Computational Methods and Function Theory
Volume 2 (2002), No. 2, 337–351

A Möbius-Invariant Family of Conformal Maps

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(Communicated by Mario Bonk)

Abstract. Let f be a conformal map of the unit disk $\mathbb{D}$ into $\hat{\mathbb{C}}$ and let

$$Q_f(z,\zeta) = \frac{(1-|z|^2)|f'(z)|(1-|\zeta|^2)|f'(\zeta)|}{|f(z)-f(\zeta)|^2}\lambda_{\mathbb{D}}(z,\zeta)^2,$$

where $\lambda_{\mathbb{D}}$ denotes the hyperbolic distance. We introduce the family ML of all conformal maps f for which $Q_f(z,\zeta)$ remains bounded. It contains all maps f that have a quasi-conformal extension to $\hat{\mathbb{C}}$ but also some functions for which $f(\mathbb{D})$ has outward-pointing cusps. We show that f has a continuous extension to $\overline{\mathbb{D}}$ and study multiple boundary points and the Schwarzian derivative.

Keywords. Möbius-invariant, conformal map, quasi-conformal extension, multiple points, cusp, Schwarzian derivative.

2000 MSC. Primary 30C55; Secondary 30C45, 30C62.

1. Möbius-invariant families

Let Möb denote the group of all (conformal) Möbius transformations of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and Möb($\mathbb{D}$) the subgroup of Möbius transformations that map the unit disk $\mathbb{D}$ onto itself.

A family $\mathcal{F}$ of meromorphic functions in $\mathbb{D}$ will be called *Möbius-invariant* if

$$f \in \mathcal{F},\ \sigma \in \text{Möb},\ \tau \in \text{Möb}(\mathbb{D}) \quad \Rightarrow \quad \sigma \circ f \circ \tau \in \mathcal{F}. \tag{1.1}$$

The name "Möbius-invariant" is perhaps not quite standard. For instance in [5, p. 162], "invariant family" only means that $f \in \mathcal{F}$, $\tau \in$ Möb($\mathbb{D}$) implies $f \circ \tau \in \mathcal{F}$. We say that $f \in \mathcal{F}$ is *normalized* if

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0. \tag{1.2}$$

For every $f \in \mathcal{F}$ we can find $\sigma \in$ Möb such that $\sigma \circ f$ is normalized.

Received October 29, 2002.
The research was supported by Colciencias and Deutsche Forschungsgemeinschaft (DFG).

ISSN 1617-9447/\$ 2.50

The basic Möbius-invariant differential operator is the Schwarzian derivative

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

It satisfies $S_{\sigma\circ f\circ\tau}(z) = S_f(\tau(z))\tau'(z)^2$ for $\sigma \in$ Möb, $\tau \in$ Möb$(\mathbb{D})$ and thus

$$(1 - |z|^2)^2\,|S_{\sigma\circ f\circ\tau}(z)| = (1 - |\tau(z)|^2)^2\,|S_f(\tau(z))|. \tag{1.3}$$

Hence the *Schwarzian norm* [8, p. 54]

$$\|S_f\| = \sup_{z\in\mathbb{D}}(1 - |z|^2)^2\,|S_f(z)| \tag{1.4}$$

satisfies $\|S_{\sigma\circ f\circ\tau}\| = \|S_f\|$.

We now mention three Möbius-invariant families.

(a) Let $0 \le \alpha < \infty$. The functions f that are meromorphic and locally univalent in $\mathbb{D}$ and satisfy $\|S_f\| \le \alpha$ form a Möbius-invariant family. If $\alpha \le 2$ then these functions are univalent by the Nehari criterion [10]. Of particular interest is the case $\alpha = 2$, the *Nehari class*. An important example is the function

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z} = z + \frac{1}{3}z^3 + \cdots, \qquad z \in \mathbb{D}, \tag{1.5}$$

with $\|S_L\| = 2$ which maps $\mathbb{D}$ onto the strip $\{|\operatorname{Im} w| < \pi/4\}$.

(b) Let $0 \le \kappa < 1$. The conformal maps of $\mathbb{D}$ into $\hat{\mathbb{C}}$ with a κ-quasiconformal extension to $\hat{\mathbb{C}}$ form a Möbius-invariant family. They satisfy [6, 7]

$$\|S_f\| \le 6\kappa, \tag{1.6}$$

and every function f with $\|S_f\| \le 2\kappa$ belongs to this family [1].

(c) We shall introduce the Möbius-invariant family ML ("Möbius-invariant with logarithm"). It contains the Nehari class in (a) and also all families described in (b). All functions $f \in ML$ are univalent and have a continuous extension to $\bar{\mathbb{D}}$; see Theorem 3. The image domain $G = f(\mathbb{D})$ may have outward-pointing cusps but no inward-pointing cusps; see Theorem 13. The boundary ∂G may have multiple points but these have to be of a rather restricted type; see Section 4. In Section 6 we shall present some examples.

2. The families ML(q)

The hyperbolic metric in $\mathbb{D}$ is given by

$$\lambda_{\mathbb{D}}(z,\zeta) \equiv \lambda(z,\zeta) = \frac{1}{2}\log\frac{1 + \left|\frac{z-\zeta}{1-\bar{z}\zeta}\right|}{1 - \left|\frac{z-\zeta}{1-\bar{z}\zeta}\right|}. \tag{2.1}$$

Let f be meromorphic and univalent in $\mathbb{D}$. For $z, \zeta \in \mathbb{D}$ we define

$$Q_f(z,\zeta) = \frac{(1-|z|^2)|f'(z)|(1-|\zeta|^2)|f'(\zeta)|}{|f(z)-f(\zeta)|^2}\lambda_{\mathbb{D}}(z,\zeta)^2 \tag{2.2}$$

taking the limit at poles of f. In particular we define $Q_f(z,z) = 1$. It is easy to verify that

$$Q_{\sigma\circ f\circ\tau}(z,\zeta) = Q_f(\tau(z),\tau(\zeta)) \qquad \text{for } \sigma \in \text{Möb},\ \tau \in \text{Möb}(\mathbb{D}). \tag{2.3}$$

For $1 \leq q < \infty$, let $ML(q)$ denote the family of all functions univalent in $\mathbb{D}$ which satisfy

$$Q_f(z,\zeta) \leq q \qquad \text{for } z, \zeta \in \mathbb{D}. \tag{2.4}$$

Furthermore let ML denote the union of all families $ML(q)$, $1 \leq q < \infty$. It follows from (2.3) that the families $ML(q)$ and ML are Möbius-invariant in the strong sense of (1.1). Let $ML_0(q)$ and ML_0 denote the (non-invariant) subfamilies of normalized functions; see (1.2).

The definition of the families $ML(q)$ is motivated by the fact that the expression $Q_f(z,\zeta)$ occurs in a natural way in two different contexts.

Proposition 1 ([3, Th. 2])**.** *The family $ML(1)$ is identical to the Nehari class of the functions f with $\|S_f\| \leq 2$.*

Proposition 2 ([9, Th. 1])**.** *The conformal maps of $\mathbb{D}$ onto a hyperbolically convex subdomain belong to $ML(q)$ for some universal constant $q > 1$.*

Now we show that ML contains all univalent functions with a quasiconformal extension to $\hat{\mathbb{C}}$. It would be interesting to know whether ML is quasiconformally invariant: Let ϕ be a quasiconformal selfmap of $\hat{\mathbb{C}}$ and let $f \in ML$. Is it true that the conformal maps of $\mathbb{D}$ onto $\phi(f(\mathbb{D}))$ belong to ML?

Theorem 1. *Let f be meromorphic and univalent in $\mathbb{D}$ and suppose that f has a κ-quasiconformal extension to $\hat{\mathbb{C}}$.*

(i) *If $0 \leq \kappa \leq 1/3$ then $f \in ML(1)$.*
(ii) *If $1/3 < \kappa < 1$ then $f \in ML(q)$ where*

$$q = \sup_{0 \leq t < 1} (1-t^2)^{1-\kappa}\left(\frac{L(t)}{t}\right)^2 < \frac{4}{(1-\kappa)^2}.$$

Here $L(r)$ is defined by (1.5). Since $ML(1)$ is the Nehari class, the assertion (i) also follows from the Kühnau-Lehto estimate (1.6).

Proof. We obtain from [12, Th. 9.13] and from [12, Lemma 9.9] with $n = 2$, $\gamma_1 = 1$, $\gamma_2 = -1$ that

$$\left|\log\left(f'(z)f'(\zeta)\left(\frac{z-\zeta}{f(z)-f(\zeta)}\right)^2\right)\right| \leq \kappa \log\frac{|1-\bar{z}\zeta|^2}{(1-|z|^2)(1-|\zeta|^2)}.$$

This also follows from the Lehto Majorant Principle [8, p. 77] because the estimate with $\kappa = 1$ holds for all univalent functions by the Golusin inequality [12, Th. 3.3]. By (2.2) we therefore have

$$Q_f(z,\zeta) \le \left(\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar z\zeta|^2}\right)^{1-\kappa}\left(\frac{|1-\bar z\zeta|}{|z-\zeta|}\lambda(z,\zeta)\right)^2.$$

With $t = |z-\zeta|/|1-\bar z\zeta|$ it follows by (2.1) that

$$Q_f(z,\zeta)^{1/2} \le (1-t^2)^{(1-\kappa)/2}\,\frac{L(t)}{t}. \tag{2.5}$$

If $0 \le \kappa \le 1/3$ then the right-hand side is decreasing in $t \in [0,1]$ and therefore ≤ 1. This proves (i).

Now let $1/3 < \kappa < 1$. The first estimate in (ii) is clear by (2.5). To prove the second estimate we write $y = L(t)$ and note that $L(t)/t \le L(t) + 1$. Hence

$$(1-t^2)^{(1-\kappa)/2}\,\frac{L(t)}{t} \le 2^{1-\kappa}e^{-(1-\kappa)y}(y+1) \le \frac{2^{1-\kappa}e^{-\kappa}}{1-\kappa} < \frac{2}{1-\kappa}.$$

■

It is easy to obtain a geometric characterization of the family ML, which however is rather implicit.

Theorem 2. *Let f map $\mathbb{D}$ conformally onto $G \subset \mathbb{C}$ and write*

$$\delta(w) = \operatorname{dist}(w, \partial G) \qquad \text{for } w \in G. \tag{2.6}$$

Then $f \in ML$ holds if and only if there exists a constant q^* such that

$$\inf_C \int_C \frac{|dw|}{\delta(w)} \le \frac{q^*|w_1 - w_2|}{\sqrt{\delta(w_1)\delta(w_2)}} \qquad \text{for } w_1, w_2 \in G, \tag{2.7}$$

where C runs through all curves in G from w_1 to w_2.

The infimum in (2.7) is, by definition, the *quasi-hyperbolic metric* $\lambda^*_G(w_1, w_2)$ in G. It satisfies [13, p. 92]

$$\lambda_{\mathbb{D}}(z_1, z_2) \le \lambda^*_G(w_1, w_2) \le 4\lambda_{\mathbb{D}}(z_1, z_2) \tag{2.8}$$

for $w_j = f(z_j)$. This easily follows from the well-known fact [13, p. 9] that

$$\delta(w) \le (1-|z|^2)\,|f'(z)| \le 4\delta(w) \qquad \text{for } w = f(z),\ z \in \mathbb{D}. \tag{2.9}$$

Proof of Theorem 2. First let $f \in ML(q)$. Then, by (2.8) and (2.2),

$$\lambda^*_G(w_1, w_2) \le \frac{4\sqrt{q}|w_1 - w_2|}{\sqrt{(1-|z_1|^2)|f'(z_1)|(1-|z_2|^2)|f'(z_2)|}} \le \frac{4\sqrt{q}|w_1 - w_2|}{\sqrt{\delta(w_1)\delta(w_2)}}$$

because of (2.9). Thus (2.7) holds with $q^* = 4\sqrt{q}$. Now let (2.7) hold. Then, by (2.8),

$$\lambda_{\mathbb{D}}(z_1, z_2) \le \frac{q^*|w_1 - w_2|}{\sqrt{\delta(w_1)\delta(w_2)}} \le \frac{4q^*|w_1 - w_2|}{\sqrt{(1-|z_1|^2)|f'(z_1)|(1-|z_2|^2)|f'(z_2)|}}$$

because of (2.9). Thus (2.2) holds with $q = 16q^{*2}$. ■

Theorem 3. *Every function in ML has a continuous extension to $\bar{\mathbb{D}}$. For each $f \in ML_0(q)$ the spherical distance satisfies*

$$d^{\#}(f(z_1), f(z_2)) < \frac{19q}{\log \frac{4}{|z_1 - z_2|}} \qquad \text{for } z_1, z_2 \in \bar{\mathbb{D}}. \tag{2.10}$$

It follows from (2.10) that the normalized family $ML_0(q)$ is equicontinuous. The full family $ML(q)$ is not equicontinuous because it contains the whole family Möb. We need the following estimate for normalized univalent meromorphic functions.

Lemma 1. *If f is univalent in $\mathbb{D}$ and if $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$, then*

$$f^{\#}(z) := \frac{|f'(z)|}{1 + |f(z)|^2} \leq \frac{36}{11(1 - |z|^2)} \qquad \text{for } z \in \mathbb{D}.$$

Proof. The univalent function $g = 1/f$ has the form $g(z) = z^{-1} + b_1 z + \cdots$. Hence [12, p. 65], [14, p. 25]

$$|g'(z)| \leq \frac{1}{r^2(1 - r^2)}, \qquad |g(z) - z^{-1}| < 3r$$

for $0 < |z| = r < 1$. Hence

$$f^{\#}(z) = g^{\#}(z) \leq \frac{1}{(1 - r^2)(1 - 5r^2 + 9r^4)} \leq \frac{36}{11(1 - r^2)}.$$

■

Proof of Theorem 3. Since spherical continuity is Möbius-invariant, we may assume that $f \in ML_0(q)$ and it suffices to show (2.10) only for $|z_1| = r_1 \leq |z_2| = r_2 < 1$. We write $\eta = |z_1 - z_2|$. The case $\eta \geq 1/16$ is trivial because $q \geq 1$ and the spherical distance is always $\leq \pi$. Hence we may assume that $\eta < 1/16$ so that $r := 1 - \sqrt{\eta}/2 > 0$.

First we consider the case $r_1 \leq r$. Then $r_2 \leq r_1 + \eta \leq r + \eta < 1 - \sqrt{\eta}/4$ and thus, by Lemma 1,

$$\begin{aligned} d^{\#}(f(z_1), f(z_2)) &\leq \eta \max\{f^{\#}(z) : z \in [z_1, z_2]\} \\ &\leq \frac{36}{11} \frac{\eta}{1 - (1 - \frac{\sqrt{\eta}}{4})^2} \leq 8\sqrt{\eta} < \frac{11}{\log \frac{4}{\eta}}. \end{aligned}$$

Now we consider the case $r_1 > r$. Then $r_2 \geq r_1 > r$. With $\zeta_j = z_j/r_j$ we obtain as above from Lemma 1 that

$$d^{\#}(f(r\zeta_1), f(r\zeta_2)) < \frac{11}{\log \frac{4}{\eta}}. \tag{2.11}$$

Furthermore we obtain from (2.4) with $z = 0$, $\zeta = r\zeta_j$ that

$$\left|\frac{d}{dr}\frac{1}{f(r\zeta_j)}\right| = \frac{|f'(r\zeta_j)|}{|f(r\zeta_j)|^2} \leq \frac{q}{(1-r^2)L(r)^2} = -\frac{d}{dr}\frac{q}{L(r)}$$

for $j = 1, 2$. Integrating we conclude that

$$d^{\#}(f(z_j), f(r\zeta_j)) \leq \left|\frac{1}{f(r_j\zeta_j)} - \frac{1}{f(r\zeta_j)}\right| \leq \frac{q}{L(r\zeta_j)}$$

which is less than $4q/\log(4/\eta)$. Hence (2.10) follows from (2.11). ∎

3. Some estimates

First we prove a lower estimate of the hyperbolic metric. Let again $L(r) = \lambda(r, 0)$; see (1.5).

Lemma 2. *If $r < 1$, $\rho < 1$ and $0 < |t - \theta| \leq \pi$ then*

$$\lambda_{\mathbb{D}}(re^{it}, \rho e^{i\theta}) \geq L(r) + L(\rho) + \log\sin\frac{|t-\theta|}{2}. \tag{3.1}$$

Proof. Writing $z = re^{it}$, $\zeta = \rho e^{i\theta}$, we obtain from (2.1) that

$$\lambda(z, \zeta) = \log(|1 - \bar{z}\zeta| + |z - \zeta|) - \frac{1}{2}\log(1 - r^2) - \frac{1}{2}\log(1 - \rho^2). \tag{3.2}$$

With $a = \sin(|t - \theta|/2)$ we have

$$\frac{|1 - \bar{z}\zeta| + |z - \zeta|}{a} = \sqrt{\left(\frac{1 - \rho r}{a}\right)^2 + 4\rho r} + \sqrt{\left(\frac{r - \rho}{a}\right)^2 + 4\rho r}.$$

This expression becomes minimal for $a = 1$, so that

$$|1 - \bar{z}\zeta| + |z - \zeta| \geq a(1 + r)(1 + \rho),$$

and (3.1) follows from (3.2). ∎

Many estimates take a simpler form if we send the boundary point of interest to ∞. If $f \in ML(q)$ and $f(0) = 0$, $f'(0) = 1$ then

$$g(z) = \frac{f(z)}{1 - \frac{f(z)}{f(e^{i\theta})}}, \qquad z \in \mathbb{D}, \tag{3.3}$$

satisfies $g(0) = 0$, $g'(0) = 1$, $g(e^{i\theta}) = \infty$ and $g \in ML(q)$.

Theorem 4. *Let $f \in ML(q)$, $f(0) = 0$ and $f(e^{i\theta}) = \infty$ and write*

$$s = \log\frac{1}{\sin\frac{|t-\theta|}{2}}, \qquad 0 < |t - \theta| \leq \pi, \tag{3.4}$$

and $z = re^{it}$, $\zeta = \rho e^{i\theta}$. *If* $L(\rho) \geq s$ *then*

$$\frac{(1-r^2)|f'(z)|}{|f(z)-f(\zeta)|} \leq \frac{q}{L(r)+L(\rho)-s}, \tag{3.5}$$

$$\left|\frac{f(z)-f(\zeta)}{f(\zeta)}\right| \geq \left(\frac{L(\rho)-s}{L(r)+L(\rho)-s}\right)^q. \tag{3.6}$$

Proof. By (2.2) and (2.4), we have

$$\left|\frac{\partial}{\partial \rho}\frac{(1-r^2)f'(z)}{f(z)-f(\zeta)}\right| = \frac{(1-r^2)|f'(z)f'(\zeta)|}{|f(z)-f(\zeta)|^2} \leq \frac{q}{1-\rho^2}\frac{1}{\lambda(z,\zeta)^2}$$

and by Lemma 2 we obtain

$$\frac{q}{1-\rho^2}\frac{1}{\lambda(z,\zeta)^2} \leq \frac{q}{1-\rho^2}\frac{1}{(L(r)+L(\rho)-s)^2} = -\frac{\partial}{\partial\rho}\frac{q}{L(r)+L(\rho)-s}.$$

Now (3.5) follows by integration from ρ to 1 because $L(1) = \infty$ and $f(e^{i\theta}) = \infty$. Furthermore we have

$$\left|\frac{\partial}{\partial r}\log\frac{f(\zeta)}{f(\zeta)-f(z)}\right| = \frac{|f'(z)|}{|f(z)-f(\zeta)|} \leq \frac{q}{1-r^2}\frac{1}{L(r)+L(\rho)-s}$$

by (3.5). We integrate from 0 to r using $f(0) = 0$, and (3.6) follows by exponentiation. ∎

Theorem 5. *Let* $f \in ML(q)$ *and* $f(0) = 0$, $f'(0) = 1$, $f(e^{i\theta}) = \infty$. *Then*

$$|f(\rho e^{i\theta})| \geq \frac{L(\rho)}{q} \qquad \text{for } 0 \leq \rho < 1. \tag{3.7}$$

If furthermore

$$a := \liminf_{\rho\to 1}\frac{|f(\rho e^{i\theta})|}{L(\rho)} < \infty$$

then, for $z \in \mathbb{D}$,

$$|f(z)| \leq aqL(|z|), \qquad (1-|z|^2)|f'(z)| \leq aq. \tag{3.8}$$

Proof. We may assume that $\theta = 0$. Since $f(0) = 0$ and $f'(0) = 1$, we obtain from (2.4) that

$$\frac{d}{dr}\left|\frac{1}{f(r)}\right| \leq \frac{|f'(r)|}{|f(r)|^2} \leq \frac{q}{1-r^2}\frac{1}{L(r)^2} = -\frac{d}{dr}\frac{q}{L(r)}.$$

Hence (3.7) follows by integration from ρ to 1 because $f(1) = \infty$.

Now let $a < \infty$. Then there exists (ρ_n) such that $\rho_n \to 1$ and $|f(\rho_n)| \sim aL(\rho_n)$ as $n \to \infty$. Hence it follows from (3.5) with $\zeta = \rho_n$ for $n \to \infty$ that

$$(1-r^2)|f'(r)| \leq aq \qquad \text{for } 0 \leq r < 1,$$

and the first inequality in (3.8) follows by integration because $f(0) = 0$. ∎

The next estimate is a weak "differentiation" of (3.7) and goes in the direction opposite to (3.8).

Theorem 6. *Let $f \in ML(q)$ and $f(0) = 0$, $f'(0) = 1$, $f(e^{i\theta}) = \infty$. Then*

$$\limsup_{\rho \to 1}(1-\rho^2)|f'(\rho e^{i\theta})| \geq \frac{1}{q}, \qquad \limsup_{\rho \to 1} \frac{(1-\rho^2)|f'(\rho e^{i\theta})|L(\rho)}{|f(\rho e^{i\theta})|} \geq 1. \tag{3.9}$$

Proof. Suppose that the first inequality is false. Then there exists $b < 1/q$ such that

$$\frac{d}{d\rho}|f(\rho e^{i\theta})| \leq |f'(\rho e^{i\theta})| < \frac{b}{1-\rho^2} = bL'(\rho)$$

for ρ close to 1, and integration gives $|f(\rho e^{i\theta})| < c_1 + bL(\rho)$ with some constant c_1. But this contradicts (3.7). If the second inequality in (3.9) is false then we get

$$\log|f(\rho e^{i\theta})| < c_2 + \log[L(\rho)^b]$$

with $b < 1$ which again contradicts (3.7). ∎

4. Multiple boundary points

Let $f \in ML(q)$ map $\mathbb{D}$ onto G. We now consider points $\omega \in \partial G$ such that

$$f(\zeta_\nu) = \omega, \qquad \nu = 1, \dots, n, \quad \text{with different } \zeta_\nu \in \mathbb{T}. \tag{4.1}$$

We shall see that this puts severe restrictions on G near ω. First we consider the case $\omega = \infty$.

Theorem 7. *Let $f \in ML(q)$ and suppose that $f(0) = 0$ and $f(\zeta_1) = f(\zeta_2) = \infty$ with $\zeta_1 \neq \zeta_2$. Then*

$$\limsup_{|z| \to 1} \frac{(1-|z|^2)|f'(z)|}{|f(z)|}L(|z|) \leq q, \tag{4.2}$$

$$|f(z)| < L(|z|)^{q+\varepsilon} \quad \textit{for } \varepsilon > 0,\ r_0(\varepsilon) < |z| < 1. \tag{4.3}$$

Proof. Let $z \to \zeta \in \mathbb{T}$. First let $f(\zeta) \neq \infty$. Then, by (2.2),

$$\frac{(1-|z|^2)|f'(z)|}{|f(z)|}L(|z|) \leq \frac{q|f(z)|}{|f'(0)|L(|z|)} \to 0 \qquad \text{as } z \to \zeta.$$

Now let $f(\zeta) = \infty$. We can write $\mathbb{T} = T_1 \cup T_2$ where the T_j are overlapping closed arcs with $\zeta_j = e^{i\theta_j} \notin T_j$. Then the value s in (3.4) remains bounded by some s_j for $\theta = \theta_j$, $e^{it} \in T_j$. Hence it follows from (3.5) that

$$\frac{(1-r^2)|f'(re^{it})|}{|f(re^{it}) - f(\rho_j\zeta_j)|} \leq \frac{q}{L(r)} \qquad \text{for } e^{it} \in T_j$$

if ρ_j is chosen such that $L(\rho_j) > s_j$. This implies (4.2) because ζ is an inner point of some arc T_j and $f(z) \to f(\zeta) = \infty$ as $z \to \zeta$. Furthermore it follows from (4.2) that

$$\frac{d}{dr}\left[\log|f(re^{it})| - (q+\varepsilon)\log L(r)\right] \leq \left|\frac{f'(re^{it})}{f(re^{it})}\right| - \frac{q+\varepsilon}{(1-r^2)L(r)} < 0$$

for $r > r_0(\varepsilon)$, which implies (4.3). ■

Now we interpret Theorem 7 geometrically for the case $\omega \in \mathbb{C}$. We shall see that G is cusp-like near the multiple boundary point ω and, in particular, cannot have a corner of any positive angle at ω. Let $\delta(w)$ again be defined by (2.6).

Theorem 8. *Let $f \in ML(q)$ map $\mathbb{D}$ onto G and suppose that $\omega = f(\zeta_1) = f(\zeta_2) \neq \infty$ with $\zeta_1 \neq \zeta_2$. Then, for every $\varepsilon > 0$,*

$$\delta(w) = \mathcal{O}(|w-\omega|^{1-\varepsilon-1/q}) \qquad \text{as } w \to \omega,\ w \in G. \tag{4.4}$$

Proof. We may assume $f(0) = 0$. We write $w = f(z)$, $r = |z|$ and, using the transformation (3.3) and also (2.9), we obtain from Theorem 7 that

$$\limsup_{w\to\omega} \frac{|\omega|\delta(w)L(r)}{|w||w-\omega|} \leq q, \qquad \left|\frac{w\omega}{w-\omega}\right| < L(r)^{q+\varepsilon}, \qquad r > r_o(\varepsilon),$$

and thus, with suitable constants c_1 and c_2,

$$\delta(w) < c_1\frac{|w-\omega|}{L(r)}, \qquad \frac{1}{|w-\omega|} < c_2 L(r)^{q+\varepsilon}$$

for $|w-\omega| < \eta(\varepsilon)$, $w \in G$. It follows that

$$\delta(w) < c_3|w-\omega|^{1+1/(q+\varepsilon)}$$

for some constant c_3, which implies (4.4). ■

Now we prove that the multiplicity of multiple points is bounded by a constant depending only on q.

Theorem 9. *Let $f \in ML(q)$. If (4.1) holds for some $\omega \in \hat{\mathbb{C}}$ then*

$$n \leq \frac{\pi}{\arcsin(2^{-q-1})} \tag{4.5}$$

Proof. We may assume that $f(0) = 0$ and $\omega = \infty$. For every $\rho > 0$ there exist r_ν such that

$$|f(r_\nu\zeta_\nu)| = \rho \qquad \text{for } \nu = 1, \dots, n. \tag{4.6}$$

We may assume that the points ζ_ν are cyclically ordered on $\mathbb{T}$ where $\zeta_0 = \zeta_n$. Now we choose ν such that $|f(r_\nu\zeta_\nu) - f(r_{\nu-1}\zeta_{\nu-1})|$ is minimal among these n values. Then it follows from (4.6) that

$$|f(r_\nu\zeta_\nu) - f(r_{\nu-1}\zeta_{\nu-1})| \leq 2\rho\sin\frac{\pi}{n}.$$

Let $r_{\nu-1} \le r_\nu$, say. We deduce from (4.6) and from (3.6) in Theorem 4 that

$$2\sin\frac{\pi}{n} \ge \left(\frac{L(r_\nu)-s}{L(r_\nu)+L(r_{\nu-1})-s}\right)^q \ge \left(\frac{L(r_\nu)-s}{2L(r_\nu)-s)}\right)^q$$

which converges to 2^{-q} as $\rho \to \infty$. This implies (4.5). ■

The bound for n seems to be much too big. For $q = 1$ it gives $n \le 12$ whereas the real value is $n = 2$. This also holds for q slightly larger than 1.

Theorem 10. *There exists $q^* > 1$ such that every function $f \in ML(q)$ with $1 \le q < q^*$ assumes the same boundary value at most twice.*

Proof. Let q^* denote the infimum of all q such that there are $f \in ML(q)$ satisfying (4.1) with $n = 3$. Then there are $q_k \to q^*$ and $f_k \in ML(q_k)$ such that $f_k(z_{k,1}) = f_k(z_{k,2}) = f_k(z_{k,3})$. We choose $\tau_k \in$ Möb$(\mathbb{D})$ such that $z_{k,j} = \tau_k(e^{2\pi ij/3})$ for $j = 1, 2, 3$ and then $\sigma_k \in$ Möb such that

$$g_k := \sigma_k \circ f_k \circ \tau_k \in ML_0(q_k) \subset ML_0(q^* + 1)$$

if k is large. Since $ML_0(q^*+1)$ is equicontinuous by Theorem 3, we may assume that $g_k \to g$ as $k \to \infty$ uniformly in $\overline{\mathbb{D}}$. It follows from (2.2) and (2.4) that $Q_g(z,\zeta) \le q^*$, furthermore $g(e^{2\pi i/3}) = g(e^{4\pi i/3}) = g(1)$. Hence the infimum q^* is attained and we have $q^* > 1$ because every function in the Nehari class $ML(1)$ takes every value at most twice [4]. ■

A particularly simple case of multiple boundary points is given by *n-symmetric functions*, that is functions satisfying

$$f(e^{2\pi i/n}z) = e^{2\pi i/n} f(z) \qquad \text{for } z \in \mathbb{D}. \tag{4.7}$$

Compare Example 2.

Theorem 11. *If $f \in ML(q)$ is n-symmetric and unbounded then $q \ge 1/\sin^2(\pi/n)$, more precisely*

$$1 \le \limsup_{|z|\to 1} \frac{(1-|z|^2)|f'(z)|}{|f(z)|} L(|z|) \le \sqrt{q}\sin\frac{\pi}{n}. \tag{4.8}$$

Proof. We obtain from (2.2) and (4.7) that

$$\sqrt{Q_f(z, e^{2\pi i/n}z)} = \frac{(1-|z|^2)|f'(z)|}{|1-e^{2\pi i/n}||f(z)|}\lambda(z, e^{2\pi i/n}z) \le \sqrt{q}.$$

We have $\lambda(z, e^{2\pi i/n}z) \sim 2L(|z|)$ as $|z| \to 1$ by Lemma 2, furthermore $|1-e^{2\pi i/n}| = 2\sin(\pi/n)$. Hence we obtain the second inequality (4.8) by letting $|z| \to 1$ suitably. The first inequality (4.8) holds by Theorem 6. ■

5. The Schwarzian derivative

The norm $\|S_f\|$ was defined in (1.4). By definition we have $\|S_f\| \leq 2$ for the Nehari class $ML(1)$, and $\|S_f\| \leq 6$ holds for all univalent functions [12, p. 68], with equality e.g. for the Koebe function. Now we show that the upper bound of $\|S_f\|$ for $f \in ML(q)$ approaches these values 2 and 6 as $q \to 1$ and $q \to \infty$, respectively.

Theorem 12. *Let $f \in ML(q)$ and $1 \leq q < \infty$. Then*

$$\|S_f\| \leq 2(1 + c\sqrt{q-1}) \tag{5.1}$$

where c is an absolute constant and furthermore, for every $\varepsilon > 0$,

$$\|S_f\| \leq 6(1 - e^{-(1+\varepsilon)\sqrt{q}}) \qquad \text{for } q \geq q_0(\varepsilon). \tag{5.2}$$

Proof. Let $z_0 \in \mathbb{D}$ and write $\tau(z) = (z + z_0)/(1 + \bar{z}_0 z)$. We can choose $\sigma \in$ Möb such that

$$g(z) := \sigma \circ f \circ \tau(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \tag{5.3}$$

with $b_1 \leq 0$. It follows from (1.3) that

$$(1 - |z_0|^2)^2 \, |S_f(z_0)| = |S_g(0)| = 6|b_1|. \tag{5.4}$$

To prove (5.1) we may assume that $1 \leq q \leq 17/16$. Let $0 < r \leq 1/2$. By (5.3) and (2.2) we have

$$\begin{aligned} \left| -\frac{1}{r^2} + b_1 \pm 2b_2 r + \sum_{n=3}^{\infty} n b_n (\pm r)^{n-1} \right| &= |g'(\pm r)| = \frac{Q_{1/g}(0, \pm r)}{(1 - r^2)L(r)^2} \\ &\leq \frac{q}{(1 - r^2)L(r)^2} \leq \frac{q}{r^2} + \frac{q}{3} + c_1 r^2. \end{aligned} \tag{5.5}$$

It follows from the Schwarz inequality and the Area Theorem [12, p. 18] that

$$\sum_{n=3}^{\infty} n|b_n| r^{n-1} \leq \left(\sum_{n=1}^{\infty} n|b_n|^2 \sum_{n=3}^{\infty} n r^{2n-2} \right)^{1/2} \leq c_2 r^2. \tag{5.6}$$

Since $b_1 \leq 0$ we thus obtain from (5.5) that

$$\frac{1}{r^2} + |b_1| \mp 2r \operatorname{Re} b_2 \leq \frac{q}{r^2} + \frac{q}{3} + c_3 r^2.$$

Choosing the sign suitably we deduce that

$$|b_1| \leq \frac{q-1}{r^2} + \frac{q}{3} + c_3 r^2.$$

Now (5.1) follows from (5.4) if we choose $r = (q-1)^{1/4}$.

To prove (5.2) we write $b_1 = \delta - 1$ where $0 \leq \delta \leq 1$. If $0 < x < 1$ then, by (5.3),

$$\frac{1}{2}\left(g(x) - g(-x)\right) = \frac{1}{x} - x + \delta x + \frac{1}{2}\sum_{n=2}^{\infty} b_n \left(x^n - (-x)^n\right).$$

By the Area Theorem we have

$$\begin{aligned}\left(\sum_{n=2}^{\infty} |b_n| x^n\right)^2 &\leq \sum_{n=2}^{\infty} n|b_n|^2 \sum_{n=2}^{\infty} \frac{1}{n} x^{2n} \\ &\leq (1 - |b_1|^2)\log\frac{1}{1-x^2} \leq 2\delta \log\frac{1}{1-x^2}\end{aligned}$$

and therefore

$$\frac{1}{2}|g(x) - g(-x)| \leq \frac{1-x^2}{x} + \delta x + \sqrt{2\delta}\left(\log\frac{1}{1-x^2}\right)^{1/2}. \tag{5.7}$$

As in (5.6) we obtain from the Area Theorem that

$$|g'(\pm x)| = \left|\frac{1}{x^2} + 1 - \delta - \sum_{n=2}^{\infty} n b_n (\pm x)^{n-1}\right| \geq \frac{1+x^2}{x^2} - \delta - \frac{\sqrt{2\delta}}{1-x^2}. \tag{5.8}$$

We deduce from (5.7), (5.8) and (2.2) that

$$\varphi(x, \delta) := \frac{\left(\frac{1+x^2}{x^2} - \delta - \frac{\sqrt{2\delta}}{1-x^2}\right)(1-x^2)L(x)}{\frac{1-x^2}{x} + \delta + \sqrt{2\delta}\left(\log\frac{1}{1-x^2}\right)^{1/2}} \leq \sqrt{Q_g(x, -x)} \leq \sqrt{q}.$$

We set $\delta = e^{-p}$ and choose $x = 1 - pe^{-p/2}$. Since $\varphi(1 - pe^{-p/2}, e^{-p}) \sim p$ as $p \to \infty$, it follows that

$$1 - |b_1| = \delta = e^{-p} > e^{-(1+\varepsilon)\sqrt{q}} \qquad \text{for } q \geq q_0(\varepsilon),$$

which implies (5.2) by (5.4). ∎

Estimates for the Schwarzian norm imply estimates for f and f'. Thus we obtain the following theorem from [11] and from a result of Chuaqui and Osgood [2].

Theorem 13. *Let $f \in ML(q)$ and $f(0) = 0$, $f'(0) = 1$. If $\varepsilon > 0$ then*

$$\alpha := \sqrt{1 + \frac{1}{2}\|S_f\|} < 2 - e^{(1+\varepsilon)\sqrt{q}} \qquad \textit{for } q \geq q_0(\varepsilon). \tag{5.9}$$

(i) *If $f''(0) = 0$ then, for $|z| = r < 1$,*

$$\frac{(1+r)^\alpha - (1-r)^\alpha}{(1+r)^\alpha + (1-r)^\alpha} \leq |f'(z)|.$$

(ii) *If f has no poles then* $|f''(0)| \le 2\alpha$ *and, for* $0 < r < 1$,

$$\frac{r}{2\alpha} < |f(z)| < \frac{1}{(1-r)^{\alpha}}, \tag{5.10}$$

$$\left(\frac{1-r}{1+r}\right)^{\alpha} \le (1-|z|^2)|f'(z)| \le \left(\frac{1+r}{1-r}\right)^{\alpha}. \tag{5.11}$$

In particular (5.11) shows that, for functions in ML, the image domain has no inward-pointing cusps, because for an inward-pointing (Dini-smooth) cusp [13, Th. 3.9] at $f(\zeta)$, we have $|f'(r\zeta)| \sim c(1-r)$ as $r \to 1$ with $0 < c < \infty$.

Proof. The estimate (5.9) follows at once from (5.2) and then (i) from [2, Th. 1] where the lower bounds holds also for $t \ge 1$.

Now we consider the family $ML_1(q)$ of all analytic functions $f(z) = z + a_2 z^2 + \cdots$ in $ML(q)$. This family is "linearly invariant" in the sense that, if $f \in ML_1(q)$ and $\tau \in \text{Möb}(\mathbb{D})$, then

$$g(z) = \frac{f(\tau(z)) - f(\tau(0))}{\tau'(0) f'(\tau(0))}, \qquad z \in \mathbb{D},$$

also belongs to $ML_1(q)$. By [11, Folg. 2.3] the "order" of $ML_1(q)$ satisfies

$$\sup\{|a_2| : f \in ML_1(q)\} \le \sqrt{1 + \frac{1}{2}\|S_f\|} = \alpha.$$

Now (5.10) and (5.11) follow from [11, Satz 1.1] because f is univalent in $\mathbb{D}$. ∎

6. Three examples

Example 1. Let $a \in \mathbb{C}$, $a \ne 0$ and

$$f(z) = \left(\frac{1+z}{1-z}\right)^{a}, \qquad z \in \mathbb{D}.$$

The Schwarzian derivative is $S_f(z) = 2(1-a^2)/(1-z^2)^2$. If $|a \pm 1| < 1$ then $f(\mathbb{D})$ is the Jordan domain between two rays (if $a \in \mathbb{R}$) or two logarithmic spirals (if $a \notin \mathbb{R}$) and f has a quasiconformal extension to $\hat{\mathbb{C}}$. Hence $f \in ML$. In particular, if $|a^2 - 1| \le 1$ then $\|S_f\| \le 2$ so that $f \in \text{ML}(1)$ by Proposition 1. If $|a \pm 1| = 1$ then $f(\mathbb{D})$ is the plane slit along a single logarithmic spiral so that $f \notin \text{ML}$ by Theorem 8.

Example 2. Let $m = 2, 3, \dots$ and

$$f(z) = \frac{1}{m} \sum_{\nu=0}^{m-1} e^{-i\pi\nu/m} L(e^{i\pi\nu/m}) = \sum_{k=0}^{\infty} \frac{1}{2mk+1} z^{2mk+1}.$$

This function satisfies $f'(z) = 1/(1-z^{2m})$. Hence $\operatorname{Re} f'(z) > 1/2 > 0$ so that f is univalent in $\mathbb{D}$. Furthermore f is $2m$-symmetric; see (4.7). Near ∞ the image domain consists of $2m$ parts which are asymptotically strips of width $\pi/(2m)$. In a later paper we will show that $f \in ML$.

Now let $m = 2$. We prove that

$$Q_f(x, 0) \le 2, \quad Q_f(x, ix) \le 2 \qquad \text{for } 0 \le x < 1. \tag{6.1}$$

By (2.2) the first inequality is equivalent to

$$\varphi_1(x) := \sqrt{2(1+x^2)}f(x) - L(x) \ge 0,$$

which holds because $\varphi_1'(x) \ge 0$ and $\varphi_1(0) = 0$. The second inequality (6.1) is equivalent to

$$\varphi_2(x) = 2(1+x^2)f(x) - \lambda(x, ix) \ge 0,$$

which holds because $\varphi_2(0) = 0$ and

$$\varphi_2'(x) = 4xf(x) + \frac{2(1+x^2)}{1-x^4} - \frac{\sqrt{2}}{\sqrt{1+x^4}}\frac{1+x^2}{1-x^2} \ge 0.$$

In view of (6.1), it is reasonable to conjecture that $f \in ML(2)$. This would show that Theorem 11 is best possible for $n = 4$.

Example 3. Now we want to give an example where $f(\mathbb{D})$ is much broader than a strip at ∞ in both directions. Let $0 < a < 16/(3\pi^2)$ and

$$f(z) = L(z) + aL(z)^3, \qquad z \in \mathbb{D}.$$

Since $L \in ML(1)$ we have, by (2.2),

$$Q_L(z_1, z_2) = \frac{(1-|z_1|^2)(1-|z_2|^2)\lambda(z_1,z_2)^2}{|1-z_1{}^2|\,|1-z_2{}^2|\,|L(z_1)-L(z_2)|^2} \le 1.$$

We write $L(z_j) = w_j = u_j + iv_j$ and therefore obtain, by (2.2), that

$$Q_f(z_1, z_2) \le \frac{|1+3aw_1{}^2|\,|1+3aw_2{}^2|}{|1+3(w_1{}^2+w_1w_2+w_2{}^2)|^2}.$$

The real part of the square root of the denominator is at least

$$\begin{aligned} &1 + a(u_1{}^2 - v_1{}^2 + u_1u_2 - v_1v_2 + u_2{}^2 - v_2{}^2) \\ &\ge \left(1 - 3\pi^2\frac{a}{16}\right) + \frac{a}{2}(u_1{}^2 + u_2{}^2) > 0 \end{aligned}$$

for $|v_j| \le \pi/4$ and the nominator is less than $16(1+u_1{}^2)(1+u_2{}^2)$. Hence $Q_f(z_1, z_2)$ is bounded so that $f \in ML$.

Acknowledgement. We want to thank the referee for carefully reading our manuscript.

References

1. L. V. Ahlfors and G. Weill, A uniqueness theorem for Beltrami equations, *Proc. Amer. Math. Soc.* **13** (1962), 975–978.
2. M. Chuaqui and B. Osgood, Sharp distortion theorems associated with the Schwarzian derivative, *J. London Math. Soc.* **48** (1993), 289–298.
3. M. Chuaqui and Ch. Pommerenke, Characteristic properties of Nehari functions, *Pacific J. Math.* **188** (1999), 83–94.
4. F. Gehring and Ch. Pommerenke, On the Nehari univalence criterion and quasicircles, *Comment. Math. Helv.* **59** (1984), 226–242.
5. W. K. Hayman, *Meromorphic Functions*, University Press, Oxford, 1964.
6. R. Kühnau, Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen, *Math. Nachr.* **48** (1971), 77–105.
7. O. Lehto, Schlicht functions with a quasiconformal extension, *Ann. Acad. Sci. Fenn. Math.* **500** (1971), 1–10.
8. ———, *Univalent Functions and Teichmüller Spaces*, Springer, New York, 1987.
9. D. Mejía and Ch. Pommerenke, On the derivative of hyperbolically convex functions, *Ann. Acad. Sci. Fenn. Math.* **27** (2002), 47–56.
10. Z. Nehari, The Schwarzian derivative and schlicht functions, *Bull. Amer. Math. Soc.* **55** (1949), 545–551.
11. Ch. Pommerenke, Linear-invariante Familien analytischer Funktionen I, *Math. Ann.* **155** (1964), 108–154.
12. ———, *Univalent Functions (with a chapter on quadratic differentials by Gerd Jensen)*, Vandenhoeck & Ruprecht, Göttingen, 1975.
13. ———, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, 1992.
14. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis II*, Springer, Berlin, 1925.

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