General Relativity and Gravitation, Vol. 25, No. 1, 1993

Trapping and Completeness of a Spherical Light-Cone

Sean A. Hayward[1]

Received May 26, 1992

This article concerns an outgoing spherically symmetric light-cone evolving according to the Einstein equations. The relevant equations are solved explicitly as integrals of the infalling matter, which is unspecified apart from energy conditions and a consistency condition. A condition on the matter is derived which is necessary and sufficient for the cone to be trapped. If the cone is not trapped it is complete, and if it is trapped it will recollapse to a point. In either case, the asymptotic behaviour of the matter is consequently constrained.

In General Relativity, dense concentrations of energy can produce trapped surfaces, and consequently black holes. Making this statement precise requires a condition for the trapping of a surface in terms of an integral of the matter density and gravitational radiation inside it. Sufficient conditions for trapping without symmetry assumptions have been found [1]. Obtaining a necessary and sufficient condition is generally thought impossible, since any integration necessarily loses almost all information about the distribution of the matter. Whilst this appears to be true for the trapping of part of a Cauchy surface, the trapping of a light-cone can in fact be determined in terms of the matter and radiation falling through it. The general result will be given elsewhere, with this article being devoted to the spherically symmetric case, which can be treated more explicitly and

[1] Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Straße 1, D-8046 Garching-bei-München, Germany

0001-7701/93/0100-0095$07.00/0 © 1993 Plenum Publishing Corporation

be compared with previous work.

Recall first the usual approach to finding conditions for trapped surfaces, which is to form an integral of the matter density over a compact spatial 3-surface bounded by a sphere of symmetry S. Various results are known [2–4] which give sufficient conditions or necessary conditions for the trapping of S, usually with other assumptions such as maximal slicing or a moment of time symmetry.

An alternative approach is to measure the amount of matter entering the inward past light-cone through S. By causality, this is the same matter that passes through a Cauchy surface inside S. In fact, it is unnecessary to consider only a single surface S, since the entire light-cone can be treated just as easily. Take an event at the centre of symmetry, and consider its future light-cone. The question is whether the cone becomes trapped, in the sense of containing a future or outer trapped surface [5]. Also of interest is whether the cone is complete or reaches a caustic. Both questions can be answered precisely by a necessary and sufficient condition on an energy integral E of the infalling matter.

The spherically symmetric line-element may be written in terms of null coordinates (ξ, η) as

$$ds^2 = -2e^{-m} d\xi \, d\eta + (A/4\pi) d\Omega^2, \tag{1}$$

where $d\Omega^2$ is a metric on the unit 2-sphere, $A(\xi, \eta)$ is the area of the sphere $S(\xi, \eta)$, and $m(\xi, \eta)$ is the null scaling function. The relevant first-order variables are taken to be

$$\theta = \frac{1}{A}\frac{\partial A}{\partial \xi}, \qquad \tilde{\theta} = \frac{1}{A}\frac{\partial A}{\partial \eta}, \qquad n = \frac{\partial m}{\partial \eta}, \tag{2}$$

where θ and $\tilde{\theta}$ are the expansions in the null directions. The boost freedom $(\xi, \eta) \mapsto (\kappa\xi, \kappa^{-1}\eta)$ is left open.

Consider a light-cone given by constant η, foliated by spheres $S(\xi)$ of area $A(\xi)$, with apex $\xi = 0$. The cone is said to develop a caustic if A becomes zero, other than at the apex. In this case the outward expansion θ becomes infinite. The cone is said to be trapped if θ becomes negative. If the inward expansion $\tilde{\theta}$ is also negative, then the corresponding slice S is a future trapped surface.

Projecting the Einstein equations onto the cone yields

$$2\theta' + \theta^2 + 2\theta m' = -2\phi, \tag{3}$$

$$\tilde{\theta}' + \theta\tilde{\theta} + 4\pi A^{-1} e^{-m} = \rho, \tag{4}$$

$$2n' + \theta\tilde{\theta} + 8\pi A^{-1} e^{-m} = p, \tag{5}$$

where the prime denotes differentiation with respect to ξ, and ϕ is the (η,η) component, ρ the (ξ,η) component and p the (S,S) component of the energy tensor of the matter. Thus ϕ is the infalling energy density and ρ the momentum density as measured on a null curve normal to the cone, and p is the pressure on S. Evolution off the cone depends on the choice of matter and will not be considered here, except to note that for reasonable matter, a solution is expected to exist in a spacetime neighbourhood given $(\phi,\rho,p,m,A,\theta,\tilde{\theta},n)$ as initial data satisfying (3)–(5). In fact, the usual existence proofs [6–8] break down at the apex, but this is presumably a technical problem since no physical problems are expected there.

It will be assumed throughout that (ϕ,ρ,p) are given as integrable functions or distributions on the cone up to any caustic that forms, and that $\phi \geq 0$, which is the null convergence condition, a consequence of the weak energy condition. There are no other assumptions on the matter. It is possible to construct matter models whose dynamical equations may force (ϕ,ρ,p) to become singular on the cone, but such singularities of the matter itself are usually regarded as indicating an inconsistent matter model. For fluid models, the initial data on the cone are free, i.e. there are no matter equations on the cone, and so the assumptions are automatically consistent. It is also expected that for well-behaved fields such as spin fields, the field equations give regular solutions up to any caustic.

The value of m on the cone can be fixed by coordinate choice, corresponding to fixing the functional rescaling of ξ. If m is constant, $\xi = \tau$ is an affine parameter along the rays generating the cone. With this choice it is easy to show that a caustic forms if and only if the expansion θ first becomes negative [5]. Considering first a region where $\phi = 0$, the solution to the focussing equation (3) is $\theta = 2(\tau - \tau_0)^{-1}$, and so if θ is initially positive it will tend to zero asymptotically, and if initially negative it will tend to minus infinity in a finite affine time. If $\phi > 0$, an initially negative θ becomes infinite in a shorter time. However, a different choice of m is needed to solve the focussing equation, and in general it is only demanded that m is continuous on the cone up to any caustic, and vanishes at the apex.

The apex conditions for the cone are

$$\lim_{\xi\to 0}\xi^{-2}A = 2\pi, \qquad \lim_{\xi\to 0}\xi^{2}\phi = 0, \qquad \lim_{\xi\to 0}\xi^{2}\rho = 0, \qquad \lim_{\xi\to 0}\xi^{2}p = 0, \quad (6)$$

where the first condition is geometrical, and the others are weak conditions expressing the regularity of the matter, which may be strengthened if desired. These conditions imply the asymptotic behaviour

$$\lim_{\xi\to 0}\xi\theta = 2, \qquad \lim_{\xi\to 0}\xi m' = 0, \qquad \lim_{\xi\to 0}\xi\tilde{\theta} = -2, \qquad \lim_{\xi\to 0}\xi n = 0, \qquad (7)$$

using (2)–(5) respectively. The focussings f and $\tilde{f}$ are defined by

$$\theta = 2\xi^{-1} - f, \qquad \tilde{\theta} = -2\xi^{-1} - \tilde{f}, \tag{8}$$

and therefore have asymptotic behaviour

$$\lim_{\xi\to 0} \xi f = 0, \qquad \lim_{\xi\to 0} \xi \tilde{f} = 0. \tag{9}$$

The equations (3)–(5) may then be explicitly integrated for $(f, \tilde{f})$ by taking the coordinate choice

$$m' = \tfrac{1}{2} f, \qquad m(0) = 0. \tag{10}$$

This is not an affine choice, and in particular any caustic ($f \to \infty$) that forms is pushed to infinite ξ. This does not cause problems since it is already known that a caustic forms if and only if θ becomes negative. The area follows from (2) as

$$A = 2\pi\xi^2 e^{-2m}, \tag{11}$$

and the equations (3), (4) can be rewritten as

$$(\xi f)' = \xi\phi, \tag{12}$$

$$(\xi^2 e^{-2m} \tilde{f})' = e^{-2m}(2e^m - 2 + 4\xi m' - \xi^2\rho). \tag{13}$$

The solution to the focussing equation (12) is

$$f(\xi) = 2\xi^{-1} E(\xi), \tag{14}$$

where the energy measure E is given by

$$E(\xi) = \tfrac{1}{2} \int_0^\xi \phi(\zeta)\zeta \, d\zeta. \tag{15}$$

Then E is non-negative ($E \geq 0$) and increasing ($E' \geq 0$), and may be interpreted as measuring the energy that has crossed the cone. The expansion follows from (8) as

$$\theta(\xi) = 2\xi^{-1}(1 - E(\xi)), \tag{16}$$

and consequently the cone is trapped if and only if E exceeds 1 at some time.

Completing the solution, m is given by (10) as

$$m(\xi) = \int_0^\xi E(\zeta)\zeta^{-1} d\zeta, \tag{17}$$

so that m is non-negative ($m \geq 0$) and increasing ($m' \geq 0$), and $\tilde{\theta}$ is given by (13) as

$$\tilde{\theta}(\xi) = -2\xi^{-1} - e^{2m(\xi)}\xi^{-2}F(\xi), \tag{18}$$

where

$$F(\xi) = \int_0^\xi e^{-2m(\zeta)}\Big(2(e^{m(\zeta)} - 1) + 4\zeta m'(\zeta) - \zeta^2\rho(\zeta)\Big)d\zeta. \tag{19}$$

If $\rho \leq 0$, then F is non-negative ($F \geq 0$) and increasing ($F' \geq 0$), and so $\tilde{\theta} < 0$, which means that if the cone is trapped then the trapped slices are future trapped surfaces. Finally, the remaining equation (5) has solution

$$n(\xi) = \int_0^\xi \Big(\zeta^{-2}\Big(2 - 2e^{m(\zeta)} - 2E(\zeta) + e^{2m(\zeta)}\zeta^{-1}(1 - E(\zeta))F(\zeta)\Big) + \tfrac{1}{2}p(\zeta)\Big)d\zeta. \tag{20}$$

From the basic assumption that (ϕ, ρ, p) are integrable for all ξ, it follows by construction that $(m, A, \theta, \tilde{\theta}, n)$ exist, and hence it is expected that the metric exists in a spacetime neighbourhood of the cone, up to any caustic. Note that continuity of the metric only requires that E, F and n are integrable to continuous functions, so that (ϕ, ρ, p) may be distributions, which corresponds physically to allowing shocks and shells.

In summary, the cone is untrapped and therefore complete if and only if

$$\lim_{\xi \to \infty} E(\xi) \leq 1, \tag{21}$$

with equality holding for an event horizon, which is asymptotically trapped. Otherwise, the cone is trapped and develops a caustic, which in spherical symmetry means that it recollapses to a point. (Recall that a caustic has vanishing area A, and so has dimension at most one, but that there are no spherically symmetric 1-surfaces.) In either case, the solution on the cone is given explicitly in terms of integrals of the initial data. If (21) holds, then by the definition (15),

$$\lim_{\xi \to \infty} \xi^2\phi(\xi) = 0, \tag{22}$$

so that ϕ must fall off faster than ξ^{-2} if the cone is to be complete. A fall-off condition for the matter is also found for a collapsing cone: if (21) does not hold, then a caustic occurs at affine time $\tau = \tau_0$, and the focussing equation (3) is found to require

$$\lim_{\tau \to \tau_0} (\tau - \tau_0)^2\phi \leq \tfrac{1}{2}, \tag{23}$$

so that ϕ cannot become arbitrarily singular at the caustic. (Thanks are due to Alan Rendall for clarifying this point.) This may be interpreted as meaning that matter cannot be arbitrarily concentrated, even at a singularity, a surprising result which appears to hold more generally.

If (ϕ, ρ, p) vanish outside a compact region, the external solution is the Schwarzschild solution, and the light-cone trapping agrees with the usual pictures [5]. The above result shows that the same picture is obtained for general fall-off conditions on the matter. The strength of the method is that using only a single light-cone eliminates all complications due to the type of matter and internal or external conditions. Quite simply, if enough matter crosses the cone as measured by $E > 1$, then the cone recollapses to a point. Otherwise, the cone expands to infinity. A more precise result has been obtained using light-cones rather than Cauchy surfaces, essentially because the null expansions that define trapped surfaces are adapted to null surfaces. More importantly, the method may be generalised to determine the trapping and completeness of a general non-symmetric light-cone.

ACKNOWLEDGEMENTS

I would like to thank Niall Ó Murchadha, Alan Rendall and Uwe Brauer for their interest and for discussions and suggestions. This research was supported by the European Science Exchange Programme.

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