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Factorisation of positive definite operators

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Abstract. In this paper we prove Reade's result for the positive definite C^1 kernels by using the factorisation method used by Kühn.

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1. Introduction. Following a method used by Kühn we prove Reade's result for the positive definite C^1 kernels. (See [5] and [6].) We prove eigenvalues λ_n of positive definite T in the case $K(x,t) \in C^1[0,1]^2$ are $o(1/n^2)$ by factorising the square root operator S via the Banach space $C[0,1]$. Explicitly we show $S : L^2[0,1] \to C[0,1]$ has approximation numbers $a_n(S) = o(1/\sqrt{n})$ by using the Fejer Kernel method. (See [4].) Then, we use the fact that the identity transformation $I : C[0,1] \to L^2[0,1]$ is 2-summing and has Weyl numbers $x_n(I) = O(1/\sqrt{n})$. (See [2], [3], or [7].)

Submultiplicativity of singular numbers then gives $\lambda_n^{1/2} = o(1/n)$, and hence $\lambda_n = o(1/n^2)$.

2. Singular numbers. We need the following two types of singular numbers.

Approximation numbers (See [1, p. 204]) The nth approximation number $a_n(T)$ of the operator $T : X \to Y$ between Banach spaces $X,\ Y$ is

$$a_n(T) = \inf_R \|T - R\|$$

taken over all operators R with rank $< n$.

Weyl numbers (See [2]) The nth Weyl number $x_n(T)$ is

$$x_n(T) = \sup_A a_n(TA)$$

taken over all operators $A : H \to X$ where H is a Hilbert space and $\|A\| \leq 1$.

These are all equal to the eigenvalues in the case $X = Y = H$ a Hilbert space. They are all subadditive in the sense that

$$a_{m+n+1}(T+U) \leq a_{m+1}(T) + a_{n+1}(U)$$

for $T, U : X \to Y$. They are all submultiplicative in the sense that

$$a_{m+n+1}(TU) \leq a_{m+1}(T)a_{n+1}(U)$$

for $U : X \to Y,\ T : Y \to Z$ between the Banach spaces X, Y, Z.

We refer the reader to [1] and [2] for proofs of these properties.

3. Square roots. Any positive definite Fredholm operator

$$Tf(x) = \int_0^1 K(x,t)f(t)dt$$

with $K(x,t) \in C^1[0,1]^2$ has a unique positive square root

$$Sf(x) = \int_0^1 J(x,t)f(t)dt$$

where $J(x,t) \in L^2[0,1]^2$. If the eigenvalues and eigenfunctions of T are λ_n, ϕ_n then by Mercer's theorem we have

$$\sum_1^\infty \lambda_n < \infty.$$

Therefore

$$J(x,t) = \sum_1^\infty \lambda_n^{1/2} \phi_n(x) \overline{\phi_n(t)}$$

is a Hilbert-Schmidt kernel.

Lemma 3.1. *If $T : L^2[0,1] \to L^2[0,1]$ is a Mercer operator (positive definite with continuous kernel), then its square root $S : L^2[0,1] \to L^2[0,1]$ is into $C[0,1]$.*

Proof. If the eigenvalues and eigenfunctions of T are λ_n, ϕ_n then we have

$$Tf(x) = \sum_1^\infty \lambda_n \langle f, \phi_n \rangle \phi_n(x),$$

$$Sf(x) = \sum_1^\infty \lambda_n^{1/2} \langle f, \phi_n \rangle \phi_n(x).$$

This series is uniformly convergent over $[0,1]$ since

$$\left|\sum_{M}^{N} \lambda_n^{1/2} \langle f, \phi_n \rangle \phi_n(x)\right|^2 \leq \sum_{M}^{N} \lambda_n \left|\phi_n(x)\right|^2 \sum_{M}^{N} \left|\langle f, \phi_n \rangle\right|^2 .$$

Both summations on the right hand side $\to 0$ uniformly in x since

$$\sum_{1}^{\infty} \lambda_n \left|\phi_n(x)\right|^2 = K(x,x)$$

uniformly in x by Mercer's theorem and

$$\sum_{1}^{\infty} \left|\langle f, \phi_n \rangle\right|^2 \leq \|f\|_2^2$$

by Bessel's inequality. □

Lemma 3.2. *$S : L^2[0,1] \to C[0,1]$ defined by*

$$Sf(x) = \int_0^1 J(x,t) f(t) dt$$

has $a_n(S) = o(1/\sqrt{n})$.

Proof. Suppose $\|f\|_2 \leq 1$ and $\varepsilon > 0$ are given. Choose $\delta > 0$ such that

$$|\partial K/\partial x(x,t) - \partial K/\partial x(y,u)| < \varepsilon^2$$

for all $|x-y| < \delta$, $|t-u| < \delta$.

Working with the Fejer Kernel on the interval $[-\pi, \pi]$, if we let

$$R_n f(x) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} \frac{\sin^2 nt/2}{\sin^2 t/2} f(x-t) dt,$$

then

$$R_n Sf(x) - Sf(x) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} \frac{\sin^2 nt/2}{\sin^2 t/2} \left(Sf(x-t) - Sf(x)\right) dt$$

where

$$\begin{aligned} |Sf(x-t) - Sf(x)|^2 &= \left| \int_0^1 \left(J(x-t,u) - J(x,u)\right) f(u) du \right|^2 \\ &\leq \int_0^1 |J(x-t,u) - J(x,u)|^2 \, du \int_0^1 |f(u)|^2 \, du \end{aligned}$$

$$\le \int_0^1 (J(x-t,u)-J(x,u))\,\overline{(J(x-t,u)-J(x,u))}du$$
$$= \int_0^1 (J(x-t,u)-J(x,u))\,(J(u,x-t)-J(u,x))\,du$$
$$\le |K(x-t,x-t)-K(x,x-t)-K(x-t,x)+K(x,x)|$$

since

$$\int_0^1 J(x,u)J(u,t)du = K(x,t).$$

We also have

$$|K(x-t,x-t)-K(x,x-t)-K(x-t,x)+K(x,x)| = \left|\partial K/\partial x(x-\theta t,x-t)-\partial K/\partial x(x-\theta^{'}t,x)\right||t|$$

for some $0<\theta$, $\theta^{'}<1$ by the mean value theorem. Therefore

$$|Sf(x-t)-Sf(x)|^2 < \begin{cases} \varepsilon^2|t| & \text{if } |t|<\delta, \\ 2\|\partial K/\partial x\|_\infty |t| & \text{otherwise.}\end{cases}$$

It follows that if we write

$$R_nSf(x)-Sf(x) = \frac{1}{2\pi n}\left(\int_{-\pi}^{-\delta}+\int_{-\delta}^{\delta}+\int_{\delta}^{\pi}\right)\frac{\sin^2 nt/2}{\sin^2 t/2}\,(Sf(x-t)-Sf(x))\,dt,$$

then

$$\left|\frac{1}{2\pi n}\int_{-\delta}^{\delta}\frac{\sin^2 nt/2}{\sin^2 t/2}\,(Sf(x-t)-Sf(x))\,dt\right| \le \frac{1}{2\pi n}\int_{-\delta}^{\delta}\frac{\sin^2 nt/2}{\sin^2 t/2}\,|Sf(x-t)-Sf(x)|\,dt$$
$$< \frac{\varepsilon}{2\pi n}\int_{-\delta}^{\delta}\frac{\sin^2 nt/2}{\sin^2 t/2}\,|t|^{1/2}\,dt \le \frac{\varepsilon}{2\pi n}\int_{-\pi}^{\pi}\frac{\sin^2 nt/2}{\sin^2 t/2}\,|t|^{1/2}\,dt = \frac{\varepsilon}{\pi n}\int_{0}^{\pi}\frac{\sin^2 nt/2}{\sin^2 t/2}t^{1/2}dt$$
$$\le \frac{\varepsilon\pi}{n}\int_0^\pi \frac{\sin^2 nt/2}{t^{3/2}}dt \le \frac{\varepsilon\pi}{\sqrt{2n}}\int_0^{n\pi/2}\frac{\sin^2 u}{u^{3/2}}du \le \frac{\varepsilon\pi}{\sqrt{2n}}\int_0^{\infty}\frac{\sin^2 u}{u^{3/2}}du.$$

Also

$$
\begin{aligned}
&\left| \frac{1}{2\pi n} \int\limits_{\delta}^{\pi} \frac{\sin^2 nt/2}{\sin^2 t/2} \left(Sf(x-t) - Sf(x) \right) dt \right| \\
&\quad \leq \frac{1}{2\pi n} \int\limits_{\delta}^{\pi} \frac{\sin^2 nt/2}{\sin^2 t/2} \left| Sf(x-t) - Sf(x) \right| dt \\
&\quad < \frac{\sqrt{2 \left\| \partial K / \partial x \right\|_\infty}}{2\pi n} \int\limits_{\delta}^{\pi} \frac{\sin^2 nt/2}{\sin^2 t/2} t^{1/2} dt \\
&\quad < \frac{\sqrt{2 \left\| \partial K / \partial x \right\|_\infty}}{2\pi n \sin^2 \delta/2} \int\limits_{0}^{\pi} t^{1/2} dt \\
&\quad < \frac{\varepsilon}{\sqrt{n}}
\end{aligned}
$$

for sufficiently large n.
The first integral is handled similarly. Hence the result follows. □

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