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On the global attraction for the generalized Liénard equation ☆

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Abstract

This paper deals with the global attraction and global weak attraction of the origin for the generalized Liénard equation. The problem proposed by Jiang is solved by a negative answer and some new criteria for the globally asymptotically stability are obtained by the ways different from the usual Filippov's conditions.
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1. Introduction

This paper is devoted to the conditions of the global attraction and global weak attraction for the system

$$\begin{cases} \dot{x} = \dfrac{1}{a(x)}\big(\varphi(y) - F(x)\big), \\ \dot{y} = -a(x)g(x), \end{cases} \tag{E}$$

which is the equivalent system of the following generalized Liénard equation:

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0. \tag{1.1}$$

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In fact, using the transformation $y = a(x)\dot{x} + F(x)$, one can change (1.1) into (E) with $\varphi(y) \equiv y$, $a(x) = \exp(\int_0^x f_2(s)\,ds)$ and $F(x) = \int_0^x a(s) f_1(s)\,ds$. The single singular point $(0,0)$ is said to be a global weak attractor if

$$(0,0) \in \omega(P) \quad \forall P \in R^2, \quad \text{and} \quad \exists P_0 \quad \text{such that } \omega(P_0) \neq \{(0,0)\}.$$

The single singular point is said to be a global attractor if

$$\omega(P) = \{(0,0)\} \quad \text{for any } P \in R^2,$$

where $\omega(P)$ denotes the positive limit set of the orbit $\mathcal{O}(P)$. This topic is motivated by the problem proposed in [1] by Jiang. Let us state the problem using the same notation as in [1].

Jiang's problem. [1] Suppose that (E) satisfies the assumptions (A_0)–(A_2), $(A_4^-)_1$ or $(A_4^-)_2$, $(A_4^+)_1$ or $(A_4^+)_2$, (1.2) and (1.3) defined as follows, whether the origin is globally attractive or not?

- (A_0) The functions $F(x), g(x)$ and $a(x)$ are continuous on R with $F(0) = 0$, $xg(x) > 0$ $(x \neq 0)$ and $a(x) > 0$ $(x \in R)$.
- (A_1) $\varphi(y)$ is locally Lipschitz continuous and strictly increasing on R with $\varphi(0) = 0$ and $\varphi(\pm\infty) = \pm\infty$.
- (A_2) $F(G_0^{-1}(-z)) \leqslant F(G_0^{-1}(z))$ for $z \in (0, \min\{G_0(-\infty), G_0(+\infty)\})$ and $F(G_0^{-1}(-z)) \not\equiv F(G_0^{-1}(z))$ for $0 < z \ll 1$ where $G_0(x) = \int_0^x a^2(s)|g(s)|\,ds$.
- $(A_4^+)_1$ $\limsup_{x\to+\infty} F(x) > -\infty$.
- $(A_4^+)_2$ There exist constants $y_0 < 0$, μ, N and $\beta > 0$ with $\alpha = \mu\beta > 1/4$ such that $\varphi(y)$ is differentiable on $(-\infty, y_0]$, $\varphi'(y) \geqslant \mu$ for $y \leqslant y_0$, $F(x) < 0$ for all $x \geqslant N$ and for any $b > N$, there exists $\tilde{b} > b$ satisfying

$$\int_b^x \frac{g(s)a^2(s)}{F(s)}\,ds \leqslant \beta F(x) \quad \text{for all } x \geqslant \tilde{b}.$$

- $(A_4^-)_1$ $\liminf_{x\to-\infty} F(x) < +\infty$.
- $(A_4^-)_2$ There exist constants y_0, μ, N and $\beta > 0$ with $\alpha = \mu\beta > 1/4$ such that $\varphi'(y) \geqslant \mu$ for $y \geqslant y_0$, $F(x) > 0$ for all $x \leqslant -N$ and for any $b > N$, there exists $\tilde{b} > b$ satisfying

$$\int_{-b}^x \frac{g(s)a^2(s)}{F(s)}\,ds \geqslant \beta F(x) \quad \text{for all } x \leqslant -\tilde{b},$$

$$\int_0^{+\infty} \frac{g(s)a^2(s)}{1 + F_-(s)}\,ds = +\infty \quad \text{or} \quad \limsup_{x\to+\infty} F(x) = +\infty, \tag{1.2}$$

$$\int_0^{-\infty} \frac{g(s)a^2(s)}{1 + F_+(s)}\,ds = +\infty \quad \text{or} \quad \liminf_{x\to-\infty} F(x) = -\infty, \tag{1.3}$$

where $F_-(x) = \max\{0, -F(x)\}$, $F_+(x) = \max\{0, F(x)\}$.

The paper is organized as follows. In Section 2, we first obtain some necessary and sufficient conditions for the origin to be a global attractor and a global weak attractor and then give a negative answer to Jiang's problem. In Section 3 we discuss the conditions for the origin to be a global attractor.

In the sequel, as usual, we will assume that system (E) satisfies (A_0) and (A_1) so that for any initial point $P(x_0, y_0)$, (E) has a unique orbit through P [1].

2. A criterion and the answer to Jiang's problem

First we introduce some notation and definitions.

Let $\mathcal{O}^+(P)$ ($\mathcal{O}^-(P)$) be the positive (negative) semi-orbit passing through the point P. We call the curve $\varphi(y) = F(x)$ the characteristic curve of (E) and define the regions and curves as follows:

$$D_1 = \{(x, y)\colon \varphi(y) > F(x)\ (x \geqslant 0)\}, \qquad D_2 = \{(x, y)\colon \varphi(y) < F(x)\ (x \geqslant 0)\},$$
$$D_3 = \{(x, y)\colon \varphi(y) < F(x)\ (x \leqslant 0)\}, \qquad D_4 = \{(x, y)\colon \varphi(y) > F(x)\ (x \leqslant 0)\},$$
$$\Gamma^+ = \{(x, y)\colon \varphi(y) = F(x)\ (x > 0)\}, \qquad \Gamma^- = \{(x, y)\colon \varphi(y) = F(x)\ (x < 0)\}.$$

System (E) is said to have the property $(\mathcal{H}^+)$ if for any $P \in D_1$, $\mathcal{O}^+(P)$ intersects the characteristic curve Γ^+. System (E) is said to have the property $(\mathcal{H}^-)$ if for any $P \in D_3$, $\mathcal{O}^+(P)$ intersects Γ^-. A homoclinic orbit $\mathcal{O}(P)$ passing through $P \neq (0, 0)$ is an orbit satisfying $\omega(P) = \alpha(P) = (0, 0)$. Noting the uniqueness of the origin and Poincaré–Bendixson theorem, we can conclude

Proposition. [2] *If $\mathcal{O}(P)$ is a homoclinic orbit of (E), then every orbit passing through a point in the region D, which is surrounded by $\mathcal{O}(P) \cup \{(0, 0)\}$, is a homoclinic orbit.*

For homoclinic orbits $\mathcal{O}(P_1)$ and $\mathcal{O}(P_2)$, if $\mathcal{O}(P_1)$ is contained in the region surrounded by $\mathcal{O}(P_2) \cup \{(0, 0)\}$, we call $\mathcal{O}(P_1)$ and $\mathcal{O}(P_2)$ in the same class. By a maximal elliptic sector we mean the closure of the region consisting of all the homoclinic orbits in the same class. It is proved in [2] that (E) has at most one maximal elliptic sector for $\varphi(y) \equiv y$.

Lemma 1. *Suppose that (A_2) hold. Then all the solutions of (E) are positively bounded if and only if $(\mathcal{H}^+)$ and $(\mathcal{H}^-)$ hold.*

Proof. *Sufficiency*. By $(\mathcal{H}^+)$, for any point $P \in \{D_1 \cup \Gamma^+ \cup D_2\}$, $\mathcal{O}^+(P)$ either tends to the origin or intersects with the negative y-axis. By $(\mathcal{H}^-)$, for any point $P \in \{D_3 \cup \Gamma^- \cup D_4\}$, $\mathcal{O}^+(P)$ either tends to the origin or intersects with the positive y-axis. Hence we just prove that for any point $P(0, y_0)$, $\mathcal{O}^+(P)$ is bounded. For $y_0 > 0$, $\mathcal{O}^+(P)$ either tends to the origin or intersects with the positive y-axis at the point $P'(0, y')$ at the second time. It follows from (A_2) that $y' < y_0$. So $\mathcal{O}^+(P)$ is bounded by the uniqueness of the orbit. A similar argument yields the conclusion for the case of $y_0 < 0$.

Necessity. Suppose that the statement is false. For example, $(\mathcal{H}^+)$ does not hold. Then there exists a point $P_0(x_0, y_0) \in D_1$ such that $\mathcal{O}^+(P_0)$ lies above Γ^+ all the times. So $x(t) > x_0 \geqslant 0$ for all $t > 0$ and $\lim_{t\to+\infty} x(t) = +\infty$, contradicting the assumption that $(0, 0) \in \omega(P_0)$. This completes the proof. $\square$

Theorem 1. *For system* (E)*, the following results hold*:

(i) *The necessary condition for the origin to be a global attractor* (*or a global weak attractor*) *is that all the solutions of* (E) *are positively bounded and there does not exist non-trivial closed orbit.*

(ii) *Suppose that all the solutions are positively bounded and there does not exist non-trivial closed orbit. Then the origin is a global weak attractor if and only if both* (a) *and* (b) *hold*:
   (a) *there exists a bounded maximal elliptic sector* S^*;
   (b) *there exists an outer neighborhood* U *of* S^* *such that for any point* $P \in U$, $\mathcal{O}^+(P)$ *spirals around* S^* *infinitely.*

   The origin is a global attractor if and only if (c) *or* (c′) *holds*:
   (c) *there does not exist any maximal elliptic sector*;
   (c′) *there exists a maximal elliptic sector* S^**, but* $\mathcal{O}^+(P)$ *does not spiral around* S^* *infinitely for the point* P *in any outer neighborhood of* S^*.

Proof. (i) Suppose that the result is true, but there is a point $P \in R^2$ such that $\mathcal{O}^+(P)$ is unbounded. Then $\mathcal{O}^+(P)$ either does not intersect with the characteristic curve or tends to infinity spirally. In both cases there is a neighborhood U of $(0,0)$ such that $\mathcal{O}^+(P)$ cannot locate in U infinitely, which implies that $\{(0,0)\} \notin \omega(P)$. This contradicts with the assumptions. If all the solutions are positively bounded, then by Poincaré–Bendixson theorem, for any point P, $\omega(P)$ is non-empty and either $(0,0) \in \omega(P)$ or $(0,0) \notin \omega(P)$. For the latter case $\omega(P)$ is a closed orbit. Therefore, there does not exist any closed orbit if the origin is a global attractor (or a global weak attractor).

(ii) The second assertion can be obtained by the definitions of the global attractor (or the global weak attractor) and the properties of system (E). This completes the proof. □

Theorem 2. *Suppose that* (E) *satisfies the assumptions in Jiang's problem. Then the following results hold*:

(i) *All the solutions are positively bounded.*

(ii) *There exists the system satisfying the assumptions in Jiang's problem and having both the properties* (a) *and* (b)*. So the origin is a global weak attractor.*

Proof. (i) It follows from [1] that (E) has the properties $(\mathcal{H}^+)$ and $(\mathcal{H}^-)$ under the assumptions $(A_4^+)_1$ or $(A_4^+)_2$, $(A_4^-)_1$ or $(A_4^-)_2$, (1.2) and (1.3). It follows from Lemma 1 that all the solutions are positively bounded.

(ii) It follows from (i) that for any point $P \in R^2$, the positive semi-orbit $\mathcal{O}^+(P) \subset B(r)$ for some $r > 0$ and $\omega(P)$ is non-empty closed set in $B(r)$. By [3, Theorem 3.1], there are no non-trivial closed orbits of (E). So the Poincaré–Bendixson theorem shows that $(0,0) \in \omega(P)$. In order to show that $\omega(P) \neq \{(0,0)\}$ for some $P \in R^2$, let us consider the following system:

$$\begin{cases} \dot{x} = y - F_\delta(x), \\ \dot{y} = -x^3, \end{cases} \tag{2.1$_\delta$}$$

where $F_\delta(x) = (1-\delta)F_0(x) + \delta F(x)$, $F(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2$, $\delta \in (0,1)$ and

$$F_0(x) = \begin{cases} \frac{1}{3}x^3 - \frac{3}{2}x^2, & x \geqslant 0, \\ -\frac{1}{3}x^3 - \frac{3}{2}x^2, & x < 0. \end{cases}$$

It follows from [2] that there exists $\delta_0 \in (0,1)$ such that system $(2.1)_\delta$ has the maximal elliptic sector S^* and every orbit outside of S^* surrounds S^* and tends to ∂S^*. So there exists $P_0 \in R^2$ such that $\omega(P_0) \neq \{(0,0)\}$. Hence the origin is a global weak attractor. Furthermore, it is easy to check that system $(2.1)_\delta$ satisfies all the assumptions. Here we just check that (A_2) holds. Indeed,

$$\begin{aligned}
&z = G_0(x) = \int_0^x s^3\,ds = \frac{1}{4}x^4,\\
&F\big(G_0^{-1}(z)\big) = \frac{1}{3}(4z)^{\frac{3}{4}} - \frac{3}{2}(4z)^{\frac{1}{2}},\\
&F\big(G_0^{-1}(-z)\big) = -\frac{2}{3}\delta(4z)^{\frac{3}{4}} + F\big(G_0^{-1}(z)\big),\\
&F\big(G_0^{-1}(-z)\big) - F\big(G_0^{-1}(z)\big) = -\frac{2}{3}\delta(4z)^{\frac{3}{4}} < 0 \quad \text{for } \delta \in (0,1).
\end{aligned}$$

The proof is over. □

Corollary 1. *Suppose that system (E) satisfies (A_i), $i = 0, 1, 2$, and all the solutions are positively bounded. Then the origin is a global weak attractor.*

Remark 1. Theorem 2 gives a negative answer to Jiang's problem. This indicates that the Filippov's condition (A_2) cannot exclude the existence of the maximal elliptic sector S^* and the point P with $\mathcal{O}^+(P)$ spiralling around S^* infinitely. However, Yang and Zhu [4] give an affirmative answer to Jiang's problem due to the negligence of the case (i) in Theorem 1.

3. The global attractor

In this section we discuss the conditions for the origin to be a global attractor.

Theorem 3. *Suppose that (E) satisfies the following conditions:*

(i) *$\exists a \geqslant 0$, $b \in R$ and $r(x) \in C((-\infty, -a])$ such that for $x \leqslant -a$,*

$$r(x) > F(x), \qquad \varphi^{-1}\big(r(x)\big) + \int_x^{-a} \frac{a^2(x)g(x)}{r(x) - F(x)}\,dx \leqslant b. \tag{3.1}$$

(ii) *$\exists a_- < 0 < a_+$ such that $G(a_\pm) > G(-a) + \Phi(b)$ and $xF(x) > 0$ for $x \in (a_-, 0) \cup (0, a_+)$, where $\Phi(y) = \int_0^y \varphi(s)\,ds$, $G(x) = \int_0^x a^2(s)g(s)\,ds$.*

Then the origin is a global attractor if and only if system (E) has the properties $(\mathcal{H}^+)$ and $(\mathcal{H}^-)$. Particularly, the zero solution is globally asymptotically stable.

Proof. Lemma 1 yields the necessity. We accomplish the proof of the sufficiency by two steps.

(I) We first prove that there exists a point $P_0 \in D_4$ such that $\mathcal{O}^-(P_0)$ tends to infinity and $\mathcal{O}^+(P_0)$ tends to the origin. Therefore, there are no closed orbits for system (E) in view of the uniqueness of the orbit.

In fact, since

$$\lim_{y\to 0+}\big(G(a_+)-G(-a)-\Phi(b+y)\big)=G(a_+)-G(-a)-\Phi(b)>0,$$
$$\lim_{y\to 0+}\big(G(a_-)-G(-a)-\Phi(b+y)\big)=G(a_-)-G(-a)-\Phi(b)>0,$$

there exists $0<\epsilon_0\ll 1$ such that $G(a_\pm)>G(-a)+\Phi(b+\epsilon_0)$. Let $y_0=b+\epsilon_0$. Then the point $P_0(-a,y_0)$ lies above the point $(-a,\varphi^{-1}(r(-a)))$. We assert that $\mathcal{O}^-(P_0)$ lies above the curve $y=\varphi^{-1}(F(x))$ at all times. Otherwise, $\mathcal{O}^-(P_0)$ must intersect with the curve $y=\varphi^{-1}(r(x))$ at the point $(x_1,\varphi^{-1}(r(x_1)))$ with $x_1<-a$ and $y(x)>\varphi^{-1}(r(x))$ for $x\in(x_1,-a)$, where $y(x)$ expresses the trajectory departing from the point P_0. Noting that

$$\frac{dy}{dx}=\frac{-a^2(x)g(x)}{\varphi(y)-F(x)},\tag{3.2}$$

we integrate (3.2) from x_1 to $-a$. Thus

$$y(-a)-y(x_1)=\int_{-a}^{x_1}\frac{-a^2(x)g(x)}{\varphi(y(x))-F(x)}\,dx\leqslant\int_{x_1}^{-a}\frac{a^2(x)g(x)}{r(x)-F(x)}\,dx.$$

By (3.1), we have

$$y(-a)=b+\epsilon_0\leqslant\int_{x_1}^{-a}\frac{a^2(x)g(x)}{r(x)-F(x)}\,dx+y(x_1)\leqslant b,$$

which implies that $\epsilon_0\leqslant 0$. This is a contradiction. Taking account of (E) has the unique finite singular point $(0,0)$, we conclude that $\mathcal{O}^-(P_0)$ tends to infinity as $t\to-\infty$.

Let us deal with $\mathcal{O}^+(P_0)$. Define a family of closed curves as follows:

$$V(x,y)=G(x)+\Phi(y)=v,\quad 0<v\leqslant G(-a)+\Phi(b+\epsilon_0).$$

By (ii), for every point (x,y) on these closed curves, it satisfies $x\in(a_-,a_+)$. The conclusion derived from the following calculations:

$$\left.\frac{dV}{dt}\right|_{(E_1)}=-a^2(x)g(x)F(x)<0,\quad x\in(a_-,a_+)/\big\{(0,0)\big\}.$$

(II) Next, we prove that $\mathcal{O}^+(P)$ is bounded for any P. Indeed, for any $P\in R^2$, without loss of generality, assume that $P\in D_4$ and P is above the negative semi-orbit of $\mathcal{O}^-(P_0)$. Since system (E) has the property $(\mathcal{H}^+)$, $\mathcal{O}^+(P)$ must intersect with the positive y-axis and then intersect with the characteristic curve Γ^+. Therefore it either tends to the origin or transverses the negative y-axis and intersects with Γ^- at the point P_- because the system has the property $(\mathcal{H}^-)$. It follows from the uniqueness of the orbit that $\mathcal{O}^+(P_-)$ cannot intersect with the negative semi-orbit departing from the P_0. Thus $\mathcal{O}^+(P_-)$ either tends to $(0,0)$ or intersects with the line $x=-a$ at the point P_-^- which is under the point P_0. This indicates that P_-^- locates in the attracting region of the origin and hence $\mathcal{O}^+(P)$ is bounded. Noting that there is no homoclinic orbits by (ii), the results follows from the case of (ii)(c) in Theorem 1. Furthermore, it is clear from the above process that for the autonomous system (E), every positive semi-orbit tends to the origin as $t\to+\infty$ and there are no singular closed orbits. Hence the zero solution of system (E) is globally asymptotically stable. This completes the proof. $\square$

Remark 2. Theorem 3 establishes a new criterion for the globally asymptotically stability of system (E) by the ways different from the classical Filippov's conditions [1,5,6].

It is known that for any $P \in D_4$, $\mathcal{O}^-(P)$ must intersect the curve Γ^- if $\limsup_{x\to-\infty} F(x) = +\infty$ or $G(-\infty) = +\infty$. Theorem 4 gives a criterion for the origin to be a global attractor under the assumptions that both $\limsup_{x\to-\infty} F(x)$ and $G(-\infty)$ are bounded. It can be applied to the case that system (E) has the homoclinic orbits.

Lemma 2. *Suppose that* (E) *satisfies the following conditions*:

(i) $xg(x) > 0$ $(x \neq 0)$, $y\varphi(y) > 0$ $(y \neq 0)$.
(ii) $\sup_{x\leqslant 0} F(x) = M$ $(0 \leqslant M < +\infty)$, $G(-\infty) < +\infty$.
(iii) $\exists \eta_0 > 0$ *and* $y_0 > 0$ *such that* $\varphi(y_0) = M + \eta_0$ *and* $\varphi(y) \geqslant M + \eta_0$ *for* $y > y_0$.

Then $\forall \epsilon > 0$, $\mathcal{O}^-((0, y_0 + \epsilon + \frac{1}{\eta_0} G(-\infty)))$ *tends to infinity as* $t \to -\infty$.

Proof. By (ii), $\forall \epsilon > 0$, $\exists -N < 0$ such that

$$\int_{-N}^{-\infty} a^2(x)g(x)\,dx < \frac{1}{2}\epsilon\eta_0.$$

Let us investigate the negative orbit departing from the point $(-N, y_1)$ with $y_1 = y_0 + \epsilon$. Since

$$\left.\frac{dy}{dt}\right|_{t=0} = -a^2(-N)g(-N) > 0,$$

there exists $\delta > 0$ such that $\frac{dy}{dt} > 0$ and $y(t) \geqslant y_0 + \frac{\epsilon}{2}$ for $t \in (-\delta, +\delta)$. We assert that $y(t) > y_0$ for all $t \leqslant 0$. Otherwise, there exists $T < 0$ such that $y(T) = y_0$ and $y(t) > y_0$ for $t \in (T, 0)$. Noting $\varphi(y(t)) \geqslant M + \eta_0$ for $t \in (T, 0)$, we get

$$\begin{aligned} y(-N) &= y_0 - \int_{x(T)}^{-N} \frac{a^2(x)g(x)}{\varphi(y(x)) - F(x)}\,dx \\ &\leqslant y_0 + \frac{1}{\eta_0}\int_{-N}^{-\infty} a^2(x)g(x)\,dx < y_0 + \frac{1}{2}\epsilon. \end{aligned}$$

This contradicts $y(-N) = y_0 + \epsilon$. Thus for all $t \leqslant 0$,

$$\frac{dx}{dt} = \varphi(y) - F(x) \geqslant M + \eta_0 - M = \eta_0 \quad \text{and} \quad \lim_{t\to-\infty} x(t) = -\infty. \tag{3.3}$$

Now we study the positive semi-orbit departing from $(-N, y_1)$. Denote $t^* = \max\{t \geqslant 0, g(x(t)) = 0\}$. Then for $t \in (0, t^*)$ we have $\frac{dy}{dt} > 0$, $y(t) > y_1 > y_0$ and $\frac{dx}{dt} > \eta_0 > 0$. So the orbit $y(x)$ is strictly increasing for $x \in (-N, 0)$. Since

$$\left|\frac{-a^2(x)g(x)}{\varphi(y(x)) - F(x)}\right| \leqslant \frac{G^*}{\eta_0}, \quad G^* = \max_{x\in[-N,0]}\left|-a^2(x)g(x)\right|,$$

for $x \in [-N, 0]$, $\mathcal{O}^+(-N, y_1)$ must intersect with the positive y-axis at the point $B(0, y_B)$ with $y_B < y_0 + \epsilon + \frac{1}{\eta_0} G(-\infty)$. The results follows from the uniqueness of the orbit. The proof is over. □

Theorem 4. *Suppose that system* (E) *satisfies the following conditions*:

(i) $\sup_{x \leqslant 0} F(x) = M$ $(0 \leqslant M < +\infty)$, $G(-\infty) < +\infty$.
(ii) $\varphi(y)$ *is concave for* $y > 0$. *Moreover,* $\exists a > 0$ *and* $r(x) \in C$ *such that for* $x \in (0, a]$,

$$F(x) \geqslant r(x) > 0 \quad \textit{and} \quad \frac{1}{\varphi^{-1}(r(x))} \int_{0^+}^{x} \frac{a^2(u)g(u)}{r(u)}\, du \leqslant \frac{1}{4}. \tag{3.4}$$

(iii) $\beta > \varphi^{-1}(M+1) + G(-\infty)$, *where* $(0, \beta)$ *is the intersection point of* $\mathcal{O}^-(a, \varphi^{-1}(F(a)))$ *with the positive* y*-axis.*

Then the origin is a global attractor if and only if system (E) *has the properties* $(\mathcal{H}^+)$ *and* $(\mathcal{H}^-)$.

Proof. The proof is divided into two parts.

(I) We first prove that $\mathcal{O}^+(a, \varphi^{-1}(F(a)))$ tends to the origin as $t \to +\infty$ and $\mathcal{O}^-(a, \varphi^{-1}(F(a)))$ tends to infinity as $t \to -\infty$.

If the former statement is false, then $\mathcal{O}^+(a, \varphi^{-1}(F(a)))$ must intersect the negative y-axis at the point $B(0, y_B)$ with $y_B < 0$. Assume that it intersects with the x-axis at the point $(0, x_1)$ with $0 < x_1 < a$. Consider the following continuous function:

$$W(x) = \frac{\varphi^{-1}(F(x)) - y(x)}{\varphi^{-1}(F(x))}, \quad x \in (0, a], \tag{3.5}$$

where $y(x)$ expresses the orbit of (E). Since $W(a) = 0$, $W(x_1) = 1$ and $W(x) > 1$ for $x \in (0, x_1)$, there exists $x_2 \in (x_1, a)$ such that $W(x_2) = \frac{1}{2}$ and $W(x) > \frac{1}{2}$ for $x \in (0, x_2)$. This means that for $x \in (0, x_2)$, we have

$$y(x_2) = \frac{1}{2}\varphi^{-1}\big(F(x_2)\big), \qquad y(x) < \frac{1}{2}\varphi^{-1}\big(F(x)\big). \tag{3.6}$$

By (ii), $\varphi^{-1}(y)$ is convex for $y > 0$. Hence, for $x \in (0, x_2)$,

$$y(x) < \frac{1}{2}\varphi^{-1}\big(F(x)\big) = \frac{\varphi^{-1}(F(x)) + \varphi^{-1}(F(0))}{2} \leqslant \varphi^{-1}\left(\frac{1}{2}F(x)\right). \tag{3.7}$$

So, for $0 < \epsilon \ll 1$, we get

$$y(\epsilon) - y(x_2) = \int_{x_2}^{\epsilon} \frac{-a^2(x)g(x)}{\varphi(y(x)) - F(x)}\, dx \geqslant \int_{x_2}^{\epsilon} \frac{2a^2(x)g(x)}{r(x)}\, dx$$

and

$$\lim_{\epsilon \to 0+} y(\epsilon) - y(x_2) \geqslant \int_{x_2}^{0+} \frac{2a^2(x)g(x)}{r(x)}\, dx.$$

That is

$$y_B - y(x_2) \geqslant \int_{x_2}^{0+} \frac{2a^2(x)g(x)}{r(x)}\,dx \geqslant -\frac{1}{2}\varphi^{-1}\bigl(r(x_2)\bigr),$$

which implies that $y_B \geqslant 0$. This is a contradiction.

Now we consider $\mathcal{O}^-(a, \varphi^{-1}(F(a)))$. By (iii), there exists $\epsilon_0 > 0$ such that $\beta > \varphi^{-1}(M+1) + \epsilon_0 + G(-\infty)$. It follows from Lemma 2 that the negative semi-orbit departing from the point $(0, \varphi^{-1}(M+1) + \epsilon_0 + G(-\infty))$ tends to infinity as $t \to -\infty$. We assert that $\mathcal{O}^-(a, \varphi^{-1}(F(a)))$ tends to infinity as $t \to -\infty$ by the uniqueness of the orbits. To sum up, we can conclude that there are no non-trivial closed orbits for system (E).

(II) Next, we prove that all the solutions for system (E) are positively bounded.

In fact, since system (E) has the properties $(\mathcal{H}^+)$ and $(\mathcal{H}^-)$, we can prove that $\mathcal{O}^+(P)$ tends to the origin for any $P \in R^2$. Noting the properties of the orbit departing from $(a, \varphi^{-1}(F(a)))$, we know that, for any $P \in R^2$, $\mathcal{O}^+(P)$ cannot spiral the origin infinitely. It follows from Theorem 1(ii)(c′) that $(0, 0)$ is a global attractor. This completes the proof. □

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