Z}_1 = \dim Z_2$.

Suppose $\bar{Z}_1 \cap Z_2 = \{\mathbf{0}\}$. Then by $\dim \bar{Z}_1 = \dim Z_2 > \dim Z_1$, we infer

$$\dim(\bar{Z}_1 + Z_2) = \dim \bar{Z}_1 + \dim Z_2 > \dim Z_1 + \dim Z_2 \geqslant \dim(Z_1 + Z_2).$$

The latter contradicts the fact: $\bar{Z}_1 + Z_2 \subset Z_1 + Z_2$.

Thus, there is $\mathbf{0} \neq \mathbf{S} \in \bar{Z}_1 \cap Z_2$. Since both Z_1 and $Z_1 + Z_2$ are invariant under G, we infer that the orthocomplement $\bar{Z}_1$ is also invariant under G. Consequently,

$$\operatorname{span}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\} \subset \bar{Z}_1, \quad \operatorname{span}\{\mathbf{QSQ}^T \mid \mathbf{Q} \in G\} = Z_2.$$

Then, by the latter and $\dim \bar{Z}_1 = \dim Z_2$ we infer $\bar{Z}_1 = Z_2$.

This completes the proof for Theorem 3.

In the following, we show how to construct the given ICRs by fulfilling the procedures given in Section 4(c) through an example.

The ICR of the classical elasticity tensor $\mathbf{E}$ under the trigonal crystal class S_6. Let $(\mathbf{e}_1^0, \mathbf{e}_2^0, \mathbf{n})$ be a chosen orthonormal system. Under this system the tensor $\mathbf{S} \in$ Sym is of the form

$$\mathbf{S} = S_{ij}\mathbf{e}_i^0 \otimes \mathbf{e}_j^0, S_{ij} = S_{ji}, \mathbf{e}_3^0 = \mathbf{n}.$$

Let

$$\mathbf{S}_\sigma = \mathbf{Q}_\sigma \mathbf{S} \mathbf{Q}_\sigma^T, \sigma = 1, 2, \mathbf{Q}_1 = \mathbf{R}_{\mathbf{n}}^{2\pi/3}, \mathbf{Q}_2 = \mathbf{R}_{\mathbf{n}}^{4\pi/3}.$$

Then the following holds:

$$\mathbf{S} + \mathbf{S}_1 + \mathbf{S}_2 = 3S_{33}\mathbf{n} \otimes \mathbf{n} + \tfrac{3}{2}(S_{11} + S_{22})(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}). \tag{A1}$$

The condition (4.8), where $G = S_6$, yields

$$\mathbf{S} = a\mathbf{I} + b\mathbf{n} \otimes \mathbf{n}, a^2 + b^2 \neq 0.$$

Hence all one-dimensional S_6-invariant subspaces of Sym are given by

$$\operatorname{span}\{a\mathbf{I} + b\mathbf{n} \otimes \mathbf{n}\}, a^2 + b^2 \neq 0.$$

The sum of all one-dimensional S_6-invariant subspaces of Sym is a two-dimensional S_6-invariant subspace, given by span$\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}\} = Z^{\mathrm{I}}$, and therefore the orthocomplement of Z^{I} is four-dimensional, given by

$$\bar{Z}^{\mathrm{I}} = \mathrm{span}\{\mathbf{e}_1^0\otimes\mathbf{e}_1^0 - \mathbf{e}_2^0\otimes\mathbf{e}_2^0, \mathbf{e}_1^0\vee\mathbf{e}_2^0, \mathbf{n}\vee\mathbf{e}_1^0, \mathbf{n}\vee\mathbf{e}_2^0\}.$$

By using (A1) we infer

$$\mathrm{rank}\{\mathbf{QSQ}^T \mid \mathbf{Q}\in S_6\} = \mathrm{rank}\{\mathbf{S}, \mathbf{S}_1\} = 2 \quad \text{for any } \mathbf{0}\neq\mathbf{S}\in\bar{Z}^{\mathrm{I}}.$$

The condition (4.9), where $G = S_6$, is satisfied, and hence all two-dimensional irreducible S_6-invariant subspaces of Sym are given by

$$\mathrm{span}\{\mathbf{S}, \mathbf{S}_1\} \text{ for every } \mathbf{S}\in\bar{Z}^{\mathrm{I}}, \mathbf{S}\neq\mathbf{0}.$$

Thus, general forms of the irreducible S_6-invariant subspaces which constitute orthogonal decomposition of Sym are as follows:

$$\mathrm{span}\{\mathbf{W}_1\}, \mathrm{span}\{\mathbf{W}_2\}, Z_1^{\mathrm{II}} = \mathrm{span}\{\bar{\mathbf{W}}_3, \bar{\mathbf{W}}_4\}, Z_2^{\mathrm{II}} = \mathrm{span}\{\bar{\mathbf{W}}_5, \bar{\mathbf{W}}_6\},$$

where $\mathbf{W}_1, \ldots, \bar{\mathbf{W}}_6$ are orthonormal; $\mathbf{W}_1$ and $\mathbf{W}_2$ are given by (5.13) and $\bar{\mathbf{W}}_3, \ldots, \bar{\mathbf{W}}_6$ are given by

$$\begin{cases} \bar{\mathbf{W}}_{2\sigma+1} = u_\sigma(\mathbf{e}_1^0\otimes\mathbf{e}_1^0 - \mathbf{e}_2^0\otimes\mathbf{e}_2^0) + v_\sigma\mathbf{e}_1^0\vee\mathbf{e}_2^0 + p_\sigma\mathbf{n}\vee\mathbf{e}_1^0 + q_\sigma\mathbf{n}\vee\mathbf{e}_2^0, \\ \bar{\mathbf{W}}_{2\sigma+2} = v_\sigma(\mathbf{e}_1^0\otimes\mathbf{e}_1^0 - \mathbf{e}_2^0\otimes\mathbf{e}_2^0) - u_\sigma\mathbf{e}_1^0\vee\mathbf{e}_2^0 - q_\sigma\mathbf{n}\vee\mathbf{e}_1^0 + p_\sigma\mathbf{n}\vee\mathbf{e}_2^0, \end{cases}$$

$$\begin{cases} u_1v_2 - v_1u_2 - p_1q_2 + q_1p_2 = 0, \\ u_\sigma u_\tau + v_\sigma v_\tau + p_\sigma p_\tau + q_\sigma q_\tau = \delta_{\sigma\tau}/2, \end{cases} \quad \sigma,\tau = 1,2, \tag{A3}$$

and the eigenprojections are given by $\mathbf{W}_1\otimes\mathbf{W}_1$, $\mathbf{W}_2\otimes\mathbf{W}_2$, and

$$P(Z_\sigma^{\mathrm{II}}) = \bar{\mathbf{W}}_{2\sigma+1}\otimes\bar{\mathbf{W}}_{2\sigma+1} + \bar{\mathbf{W}}_{2\sigma+2}\otimes\bar{\mathbf{W}}_{2\sigma+2}, \sigma = 1,2.$$

In the following, we prove that the latter can be simplified. We prove that under the orthonormal system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$, where $(\mathbf{e}_1, \mathbf{e}_2)$ are given by (5.5) and

$$tg3x = (v_1q_1 - u_1p_1)/(u_1q_1 + v_1p_1) = (v_2q_2 - u_2p_2)/(u_2q_2 + v_2p_2), \tag{A4}$$

the tensors $(\mathbf{W}_3, \mathbf{W}_4)$ and $(\mathbf{W}_5, \mathbf{W}_6)$ given by (5.14) and (5.15) are orthonormal bases of Z_1^{II} and Z_2^{II}, respectively.

It suffices to prove that there exist $c_\sigma, d_\sigma \in R$, $c_\sigma^2 + d_\sigma^2 \neq 0$, such that

$$\begin{cases} \mathbf{W}_{2\sigma+1} = c_\sigma\bar{\mathbf{W}}_{2\sigma+1} + d_\sigma\bar{\mathbf{W}}_{2\sigma+2}, \\ \mathbf{W}_{2\sigma+2} = d_\sigma\bar{\mathbf{W}}_{2\sigma+1} - c_\sigma\bar{\mathbf{W}}_{2\sigma+2}, \end{cases} \quad \sigma = 1,2,$$

i.e.

$$\begin{cases} \mathbf{e}_1 \cdot \mathbf{W}_{2\sigma+1}\mathbf{e}_2 = n \cdot \mathbf{W}_{2\sigma+1}\mathbf{e}_1 = 0, \\ \mathbf{e}_1 \cdot \mathbf{W}_{2\sigma+2}\mathbf{e}_1 = n \cdot \mathbf{W}_{2\sigma+2}\mathbf{e}_2 = 0, \end{cases} \quad \sigma = 1, 2,$$

i.e.

$$\begin{cases} (u_\sigma \sin 2x - v_\sigma \cos 2x)c_\sigma + (v_\sigma \sin 2x + u_\sigma \cos 2x)d_\sigma = 0, \\ (p_\sigma \cos x + q_\sigma \sin x)c_\sigma + (p_\sigma \sin x - q_\sigma \cos x)d_\sigma = 0, \end{cases} \quad \sigma = 1, 2.$$

The above two systems of equations have nontrivial solutions $c_\sigma^2 + d_\sigma^2 \neq 0$ iff (A4) holds. Moreover, by using (A3) we can infer that the second equality of (A4) holds.

Thus, we conclude that the ICR of $\mathbf{E}$ under S_6 is given by (5.12)–(5.15).

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