ON THE DIMENSION OF GRADED ALGEBRAS

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UDC 512

To each graded algebra R with a finite number of generators we associate the series T(R, z) = $\Sigma d_n z^n$, where d_n is the dimension of the homogeneous component of R. It is proved that if the dimensions d_n have polynomial growth, then the Krull dimension of R cannot exceed the order of the pole of the series T(R, z) for z = 1 by more than 1.

We shall mainly consider the graded algebra $R=\sum_{i=0}^{\infty}R_i$ over an arbitrary field k having a finite number of generators and a finite number of defining relations. However, we shall require a somewhat more general formulation of the main hypothesis.

Let $M=\sum_{i=0}^{\infty}M_i$ be a graded bimodule over a free algebra $k(x_1, \ldots, x_s)$ having a finite number of generators and a finite number of defining relations. Let the symbol $d_i(M)$, or simply d_i, denote the dimension of M_i as a vector space over k. The main hypothesis has three equivalent formulations.

1. There exists a linear recurrence relation with integral coefficients λ_i:

$$d_n=\sum_{i=1}^{k}\lambda_i d_{n-i}, \tag{1}$$

which is valid for all sufficiently large n. The polynomial $Q(M, z) = z^k - \Sigma\lambda_i z^{k-i}$ will be called the characteristic polynomial of the module M.

2. For all sufficiently large n

$$d_n = \Sigma P_i(n)\,\alpha_i^n, \tag{2}$$

where the α_i are algebraic integers and the $P_i(n)$ are polynomials with rational coefficients.

3. The series

$$T(M,z)=\sum_{n=0}^{\infty}d_n z^n \tag{3}$$

is a rational function with integral coefficients and lowest term in the denominator equal to 1.

The proof that these formulations are equivalent can be found in [1], Chap. 3.

The foregoing quantities are related as follows. The α_i are the roots of the characteristic polynomial Q(M, z); $P_i(n)$ is a polynomial of degree k_i, where $k_i + 1$ is the multiplicity of α_i; $T(M, z) = f(z)g^{-1}(z)$, where $g(z) = z^kQ(M, z^{-1})$, and the degree of $f(z)$ is less than the number starting with which the recurrence relation holds for all following numbers.

In the general case the main hypothesis remains unproved. A proof exists in the following special cases: modules over commutative algebras ([2], Theorem 15.2); algebras defined by a finite collection of words ([3], Theorem 2); algebras having global dimension less than 3 ([3], Theorem 3). Theorem 1 below proves the main hypothesis for a rather narrow class of algebras; however, it is interesting because it establishes a connection between the coefficients of recurrence relation (1) and the Möbius function for a monoid (see [4]) or a partially ordered set (see [1], 2.2).

Moscow Institute of Electronic Machine Building. Translated from Matematicheskie Zametki, Vol. 14, No. 2, pp. 209-216, August, 1973. Original article submitted January 3, 1972.

Following Cartier and Foata [4], we introduce the following definitions. Let M be a monoid and M^* the set of nonidentity elements of M. By a factorization of x in M we mean any sequence $(x_1, x_2, \ldots, x_n)$ of elements of M^* such that $x = x_1x_2 \ldots x_n$. The number n is called the length of the factorization. We consider a monoid in which any element admits only a finite number of distinct factorizations. The number of factorizations of x is denoted by d(x). The number of factorizations of even length is denote by $d_+(x)$ and of odd length by $d_-(x)$. It is obvious that $d(x) = d_+(x) + d_-(x)$. The function $\mu(x) = d_+(x) - d_-(x)$ is called the Möbius function of M.

Let A be the set of functions on the monoid M. We introduce a multiplication operation on A as follows:

$$fg(x) = \Sigma f(x_1)\, g(x_2), \tag{4}$$

which turns A into a monoid. The function $\varepsilon(x)$, $\varepsilon(1) = 1$, and $\varepsilon(x) = 0$ for $x \neq 1$ is the identity of A. The sum in (4) can be extended to all pairs (x_1, x_2) such that $x = x_1x_2$; among these pairs are (1, x) and (x, 1). Let $\zeta(x) = 1$ for all $x \in M$. Then (see [4])

$$\zeta\mu = \mu\zeta = \varepsilon.$$

THEOREM 1. Let M be a graded monoid without zero and with a finite number of generators. Let $\mu(x)$ be such that $\mu(x) \neq 0$ for only a finite number of elements of M. Then the semigroup algebra k(M) satisfies the main hypothesis.

Proof. Consider the following series product:

$$\sum_{x \in M} \mu(x)\, x \sum_{y \in M} \zeta(y)\, y = \sum_{z \in M} \varepsilon(z)\, z = 1.$$

Since there is no zero in M,

$$\sum \mu(x)\, t^{|x|} \sum \zeta(y)\, t^{|y|} = 1,$$

where $|x|$ is the degree of x. Collecting the coefficients of t^N, we obtain

$$\Sigma \mu(x)\, \zeta(y) = 0, \tag{5}$$

where $|x| + |y| = N > 0$.

Let $\lambda_i = \Sigma_{|x|=i}\ \mu(x)$ and let $d_j = \Sigma_{|y|=j}\ \zeta(y)$ be the number of elements of M having degree j. These elements can be taken as a basis for the j-th homogeneous component of the semigroup algebra k(M). Therefore d_j coincides with $d_j(k(M))$.

Equality (5) can now be rewritten in the form

$$\sum_{|xy| = N} \mu(x)\, \zeta(y) = \sum_{|x| < N} \mu(x) \sum_{|y| = N - |x|} \zeta(y) = \sum_{i=0}^{N} \lambda_i d_{N-i} = 0. \tag{6}$$

Since almost all $\lambda_i = 0$, by the condition of the theorem, equality (6) becomes recurrence relation (1).

COROLLARY. Let an algebra R be defined by a finite number of generators $x_1, x_2, \ldots, x_s$ and relations of the form $x_ix_j - x_jx_i$ for some i and j. Then R satisfies the main hypothesis.

Proof. The algebra R satisfies all the conditions of Theorem 1 (see [4]).

The basic ring-theoretic operations, when applied to algebras satisfying the main hypothesis, again result in algebras satisfying it.

a) For any exact sequence

$$0 \to M_1 \to M \to M_2 \to 0,$$
$$T(M, z) = T(M_1, z) + T(M_2, z).$$

b) It is obvious that

$$T(M_1 \oplus_k M_2, z) = T(M_1, z) \cdot T(M_2, z).$$

c) If R_1 and R_2 are graded algebras and $R_1 * R_2$ is their free product, then

$$T(R_1 * R_2, z) = [T^{-1}(R_1, z) + T^{-1}(R_2, z) - 1]^{-1}. \tag{7}$$

Viewing R_1, R_2, and $R_1 * R_2$ as k-modules, we obtain

$$R_1 * R_2 = k \dotplus I_1 \dotplus I_2 \dotplus I_1 \otimes I_2 \dotplus I_2 \otimes I_1 \dotplus I_1 \otimes I_2 \otimes I_1 \dotplus \dots,$$

where $I_1(I_2)$ is the augmentation ideal of the algebra $R_1(R_2)$. Let $T(I_1, z) = a_1$ and $T(I_2, z) = a_2$. Then

$$T(R_1 * R_2, z) = 1 + a_1 + a_2 + 2a_1a_2 + a_1^2a_2 + a_1a_2^2 + \dots =$$
$$= 1 + (a_1 + a_2 + 2a_1a_2)(1 + a_1a_2 + a_1^2a_2^2 + \dots) = 1 + (a_1 + a_2 + 2a_1a_2)(1 - a_1a_2)^{-1}.$$

Substituting $T(R_1, z) = a_1 + 1$ and $T(R_2, z) = a_2 + 1$ into the resulting equation, we obtain Eq. (7).

If we know the expressions for $T(R_1 * R_2, z)$, it is easy to obtain the characteristic polynomial for $R_1 * R_2$. An especially simple formula is obtained when recurrence relation (1) is valid for all numbers starting with the first. In this case

$$T(R_i, z) = Q^{-1}(R_i, z^{-1}) z^{-k}$$

and

$$T(R_1 * R_2, z) = [Q(R_1, z^{-1}) z^k + Q(R_2, z^{-1}) z^k - 1]^{-1},$$

i.e.,

$$Q(R_1 * R_2, z) = Q(R_1, z) + Q(R_2, z) - z^k.$$

This means that the right sides of the recurrence relations can be added.

d) For the algebra of polynomials over the algebra A we obtain

$$T(A[x], z) = T(A)(1 - z)^{-1}. \tag{8}$$

Indeed, the algebra $A[x]$, viewed as a k-module, can be decomposed into the direct sum $A[x] = A + Ax + Ax^2 + \dots$. Therefore

$$T(A[x], z) = T(A) + zT(A) + z^2T(A) + \dots = T(A)(1 - z)^{-1}.$$

LEMMA 1. If r is the radius of convergence of the series $T(M, z)$, then $T(M, z)$ has a pole at the point $z = r$, and all the poles of this series on the circle $|z| = r$ have order not exceeding the order of the pole at $z = r$.

Proof. Since the terms of the series are positive, the existence of a pole at $z = r$ is obvious. The comparison of the poles is obtained from the inequality

$$\left|\sum d_i u^i\right| \leqslant \sum d_i |u|^i.$$

LEMMA 2. If the series $T(M, z)$ has radius of convergence equal to 1, then all of its poles are situated at roots of 1.

Proof. Let $T(M, z) = f(z)g^{-1}(z)$, where $g(z) = 1 + a_1z + \dots + a_kz^k$. Since the product of the roots of $g(z)$ is equal to 1 and the smallest has modulus equal to unity, all the roots of $g(z)$ are equal to 1 in modulus. If all the conjugate algebraic integers are equal to 1 in modulus, then they are roots of 1 of integral degree [5].

All modules that satisfy the main hypothesis can be divided into two classes:

a) modules of polynomial type, i.e., $T(M, z)$ has poles only on the circle $|z| = 1$;

b) modules of exponential type, i.e., $T(M, z)$ has at least one pole for $z = r < 1$.

It is obvious that for any module M of polynomial type

$$T(M, z) = f(z) \Pi_i (1 - z^{m_i}).$$

Let R be a graded algebra and R[x] the algebra of polynomials over R in the commuting variable x. Let us establish a correspondence between ideals of the ring R and homogeneous ideals of the ring R[x]. This correspondence is established as in [6], Chap. 7, § 5. Since we need to perform the same construction in a more general case, we shall verify these properties.

Let $a = a_k + a_{k+1} + \ldots + a_n \in R$, where a_i is a homogeneous element of degree i in R and $a_k \neq 0$. Set $\varphi(a) = a_k x^{n-k} + a_{k+1}x^{n-k-1} + \ldots + a_n$, and define $\psi: R[x] \to R$, by setting $\psi(x) = 1$ and extending it further as a ring homomorphism. The mapping φ is not a homomorphism; the mapping ψ is a homomorphism, but we shall consider it only on homogeneous elements. Obviously,

$$\psi\varphi(a) = a, \quad \varphi\psi(b) = x^{-s}b, \tag{9}$$

where x^s is the highest degree of s dividing b.

These mappings can be extended to ideals. Small Gothic letters will denote ideals of R, and capital letters ideals of R[x]. If $\mathfrak{a} \in R$, then $\varphi(\mathfrak{a})$ is not an ideal in R[x], since it does not contain elements divisible by x. However, it is easy to verify that the set of elements of the form $x^s\varphi(a)$, where $a \in \mathfrak{a}$, and s is any natural number, is an ideal in R[x], which we shall henceforth denote by $\varphi(\mathfrak{a})$.

Equalities (9) extended to ideals become the inclusions

$$\psi\varphi(\mathfrak{a}) = \mathfrak{a}, \quad \varphi\psi(\mathfrak{A}) \supset \mathfrak{A}. \tag{10}$$

Since the ideal $\varphi(\mathfrak{a})$ is generated by the elements $\varphi(a)$, where $a \in \mathfrak{a}$, we have $\varphi(\mathfrak{a}\mathfrak{b}) = \varphi(\mathfrak{a})\varphi(\mathfrak{b})$, and if $\mathfrak{a} \subset \mathfrak{b}$, then $\varphi(\mathfrak{a}) \subset \varphi(\mathfrak{b})$. Let us prove that prime ideals are carried to prime ideals. Indeed, let $\mathfrak{A}\mathfrak{B} \subset \varphi(\mathfrak{p})$ where $\mathfrak{p}$ is prime. Then $\psi(\mathfrak{A})\psi(\mathfrak{B}) \subset \psi\varphi(\mathfrak{p}) = (\mathfrak{p})$. Let $\psi(\mathfrak{A}) \subset \mathfrak{p}$. Then $\varphi\psi(\mathfrak{A}) \subset \varphi(\mathfrak{p})$. Using (10), we obtain $\mathfrak{A} \subset \varphi\psi(\mathfrak{A}) \subset \varphi(\mathfrak{p})$, i.e., $\varphi(\mathfrak{p})$ is prime.

<u>THEOREM 2.</u> Let R be a graded algebra of polynomial type, and let the series T(R, z) have a pole of order s at the point z = 1. Then the Krull dimension of R does not exceed s + 1.

<u>Proof.</u> Let $\mathfrak{q}_0 \subset \mathfrak{q}_1$ be homogeneous prime ideals of R. Replacing $R/\mathfrak{q}_0$ by A, we can assume that $\mathfrak{q}_0 = 0$. For any left ideal $\mathfrak{q}$ of A we have $0 \neq \mathfrak{q}_1\mathfrak{q} \in \mathfrak{q}_1 \cap \mathfrak{q}$, whence A is an essential extension of $\mathfrak{q}_1$. Therefore, $\mathfrak{q}_1$ contains an element a which is not a divisor of zero (see [7]). The left A-modules A and Aa are isomorphic, and so $T(A, z) = z^k T(Aa, z)$, where k is the degree of a. Hence we obtain the coefficient inequality $T(\mathfrak{q}_1, z) \geq z^k T(A, z)$, which remains valid upon comparing the values of the series for positive values of z not exceeding 1. Therefore

$$T(A/\mathfrak{q}_1, z) = T(A, z) - T(\mathfrak{q}_1, z) \leqslant T(A, z) - z^k T(A, z) = (1 - z)h(z)T(A, z),$$

where h(z) is a polynomial. It is apparent from the resulting inequality that the series $T(A/\mathfrak{q}_1, z)$ has a pole of smaller order than the series T(A, z). If the series T(A, z) does not have z = 1 as a singular point then it is a polynomial. In this case the algebra A is finite-dimensional over the field k, and so it cannot contain a chain of prime ideals of nonzero length. Using induction, we obtain that the algebra R cannot contain a chain of homogeneous prime ideals of length bigger than s.

The series T(R[x], z) according to formula (8), has a pole of order s + 1 at the point z = 1. Using the correspondence φ between prime ideals of R and homogeneous prime ideals of R[x], we obtain the assertion of the theorem.

Commutative algebras with a finite number of generators are of polynomial type and are noetherian. It is apparent from the proof of Theorem 2 that the ascending chain condition on ideals in a ring of polynomial type holds if there exist nondivisors of zero in a sufficient number of factors of this series. The question arises of whether all algebras of polynomial type are noetherian. It turns out that this is false. The algebra $R = k(xy)/(x^2, xyx, yxy)$, serves as an example. If $\varphi: k(x, y) \to R$ is the canonical homomorphism and $\varphi(x) = a$, $\varphi(y) = b$, then as a basis for the homogeneous component R_n we can take the four elements b^n, ab^{n-1}, $b^{n-1}a$, and $ab^{n-2}a$. It is easy to compute $T(R, z) = (1 + z + z^2 + z^3)(1 - z)^{-1}$; however, R contains an ideal $\mathfrak{a}$, generated by the infinite collection of elements ab^na (n = 1, 2, . . .).

The ideal $\mathfrak{a}$ splits into an infinite direct sum, since $Rab^naR = kab^nak$, i.e., has infinite dimension in the sense of Goldie; therefore (see [8]) it has an infinite Krull dimension in the sense of Rentschler and Gabriel [9].

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