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Geometry as an Aspect of Dynamics

A. L. L. Videira,[1] A. L. Rocha Barros,[2] and N. C. Fernandes[2]

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Contrary to the predominant way of doing physics, we claim that the geometrical structure of a general differentiable space-time manifold can be determined from purely dynamical considerations. Any n*-dimensional manifold* V_n *has associated with it a symplectic structure given by the* 2n *numbers* **p** *and* **x** *of the* 2n*-dimensional cotangent fiber bundle* TV_n. *Hence, one is led, in a natural way, to the Hamiltonian description of dynamics, constructed in terms of the covariant momentum* **p** *(a dynamical quantity) and of the contravariant position vector* **x** *(a geometrical quantity). That is, the Hamiltonian description furnishes a natural way of relating dynamics and geometry. Thus, starting from the Hamiltonian state function (for a particle)—taken as the fundamental dynamical entity—we show that general relativistic physics implies a general pseudo-Riemannian geometry, whereas the physics of the special theory of relativity is tied up with Minkowski space-time, and nonrelativistic dynamics is bound up to Newton–Cartan space-time.*

1. INTRODUCTION

It has been claimed(1) that perhaps the most important consequence of relativity theory is that space and time are not concepts which can be considered independently of each other, but that they must be combined in such a fashion as to give a four-dimensional description of physical phenomena. In this context, it has been stated, then, that *dynamics becomes an aspect of geometry.* This establishes an intimate relation between dynamics and geometry which can be considered even in prerelativistic theories.

In the usual manner of describing the natural world, this association

[1] Departamento de Física, Pontifícia Universidade Católica, Cx.P. 38071, Rio de Janeiro, RJ, Brasil.

[2] Instituto de Física, Universidade de São Paulo, São Paulo, Brasil.

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between dynamics and geometry is customarily considered as follows. When building a dynamical model of Nature one always begins by postulating a certain space-time and from there proceeds to develop a certain physics in that arena, which is, then, considered as a substratum to the physical world. That is, one usually starts from a given, preestablished geometry, upon which a consequential dynamics is established, and it is well known that the choice of the geometry (of the postulated space-time) uniquely determines the physics that can be constructed in that postulated space-time. Thus, just as the only dynamics compatible with the absolute space-time of Newton is precisely Newtonian dynamics, correspondingly, in Minkowski space-time, only the dynamics of special relativity can be naturally built.

In the present work, we intend to show how the introduction from the onset, in a general differentiable space-time manifold, of a certain well-defined minimal set of fundamental dynamical quantities allows the specific geometric structure of that manifold to be fixed. This view is basically contrary to the usual one and will be detailed below.

Before turning to the point of view to be developed here, let us present the special relativistic and Newtonian cases, as they are usually stated. The four-dimensional space-time manifold of Minkowski consists of a three-dimensional spatial hypercone with time pointing along its symmetry axis. The geometry of this manifold has as its invariance group the full Lorentz group (or group of Poincaré)

$$x'^{\mu} = L^{\mu}_{\nu} x^{\nu} + a^{\mu} \tag{1}$$

with Greek indices running from 1 to 4. Here, (L^{μ}_{ν}) is a (4×4) orthogonal matrix and a^{μ} is an arbitrary (constant) 4-vector.

Since it is perhaps somewhat less familiar than its Minkowski counterpart, let us dwell—although still in a cursory fashion—with the Newtonian case in a little more detail. In the Newtonian case, the four-dimensional space-time manifold was first introduced by E. Cartan[(2)] as an affine manifold E_4, consisting of a three-dimensional spacelike hypersurface, orthogonal to the absolute time axis. This geometry fixes the group of symmetry

$$x'^{\alpha} = G^{\alpha}_{\beta} x^{\beta} + k^{\alpha} \tag{2}$$

Here, the matrix (G^{α}_{β}) has the $(3+1) \times (3+1)$ block form:

$$(G^{\alpha}_{\beta}) = \begin{pmatrix} G & \mathbf{v} \\ 0 & 1 \end{pmatrix} \tag{3}$$

where G is a (3×3) orthogonal matrix and the (3×1) column vector $\mathbf{v}$ is arbitrary. This geometry (and its related symmetry group) determines both the absolute kinematical and dynamical entities, that is, those entities which are left invariant by the transformations (2).

The matrix (G^{α}_{β}) can be diagonalized and put in the form

$$\begin{pmatrix} GG^T & \mathbf{v} \\ 0 & 0 \end{pmatrix}$$

From this, it is seen that the metric (or fundamental) tensor ${}^{(n)}g_{\alpha\beta} = {}^{(n)}\eta_{\alpha\beta}$ of the affine Newtonian space-time E_4 is *singular.*[3,4] This fact immediately distinguishes Newtonian space-time from its special-relativistic counterpart. In fact, while in this latter case one can introduce dual metric tensors ${}^{(r)}g_{\alpha\beta}$ and ${}^{(r)}g^{\alpha\beta}$, one being the inverse of the other, this cannot be done in E_4, since there the inverse does not exist. Therefore, it is precisely in E_4, the Newtonian space-time, where the distinction between covariant and contravariant 4-vectors will be expected to be more fundamental than in the special relativistic case, where there exists a complete transposition between contravariant and covariant quantities. This, of course, should not be taken as meaning that in the three-dimensional spacelike hypersurface E_3 of E_4 this raising or lowering of indices is not fully justified, since that submanifold E_3 is Euclidean. This last fact leads to the consideration made a long time ago by E. Cartan[3] that E_4 is *not* an Euclidean manifold, but its affine connection, and ${}^{(n)}V_4$ is Euclidean, which is just another way of saying that the metric tensor of E_4 is singular.

2. CONTRAVARIANT AND COVARIANT VECTORS

When examining the interconnection between physics and geometry, it is of paramount importance to establish the essential distinction that exists between contravariant and covariant entities. A very striking aspect of this distinction, as has been repeatedly pointed out,[5] is that, while the contravariant vectors are the ones which are more intimately related with geometry, the covariant vectors are the ones which are more closely connected with physics. In this regard, two instances come immediately to mind: the position vector $\mathbf{x}$, which is essentially contravariant, and the momentum $\mathbf{p}$, which is essentially covariant. In this section, we discuss some aspects which manifest this distinction.

Given the vector affine space E_n, the linear mapping ω: $E_n \to R$ of E_n over R defines a linear form over E_n. The vectors of E_n are the *contravariant* vectors $\mathbf{x}$, which, in a given basis $\{\mathbf{e}_i\}$, are written as

$$\mathbf{x} = x^i \mathbf{e}_i \tag{4}$$

The linear forms over E_n belong to another vector affine space E_n^*, the dual of E_n. The vectors $\mathbf{x}^* \in E_n^*$ are the *covariant* vectors $\omega(\mathbf{x})$:

$$\mathbf{x}^* = \omega(\mathbf{x}) = \omega(\mathbf{e}_i)\, \mathbf{x}^i = a_i \mathbf{x}^i \tag{5}$$

where we can consider the $a_i \equiv \omega(\mathbf{e}_i)$ as the components of the covariant vector ω in the dual basis $\{\mathbf{x}^i\} \equiv \{\mathbf{e}^i\}$, i.e., we may write a covariant vector $\mathbf{x}^* \in E_n^*$ as

$$\mathbf{x}^* = x_i \mathbf{e}^i \tag{6}$$

with $x_i \equiv a_i$. (While the x^i are considered as vector components in E_n, in the dual space E_n^* they are linearly independent one-forms.)

The geometrical meaning of the contravariant and covariant vectors is obtained through the introduction of an affine space $(0, E_n) \equiv \mathscr{E}_n$, which is a space of points having a structure of a vector space depending on the point 0, taken as the origin.[6] It should be noticed that neither a metric was defined in E_n, nor a distance in $\mathscr{E}_n$.

The contravariant vector $\mathbf{x} = x^i \mathbf{e}_i \in E_n$ is represented geometrically by an oriented line, whereas the covariant vector $\mathbf{x}^* = x_i \mathbf{e}^i \in E_n^*$ is represented by two parallel hyperplanes, since we have a family $\mathbf{x}^* = x_i \mathbf{e}^i = \omega(\mathbf{x}) = a_i x^i = k$ of parallel hyperplanes, depending on the parameter k. Since the coordinate axes are intercepted at $x^i = k/a_i$, the components of a contravariant vector have dimensions of length—an *extensive* quantity—while the covariant vector components have dimensions of the inverse of a length—an *intensive* quantity.

As appropriate examples, we notice that the *position* vector $\mathbf{x}$ is essentially contravariant, while the gradient $\partial\phi/\partial\mathbf{x}$ of a scalar function $\phi(\mathbf{x})$ of position is essentially covariant. Recalling that in physics the dynamical quantity *momentum* $\mathbf{p}$ is defined as $\propto \partial\phi/\partial\mathbf{x}$, this definition makes momentum a covariant vector, and hence it is much more appropriate to write the fundamental equation of Newtonian dynamics as $\mathbf{f} = d\mathbf{p}/dt$, rather than in the form $\mathbf{f} = m\, d^2\mathbf{x}/dt^2$.

With contravariant and covariant vectors, many different kinds of algebras can be built.[7] Thus, let the contravariant vector $\mathbf{V} = V^j \mathbf{I}_j$ and the covariant vector $\mathbf{U} = U_j \mathbf{I}^j$ be written in the reciprocal bases $\mathbf{I}_j$ and $\mathbf{I}^j$ of a certain n-dimensional affine space. The invariant $U_j V^j$ is denoted here by $\langle \mathbf{U}, \mathbf{V} \rangle$. Introducing the symbols $(\mathbf{V})$ and $(\mathbf{U})$ associated to the vectors $\mathbf{V}$ and $\mathbf{U}$ by the anticommutation rules

$$\begin{aligned} [(\mathbf{V}), (\mathbf{V}')]_+ &= 0 \\ [(\mathbf{U}), (\mathbf{U}')]_+ &= 0 \\ [(\mathbf{V}), (\mathbf{U})]_+ &= \langle \mathbf{V}, \mathbf{U} \rangle\, 1_{G_n} \end{aligned} \tag{7}$$

we obtain the *Grassmann algebra* G_n (1_{G_n} is the unit of G_n). This algebra is generated by the elements $(\mathbf{I}_j)$ and $(\mathbf{I}^j)$ through the anticommutation rules

$$\begin{aligned} [(\mathbf{I}_j), (\mathbf{I}_k)]_+ &= 0 \\ [(\mathbf{I}^j), (\mathbf{I}^k)]_+ &= 0 \\ [(\mathbf{I}_j), (\mathbf{I}^k)]_+ &= \delta_j^k 1_{G_n} \\ \langle \mathbf{I}_j, \mathbf{I}^k \rangle &= \delta_j^k \end{aligned} \tag{8}$$

Equations (8) show that, although G_n is an algebra of an n-dimensional space, it has the structure of a *Clifford algebra* C_{2n} of a $2n$-dimensional space. The theory of G_n is, essentially, that of the spinors of E_{2n}. The Grassmann algebra G_n, taken over the complex numbers, is equivalent to an n-dimensional Jordan–Wigner algebra. Taking the adjoint $(\mathbf{I}^j) = (\mathbf{I}_j)^\dagger$, the anticommutation rules (8) become the n-dimensional equivalent to emission and absorption operators of the second quantization for fermions.[8]

Similarly, one can define an associative algebra L_n, with elements denoted by $\{\mathbf{V}\}$ and $\{\mathbf{U}\}$, satisfying the commutation rules

$$\begin{aligned} [\{\mathbf{V}\}, \{\mathbf{V}'\}] &= 0 \\ [\{\mathbf{U}\}, \{\mathbf{U}'\}] &= 0 \\ [\{\mathbf{V}\}, \{\mathbf{U}\}] &= \langle \mathbf{V}, \mathbf{U} \rangle\, 1_{L_n} \end{aligned} \tag{9}$$

(1_{L_n} being the unit element of L_n), and the generators of L_n satisfying the commutation rules

$$\begin{aligned} [\{\mathbf{I}_j\}, \{\mathbf{I}_k\}] &= 0 \\ [\{\mathbf{I}^j\}, \{\mathbf{I}^i\}] &= 0 \\ [\{\mathbf{I}_j\}, \{\mathbf{I}^k\}] &= \delta_j^k 1_{L_n} \end{aligned} \tag{10}$$

Equations (10) provide the Heisenberg commutation rules for the coordinate $\mathbf{Q} = Q^j \mathbf{q}_j$ and momentum operators $\mathbf{P} = P_j \mathbf{p}^j$, the generators of which are given by $\mathbf{q}_j = \{\mathbf{I}^j\}$ and $\mathbf{p}^j = i\hbar^{-1}\{\mathbf{I}^j\}$, where $\hbar$ is Planck's constant. Thus, L_n over the complex numbers is equivalent to the *Heisenberg algebra* for the operators $\mathbf{Q}$ and $\mathbf{P}$ of a quantum system with n degrees of freedom. It can also he shown that quantum kinematics is related to the *symplectic geometry* of the phase space of Hamiltonian classical mechanics through its symplectic algebra L_n.[9] Besides, the algebra L_n over the complex numbers provides the n-dimensional equivalent to the *Dirac–Jordan–Klein* algebra for the emission and absorption operators of the second quantization for bosons. In four-dimensional space, the action algebra, obtained from $dV = dp_i\, dx^i$, $i = 1, 2, 3, 4$, provides a quadratic form in eight variables. This is the only instance in which there is a triality: one vector and two half-spinors, all with eight components and with similar properties.[9,10]

3. BASIC POSTULATE

Having presented the above considerations upon the different algebraic structures generated by covariant and contravariant vectors, we may begin to assign a dynamical meaning to some of these vectors.

As we already said, the usual way of building physical models and/or theories consists in postulating a given space-time manifold, which is almost always metric (it can be shown that a differentiable manifold always admits a Riemannian metric[11,12]) and where that metric is always fixed *ab initio*. This is the fixed space-time framework upon which a certain theory is built.

Our starting point here is just the opposite: we try to determine the geometry by means of the introduction of a certain minimal number of fundamental dynamical objects. This point of view (Leibniz inspired) opposes the usual epistemological stand, which begins with the notion of space (of Aristotle, Newton, Minkowski, Riemann, Weyl, etc.) as the basic entity in Nature.

The only way a physicist has of interacting with Nature is by means of measuring processes (observations transmitted first to the senses and from those to the brain). The only manner of an interaction reaching the senses (and thence the brain) is by means of a signal which transfers information from the system to the observer. For this, a physical field is needed, to which certain energy and momentum densities may be ascribed, and which are the physical agents for the transmission of the signal. Therefore, it is only through the transfer of energy and momentum that a certain knowledge of the world, that is, of natural phenomena, may be obtained—in particular, a certain knowledge of its space-time features. In other words, *the very notion of space-time is strictly dependent on the notion of energy-momentum*. In the field equations of general relativity, this two-way street interplay between geometry and dynamics is already clearly present, and in the very cosmological model most widely accepted nowadays (and derived from those field equations)—the big-bang model—the creation (expansion) of space-time is inextricably associated to the total initial energy-momentum density of the universe. That is, the initial dynamical content is the only determinant of how the geometric structure unfolds.

Thus, let us consider the antisymmetrical bilinear form $dV = dp_\mu\, dx^\mu$, built up with the covariant momentum 4-vector p_μ and the contravariant position 4-vector x^μ. The hypervolume dV (physically, the *action*) is constant with respect to a variation of a parameter λ (which may be identified with the cosmological time). The universe's initial conditions were such that, for $\lambda = 0$, the momentum content was extremely high, whereas the space-time content was extremely low. We have here the most basic and fundamental observation referred to above: that the covariant vectors characterize the dynamical aspects, whereas the contravariant ones characterize the geometrical aspects.

With the aim in mind, then, of trying to determine a certain geometry (i.e., a certain metric) starting from a minimal number of dynamical objects, we begin by postulating the existence of a space-time manifold, the most general possible, with the least number of predetermined geometrical properties. Next, we shall populate the naked manifold with certain dynamical objects, taken as fundamental, trying then to determine what kind of manifold is compatible with these dynamical objects.

We shall take, then, as basic postulate of all our future considerations, the following one.

Existence of a Differentiable Manifold. There is a 4-dimensional differentiable manifold, $V_4(x^\mu)$, homogeneous in the (contravariant) space-time coordinates x^μ, which constitute a local system of coordinates (a chart). This parametrization need not cover the whole manifold V_4.

Now, to any n-dimensional differentiable manifold V_n there are naturally associated two vector spaces, namely (1) a *tangent* vector space, $T_x V_n$, tangent to V_n at point $\mathbf{x} \in V_n$, and locally spanned by the *contravariant* vectors $\mathbf{q} = q^i(\partial/\partial \mathbf{x}^i) \equiv q^i \partial_i$, $i = 1,..., n$, where $\{\partial_i\}$ is the natural basis of $T_x V_n$, and (2) a *cotangent* vector space, $T_x^* V_n$, *locally spanned by the covariant* vectors $\boldsymbol{\omega} = \omega_i \, dx^i$, where the natural local basis $\{d\mathbf{x}^i\}$ of $T_x^* V_n$ is the dual of $\{\partial_i\}$. These dual bases are, of course, special cases of $\{\mathbf{e}_i\}$, the basis of the vector affine space E_n, and of $\{\mathbf{e}^i\}$, the basis of its dual vector space E_n^*, introduced in Section 2.

The set formed by the union of the tangent spaces $T_x V_n$ for all the points $\mathbf{x} \in V_n$ is the tangent fiber bundle $TV_n = \bigcup_{\mathbf{x} \in V_n} T_x V_n$, a $2n$-dimensional space whose elements are specified by the $2n$ components $(\mathbf{q}, \mathbf{x})$. Also, the set $T^* V_n = \bigcup_{\mathbf{x} \in V_n} T_x^* V_n$, the cotangent fiber bundle, is a $2n$-dimensional space whose elements are specified by the $2n$ components $(\boldsymbol{\omega}, \mathbf{x})$. These two fiber bundles are self-dual.

The cotangent fiber bundle has a natural structure of a $2n$-dimensional manifold: a point of $T^* V_n$ is a one-form $\mathbf{p}$ over the tangent space to V_n in one of its points (by definition). The $2n$ numbers $\mathbf{p}$ and $\mathbf{q}$ form a set of local coordinates of a point of $T^* V_n$, which, then, has a *natural symplectic* *structure* given in the chart above by the exterior product $\boldsymbol{\omega} = d\mathbf{p} \wedge d\mathbf{q}$. Following our previous consideration of Section 2, we identify the covariant vector $\mathbf{p}$ as a *dynamical* object and the contravariant vector $\mathbf{q}$ as a *geometrical* object.

These considerations lead us in a natural way to the *Hamiltonian description of dynamics*, that is, to a description in terms of the Hamiltonian state function $H(\mathbf{p}, \mathbf{q})$, constructed in terms of the covariant vector $\mathbf{p}$ (a dynamical object) and in terms of the contravariant vector $\mathbf{q}$ (a geometrical object). That is: *the Hamiltonian description* (based on an even-dimensional manifold with a symplectic structure on it) *furnishes a natural way of relating dynamics and geometry.*[3]

This built- in relationship between dynamics and geometry—allowed by the Hamiltonian description—is fundamental for the establishment of the foundations of both classical and quantum mechanics, and recalls to mind the claim made by some people that perhaps the most important lesson of all from Einstein is that *geometry has its own Hamiltonian.*

Before starting, we would like to observe that, in what follows, we take *particle* dynamics[(13)] as the foundations of the world geometry. However, attention should be called to a recent paper[(14)] where a quantum field theory is formulated which "abandons as superfluous the notion of the four-dimensional space-time continuum." In their own words, Kaplunovski and Weinstein developed "a framework which allows the treatment of the topology and dimension of the space-time continuum as dynamically generated." Actually, what they did was to present examples of "quantum systems which start out with a well-defined notion of time but no notion of space, and dynamically undergo a transition to a space-time phase—a phase in which the physics of the low-energy degrees of freedom of the system are best described by an effective Lagrangian written in terms of conventional relativistic fields. In this sense, the notion of the four-dimensional space-time continuum as the arena within which the game of field theory is to be played is replaced by the notion of the space-time continuum as an illusion of low-energy dynamics."

Let us, then, introduce into our "naked" n-dimensional differentiable manifold $V_n(x^\mu)$ a particle of momentum p_μ, describing a world-line Γ characterized by x^μ, $\mu = 1, 2, \ldots, n$. The manifold V_n is "naked," *ab initio*, due to the absence of dynamical quantities besides the momentum p_μ and to the absence of any geometrical structure besides the existence of coordinates x^μ, both of which are naturally attached, as stated above, to the self-dual fiber bundles of V_n, T^*V_n, and TV_n, respectively.

[3] In the suggestive words of V. I. Arnold:[(9)] "Hamiltonian mechanics is geometry in phase space."

Even in a manifold without metric (say in a affine manifold) we can introduce the mathematical operation of tensor product. That is, given the vector affine space of dimension n^r, $\otimes^r T_x V_n$ of the r contravariant tensors and the vector affine space of dimension n^s, $\otimes^s T_x^* V_n$ (dual of $\otimes^s T_x V_n$) of the covariant tensors, we can always consider the tensor product operation $(\otimes T_x^* V_n)^s \otimes (\otimes T_x V_n)^r$, by which another class of tensors is constructed, the class of the mixed tensors—s times covariant and r times contravariant—defined as multilinear forms over the tensor product above. Making some of the upper (contravariant) indices of a mixed tensor, in a certain basis, equal to some of the lower (covariant) indices, the so-called contraction operation is introduced (which, although an intrinsic operation, is more easily treated this way, with the components expressed in a given basis).

Then, if $\{\mathbf{e}_\mu\} \equiv \{\partial/\partial \mathbf{x}^\mu\}$ is the natural frame of $T_x V_n$, and if $\{\mathbf{e}^\mu\} \equiv \{d\mathbf{x}^\mu\}$ is the natural frame of $T_x^* V_n$, we can build the following second-order tensors: $g_{\mu\nu}(x^\lambda)\, \mathbf{e}^\mu \otimes \mathbf{e}^\nu$, $g^{\mu\nu}(x^\lambda)\, \mathbf{e}_\mu \otimes \mathbf{e}_\nu$, $g^\mu_\nu(x^\lambda)\, \mathbf{e}_\mu \otimes \mathbf{e}^\nu$, where, as of now, none of these tensors has any preassigned geometrical meaning.

Recalling, next, our above considerations about the Hamiltonian description as being the natural way of relating dynamics and geometry, we require that the dynamical state of a particle of momentum p_μ, moving along a world-line $\Gamma(x^\lambda)$ of V_n, be characterized uniquely by a scalar function H, which is constructed by the contraction of the second-order symmetrical contravariant tensor $g_{\mu\nu}(x^\lambda)\, \mathbf{e}^\mu \otimes \mathbf{e}^\nu$, with $x^\lambda \in \Gamma$, by the first-order covariant momenta $p_\mu \mathbf{e}^\mu$, $p_\nu \mathbf{e}^\nu$:

$$2H(p_\rho, x^\rho) = g^{\mu\nu}(x^\lambda)\, p_\mu p_\nu \tag{11}$$

Since this function depends only on the canonical pairs (p_μ, x^μ), we attribute to H the dynamical meaning of the *Hamiltonian state function* of the particle.

In what follows, we shall analyze, separately, the cases of relativistic mechanics (both the general and the special theories) and of nonrelativistic mechanics.

4. RELATIVISTIC MECHANICS AND RELATED GEOMETRIES

4.1. General Relativity

We now take ${}^{(r)}V_4$ for our manifold, and as we want to start imposing general relativity (GR)—that is, the physics of gravitation—*all* the gravitational effects must, somehow, be described by the functions $g^{\mu\nu}(x^\lambda)$ in Eq. (11), although, we insist, no definite geometrical meaning has, as yet, been assigned to these "potentials."

The *dynamical* introduction of a metric (*dynamical metrization*) into our manifold ${}^{(r)}V_4$ may now be accomplished by means of a contravariant 4-vector p^μ (with no dynamical meaning *a priori*), which must be introduced in terms of the only dynamical entity describing the (gravitational) physics—the Hamiltonian state function (11). Thus, we *define*

$$p^\mu \equiv \frac{\partial}{\partial p_\mu} H(p_\rho, x^\rho) \tag{12}$$

with Greek indices now running from 0 to 3. Then, from (11) and (12) we immediately get

$$p^\rho = \frac{\partial H}{\partial p_\rho} = \tfrac{1}{2} g^{\mu\nu}(\delta^\rho_\mu p_\nu + \delta^\rho_\nu p_\mu) = \tfrac{1}{2}(g^{\rho\nu} p_\nu + g^{\mu\rho} p_\mu) = g^{\rho\mu} p_\mu$$

That is,

$$p^\rho = g^{\rho\mu} p_\mu \tag{13}$$

Consequently, we can write the Hamiltonian of GR as the inner product

$$2H = g^{\mu\nu}(x^\lambda)\, p_\mu p_\nu = p_\mu p^\mu = p^\mu p_\mu \equiv p^2 \tag{14}$$

and, therefore, the attribution of this inner product to ${}^{(r)}V_4$ endows $g^{\mu\nu}(x^\lambda)$ with the geometrical meaning of a *metric* tensor, that is, it makes this manifold both *metric* and *Riemannian*. This, of course, is exactly what was desired, since in GR the specification of the space-time geometry must be due entirely to the metric coefficients $g^{\mu\nu}$ (which in that theory play the role of gravitational "potentials").

Moreover, since we imposed that the inner product (14) must represent an invariant (the energy scalar function) of the GR group, the metric has to be indefinite, with signature of absolute value 2; in other words, the metric of ${}^{(r)}V_4$ has to be *pseudo*-Riemannian.

Since in GR there are no external potentials, the inertial motion of (a free) particle along a world-line Γ—a geodesic of ${}^{(r)}V_4$—is characterized by the only condition that the covariant 4-vector p_μ has to be a constant of motion along Γ:

$$p_\mu \mid_\Gamma = \text{const} \tag{15}$$

Next, we observe that we still have open the possibility of differentiating the particle's Hamiltonian state function with respect to the canonical contravariant 4-vector x^μ, defining the following covariant 4-vector:

$$\phi_\mu = \frac{\partial H}{\partial x^\mu} \tag{16}$$

which, in GR, has no obvious dynamical meaning.[4]

4.2. Special Relativity

The situation now in special relativity (SR) will be considered as a particular case of the former general relativistic one, and so both the procedures and the conclusions reached in Section 4.1 will be presently retained. Thus, here in SR, we already have a *dynamically metrized* (pseudo-Riemannian, with signature of absolute value 2) manifold ${}^{(r)}V_4$ to start with, and the further specialization of this manifold will have to be dynamically accomplished also.

As said above, in GR, a particle moving in a given gravitational field is always an inertial system, that is, it always follows a geodesic of the geometry related to that field; there are no external potentials acting on the particle. The situation is quite different in SR, though, where the equations of motion are still expressed in terms of forces.

However, again, as was the case for the dynamical metrization accomplished in GR via the definition of a contravariant 4-vector p^μ—derived from the basic dynamical function H, which has to determine all the physics—here, in SR, all the physics will also have to come out from the particle's Hamiltonian state function, Eq. (11). With this in mind, we then attribute to the covariant 4-vector ϕ_μ, defined in Eq. (16), the meaning of an *external potential function* to which a particle may be submitted.

As we are imposing Hamiltonian dynamics, the state function (11) has now, in SR, to independently satisfy the canonical pair of conjugate equations of motion:

$$\frac{d}{ds} x^\mu = \frac{\partial H}{\partial p_\mu} \tag{17}$$

$$-\frac{d}{ds} p_\mu = \frac{\partial H}{\partial x^\mu} = \phi_\mu \tag{17'}$$

where ds is an element along any world-line Γ. As before in GR, we characterize inertial motion by the condition (15), that is, by $p_\mu|_\Gamma =$ constant of

[4] With the aid of Eq. (16), it may easily be shown (see Appendix) that at any point x^μ of a world-line of the particle, the contravariant vector p^μ is tangent to that world-line.

motion, which, by (17′), is equivalent to the condition $\phi_\mu \,|_\Gamma = \text{const}$. Then, from (14) and (16),

$$\phi_\mu \,|_\Gamma = \frac{\partial H}{\partial x^\mu}\bigg|_\Gamma = \frac{\partial}{\partial x^\mu}\,(g^{\rho\sigma}(x^\lambda)\, p_\rho\, p_\sigma)\bigg|_\Gamma$$
$$= \frac{\partial g^{\rho\sigma}(x^\lambda)}{\partial x^\mu}\bigg|_\Gamma\, p_\rho\, p_\sigma = \text{const} = 0 \tag{18}$$

which means that the metric coefficients $g^{\rho\sigma}$ must be constant over the particle's world-line Γ, and since this Γ is arbitrary, this, in turn, implies that the $g^{\mu\nu}$ must be constant over all the manifold ${}^{(r)}V_4$. That is, starting from the Hamiltonian state function (11), using the external potential (16), and imposing the free motion condition (15), we conclude that the geometry of the pseudo-Riemannian manifold ${}^{(r)}V_4$ has to be *flat*, with signature of absolute value 2. This also corresponds to having $H = \text{const} = E$ over any Γ of ${}^{(r)}V_4$ (that is, over all the manifold). This implies that we can define the energy E over all of ${}^{(r)}V_4$, which, in turn, is equivalent to stating that we can now build, in SR, a *global* inertial frame over all of ${}^{(r)}V_4$.

On the other hand, if we had admitted in our flat manifold that $2H = g^{\mu\nu} p_\mu p_\nu$ had a *positive definite* metric, it can be easily shown[4,15] that this is equivalent to admitting that there is no upper bound to the velocity: an infinite value for the speed of particles would be physically realizable. This, in turn, is equivalent to admitting that the spacelike and timelike components of the 4-momentum are entirely interchangeable, a possibility which is completely foreign to our experience. We must, therefore, impose the dynamical principle that there is a limiting speed for the propagation of physical signals.

5. NONRELATIVISTIC MECHANICS AND RELATED GEOMETRY

Here, in the nonrelativistic case, we shall take ${}^{(nr)}V_4$ as our differentiable 4-dimensional space-time manifold. According to the basic postulate, let us introduce again into this manifold a particle of 3-momentum p_i, $i = 1, 2, 3$. Let this particle be moving with 3-velocity defined by $\dot{x}^i \equiv dx^i/dx^4$, where x^i are the space variables and x^4 is the time variable.

Once more, as we did in the relativistic section, we begin by considering, associated to the moving particle, its Hamiltonian state function as our fundamental dynamical entity. However, in the nonrelativistic situation, we replace the relativistic 4-momentum p_μ of the particle by its 3-momentum p_i and by an additional parameter m, with units of mass:

$$2H = m^{-1} g^{ij}(x^k, x^4)\, p_i\, p_j \tag{19}$$

where the functions $g^{ij}(x^k, x^4)$, as before, have no preassigned geometrical meaning.

Next, following what was done in the previous section, we define the following two 3-vectors (with no preassigned meaning), namely a contravariant one,

$$p^k \equiv m \frac{\partial H}{\partial p_k} \tag{20}$$

and a covariant one,

$$\phi_k = \frac{\partial H}{\partial x^k} \tag{21}$$

From (19) and (20),

$$p^k = \tfrac{1}{2} g^{ij}(\delta_i^k\, p_j + \delta_j^k\, p_i) = \tfrac{1}{2}(g^{kj} p_j + g^{ik} p_i) = g^{kj} p_j \tag{22}$$

and, hence we may also write here the Hamiltonian (19) as an inner product:

$$2H = m^{-1} g^{ij}(x^k, x^4)\, p_i\, p_j = m^{-1} p_i\, p^i = m^{-1} p^i p_i = m^{-1} p^2 \tag{23}$$

That is, the three-dimensional hypersurface ${}^{(nr)}V_3$ is *metric*, and so must also have g_{ij}, satisfying $g_{ij}\, g^{jk} = \delta_i^k$, such that

$$p_i = g_{ij}\, p^j \tag{24}$$

From (19) and (21), we get

$$\phi_k = (2m)^{-1} \frac{\partial g^{ij}}{\partial x^k}\, p_i\, p_j \tag{25}$$

Since we are imposing Hamiltonian dynamics, the equations of motion of the particle are independently given by the Hamiltonian conjugate pair

$$\dot{x}^k = \frac{\partial H}{\partial p_k} \tag{26}$$

$$-\dot{p}_k = \frac{\partial H}{\partial x^k} \tag{27}$$

From (20) and (26) we see that we can relate p^i with m through the expression

$$p^i = m\dot{x}^i \tag{28}$$

and, therefore, m may be identified with what Poincaré[16] called the *mass of Maupertuis* of the moving particle. Also, from (21) and (27), we get

$$\phi_k = -\dot{p}_k \tag{29}$$

that is, ϕ_k (as expected) is the external force applied to a particle of momentum p_i and of mass m [by Eq. (28)].

In both relativistic situations (GR and SR), we had inertial motion characterized by the condition that the 4-momentum $p = (p_i, E/c)$, with E being the total energy of the particle and c the velocity of light, be a constant of motion over any world-line Γ of the manifold. Here, in the nonrelativistic case, instead, we are dealing with the 3-momentum p_i and the mass m, and so we shall characterize free motion of our particle by the requirements that both p_i and the Hamiltonian H be constants of motion over any world-line Γ of the manifold ${}^{(nr)}V_4$:

$$p_i \,|_\Gamma = \text{const} \to \left.\frac{dp_i}{dx^4}\right|_\Gamma = 0 \tag{30}$$

$$H \,|_\Gamma = \text{const} \equiv E \to \left.\frac{dH}{dx^4}\right|_\Gamma = 0 \tag{31}$$

Then, from (30) and (29), $\phi_k \,|_\Gamma = 0$ and, from (25),

$$\left.\frac{\partial g^{ij}}{\partial x^k}\right|_\Gamma p_i p_j = 0 \Rightarrow \left.\frac{\partial g^{ij}}{\partial x^k}\right|_\Gamma = 0$$

and since Γ is arbitrary, this implies that ${}^{(nr)}V_3$ is *flat*.

Also, from the equations of motion (26) and (27), we know that $dH/dx^4 = \partial H/\partial x^4$, and so, from (31),

$$0 = \left.\frac{\partial H}{\partial x^4}\right|_\Gamma = (2m)^{-1} \left.\frac{\partial g^{ij}}{\partial x^4}\right|_\Gamma p_i p_j \Rightarrow \left.\frac{\partial g^{ij}}{\partial x^4}\right|_\Gamma = 0$$

which means that g^{ij} is also *time-independent* over ${}^{(nr)}V_3$.

We notice that in the basic dynamical equations which we considered—Hamilton equations of motion—the time coordinate x^4 plays the role of an independent parameter with respect to the space coordinates x^i. This means that the time axis has to be orthogonal to the three-dimensional spatial flat metric manifold ${}^{(nr)}V_3$. In other words, we obtained the so-called Newton–Cartan space-time geometry.[3]

6. CONCLUSIONS

Contrary to the customary way of doing physics, we were presently able to show that starting from given dynamical quantities—basically the Hamiltonian state function, which naturally comes out associated to a differentiable manifold—we can arrive at certain specific geometries. Thus, general relativistic physics implies general Riemannian geometry (Einstein space-time), while the physics of the special theory of relativity is tied up with a flat Riemannian manifold (Minkowski space-time). Finally, nonrelativistic particle dynamics is bound to Newton–Cartan space-time.

What this clearly seems to indicate is that the connection between physics and geometry is even more profound than it is commonly considered. By this, we mean that not only particle dynamics and certain space-times are closely interconnected, as stated above, but, more important, that the point of view taken here is perhaps the most fundamental. Namely, that instead of departing from a given postulated space-time and then inferring the associated particle dynamics, we should start by imposing a certain physics and then try to determine its related geometry. In other words, geometry should be considered as an aspect of dynamics. This point of view reminds us of Leibniz's conception of dynamics.[17]

APPENDIX

We recall that from the theory of first-order partial differential equations there is the Cauchy system of ordinary differential equations for the characteristic lines associated to those equations. In the case of Eq. (15), $p_\mu p^\mu = p^\mu p_\mu = 2H(p_\rho, x^\rho)$, the Cauchy system is

$$\frac{dx^\mu}{p^\mu} = -\frac{dp_\mu}{\phi_\mu} \equiv d\sigma \tag{A.1}$$

with $d\sigma$ denoting the common value of the eight ratios above. The infinitesimal proper time ds, corresponding to the infinitesimal displacement dx^μ on the world-lines Γ of the particle, is defined as $ds^2 \equiv g_{\mu\nu}(x^\lambda)\, dx^\mu\, dx^\nu$, which, according to the Cauchy system (A.1), may be written as

$$ds^2 = g_{\mu\nu}\, p^\mu p^\nu\, d\sigma^2 \tag{A.2}$$

or, by Eq. (15), as

$$ds^2 = 2H\, d\sigma^2 \tag{A.3}$$

Hence, it follows from (A.1) and (A.3) that, in any point x^μ of a world-line Γ of the particle, the contravariant vector p^μ is tangent to that world line:

$$p^\mu|_\Gamma = \frac{dx^\mu}{d\sigma}\bigg|_\Gamma = \frac{dx^\mu}{ds}\frac{1}{\sqrt{2H}}\bigg|_\Gamma \tag{A.4}$$

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