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Non-constant positive steady-states of a diffusive predator–prey system in homogeneous environment

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Abstract

In this paper, we investigate the existence and non-existence of non-constant positive steady-states of a diffusive predator–prey interaction system under homogeneous Neumann boundary condition. In homogeneous environment, we show that the predator–prey model with Leslie–Gower functional response has no non-constant positive solution, but the system with a general functional response may have at least one non-constant positive steady-state under some conditions.

Keywords: Non-constant positive solution; Locally/globally asymptotically stable; Functional response

1. Introduction

In this paper, we study the following diffusive predator–prey system of Holling–Tanner type:

$$\begin{cases} u_t - d_1\Delta u = ug(u) - p(u)v, & \\ v_t - d_2\Delta v = v\left(\delta - \beta\dfrac{v}{u}\right) & \text{in } (0,\infty)\times\Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; the given coefficients δ, β, d_1 and d_2 are positive constants; ν is the outward directional derivative normal to $\partial\Omega$. Moreover, the C^1-functions $g(u)$ and $p(u)$ are assumed to satisfy the following hypotheses throughout this paper:

(H1) There exist positive constants K and $\tilde{g}$ such that $g(K) = 0$ and $g_u(u) \leqslant -\tilde{g}$ for all $u > 0$.
(H2) $p(0) = 0$, and there exists a positive constant M such that $0 < p_u(u) \leqslant M$ for all $u > 0$.

In the system (1.1), u and v represent the densities of prey and predator in the spatial region Ω; δ stands for an intrinsic growth rate of predator v; β/δ is the number of prey required to support one predator; and the carrying capacity of predator is proportional to the densities of prey.

The following corresponding ODE system to (1.1)

$$\begin{cases} x_t = xg(x) - p(x)y, & x(0) > 0, \\ y_t = y\left(\delta - \beta \dfrac{y}{x}\right), & y(0) > 0, \end{cases}$$

can be classified into four types depending on the functional response $p(x)$ when $g(x) = 1 - x$:

$$\begin{aligned} &\text{Type 1:} \quad p(x) = x, \\ &\text{Type 2:} \quad p(x) = \frac{x}{x+a}, \\ &\text{Type 3:} \quad p(x) = \frac{x^2}{(a+x)(b+x)}, \\ &\text{Type 4:} \quad p(x) = 1 - e^{-ax}, \end{aligned}$$

where a and b are positive constants. Types 1–4 are respectively called Leslie–Gower, Holling–Tanner, the sigmoidal and Ivlev functional response [4,5,12]. Note that Types 1–4 satisfy hypotheses (H1) and (H2).

In [1], Y. Du and S.B. Hsu considered the following diffusive predator–prey model with Leslie–Gower functional response (Type 1):

$$\begin{cases} u_t - d_1 \Delta u = u[\lambda - \alpha u - \beta v], & \\ v_t - d_2 \Delta v = v\mu\left(1 - \dfrac{v}{u}\right) & \text{in } (0,\infty) \times \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega. \end{cases} \tag{LG}$$

They showed that the steady-states of (LG) have no non-constant positive solution when the given coefficients λ, α, β and μ are all constants (i.e., in homogeneous environment) and satisfy suitable conditions, while a non-constant positive solution can be created when the species concentrate on some region of spatial habitat Ω (i.e., in heterogeneous environment). In details, by choosing a suitable coefficient function which vanishes in a subdomain of Ω, they showed that certain patterned solutions can be obtained in heterogeneous environment.

In [10], the system (1.1) with Holling–Tanner functional response (Type 2) was considered and the existence and non-existence of non-constant positive steady-states in homogeneous environment were studied.

The main part of this article is concerned with the positive steady-states of (1.1), i.e., we investigate the existence and non-existence of non-constant positive solutions to the following elliptic system in homogeneous environment:

$$\begin{cases} -d_1\Delta u = ug(u) - p(u)v, & \\ -d_2\Delta v = v\left(\delta - \beta\dfrac{v}{u}\right) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In view of our main results, the predator–prey model (1.2) with Leslie–Gower functional response (Type 1) has no non-constant positive solution as in [1], but the other type predator–prey models may have at least one non-constant positive steady-state under some conditions. Note that (1.1), and so (1.2), has a unique constant positive equilibrium point $\mathbf{e}_* = (u_*, v_*)$ under the assumptions (H1) and (H2), where $g(u_*) = \frac{\delta}{\beta}p(u_*)$ and $v_* = \frac{\delta}{\beta}u_*$. In fact, one can easily see that the function $f(u) := g(u) - \frac{\delta}{\beta}p(u)$ has a unique positive root u_* since $f(0) = g(0) > 0$, $f(K) = -\frac{\delta}{\beta}p(K) < 0$ and $f_u(u) < 0$ for $u > 0$.

This paper is organized as follows. In Section 2, we investigate the local and global stability of positive constant solution $\mathbf{e}_* = (u_*, v_*)$. In Section 3, we show the existence and non-existence of non-constant positive solutions of (1.1) for some parameter ranges.

2. Global and local stability of the positive constant solution

In this section, we study the local and global stability of positive constant solution of (1.1).

First, we discuss the global stability of $\mathbf{e}_* = (u_*, v_*)$ which implies the non-existence of non-constant positive solutions. To this end, we impose the following additional hypothesis:

(H2*) $-\tilde{p} \leqslant \frac{d}{du}(\frac{p(u)}{u}) \leqslant 0$ for $u > 0$ and some positive constant $\tilde{p}$.

For simplicity, we denote $f_1(u, v) := ug(u) - p(u)v$ and $f_2(u, v) := v(\delta - \beta\frac{v}{u})$ throughout this paper.

Theorem 2.1. *Assume that* (H2*) *holds. If $\frac{K}{2} < u_* \leqslant \frac{\tilde{g}}{\tilde{p}}\frac{\beta}{\delta}$, then the positive constant solution $\mathbf{e}_*$ is globally asymptotically stable, that is to say, (u_*, v_*) attracts every positive solution of* (1.1).

Proof. Let $(u(t, x), v(t, x))$ be a positive solution of (1.1). As in [5], define the Lyapunov function

$$E(t) = \int_\Omega W(u, v)\,dx,$$

where

$$W(u, v) := \int_{u_*}^{u} \frac{u - u_*}{up(u)}\,du + A\int_{v_*}^{v} \frac{v - v_*}{v}\,dv$$

for some positive constant A which will be chosen later. Then we have

$$\begin{aligned}
E'(t) &= \int_\Omega (W_u u_t + W_v v_t)\,dx \\
&= \int_\Omega \left(\frac{u-u_*}{up(u)} d_1 \Delta u + A\frac{v-v_*}{v} d_2 \Delta v\right) dx + \int_\Omega \big(W_u f_1(u,v) + W_v f_2(u,v)\big)\,dx \\
&= I_1(t) + I_2(t),
\end{aligned}$$

where

$$I_1(t) := -\int_\Omega \left[\frac{d_1}{u^2 p^2(u)}\big(up(u) - (u-u_*)\big(p(u)+up_u(u)\big)\big)|\nabla u|^2 + Ad_2\frac{v_*}{v^2}|\nabla v|^2\right] dx$$

and

$$I_2(t) := \int_\Omega \big(W_u f_1(u,v) + W_v f_2(u,v)\big)\,dx.$$

Using the comparison argument for parabolic problem, one can easily see $0 < u(t,x) \leqslant U(t,x)$ for all $(t,x) \in (0,\infty)\times\Omega$, where U is the unique solution of

$$\begin{cases}
U_t - d_1\Delta U = Ug(U) & \text{in } (0,\infty)\times\Omega, \\
\dfrac{\partial U}{\partial \nu} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\
U(0,x) = u(0,x) & \text{in } \Omega.
\end{cases}$$

Then we can find a large T such that $u(t,x) \leqslant K+\epsilon$ in $[T,\infty)\times\Omega$ for any positive constant ϵ with $\epsilon \leqslant 2u_* - K$ from the well-known fact that $u(t,x) \leqslant U(t,x) \to K$ as $t\to\infty$, where $K>0$ is the constant which satisfies (H1).

Claim 1. $I_1(t) \leqslant 0$ for $t \geqslant T$.

Claim 2. $I_2(t) \leqslant 0$ for $A := \frac{\beta^2}{\delta}$.

If Claim 1 and 2 hold, then $E'(t) \leqslant 0$ for all $t \geqslant T$ which implies the desired result since the equality holds only when $(u,v) = (u_*, v_*)$.

Proof of Claim 1. Since $\frac{d}{du}(\frac{p(u)}{u}) = \frac{up_u(u)-p(u)}{u^2} < 0$ for $u>0$ from (H2*), $up_u(u) < p(u)$ for $u>0$. Using this fact and the assumption $\frac{K}{2} < u_*$, we have

$$\begin{aligned}
up(u) - (u-u_*)\big(p(u)+up_u(u)\big) &= -u^2 p_u(u) + u_*\big(p(u)+up_u(u)\big) \\
&\geqslant -u^2 p_u(u) + u_*\big[2up_u(u)\big] \\
&= up_u(u)[2u_* - u] \\
&\geqslant up_u(u)[2u_* - K - \epsilon] \geqslant 0
\end{aligned}$$

for $t \geqslant T$ which derives the result.

Proof of Claim 2. Since $f_1(u_*, v_*) = f_2(u_*, v_*) = 0$, the integral $I_2(t)$ becomes

$$\begin{aligned}
&\int_\Omega \big(W_u f_1(u,v) + W_v f_2(u,v)\big)\,dx \\
&\quad = \int_\Omega \left[\frac{u-u_*}{p(u)}\left(\frac{f_1(u,v)}{u} - \frac{f_1(u_*,v_*)}{u_*}\right) + A(v-v_*)\left(\frac{f_2(u,v)}{v} - \frac{f_2(u_*,v_*)}{v_*}\right)\right]dx \\
&\quad = \int_\Omega \left[\frac{u-u_*}{p(u)}\left(g(u) - g(u_*) - \left(\frac{p(u)}{u}v_* - \frac{p(u_*)}{u_*}v_*\right) - \left(\frac{p(u)}{u}v - \frac{p(u)}{u}v_*\right)\right)\right. \\
&\qquad \left. + A\beta(v-v_*)\left(-\frac{v}{u} + \frac{v_*}{u} - \frac{v_*}{u} + \frac{v_*}{u_*}\right)\right]dx \\
&\quad = \int_\Omega \left[\frac{1}{p(u)}\left(g_u(\xi) - v_*\frac{d}{du}\left(\frac{p(u)}{u}\right)\bigg|_{u=\eta}\right)(u-u_*)^2\right. \\
&\qquad \left. + \frac{(u-u_*)(v-v_*)}{u}\left(-1 + A\beta\frac{v_*}{u_*}\right) + \frac{(v-v_*)^2}{u}(-\beta A)\right]dx
\end{aligned}$$

for some ξ and η. Note that $-1 + A\beta\frac{v_*}{u_*} = 0$ for $A = \frac{1}{\delta}$, and thus $I_2(t) \leqslant 0$ since

$$g_u(\xi) - v_*\frac{d}{du}\left(\frac{p(u)}{u}\right)\bigg|_{u=\eta} \leqslant -\tilde{g} + \tilde{p}v_* = -\tilde{g} + \tilde{p}\frac{\delta}{\beta}u_* \leqslant 0 \tag{2.1}$$

from the hypotheses (H1) and (H2*). □

Remark 2.2. In Theorem 2.1, if $\frac{d}{du}(\frac{p(u)}{u}) \equiv 0$, then (2.1) is always satisfied since $g_u(\xi) < 0$, and so the same result holds only if we assume $\frac{K}{2} < u_*$. We point out that the predator–prey models with Leslie–Gower functional response (Type 1) satisfy the condition $\frac{d}{du}(\frac{p(u)}{u}) \equiv 0$.

Now we investigate the local stability for the positive equilibrium point $\mathbf{e}_* = (u_*, v_*)$ without the hypothesis (H2*).

Notation 2.3.

(i) $0 = \mu_0 < \mu_1 < \mu_2 < \cdots \to \infty$ are the eigenvalues of $-\Delta$ on Ω under homogeneous Neumann boundary condition.
(ii) $S(\mu)$ is the set of eigenfunctions corresponding to μ.
(iii) $\mathbf{X}_{ij} := \{\mathbf{c}\cdot\varphi_{ij}\colon\ \mathbf{c} \in \mathbb{R}^2\}$, where $\{\varphi_{ij}\}$ are orthonormal basis of $S(\mu_i)$ for $j = 1, \ldots, \dim[S(\mu_i)]$.
(iv) $\mathbf{X} := \{(u,v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\colon\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\}$, and so $\mathbf{X} = \bigoplus_{i=0}^{\infty}\bigoplus_{j=1}^{\dim[S(\mu_i)]}\mathbf{X}_{ij}$.

Using the above notations, the linearization of (1.1) at the positive constant solution $\mathbf{e}_*$ can be expressed by

$$\mathbf{e}_t = \big(\mathbf{D}\Delta + \mathbf{F_e}(\mathbf{e}_*)\big)\mathbf{e},$$

where $\mathbf{e} = (u(t,x), v(t,x))^T$, $\mathbf{F} = (ug(u) - p(u)v, v(\delta - \beta\frac{v}{u}))$,

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \quad \text{and} \quad \mathbf{F_e}(\mathbf{e}_*) = \begin{pmatrix} g(u_*) + u_* g_u(u_*) - p_u(u_*)v_* & -p(u_*) \\ \frac{\delta^2}{\beta} & -\delta \end{pmatrix}.$$

For $i \geqslant 0$, observe that $\bigoplus_{j=1}^{\dim[S(\mu_i)]} \mathbf{X}_{ij}$ is invariant under the operator $\mathbf{D}\Delta + \mathbf{F_e}(\mathbf{e}_*)$; and λ is an eigenvalue of $\mathbf{D}\Delta + \mathbf{F_e}(\mathbf{e}_*)$ on $\bigoplus_{j=1}^{\dim[S(\mu_i)]} \mathbf{X}_{ij}$ if and only if λ is an eigenvalue of the matrix $-\mu_i \mathbf{D} + \mathbf{F_e}(\mathbf{e}_*)$. Moreover,

$$\det\bigl(\lambda \mathbf{I} + \mu_i \mathbf{D} - \mathbf{F_e}(\mathbf{e}_*)\bigr) = \lambda^2 + \operatorname{trace}\bigl(\mu_i \mathbf{D} - \mathbf{F_e}(\mathbf{e}_*)\bigr)\lambda + \det\bigl(\mu_i \mathbf{D} - \mathbf{F_e}(\mathbf{e}_*)\bigr),$$

where

$$\operatorname{trace}\bigl(\mu_i \mathbf{D} - \mathbf{F_e}(\mathbf{e}_*)\bigr) = \mu_i(d_1 + d_2) - \bigl(g(u_*) + u_* g_u(u_*) - p_u(u_*)v_*\bigr) + \delta$$

and

$$\begin{aligned} \det\bigl(\mu_i \mathbf{D} - \mathbf{F_e}(\mathbf{e}_*)\bigr) &= d_1 d_2 \mu_i^2 + \bigl(d_1\delta - d_2\bigl(g(u_*) + u_* g_u(u_*) - p_u(u_*)v_*\bigr)\bigr)\mu_i \\ &\quad - \bigl(g(u_*) + u_* g_u(u_*) - p_u(u_*)v_*\bigr)\delta + p(u_*)\frac{\delta^2}{\beta}. \end{aligned}$$

If $g(u_*) + u_* g_u(u_*) \leqslant p_u(u_*)v_*$, then $\det(\mu_i \mathbf{D} - \mathbf{F_e}(\mathbf{e}_*)) > 0$ and $\operatorname{trace}(\mu_i \mathbf{D} - \mathbf{F_e}(\mathbf{e}_*)) > 0$, and thus the two eigenvalues of the matrix $-\mu_i \mathbf{D} + \mathbf{F_e}(\mathbf{e}_*)$ have negative real parts for $i \geqslant 0$. Therefore, Theorem 5.1.1 in [3] concludes the following result.

Theorem 2.4. *If $g(u_*) + u_* g_u(u_*) \leqslant p_u(u_*)v_*$, then the constant positive equilibrium solution $\mathbf{e}_*$ of* (1.1) *is locally asymptotically stable.*

Remark 2.5. In view of Theorems 2.1 and 2.4, we do not expect the existence of non-constant positive steady-states for predator–prey models with Leslie–Gower functional response (Type 1). More precisely, if $p(u) = u$, then (H2*) is clearly satisfied and it is easy to check that $g(u_*) + u_* g_u(u_*) \leqslant p_u(u_*)v_*$.

3. Non-constant positive steady-states

In this section, we study the existence and non-existence of non-constant positive solutions of (1.2) by using the index theory. To do this, we first obtain an a priori bound for positive solutions of (1.2).

3.1. An a priori bound

The following two lemmas can be found in [6,7], respectively.

Lemma 3.1 *(Maximum principle). Suppose that $h \in C(\overline{\Omega} \times \mathbb{R})$.*

(i) *Assume that $\phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta\phi + h\bigl(x, \phi(x)\bigr) \geqslant 0 \quad \text{in } \Omega, \qquad \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

If $\phi(x_0) = \max_{\overline{\Omega}} \phi$, then $h(x_0, \phi(x_0)) \geqslant 0$.

(ii) *Assume that $\phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\Delta\phi + h\big(x, \phi(x)\big) \leqslant 0 \quad \text{in } \Omega, \qquad \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

If $\phi(x_0) = \min_{\overline{\Omega}} \phi$, then $h(x_0, \phi(x_0)) \leqslant 0$.

Lemma 3.2 *(Harnack inequality). Let $\phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to $\Delta\phi + c(x)\phi = 0$ in Ω subject to homogeneous Neumann boundary condition with $c(x) \in C(\overline{\Omega})$. Then there exists a positive constant $C_* = C_*(\|c\|_\infty)$ such that*

$$\max_{\overline{\Omega}} \phi \leqslant C_* \min_{\overline{\Omega}} \phi.$$

Note that the positive solutions of (1.2) are contained in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ by the standard regularity theorem for elliptic equations [2,11], and so Lemma 3.2 can be applied to system (1.2). For simplicity, denote $\Gamma := (K, \delta, \beta)$.

Theorem 3.3. *Let d be a fixed positive constant. Then for $d_1, d_2 \geqslant d$, there exists a positive constant $\widetilde{C}(\Gamma, d)$ such that any positive solution (u, v) of (1.2) satisfies*

$$\widetilde{C} \leqslant u, v \leqslant K \max\left\{1, \frac{\delta}{\beta}\right\}.$$

Proof. It is easy to see that a simple comparison argument for elliptic problem yields

$$0 < u \leqslant K \quad \text{and} \quad 0 < v \leqslant \frac{\delta}{\beta}\|u\|_\infty \leqslant \frac{\delta}{\beta} K$$

which imply $u, v \leqslant K \max\{1, \frac{\delta}{\beta}\}$. To prove the existence of lower bound $\widetilde{C}$, let $u(x_0) = \min_{\overline{\Omega}} u(x)$, $v(y_0) = \min_{\overline{\Omega}} v(x)$ and $v(y_1) = \max_{\overline{\Omega}} v(x)$, then by Lemma 3.1, we obtain

$$u(x_0) g\big(u(x_0)\big) - p\big(u(x_0)\big) v(x_0) \leqslant 0, \qquad \delta - \beta \frac{v(y_0)}{u(y_0)} \leqslant 0 \quad \text{and} \quad \delta - \beta \frac{v(y_1)}{u(y_1)} \geqslant 0,$$

and so

$$\begin{cases} g\big(u(x_0)\big) \leqslant \dfrac{p(u(x_0))}{u(x_0)} v(x_0), \\ \dfrac{\delta}{\beta} u(x_0) \leqslant \dfrac{\delta}{\beta} u(y_0) \leqslant v(y_0), \\ v(y_1) \leqslant \dfrac{\delta}{\beta} u(y_1) \leqslant \dfrac{\delta}{\beta} \max\limits_{\overline{\Omega}} u(x). \end{cases} \tag{3.1}$$

Since $p(0) = 0$ and $0 < p_u(u) \leqslant M$ for all $u > 0$ from (H2), $p(u) \leqslant Mu$ for all $u > 0$. Using this fact and the hypothesis (H1), we have

$$\begin{aligned} -\big(u(x_0) - K\big)\tilde{g} &\leqslant \big(u(x_0) - K\big) g_u(\xi) = g\big(u(x_0)\big) - g(K) = g\big(u(x_0)\big) \\ &\leqslant \frac{p(u(x_0))}{u(x_0)} v(x_0) \leqslant M v(x_0) \leqslant M \max_{\overline{\Omega}} v(x) \leqslant M \frac{\delta}{\beta} \max_{\overline{\Omega}} u(x) \end{aligned}$$

for some ξ from the first and third inequalities in (3.1), and thus

$$\tilde{g} K \leqslant \tilde{g} \min_{\overline{\Omega}} u(x) + M \frac{\delta}{\beta} \max_{\overline{\Omega}} u(x). \tag{3.2}$$

Applying Lemma 3.2 to the following single equation:

$$\begin{cases} \Delta u + \dfrac{1}{d_1}\big(ug(u) - p(u)v\big) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

we see that $\max_{\overline{\Omega}} u(x) \leqslant C_* \min_{\overline{\Omega}} u(x)$ for some positive constant C_*. By combining this with (3.2),

$$\frac{\tilde{g}K}{\tilde{g} + M\frac{\delta}{\beta}C_*} \leqslant \min_{\overline{\Omega}} u(x) = u(x_0).$$

Moreover, the second inequality in (3.1) yields

$$\frac{\delta}{\beta}\frac{\tilde{g}K}{\tilde{g} + M\frac{\delta}{\beta}C_*} \leqslant \min_{\overline{\Omega}} v(x) = v(y_0).$$

Therefore, the desired result follows by taking $\widetilde{C} := \frac{\tilde{g}K}{\tilde{g}+M\frac{\delta}{\beta}C_*}\min\{1, \frac{\delta}{\beta}\}$. □

3.2. Non-existence of non-constant positive steady-states

Now we show the non-existence of non-constant positive solutions of (1.2) by the effect of diffusions.

Theorem 3.4. *Let D_2 be a fixed positive constant with $D_2 > \frac{\delta}{\mu_1}$. Then there exists a positive constant $D_1(\Gamma, D_2)$ such that* (1.2) *has no non-constant positive solution provided that $d_1 \geqslant D_1$ and $d_2 \geqslant D_2$.*

Proof. Let $\overline{\varphi} = \frac{1}{|\Omega|}\int_\Omega \varphi\,dx$ for any $\varphi \in L^1(\Omega)$. By multiplying $(u - \bar{u})$ and $(v - \bar{v})$ to the first and second equations in (1.2) respectively, and then integrating on Ω, we have

$$\begin{aligned} &\int_\Omega \big[d_1|\nabla u|^2 + d_2|\nabla v|^2\big] \\ &\quad= \int_\Omega \big[(u-\bar{u})\big(f_1(u,v) - f_1(\bar{u},\bar{v})\big) + (v-\bar{v})\big(f_2(u,v) - f_2(\bar{u},\bar{v})\big)\big] \\ &\quad= \int_\Omega \bigg[(u-\bar{u})\big((u-\bar{u})g(u) + \bar{u}\big(g(u) - g(\bar{u})\big) - \bar{v}\big(p(u) - p(\bar{u})\big) - p(u)(v-\bar{v})\big) \\ &\qquad + (v-\bar{v})\bigg(\delta(v-\bar{v}) + \frac{\beta\bar{v}^2}{\bar{u}} - \frac{\beta\bar{v}^2}{u} - \frac{\beta v^2}{\bar{u}} + \frac{\beta\bar{v}^2}{u}\bigg)\bigg] \\ &\quad= \int_\Omega \bigg[(u-\bar{u})^2\big(g(u) + \bar{u}g_u(\xi) - \bar{v}p_u(\eta)\big) + (u-\bar{u})(v-\bar{v})\bigg(-p(u) + \frac{\beta\bar{v}^2}{u\bar{u}}\bigg) \\ &\qquad + (v-\bar{v})^2\bigg(\delta - \frac{\beta(v+\bar{v})}{u}\bigg)\bigg] \end{aligned} \tag{3.3}$$

for some ξ and η. Since u has a uniform upper bound by Theorem 3.3, the last integral in (3.3) is smaller than or equal to the following:

$$\int_\Omega \left[(u-\bar u)^2 g(0) + L|u-\bar u||v-\bar v| + \delta(v-\bar v)^2\right]$$
$$\leqslant \int_\Omega \left[(u-\bar u)^2\left(g(0)+\frac{L}{2\epsilon}\right) + (v-\bar v)^2\left(\delta + \frac{\epsilon L}{2}\right)\right] \tag{3.4}$$

for some positive constant L and an arbitrary positive constant ϵ. Note that the last inequality follows from the fact

$$2L|u-\bar u||v-\bar v| = 2\sqrt{\frac{L}{\epsilon}}|u-\bar u|\cdot\sqrt{\epsilon L}|v-\bar v| \leqslant \frac{L}{\epsilon}|u-\bar u|^2 + \epsilon L|v-\bar v|^2.$$

Synthetically, we have

$$\int_\Omega d_1|\nabla u|^2 + d_2|\nabla v|^2 \leqslant \int_\Omega \left[(u-\bar u)^2\left(g(0)+\frac{L}{2\epsilon}\right) + (v-\bar v)^2\left(\delta+\frac{\epsilon L}{2}\right)\right], \tag{3.5}$$

and so

$$\int_\Omega d_1\mu_1(u-\bar u)^2 + d_2\mu_1(v-\bar v)^2 \leqslant \int_\Omega \left[(u-\bar u)^2\left(g(0)+\frac{L}{2\epsilon}\right) + (v-\bar v)^2\left(\delta+\frac{\epsilon L}{2}\right)\right]$$

by using Poincaré inequality. Since $d_2\mu_1 > \delta$ from the assumption, we can find a sufficiently small ϵ_0 such that $d_2\mu_1 \geqslant \delta + \frac{\epsilon_0 L}{2}$. Finally, by taking $D_1 := \frac{1}{\mu_1}(g(0)+\frac{L}{2\epsilon_0})$, one can conclude that $u=\bar u$ and $v=\bar v$ which complete the proof. □

3.3. *Existence of non-constant positive steady-states*

To show the existence of non-constant positive solutions, we use Leray–Schauder degree theory. For the sake of convenience, define a compact operator $\mathcal{F}:\mathbf{X}\to\mathbf{X}$ by

$$\mathcal{F}(\mathbf{e}) := \begin{pmatrix} (I-d_1\Delta)^{-1}[f_1(u,v)+u] \\ (I-d_2\Delta)^{-1}[f_2(u,v)+v] \end{pmatrix},$$

where $\mathbf{e}=(u(x),v(x))^T$. Then the system (1.2) is equivalent to the equation $(\mathbf{I}-\mathcal{F})\mathbf{e}=0$. To apply the index theory, we investigate the eigenvalue of the problem:

$$-\big(\mathbf{I}-\mathcal{F}_{\mathbf{e}}(\mathbf{e}_*)\big)\Psi = \lambda\Psi, \quad \Psi\neq\mathbf{0}, \tag{3.6}$$

where $\Psi=(\psi_1,\psi_2)^T$ and $\mathbf{e}_*=(u_*,v_*)$. If 0 is not an eigenvalue of (3.6), then the Leray–Schauder theorem [8, Theorem 2.8.1] implies

$$\operatorname{index}(I-\mathcal{F},\mathbf{e}_*) = (-1)^\gamma, \tag{3.7}$$

where $\gamma=\sum_{\lambda>0} n_\lambda$ and n_λ is the algebraic multiplicity of the positive eigenvalue λ of (3.6). After some calculation, (3.6) can be rewritten as

$$\begin{cases} -d_1(\lambda+1)\Delta\psi_1 + \left[\lambda - \big(g(u_*)+u_*g_u(u_*) - p_u(u_*)v_*\big)\right]\psi_1 + p(u_*)\psi_2 = 0, \\ -d_2(\lambda+1)\Delta\psi_2 - \dfrac{\delta^2}{\beta}\psi_1 + (\lambda+\delta)\psi_2 = 0 \quad \text{in } \Omega, \\ \dfrac{\partial\psi_1}{\partial\nu} = \dfrac{\partial\psi_2}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \\ \psi_i\neq 0 \quad \text{for } i=1,2. \end{cases} \tag{3.8}$$

Observe that (3.8) has a non-trivial solution if and only if $P_k(\lambda) = 0$ for some $\lambda \geqslant 0$ and $k \geqslant 0$, where

$$P_k(\lambda) := \det \begin{pmatrix} \lambda + \frac{d_1\mu_k - (g(u_*)+u_*g_u(u_*)-p_u(u_*)v_*)}{1+d_1\mu_k} & \frac{p(u_*)}{1+d_1\mu_k} \\ -\frac{1}{1+d_2\mu_k}\frac{\delta^2}{\beta} & \lambda + \frac{d_2\mu_k+\delta}{1+d_2\mu_k} \end{pmatrix}.$$

That is to say, λ is an eigenvalue of (3.6), and so (3.8), if and only if λ is a positive root of the characteristic equation $P_k(\lambda) = 0$ for $k \geqslant 0$. Therefore, if $P_k(0) \neq 0$ for all $k \geqslant 0$, we can see that

$$\operatorname{index}(I - \mathcal{F}, \mathbf{e}_*) = (-1)^\gamma, \quad \gamma = \sum_{k \geqslant 0} \sum_{\lambda_k > 0} m_{\lambda_k} \dim\bigl[S(\mu_k)\bigr],$$

where m_{λ_k} is the multiplicity of λ_k as a positive root of $P_k(\lambda) = 0$. For more details on the verification of the above formula, one can refer to [9,10].

In view of Theorem 2.4, we see that there might be no non-constant positive solution of (1.2) if $\alpha := g(u_*) + u_*g_u(u_*) - p_u(u_*)v_* \leqslant 0$, and so it is natural to assume $\alpha > 0$ to investigate the non-constant positive solutions of (1.2).

Lemma 3.5. *Assume that $\delta > \alpha := g(u_*) + u_*g_u(u_*) - p_u(u_*)v_* > 0$. Then there exists a positive constant $\widehat{D}_1 := \widehat{D}_1(\Gamma, d_2)$ such that* $\operatorname{index}(I - \mathcal{F}, \mathbf{e}_*) = 1$ *provided that $d_1 \geqslant \widehat{D}_1$.*

Proof. First, note that

$$-\delta\alpha + \frac{\delta^2}{\beta} p(u_*) = -\delta u_* g_u(u_*) + \delta p_u(u_*)v_* > 0$$

since $g(u_*) = \frac{\delta}{\beta} p(u_*)$. If $k = 0$ (i.e., $\mu_0 = 0$), then we have

$$P_0(\lambda) = \lambda^2 + (\delta - \alpha)\lambda - \delta\alpha + p(u_*)\frac{\delta^2}{\beta} > 0$$

for all $\lambda \geqslant 0$. In the case of $k \geqslant 1$ (i.e., $\mu_k > 0$), the polynomial $P_k(\lambda)$ has the form

$$P_k(\lambda) = (\lambda + 1)\left(\lambda + \frac{d_2\mu_k + \delta}{1 + d_2\mu_k}\right) + \mathcal{O}\left(\frac{1}{d_1}\right),$$

and thus there exists a large positive constant $\widehat{D}_1(\Gamma, d_2)$ such that $P_k(\lambda) > 0$ for all $d_1 \geqslant \widehat{D}_1$ and $\lambda \geqslant 0$. Therefore, one can conclude that $\gamma = \sum_{k \geqslant 0} \sum_{\lambda_k > 0} n_{\lambda_k} = 0$ for all $d_1 \geqslant \widehat{D}_1$ which implies the desired result. □

Now we prove the existence of non-constant positive solutions of (1.2) for some d_1 when d_2 is sufficiently large.

Theorem 3.6. *Assume that $\delta > \alpha > 0$, $d_1 \in (\frac{\alpha}{\mu_{k_0+1}}, \frac{\alpha}{\mu_{k_0}})$ and $\sum_{k=1}^{k_0} \dim[S(\mu_k)]$ is odd for some $k_0 \geqslant 1$. Then there exist a positive constant $\widehat{D}_2(\Gamma, d_1)$ such that (1.2) has at least one non-constant positive solution provided that $d_2 \geqslant \widehat{D}_2$.*

Proof. Similarly as in the proof of Lemma 3.5, $P_0(\lambda) > 0$ for all $\lambda \geqslant 0$ and $P_k(\lambda)$ has the form:

$$P_k(\lambda) = (\lambda + 1)\left(\lambda + \frac{d_1\mu_k - \alpha}{1 + d_1\mu_k}\right) + \mathcal{O}\left(\frac{1}{d_2}\right)$$

for $k \geqslant 1$. Since $d_1 \in (\frac{\alpha}{\mu_{k_0+1}}, \frac{\alpha}{\mu_{k_0}})$ from the assumption, one can see that $(\lambda+1)(\lambda+\frac{d_1\mu_k-\alpha}{1+d_1\mu_k})=0$ has only one positive root for $1 \leqslant k \leqslant k_0$, but the roots are all negative for $k > k_0$. Thus we can find a positive constant $\widehat{D}_2$ such that the polynomial $P_k(\lambda)=0$ has only one simple positive root for $1 \leqslant k \leqslant k_0$, while all roots of $P_k(\lambda)=0$ have negative real parts for $k > k_0$, when $d_2 \geqslant \widehat{D}_2$. Therefore if $d_2 \geqslant \widehat{D}_2$, then we have

$$\operatorname{index}(I-\mathcal{F},\mathbf{e}_*)=(-1)^{\sum_{k=1}^{k_0}\dim[S(\mu_k)]}=-1.$$

To finish the proof, a contradiction argument will be used by assuming that (1.2) has no non-constant positive solution. For $\theta \in [0,1]$ and $\widetilde{D}_1 \geqslant \max\{D_1, \widehat{D}_1\}$, define a homotopy

$$\mathcal{F}_\theta(\mathbf{e}) := \begin{pmatrix} (I-\theta d_1\Delta-(1-\theta)\widetilde{D}_1\Delta)^{-1}[f_1(u,v)+u] \\ (I-\theta d_2\Delta-(1-\theta)(\frac{\delta}{\mu_1}+1)\Delta)^{-1}[f_2(u,v)+v] \end{pmatrix},$$

where D_1 and $\widehat{D}_1$ are positive constants defined in Theorem 3.4 and Lemma 3.5, respectively. By Theorem 3.3, the positive solutions of problem $\mathcal{F}_\theta(\mathbf{e})=\mathbf{e}$ are contained in $\Lambda := \{\mathbf{e}\in\mathbf{X}\colon \widetilde{C}/2<u,v<2K\max\{1,\frac{\delta}{\beta}\}\}$. Since $\mathcal{F}_\theta(\mathbf{e}) \neq \mathbf{e}$ for all $\mathbf{e}\in\partial\Lambda$ and $\mathcal{F}_\theta(\mathbf{e}) : \Lambda\times[0,1]\to X$ is compact, one can see that the degree $\deg(I-\mathcal{F}_\theta(\mathbf{e}),\Lambda,0)$ is well defined. Moreover, using the homotopy invariance property of the degree, $\deg(I-\mathcal{F}_0(\mathbf{e}),\Lambda,0)=\deg(I-\mathcal{F}_1(\mathbf{e}),\Lambda,0)$. Since $\widetilde{D}_1 > D_1$, $\mathcal{F}_0(\mathbf{e})=\mathbf{e}$ has no non-constant positive solution by Theorem 3.4, and so $\deg(I-\mathcal{F}_0(\mathbf{e}),\Lambda,0)=\operatorname{index}(I-\mathcal{F}_0,\mathbf{e}_*)=1$ by Lemma 3.5. On the other hand, since we assume that there is no non-constant positive solution of (1.2), we have $\deg(I-\mathcal{F}_1(\mathbf{e}),\Lambda,0)=\operatorname{index}(I-\mathcal{F},\mathbf{e}_*)=-1$ which derives a contradiction. □

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