Integr. equ. oper. theory 55 (2006), 487–496

0378-620X/040487-10, *published online* December 20, 2005
DOI 10.1007/s00020-005-1399-8

Integral Equations and Operator Theory

Strictly Singular and Regular Integral Operators

Julio Flores

Abstract. In the setting of Köthe function spaces we present some sufficient conditions for a regular integral operator to be strictly singular.

Mathematics Subject Classification (2000). 47B65.

1. Introduction

The class of integral operators between function spaces has been widely studied by many authors (cf. [7], [3], [19]) as well as its relation with other classes of operators such as compact operators (cf. [17], [14], [2]). Following along this line Dodds and Schep provide in [6] some sufficient conditions under which every integral operator between two Köthe function spaces is compact. Motivated by this, we consider in this note the existence of similar results when we replace the class of compact operators by the closely related class of strictly singular operators. Recall that a bounded operator T between two Banach spaces X and Y is said to be *strictly singular* (or Kato) if the restriction of T to any infinite-dimensional (closed) subspace of X is not an isomorphism.

More specifically our motivation arises from Dodds and Schep's result together with other result due to Johnson ([12]), concerning the characterization of compactness of operators by means of factorization through certain spaces. When these two results are combined they yield a sufficient condition under which every regular integral operator from $L^p(\mu)$ to $L^q(\nu)$, $1 < p < \infty, 1 \leq q < \infty, p \neq q$ is strictly singular (see Proposition 2.6 below). From this point we focus on obtaining analogous results in a more general setting. Thus, for Köthe function spaces we obtain the following:

Theorem 1.1. *Let (Ω, Σ, μ) and (Ω', Σ', ν) be probability spaces and let $E(\mu)$ and $F(\nu)$ be Köthe function spaces such that $E(\mu)$ is reflexive and satisfies the*

Partially supported by BFM2001-1284.

subsequence splitting property, and $F(\nu)$ is order continuous. Let us assume that $[s(E(\mu)), \sigma(E(\mu))] \cap [s(F(\nu)), \sigma(F(\nu))] = \varnothing$. Then every regular integral operator $T : E(\mu) \to F(\nu)$ is strictly singular.

The paper is divided in two parts. In the first one we present Proposition 2.6 and a generalization of it to Köthe function spaces (Theorem 1.1). In the second part we give some results on stability by duality of the property "being strictly singular".

We refer the reader to ([1]), ([16]) and ([18]) for any unexplained terms concerning Banach lattice and regular operators theory.

2. Strict singularity of regular integral operators between Köthe function spaces

Just to fix things let us recall some definitions and basic facts. All measure spaces in the sequel are probability spaces.

Definition 2.1. (*cf.* [18, 3.3]) Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces on (Ω, Σ, μ) and (Ω', Σ', ν) respectively, and $T : E(\mu) \to F(\nu)$ a bounded operator. T is *integral* if there is a real function K (kernel) defined on $\Omega \times \Omega'$ which is $\Sigma \times \Sigma'$-measurable, such that

$$Tf(t) = \int_{\Omega} K(s,t) f(s) \mu(ds)$$

for every $f \in E$ and ν-almost every $t \in \Omega'$.

We say that the operator T is *regular integral* if it is integral and it is the difference of two positive operators (an operator T is *positive* if it transforms positive elements in positive elements). It is well known that an integral operator T is positive if and only if its kernel satisfies $K(s,t) \geq 0$ $\mu \times \nu$-a.e. If T is an integral operator on $E(\mu)$ and with values in $F(\nu)$, with kernel $K_T(s,t)$, then T is regular if and only if the operator S defined by means of the kernel $|K_T(s,t)|$ has its range in $F(\nu)$. In this case the modulus operator $|T|$ equals S (cf. [18, Thm. 3.3.5]).

One of the reasons why we work in the setting of regular operators is that the adjoint operator of a regular integral operator is integral. This is not true in general.

Proposition 2.2. (cf. [18, Prop. 3.3.2]) *Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces and let $T : E(\mu) \to F(\nu)$ be a regular integral operator with kernel K. If $F(\nu)$ is order continuous then the adjoint operator $T' : F(\nu)' \to E(\mu)'$ is integral with kernel $K'(t,s) = K(s,t)$ for every $(s,t) \in \Omega \times \Omega'$.*

Recall that a Banach lattice E satisfies an *upper* (resp. *lower) q-estimate* if there exists a constant $M > 0$ such that $\left\| \sum_{i=1}^{n} x_i \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^q \right)^{\frac{1}{q}}$ (resp.

$\left\|\sum_{i=1}^{n} x_i\right\| \geq M\left(\sum_{i=1}^{n}\|x_i\|^q\right)^{\frac{1}{q}}$) for every disjoint sequence $(x_i)_{i=1}^n$ in E and every natural n . The lower (resp. upper) index of E is defined as $s(E) = \sup\{q \geq 1 : E$ satisfies an upper q-estimate$\}$ (resp. $\sigma(E) = \inf\{q \geq 1 : E$ satisfies a lower q-estimate$\}$). It is well known that $1 \leq s(E) \leq \sigma(E) \leq \infty$ and that $\frac{1}{s(E)}+\frac{1}{\sigma(E')} = 1$ and $\frac{1}{\sigma(E)} + \frac{1}{s(E')} = 1$ (cf. [22, pag. 563]).

Proposition 2.3. ([6]) *Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces and $T : E(\mu) \to F(\nu)$ be an integral operator. Then T is compact if one of the following conditions holds:*

i) *$E(\mu)'$ is order continuous and $F(\nu) = L^1(\nu)$.*
ii) *T' is integral, $E(\mu)$ and $F(\nu)$ are order continuous and $\sigma(F(\nu)) < s(E(\mu))$.*

Note that condition ii) in the previous proposition is fulfilled whenever T is regular by Proposition 2.2.

The second result about factorization is due to Johnson :

Proposition 2.4. ([12]) *Let (Ω, Σ, μ) be a separable atomless probability space, and a Banach space X.*

i) *Let $T : X \to L^q(\mu)$ be a bounded operator and $2 < q < \infty$. Then T factorizes through l^q if and only if $T : X \to L^2(\mu)$ is compact.*
ii) *Let $T : L^p(\mu) \to X$ be a bounded operator and $1 < p < 2$. Then T factorizes through l^p if and only if the restriction operator $T : L^2(\mu) \to X$ is compact.*

These two results give as a consequence the following:

Proposition 2.5. *Let (Ω, Σ, μ) be a separable atomless probability space and $E(\nu)$ a Köthe function space.*

i) *If $1 < p < s(E(\nu)) \leq \sigma(E(\nu)) < 2$ and $T : L^p(\mu) \to E(\nu)$ is a regular integral operator then T is strictly singular.*
ii) *If $2 < s(E(\nu)) \leq \sigma(E(\nu)) < p < \infty$ and $T : E(\nu) \to L^p(\mu)$ is a regular integral operator then T is strictly singular.*

Proof. Consider the restriction $T : L^2(\mu) \to E(\nu)$, which is compact, since $\sigma(E(\nu)) < 2$, by Proposition 2.3 ii). It follows, by Proposition 2.4, that $T : L^p(\mu) \to E(\nu)$ factorizes through l^p, that is there are two operators $T_1 : L^p(\mu) \to l^p$ and $T_2 : l^p \to E(\nu)$ such that $T = T_2T_1$. The operator $T_2 : l^p \to E(\nu)$ is strictly singular since $p < s(E(\nu)) < 2$ (cf. [16, Thm. 1.d.7]). Hence $T : L^p(\mu) \to E(\nu)$ is also strictly singular.

ii) The operator $T : E(\nu) \to L^2(\mu)$ is compact, since $s(E(\nu)) > 2$, by Proposition 2.3. It follows by Proposition 2.4, that the operator $T : E(\nu) \to L^p(\mu)$ is the composition of two operators $T_1 : E(\nu) \to l^p$ and $T_2 : l^p \to L^p(\mu)$. Now the operator $T : E(\nu) \to l^p$ is strictly singular since $2 < \sigma(E(\nu)) < p$ (cf. [16, Thm. 1.d.7]). Hence $T : E(\nu) \to L^p(\mu)$ is also strictly singular. □

We will use the well-known Kadec-Pelczynski's disjointification technique (cf. [16, Prop. 1.c.8]). If $F(\mu)$ is a Köthe function space on a probability space (Ω, Σ, μ) we consider the sets $M_{F(\mu)}(\varepsilon) = \{y \in F(\mu) : \mu(\sigma(y, \varepsilon)) \geq \varepsilon\}$, where $\varepsilon > 0$ and $\sigma(y, \varepsilon) = \{t \in \Omega : |y(t)| \geq \varepsilon \|y\|\}$. Recall too that a Banach space X is *subprojective* ([21]) if every infinite dimensional closed subspace M in X contains an infinite dimensional closed subspace N which is complemented in X. Examples of subprojective function spaces are $L^p[0,1]$, $p \geq 2$.

Proposition 2.6. *Let (Ω, Σ, μ) and (Ω', Σ', ν) be separable atomless probability spaces, and $1 < p < \infty$, $1 \leq q < \infty$ with $p \neq q$. Then every regular integral operator $T : L^p(\mu) \to L^q(\nu)$ is strictly singular.*

Proof. Case $1 \leq q < p$ follows from Proposition 2.3 i). As for case $1 < p < q$ we can restrict ourselves, by Proposition 2.5, to prove the cases $1 < p \leq 2 < q$ and $1 < p < q = 2$.

(i) If $1 < p \leq 2 < q$ we consider the operator $T : L^p(\mu) \to L^1(\nu)$ which is compact by Proposition 2.3 ii). Assume that $T : L^p(\mu) \to L^q(\nu)$ is not strictly singular; then there is a subspace N of $L^p(\mu)$ such that T is invertible on N. Let $T(N) \subset L^q(\nu)$. If $T(N) \subset M_q(\varepsilon)$ for some Kadec-Pelczynski set, then $T(N)$ would be a closed subspace of $L^1(\nu)$ (cf. [16, Prop. 1.c.8]), which is a contradiction with the fact that $T : L^p(\mu) \to L^1(\nu)$ is compact.

Assume then that $T(N)$ is not included in any $M_q(\varepsilon)$; then, by [16, Prop. 1.c.8], there are a normalized sequence $(y_n)_n$ in $T(N)$ and a disjoint sequence $(z_n)_n$ in $L^q(\nu)$ which are basic equivalent sequences. It follows that $L^p(\mu)$ contains an isomorphic copy of the span $[z_n]$ or equivalently an isomorphic copy of l^q which is impossible in our case.

(ii) If $1 < p < q = 2$, then the operator $T' : L^2(\nu) \to L^{p'}(\mu)$, $2 < p' < \infty$, is also regular and integral, and hence strictly singular by part (i). It follows that $T : L^p(\mu) \to L^2(\nu)$ is strictly singular by [21, Thm. 2.2]. □

Note that the previous result holds trivially true if the probability spaces are purely atomic. In fact, as we look for a generalization of Proposition 2.6 we will see that the separability and lack of atoms actually play no role.

Recall that an order continuous Köthe function space $E(\mu)$ satisfies the *subsequence splitting property* (cf. [20]) if for every bounded sequence $(f_n)_n$ included in $E(\mu)$ there is a subsequence $(n_k)_k$ and sequences $(g_k)_k$, $(h_k)_k$ in E with $|g_k| \wedge |h_k| = 0$ and $f_{n_k} = g_k + h_k$ such that (i) $(g_k)_k$ is uniformly integrable and (ii) $|h_k| \wedge |h_l| = 0$ if $k \neq l$. As usual $(g_k)_k$ is uniformly integrable means that for every $\varepsilon > 0$ there is some $\delta > 0$ such that $\mu(A) < \delta$ implies $\|g_k \chi_A\|_{E(\mu)} < \varepsilon$ for all $k \in \mathbb{N}$.

Notice that every Köthe function space which does not uniformly contain copies of l_n^∞ for all natural n satisfies the subsequence splitting property (cf. [13], [8]). This is the case for every space $E(\mu)$ with finite upper index $\sigma(E(\mu))$. Also every rearrangement invariant function space which contains no isomorphic copy of c_0 has the subsequence splitting property (cf. [20]).

The following result is essential in the proof of Theorem 1.1. It involves the use of a lattice version of the class of strictly singular operators, introduced by Hernández and Rodríguez-Salinas in [11]. Precisely, a bounded operator T from a Banach lattice E into a Banach space Y is said to be *disjointly strictly singular* if there is no disjoint sequence of non-null vectors $(x_n)_n$ in E such that the restriction of T to the subspace $[x_n]$ spanned by the sequence $(x_n)_n$ is an isomorphism. The class of all disjointly strictly singular operators clearly contains every strictly singular operator but not conversely (f.i the natural inclusion $j : L^p[0,1] \to L^q[0,1]$ with $\infty > p > q \geq 1$).

Proposition 2.7. *Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces such that $E(\mu)$ is reflexive and satisfies the subsequence splitting property, and $F(\nu)$ is order continuous. Let $T : E(\mu) \to F(\nu)$ be a regular integral operator. Then T is disjointly strictly singular if and only if T is strictly singular.*

Proof. If we assume that $T : E(\mu) \to F(\nu)$ is not strictly singular, then there is an infinite dimensional subspace N of $E(\mu)$ such that the restriction $T|_N : N \to F(\nu)$ is an isomorphism onto its image, and yet the restriction $T|_N : N \to L^1(\nu)$ is compact by Proposition 2.3. Note that N is reflexive by the assumption on $E(\mu)$. Hence we can choose a normalized weakly null sequence $(f_n)_n$ in N. Since $E(\mu)$ satisfies the subsequence splitting property we can extract a subsequence, still denoted by $(f_n)_n$, such that $f_n = g_n + h_n$, where $(g_n)_n$ is uniformly integrable and $(h_n)_n$ is pairwise disjoint and $|g_n| \wedge |h_n| = 0$, for every $n \in \mathbb{N}$. From this point we proceed as in the proof of [9, Thm. 3.1] to conclude that T is invertible on the span $[h_n]$. Contradiction. $\square$

Remark 2.8. A careful reading of the proof of [9, Thm. 3.1] reveals that the regularity of the operator S there is used to justify that S transforms uniformly integrable sets into uniformly integrable sets.

Proposition 2.9. *Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces such that $E(\mu)'$ and $F(\nu)$ are order continuous. Assume that $[s(E(\mu)), \sigma(E(\mu))] \cap [s(F(\nu)), \sigma(F(\nu))] = \varnothing$. Then every integral operator from $E(\mu)$ to $F(\nu)$ is disjointly strictly singular.*

Proof. Assume that there is some integral operator $T : E(\mu) \to F(\nu)$ which is not disjointly strictly singular. Then there exist a normalized disjoint sequence $(x_n)_n \subset E(\mu)$ and a constant $\alpha > 0$ such that $\|Tx\| \geq \alpha \|x\|$ for all $x \in [x_n]$. Note that the operator $T : E(\mu) \to L^1(\nu)$ is compact by Proposition 2.3. Assume that $(Tx_n)_n \subset M_{F(\nu)}(\varepsilon)$ for some $\varepsilon > 0$, then $\|Tx_n\|_1 \geq \varepsilon^2 \|Tx_n\|_{F(\nu)}$; it follows that $\|Tx_n\|_{F(\nu)} \to 0$ which is impossible since $\|x_n\| = 1$ for all n and T is invertible on $[x_n]$. Hence we may assume that $[Tx_n] \nsubseteq M_{F(\nu)}(\varepsilon)$ for all $\varepsilon > 0$; in such case there are a subsequence $(n_j)_j$ and a disjoint sequence $(z_j)_j \subset F(\nu)$ such that $(Tx_{n_j})_j$ and $(z_j)_j$ are equivalent basic sequences (cf. [16, Prop. 1.c.8]). Now, since $(x_{n_j})_j$ and $(Tx_{n_j})_j$ are equivalent it follows that the disjoint sequences $(x_{n_j})_j \subset E(\mu)$ and $(z_j)_j \subset F(\nu)$ are also equivalent; however this is a contradiction with the assumption $[s(E(\mu)), \sigma(E(\mu))] \cap [s(F(\nu)), \sigma(F(\nu))] = \varnothing$. $\square$

Remark 2.10. In the previous result the operator T is not required to be regular. In contrast, Proposition 2.9 holds true for arbitrary regular operators. Indeed, note that under those conditions every regular operator transforms norm bounded disjoint sequences in sequences converging to zero in the $\| \|_{F(\nu)}$-norm.

Proof of Theorem 1.1. Just put together Propositions 2.7 and 2.9. □

Note that Proposition 2.6 is a particular case of Theorem 1.1.

We show now that regularity in Theorem 1.1 is essential. Indeed, let $(g_n)_n$ be a pairwise disjoint normalized sequence of positive functions in $L^2[0,1]$. Let $P : L^2[0,1] \to L^2[0,1]$ be the natural projection $P(f) = \sum_{n=1} (\int f g_n) g_n$. Since the sequence $(g_n)_n$ is equivalent to the canonical basis $(e_n)_n$ of l^2 we can consider the operator $\tilde{P} : L^2[0,1] \to l^2$ defined by $\tilde{P}(f) = \sum_{n=1} (\int f g_n) e_n$ which is clearly positive and therefore order bounded. Note also that the operator $\tilde{P}$ is integral with kernel $K(n,t) = g_n(t)$. Using for instance the Rademacher functions we can choose an operator $\varphi : l^2 \to L^1[0,1]$ which is an isomorphism onto its image. Besides φ is integral with kernel $K(n,t) = (\varphi(e_n))(t)$. Observe that the operator $\varphi\tilde{P} : L^2[0,1] \to L^1[0,1]$ is also integral. Indeed, take an arbitrary order bounded sequence $(f_n)_n$ in $L^2[0,1]$ converging to zero in measure. Since $\tilde{P}$ is integral the sequence $(\tilde{P}f_n)_n$ converges to zero almost everywhere by Bukhvalov's theorem (cf. [18, 3.3.11]), hence it converges to zero in measure on every subset of finite measure. Moreover $(\tilde{P}f_n)_n$ is order bounded since $\tilde{P}$ is so. Since the operator φ is also integral we equally obtain that the sequence $(\varphi\tilde{P}f_n)_n$ converges to zero almost everywhere. It suffices to use Bukhvalov's theorem again to conclude that the product $\varphi\tilde{P} : L^2[0,1] \to L^1[0,1]$ is integral.

On the other hand the operator $\varphi\tilde{P}$ is not disjointly strictly singular since it is an isomorphism when restricted to the span $[g_n]$. Finally the operator $\varphi\tilde{P}$ is not regular by Remark 2.10.

What has been seen so far makes one wonder about the situation in the case $p = q$. Contrarily to what might be expected Proposition 2.6 does not hold in the case $p = q$. To see this we will make use of the next result due to Caselles and González ([5]).

Proposition 2.11. *Let $T : L^p(\mu) \to L^p(\nu)$ be a regular operator, with $1 < p < \infty$. Then T is strictly singular if and only if T is compact.*

Proposition 2.11 is not true for non-regular operators: take $2 < p < \infty$, a pairwise disjoint normalized sequence of positive vectors $(y_n)_n \subset L^p[0,1]$ and write $\varphi_2 : l^p \to [y_n]$ for the positive isometry which transforms every element e_n into the element y_n. Call $P : L^p[0,1] \to [r_n]$ the projection on the subspace spanned by the sequence $(r_n)_n$ of Rademacher functions and $\varphi_1 : [r_n] \to l^2$ the isomorphism which takes every r_n to the n^{th} element of the canonical basis of l^2. If $J : l^2 \hookrightarrow l^p$ is the canonical inclusion then the operator $T : \varphi_2 J \varphi_1 P : L^p[0,1] \to L^p[0,1]$ is strictly singular but it is not compact.

Assume now that every regular integral operator on $L^p(\mu)$, $1 < p < \infty$, is strictly singular. Then every operator belonging to this class would be compact by Proposition 2.11; however the potential-type operators studied by Krasnoselski et. al. (cf. [14, Pag. 84]) provide examples that this is not the case.

Notice that from Proposition 2.7 we actually obtain a sharpened version of Proposition 2.11 when we restrict ourselves to the class of regular integral operators:

Corollary 2.12. *Let $T : L^p(\mu) \to L^p(\nu)$ be a regular integral operator, with $1 < p < \infty$. Then T is compact if and only if T is disjointly strictly singular.*

Remark 2.13. When $p = 1$ it is possible to find regular integral operators which are strictly singular and yet not compact (cf. [10, III Prop. 3.10]).

3. Stability of strict singularity under duality

It is known that the strict singularity is not stable by duality. However some conditions on partial stability have been given in [21]. We want to show that when we work with regular integral operators the situation improves in some cases.

A natural approach is to consider the relations $\frac{1}{s(E)} + \frac{1}{\sigma(E')} = 1$ and $\frac{1}{\sigma(E)} + \frac{1}{s(E')} = 1$, Proposition 2.2 and Theorem 1.1.Thus we obtain the following

Proposition 3.1. *Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces with indices satisfying $1 < s(E(\mu)) \leq \sigma(E(\mu)) < \infty$ and $1 < s(F(\nu)) \leq \sigma(F(\nu)) < \infty$. Assume that $[s(E(\mu)), \sigma(E(\mu))] \cap [s(F(\nu)), \sigma(F(\nu))] = \varnothing$. If $T : E(\mu) \to F(\nu)$ is a regular integral operator then both T and T' are strictly singular.*

Note that this result does not apply to endomorphisms in $L^p(\mu)$. To consider this case we introduce the following

Definition 3.2. We say that a Banach lattice E is *disjointly subprojective* if for every pairwise disjoint sequence $(x_n)_n$ in E there is a subsequence $(n_k)_k$ such that the span $[x_{n_k}]$ is complemented in E.

This terminology extends in some sense to the lattice setting the notion of subprojective Banach spaces referred to above. Recall that $L^p[0,1]$-spaces are subprojective when $p \geq 2$; in contrast $L^p(\mu)$-spaces are disjointly subprojective for every $1 \leq p < \infty$; other disjointly subprojective spaces are the Lorentz function spaces $L_{p,q}[0,1]$ (cf. [4]) and some Orlicz function spaces $L^{x^p log^q(1+x)}[0,1]$, $1 < p < \infty, -\infty < q < \infty$.

Recall that an operator $T : E(\mu) \to F(\nu)$ is *compact in measure* if for every norm-bounded sequence $(f_n)_n$ in $E(\mu)$ there is some subsequence $(n_k)_k$ such that $(Tf_{n_k})_k$ is convergent in ν-measure.

Proposition 3.3. *Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces such that $F(\nu)$ is order continuous and disjointly subprojective. Let $T : E(\mu) \to F(\nu)$ be a regular operator such that T' is strictly singular. Then T is strictly singular if one of the following conditions holds:*

a) *$T : E(\mu) \to L^1(\nu)$ is strictly singular.*
b) *$E(\mu)'$ is order continuous and T is compact in measure.*

Proof. Assume that T is not strictly singular; then there is a closed subspace $N \subset E(\mu)$ isomorphic to $T(N)$. Suppose that $T(N) \subset M_{F(\nu)}(\varepsilon)$ for some $\varepsilon > 0$, where $M_{F(\nu)}(\varepsilon)$ is a Kadec-Pelczynski set. Then $T(N)$ is a closed subspace of $L^1(\nu)$. If we assume condition a) we arrive to a contradiction. Alternatively, if we assume b), then by the dominated convergence theorem we get that $T[-x, x]$ is relatively compact in $L^1(\nu)$ for every positive element $x \in E(\mu)$; it follows that $T : E(\mu) \to L^1(\nu)$ is weakly sequentially precompact (cf. [18, Thm. 3.4.18]) and hence that T does not preserve an isomorphic copy of l^1. Thus we can assume that N does not contain an isomorphic copy of l^1 and we can choose (by Rosenthal's dichotomy theorem) a normalized weakly null sequence $(x_n)_n$ in N. Now the sequence $(Tx_n)_n$ converges to zero in the weak topology of $L^1(\nu)$ and, passing to a subsequence, we can assume by the assumption b) that $(Tx_n)_n$ converges to zero in ν-measure. Since we supposed that $T(N) \subset M_{F(\nu)}(\varepsilon)$, we get that $(Tx_n)_n$ converges to zero in $F(\nu)$. This is a contradiction with the fact that $(x_n)_n$ is normalized and $T(N)$ is isomorphic to N. Summing up, if $T(N) \subset M_{F(\nu)}(\varepsilon)$ for some $\varepsilon > 0$, then both a) and b) lead to a contradiction.

Thus we can assume that $T(N) \not\subseteq M_{F(\nu)}(\varepsilon)$ for every $\varepsilon > 0$. Then there is a normalized sequence $(y_n)_n$ in $T(N)$ and a disjoint sequence $(w_n)_n$ in $F(\nu)$ such that $\|Ty_n - w_n\| \to 0$ (cf. [16, Prop. 1.c.8]). Since $F(\nu)$ is disjointly subprojective we get by a standard perturbation result (cf. [15, Prop. 1.a.9])) that there is a subsequence, still denoted by $(Ty_n)_n$ such that the span $[Ty_n]$ is complemented in $F(\nu)$ by some projection P. We want to show now that the adjoint operator T' is invertible on the subspace $K = \{y' \in F(\nu)' : y'(y) = y'(Py)\}$ and hence that T' is not strictly singular. This contradiction will conclude the proof.

Indeed, since the sequence $(w_n)_n$ is disjoint and normalized we have $|a_k| \leq \left\| \sum_{n=1}^{\infty} a_n w_n \right\|$ for every k. Also, by passing to a subsequence, we may assume that

$$\sum_{n=1}^{\infty} \|Ty_n - w_n\| < \varepsilon < \frac{M_1}{2M_2},$$

where M_1 and M_2 are such that

$$M_1 \left\| \sum_{n=1}^{\infty} a_n y_n \right\| \leq \left\| \sum_{n=1}^{\infty} a_n w_n \right\| \leq M_2 \left\| \sum_{n=1}^{\infty} a_n y_n \right\|.$$

Hence, if $y' \in K$ we have

$$\|T'y'\| = \sup\{|y'(Tx)| : x \in B_E\} \geq \sup\{|y'(Tx)| : x \in B_{[y_n]}\}$$
$$= \sup\left\{\left|y'\left(\sum_{n=1}^{\infty} a_n T y_n\right)\right| : \left\|\sum_{n=1}^{\infty} a_n y_n\right\| \leq 1\right\}$$
$$\geq \sup\left\{\left|y'\left(\sum_{n=1}^{\infty} a_n w_n\right)\right| : \left\|\sum_{n=1}^{\infty} a_n y_n\right\| \leq 1\right\}$$
$$- \sup\left\{\left|y'\left(\sum_{n=1}^{\infty} a_n (w_n - T y_n)\right)\right| : \left\|\sum_{n=1}^{\infty} a_n y_n\right\| \leq 1\right\}$$
$$\geq \sup\left\{\left|y'\left(\sum_{n=1}^{\infty} a_n w_n\right)\right| : \left\|\sum_{n=1}^{\infty} a_n w_n\right\| \leq M_1\right\}$$
$$- \sup\left\{\left|y'\left(\sum_{n=1}^{\infty} a_n (w_n - T y_n)\right)\right| : \left\|\sum_{n=1}^{\infty} a_n w_n\right\| \leq M_2\right\}$$
$$\geq \sup\left\{|y'(Py)| : \|y\| \leq M_1\right\}$$
$$- \sup\left\{\left|y'\left(\sum_{n=1}^{\infty} a_n (w_n - T y_n)\right)\right| : \left\|\sum_{n=1}^{\infty} a_n w_n\right\| \leq M_2\right\}$$
$$\geq M_1\|y'P\| - \sup\left\{\|y'\|\left\|\sum_{n=1}^{\infty} a_n w_n\right\|\left(\sum_{n=1}^{\infty} \|T y_n - w_n\|\right) : \left\|\sum_{n=1}^{\infty} a_n w_n\right\| \leq M_2\right\}$$
$$\geq M_1\|y'P\| - M_2\|y'\|\varepsilon > M_1\|y'\| - \frac{M_1}{2}\|y'\| = \frac{M_1}{2}\|y'\|.$$

Thus we have shown that T' is invertible on K as desired. □

Again from Proposition 2.2 and Proposition 3.3 we obtain the following

Proposition 3.4. *Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces such that $E(\mu)$ is reflexive, $F(\nu)$ is order cotinuous and both $E(\mu)$ and $F(\nu)$ are disjointly subprojective. Let $T : E(\mu) \to F(\nu)$ be a regular integral operator. Then T is strictly singular if and only if T' is strictly singular.*

Proof. T'' is strictly singular since T is so and $E(\mu)$ is reflexive. Since the adjoint operator $T' : F(\nu)' \to E(\mu)'$ is regular and integral, the operator $T' : F(\nu)' \to L^1(\mu)$ is compact by Proposition 2.3, and T' is strictly singular by Proposition 3.3. The converse is equally clear using that $T : E(\mu) \to L^1(\nu)$ is compact and Proposition 3.3. □

References

[1] C.D. Aliprantis and O. Burkinshaw, *Positive operators*, Academic Press, 1985.

[2] T. Andô, *On compactness of integral operators*, Indag. Math. **24** (1962), 235–239.

[3] A. V. Bukhvalov, *Integral representations of linear operators*, J. Soviet. Math. **8** (1978), 129–137.

[4] N.L. Carothers and S.J. Dilworth, *Geometry of Lorentz Spaces via Interpolation*, Longhorn Notes. The University of Texas at Austin. Functional Analysis Seminar. (1985-1986), 107–133.

[5] V. Caselles and M. González, *Compactness properties of strictly singular operators in Banach lattices*, Semesterbericht Funktionalanalysis. Tübingen, Summersemester. (1987), 175–189.

[6] P.G Dodds and A.R. Schep, *Compact integral operators on Banach function spaces*, Math. Z. **180** (1982), 249–255.

[7] N. Dunford and B.J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392.

[8] T. Figiel, N. Ghoussoub, and W.B. Johnson, *On the structure of nonweakly compact operators on Banach lattices*, Math. Ann. **257** (1981), 317–334.

[9] J. Flores and F. L. Hernández, *Domination by positive strictly singular operators*, J. London Math. Soc **66** (2002), no. 2, 433–452.

[10] S. Goldberg, *Unbounded linear operators*, Dover, New York, 1966.

[11] F. L. Hernández and B. Rodríguez-Salinas, *On l^p-complemented copies in Orlicz spaces II*, Israel J. of Math. **68** (1989), 27–55.

[12] W.B. Johnson, *Operators into L_p which factor through l_p*, J. London Math. Soc. **2** (1976), no. 14, 333–339.

[13] W.B Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces.*, Memoirs Amer. Math. Soc. **19** (1979).

[14] M.A Krasnoselskii, P.P Zabreiko, E.I. Pustylnik, and E.B. Sbolevskii, *Integral operators in spaces of summable functions.*, Nordhoff, Leyden, 1976.

[15] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, 1977.

[16] ______, *Classical Banach Spaces II*, Springer-Verlag, 1979.

[17] W.A.J Luxemburg and A.C Zaanen, *Compactness of integral operators in Banach function spaces*, Math. Ann. **149** (1963), 150–180.

[18] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, 1991.

[19] A.R Schep, *Kernel operators*, Ph.D. thesis, University Leiden, Netherlands, 1977.

[20] L. Weis, *Banach lattices with the subsequence splitting property*, Proc. Amer. Math. Soc. **105** (1989), 87–96.

[21] R.J. Whitley, *Strictly singular operators and their conjugates*, Trans. Amer. Math. Soc. **113** (1964), 252–261.

[22] A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983.

Julio Flores
Área de Matemática Aplicada, Escet, URJC, 28933 Móstoles, Madrid, Spain
e-mail: `julio.flores@urjc.es`

Submitted: January 22, 2003
Revised: June 17, 2005