Mh. Math. 115, 13—26 (1993)

Monatshefte für
Mathematik

Real Moduli of Complex Objects: Surfaces and Bundles

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(Received 10 October 1990; in revised form 30 March 1992)

Abstract. Here we study the real locus (i.e. the fixed locus by the conjugation) of a few moduli spaces (defined over $\boldsymbol{R}$) of complex objects (essentially moduli spaces of surfaces of general type or of vector bundles on curves and $\boldsymbol{P}^2$).

In algebraic geometry (over the complex number field $\boldsymbol{C}$) several classifications problems depend on moduli and one obtains a coarse moduli scheme, M, which, in particular, parametrizes the objects under considerations. Often the set of these objects (hence M) is defined over a subfield of $\boldsymbol{C}$, usually $\boldsymbol{Q}$. In particular M is defined over $\boldsymbol{R}$ and there is the complex conjugation σ acting on M. It is very natural to study the σ-invariant locus $M(\boldsymbol{R})$ of M (at least once it appeared "in nature": the moduli space of real instanton bundles on $\boldsymbol{P}^3$). This study will be done in this paper for many moduli problems. In § 4 and § 5 we take as M the coarse moduli scheme $M(r, c_1, c_2)$ of rank-r stable vector bundles on $\boldsymbol{P}^2$ with Chern classes c_1 and c_2. The main result of this paper is Theorem 5.2, stating that $M(3, c_1, c_2)(\boldsymbol{R})$ is connected (when $M(3, c_1, c_2) \neq \emptyset$). For $M(2, c_1, c_2)(\boldsymbol{R})$ we have only a much weaker result (see 5.3). The more interesting part of 5.2 (and § 4 and § 5) are the proofs, since, thanks to [B], the proofs can give the same results on other rational surfaces, on which we do not have the very fine knowledge of the structure of the moduli space that we have for the case of $\boldsymbol{P}^2$ (thanks to the fundamental works of Drezet and Le Potier). In § 3 we consider the (easier) case of vector bundles on a real curve with a real point. In § 2 we collect a few results (essentially non-emptyness ones) on the real part of the moduli scheme of surfaces of general type with given Chern numbers.

A problem related to the study of the σ-invariant part $M(\boldsymbol{R})$ of a coarse moduli scheme is to know for what $t \in M(\boldsymbol{C})$ the object parametrized by t is defined over Spec $(\boldsymbol{R})$; if M is a fine moduli scheme the two problems are equivalent (see remark 1.1), but not in general, as shown even for the moduli scheme M_g of curves of genus g (see [Se] for a detailed study of this case).

The second version of the paper was much influenced by the reading of [Hu] (in preprint form), [Se], and [Si], and by several conversations with A. Tognoli; thanks to all the mathematicians involved. Part of this work was done while the author was in the warm atmosphere of Max-Planck-Institut für Mathematik in Bonn: thanks!

1. Fix a scheme T. For any scheme A over T (i.e. in the category Schemes/T of schemes with a morphism to T), let h_A: Schemes/$T \to$ $\to$ Sets be the contravariant functor given by $h_A(B) = \mathrm{Mor}_T(B, A)$. Consider a contravariant functor F: Schemes/$T \to$ Sets (or a functor defined on a subcategory $\boldsymbol{A}$ of Schemes/T); a scheme M over T (or in the subcategory $\boldsymbol{A}$) is called a coarse moduli scheme for F if there is a natural transformation τ: $F \to h_M$ such that for every $Y \in \boldsymbol{A}$, every natural transformation $F \to h_Y$ factors uniquely through τ and such that τ induces a bijection when evaluated on the spectrum of any algebraically closed field (restricted to be over T and in $\boldsymbol{A}$, of course). Many moduli functors have a coarse moduli scheme; for instance this is the case if F is representable (i.e. equivalent to some h_M; then M is called a fine moduli scheme) but also in many other natural moduli problems: moduli of curves and of surfaces of general type, moduli of vector bundles. Fix $\boldsymbol{A}$, F and a coarse moduli scheme M for F. Fix a subcategory $\boldsymbol{B}$ of $\boldsymbol{A}$; we will say that M is $\boldsymbol{B}$-fine if the restriction of F to $\boldsymbol{B}$ is represented by the restriction of h_A to $\boldsymbol{B}$. We will say that M is birationally $\boldsymbol{B}$-fine if there is a Zariski open (scheme-theoretically) dense subset U of M such that U is $\boldsymbol{B}$-fine for the "open subfunctor of F counterimage of U by the map $F \to h_M$". We will be interested only in the case $T = \mathrm{Spec}(\boldsymbol{K})$ with $\boldsymbol{K} = \boldsymbol{C}$, $\boldsymbol{R}$, or $\boldsymbol{Q}$, and $\boldsymbol{B} = \mathrm{Spec}(\boldsymbol{R})$; we will say also $\boldsymbol{R}$-fine instead of Schemes/Spec $(\boldsymbol{R})$-fine. The following remark is trivial (it is just the definition of representable functor) and certainly very well-known, but it is very useful: for instance it explains why a curve with real moduli and no non trivial automorphism is defined over $\boldsymbol{R}$ ([Se], Lemma 4.3).

Remark 1.1. Every fine moduli problem is $\boldsymbol{R}$-fine. Every birationally fine moduli problem is birationally $\boldsymbol{R}$-fine; furthermore it is $\boldsymbol{R}$-fine at least over the open subfunctor on which it is fine.

We were lucky and the foundational results (geometric invariant theory [Mu] and the existence of moduli schemes for torsion-free stable sheaves [M]) were proved over any base field of characteristic zero.

A different problem is the determination of the different real objects which become isomorphic after the base change $\operatorname{Spec}(\boldsymbol{C}) \to \operatorname{Spec}(\boldsymbol{R})$. However this problem is vacuous in the cases we are interested in § 3, § 4, and § 5 (vector bundles over a complete variety) for the following reason; fix a proper algebraic scheme $X \to \operatorname{Spec}(\boldsymbol{K})$, $\boldsymbol{K}$ a field, and let $\boldsymbol{L}$ be an extension field of $\boldsymbol{K}$; let F be a coherent sheaf on X; let X_L and F_L be the objects obtained by base change $\operatorname{Spec}(\boldsymbol{L}) \to \operatorname{Spec}(\boldsymbol{K})$; by ([Ha], p. 255) (and an inductive limit as in E.G.A. IV, § 8, if $\boldsymbol{L}$ is not finite over $\boldsymbol{K}$) $H^i(X_L, F_L) \cong H^i(X, F) \otimes_K \boldsymbol{L}$; for any two sheaves A, B on X, apply this formula for $i = 0$ and $F = \operatorname{Hom}(A, B)$ to show that if A_L and B_L are isomorphic, then A and B are isomorphic.

In this paper σ denotes the action of any antiholomorphic involution induced on any scheme M over $\boldsymbol{C}$ (and on $M(\boldsymbol{C})$, too) by the complex conjugation of $\boldsymbol{C}$. A scheme Y over $\operatorname{Spec}(\boldsymbol{C})$ is called real if it comes by base change from a scheme, Y', defined over $\operatorname{Spec}(\boldsymbol{R})$; thus on a real scheme Y there is an antiholomorphic involution σ and a sheaf F over Y is called σ-invariant if $F \cong \sigma^*(F)$, while it is called real if it comes from a sheaf on Y' defined over $\operatorname{Spec}(\boldsymbol{R})$.

2. In this section we will consider moduli of surfaces (from the point of view of $\operatorname{Spec}(\boldsymbol{R})$). Fix a smooth complete surface X (over $\boldsymbol{C}$). We use the standard notations $\chi(X) := \chi(\boldsymbol{O}_X)$, $c_2(X)$ and $c_1^2(X)$ for its numerical invariants. By Noether's formula, the invariants $\chi(X)$, $c_1^2(X)$ and $c_2(X)$ are linked by: $12\chi(X) = c_1^2(X) + c_2(X)$. One of the main problems in surface theory is to know when a pair (c_1^2, c_2) occurs as pair of Chern classes of minimal surfaces of general type. Furthermore, for such (c_1^2, c_2) there is a quasiprojective coarse moduli scheme $M(c_1^2, c_2)$, which is defined over $\boldsymbol{Q}$; it is often reducible (see [C1]), not reduced and/or disconnected; anyway it has the conjugation σ. In the extremal case $c_2 > 0$, $c_1^2 = 3c_2$ (i.e. when the Miyaoka-Yau inequality is an equality) $M(c_1^2, c_2)_{\mathrm{red}}$ consists of finitely many points which are permuted by σ; hence some of them will (perhaps) be σ-invariant while the others will be divided into conjugate pairs; here it is very natural to ask when there is a σ-invariant point.

Proposition 2.1. *For all integers x, y, with $2x - 6 \leqslant y \leqslant 8x - 126$, there exists a minimal real surface X of general type with $c_1^2(X) = y$, $\chi(X) = x$.*

For 2.1 it is sufficient to note two very easy facts. First, the constructions of [P] made to prove ([P], Prop. 3.19, 3.23, and Th. 2′) can be done over $\boldsymbol{R}$. In [P] double coverings and genus 2 fibrations are used. To obtain genus 2 fibrations over $\boldsymbol{R}$ the results of [P] must be modified only very slightly. The second fact is that the few cases left open in [P] and covered in [X] can be done easily over $\boldsymbol{R}$, too. The content of 2.1 was remarked also in the introduction to [Si]. Now we review (following the exposition in [Hu]) other known methods to construct surfaces of general type and state when the methods work over $\boldsymbol{R}$. SOMMESE's method [So] (again essentially coverings plus fibre products) works verbatim over $\boldsymbol{R}$ (with a good choice of real curves, real ramification points, ...); see also the proof of the corresponding result in ([Hu], 5.2.4). Thus SOMMESE's method gives the following result.

Proposition 2.2. *For all rational numbers r with $1/5 \leqslant r \leqslant 3$, there is a minimal surface of general type X defined over $\boldsymbol{R}$ with $c_1^2(X)/\chi(X) = r$.*

To obtain the full force of 2.2, by the proof of ([Hu], 5.2.4) it is sufficient to have over $\boldsymbol{R}$ two suitable surfaces S, S' with $c_1(S)^2 = 3c_2(S)$, $5c_1(S')^2 = c_2(S')$, S and S' fibered over $\boldsymbol{R}$ over a curve of genus > 1. The surface S is the surface H, constructed (see [H]) as *real* arrangement of lines; this surface is explained in detail in ([So], § 1); also the 4 fibrations of S over a curve of genus 6 are defined over $\boldsymbol{R}$ since they come as Stein factorization essentially from the projection of $\boldsymbol{P}^2$ onto $\boldsymbol{P}^1$ from a real point. A surface S' (with $c_1^2(S') = 12$, $c_2(S') = 60$) is constructed over $\boldsymbol{R}$ in the second part of the proof of ([So], Th. 2.3). The existence (over $\boldsymbol{R}$) of the surface S is interesting for the discussion given before 2.1.

Another method (due to MIYAOKA [Mi]; see ([Hu], Part 1, §2)) gives interesting surfaces over $\boldsymbol{R}$ (one can make the projections over $\boldsymbol{R}$, hence the branch curve and the associated Galois covering are defined over $\boldsymbol{R}$).

The more powerful method of construction is due to Xiao (see [Ch]); the construction of [Ch] depends upon the choice of a finite group G, $G \subset SO(3) \subset PGL(2, \boldsymbol{C})$, G the symmetry group of a platonic solid; when G is invariant under conjugation (octahedral group O_{24}) everything is defined over $\boldsymbol{R}$; but apparently the construction is not real in the other cases (e.g. the icosahedral group I_{60}); thus I do not know the existence over $\boldsymbol{R}$ of surfaces with all the Chern invariants considered in [Ch].

In all the cases considered, it is easy to construct real surfaces whose real part has various numbers of connected components (indeed the number can be unbounded, increasing c_2); for certain invariants c_i, this is however obvious in some cases simply taking as surfaces the product of curves of suitable real part. Other *real* constructions; are the bidouble covers of (C1] and the case of regular surfaces ($q = 0$) of the fundamental paper [C] (extended to certain higher dimensional projective varieties in [C2]). In [C] it is given over $\boldsymbol{R}$ a parameter space for all regular (i.e. $q = 0$) surfaces with given Chern numbers; in a few cases (e.g. [CD]) it is possible to describe this parameter space and deduce the number of irreducible components, their unirationality (over $\boldsymbol{R}$ and even over $\mathrm{Spec}(\boldsymbol{Q})$), the density (both in the Zariski topology and in the transcendental topology) of the real surfaces, and so on.

3. In this section we will consider the case of coherent sheaves and vector bundles on real (i.e. defined over $\mathrm{Spec}(\boldsymbol{R})$) curves. Fix a smooth complete real curve C. The case in which the curve C has a real point (i.e. a point with $\boldsymbol{R}$ as quotient field) is easy.

Theorem 3.1. *Assume that the real curve C has a real point. Then every σ-invariant coherent sheaf on C is defined over $\boldsymbol{R}$.*

Proof. Fix a σ-invariant sheaf F and use induction on the rank r of F. First assume $r = 0$, i.e. F torsion. Decomposing $\mathrm{Supp}(F)$, we may assume that $\mathrm{Supp}(F)$ is either a real point of C or a pair $P = \{a, \sigma(a)\}$ of conjugate points; just to fix the notations, we assume the latter case. Let $\boldsymbol{O}$ be the semilocal ring of P and $\boldsymbol{m}$ the intersection of its 2 maximal ideals. By uniquess the $\boldsymbol{m}$-filtration $0 \subset \boldsymbol{m}^k F \subset \boldsymbol{m}^{k-1} F \subset \dots \boldsymbol{m} F \subset F$ is σ-invariant and its graded pieces are just 2 conjugate copies of vector spaces on a and $\sigma(a)$ with trivial multiplication (hence defined over $\boldsymbol{R}$). By the σ-invariance of the extension of $\boldsymbol{m}^{t-1} F$ by $\boldsymbol{m}^{t-1} F/\boldsymbol{m}^t F$ inducing $\boldsymbol{m}^t F$ in the vector space defined over $\boldsymbol{R}$ of all such extensions, we get that F is real.

Now assume $r > 0$. Since C is smooth, we have $F \cong F' \oplus F''$ with F' locally free and F'' torsion; by the uniqueness of the decomposition (Krull-Remak-Schmidt theorem) and the case $r = 0$ just done, we may assume F locally free. Up to a twist with a finite number of sufficiently general real points of C, we may assume that $0 < h^0(F) \leqslant r$; $H^0(F)$

spans a subsheaf G of F with rank $(G) \leqslant r$; G is σ-invariant, hence so is F/G and the following exact sequence

$$0 \to G \to F \to F/G \to 0 \tag{1}$$

is σ-invariant. If rank $(G) < r$, we get rank $(F/G) < r$, G and F/G real; since the set of extensions as (1) comes from the vector space with real structure $\mathrm{Ext}^1(F/G, G)$, we see that the σ-invariant extension giving F is real; thus F is real. If rank $(G) = r$, then G is the trivial rank-r vector bundles, hence real, and G is torsion, hence real. Again we see that F is real. ■

Fix a smooth projective curve C of genus $g \geqslant 2$. For any $L \in \mathrm{Pic}(C)$, we denote by $M(r, L)$ the moduli space of rank-r stable vector bundles on C with determinant isomorphic to L; although C does not appear in this notations, this abuse of notations should give no trouble. It is well-known that $M(r, L)$ is smooth, integral and of dimension $(r^2 - 1)(g - 1)$. In particular if C and L are defined over $\boldsymbol{R}$, then $M(r, L)(\boldsymbol{R})$ is smooth and pure-dimensional (see e.g. [Si], Ch. I (1.14)). If r and $\deg(L)$ are coprime, $M(r, L)$ is a fine moduli space and it is complete. However this is not the case without the assumption on r and $\deg(L)$; for any r, L we denote by $S(r, L)$ its natural compactification by equivalence classes of semistable sheaves (see 3.3 for more details when $r = 2$) and set $B(r, L) := S(r, L) \setminus M(r, L)$. Now we want to give a geometric proof of the following result.

Proposition 3.2. *Fix a real curve C of genus $g \geqslant 6$ with a real point and $L \in \mathrm{Pic}(C)(\boldsymbol{R})$; Then $M(2, L)(\boldsymbol{R})$ is connected.*

Proof. Since $M(2, L)$ is smooth and pure-dimensional, it is sufficient to find a Zariski open dense subset Ω of $M(2, L)$, Ω defined over $\boldsymbol{R}$, such that $\Omega(\boldsymbol{R})$ is connected. Fix a general $E \in M(2, L)$. We distinguish two cases according to the parity of d.

(a) Assume d odd. Up to a twist by a multiple of a real point of C, we may assume $d = 2g - 1$, i.e. by Riemann-Roch $\chi(E) = 1$. By the generality of E we have $h^0(E) = 1$ while $h^0(E(-P)) = 0$ for every $P \in C$; this the only "generality assumption" we will use. Fix $s \in H^0(E)$, $s \neq 0$. By the assumption on $h^0(E(-P))$, s gives the exact sequence

$$0 \to \boldsymbol{O}_C \to E \to L \to 0. \tag{2}$$

Since L is real, the vector space $V := \mathrm{Ext}^1(L, \boldsymbol{O}_C)$ is the complexification of a real vector space. V has dimension $3g - 2$. Let U be the

Zariski dense open subset of V parametrizing stable bundles. Since $M(2, L)(\boldsymbol{R})$ is smooth and pure-dimensional, it is sufficient to check that $V \setminus U$ has codimension > 1 in V. Fix E' fitting in (2) and assume that it has a subline bundle M with $\deg(M) \geqslant g$; we get the exact sequence

$$0 \to M \to E' \to L \otimes M^{-1} \to 0. \tag{3}$$

Note that $h^1(M^{\otimes 2} \otimes L^{-1}) = h^0(K_C \otimes L \otimes (M^{\otimes 2})^{-1}) \leqslant g - 1$ with equality if and only if $\deg(M) = g$ and $h^0(M^{\otimes 2} \otimes L^{-1}) > 0$; thus each M gives at most a $g - 1$ dimensional family of non isomorphic bundles which can be fitted in (3). Now to see how many times each E' in (3) can give a bundle E fitting in (2), we have only to bound $h^0(E')$. Set $x := \deg(M)$.

(a1) First we will assume $\deg(L \otimes M^{-1}) \geqslant 2$. Given $L \otimes M^{-1}$, M is uniquely determined and by Clifford's theorem $h^0(L) \leqslant 1 + (x/2)$; by Marten's theorem ([A4], Ch. 4, Th. 5.1), the set W of $L \otimes M^{-1}$ with $h^0(L \otimes M^{-1}) = k + 1 \geqslant 2$ has dimension at most $d - 2k - x$. Thus we conclude.

(a2) Now assume $\deg(L \otimes M^{-1}) < 0$. Thus $h^1(M^{\otimes 2} \otimes L^{-1}) = 0$. Hence (3) splits. Since $h^0(L \otimes M^{-1}) = 0$, every section of E' vanishes at least at a point of C. Hence E' cannot fit in the exact sequence (2).

(a3) Now assume $0 \leqslant \deg(L \otimes M^{-1}) \leqslant 1$. Since $h^0(M) \leqslant g$, $h^0((L \otimes M^{-1}) \leqslant 1$, and $h^1(M^{\otimes 2} \otimes L^{-1}) \leqslant 2$, we conclude.

(b) Now assume d even. Twisting by a real point we may assume $d = 2g$, i.e. $\chi(E) = 2$; again for general E we have $h^0(E) = 2$ and that $H^0(E)$ defines an exact sequence

$$0 \to 2\boldsymbol{O} \to E \to \boldsymbol{O}_D \to 0 \tag{4}$$

for some positive divisor D with $|D| = L$. For fixed D, the set of extensions now has dimension $4g$ and again we conclude (of course, if $\deg(M) = g$, we note that $h^0(M^{\otimes 2} \otimes L^{-1}) \leqslant g - 1$, except for the finitely many M with $M^{\otimes 2} \cong L$, and apply again Marten's theorem to $K \otimes (L \otimes M^{-1})^{-1}$). ■

3.3. Here we will check what happens at the boundary $B(2, d)$ of $M(2, d)$ when d is even (without assuming that C has a real point). Each element of the boundary represents an equivalence class of semistable bundles (each equivalence class being uniquely determined by the graduation associated to the Harder-Narasimhan filtration). Fix an equivalence class $T \in B(2, d)(\boldsymbol{R})$. By the construction of the moduli space of semi-stable bundles as a quotient using geometric invariant theory, we know that the decomposable bundle in T, say $M \oplus N \in T$, must be σ-invariant; we have $\deg(M) = \deg(N)$.

First assume M not isomorphic to N. Then, by the Krull-Remak-Schmidt uniqueness of decomposition, N is the conjugate $\sigma(M)$ of N; to check that $M \oplus \sigma(M)$ is defined over $\boldsymbol{R}$, make the same coordinate change as the one giving the isomorphism of $SO(2, \boldsymbol{R})$ and $U(1, \boldsymbol{C})$. We note that no other bundle $F \in T$ is real; indeed F is given by a non-split extension of M and N; by the uniqueness of such an extension if F is real, then so in M, hence N.

Now assume $M \cong N$. If M is real $M \oplus M$ is real (as well many other bundles in the same class: those given by real extensions of M by itself). If M is not real, $M \otimes M$ cannot be isomorphic to $\sigma(M) \oplus \sigma(M)$ by the Krull-Remak-Schmidt theorem. ■

Remark 3.4. If C has no real point, there are some problems (both for 3.1 and 3.2). To carry over the proof of 3.2 and obtain also that every stable σ-invariant rank-2 bundle of even degree is the limit of a family of real ones, there is only one problem: the twisting to reduce to the case $\deg(L) = 2g$; indeed twisting $E \in M(r, L)$ by a multiple of a pair of conjugate points change the degree of E by multiples of $2r$. Thus the proof of 3.2 works only if $\deg(L) - 2g$ is divisible by 4. For 3.1 we have an analogous problem with twistings; for 3.1 this problem is aggravated by the inductive proof of 3.1.

4. In this section and the next one we will consider the case of the moduli space $M(r, c_1, c_2)$ of stable rank-2 vector bundles on $\boldsymbol{P}^2$ with Chern classes $c_i's$. A very similar study leading to very similar results could be done in the case of other rational surfaces and higher rank, in particular certainly in the setting of [B] (i.e. either very particular rational surfaces (e.g. with low Picard number) or any rational surface but very good polarization). In certain cases a similar problem can be handled for suitable strata of "natural" stratifications of the moduli space (for the extension from [B] to the "stratification" case (over $\boldsymbol{C}$) see [BB]). Set $S = \boldsymbol{P}^2$, $\boldsymbol{O} = \boldsymbol{O}_S$, and $L := c_1$; for every closed subscheme Z of S, let $\boldsymbol{I}_Z$ be its ideal sheaf; furthermore, we will often identify Chern classes on $\boldsymbol{P}^2$ and integers; the identification of $\mathrm{Pic}(S)$ and $\boldsymbol{Z}$ we choose sends the line bundle $\boldsymbol{O}(t)$ to its degree, t. Let $M(r, c_1, c_2)$ be the moduli scheme of rank-r stable vector bundles on $\boldsymbol{P}$ with Chern classes

c_i. It is known (see [E]) that $M(r, c_1, c_2)$ is irreducible (when it is not empty). We want to stress that the main point of the paper is the proof, not the statement, of 5.2 (and the related proofs of 4.1 and 5.3); indeed on $\boldsymbol{P}^2$, thanks to fundamental works of Drezet and Le Potier, we have a very fine knowledge of the cohomology of $M(r, c_1, c_2)$, which however is still missing for other rational surfaces; the method chosen for the proofs in § 4 and § 5, thanks to [B], gives immediately the same informations for certain moduli spaces of vector bundles on other rational surfaces; exploiting the full force of the elementary transformations used in [B] something can be said also for the other moduli spaces on the same surfaces (certainly for all moduli spaces when the rank is 2).

Theorem 4.1. *For all c_1, c_2 with $M(2, c_1, c_2) \neq \emptyset$, $M(2, c_1, c_2)(\boldsymbol{R})$ is the closure of the set of real bundles.*

Proof. If $M(2, c_1, c_2)$ is a fine moduli space (it is known that this is the case if either c_1 is odd or c_1 is even and $c_1^2 - 4c_2 \equiv 0 \bmod (8)$) this is of course contained in 1.1, but we will not use this remark. Fix c_1, c_2. Set $M := M(2, c_1, c_2)$. Since M is smooth and irreducible, $M(\boldsymbol{R})$ is a smooth equidimensional differential manifold (of dimension $\dim(M)$ or empty) ([Si], Ch. I, 1.14); hence we may replace M by any of its Zariski dense σ-invariant open subsets. Take any rank 2 vector bundle W on $\boldsymbol{P}^2$ and any integer t. Note that $\deg(c_1(F(t)) = \deg(c_1(W)) + 2t$ and $c_2(W(t)) = c_2(W) + t(\deg(c_1(W)) + t^2$. Hence the natural transformation which sends W into $W(t)$ induces an isomorphism between $M(2, c_1, c_2)$ and $M(2, c_1 + 2t, c_2 + tc_1 + t^2)$. Hence, up to a twist, we may assume $\chi(E) > 0$ and $\chi(E(-1)) \leqslant 0$ for any $E \in M(2, c_1, c_2)$; to check this triviality one can use for instance the vanishing theorem in the form introduced by Serre or the very useful explict formula $\chi(E) = 1 - c_2 + ((c_1 + 1)(c_1 + 2)/2)$ for Riemann-Roch Theorem (see e.g. [Br], just before Th. 5.1)). The essential content of the proof of ([Br], Th. 5.1), is that with this normalization, a general $E \in M$ fits in an exact sequence

$$0 \to \boldsymbol{O} \to E \to \boldsymbol{I}_T(c_1) \to 0, \tag{5}$$

in which T is a general subset of S with $\mathrm{card}(T) = c_2$ and that, viceversa, given any "sufficiently general" $T \subset S$ with $\mathrm{card}(T) = c_2$ there is a stable bundle E with c_i as Chern classes, E fitting in the exact sequence (5); here as meaning of "sufficiently general" one can take "with $h^0(S, \boldsymbol{I}_T(c_1 - 1)) = 0$"; this condition is satisfied for T general in

$\boldsymbol{P}^2$ (hence for T general in $\boldsymbol{P}^2(\boldsymbol{R})$) by the cohomology of line bundles on $\boldsymbol{P}^2$, Riemann-Roch Formula and a theorem of Barth which states that $c_1^2 - 4c_2 < 0$ if $M \neq \emptyset$. We will return on the proof of [Br], Th. 5.1) in part (b) below. Fix a general $E \in M$. By the statement of ([Br], Th. 5.1) (or just the exact sequence (5)) we have $h^1(E) = h^0(E(-1)) = 0$ and this is true for all E in a Zariski open set M' of M. Set $2t + e = \chi(E)$ with $t \geqslant 0$ and $1 \leqslant e \leqslant 2$. Fix a general subset A of $\boldsymbol{P}^2(\boldsymbol{R})$ with $\mathrm{card}(S) = t$. By the exact sequence (5), the discussion made around (5) and the fact that $\boldsymbol{P}^2(\boldsymbol{R})$ is Zariski dense in $\boldsymbol{P}^2(\boldsymbol{C})$, we have the existence of a Zariski open dense subset M'' of M' such that for every $E \in M''$, $h^0(E \otimes \boldsymbol{I}_A) = e$.

(a) Assume $e = 1$. Fix $s \in H^0(E \otimes \boldsymbol{I}_A)$, $s \neq 0$. Restricting if necessary M'', we see (as in the constructive proof of [Br], Th. 5.1): start with the zero-locus $(s)_0$ with $A \subset (s)_0$ and built E as an extension) that the zero-locus $T := (s)_0$ is reduced (hence $\mathrm{card}(T) = c_2$), and that for general E we can take T general. Thus s induces an exact sequence Since $h^0(E \otimes \boldsymbol{I}_A) = 1$, (5) is uniquely determined; thus if E is σ-invariant, so is (5) and T. As in the proof of 3.1, we get that E is real.

(b) Assume $e = 2$. Fix a basis s, s' of $H^0(E \otimes \boldsymbol{I}_A)$. For a "definitive" discussion on the Cayley-Bacharach theorem that we will use below, see [C3], which generalizes and states carefully the results proved in [GH1] and ([GH2], pp. 726—731); just after [GH1] the theorem was proved independently by J. BRUN (essentially it is in ([Br], § 1) for reduced 0-loci) and H. Lange. We claim that for general E the sections s and s' have no common zero; indeed s gives (5); fix a common zero $P \in T$; by the proofs in ([Br], § 5), (i.e. essentially by the Cayley-Bacharach property and the negativity of the canonical bundle of $\boldsymbol{P}^2$) we can move freely T (in a Zariski open set of the symmetric product of c copies of $\boldsymbol{P}^2$) and get always a bundle; since $\boldsymbol{P}^2$ is irreducible, we would get that for general E we have $(s)_0 = (s')_0$; it is easy to find bundles for which this is not true (e.g. as an extension like (6) below). Thus s and s' are linearly independent except on a divisor, D, with $\deg(D) = c_1$, and at each point of D one of them is not 0. Thus $H^0(E \otimes \boldsymbol{I}_A)$ induces the following exact sequence

$$0 \to 2\boldsymbol{O} \to E \to \boldsymbol{O}_D \to 0, \tag{6}$$

which is uniquely determined by E (since it does not depend, with the more invariant notation "$H^0(E \otimes \boldsymbol{I}_A) \otimes \boldsymbol{O}$" instead of "$2\boldsymbol{O}$" from the choice of s and s'). Hence if E is σ-invariant, so are (6) and D. Again we find that E is real. ■

5. Again we use the notations of § 4 (the surface is $\boldsymbol{P}^2$ and we write: $M(r, c_1, c_2)$, $\boldsymbol{O}$, $\boldsymbol{I}_Z$). Now we consider $M(3, c_1, c_2)$. Since [Br] was written only for rank-2 stable bundles, we will write (see 5.1) the rank-3 case. First in 5.1 we will construct a good family of bundles for every rank r. The same construction was done (in a more general setting) in joint work with R. BRUSSEE ([BB]); we include it here in (5.1) to fix the notations, to show that in the case of $\boldsymbol{P}^2$ explicit good bounds for c_2 come easily from the construction, and to show that indeed we get in this way when $r = 3$ many rank 3 bundles with "high order of stability"; moreover, working on $\boldsymbol{P}^2$ it is also easy to give a quantitative description of the meaning of the words "high order of stability".

5.1. Fix integers $r \geqslant 2$ and n, (r = rank, $n = c_1$). Set $L = \boldsymbol{O}(n)$ with $n \geqslant 2$. Fix an integer x such that $(n(n+1)/2) - (r-2) \leqslant x \leqslant$ $\leqslant (n+2)(n+1)/2$ and $Z \subset \boldsymbol{P}^2$, $\operatorname{card}(Z) = x$; if $n \leqslant 5$, assume $x \geqslant n(n+1)/2$. Consider an exact sequence

$$0 \to (r-1)\,\boldsymbol{O} \to E \to \boldsymbol{I}_Z(n) \to 0. \tag{7}$$

By semicontinuity, $\dim(\operatorname{Ext}^1(\boldsymbol{I}_Z, (r-1), \boldsymbol{O})$ is constant for Z in a Zariski open subset U of $S^x(\boldsymbol{P}^2)$. Thus over U the relative Ext^1 of [BPS] is a vector bundle V over U, and we get a family of torsion-free rank r sheaves on X parametrized by V. Since $H^0(\boldsymbol{I}_Z(n-3)) = 0$ for general x by the choice of x, a general such extension gives a vector bundle ("Cayley-Bacharach property": same proof as in [C3] or ([GH2], pp. 726—731), for the case $r = 2$).

(a) First we check that if E is given by (7) and is locally free, then its local deformation space is smooth of dimension $h^1(E \otimes E^*)$. By local deformation theory it is sufficient to check that $h^2(E \otimes E^*) = 0$. Tensoring (7) by E^*, we see that it is sufficient to check that $h^2(E^*) =$ $= h^2(E^*(n)) = 0$; by Serre duality it is sufficient to check that $h^0(E(-3)) = 0$; this follows from the assumptions on x and Z, and from (7). The proof uses only that $p_a(\boldsymbol{P}^2) = 0$, plus that $h^0(L \otimes K^{-1}) > 0$.

(b) Now we check that if $r = 3$ the general such bundle E is stable (in general for $r > 3$ one gets only that E is not destabilized by subsheaves of rank 1 or $r-1$; this seems an interesting notion: it shows when the twists of E or its dual have sections). Fix line bundles M, N on $\boldsymbol{P}^2$. By (7) we get at once that M is not a subsheaf of E if $\deg(M) \geqslant n/3$ (and indeed if $\deg(M) > 1$ when $n \geqslant 6$). Dualizing (7) we get

$$0 \to \boldsymbol{O}(-n) \to E \to 2\,\boldsymbol{O} \to \boldsymbol{O}_Z \to 0. \tag{8}$$

If the map $2\boldsymbol{O} \to \boldsymbol{O}_Z$ in (8) is sufficiently general (i.e. if E is general) we see that E cannot have any $N \in \mathrm{Pic}(\boldsymbol{P}^2)$ with $2h^0(N^*) \leqslant x$ as a subsheaf.

(c) We record here that we proved in (b) that for a general E as in (7), $h^0(E(-2)) = 0$ and $h^0(E(-1)) = 0$ if $x \geqslant n(n+1)/2$. Furthermore by construction the dependency locus of $r-1$ sections of E can be as general as we need. ■

Theorem 5.2. *For all c_1, c_2 with $M(3, c_1, c_2) \neq \emptyset$ and $c_1 \geqslant 6$, $M(3, c_1, c_2)(\boldsymbol{R})$ is connected.*

Proof. Set $M = M(3, c_1, c_2)$. By Riemann-Roch (whose explicit formula can be easily found using (7)) we may assume, up to a twist, $\chi(E(-1)) \leqslant 1$ and $\chi(E) \geqslant 2$, i.e. that $x := c_2$ and $n := c_1$ satisfy the numerical assumptions of 5.1. Since M is smooth, as in the proof of 4.2, it is sufficient to check the connectedness of $U(\boldsymbol{R})$ for a Zariski open (not empty) subset of M. Furthermore it would be sufficient to check the connectedness of the real part of some space V' with a dominant map $V' \to M$. As a very good approximation we will consider the family given in 5.1. Even if on no Zariski open non empty subset of M there is a universal family of bundles (i.e. even if M is not birationally fine), the use of suitable V' allows us to work with families of bundles.

(1) Fix Z as in 5.1, Z sufficiently general, Z σ-invariant (i.e. real since the Hilbert scheme is a representable functor). Consider the family V of stable vector bundles given by (7). The set of extensions of type (7) is a complex vector space W defined over $\boldsymbol{R}$; $W \neq \emptyset$ by 5.1(a). In W by Cayley-Bacharach (5.1(b)) the set of bundles is the complements of finitely many linear spaces (whose union is σ-invariant); each of these linear spaces has codimension 2. Each extension given by (2) has $h^0(E(-1)) \leqslant 1$ and $h^0(E(-2)) = 0$, i.e. one half of the condition of stability comes free (and in a very strong form). Again by the proof of 5.1(a), the other half of the stability condition holds outside codimension 2. Thus the real part of the corresponding family of bundles is connected.

(2) Now we move Z in the Zariski open subset of Hilb(X) on which all the conditions used in part (1) are satisfied. The real part of the corresponding family V of bundles is not connected; its connected components are given exactly fixing the number of points of $Z \cap \boldsymbol{P}^2(\boldsymbol{R})$. Since $M(\boldsymbol{R})$ is smooth and equidimensional, it is sufficient to do the following trick. Consider two connected components A, A' of $V(\boldsymbol{R})$, one corresponding to all subsets $Z's$ with, say, t real points, and the other corresponding to all subsets $Z's$ with $t+2$ real points. We may deform a family of $Z's$ in A and a family of points in A' to the same configuration Z'', with Z'' given by $(c_2 - t - 2)/2$ pairs of

conjugate points (sufficiently general), t (sufficiently general points) of $X(\boldsymbol{R})$ and a double point with support at a point of $X(\boldsymbol{R})$ and real tangent line. We check the dimension of the vector space of extensions as in (7) for Z'' and we get the same answer as for a general Z. Furthermore the same calculations as in 5.1 give that the general such extension corresponds to a stable bundle; for rank 2 one can find the correct extension of the Cayley-Bacharach property to unreduced subschemes like Z'' in [C3] (or for this particular case in [BFS]); a similar proof works for rank 3. By the theory of relative Ext [BPS] we get a family of sheaves parametrized by a vector bundle over a Zariski open subset of $\mathrm{Hilb}(\boldsymbol{P}^2)$ containing Z''. Thus the closure of the images of the various components of $V(\boldsymbol{R})$ is connected. ∎

Proposition 5.3. *For all c_1, c_2 with $M(2, c_1, c_2) \neq \emptyset$, $M(2, c_1, c_2)(\boldsymbol{R})$ is semialgebraic and its closure in the complete moduli space of semistable rank-2 torsion-free sheaves with Chern classes c_i is connected.*

Proof. The proof is very similar to (but easier than) the proof of 5.2. Here we set $n := c_1$, $x := c_2$. Up to a twist we may assume $n(n+1)/2 < x \leqslant (n+1)(n+2)/2$. We may repeat the proof of 5.2, quoting ([Br], Th. 5.1), instead of our 5.1. The thing which fails now is that the condition that E is locally free at a real point of Z holds only outside a hyperplane; thus (with the notations of the proof of 5.2) the only component of $V(\boldsymbol{R})$ which now gives no trouble is the one (existing only if x is even) in which Z has no real point: over such Z the bad set of extensions, as union of intersections of pairs of conjugate (not real) hyperplanes has codimension two in the set of all real extensions. For the other subschemes Z there is no problem if we allow sheaves with mild singularities; in rank 2 the stability condition is always satisfied by every sheaf fitting in (7) (with $r = 2$). ∎

Note that the proof of 5.3 shows that $M(2, c_1, c_2)(\boldsymbol{R})$ may be made connected adding only stable torsion-free sheaves with very mild singularities. We do not know if $M(2, c_1, c_2)(\boldsymbol{R})$ is connected.

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