Invent. math. 76, 15-32 (1984)

Inventiones mathematicae

Natural coefficients and invariants for Markov-shifts

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1. Introduction

Unlike abstract ergodic theory, where the notion of isomorphism is straightforward due to the restriction to metric properties of the underlying dynamical systems, the theory of Markov-shifts admits a plethora of definitions of isomorphism, ranging from topological conjugacy to finitary isomorphism of various strengths. In [5] M. Keane and M. Smorodinsky showed that two finite state Markov shifts with the same period and the same entropy are finitarily isomorphic. On the other hand it was shown in [9] and [6] that such isomorphisms cannot, in general, have finite expected code-lengths. The results in [9] imply that in many cases the future code-length of either the isomorphism or of its inverse has infinite expectation, and in [6] it was proved that the total code-length of the isomorphism itself has, in general, infinite expectation. Both [9] and [6] use cohomological methods: in [9] W. Parry considers numerical invariants associated with the information function, whereas in [6] W. Krieger studies the behaviour of the Radon-Nikodym derivative of the shift invariant measure under the group of uniformly finite dimensional homeomorphisms (the transverse flow) and obtains an invariant group, denoted by Δ.

For finitary isomorphisms with finite expected total code lengths and inverse code lengths we shall reconstruct Krieger's invariant Δ and construct a further invariant group $\Gamma \supset \Delta$ such that Γ/Δ is cyclic. Moreover we shall show that there exists a distinguished generator $c\Delta$ of Γ/Δ which is again an invariant. From the triple $(\Gamma, \Delta, c\Delta)$ one can then derive a variety of further, numerical invariants.

The remainder of this paper is devoted to showing that Γ, Δ and $c\Delta$ are invariants of other types of isomorphism which involve the topology of the shift spaces. When one considers the strictest form of isomorphism - block isomorphism - between two Markov shifts T_P and T_Q one confronts the problem of solving the following matrix equations:

$$U(t)\,P^t = Q^t\,U(t) \qquad U(t)\,V(t) = (Q^t)^{(n)}$$
$$P^t\,V(t) = V(t)\,Q^t \qquad V(t)\,U(t) = (P^t)^{(n)},$$

where $P^t(Q^t)$ is obtained by raising every non-zero entry of $P(Q)$ to the power $t \geqq 0$, and where $(P^t)^{(n)}$ $((Q^t)^{(n)})$ denotes the n-th power of $P^t(Q^t)$ in the usual sense. The problem is to find matrices $U(t)$, $V(t)$ over the semi-ring $\mathbb{Z}^+(\exp)$ consisting of non-negative integral combinations of exponentials γ^t, $\gamma > 0$. One can show (cf. [13]) that the bases γ of these exponentials need only be sought in the multiplicative group generated by the non-zero entries of P and Q. The entries of $U(t)$ and $V(t)$ thus have to be chosen from a sub-semi-ring of $\mathbb{Z}^+(\exp)$ which appears to depend on both P and Q. S. Tuncel posed the problem of how to overcome this unsatisfactory situation and find a "natural" coefficient ring which is common to both P and Q ([13], cf. also [11]). We solve this problem by showing that the non-zero entries of $U(t)$ and $V(t)$ can always be chosen to be non-negative integral combinations of exponentials γ^t with $\gamma \in \Gamma$, the group mentioned earlier. S. Tuncel's problem was one of the principal motivations for the work presented here and we would like to thank him for posing a variety of searching questions in this direction. Thanks are also due to A. Ocneanu, who discussed Krieger's paper [6] with the authors.

To aid the reader's understanding of the relationships between some of the various types of isomorphisms dealt with in this paper we include the following diagram:

$$\begin{tikzcd}
& & \text{Block isomorphism} \arrow[d, Leftrightarrow] & & \\
& & \text{Strong shift equivalence} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
& \text{Almost block isomorphism} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \text{Shift equivalence} \arrow[dr, Rightarrow] & \\
\text{Finite equivalence} \arrow[d, Rightarrow] & & \begin{array}{c}\text{Finitary equivalence with}\\ \text{finite expected code length}\end{array} & & \text{Identical zeta functions} \arrow[d, Rightarrow] \\
\text{Identical } \beta\text{-functions} & & & & \text{Identical } \beta\text{-functions}
\end{tikzcd}$$

The definitions of these terms are given in Sect. 2 and 7, with the exceptions of *strong shift equivalence* and *finite equivalence*. These terms are defined in [11] and are included in the diagram only for the sake of completeness.

2. Preliminaries

Let P be an irreducible, non-negative $k \times k$ matrix whose maximal eigenvalue is 1, and let $\bar{p} > 0$, $p > 0$ be corresponding left and right eigenvectors with $\sum_{i=1}^{k} \bar{p}(i)\,p(i) = 1$. Let X_P be the space of doubly infinite sequences $x = (x_n)$ such that $P(x_n, x_{n+1}) > 0$ for every $n \in \mathbb{Z}$, and let $T_P \colon X_P \to X_P$ denote the shift transformation $(T_P x)_n = x_{n+1}$. X_P is a zero-dimensional, compact space, and we define a T_P-invariant Markov probability measure m_P on X_P by setting, for every cylinder set

$$[i_{-m}, \dots, i_n]_{-m} = \{x \in X_P: x_r = i_r \text{ for } -m \leqq r \leqq n\},$$
$$m_P[i_{-m}, \dots, i_n]_{-m} = \bar{p}(i_{-m}) P(i_{-m}, i_{-m+1}) \dots (P(i_{n-1}, i_n) p(i_n).$$

The measure preserving automorphism T_P of (X_P, m_P) is called a *Markov shift*. Two Markov shifts T_P and T_Q are *isomorphic* if there exists an invertible measure preserving transformation $\phi: X_P \to X_Q$ such that $\phi T_P = T_Q \phi$ a.e. The isomorphism ϕ is called *finitary* if it has the additional property that $\phi[i]_0$ is - up to a null set - a countable union of cylinder sets in X_Q for each state $i = 1, \dots, k$ of P, and $\phi^{-1}[j]_0$ is (again up to a set of measure zero) a countable union of cylinder sets in X_P for every state j of Q. The significance of this definition lies in the fact that for such an isomorphism it is possible (almost surely) to specify $\phi(x)_0$ and $\phi^{-1}(y)_0$ by knowing only finitely many coordinates of $x \in X_P$ and $y \in X_Q$. In other words there exist a set $E \subset X_P$ of measure zero and two non-negative integer valued measurable functions m_ϕ and a_ϕ on X_P such that if $x, x' \in X_P \setminus E$ and $x_i = x'_i$ for $-m_\phi(x) \leqq i \leqq a_\phi(x)$ then $\phi(x)_0 = \phi(x')_0$, and a similar statement holds for ϕ^{-1}. The basic result in the theory of finitary isomorphisms is due to M. Keane and M. Smorodinsky ([5]):

2.1. Proposition. *If T_P and T_Q have the same entropy and if P and Q have the same period then T_P and T_Q are finitarily isomorphic.*

In order to describe the role played by the assumption of finite expected code length we follow W. Krieger's ideas in [6]. To specify the coordinates $\phi(x)_{-n}, \dots, \phi(x)_0$ we need to know the coordinates x_i of x for

$$-\bigvee_{r=0}^{n} (m_\phi(T_P^r x) + r) \leqq i \leqq \bigvee_{r=0}^{n} (a_\phi(T_P^{-r} x) - r),$$

where $\bigvee$ denotes the maximum. The past $\{\phi(x)_{-n}: n = 0, 1, \dots\}$ of $\phi(x)$ is thus determined by the coordinates x_i with

$$-\infty < i \leqq a_\phi^*(x) = \bigvee_{r=0}^{\infty} (a_\phi(T_P^{-r} x) - r).$$

In a similar way one sees that the future $\{\phi(x)_n: n = 0, 1, \dots\}$ of $\phi(x)$ is given by

$$\left\{x_i: -m_\phi^*(x) = -\bigvee_{r=0}^{\infty} (m_\phi(T^r x) - r) \leqq i < \infty\right\}.$$

These statements are, of course, only true outside some set of measure zero, and analogous assertions can be made about ϕ^{-1} and the functions $a_{\phi^{-1}}^*$, $m_{\phi^{-1}}^*$.

It is at this point that one encounters a problem: the existence of a finitary isomorphism (under appropriate conditions) gives no guarantee that the functions a_ϕ^*, m_ϕ^*, $a_{\phi^{-1}}^*$, $m_{\phi^{-1}}^*$ are finite a.e., as would be desirable for an interpretation of ϕ and ϕ^{-1} as codes respecting the sequential structure of the processes. One of the crucial observations in [6] is that a_ϕ^*, m_ϕ^*, $a_{\phi^{-1}}^*$ and $m_{\phi^{-1}}^*$ are finite a.e. under the following

2.2. Hypothesis F.E. $\int (a_\phi + m_\phi)\, dm_P < \infty$ and $\int (a_{\phi^{-1}} + m_{\phi^{-1}})\, dm_Q < \infty$.

From the finiteness of a^*_ϕ, m^*_ϕ, $a^*_{\phi^{-1}}$ and $m^*_{\phi^{-1}}$ under the hypothesis F.E. Krieger draws some important conclusions:

2.3. Proposition. *Under the hypothesis F.E. there exists a set $E \subset X_P$ of measure zero with the following properties.*

(1) *If $x, x' \in X_P \setminus E$ and $x_i = x'_i$ for $i \geqq N$ for some $N \in \mathbb{Z}$ then there exists an integer M, depending on x and N, with $\phi(x)_i = \phi(x')_i$ for $i \geqq M$;*

(2) *similarly, if $x, x' \in X_P \setminus E$ satisfy $x_i = x'_i$ for $i \leqq N$ there exists $M = M(N, x) \in \mathbb{Z}$ with $\phi(x)_i = \phi(x')_i$ for $i \leqq M$;*

(3) *if $x, x' \in X_P \setminus E$ satisfy $x_i = x'_i$ for $|i| \geqq N \geqq 0$ there exists $M' = M'(N, x)$ with $\phi(x)_i = \phi(x')_i$ for $|i| \geqq M'$.*

Analogous statements hold for ϕ^{-1}.

We define an equivalence relation $\sim$ on $X_P (X_Q)$ by setting $x \sim x'$ if $x_n = x'_n$ for all but finitely many $n \in \mathbb{Z}$. Assertion (3) in Proposition 2.3 can then be restated in terms of this equivalence relation: there exist sets of measure zero $E \subset X_P$, $F \subset X_Q$ such that, for $x, x' \in X_P \setminus E$ and $\phi(x)$, $\phi(x') \in X_Q \setminus F$, we have $x \sim x'$ if and only if $\phi(x) \sim \phi(x')$. In [6], Krieger expresses this fact in yet another way: call a non-singular automorphism V of (X_P, m_P) *locally finite dimensional* if, for m_P-a.e. x, $Vx \sim x$, and denote by F_P and F_Q the group of all locally finite dimensional automorphisms of (X_P, m_P) and (X_Q, m_Q), respectively. Then the hypothesis F.E. implies that $\phi F_P \phi^{-1} = F_Q$. Let Δ_P denote the multiplicative subgroup of $\mathbb{R}^+$ consisting of all possible ratios

$$\frac{P(i_0, i_1) \dots P(i_{n-1}, i_0)}{P(i_0, j_1) \dots P(j_{n-1}, i_0)} \quad n \geqq 2,\ i_r, j_r \in \{1, \dots, k\},$$

and involving only non-zero entries of P, and define Δ_Q analogously. Obviously we have

$$\frac{dm_P \cdot V}{dm_P}(x) \in \Delta_P \qquad m_P\text{-a.e.},$$

for every $V \in F_P$, and can find, for every $\alpha \in \Delta_P$, an element $V \in F_P$ with

$$m_P\left(\left\{x \colon \frac{dm_P \cdot V}{dm_P}(x) = \alpha\right\}\right) > 0.$$

Since $\phi F_P \phi^{-1} = F_Q$ and $m_P \phi^{-1} = m_Q$ one obtains immediately that $\Delta_P = \Delta_Q$ (cf. [6]). This fact makes it possible to find many examples of Markov shifts which are finitarily isomorphic, but for which the isomorphism cannot satisfy the hypothesis F.E.

We conclude this section by remarking that the hypothesis F.E. arises naturally in the context of certain topological maps between Markov shifts. Two Markov shifts T_P and T_Q are said to be *almost block-isomorphic* if there exists a Markov shift T_R and continuous, surjective, measure preserving and a.e. one-to-one maps $\phi \colon X_R \to X_P$, $\psi \colon X_R \to X_Q$ with $\phi T_R = T_P \phi$ and $\psi T_R = T_Q \psi$. In [1, p. 78] it is shown that, for almost block isomorphic T_P and T_Q, the isomorphism $\phi \cdot \psi^{-1} \colon X_P \to X_Q$ satisfies the hypothesis F.E.

3. The information cocycle

In this section we describe the framework in which all the invariants in [6] and [9] are derived from a common underlying cocycle-coboundary relation. Let T_P and T_Q be ergodic Markov shifts with a finitary isomorphism $\phi: X_P \to X_P$ satisfying the hypothesis F.E. We denote by α_P and α_Q the time-zero state partitions of X_P and X_Q, respectively, and write $\mathscr{A}_P$ and $\mathscr{A}_Q$ for the past σ-algebras generated by

$$\bigcup_{n=0}^{\infty} T_P^{-n}\alpha_P \quad \text{and} \quad \bigcup_{n=0}^{\infty} T_Q^{-n}\alpha_Q.$$

The conditional information functions $I_P = I(\mathscr{A}_P \mid T_P^{-1}\mathscr{A}_P)$, $I_Q = I(\mathscr{A}_Q \mid T_Q^{-1}\mathscr{A}_Q)$ are of the form

$$I_P(x) = \log(\bar{p}(x_1)/\bar{p}(x_0)\, P(x_0, x_1)), \qquad x \in X_P$$

and

$$I_Q(y) = \log(\bar{q}(y_1)/\bar{q}(y_0)\, Q(y_0, y_1)), \qquad y \in X_Q, \tag{3.1}$$

and thus assume only finitely many values. From [9] we know that there exists a measurable function $f: X_P \to \mathbb{R}$ satisfying

$$I_P = I_Q \circ \phi + f \circ T_P - f. \tag{3.2}$$

For reasons which will become clear in Sect. 5 we need to prove that f assumes (almost surely) only a countable number of values, or, equivalently, that f is constant on some set of positive measure. Our proof of this fact relies on a canonical extension of the information function to a large group of automorphisms of (X_P, m_P) and (X_Q, m_Q), respectively. This extension, the so-called information cocycle, was introduced by R. Butler and K. Schmidt in [2], and we recall briefly some of its relevant aspects.

Let $(X, \mathscr{S}, \mu)$ be a Lebesgue probability space and $\mathscr{A}$, $\mathscr{B} \subset \mathscr{S}$ two sub-σ-algebras. We say that $\mathscr{A} \underset{I}{\sim} \mathscr{B}$ if $I(\mathscr{A} \mid \mathscr{B}) < \infty$ and $I(\mathscr{B} \mid \mathscr{A}) < \infty$ μ-a.e. For a fixed σ-algebra $\mathscr{A} \subset \mathscr{S}$, let G be a group of measure preserving automorphisms of $(X, \mathscr{S}, \mu)$ with $\mathscr{A} \underset{I}{\sim} g^{-1}\mathscr{A}$ for every $g \in G$, and define, for every $g \in G$, $J(\mathscr{A}, g): X \to \mathbb{R}$ by

$$J(\mathscr{A}, g) = I(\mathscr{A} \mid g^{-1}\mathscr{A}) - I(g^{-1}\mathscr{A} \mid \mathscr{A}). \tag{3.3}$$

In [2] the following facts are proved:

(1) $J(\mathscr{A}, \cdot)$ is a cocycle for G, i.e.

$$J(\mathscr{A}, gh) = J(\mathscr{A}, g) \circ h + J(\mathscr{A}, h) \qquad \mu\text{-a.e.}, \tag{3.4}$$

for every $g, h \in G$;

(2) If $\mathscr{B} \subset \mathscr{S}$ is a σ-algebra with $\mathscr{A} \underset{I}{\sim} \mathscr{B}$, then $J(\mathscr{A}, \cdot)$ and $J(\mathscr{B}, \cdot)$ are cohomologous cocycles: there exists a measurable function $f: X \to \mathbb{R}$ (in fact, $f = I(\mathscr{A} \mid \mathscr{B}) - I(\mathscr{B} \mid \mathscr{A})$) such that

$$J(\mathscr{A}, g) - J(\mathscr{B}, g) = f \circ g - f \quad \mu\text{-a.e.}, \tag{3.5}$$

for every $g \in G$.

We shall apply these results to the measure space (X_P, m_P), the σ-algebras $\mathscr{A}=\mathscr{A}_P$, $\mathscr{B}=\phi^{-1}\mathscr{A}_Q$, and the group $G_P=\langle F_P^0, T_P\rangle$, where F_P^0 is the group of measure preserving elements in F_P and G_P the group generated by F_P^0 and T_P. If F_Q^0 and G_Q are the analogous groups on (X_Q, m_Q), the remarks at the end of Sect. 2 imply that $\phi F_P^0 \phi^{-1}=F_Q^0$ and $\phi G_P \phi^{-1}=G_Q$, and it is not difficult to see that $g^{-1}\mathscr{A}_P \underset{I}{\sim} \mathscr{A}_P$ and $h^{-1}\mathscr{A}_Q \underset{I}{\sim} \mathscr{A}_Q$ for all $g\in G_P$ and $h\in G_Q$. Since $\mathscr{A}_P \underset{I}{\sim} \phi^{-1}\mathscr{A}_Q$ (cf. [2, 9]), it follows that $g^{-1}\phi^{-1}\mathscr{A}_Q \underset{I}{\sim} \phi^{-1}\mathscr{A}_Q$ for every $g\in G_P$, and we can use the assertions (1) and (2) above to see that, for every $g\in G_P$,

$$J(\mathscr{A}_P, g)-J(\phi^{-1}\mathscr{A}_Q, g)=f\circ g-f, \tag{3.6}$$

where

$$f=I(\mathscr{A}_P\,|\,\phi^{-1}\mathscr{A}_Q)-I(\phi^{-1}\mathscr{A}_Q\,|\,\mathscr{A}_P) \tag{3.7}$$

is the function in which we are interested. The explicit form of $J(\mathscr{A}_P, g)$ can be computed quite easily from (3.3) and (3.4). For $g=T_P$ we have $J(\mathscr{A}_P, T_P)=I_P$ (cf. (2.1)), and a short computation (cf. [2, (4.2)]) shows that, for every $V\in F_P^0$,

$$J(\mathscr{A}_P, V)(x)=\log\left[\frac{\bar{p}((Vx)_0}{\bar{p}(x_0)}\cdot\prod_{i=0}^{\infty}\frac{P((Vx)_i, (Vx)_{i+1})}{P(x_i, x_{i+1})}\right]. \tag{3.8}$$

Note that the infinite product on the right hand side of (3.8) terminates after a finite number of terms.

4. The function f

Let $C=[i_{-M}, \ldots, i_0, \ldots, i_N]_{-M}$ be a cylinder in X_P, and define a subgroup H_C of F_P^0 as follows:

$$H_C=H_C^+\cdot H_C^-,$$

where

$$H_C^+=\{V\in F_P^0\colon (Vx)_n=x_n \text{ for a.e. } x \text{ and every } n\leqq N\}$$

and

$$H_C^-=\{V\in F_P^0\colon (Vx)_n=x_n \text{ for a.e. } x \text{ and every } n\geqq -M\}.$$

Evidently H_C leaves C invariant, and we wish to prove that H_C acts ergodically on C. To do this we need the following result due to E. Hewitt and L.J. Savage ([3]).

4.1. Proposition. *Let $(X, \mathscr{S}, \mu)=\prod_{n=-\infty}^{\infty}(X_n, \mathscr{S}_n, \mu_n)$ be an infinite cartesian product of identical probability spaces $(X_n, \mathscr{S}_n, \mu_n)=(X_0, \mathscr{S}_0, \mu_0)$, and let S^∞ denote the group of all finite permutations of $\mathbb{Z}$* (i.e. *each element of S^∞ leaves all but a finite number of integers fixed*). *Consider the action of S^∞ on $(X, \mathscr{S}, \mu)$ given by $(\pi x)_n=x_{\pi(n)}$, $x\in X$, $\pi\in S^\infty$. This action is measure preserving and ergodic.*

As an immediate consequence of Proposition 4.1 we obtain

4.2. Corollary. *Let $S' \subset S$ denote the subgroup of finite permutations of $\mathbb{Z}$ satisfying $\pi(i) \geqq 0$ for every $i \geqq 0$ and $\pi(i) < 0$ for every $i < 0$. Then the action of S' on $(X, \mathcal{S}, \mu)$ described in Proposition* 4.1 *is ergodic.*

We can now prove the ergodicity of H_C.

4.3. Proposition. *Let m_C be the probability measure on C given by $m_C(A) = m_P(A \cap C)/m_P(C)$. The action of H_C on the probability space (C, m_C) is ergodic.*

Proof. As in Sect. 3 we denote by $\alpha_P = \{A_1, \ldots, A_k\}$ the time zero partition of X_P, where $A_i = \{x \in X_P \colon x_0 = i\}$, and put $\beta = \bigvee_{i=-M}^{N} T^{-i}\alpha_P$. Clearly, $C \in \beta$, and we can partition (almost all of) C into a countable number of subsets of the form

$$C \cap T^{-1} B_1 \cap \ldots \cap T^{-(n-1)} B_{n-1} \cap T^{-n} C,$$

where $n \geqq 1$ and $B_i \in \beta$ with $B_i \neq C$ for $1 \leqq i < n$. We denote this partition of C by γ, choose an enumeration $\gamma = \{\gamma_i \colon i \in \mathbb{N}\}$ of γ, and define a probability measure ν on $\mathbb{N}$ by setting $\nu(i) = m_C(\gamma_i)$, $i \in \mathbb{N}$. It is a well known consequence of the strong Markov property that (C, m_C) can be written as an infinite cartesian product of identical copies of the probability space $(\mathbb{N}, \nu)$: $(C, m_C) = \prod_{n=-\infty}^{\infty} (\mathbb{N}, \nu)$. Using Corollary 4.2 we see that the action induced by S' on (C, m_C) is ergodic, and it is not difficult to see that every element $\pi \in \mathcal{S}'$ is, in fact, represented in H_C. □

As we have explained in Sect. 2, the functions a_ϕ^* and m_ϕ^* are finite a.e., and we can find a positive integer M and a cylinder

$$C = [i_{-M}, \ldots, i_0, \ldots, i_M]_{-M} \subset X_P$$

such that

$$D = C \cap \{x \colon a_\phi^*(x) \leqq M \text{ and } m_\phi^*(x) \leqq M\}$$

has positive measure.

4.4. Proposition. f is constant a.e. on D.

Proof. Choose $\alpha \in \mathbb{R}$ such that $A_\varepsilon = \{x \colon |f(x) - \alpha| < \varepsilon\} \cap D$ has positive measure for every $\varepsilon > 0$ and put $B_\varepsilon = D \setminus A_\varepsilon$. Assume that, for some $\varepsilon > 0$, $m_P(B_\varepsilon) > 0$. In this case we can use Proposition 4.3 to find automorphisms $V^+ \in H_C^+$, $V^- \in H_C^-$ such that $V = V^+ V^-$ satisfies $m_P(VA_\varepsilon \cap B_\varepsilon) > 0$. We shall now prove that, m_P-a.e. on $A = A_\varepsilon \cap V^{-1} B_\varepsilon$, $J(\mathscr{A}_P, V) = J(\phi^{-1}\mathscr{A}_Q, V) = 0$. Formula (3.6) then yields that $f(Vx) = f(x)$ for a.e. $x \in A$, which is absurd. This contradiction will imply the constancy of f on D.

In order to compute $J(\mathscr{A}_P, V)$ and $J(\phi^{-1}\mathscr{A}, V)$ on A we first note that $(V^- x)_i = x_i$ for every $i \geqq -M$ and a.e. $x \in D$. Since $m_\phi^* \leqq M$ on D, the definition of m^* implies that $(\phi(V^- x))_i = \phi(x)_i$ for every $i \geqq 0$ and a.e. $x \in D$. A glance at (3.8) now shows that

$$J(\mathscr{A}_P, V^-) = 0 \quad \text{and} \quad J(\phi^{-1}\mathscr{A}_Q, V^-) = J(\mathscr{A}_Q, \phi V^- \phi^{-1}) \circ \phi = 0$$

a.e. on D, and, in particular, on A.

Now consider $x' \in A' = V^- A$. Since $V^+ x' \in D$ and $a_\phi^* \leqq M$ on D, we conclude as before that $x'_i = (V^+ x')_i$ for $i \leqq M$ and $\phi(x')_i = (\phi(V^+ x'))_i$ for $i \leqq 0$ and a.e. on A'. Using the fact that V^+ preserves m_P we obtain from (3.8) that

$$J(\mathscr{A}_P, V^+) = J(\phi^{-1} \mathscr{A}_Q, V^+) = 0$$

a.e. on A'. The cocycle equation (3.4) can now be applied to give

$$J(\mathscr{A}_P, V) = J(\mathscr{A}_P, V^+) \circ V^- + J(\mathscr{A}_P, V^-) = 0$$

and

$$J(\phi^{-1} \mathscr{A}_Q, V) = J(\phi^{-1} \mathscr{A}_Q, V^+) \circ V^- + J(\phi^{-1} \mathscr{A}_Q, V^-) = 0$$

a.e. on A. This completes the proof of this proposition. □

As an immediate consequence of Proposition 4.4 we have

4.5. Theorem. *There exists a set $F \subset X_P$ of measure zero such that f assumes only a countable number of values on $X_P \setminus F$.*

Proof. Equation (3.6) shows that, for a.e. $x \in D$, and for every $n \geqq 0$,

$$\begin{aligned} f(T_P^n x) &= f(x) + J(\mathscr{A}_P, T_P^n) - J(\phi^{-1} \mathscr{A}_Q, T_P^n) \\ &= f(x) + J(\mathscr{A}_P, T_P^n) - J(\mathscr{A}_Q, T_Q^n) \circ \phi. \end{aligned}$$

The ergodicity of T_P, together with the fact that the right hand side in the above equation assumes only a countable number of values, completes the proof. □

5. The invariants Γ_P and Δ_P

Let T_P be an ergodic Markov shift. As in Sect. 2 we define Δ_P to be the multiplicative subgroup of $\mathbb{R}^+$ given by

$$\begin{aligned} \Delta_P = \Big\{ \frac{P(i_0, i_1) \dots P(i_{n-1}, i_n)}{P(j_0, j_1) \dots P(j_{n-1}, j_n)} &: n \geqq 1,\ i_0 = i_n = j_0 = j_n, \\ & i_l, j_l \in \{1, \dots, k\} \\ & \text{and } P(i_l, i_{l+1}), \\ & P(j_l, j_{l+1}) > 0 \text{ for } 0 \leqq l \leqq n-1 \Big\}. \end{aligned} \tag{5.1}$$

Let furthermore Γ_P denote the multiplicative subgroup of $\mathbb{R}^+$ generated by

$$\begin{aligned} \{ P(i_0, i_1) \dots P(i_{n-1}, i_n) &: n \geqq 1,\ i_0 = i_n,\ i_l \in \{1, \dots, k\} \\ & \text{and } P(i_l, i_{l+1}) > 0 \text{ for } 0 \leqq l \leqq n-1 \}. \end{aligned} \tag{5.2}$$

5.1. Proposition. Let $d \geqq 1$ be the period of P and let $\Gamma_P^{1/d}$ denote the multiplicative subgroup of $\mathbb{R}^+$ consisting of the positive d-th roots of the elements of Γ_P. There exist numbers $r_i > 0$, $1 \leqq i \leqq k$, and $c_P \in \Gamma_P^{1/d}$ such that

(1) $P(i, j)\, r_j / r_i \in \Gamma_P^{1/d}$ and $P(i, j)\, r_j / c_P r_i \in \Delta_P$ for every $i, j \in \{1, \dots, k\}$ with $P(i, j) > 0$, and

(2) Δ_P is the group generated by $\{ P(i, j)\, r_j / c_P r_i : 1 \leqq i, j \leqq k \text{ and } P(i, j) > 0 \}$.

Proof. Let $E_1, \ldots, E_d$, $d \geqq 1$, denote the periodic classes of states of P. For every $i = 1, \ldots, k$ we put $l(i) = j$ whenever $i \in E_j$. Without loss in generality we may assume that $i \in E_i$ (i.e. $l(i) = i$) for $1 \leqq i \leqq d$, and that $P(i, i+1) > 0$ for $1 \leqq i \leqq d$. Choose a state $1' \in E_1$ with $P(d, 1') > 0$, an integer $m \geqq 0$, and sequences $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ in $\{1, \ldots, k\}$ with

$$P(1, a_1)\, P(a_1, a_2) \ldots P(a_m, 1) > 0$$

and

$$P(1', b_1)\, P(b_1, b_2) \ldots P(b_m, 1) > 0.$$

Put

$$\gamma = \frac{P(1, a_1)\, P(a_1, a_2) \ldots P(a_m, 1)}{P(d, 1')\, P(1', b_1) \ldots P(b_m, 1)\, P(1, 2) \ldots P(d-1, d)}.$$

Clearly we have $\gamma \in \Gamma_P$, and we put $c_P = \gamma^{-1/d}$. Next we select an integer $n \geqq 0$ and sequences $s_1^{(i)}, \ldots, s_n^{(i)}$, $1 \leqq i \leqq k$, in $\{1, \ldots, k\}$ such that

$$u_i = P(i, s_1^{(i)}) \ldots P(s_n^{(i)}, l(i))\, P(l(i), l(i)+1) \ldots P(d-1, d) > 0$$

for every $i = 1, \ldots, k$. Put

$$r_i = u_i\, c_P^{l(i)}, \qquad 1 \leqq i \leqq k.$$

For every i with $l(i) < d$ and every j with $P(i, j) > 0$ we observe that $l(j) = l(i) + 1$, so that for this case

$$\frac{P(i, j)\, u_j}{u_i} = \frac{P(i, j)\, P(j, s_1^{(j)}) \ldots P(s_n^{(j)}, l(i)+1)\, P(l(i)+1, l(i)+2) \ldots P(d-1, d)}{P(i, s_1^{(i)}) \ldots P(s_n^{(i)}, l(i))\, P(l(i), l(i)+1) \ldots P(d-1, d)}$$

is an element of Δ_P and hence Γ_P. Furthermore,

$$\frac{P(i, j)\, r_j}{r_i} = \frac{P(i, j)\, u_j\, c_P}{u_i} \in \Gamma_P^{1/d},$$

and

$$\frac{P(i, j)\, r_j}{c_P\, r_i} = \frac{P(i, j)\, u_j}{u_i} \in \Delta_P.$$

Turning now to the case where $l(i) = d$ we note that every j with $P(i, j) > 0$ satisfies $l(j) = 1$, and hence

$$\frac{P(i, j)\, u_j}{u_i} = \frac{P(i, j)\, P(j, s_1^{(j)}) \ldots P(s_n^{(j)}, 1)\, P(1, 2) \ldots P(d-1, d)}{P(i, s_1^{(i)}) \ldots P(s_n^{(i)}, d)} \in \Gamma_P.$$

It follows that

$$\frac{P(i, j)\, r_j}{r_i} = \frac{P(i, j)\, u_j}{c_P^{d-1}\, u_i} \in \Gamma_P^{1/d},$$

and

$$\begin{aligned} \frac{P(i, j)\, r_j}{c_P\, r_i} &= \frac{P(i, j)\, u_j}{u_i} \cdot \gamma \\ &= \frac{P(i, j)\, P(j, s_1^{(j)}) \ldots P(s_n^{(j)}, 1) \cdot P(1, a_1) \ldots P(a_m, 1)}{P(i, s_1^{(i)}) \ldots P(s_n^{(i)}, d)\, P(d, 1')\, P(1', b_1) \ldots P(b_m, 1)} \end{aligned}$$

lies again in Δ_P. This completes the proof of (1). The second assertion is obvious. $\square$

5.2. Corollary. The quotient group Γ_P/Δ_P is cyclic with generator $c_P^d \Delta_P$.

Proof. By definition, Γ_P is generated by all non-zero products of the form $P(i_0, i_1) \dots P(i_{n-1}, i_0)$, $n \geqq 0$. We note that every such n must be a multiple of d, the period of P, and apply Proposition 5.1 to see that $P(i_0, i_1) \dots P(i_{n-1}, i_0) \in c_P^n \Delta_P$ with $d|n$. Since the highest common factor of all such integers n is equal to d we obtain that $\Gamma_P \subset \{c_P^{md} \Delta_P \colon m \in \mathbb{Z}\}$, and hence $\Gamma_P = \{c_P^{md} \Delta_P \colon m \in \mathbb{Z}\}$, as claimed. $\square$

5.3. *Remark.* In the course of the proof of Proposition 5.1 we have proved the existence of positive real numbers $u_1, \dots, u_k$ such that $P(i,j)\, u_j/u_i \in \Gamma_P$ for every i,j with $P(i,j) > 0$. If P is aperiodic we may, of course, choose $u_i = r_i$, put $P'(i,j) = P(i,j)\, u_j/u_i$, and consider the matrix P' instead of P. Evidently $T_{P'} = T_P$, $\Gamma_{P'} = \Gamma_P$, $\Delta_{P'} = \Delta_P$, and we have $P'(i,j) \in c_{P'} \Delta_{P'} \subset \Gamma_{P'}$ for every i,j with $P'(i,j) > 0$. If P has period $d > 1$ we can adjust the entries of P in two different ways: put $P'(i,j) = P(i,j)\, u_j/u_i$, $P''(i,j) = P(i,j)\, r_j/r_i$ and note once again that $T_P = T_{P'} = T_{P''}$, $\Gamma_P = \Gamma_{P'} = \Gamma_{P''}$, $\Delta_P = \Delta_{P'} = \Delta_{P''}$ and $c_P \Delta_P = c_{P'} \Delta_{P'} = c_{P''} \Delta_{P''}$. The matrix P' satisfies $P'(i,j) \in \Gamma_{P'} = \Gamma_P$ for i,j with $P'(i,j) > 0$, and P'' has the property that $P''(i,j) \in c_{P''} \Delta_{P''} \subset \Gamma_{P''}^{1/d}$ whenever $P''(i,j) > 0$.

5.4. *Examples.* 1. If P is the Bernoulli matrix $\begin{pmatrix} p_1, \dots, p_n \\ \dots \\ p_1, \dots, p_n \end{pmatrix}$ then $\Gamma_P = \langle p_1, \dots, p_n \rangle$, the multiplicative group generated by $p_1, \dots, p_n$. Moreover $\Delta_P = \{p_1^{m_1} \dots p_n^{m_n} \colon m_1, \dots, m_n \in \mathbb{Z}$ and $m_1 + \dots + m_n = 0\}$. Γ_P/Δ_P may be any cyclic group depending on the values $p_1, \dots, p_n$.

2. $\Delta_P = \{1\}$ if and only if P is of maximal type i.e. $P_{i,j} = A(i,j)\, r_j/\beta r_i$ where A is a zero-one matrix, β is the maximum eigenvalue and $Ar = \beta r$.

3. $P = \begin{pmatrix} p & q \\ p & q \end{pmatrix}$, $Q = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$ then

$$\Gamma_P = \{p^m q^n \colon m, n \in \mathbb{Z}\}, \qquad \Delta_P = \{(p/q)^n \colon n \in \mathbb{Z}\}$$

as above and

$$\Gamma_Q = \{p^m q^{2n} \colon m, n \in \mathbb{Z}\}, \qquad \Delta_Q = \{(p/q)^{2n} \colon n \in \mathbb{Z}\}.$$

Having described the groups Γ_P and Δ_P and their connection with the entries of P we can now prove their invariance properties.

5.5. Theorem. *Let T_P and T_Q be ergodic Markov shifts and assume the existence of a finitary isomorphism $\phi \colon X_P \to X_Q$ satisfying the hypothesis F.E.. Then $\Gamma_P = \Gamma_Q$, $\Delta_P = \Delta_Q$, and $c_P^d \Delta_P = c_Q^d \Delta_P$, where d is the period of P (or Q).*

Proof. As we have observed in Remark 5.3 we may assume without loss in generality that $P(i,j) \in \Gamma_P \cup \{0\}$ and $Q(i',j') \in \Gamma_Q \cup \{0\}$ for all states i,j of P and i',j' of Q. From (3.1), (3.2) and (3.6), combined with Corollary 4, 5, we deduce the existence of a countable subset $\Sigma_0 \subset \mathbb{R}^+$ and of a measurable function

$g: X_P \to \Sigma_0$ with

$$P(x_0, x_1) = Q(\phi(x)_0, \phi(x)_1)\, g(T_P x)/g(x) \quad \text{a.e..} \tag{5.3}$$

Let Σ be the multiplicative subgroup of $\mathbb{R}^+$ generated by Σ_0, Γ_P and Γ_Q, considered as a countable, discrete group, and denote by $\hat{\Sigma}$ its character group. For $\chi \in \Gamma_Q^\perp = \{\psi \in \hat{\Sigma}: \psi(\alpha) = 1 \text{ for all } \alpha \in \Gamma_Q\}$, (5.3) implies that

$$\chi(P(x_0, x_1)) = \chi(g(T_P x))/\chi(g(x)) \quad \text{a.e..}$$

We thus have a function of two coordinates $u(x_0, x_1) = \chi(P(x_0, x_1))$ which satisfies $u(x_0, x_1) = v(T_P x)/v(x)$ for some measurable function v of modulus 1. By [8] v must be a function of a single coordinate, $v(x) = v'(x_0)$ and $\chi(P(x_0, x_1)) = v'(x_1)/v'(x_0)$ everywhere. If we now take one of the non-zero products $P(i_0, i_1) \dots P(i_{n-1}, i_0)$ which generate Γ_P we find that

$$\chi(P(i_0, i_1) \dots P(i_{n-1}, i_0)) = v'(i_0)/v'(i_0) = 1.$$

Hence $\chi \in \Gamma_P^\perp$, and $\Gamma_P \subset \Gamma_Q$. A symmetrical argument yields $\Gamma_Q \subset \Gamma_P$, so that $\Gamma_P = \Gamma_Q$.

Next we turn to Δ_P and Δ_Q. Let P'', Q'' be defined as in Remark 5.3. Formula (5.3) shows the existence of a measurable function $h: X_P \to \Sigma^{1/d}$, the group of positive d-th roots of elements of Σ, such that

$$P''(x_0, x_1) = Q''(\phi(x)_0, \phi(x)_1)\, h(T_P x)/h(x) \quad \text{a.e..} \tag{5.4}$$

We divide both sides of (5.4) by c_Q, note that $Q''(\phi(x_0), \phi(x)_1)/c_Q \in \Delta_Q$ (cf. Proposition 5.1), and apply a character $\chi \in \Delta_Q^\perp \subset \widehat{(\Sigma^{1/d})}$ to this new equation to obtain

$$\chi(P''(x_0, x_1)/c_Q) = \chi(h(T_P x))/\chi(h(x)) \quad \text{a.e.}$$

Exactly as before we conclude the existence of a function v'' on $\{1, \dots, k\}$ of modulus 1 with

$$\chi(P''(x_0, x_1)/c_Q) = v''(x_1)/v''(x_0) \quad \text{everywhere.}$$

If $P''(i_0, i_1) \dots P''(i_{n-1}, i_0) = P(i_0, i_1) \dots P(i_{n-1}, i_0) > 0$, we have

$$\chi(c_Q)^{-n}\, \chi(P(i_0, i_1) \dots P(i_{n-1}, i_0)) = 1 \tag{5.5}$$

and by considering ratios of such expressions we see that $\chi \in \Delta_Q^\perp$, i.e. that $\Delta_P \subset \Delta_Q$. Symmetry yields therefore that $\Delta_P = \Delta_Q$. The integer n in (5.5) is divisible by d, and $c_Q^d \in \Gamma_Q$ by Proposition 5.1. Again by Proposition 5.1, $P''(i,j)/c_P \in \Delta_P = \Delta_Q$, and we can re-write (5.5) in the form

$$\chi(c_P/c_Q)^n\, \chi(P(i_0, i_1) \dots P(i_{n-1}, i_0)/c_P^n) = \chi(c_P/c_Q)^n = 1.$$

Since d is the highest common factor of all such integers n, we have $\chi(c_P^d/c_Q^d) = 1$ for every $\chi \in \Delta_P^\perp$, so that $c_P^d\, \Delta_P = c_Q^d\, \Delta_Q$, as claimed. The proof of the theorem is now complete. □

6. Further invariants

Throughout this section we shall assume T_P and T_Q to be ergodic Markov shifts and ϕ: $X_P \to X_Q$ a finitary isomorphism satisfying hypothesis F.E.. From Theorem 5.5 we know that $\Gamma_P = \Gamma_Q = \Gamma$, $\Delta_P = \Delta_Q = \Delta$, and Remark 5.3 allows us to assume that $P(i,j) \in \Gamma \cup \{0\}$ and $Q(i',j') \in \Gamma \cup \{0\}$ for all states i, j of P and i', j' of Q, respectively. Equation (5.3) now implies the existence of a measurable function $g: X_P \to \Gamma$ with

$$P(x_0, x_1) = Q(\phi(x)_0, \phi(x)_0, \phi(x)_1)\, g(T_P x)/g(x) \quad \text{a.e..} \tag{6.1}$$

From (6.1) one can derive several further invariants. The first of these was introduced in [8] and is related to problems discussed in [7]. Let d be the common period of P and Q and define c_P, $c_Q \in \Gamma^{1/d}$ according to Proposition 5.1. For every $\chi \in \Delta^\perp \subset \widehat{\Gamma^{1/d}}$ we have

$$\chi(c_P) = \chi(c_Q),$$

and there exist measurable functions u, v of modulus 1 on X_P with

$$\chi(P(x_0, x_1)) = \chi(c_P)\, u(T_P x)/u(x)$$

and

$$\chi(Q(\phi(x)_0, \phi(x)_1)) = \chi(c_Q)\, v(T_P x)/v(x)$$

a.e. on X_P. On the other hand, if $\chi \in \widehat{\Gamma^{1/d}}$ satisfies

$$\chi(P(x_0, x_1)) = \alpha\, u(T_P x)/u(x) \quad \text{a.e.} \tag{6.2}$$

for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and some measurable function $u: X_P \to \mathbb{C}$ of modulus 1, we can once again apply [8] to conclude that u must be a function of a single coordinate: $u(x) = u(x_0)$. Equation (6.2) thus yields

$$\chi(P(x_0, x_1)) = \alpha\, u(x_1)/u(x_0)$$

which in turn is easily seen to imply that $\chi \in \Delta^\perp$ and $\alpha = \chi(c_P)$. Since the situation is symmetrical in P and Q we obtain the following result.

6.1. Theorem. *Let T_P be an ergodic Markov shift with period $d \geqq 1$ and let $\Lambda_P = \{(\alpha, \chi)$: $\alpha \in \mathbb{C}$, $|\alpha| = 1$, $\chi \in \widehat{\Gamma^{1/d}}$, and there exists a measurable map $u: X_P \to \mathbb{C}$ with $\chi(P(x_0, x_1)) = \alpha u(T_P x)/u(x)$ a.e.$\}$. Then*

$$\Lambda_P = \{(\chi(c_P), \chi): \chi \in \Delta_P^\perp \subset \widehat{\Gamma^{1/d}}\}.$$

Furthermore, if T_Q is another Markov shift, and if there exist a finitary isomorphism ϕ: $X_P \to X_Q$ satisfying the hypothesis F.E., then $\Lambda_P = \Lambda_Q$.

The group invariant Λ_P in [8] can thus be expressed explicitly in terms of $(\Gamma_P, \Delta_P, c_P \Delta_P)$. In order to obtain further numerical invariants from (6.1) we denote by $\mathrm{Hom}(\Gamma, \mathbb{C})$ the set of all homomorphisms from Γ into the additive group of complex numbers.

6.2. Theorem. *Let T_P, T_Q be ergodic Markov shifts, and assume that there exists a finitary isomorphism ϕ: $X_P \to X_Q$ satisfying the hypothesis F.E.. Then $\Gamma_P = \Gamma_Q = \Gamma$ and, for every $\eta \in \mathrm{Hom}(\Gamma, \mathbb{C})$,*

$$h_{P,\eta} = \int \eta(P(x_0, x_1))\, dm_P(x) = \int \eta(Q(y_0, y_1))\, dm_Q(y) = h_{Q,\eta} \tag{6.3}$$

and

$$\sigma^2_{P,\eta} = \sigma^2_{Q,\eta},$$

where

$$\sigma^2_{P,\eta} = \lim_{n\to\infty} \frac{1}{n} \int |\eta(P(x_0, x_1)) + \ldots + \eta(P(x_{n-1}) x_n)) - n h_{P,\eta}|^2\, dm_P(x)$$

and where $\sigma^2_{Q,\eta}$ is analogously defined (we are assuming without loss of generality that $P(x_0, x_1) \in \Gamma$ and $Q(y_0, y_1) \in \Gamma$ - cf. Remark 5.3).

Proof. We apply η to both sides of (6.1) and obtain

$$\eta(P(x_0, x_1)) = \eta(Q(\phi(x)_0, \phi(x)_1) + \eta(g(T_P x)) - \eta(g(x)) \quad \text{a.e..}$$

Hence $\int \eta(P(x_0, x_1))\, dm_P(x) = \int \eta(Q(y_0, y_1))\, dm_Q(y)$. To prove (6.4) we recall the central limit theorem for functions of finitely many variables on a Markov shift space (cf. [4]). Since

$$\begin{aligned}&\eta(P(x_0, x_1)) + \ldots + \eta(P(x_{n-1}, x_n))\\ &\quad = \eta(Q(\phi(x)_0, \phi(x)_1)) + \ldots + \eta(Q(\phi(x)_{n-1}, \phi(x)_n)) + \eta(g(T_P^n x)) - \eta(g(x))\end{aligned}$$

and $(\eta(g(T_P^n x)) - \eta(g(x)))/\sqrt{n} \to 0$ in measure it is easy to see that $\sigma^2_{P,\eta} = \sigma^2_{Q,\eta}$. □

7. The zeta function and block maps

So far we have discussed invariants for finitary isomorphisms satisfying *F.E.* Our next object is to show how these invariants Γ_P, Δ_P, $c_P \Delta_P$, $h_{P,\eta}$ and $\sigma^2_{P,\eta}$ relate to other forms of isomorphism.

If P, Q are irreducible 0-1-matrices we say that T_P, T_Q are *block-isomorphic* if there exists a measure preserving homeomorphism ϕ such that $\phi T_P = T_Q \phi$. Necessary and sufficient conditions for block isomorphism have been given in [11]. A necessary condition is that P and Q be *shift equivalent*, i.e. that there exist matrices $U(t)$, $V(t)$ over $\mathbb{Z}^+(\exp)$ (defined in Section 1), $t \geqq 0$, and an integer $n \geqq 1$ such that

$$\begin{aligned} U(t) P^t &= Q^t U(t) \qquad & U(t) V(t) &= (Q^t)^{(n)} \\ P^t V(t) &= U(t) Q^t \qquad & V(t) U(t) &= (P^t)^{(n)} \end{aligned}$$

for every $t \geqq 0$, where $P^t(i,j)$ is $P(i,j)$ raised to the power t and $(P^t)^{(n)}$ denotes the n-th power of P^t in the usual sense. The *zeta-function* of T_P is defined by

$$\zeta_P(s,t) = \exp \sum_{n=1}^{\infty} \frac{s^n}{n} \sum_{T_P^n x = x} P^t(x_0, x_1) \ldots P^t(x_{n-1}, x_n).$$

7.1. Proposition [11]. *If T_P and T_Q are block-isomorphic then P and Q are shift equivalent. If P and Q are shift equivalent they have the same zeta function. The generators of Γ_P in (5.2) appear as coefficients in the zeta function, so that Γ_P is an invariant of the zeta function and hence of shift equivalence and, of course, of block-isomorphism.*

The following proposition shows how Γ_P and Δ_P are affected by continuous finite-to-one maps.

7.2. Proposition. *If ϕ is a finite-to-one block-homomorphism between T_P and T_Q (i.e. a continuous, finite-to-one, surjective, measure preserving map such that $\phi T_P = T_Q \phi$) then $\Gamma_P \subset \Gamma_Q$, $\Delta_P \subset \Delta_Q$ and Γ_Q/Γ_P, Δ_Q/Δ_P are finite groups.*

Proof. In view of the invariance of the groups Γ_P, Δ_P under block-isomorphism we may assume ϕ to be a one-block-map. If $P(x_0, x_1) \dots P(x_{n-1}, x_0) \neq 0$ then $x = (\dots, x_0, x_1, \dots, x_{n-1}, x_0, x_1 \dots)$ is a fixed point of T_P^n, and $\phi(x) = y = (\dots, y_0, y_1, \dots, y_{n-1}, y_0, \dots)$ is a fixed point for T_Q^n. Moreover there exist real numbers $r_j > 0$ such that $P(i,j) = Q(\phi(i), \phi(j))\, r_j/r_i$ whenever $P(i,j) \neq 0$. Hence $P(x_0, x_1) \dots P(x_{n-1}, x_0) = Q(y_0, y_1) \dots Q(y_{n-1}, y_0) \in \Gamma_Q$, i.e. $\Gamma_P \subset \Gamma_Q$. If, on the other hand, $Q(y_0, y_1) \dots Q(y_{n-1}, y_0) \in \Gamma_Q$ and if ϕ is at most l-to-1 then each $x \in X_P$ with $\phi(x) = y = (\dots, y_0, y_1, \dots, y_{n-1}, y_0, \dots)$ is T_P-periodic with a period $l'n$, $1 \leqq l' \leqq l$. Thus $P(x_0, x_1) \dots P(x_{l'n-1}, x_0) = [Q(y_0, y_1) \dots Q(y_{n-1}, y_0)]^{l'}$. Hence each generator of Γ_P has order $l' \leqq l$ modulo Γ_P, so that Γ_Q/Γ_P is finite. The assertions concerning Δ_P and Δ_Q are proved in a similar manner.

We recall the definition of almost block-isomorphism in Sect. 2. Our next result is a consequence of Theorem 5.5 and of the short discussion at the end of Sect. 2.

7.3. Proposition. *If T_P and T_Q are almost block-isomorphic then $\Gamma_P = \Gamma_Q$, $\Delta_P = \Delta_Q$, and $c_P^d \Delta_P = c_Q^d \Delta_Q$, where d denotes the period of P and Q.*

The remainder of this section is devoted to a discussion of the connection between the β-function β_P of T_P and the groups Γ_P and Δ_P. For every $t \in \mathbb{R}$, $\beta_P(t)$ is defined as the maximum eigenvalue of the matrix P^t or, equivalently, as the reciprocal of the smallest positive pole of $\zeta_P(\cdot, t)$. In [12], S. Tuncel proved that β_P is an invariant under various kinds of isomorphisms involving the topology of the Markov shift spaces. It seems tempting to conjecture that β_P is an invariant of finitary isomorphisms with hypothesis *F.E.*, but we have been unable to prove it.[1] There is, however, an intimate connection between β_P and the group Γ_P.

7.4. Theorem. *Let T_P and T_Q be ergodic Markov shifts. If $\beta_P = \beta_Q$ then $\Gamma_P \cap \Gamma_Q$ has finite index both in Γ_P and in Γ_Q, or, equivalently, both Γ_P and Γ_Q have finite index in the group $\Gamma^* = \langle \Gamma_P, \Gamma_Q \rangle$ generated by Γ_P and Γ_Q in $\mathbb{R}^+$,*

Proof. According to Remark 5.3 we may assume that $P(i,j) \in \Gamma_P \cup \{0\}$ and $Q(i',j') \in \Gamma_Q \cup \{0\}$ for every pair of states i,j of P and i',j' of Q, respectively.

Γ_P has finite index in a group G which is a direct summand of Γ^*. Let $\gamma_1, \dots, \gamma_m$ be an independent basis for G and let $\gamma_1, \dots, \gamma_m, \eta_1, \dots, \eta_n$ be an independent basis for Γ^*. In [10] it was shown that there is a matrix $U(t)$ with

[1] This problem has since been answered affirmatively by the second named author (cf. pp. 33-40 of this issue)

entries from $\mathbb{Z}^+(\Gamma^*)$, the semi-ring of non-negative integral combinations of exponentials γ^t with $\gamma \in \Gamma^*$, such that $U(0) > 0$ and

$$U(t)\, P^t = Q^t\, U(t).$$

We can therefore construct matrices $U(x, y)$, $P(x)$, $Q(x, y)$, $(x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n))$ over the ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$ such that the entries of P, Q are monomials in $x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}$ with

$$U(t) = U(x, y), \qquad P^t = P(x), \quad Q^t = Q(x, y) \tag{7.1}$$

and

$$U(x, y)\, P(x) = Q(x, y)\, U(x, y) \tag{7.2}$$

when $x_1 = \gamma_1^t, \ldots, x_m = \gamma_m^t$, $y_1 = \eta_1^t, \ldots, y_n = \eta_n^t$, $t \in \mathbb{R}$.

Since $\gamma_1, \ldots, \gamma_m$, $\eta_1, \ldots, \eta_n$ are independent it follows that (7.2) is valid whenever $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We can therefore assign the values $x_1 = 1, \ldots, x_m = 1$ to obtain the equation

$$U(1, y)\, P(1) = Q(1, y)\, U(1, y) \tag{7.3}$$

valid for $1 = (1, \ldots, 1)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

The matrix $P(1) = P^0$ is the matrix of zeros and ones with ones in the non-zero places of P and $P(1)$ has maximum eigenvalue $\beta(0)$. Thus $Q(1, y)$ has maximum eigenvalue $\beta(0)$ for all y in a neighbourhood of $y_1 = 1, \ldots, y_n = 1$. For each such y the matrix $Q(1, y)$ with entries $Q_{i,j}(1, y)$ can be stochasticized to obtain $\{Q_{i,j}(1, y)\, r_j / \beta(0)\, r_i\} = Q_y$ where

$$\sum_j Q_{i,j}(1, y)\, r_j = \beta(0)\, r_i.$$

Moreover the maximum eigenvalue of Q_y^t is $\beta(0)^{1-t}$, the negative of whose derivative at $t = 1$ is $\log \beta(0)$, which, by [11], is the entropy of the Markov measure defined by Q_y. Hence this Markov measure is of maximal type and we conclude that

$$Q_{i,j}(1, y)\, q_j(y) / q_i(y) = Q_{i,j}(1, 1)$$

for a suitable positive vector $q(y)$.

Since the entries of $Q(x, y)$ are monomials we deduce that

$$Q_{i,j}(x, y)\, q_j(y) / q_i(y) = Q_{i,j}(x, 1)$$

and hence

$$Q_{i_0, i_1} \cdots Q_{i_{n-1}, i_0} \in G$$

when this product is positive. Therefore $\Gamma_Q \subset G$ i.e. $\Gamma^* = \langle \Gamma_P, \Gamma_Q \rangle = G$. However, Γ_P has finite index in G. In a similar way one proves that Γ_Q also has finite index in $\langle \Gamma_P, \Gamma_Q \rangle$. $\square$

8. The natural coefficient ring $\mathbb{Z}(\Gamma_P)$

Let P and Q be non-negative, irreducible matrices which are shift equivalent. If one tries to show that the corresponding Markov shifts are block-isomorphic one has to solve the matrix equations

$$\begin{array}{ll} U(t)\,P^t = Q^t\,U(t) & U(t)\,V(t) = (Q^t)^{(n)} \\ P^t\,V(t) = V(t)\,Q^t & V(t)\,U(t) = (P^t)^{(n)} \end{array} \tag{8.1}$$

which have already been described in Sect. 7. The entries of $U(\cdot)$, $V(\cdot)$ must lie in the semi-ring $\mathbb{Z}^+(\exp)$ of finite non-negative integral combinations of exponential functions c^t, $c>0$. If the matrices $U(\cdot)$, $V(\cdot)$ exist they may, in fact, be assumed to have entries in $\mathbb{Z}^+(G)$, the semi-ring of finite non-negative integral combinations of functions c^t, $c\in G$, where G is the multiplicative group generated by the non-zero entries of P and Q (cf. [13]). One serious shortcoming of this procedure is the fact that G, and hence the coefficient ring $\mathbb{Z}^+(G)$, depends on *both* P and Q. We illustrate this with an example from Tuncel [13].

$$Q^t = \begin{pmatrix} 0 & p^{2t} & q^t(1+p)^t \\ 1 & 0 & 0 \\ \dfrac{p^t}{(1+p)^t} & \dfrac{p^{2t}}{(1+p)^t} & q^t \end{pmatrix} = U(t)\,V(t)$$

$$P^t = \begin{pmatrix} 0 & p^{2t} & p^t q^t & q^t \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & p^{2t} & p^t q^t & q^t \end{pmatrix} = V(t)\,U(t)$$

where $0<p,\ q<1,\ p\neq q$,

$$U(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{p^t}{(1+p)^t} & \dfrac{1}{(1+p)^t} \end{pmatrix},$$

and

$$V(t) = \begin{pmatrix} 0 & p^{2t} & q^t(1+p)^t \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & p^{2t} & q^t(1+p)^t \end{pmatrix}.$$

In this example, P and Q are strongly shift-equivalent with "lag" 1. The entries of both $U(\cdot)$ and $V(\cdot)$ lie in $\mathbb{Z}^+(G_Q)$, where G_Q is the group generated by the non-zero entries of Q, but not in $\mathbb{Z}^+(G_P)$.

We now recall that the shift equivalence of P and Q implies that $\Gamma_P = \Gamma_Q$ (cf. the brief discussion following Proposition 7.1). Using Remark 5.3 we may modify the entries of P and Q to satisfy $P(i,j) \in \Gamma_P \cup \{0\}$, $Q(i',j') \in \Gamma_P \cup \{0\}$, so that, after this modification, $G = \Gamma_P = \Gamma_Q$, and the coefficient ring $\mathbb{Z}(\Gamma_P) = \mathbb{Z}(\Gamma_Q)$ is determined by P and Q separately. This discussion shows that, if the required matrices $U(\cdot)$ and $V(\cdot)$ exist, their entries may be assumed to lie in the 'natural coefficient ring' $\mathbb{Z}(\Gamma_P)$. In the terminology used for the proof of Theorem 7.4 we may rephrase this last statement as follows: the equations (8.1) have solutions $U(\cdot)$ and $V(\cdot)$ with entries in $\mathbb{Z}^+(\exp)$ if and only if there exists matrices $\tilde{U}$ and $\tilde{V}$ with entries in $\mathbb{Z}^+(\Gamma_P)$, such that

$$\begin{aligned} \tilde{U} \cdot \tilde{P} &= \tilde{Q} \cdot \tilde{U} \qquad & \tilde{U} \cdot \tilde{V} &= \tilde{Q}^{(n)} \\ \tilde{P} \cdot \tilde{V} &= \tilde{V} \cdot \tilde{Q} \qquad & \tilde{V} \cdot \tilde{U} &= \tilde{P}^{(n)} \end{aligned} \tag{8.2}$$

This restriction on the coefficients of $\tilde{U}$ and $\tilde{V}$ offers considerable conceptual and computational advantages, but the problem of the existence of $\tilde{U}$ and $\tilde{V}$ still appears to be difficult.

Finally we remark that, under the assumption of *strong shift equivalence*, the required matrices U and V do exist and can therefore be chosen to satisfy (8.2) above (for the definition of strong shift equivalence and its properties we refer to [11]).

We conclude this paper with some open problems. Krieger's proof that Δ is invariant under the hypothesis *F.E.* actually gives more than what we have emphasized, namely, that if ϕ is finitary and if a_ϕ, m_ϕ have finite expectation (with no assumption on $a_{\phi^{-1}}, m_{\phi^{-1}}$) then $\Delta_P \subset \Delta_Q$. Under the same conditions is it true that $\Gamma_P \subset \Gamma_Q$? We seem to have made essential use of the restricted cocycle-coboundary equation, which is valid under the hypothesis F.E. Can this hypothesis be weakened so that the restricted cocycle-coboundary equation is still valid?

When $\beta_P = \beta_Q$ it was conjectured in [10] that T_P, T_Q are finitely equivalent and the first steps towards a proof were provided. It was also shown in [10] that $\beta_P = \beta_Q$ does not imply that T_P, T_Q are almost block isomorphic, the most obvious obstruction to such a conjecture being $\Delta_P \neq \Delta_Q$. A possible conjecture is that the invariants β_P, Γ_P, Δ_P, $c_P^d \Delta_P$ are complete with respect to almost block isomorphism.

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Oblatum 20-VII-1983