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Symmetry Adapted Functions Belonging to the Dirac Groups

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Abstracts

An earlier analysis of the canonical form of a pair of invertible operators obeying the exchange rule

$$AB = \omega BA$$

is extended to cover a set of operators, between each pair of which a relation of this type exists; and for which a power of each operator is the unit matrix. Such relations define a system which may be regarded as a generalization of the Dirac matrices of relativistic quantum mechanics. We concentrate upon the group theoretic aspects of such a system and its matrix representations. Applications arise from the fact that all projective representations of finite abelian groups take the form of a Dirac Group. In particular, the representations of the magnetic space groups, which are projective representations of the lattice groups, arise in this manner.

On généralise une analyse antérieure de la forme canonique d'une paire d'opérateurs ayant des inverses et satisfaisant à la relation

$$AB = \omega BA$$

à un ensemble d'opérateurs satisfaisant à une relation de ce type et pour lesquels une certaine puissance de chaque opérateur est l'opérateur d'identité. Ces relations définissent un système qui peut être regardé comme une généralisation des matrices de Dirac dans la mécanique quantique relativiste. Nous nous intéressons surtout aux aspects se rapportant à la théorie des groupes. Des applications proviennent du fait que toutes les représentations projectives des groupes abéliens finis prennent la forme d'un groupe de Dirac. En particulier les représentations des groupes d'espace magnétiques qui sont des représentations projectives des groupes de réseau, surgissent de cette façon-ci.

Eine vorläufige Analyse der kanonischen Form eines Paars von invertiblen Operatoren, die die Bedingung

$$AB = \omega BA$$

genügen, wird zu einer Menge von Operatoren erweitert, für welche eine solche Bedingung existiert und für welche eine Potenz jedes Operators gleich den Einheitsoperator ist.

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Solche Bedingungen definieren ein System welches als eine Verallgemeinerung der Diracschen Matrizen der relativistischen Quantenmechanik betrachtet werden kann. Wir interessieren uns vor allem für die gruppentheoretischen Aspekte eines solchen Systems. Anwendungen entstehen von der Tatsache dass alle projektiven Darstellungen endlicher, abelscher Gruppen die Form einer Diracgruppe annehmen. Insbesondere entstehen die Darstellungen der magnetischen Raumgruppen, die projektive Darstellungen der Gittergruppen sind, in dieser Weise.

1. Introduction

An earlier paper [1] introduced the idea of an "exchange relation" between two operators A and H, such a relation being said to be fulfilled when they satisfy the equation

$$AH = \omega HA \tag{1}$$

The multiplier $\omega = \omega(A, H)$ is a scalar assumed to be non-zero, which must be an mth root of unity when A and H are finite non-singular operators of order m. In case $\omega = 1$, A and H are two commuting operators; when $\omega = -1$, they anticommute. This relation has also been discussed by Ceconi [2], Cherubino [3], Kurosaki [4], and Drazin [5].

It is possible to have a set of operators $S = \{A_1, A_2, \cdots, A_\nu\}$, each pair having its own exchange factor. When S is a group, $\omega(A_i, A_j)$ is a one dimensional representation of S when either of its arguments is held fixed, and the canonical forms deduced for operators satisfying an exchange relation in reference [1] are a tool for constructing irreducible representations of S. The essence of the construction is that the representations of S are induced by the kernel of ω.

The purpose of the present paper is to illustrate such a construction, and thereby the methods of reference [1], by treating a class of groups which are defined by exchange relations. The prototype example is formed by the four Dirac matrices of relativistic quantum mechanics, or rather the group of order 16 which is their multiplicative hull. We generalize the prototype both by considering a system having an arbitrary number of anticommuting operators to illuminate the theory of Clifford algebras, and by allowing the more general exchange relation of equation (1) in place of anticommutation.

It is not to say that the results of such an investigation are not well known, both to mathematicians and to physicists. Every textbook dealing with the relativistic wave mechanics of a spinning electron since the original paper of Dirac [6] has made some mention of the four-dimensional representations of the Dirac matrices. Dirac himself introduced a matrix representation of his operators, and not long thereafter Temple [7] analyzed the group properties of such an assemblage. Since then there have been many other papers. The sophistication of the different accounts ranges from van der Waerden's [8] derivation of the Dirac matrices from the theory of semi-simple algebras and Pauli's exhaustive analysis [9] to simple and straightforward computations using only the theory of matrices [10].

In the mathematical literature, the treatment ranges from Clifford's [11] original idea of what has come to be called the Clifford algebra, whose quaternion-like operators anticommute and yield a square of -1, through Witt's [12] use of a hypercomplex anticommuting system with arbitrary scalar squares in the theory of quadratic forms, to Artin's [13] thoroughly elegant formulation of the same topic. For the representation of such systems considerable use has been made of their algebraic rather than group theoretic properties, an idea elaborated by Brauer and Weyl [14] as well as in the work of Pauli and van der Waerden already mentioned. Lee [15] has carried this idea through in detail for a system of operators having unit square, but in reviewing this latter work, Albert [16] is emphatic in his insistence that the idea of a semi-simple algebra is paramount, and that the Clifford algebras have a particularly simple structure in this regard.

Considerable impetus was given to the study of anticommuting systems by Eddington's cosmological notions and his theory of E-numbers [17], resulting in a whole series of papers by various authors, which cover the entire range of questions which one would expect to be resolved about the system. Littlewood [18] treats the construction of their irreducible representations by group theoretical methods sufficiently general to apply to the present paper, and in fact analyzes the case $\omega^3 = 1$ in detail as well as that for $\omega^2 = 1$. Eddington [19] in the restricted case of 4×4 matrices, and Newman [20] more generally, determined the number of anticommuting operators which must be real and imaginary, respectively.

At first, interest centered in obtaining representations for these systems [21], but gradually the emphasis shifted to the symmetry adapted functions and transformation properties independent of a particular representation. Eddington [22] spoke of "pure" E-numbers, but these were the principal idempotents and their "factorization" amounted to the construction of the symmetry adapted functions. A much more exhaustive treatment of this realm was given by Harish-Chandra [23] and Rao [24], the former also applying similar techniques [25] to the closely related Kemmer matrices [26].

Although Dirac's use of anticommuting operators in 1928 to factor a quadratic form into linear factors suggests a similar technique to be used for other forms, it was not until 1954 that Heerema [27] published a comprehensive analysis of the use of hypercomplex numbers to factor a binary cubic. Beyond this, most attention seems to have centered on quadratic forms, Witt's paper having appeared in 1937, at the same time as another by Mordell [28]. Nevertheless as early as 1882, Sylvester [29] was using hypercomplex number systems for similar purposes.

The fact that the groups which we are using are derived from the generators of semi-simple algebras has as its direct consequence a situation already quite apparent for the Dirac matrices; namely that their irreducible representations will consist of a few, perhaps only one, representations of high dimension, together with a plethora of one-dimensional representations. These, even for the Clifford algebra, play a fundamental role in the construction of the high-dimensional representations in spite of their general lack of utility in physical applications.

Most recently, groups of the Dirac type have been found to occur among the magnetic space groups, the multiplying factor ω resulting from the gauge transformation which accompanies a translation. Zak [30] has studied these groups in some detail, as has Brown [31]. Daltabuit and McIntosh [32] have considered the form of the crystal lattice groups in the presence of a magnetic field, taking into account rotations and reflections in addition to the translations. In another direction, Barut and Komy [33] have investigated the projective representations of a family of commuting parity operators, whereas Barut [34] has treated the subject of the present paper from the point of view of algebraic theory, imposing polynomial constraints on his non-commuting operators rather than our simple exchange relations.

The relation between the Dirac groups and projective representations of abelian groups such as the translation groups of the crystal lattices is no accident, since the formation of appropriate Dirac groups suffices to determine all the projective representations of the finite abelian groups. The principal exposition of this relationship is Frucht's paper of 1931 [35].

Although the purpose of this paper is to illustrate the group theoretical point of view, we should mention that there have been several comprehensive algebraic treatments [36].

2. Operators Satisfying the Functional Equation $HA = Af(H)$

In preparation for the application of the methods of reference [1] to the analysis of the representations of the Dirac groups, we shall make use of this opportunity for reviewing them to restate them in a more general form. In this regard, we see that the exchange relation defined in equation (1)

$$HA = \omega AH \tag{1}$$

is a special case of a more general exchange relation,

$$HA = Af(H) \tag{2}$$

for which $f(H) = \omega H$. In the ensuing discussion we shall simply call this function ω, to avoid repeating the fact that it is the special case corresponding to scalar multiplication. If Φ is an eigenfunction of the operator H,

$$H\Phi = \lambda\Phi \tag{3}$$

belonging to the eigenvalue λ, then

$$f(H)\Phi = f(\lambda)\Phi \tag{4}$$

Multiplying equation (4) on the left by A and applying equation (2), we obtain

$$H(A\Phi) = f(\lambda)(A\Phi) \tag{5}$$

This is the fundamental relation upon which the remainder of our analysis rests. Essentially, the functions Φ, $A\Phi$, $A^2\Phi, \cdots$, form a chain of eigenfunctions belonging respectively to the eigenvalues λ, $f(\lambda)$, $f^2(\lambda), \cdots$, where by $f^2(\lambda)$ we mean the composite $f(f(\lambda))$, and so on for higher powers.

Since $f(z)$ defines a mapping of the complex plane which contains the eigenvalues, we shall call a point for which

$$f(z) = z \tag{6}$$

a fixed point, and a point z for which there is an integer k such that

$$f^k(z) = z \tag{7}$$

a repetition point, of degree k. The set of complex numbers $K = \{z, f(z), f^2(z), \cdots\}$ we shall call an f-cycle of z. Also two consecutive images, $f^p(z)$ and $f^{p+1}(z)$ will be called neighbors. If f is a one-to-one function, and hence invertible, the f-cycles will be equivalence classes.

If we assume that $\mathbf{H}$ is a diagonalizable matrix, for instance if it is Hermitean or unitary, then a similarity transformation by the matrix $\mathbf{O}$ will make it a diagonal matrix $\mathbf{\Lambda}$, while transforming A into another matrix, $\mathbf{C}$. Then,

$$\mathbf{O}^{-1}\mathbf{H}\mathbf{O} = \mathbf{\Lambda} \tag{8}$$

$$\mathbf{O}^{-1}A\mathbf{O} = \mathbf{C} \tag{9}$$

whence equation (2) becomes

$$\mathbf{\Lambda}\mathbf{C} = \mathbf{C}f(\mathbf{\Lambda}) \tag{10}$$

When this equation is written in terms of its matrix elements, we find

$$\mathbf{C}_{ij}(\lambda_i - f(\lambda_j)) = 0 \tag{11}$$

The conclusion is, that the matrix elements of $\mathbf{C}$ vanish unless they connect subspaces belonging to neighboring eigenvalues of H. Equation (11) thereby defines a mutual canonical form for the operators A and H; the stable subspaces of H form a system of imprimitivity for A.

If further requirements are placed on the operator H, the conclusions regarding A can be sharpened accordingly. If we require the two operators to be jointly irreducible, the eigenvalues of H can belong to only one cycle, but in any event the vector space V upon which A and H operate can be reduced to a direct sum of subspaces V_i, each belonging to a simple f-cycle K_i of H. Fixed points of f, if any, correspond to subspaces which are stable under both A and H. If A and H are finite operators, the cycles cannot be indefinitely long, unless the subspaces corresponding to all but finitely many points of the cycle are zero. If A is invertible, it cannot annihilate any of the subspaces upon which it acts transitively, nor can it connect subspaces of different dimensionality. As a result, each point of the spectrum of H must be a repetition point, of which any fixed points are a particular instance. For example, in the case of the multiplier ω, zero was a fixed point which had to be treated somewhat differently from other points of the spectrum in reference [1].

The multiplier ω allows a certain symmetry between A and H which is lost in the more general exchange relation. Since ω could be regarded as a multiplier of either matrix, either could be diagonalized while the other was brought to the complementary form.

A natural extension from a single exchange relation is to consider families of exchange relations, the operators of which possess some further properties. In considering such an extension let us prepare by noting that

$$H^2A = A(f(H))^2 \tag{12}$$

obtained by applying equation (2) to $H(HA)$ to obtain $HAf(H)$, and then again to the factor HA appearing on the left. Similar results apparently hold for other powers;

$$H^nA = A(f(H))^n \tag{13}$$

so that if g is a function possessing an appropriate power series, or if H is an operator to which an appropriate spectral resolution may be applied, we can also conclude that

$$g(H)A = Ag(f(H)) \tag{14}$$

This result may be applied to calculate the exchange of any function of H with A. Proceeding in the other direction, we find

$$\begin{aligned} HA^2 &= (Af(H))A \\ &= A^2f(f(H)) \\ &= A^2f^2(H) \end{aligned} \tag{15}$$

or more generally

$$HA^n = A^nf^n(H) \tag{16}$$

The exchange of H with a power of A yields the nth iterate of f, whereas the exchange of a power of H with A yields the nth power of the value of f; that is

$$H^nA^m = A^m(f^m(H))^n$$

One extension of the exchange relation (2) is to suppose that there is a group of operators $\mathbf{A} = \{A_i\}$ and corresponding functions $\mathbf{F} = \{f_i\}$, such that for a fixed operator H,

$$HA_i = A_i f_i(H) \tag{17}$$

It then appears that

$$HA_iA_j = A_iA_jf_j(f_i(H)) \tag{18}$$

so that $\mathbf{F}$ is a group of functions homomorphic to $\mathbf{A}$ with respect to composition as the group multiplication. They also form a group of transformations of the complex plane into itself, as well as the spectrum of the operator H.

Since $\mathbf{A}$ operates imprimitively on the stable subspaces of H, it follows that $\mathbf{A}$ is an induced representation, induced by any subgroup of transformations for which a given point of the spectrum is a fixed point.

We could vary the theme by interchanging the roles of A and H, to suppose that there were given a group of operators $\mathbf{A} = \{A_i\}$ as well as a fixed function f and operator H, such that

$$A_iH = Hf(A_i) \tag{19}$$

Then we see that

$$A_iA_jH = Hf(A_i)f(A_j) \tag{20}$$

which allows us to compare $Hf(A_iA_j) = Hf(A_i)f(A_j)$, and to suppose that f is a homomorphism. Hence if the set of matrices $\{A_j\}$ would be the representation of a group, then the set $\{f(A_j)\}$ would be another. Schur's lemma then applies to equation (19) so that again $\mathbf{A}$ is a group of imprimitive operators and therefore an induced representation induced by that subgroup whose elements satisfy the relation

$$A_i = f(A_i) \tag{21}$$

The function f generates the star of the representation $\theta = \{A_i\}$.

3. Projective Representations [37, 38]

The multiplier $\omega(A, H)$ of the exchange relation (1) is the character of a one-dimensional representation when its arguments A and H belong to a group, and one of the two arguments is held fixed. Such a function is called a *bicharacter*, and plays an important role in the theory of projective representations. For abelian groups, the possible bicharacters determine all the possible projective representations. In order to appreciate the connection, let us briefly review the theory of projective representations.

A projective representation of a group G is a mapping D from G to a set $\mathbf{\Gamma}$ of matrices, satisfying the rule

$$\mathbf{D}(a)\mathbf{D}(b) = \lambda(a, b)\mathbf{D}(ab) \tag{22}$$

in which $\lambda(a, b)$ is a non-zero scalar multiplier.

The multipliers $\lambda(a, b)$ are subjected to constraints on account of the rules of group multiplication, the important one of which is imposed by the associative law. If we calculate alternatively $\mathbf{D}(a)[\mathbf{D}(b)\mathbf{D}(c)]$ and $[\mathbf{D}(a)\mathbf{D}(b)]\mathbf{D}(c)$, both of which result in a multiple of $D(abc)$, we find

$$\lambda(a, b)\lambda(ab, c) = \lambda(a, bc)\lambda(b, c) \tag{23}$$

One special case, obtained by setting $b = e$ is

$$\lambda(a, e)\lambda(a, c) = \lambda(a, c)\lambda(e, c) \tag{24}$$

whence

$$\lambda(a, e) = \lambda(e, c) \tag{25}$$

whatever a or c. Hence all such values of λ could be taken equal to $\lambda(e, e)$.

Conversely, any complex valued function $\lambda: G \times G \to C - \{0\}$, satisfying this identity, is a multiplier for some projective representation. The substantiation of this statement will require several steps, of which the first is to consider the totality of all such functions, which satisfy equation (23). If we take both the inverse and the transpose of both sides of equation (23), we obtain a new representation of G whose multiplier is the reciprocal of the original. We notice that, because any group has ordinary representations, the function $\lambda(a, b) = 1$ is a multiplier for a projective representation. Finally, if we have two multipliers λ_1 and λ_2 belonging to representations $\mathbf{D}_1$ and $\mathbf{D}_2$ respectively, then the product of the two multipliers, $\lambda_1\lambda_2$, is a multiplier belonging to the Kronecker product of the two representations, $\mathbf{D}_1 \times \mathbf{D}_2$. In summary, the totality of possible multipliers forms a group with respect to the pointwise multiplication of function values.

In actuality the totality of all possible multipliers is much too large, because of the possibility of making trivial modifications to a projective representation. This is done by taking any function $\rho: G \to C - \{0\}$, and multiplying it by the representation $\mathbf{D}$, to form the new set of matrices $\mathbf{D}'(a) = \rho(a)\mathbf{D}(a)$. In place of equation (22), they now satisfy the equation

$$\mathbf{D}'(a)\mathbf{D}'(b) = \frac{\rho(a)\rho(b)}{\rho(ab)}\lambda(a, b)\mathbf{D}'(ab) \tag{26}$$

Two multipliers which are related in this manner by the existence of a function ρ such that

$$\lambda(a, b) = \frac{\rho(a)\rho(b)}{\rho(ab)} \lambda'(a, b) \tag{27}$$

are called *equivalent.* It is readily enough seen that this equivalence is a congruence relation in the group of multipliers; we call its factor group the *multiplier group* of G.

Furthermore, we notice that the set of all functions satisfying equation (23) is closed with respect to the formation of inverses and products, and contains the constant function 1. Therefore we can deal with the functions satisfying equation (23) independently of whether or not we know that they arose from a projective representation, forming a group of them, and introducing the equivalence described by equation (27). In dealing with multipliers, arising from a projective representation with $n \times n$ matrices, we could divide each matrix by the nth root of its determinant. Then, taking determinants of both sides of equation (22), we could deduce that the absolute value of the multiplier was 1; moreover that the multiplier itself was an nth root of unity. It would follow that in the equivalence class of every multiplier was a multiplier whose values were roots of unity.

However, we can obtain a slightly better result in the general case. Given a function λ satisfying equation (23), we define ρ by

$$\rho(a) = \prod_{b \in G} \lambda(a, b) \tag{28}$$

and we may verify, by using equation (23), that

$$\frac{\rho(a)\rho(b)}{\rho(ab)} = \lambda(a, b)^{{}^\circ G} \tag{29}$$

This latter equation asserts that $\lambda(a, b)^{{}^\circ G}$ is equivalent to the multiplier 1, and the representation may be normalized accordingly. We then find that there is, in the equivalence class of every multiplier, a multiplier whose values are $^\circ G$th roots of unity.

We will need to deal with primitive roots of unity; that is nth roots for which n is a prime, or in general roots which are not also roots of lesser order. However, since we know that the values of the multipliers are $^\circ G$th roots of unity, and that they form a group we may take account of the fact that every subgroup of a cyclic group is also cyclic to write each value as a power of some primitive value. This latter will not necessarily be an $^\circ G$th root, but in any event will itself be a power of one.

We are now ready for the demonstration that every multiplier, as we shall call any function which satisfies equation (23), is the multiplier of some projective representation of the group G. Let us call the set of values of this multiplier C, which is a cyclic group according to the arguments of the preceding paragraph. In the cartesian product $C \times G$ we define a product according to the rule

$$(\xi, g)(\xi', g') = (\xi\xi'\lambda(g, g'), gg') \tag{30}$$

To verify that we have defined a group, we note that

$$\begin{aligned}[(\xi, a)(\xi', a')](\xi'', a'') &= (\xi\xi'\lambda(a, a'), aa')(\xi'', a'') \\ &= (\xi\xi'\xi''\lambda(a, a')\lambda(aa', a''), (aa')a'')\end{aligned} \tag{31}$$

whereas

$$\begin{aligned}(\xi, a)[(\xi', a')(\xi'', a'')] &= (\xi, a)(\xi'\xi''\lambda(a', a''), a'a'') \\ &= (\xi\xi'\xi''\lambda(a, a'a'')\lambda(a', a''), a(a'a''))\end{aligned} \tag{32}$$

so that the identity (23) is crucial in verifying the associative law. An element which serves as an identity may be seen to be

$$\left(\frac{1}{\lambda(e, e)}, e\right) \tag{33}$$

since

$$\left(\frac{1}{\lambda(e, e)}, e\right)(\xi, a) = \left(\frac{\xi\lambda(a, e)}{\lambda(e, e)}, a\right) = (\xi, a) \tag{34}$$

whereas the inverse is given by

$$(\xi, a)^{-1} = \left(\frac{1}{\xi\lambda(e, e)\lambda(a^{-1}, a)}, a^{-1}\right) \tag{35}$$

where one uses the fact that $\lambda(a, a^{-1}) = \lambda(a^{-1}, a)$ which can be verified by letting $a = c = b^{-1}$ in (23).

The set

$$M = [(\xi, a) : a = e] \tag{36}$$

forms a normal subgroup contained within the center of $\hat{G} = C \times G$, so that in any representation its elements will be represented by multiples of the identity. Unfortunately we cannot always guarantee that the multiplier will be ξ itself, rather than a homomorphic image. It is for this reason that we earlier took such pains to show that we can always work with primitive roots of unity, for we do know that there will always be at least one non-trivial representation which would then have to be faithful. Multipliers which are not primitive roots will have to be resolved into their prime factors.

Given that M is a normal subgroup,

$$\hat{G}/M \sim G \tag{37}$$

and we have the factorization

$$(\xi, e)(1, a) = (\xi, a) \tag{38}$$

assuming $\lambda(e, e) = 1$. Thus the mapping

$$(\xi, a) \to \xi D(a) \tag{39}$$

is a representation of $\hat{G}$, satisfying the rule that

$$D(1, a)D(1, b) \to \lambda(a, b)D(1, ab) \tag{40}$$

which shows that it is a projective representation of G.

The exchange factors for a Dirac group form a set of multipliers for the factor group D_n/Σ,

$$A(BC) = \omega(A, BC)\omega(B, C)CBA$$

while

$$(AB)C = \omega(AB, C)\omega(A, B)CBA$$

Since the matrices involved are non-zero, the equality of the coefficients establishes the result that $\omega(A, B)$ is a multiplier. The question of determining all the possible exchange rules for a given set of operators is then apparently equivalent to determining the bicharacters, and thus closely connected with the determination of their projective representations.

It should be noted that whereas the set of elements

$$C^* = \{(\xi, a) : \xi \in C, a = e\}$$

forms a subgroup of $\hat{G}$, the elements in

$$G^* = \{(\xi, x) : \xi = 1, x \in G\}$$

do not, even though $\hat{G} = G^*C^*$.

4. Induced Representations

Probably the most important technique for the construction of group representations is the method of induced representations. It is an inductive technique, by which the construction of a representation of a group may be reduced to finding the representation of a subgroup, from which the representation of the entire group may be found by a standard calculation. In many cases one may proceed through a chain of subgroups until one finally arrives at a cyclic subgroup, whose representations are already known, as roots of unity.

The idea of induced representations, and of imprimitive matrices, with which they are always associated, is as old as group representation theory itself, having originally been analyzed by Frobenius [39]. A more modern survey was given by Mackey [40], while the subject covers at least one chapter of the textbook of Curtis and Reiner [38]. In this section, we review the theory of induced representations, as well as the criteria for their irreducibility, which takes a particularly elegant form when the subgroup from which the representation is induced is normal.

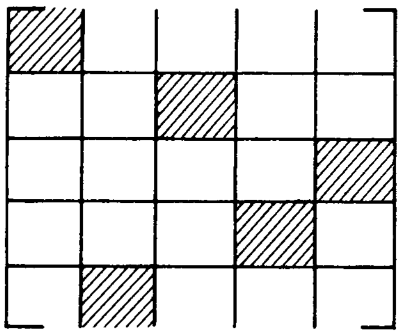
By an imprimitive matrix $\mathbf{M}$ is meant one which has the form of a superpermutation matrix. It is a linear transformation of a vector space V, which may be partitioned into subspaces V_i.

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

These possess the property that for some j, given i,

$$\mathbf{M}(V_i) \subset V_j$$

If a basis is chosen compatible to the subspaces V_i, $\mathbf{M}$ will have the form:



where the shaded blocks stand for non-zero submatrices. Thus it possess submatrices, which we may label $\mathbf{M}_{ij}$, labelled according to the subspaces V_i, for which only one submatrix in each row and column is non-zero. $\mathbf{M}_{ij} \neq 0$ when $\mathbf{M}(V_j) = V_i$.

A collection of imprimitive matrices is called transitive when, given any pair of subspaces V_i and V_j, there is a member of the set, M^k, for which $\mathbf{M}_{ij}^k \neq 0$. If such a collection is not transitive, but it forms a group, then the subspaces may be assorted into equivalence classes for which the transformations are transitive, but which cannot be mapped into one another. Thus if one assumes that he has an irreducible group of imprimitive matrices, the group must be transitive. Assuming that our collection of imprimitive matrices $\mathbf{M} = \{\mathbf{M}_z^1, \mathbf{M}_z^2, \cdots\}$ is a group, or better, a representation of a group, we see that the individual submatrices $\mathbf{M}_{ij}^k$, when non-zero, are square, invertible, and all of the same dimension. These properties may be deduced either by considering the determinants of $\mathbf{M}^k$, or by regarding its structure as a transformation of V. From the first point of view, if a submatrix were rectangular, say more columns than rows, some of the columns would be linearly dependent. Since the remaining components in $\mathbf{M}$ are zero, these columns will be dependent in $\mathbf{M}$, making it non-invertible. From the second point of view, if $\mathbf{M}_{ij}^k$ is rectangular, V_i is mapped into a space V_k of a different dimension, a mapping which is assuredly non-invertible. The other properties follow from similar reasoning.

It follows from the definition that the product of two imprimitive matrices is again an imprimitive matrix. In particular, if $\mathbf{D}(a)$ and $\mathbf{D}(b)$ are imprimitive, then their product $\mathbf{D}(a)\mathbf{D}(b)$ is also imprimitive and since this is a representation, we have

$$\mathbf{D}(a)\mathbf{D}(b) = \mathbf{D}(ab) \tag{41}$$

The representing matrix of the identity element of the group is a unit matrix in any representation; thus is always has the imprimitive form.

Considering now the block structure of the equation

$$\mathbf{D}(a)\mathbf{D}(a^{-1}) = \mathbf{D}(e) \tag{42}$$

we see that the non-zero block structure of $\mathbf{D}(a^{-1})$ is the transpose of the block structure of $\mathbf{D}(a)$. Furthermore, $\mathbf{D}(a^{-1}) = \mathbf{D}^{-1}(a)$ and, labelling the submatrices by (super) row and column indices, we see that $\mathbf{D}_{ij}(a^{-1}) = [\mathbf{D}_{ji}(a)]^{-1}$. In general we have

$$\mathbf{D}_{ij}(a)\mathbf{D}_{jk}(b) = \mathbf{D}_{ik}(ab) \tag{43}$$

since the rule is matrix multiplication for a representation, no sum appears because we have only one non-zero block per row and column.

Consider now the subset H of G such that $\mathbf{D}_{ii}(h) \neq 0$, $h \in H$. This subset is a subgroup of G.

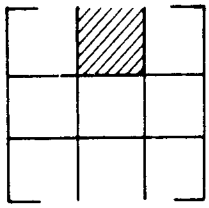
Proof: (*1*) H is non-empty and contains e, for $\mathbf{D}_{ii}(e) \neq 0$;
(*2*) H is closed, for

$$\mathbf{D}_{ii}(h)\mathbf{D}_{ii}(h') = \mathbf{D}_{ii}(hh') \qquad h, h' \in H; \tag{44}$$

(*3*) Associativity is fulfilled by the nature of the matrix product;
(*4*) $h \in H$ implies $h^{-1} \in H$, for the block structure is transposed for inverses.

For H we introduce the notation $[V^i]$, which we may call the *fixing group* of V^i, since it is the subgroup of G that keeps V^i fixed.

Let us now fix our attention on $[V']$, the subgroup that keeps the first subspace fixed. The matrix

(45) $\mathbf{D}(a) =$ 

(not saying where the other submatrices are, except of course they cannot be in the first row or in the column with the shaded block) is not in the set $[V^1]$, since it maps vectors into V^1. Upon multiplying $\mathbf{D}(a)$ by $\mathbf{D}(b) \in [V^1]$, we see that the product $D(ba)$ has a non-zero submatrix in the same column of the first row as does $\mathbf{D}(a)$. Since any element $h \in [V^1]$ could have been used, we see that all the elements of the right coset of $[V^1]$ which contains $\mathbf{D}(a)$ have a non-zero submatrix in the same column of the first row. Thus the matrices of the different right cosets of $[V^1]$ are distinguished by having a non-zero submatrix in a different column of the first row. It should be noted that the number of cosets divides the order of G and the number of subspaces divides the order of the carrier space, so if the orders are incompatible, the group cannot act as an imprimitive transitive group on this space.

Let us now select a matrix from each right coset; each has a non-zero submatrix in a different column of the first row. Having selected these arbitrarily, we can now find a basis for V such that the non-zero submatrices in the first row of these generators are unit matrices. This is accomplished by letting the submatrix in $\mathbf{D}(a)$, one of the selected matrices, be the image of the submatrix $\mathbf{D}_{11}(e)$ under a similarity transformation.

$$\mathbf{D}'(a) = \mathbf{U}\mathbf{D}(a)\mathbf{U}^{-1} \tag{46}$$

where

$$\mathbf{U} = \left[\begin{array}{c|c|c} \mathbf{D}_{11}(a) & & \\ \hline & \mathbf{D}_{12}(a) & \\ \hline & & \mathbf{D}_{13}(b) \end{array}\right] \tag{47}$$

for the case of only three right cosets. It follows that

$$\mathbf{U}^{-1} = \left[\begin{array}{c|c|c} \mathbf{D}_{11}(e) & & \\ \hline & \mathbf{D}_{12}(a^{-1}) & \\ \hline & & \mathbf{D}_{31}(b^{-1}) \end{array}\right] \tag{48}$$

since $(\mathbf{D}_{ij}(a))^{-1} = \mathbf{D}_{ji}(a^{-1})$, and under the transformation we have

$$\mathbf{D}'_{12}(a) = \mathbf{D}_{11}(e)\mathbf{D}_{12}(a)\mathbf{D}_{21}(a^{-1}) = \mathbf{D}_{11}(e) \tag{49}$$

since

$$\mathbf{D}_{12}(a)\mathbf{D}_{21}(a^{-1}) = \mathbf{D}_{11}(aa^{-1}) = \mathbf{D}_{11}(e) \tag{50}$$

In the same way $\mathbf{D}_{13}(b)$ becomes $\mathbf{D}_{11}(e)$, thus a basis has been found such that the representative of one element of each right coset of $[V^1]$ has a unit matrix in some column of the first row. From now on we may refer to these matrices having a unit submatrix in the first row as coset generators.

We now wish to show that selecting the first subspace as the one to be held fixed is not an essential part of the preceding argument, for we will now show that the representation of $[V^i]$, the subgroup that keeps V^i fixed, is equivalent to the representation of $[V^1]$.

Consider the equation

$$\mathbf{D}(a)\mathbf{D}(x)\mathbf{D}(a)^{-1} = \mathbf{D}(axa^{-1}) \tag{51}$$

where $\mathbf{D}_{1i}(a) \neq 0$, $\mathbf{D}_{ii}(x) \neq 0$, $\mathbf{D}_{i1}(a^{-1}) \neq 0$, $\mathbf{D}_{11}(axa^{-1}) \neq 0$. Doing the multiplication, we see that

$$\mathbf{D}_{1i}(a)\mathbf{D}_{ii}(x)\mathbf{D}_{i1}(a^{-1}) = \mathbf{D}_{11}(axa^{-1}) \tag{52}$$

therefore the representation of $[V^i]$ is equivalent to the representation of $[V^1]$. Furthermore, we may select $\mathbf{D}(a)$ as the generator of the ith coset; then $\mathbf{D}_{1i}(a)\mathbf{D}_{i1}(a^{-1})$ = unit matrix, and $\mathbf{D}_{ii}(x)$ is equal to $\mathbf{D}_{11}(axa^{-1})$, not merely equivalent to it.

The fact that

$$\mathbf{D}_{11}(h)\mathbf{D}_{11}(h') = \mathbf{D}_{11}(hh') \qquad h, h' \in [V^1] \tag{53}$$

defines the $\mathbf{D}_{11}$ submatrices as a representation of $[V^1]$, then $\{\mathbf{D}_{ii}(a^{-1}xa)\}$ is a representation of $[V^i]$.

$\{\mathbf{D}_{ii}(a^{-1}xa)\}$ is called a conjugate representation to $\{\mathbf{D}_{11}(x)\}$. We see that the fixing subgroups are not independent, in fact they are conjugate to one another. For our purposes, the most useful way to state this property is to note that it implies that $\mathbf{D}_{ii}(x)$ is zero unless $axa^{-1} \in [V^1]$, in which case $\mathbf{D}_{ii}(x) = \mathbf{D}_{11}(axa^{-1})$. Having established this fact, we are now in a position to evaluate any submatrix of a representative of a group element. The reasoning is the same as that just given for the evaluation of a diagonal submatrix, for there is nothing special about the diagonal submatrices. Thus, we want to evaluate $\mathbf{D}_{\eta\sigma}(a)$ where $a \in G$, and we agree that η and σ are the coset generators.

We write

$$\mathbf{D}_{\eta\sigma}(a)\mathbf{D}_{\eta\sigma}(\eta^{-1}\eta a\sigma^{-1}\sigma) \tag{54}$$

and since this is an imprimitive representation we have

$$\mathbf{D}_{\eta\sigma}(\eta^{-1}\eta a\sigma^{-1}\sigma) = \mathbf{D}_{\eta 1}(\eta^{-1})\mathbf{D}_{11}(\eta a\sigma^{-1})\mathbf{D}_{1\sigma}(\sigma) \tag{55}$$

But the first and last matrices on the right side are unit matrices, since η and σ are coset generators, therefore we have

$$\mathbf{D}_{\eta\sigma}(a) = \mathbf{D}_{11}(\eta a\sigma^{-1}) \tag{56}$$

Thus, the rule for calculating submatrices is that $\mathbf{D}_{\eta\sigma}(a)$ is zero unless $\eta a\sigma^{-1} \in [V^{-1}]$, in which case it is equal to the submatrix $\mathbf{D}_{11}(\eta a\sigma^{-1})$. This rule is clearly sufficient to construct the induced representation, given any representation of any subgroup of G.

5. Criteria for the Irreducibility and Inequivalence of Induced Representations

The criterion

$$\frac{1}{{}^{\circ}G} \sum_{a\in G} \chi(a)\chi(a^{-1}) = 1 \tag{57}$$

for the irreducibility of a representation of G whose character is χ, as well as the criterion

$$\frac{1}{{}^{\circ}G} \sum_{a\in G} \chi^1(a)\chi^2(a^{-1}) = 0 \tag{58}$$

for the inequivalence of two irreducible representations, may be applied to induced representations. Since the character of an induced representation may be calculated from that of the inducing representation, these two criteria may be stated in terms of properties of the inducing representation.

We recall the imprimitive structure of an induced representation, as well as the formula

$$\mathbf{D}_{\eta}(a) = \begin{cases} d(\xi a\eta^{-1}) & \xi a\eta^{-1} \in H \\ 0 & \xi a\eta^{-1} \notin H \end{cases}$$

for the submatrices indexed by the coset generators ξ and η. We shall adopt the custom of using capital letters, such as D, X, etc. to refer to the induced representation, while small letters d, χ, etc. will refer to the subgroup H from which the representation is induced.

The character of the induced representation is obtained by summing the diagonal elements of the representation matrices, but we have to include in the sum only those which are non-zero. Thus we have

$$X(a) = \frac{1}{{}^{\circ}H} \sum_{(\xi: a \in \xi^{-1} H \xi)} \chi(\xi a \xi^{-1}) \tag{59}$$

since $\xi^{-1}H\xi$ contains those elements whose diagonal elements are non-zero. The factor $1/{}^{\circ}H$ arises from extending the sum over all ξ's which satisfy the equation, and not merely coset generators. However, we can state the condition in terms of the characteristic function $\delta(a:\xi^{-1}H\xi)$ of the conjugate subgroup $\xi^{-1}H\xi$ by writing

$$X(a) = \frac{1}{{}^{\circ}H} \sum_{\xi \in G} \chi(\xi a \xi^{-1})\ \delta(a:\xi^{-1}H\xi) \tag{60}$$

If we apply the criterion (57) for irreducibility in terms of this formula, we find ourselves calculating

$$\begin{aligned}
&\frac{1}{{}^{\circ}G} \sum_{a \in G} \left\{ \frac{1}{{}^{\circ}H} \sum_{\xi \in G} \chi(\xi a \xi^{-1})\ \delta(a:\xi^{-1}H\xi) \right\} \left\{ \frac{1}{{}^{\circ}H} \sum_{\eta \in G} \chi(\eta a^{-1} \eta^{-1})\ \delta(a^{-1}:\eta^{-1}H\eta) \right\} \\
&\qquad = \frac{1}{{}^{\circ}G {}^{\circ}H {}^{\circ}H} \sum_{a,\xi,\eta \in G} \chi(\xi a \xi^{-1}) \chi(\eta a^{-1} \eta^{-1})\ \delta(a:\xi^{-1}H\xi)\ \delta(a^{-1}:\eta^{-1}H\eta) \\
&\qquad = \frac{1}{{}^{\circ}G {}^{\circ}H {}^{\circ}H} \sum_{a,\xi,\eta \in G} \chi(\xi a \xi^{-1}) \chi(\eta a^{-1} \eta^{-1})\ \delta(a:\xi^{-1}H\xi \cap \eta^{-1}H\eta)
\end{aligned}$$

The last step follows from the fact that the product of two characteristic functions is the characteristic function of the intersection of their sets, and from the fact that if a^{-1} belongs to the subgroup $\eta^{-1}H\eta$, so also does a. Continuing by calling $\xi a \xi^{-1} = x$, $\xi\eta^{-1} = \theta$, we have

$$\frac{1}{{}^{\circ}H {}^{\circ}H} \sum_{x,\theta} \chi(x) \chi(\theta^{-1} x^{-1} \theta)\ \delta(x:H \cap \theta H \theta^{-1})$$

The factor ${}^{\circ}G$ is cancelled by the superfluous sum over the group. We can delete another factor ${}^{\circ}H$ if the θ-sum is replaced by a coset sum, so that our criterion finally becomes, dropping the characteristic functions and restricting the range of summation for the coset of e, and separating that first term from the other,

$$1 = \frac{1}{{}^{\circ}H} \sum_{\chi \in H} \chi(x)\chi(x^{-1}) + \sum_{\substack{\text{cosets other} \\ \text{than } H}} \frac{1}{{}^{\circ}H} \sum_{\chi \in H} \chi(x)\chi(\theta^{-1}x^{-1}\theta)\ \delta(x:H \cap \theta H \theta^{-1}) \tag{61}$$

Now, the first term is already equal to 1 if χ is an irreducible representation of H; otherwise it is already greater than 1. The second term refers to the equivalence of the conjugate representations $d(b)$ and $d(\theta^{-1}b\theta)$ of the subgroup $H \cap \theta H \theta^{-1}$, which is their common range of definition. These representations must be inequivalent, otherwise they contribute a positive non-zero amount to the sum which must not exceed 1.

A special simplification occurs when H is normal, for then $\theta H \theta^{-1} = H$. In this latter case, the criterion can be summarized as follows: A representation induced from a normal subgroup is irreducible when the inducing representation is (*1*) irreducible and (*2*) inequivalent to its conjugate by elements extraneous to H.

In any event, the same conditions hold, but the second must be modified by considering the range of common definitions of the two conjugate representations.

The second criterion, for the inequivalence of two irreducible induced representations, may be handled in a similar manner, the only difference in the derivation being that the two characters involved must be subscripted. As a result the final criterion is:

$$0 = \frac{1}{{}^{\circ}H} \sum_{x \in H} \chi^1(x)\chi^2(x^{-1}) + \sum_{\substack{\text{cosets other} \\ \text{than } H}} \frac{1}{{}^{\circ}H} \sum_{x \in H} \chi(x)\chi(\theta^{-1}x^{-1}\theta)\ \delta(x : H \cap \theta H \theta^{-1}) \tag{62}$$

Again, all terms are positive, so that each must vanish to satisfy the criterion. Hence, the inducing representations must be inequivalent, and one of them must not be the conjugate of the other.

When H is normal, we may assort its representations into conjugate stars. A representation induced by one of these representations is irreducible when the inducing representation is irreducible, and each coset of H generates a distinct prong of its star. All representations in the same star generate equivalent induced representations, but distinct stars generate inequivalent representations.

If H is not normal, the appropriate restriction of the conjugates must be made, but the theorem still holds in similar terms.

6. Anticommuting Operators, Unit Square

In his attempt to factorize a general quadratic expression

$$q = x_1^2 + x_2^2 + \cdots + x_\nu^2 \tag{63}$$

into a product of linear factors, Dirac [6] introduced a system of hypercomplex numbers $\mathbf{d} = \{\gamma_1, \gamma_2, \cdots, \gamma_\nu\}$ having the property that

$$\gamma_i^2 = 1 \tag{64a}$$

$$\gamma_i \gamma_j = -\gamma_j \gamma_i \tag{64b}$$

By writing

$$l = x_1\gamma_1 + x_2\gamma_2 + \cdots + x_\nu\gamma_\nu \tag{65}$$

we find that

$$q = l^2 \tag{66}$$

since the anticommutativity of the γ's causes the cross products to disappear from the square, whereas the fact that $\gamma^2 = 1$ removes the hypercomplex numbers from the squared terms.

Although one can consider the equations (64a and b) as the defining relations of an *algebra*, namely the Clifford algebra $\mathbf{C}_n$ of degree n, it is also possible to regard them as defining a *group*, whose group algebra contains such expressions as the right-hand side of equation (65). These groups we shall call the *Dirac groups* $\mathbf{d}_n$. Since the elements of $\mathbf{d}_n$ are at most scalar multiples of the generators of $\mathbf{C}_n$, it follows that an irreducible representation of the algebra $\mathbf{C}_n$ yields an irreducible representation of the group $\mathbf{d}_n$, and conversely.

Speaking algebraically, $\mathbf{d}_n$ has a multiplicative hull composed (disregarding sign) of all the possible products of different numbers of γ's. However, if repeated factors occur in such a product, rule (64a) may be used to delete them in pairs, while rule (64b) may be used to arrange the factors always in standard order. We regard the scalar 1 as the product of no factors. Thus the number of elements of $((\mathbf{d}_n))$ is limited to the number of products of distinct γ's arranged in standard order, which is 2^n. Allowing the two possible signs for each such term finally shows that $\mathbf{d}_n$ has $2 \cdot 2^n$ elements, of which $2 \cdot \binom{n}{k}$ are homogeneous of degree k, k ranging from 0 to n.

In order to adequately describe the group multiplication table for $\mathbf{d}_n$ we shall devote the remainder of this section to a review of Artin's arguments [13], in the notation of which we omit the traditional γ, retaining only the subscripts. Thus, we define $\mathbf{d}_n$ to be the set of ordered pairs

$$\mathbf{d}_n = \{(\lambda, I) : \lambda \in \{-1, 1\} \qquad I\{1, 2, \cdots, n\}\} \tag{67}$$

for which are defined the rule of multiplication

$$(\lambda, I) \cdot (\mu, J) = \left(\lambda\mu \prod_{\substack{m \in I \\ n \in J}} (m, n), I \,\Delta J\right) \tag{68}$$

in which

$$(m, n) = \begin{array}{l} 1 m < n \\ -1 m > n \end{array}$$

and Δ denotes the symmetric difference of the sets I and J.

To verify that equations (67) and (68) define a group, we write

$$(\lambda, I) \cdot (\mu, J) \cdot (\nu, K) = \left(\lambda\mu\nu \prod_{\substack{l \in I \\ m l J}} (l, m) \prod_{\substack{l \in I \\ n \in K}} (l, n) \prod_{\substack{m \in J \\ n \in K}} (m, n), I \,\Delta J \,\Delta K\right) \tag{69}$$

This symmetric form of the threefold product makes the associative law at once apparent, and is permissible because the superfluous factors introduced by taking the products over $J \cup K$ rather than $J \Delta K$ (for instance) arise from $J \cap K$ where each occurs twice and squares to yield a factor of 1.

The identity of $\mathbf{d}_n$ is

$$(1, \varnothing) = e \tag{70}$$

where $\varnothing$ is the empty set. The inverse of an element is given by the formula

$$(\lambda, I)^{-1} = \left(\lambda^{-1} \prod_{m,n \in I} (m, n), I\right) \tag{71}$$

Thus, an inverse differs at most in sign from the element itself.

To determine the structure of $\mathbf{d}_n$ one compares the two products,

$$(\lambda, I) \cdot (\mu, J) = \left(\lambda\mu \prod_{\substack{m \in I \\ n \in J}} (n, m), I \Delta J\right)$$

$$(\mu, J) \cdot (\lambda, I) = \left(\mu\lambda \prod_{\substack{m \in I \\ n \in J}} (n, m), J \Delta I\right)$$

finding that they are identical save for a sign which is

$$\sigma(I, J) = \prod_{\substack{m \in I \\ n \in J}} (m, n)(n, m) = (-1)^{{}^\circ I {}^\circ J {}^\circ (I \cap J)}$$

a superscript $^\circ$ denoting the order of the set in question. This means that conjugates differ at most in sign, since

$$(\chi, I)(\mu, J)(\lambda, I)^{-1} = (\sigma(I, J)\mu, J) \tag{72}$$

Moreover, since I can be chosen to be a set consisting of precisely two elements, one belonging to J and one not, unless $J = \varnothing$ or $J = N(N = \{1, 2, \cdots, n\})$ the sign change is always possible for the remaining cases. It is easily seen that no sign change is possible for $J = \varnothing$ while for $J = N$, the result depends upon whether ${}^\circ N$ is even or odd. When ${}^\circ N$ (hence n) is odd, no such choice is possible, but when it is even, I can be any set with an odd number of elements.

This is our first indication of a dichotomy between N even and N odd. For N even, only the elements $(1, \varnothing)$, $(-1, \varnothing)$ sit in classes by themselves; all other classes are of order 2, containing simultaneously (λ, I) and $(-\lambda, I)$. For n odd, $(1, N)$ and $(-1, N)$ also sit in classes by themselves.

Equation (72) also tells us about the possible commutators in $\mathbf{d}_n$; if we factor its right hand side we find

$$(\sigma(I, J)\mu, J) = (\sigma(I, J), \varnothing)(\mu; J) \tag{73}$$

and thus for possible commutators we find

$$(\lambda, I)(\mu, J)(\lambda, I)^{-1} = (\sigma(I, J), \varnothing) \tag{74}$$

Since we have seen that both signs are possible for σ, and since $\{(1, \varnothing), (-1, \varnothing)\}$ already forms a subgroup, it is the commutator subgroup, W. Its index, 2^n, is the number of one-dimensional representations of $\mathbf{d}_n$.

In the factor group $\mathbf{d}_n/W$, the distinction between an element and its negative is lost. Thus the square of every element in the factor group is the identity, and such a group is a direct product of cyclic groups of order 2

$$\mathbf{d}_n/W \simeq C_2 \times C_2 \times C_2 \times \cdots \times C_2 \qquad (n \text{ factors}) \tag{75}$$

Since $\mathbf{d}_n$ is generated by the elements $(1, \{i\})$, they can also be taken as coset generators for W.

Thus the 2^n possible one-dimensional representations of $\mathbf{d}_n$ arise from the 2^n possible assignments of $+1$ or -1 to each of these generators.

The number of classes of $\mathbf{d}_n$ for n even, is $2^n + 1$ while for n odd, it is $2^n + 2$. Thus there are respectively, for n even, one irreducible representation of dimension $2^{n/2}$ and for n odd, two irreducible representations each of dimension $2^{(n-1)/2}$. According to the theory of reference [1], these representations will be induced, by the subgroups which are the kernels of the one-dimensional representations. By selecting the one-dimensional representation properly, we can make it an inductive process whereby each $\mathbf{d}_n$ is determined by $\mathbf{d}_m$, $m < n$, until we reach the trivial cases of low degree.

7. The Representations of Anticommuting Operators with Unit Square

The representations of the Dirac group composed of anticommuting operators with unit square may be determined inductively, on account of the fact that they may be induced by a subgroup of the Dirac group which is again of the same type. To do so, we commence with the generator $(1, \{n\})$ which anticommutes with all the other generators $(1, \{i\})$. Since we are going to work with representations which are not one-dimensional, not all the elements of the commutator subgroup can be represented by the unit matrix. Since our commutator subgroup is contained in the center of $\mathbf{d}_n$, it must be represented by multiples of the unit matrix, and since it is of order 2, the element $\{-1, \varnothing\}$ must be represented by $-I$. Thus the anticommutation of the defining relations persists in the large representations.

To make the ensuing discussion more specific, we introduce the following notation:

$\gamma(\lambda, J)$ for the representative of (λ, J), $(\pm i, \pm i, \cdots, \pm i)$ for the one-dimensional representation of $\mathbf{d}_n$ which makes the corresponding assignment of ± 1 to the generators $(1, \{i\})$, in that order.

Given a representation γ of $\mathbf{d}_n$, the matrix $\boldsymbol{\gamma}(1, \{n\})$ has $(-1, -1, \cdots, 1)$ as its multiplier in the exchange group. Since $\boldsymbol{\gamma}(1, \{n\})$ takes the place of the operator H in equation (1), we can write it in diagonal form. Since our exchange relation involves multiplication by the scalar -1, the spectrum of A will contain two points, $\pm\xi$. Since $\boldsymbol{\gamma}(1, \{n\})^2 = 1$, we have $\xi = 1$; were $\boldsymbol{\gamma}$ not a root of unity, ξ would have to be determined in some other way.

$$(76) \qquad \boldsymbol{\gamma}(1, \{n\}) = \begin{bmatrix} +1 & & & & & & \\ & +1 & & & & & \\ & & +1 & & & & \\ & & & \cdots & & & \\ & & & & -1 & & \\ & & & & & -1 & \\ & & & & & & -1 \end{bmatrix} = I \otimes \Omega$$

where

$$\boldsymbol{\Omega} = \begin{bmatrix} +1 & \\ & -1 \end{bmatrix}$$

The matrices $\boldsymbol{\gamma}(1, \{i\})$ generate the operators A_i of equation (17), whence we conclude that we deal with an induced representation, induced by $(-1, -1, \cdots, +1)^{-1}$. In treating this kernel, we shall have to study the structure of $\mathbf{d}_n$ further. Since $\boldsymbol{\gamma}(1, \{i\})$ anticommutes with $\boldsymbol{\gamma}(1, \{n\})$, it seems that $\boldsymbol{\gamma}(1, \{i, j\})$ commutes with $\boldsymbol{\gamma}(1, \{n\})$ for $i, j \neq n$.

In fact, the subset

$$(77) \qquad \mathbf{d}_n^+ = ((\{(\lambda, \{i, j\}) : i \neq j\}))$$

is composed of those elements (λ, J) for which J consists of an *even* number of elements, and is a normal subgroup of $\mathbf{d}_n$. It is generated by the elements of the form

$$(\lambda, \{i, n\}) \qquad i \neq n$$

since $\{i, j\}$ is either already of this form, or else $\{i, n\} + \{j, n\} = \{i, j\}$. The mapping

$$(\lambda, \{i, n\}) \to (\lambda, \{i\})$$

between the generators $\mathbf{d}_n^+$ and $\mathbf{d}_{n-1}$ defines an isomorphism of these two groups. Care must be taken with the sign in mapping elements other than the generators, and in fact, we have the correspondence for $i, j \neq n$

$$(\lambda, \{i, j\}) \to (-\lambda, \{i, j\})$$

on account of the fact that in $\mathbf{d}_n^+$

$$(\lambda, \{i, n\})(\mu, \{j, n\}) = (-\lambda_\mu, \{i, j\})$$

The fact that this assignment preserves the products of generators ensures that the entire mapping will be a homomorphism, however.

At this point, we have two choices in describing the representation of $\mathbf{d}_n$. If we deal with the entire group $\mathbf{d}_n^+$, the kernel of $(-1, -1, \cdots, +1)$ is complicated by the fact that only those elements of $\mathbf{d}_n^+$ not containing the digit n belong to the kernel, whereas $(1, \{n\})$ itself belongs to the kernel but not $\mathbf{d}_n^+$. Consequently it is slightly more convenient to restrict our attention to $\mathbf{d}_{n-1}$ (defined by dropping the nth generator), and regarding $\gamma(1, \{n\})$ as an external matrix, obeying an exchange rule with respect to all the matrices representing $\mathbf{d}_{n-1}$. Since $\mathbf{d}_{n-1}^+ \simeq \mathbf{d}_{n-2}$, the Dirac group of degree 2 less than the original is the group inducing the representation of $\mathbf{d}_n$. Thus we may determine the irreducible representations of $\mathbf{d}_n$ through a chain

$$\mathbf{d}_n \supset \mathbf{d}_{n-2} \supset \mathbf{d}_{n-4} \supset \cdots \supset \mathbf{d}_0 \text{ or } \mathbf{d}_1 \tag{78}$$

The distinction between groups of even and odd order is manifested by the fact that this chain includes every other Dirac group, starting from a given n.

The nature of induced representations is such that one selects a representation $\boldsymbol{\gamma}$ of dimension l, of subgroup H and a set of coset generators, $(\xi, \eta, \cdots)$, k in number. The dimension of the induced representations is $k \cdot l$, comprised of submatrices $\mathbf{D}_{\xi\eta}$ of dimension l, indexed according to the coset generators. If x is an element of G such that ξ and η are the coset generators for which $\xi x \eta^{-1} \in H$, then $D_{\xi\eta}(x) = \hat{\boldsymbol{\gamma}}(\xi x \eta^{-1})$; otherwise it is zero.

In our case there are two cosets, so that we might take $\xi = (1, \varnothing)$, $\eta = (1, \{n, n-1\})$. The two diagonal submatrices vanish while those on the skew diagonal are both equal to $\hat{\boldsymbol{\gamma}}(1, \{i, n-1\})$. The particular choice of η made them equal; had we set $\eta = (1, \{n-1\})$, they would have been negative. The two representations would have been equivalent, in any event.

In summary

$$\begin{aligned} \boldsymbol{\gamma}(1, \{i\}) &= \left[\begin{array}{c|c} & \hat{\gamma}(1, \{n, n-1, i\}) \\ \hline \hat{\gamma}(1, \{n, n-1, i\}) & \end{array}\right] \\ &= \hat{\gamma}(1, \{n, n-1, i\}) \otimes S \end{aligned} \tag{79}$$

where

$$S = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

The calculation is then reduced to finding the representation $\hat{\boldsymbol{\gamma}}(1, \{n, n-1, i\})$ of $\mathbf{d}_{n-2}$.

The results agree with those obtained by other methods: for instance, the result reported by Boerner [41] or in Albert's analysis of the Clifford algebra [16].

8. General Exchange Relation

The concept of a Dirac group, as we have so far discussed it, can be generalized in two ways, both of which involve modifying equation (64), which epitomizes these groups. One, already subsumed in Artin's general analysis of Clifford algebras [13], is to modify equation (64a) by allowing the γ's to have a more general square, or even some higher power N_i which is the lowest integer for which γ^{N_i} is a scalar:

$$\gamma_i^{N_i} = \xi_i \tag{80}$$

For our purposes, of constructing a finite Dirac group, it will be necessary to restrict ξ_i to be a root of unity, identifying ξ_i with the proper representative of ξ_i.

Of course, modifications to the definition of a Dirac group, such as this one of changing the power which becomes a scalar, may be fairly far reaching, involving such changes in the analysis as altering the order of the Dirac group, the size of classes, and the number of irreducible representations. From an algebraic point of view, unless one is committed to a scalar field which lacks the necessary square roots, such a change is trivial. One could always replace γ_i by $\gamma_i/(\xi_i)^{1/2}$ to obtain the earlier rules, when $N_i = 2$. Similar replacements would be possible for other values of N_i. However, the modifications wrought in a Dirac group by such a change are more significant, and may disturb appreciably the structure of the group.

The second generalization consists of replacing equation (64b) by a more general exchange relation

$$\gamma_i \gamma_j = \omega_{ij} \gamma_j \gamma_i \tag{81}$$

We shall make both changes simultaneously.

Consequently, we replace the definition of equation (67) by

$$\mathbf{d}_n = \{(\lambda, a_1, a_2, \cdots, a_n) : \lambda \in \Sigma,\ a_i \in Z/((n_i))\} \tag{82}$$

where Z is the additive group of integers, and Σ is a cyclic group which we shall call the sign group.

Since the factors ξ_i arising from powers of the generators, and the exchange factors ω_{ij} as well as any scalar coefficients are roots of unity, there exists a primitive root $\rho = e^{2\pi i/N}$, of which each of the others is a power. Σ is the hull of this element,

$$\Sigma = ((\rho)) \tag{83}$$

As an example of the use to which a generalized Dirac group could be put, recall Dirac's original use of the $\boldsymbol{\gamma}$-matrices to factor a quadratic expression into the square of a linear expression. Given a general power sum,

$$p = x_1^k + x_2^k + \cdots + x_n^k \tag{84}$$

if we introduce γ's satisfying equation (64a) and (81), with $\omega_{ij} = e^{2\pi i/k}$, we can set

$$l = x_1\gamma_1 + x_2\gamma_2 + \cdots + x_n\gamma_n \tag{85}$$

to find that

$$l^k = p$$

Having determined to generalize the definition of a Dirac group, we must pay careful attention to a suitable notation, both to define the group in question, and to specify its elements. To define the group, assuming the integers N which define ρ and hence the cyclic group $\Sigma = C_N$ to be given, we need two more items of information. One is the list

$$(N_1R_1N_2R_2 \cdots N_nR_n)$$

whose elements satisfy the equation

$$\gamma_i^{N_i} = \rho^{R_i} = \xi_i \tag{86}$$

(and thus $\rho^{N/R_i} = 1$). The other is the list

$$(\omega_{12}\omega_{13} \cdots \omega_{1n}\omega_{23} \cdots \omega_{2n} \cdots \omega_{n-1,n})$$

comprising the exchange factors satisfying equation (81) arranged in a lexicographic order. Not all ω_{ij}'s need to be specified. Obviously,

$$\gamma_i\gamma_i = \gamma_i\gamma_i$$

whence

$$\omega_{ii} = 1$$

while from

$$\gamma_i\gamma_j = \omega_{ij}\gamma_j\gamma_i$$

we deduce

$$\omega_{ij} = \omega_{ji}^{-1} \tag{87}$$

The ω's still cannot be chosen arbitrarily. From

$$\gamma_i\gamma_j^2 = \omega_{ij}^2\gamma_j^2\gamma_i$$

and so on for higher powers, we must have

$$\omega_{ij}^{nj} = \omega_{ij}^{nj} = 1 \tag{88}$$

since the scalar γ_j^{nj} commutes with γ_i.

These convey all the information required to define a specific Dirac group. We write

$$\mathbf{d}_n = (NN_1R_1N_2R_2 \cdots N_nR_n\omega_{12} \cdots \omega_{n-1,n})$$

or more concisely,

$$\mathbf{d}_n = (NRW)$$

The subscript n is redundant, as we could determine it from the number of elements in either R or W, but is nevertheless most convenient to have it stated explicitly. It is the number of generators used in the definition.

To specify an individual element of a Dirac group, we use the alternate notation:

$$(\lambda, a_1 \cdots a_n) = \rho^\lambda\gamma_1^{a_1}\gamma_2^{a_2} \cdots \gamma_n^{a_n} \tag{89}$$

The n-tuple $(\lambda, a_1 \cdots a_n)$, sometimes abbreviated (λ, u), is convenient for actual calculation; while the γ-notation is very suggestive, not only in applications, but in heuristic discussions of the structure of the Dirac group.

In defining the rule of multiplication for a Dirac group $\mathbf{d}$, let us first note that as a set, $\mathbf{d}$ is a cartesian product

$$\mathbf{d} = C_0 \times C_1 \times \cdots \times C_n$$

which is sometimes preferably abbreviated to

$$\mathbf{d} = C_0 \times \mathbf{C} \tag{90}$$

The sign group C_0 is of degree N, while the cyclic groups C_i are of degree N_i. Thus

$$°\mathbf{d}_n = N\prod_{i=1}^{n} n_i \tag{91}$$

We shall call $\mathbf{C}$ the Dirac *module* belonging to the Dirac group $\mathbf{d}$. As a direct product of additive abelian groups, it is a module with respect to the integers Z, and is used in this sense when the rule of multiplication is defined in n-tuple form. Actually, the multiplication rules are contained implicitly in the generating relations (80) and (81). Thus, it is only necessary to adopt a standard form for a product of generators to define the multiplication in the γ-notation, and to transcribe it to the n-tuple form.

Suppose that we are given any product whatsoever involving γ's and scalars. First of all, the scalars can all be multiplied together and collected at the front of the expression, since they commute with all the generators. Then, if there are two adjacent γ's whose subscripts are not in ascending order from left to right, say, their order can be reversed, yielding a factor ω_{ij} which can be multiplied into the already existing scalar coefficient. In this way, all the γ's can be collected together, and arranged in increasing order of subscripts. Repeated γ's can be written as powers, and reduced mod n_i, to yield another factor for the scalar coefficient, and a power of γ_i less than n_i.

Thus the canonical form for a γ expression is

$$\lambda\gamma_1^{a_1}\gamma_2^{a_2}, \cdots, \gamma_n^{a_n}$$

with $a_i < n_i$, whose correspondence to an n-tuple we already know.

If two such expressions are to be multiplied, we might have

$$(\lambda\gamma_1^{a_1}\gamma_2^{a_2}\cdots\gamma_n^{a_n})\cdot(\mu\gamma_1^{b_1}\gamma_2^{b_2}\cdots\gamma_n^{b_n})$$

We might start by multiplying λ and μ, then move $\gamma_1^{b_1}$ to the left. Each of the $b_1\gamma_1$'s must be exchanged with $a_n\gamma_n$'s, for which we have

$$\gamma_n^{a_n}\gamma_1^{b_1} = \omega_{n1}^{a_nb_1}\gamma_1^{b_1}\gamma_n^{a_n} \tag{92}$$

Then they must be exchanged with $a_{n-1}\gamma_{n-1}$'s, so that by the time they have passed all the way to the left we have

$$\lambda\mu\omega_{n1}^{a_nb_1}\omega_{n-1,1}^{a_{n-1}b_1}\omega_{n-2,1}^{a_{n-2}b_1}\cdots\omega_{21}^{a_2b_1}\gamma_1^{a_1+b_1}\gamma_2^{a_2}\cdots\gamma_n^{a_n}\gamma_2^{b_2}\cdots\gamma_n^{b_n}$$

Moving $\gamma_2^{b_2}$ as far to the left as $\gamma_2^{a_2}$, then $\gamma_3^{b_3}$ and so on finally yields an expression

$$\begin{aligned}\lambda\mu\omega_{n1}^{a_nb_1}\cdots\omega_{21}^{a_2b_1}\omega_{n2}^{a_nb_2}\cdots\omega_{32}^{a_3b_2}\cdots\omega_{n-1,n}^{a_{n-1}b_n}\gamma_1^{a_1+b_1}\gamma_2^{a_2+b_2}\cdots\gamma_n^{a_n+b_n}\\ = \lambda\mu\prod_{i,j}\omega_{ij}^{a_ib_j}\gamma_1^{a_1+b_1}\gamma_2^{a_2+b_2}\cdots\gamma_n^{a_n+b_n}\end{aligned} \tag{93}$$

Now it is time to reduce $a_i + b_i \bmod n_i$. Let

$$\alpha = [\alpha/n]\cdot n + (\alpha) \bmod n$$

define the symbols involved, which gives us finally

$$\begin{aligned}\lambda\mu\xi_1^{[a_1+b_1]n_1}\xi_2^{[a_2+b_2]n_2}\cdots\xi_n^{[a_n+b_n]n_n}\prod_{ij}\omega_{ij}^{a_ib_j}\gamma_1^{(a_1+b_1)\bmod n_1}\\ \cdot\gamma_2^{(a_2+b_2)\bmod n_2}\cdots\gamma_n^{(a_n+b_n)\bmod n_n}\end{aligned}$$

This then is the rule of multiplication for two elements of a Dirac group.

To rephrase this rule in the n-tuple notation, for which we are essentially dealing with logarithms, we first introduce some definitions. Let

$$\omega_{ij} = \rho^{w_{ij}}$$

(94)
$$\lambda = \rho^{l}$$

$$\mu = \rho^{m}$$

$$\xi_i = \rho^{c_i}$$

when

$$\rho = e^{2\pi i/N}$$

Then, equation (87) shows that the matrix $\mathbf{w}_{ij}$ is antisymmetric

$$\mathbf{w}_{ij} = -\mathbf{w}_{ji} \qquad (w_{ii} = 0)$$

while the consistency condition (88) requires

$$n_j\mathbf{w}_{ij} = n_i\mathbf{w}_{ij} \equiv 0 \ (\mathrm{mod}\ N)$$

Now the multiplication rule becomes

$$(la_1 \cdots a_n)(mb_2 \cdots b_n) = (l + m + \varepsilon(I, J) + \rho(I, J), (a_1 + b_1) \bmod n_1,$$

(95)
$$(a_2 + b_2) \bmod n_2 \cdots$$

where $I = (a_1 \cdots a_n)$, $J = (b_1 \cdots b_2)$,

(96)
$$\varepsilon(I, J) = \sum_{ij} a_i w_{ij} b_j$$

(97)
$$\rho(I, J) = \sum_{i} c_i[a_i + b_i]_{n_i}$$

The notation mod n_i may be dropped, if we understand that the sum refers to addition in the Dirac module; however in defining $[a_i + b_i]$, integer arithmetic must be retained.

The *exchange factor* ε, accounts for all the ω_n acquired by using the exchange rules to collect like terms into the same standard order; a consequence of equation (81), as we have seen. To *reduction factor* ρ, arises when the number of factors exceeds the maximum number allowed by equation (80).

Having deduced the rule of group multiplication, it remains to verify the group postulates. In so doing, we thereby establish that any set of parameters obeying the consistency condition defines a Dirac group. For this task we use the n-tuple notation.

Closure follows from the nature of the definition. The verification of the associative law is done by writing the formula for a threefold product in the symmetrical form

$$(98)\quad (lz_1 \cdots z_n)(mz'_1 \cdots z'_n)(nz''_1 \cdots z''_n) = \\ (l + m + n + \sum \{z_i w_{ij} z'_j + z_i w_{ik} z''_k + z'_j w_{jk} z''_k\} \\ + \sum c_i [z_i + z'_i + z''_i](z_1 + z'_1 + z''_2) \bmod n_i \cdots)$$

The consistency condition enters into the proof of associativity because we have to be assured that such relations as

$$(99)\qquad z_i w_{ij}(z'_j + z''_j) \bmod n_j = z_i w_{ij} z'_j + z_i w_{ij} z''_j$$

are valid. Their validity might be questioned because the quantities $(z'_j + z''_j)$ are reduced mod n_j before the second product is computed. In fact, the consistency condition itself arose from just this consideration. However, the two expressions would differ at most by a term of the form $z_i w_{ij}(k_{nj})$ which is congruent to zero mod N, the number system in which the coefficients are computed.

We must also verify that the bracket $[z_i + z'_i + z''_i]$ is equivalent to either $[z_i + (z'_i + z''_i) \bmod n_i] + [z'_i + z''_i]$ or $[z_i + z'_i] + [(z_i + z'_i) \bmod n_i + z''_i]$, but this is also apparent.

A similar symmetrical form may be used to describe a product of arbitrarily many factors.

The identity is

$$(100)\qquad e = (0, 0 \cdots 0)$$

since $(0\ 0\ 0\ 0)(\lambda a_1 a_2 a_3) = (\lambda a_1 a_2 a_3)$
while

$$(101)\quad (la_1 \cdots a_n)^{-1} = (-l + \sum a_i w_{ij} a_j - \sum c_i, n_1 - a_1, n_2 - a_2, \cdots n_n - a_n)$$

We could avoid the term Σc_i in the coefficient if we wrote $-a_i$ rather than $n_i - a_i$ for the negatives of the vectors in the Dirac module.

9. The Dirac Module

Having established the rule of multiplication

$$(102)\qquad (l, I)(m, J) = (l + m + \rho(I, J) + \varepsilon(I, J), I + J)$$

for two elements of a Dirac group written in n-tuple form, we next undertake to determine the structure of these groups. Since vector addition is employed in the Dirac module itself, non-commutativity in the product will be a consequence of the behaviour of the functions ρ and ξ.

On account of its definition,

$$\varepsilon(I, J) = \sum_{ij} a_i w_{ij} b_j \tag{103}$$

it is readily seen that ε is a bilinear functional, devoid of any particular symmetry properties, however. Nevertheless it obeys the rule

$$\varepsilon(I, J) + \varepsilon(I + J, K) = \varepsilon(I + J + K) + \varepsilon(J, K) \tag{104}$$

which (in the additive form) shows that it is a multiplier for a projective representation of the Dirac module. Since the sum $I + J$ is a module sum, the consistency condition on the elements w_{ij} is required here as it was in the justification of the associative law.

The function ρ

$$\rho(I, J) = \sum_{i} c_i[a_i + b_i] \tag{105}$$

is not linear, but is still a multiplier. Thus,

$$[a_i + b_i] \neq [a_i] + [b_i] \tag{106a}$$

but

$$[a_i + b_i] + [(a_i + b_i) \bmod n_i + c_i] = [a_i + b_i + c_i] \tag{106b}$$

while

$$[a_i + (b_i + c_i) \bmod n_i] + [b_i + c_i] = [a_i + b_i + c_i] \tag{106c}$$

so that

$$\rho(I, J) + \rho(I + J, K) = \rho(I, J + K) + \rho(J, K) \tag{107}$$

Since the sum of two multipliers is a multiplier, we see that the Dirac group is a projective extension of the Dirac module.

The formula for the conjugate of an element is

$$(l, I)(m, J)(l, J)^{-1} = (m + \varepsilon(I, J) - \varepsilon(J, I), J) \tag{108}$$

and thereby differs from (m, J) by only a scalar factor. The antisymmetric form $\varepsilon(I, J) - \varepsilon(J, I)$ has a representation as an inner product

$$\varepsilon(I, J) - \varepsilon(J, I) = (I, WJ) = ((I, J)) \tag{109}$$

with respect to the matrix $\mathbf{W}$ whose elements are w_{ij}. Since this matrix itself is antisymmetric, the inner product so defined is symplectic. We shall think of the

Dirac module geometrically as a lattice, having this skew inner product. The conjugate of an element (m, J) with respect to (l, I) is then

$$(m + ((I, J)), J)$$

which differs from (m, J) by the projection of I on J according to the skew form, which is added to m.

The normalizer of (m, J) is composed of all those elements (l, I) which conjugate it into itself, which in the present case means all those elements for which $(I, J) = 0$.

$$N_{(m,J)} = \{(l, I)\mathbf{d} : (I, J) = 0\} \tag{110}$$

Geometrically, the normalizer of an element is the hyperplane orthogonal to it.

The center of $\mathbf{d}$ is composed of those elements which are orthogonal to every other element (including themselves). Thus, it is a maximal isotropic subspace.

Since the commutator of two elements is given by

$$(l, I)(m, J)(l, I)^{-1} = ((I, J), 0) \tag{111}$$

we see that the commutator subgroup is a subgroup of C_0, composed of the possible values of the bilinear form (I, J).

We may summarize these results in a diagram which differs from an ordinary lattice in that we still plot the points of the Dirac module as though they were coordinates in a cartesian space, but the scalar coefficients are indicated by the vertices of an N-gon placed at each lattice point.

Suppose that one defined, using the γ-notation

$$\delta_i = \gamma_1^{a_{i1}}\gamma_2^{a_{i2}} \cdots \gamma_n^{a_{in}} \tag{112a}$$

$$\delta_j = \gamma_1^{a_{j1}}\gamma_2^{a_{j2}} \cdots \gamma_n^{a_{jn}} \tag{112b}$$

Then we have

$$\begin{aligned} \delta_i\delta_j &= \gamma_1^{a_{i1}}\gamma_2^{a_{i2}} \cdots \gamma_n^{a_{in}}\gamma_1^{a_{j1}}\gamma_2^{a_{j2}} \cdots \gamma_n^{a_{jn}} \\ &= \omega_{n1}^{a_{in}a_{j1}}\omega_{n-1,1}^{a_{in-1}a_{j1}} \cdots \gamma_1^{a_{j1}}\gamma_1^{a_{i1}} \cdots \gamma_n^{a_{in}}\gamma_2^{a_{j2}} \cdots \gamma_n^{a_{jn}} \end{aligned} \tag{113}$$

In this case, we are going to exchange completely the order of the factors δ_i and δ_j, unlike the earlier calculations in which we simply reduced the product to canonical form. Thus we get a product of factors ω_{ij} without the restriction $i \leqq j$. Thus, we continue to find

$$\delta_i\delta_j = \omega_{n1}^{a_{in}a_{j1}} \cdots \omega_{21}^{i2aa_{j1}}\omega_{n2}^{a_{in}a_{j2}} \cdots \omega_{n1}^{a_{i1}a_{jn}}\gamma_1^{a_{j1}}\gamma_2^{a_{j2}}\gamma_1^{a_{i1}}\gamma_2^{a_{i2}} \cdots$$

and finally

$$\delta_i \delta_j = \prod_{i \neq j} \omega_{pq}^{a_{ip} a_{jq}} \tag{114}$$

In the n-tuple notation, we would have

$$(\delta_0, \delta)(\varepsilon_0, \varepsilon) = (\alpha, 0)(\varepsilon_0, \varepsilon)(\delta_0, \delta) \tag{115}$$

where

$$\alpha = \sum_{p=q} w_{pq} a_{ip} a_{jq}$$

This relation has two interpretations. If we think of i and j as fixed, then α is a certain scalar. If the sum in its definition is divided into two parts, one for $p > q$ and one for $p < q$, we have, on account of the antisymmetry of **W**,

$$\alpha = \varepsilon(\delta, W\varepsilon) - \varepsilon(\varepsilon, W\delta) \tag{116a}$$

$$\alpha = ((\delta, \varepsilon)) \tag{116b}$$

Thus, the exchange factor for two arbitrary elements of the Dirac module is given by their inner product.

To obtain a second interpretation, let i and j be variable, as would be the case if we were introducing a new set of generators, δ_i in place of the γ's. With respect to these new generators we obtain a new matrix of exchange factors, related to the old by

$$\mathbf{W}' = \mathbf{AWA}^+ \tag{117}$$

where **A** is the matrix of exponents a_{st} defining the new generators. If **A** is invertible over the integers, it follows that the new generators are equally good as the old, and may be used to yield an equivalent definition of the Dirac group, as soon as their reduction factors are also found.

The power of δ_i, n_i, which is a scalar is the least common multiple of the corresponding powers n_{ij} for each of its factors

$$n_i = \underset{i=1,n}{lcm} \; (n_{i1}, n_{i2}, \cdots, n_{in}) \tag{118}$$

By raising δ_i to this power, one can ascertain the scalar c_i. We have

$$(\delta_0, \delta)^2 = (2\delta_0 + \rho_2(\delta) + \varepsilon(\delta, \delta), 2\delta)$$

$$(\delta_0, \delta)^3 = (3\delta_0 + \rho_3(\delta) + 3\varepsilon(\delta, \delta), 3\delta)$$

$$\cdots$$

$$(\delta_0, \delta)^k = \left(k\delta_0 + \rho^k(\delta) + \frac{k(k-1)}{2} \varepsilon(\delta, \delta), k\delta\right) \tag{119}$$

when $\rho_k(\delta)$ means $\sum_i c_i[kn_i]$.

10. List of Some Dirac Groups of Low Order

We tabulate here as examples all the Dirac groups of order 8, 16, and 27. The low order groups are of special interest since chains of Dirac groups, with each group a normal subgroup of the next group, may be obtained by augmenting generators to a low order group. If we include the degenerate (Abelian) Dirac groups, then all groups of orders 8, 16, and 27 are Dirac groups. We will include the degenerate cases, since non-Abelian groups of higher order may be obtained from these by augmenting generators which fail to commute with the already specified generators.

We find it convenient to use the logarithmic notation in specifying a Dirac group. Thus, if the cyclic group of scalars consists of powers of $\rho = e^{2\pi i/K}$, then we give the integer K. The exchange factors, and the scalars which are equal to (specified) powers of the generators are then integral powers of ρ. These integers, and the integral powers of the generators which yield scalars are given by defining the function

$$N = (\eta_1, \xi_1, \eta_2, \xi_2, \cdots, \eta_k, \xi_k) \tag{120}$$

and the exchange function

$$W = (w_{12}, w_{13}, \cdots, w_{k1}, w_{23}, \cdots, w_{k-1,k}) \tag{121}$$

where k is the number of generators, η_i is the power of the ith generator which yields a scalar, ξ_i is the logarithm of that scalar to the base ρ and w_{ij} is the logarithm of the exchange factor in

$$\gamma_i\gamma_j = \omega_{ij}\gamma_j\gamma_i$$

The exchange factor ω_{ji}, ji, is given by

$$\omega_{ji} = \omega_{ji}^{-1}$$

so that w_{ji} is the additive inverse modulo K of w_{ij}. The number of generators can be inferred from N or W, but we will give this number anyhow, for added clarity. As an example of the notation described here, the quaternion group, of order 8, has two generators, and the values of K, W and N are

$$K = 2, \qquad W = (1), \qquad N = (2, 1, 2, 1)$$

This way of describing a group is highly redundant, which is to say that, for a given number of generators, there are many different values of K, W, and N

which lead to the same group. In our tabulation we will list, for fixed K and W, only those values of N which lead to distinct groups. We will also list the transformations whereby groups specified by other values of N can be shown to be isomorphic to one of the cases listed. However, we will not list the "naming transformations", which merely permute the names of the generators, are always present, and usually connect two or more values of N. For example, $N = (2021)$ and $N = (2120)$ will clearly give rise to isomorphic groups. Thus we will give only the automorphism transformations, in which an element must be mapped into another element of the same order. Finally, there are additional isomorphisms between groups having different values of K and W or different numbers of generators. Our notation is not suited to writing transformations which exhibit these isomorphisms, so since we are going to list for each group the number of elements of order 1, 2, 4, 8 etc., we observe instead that, except for a single case which we will indicate in the tables, groups in the same table having the same numbers of elements of different orders are isomorphic. The tables of degenerate groups are given first, since the headings are slightly different (W does not have to be specified).

TABLE I. Degenerate cases of order eight. Three distinct groups

#	Generators	K	N	Elements of order 1	2	4	8	Transformations and comments
1	3	0	(2, 0, 2, 0, 2, 0)	1	7			$C_2 \times C_2 \times C_2$
2	2	0	(4, 0, 2, 0)	1	3	4		$C_2 \times C_4$
3		2	(2, 0, 2, 0)	1	7			$C_2 \times C_2 \times C_2$
4			(2, 0, 2, 1)	1	3	4		$C_2 \times C_4$ $T = \{\gamma_1 \to \gamma_1\gamma_2, \gamma_2 \to \gamma_2\}$
5	1	0	(8, 0)	1	1	2	4	C_8
6		2	(4, 0)	1	3	4		$C_2 \times C_4$
7			(4, 1)	1	1	2	4	C_8
8		4	(2, 1)	1	1	2	4	C_8
9			(2, 2)	1	3	4		$C_2 \times C_4$ $T = \{\gamma \to i\gamma\}$
10	0	8		1	1	2	4	C_8

TABLE II. Degenerate cases of order 16. Five distinct groups

#	Generators	K	K	Elements of order 1	2	4	8	16	Transformations and comments
1	4	0	(2, 0, 2, 0, 2, 0, 2, 0)	1	15				$C_2 \times C_2 \times C_2 \times C_2$
2	3	0	(2, 0, 2, 0, 4, 0)	1	7	8			$C_2 \times C_2 \times C_4$
3		2	(2, 0, 2, 0, 2, 0)	1	15				$C_2 \times C_2 \times C_2 \times C_2$
4			(2, 0, 2, 0, 2, 1)	1	7	8			$C_2 \times C_2 \times C_4$
									$T = \{\gamma_1 \rightarrow \gamma_3, \gamma_2 \rightarrow \gamma_1\gamma_3,$
									$\gamma_3 \rightarrow \gamma_2\}$
5	2	0	(4, 0, 4, 0)	1	3	12			$C_4 \times C_4$
6		2	(2, 0, 4, 0)	1	7	8			$C_2 \times C_2 \times C_4$
7			(2, 0, 4, 1)	1	3	4	8		$C_2 \times C_8$
									$T_1 = \{\gamma_1 \rightarrow \gamma_1^2, \gamma_2^2 \rightarrow \gamma_2\}$
									$T_2 = \{\gamma_1 \rightarrow \gamma_1\gamma_2^2, \gamma_2 \rightarrow \gamma_2\}$
8		4	(2, 0, 2, 0)	1	7	8			$C_2 \times C_2 \times C_4$
9			(2, 0, 2, 1)	1	3	4	8		$C_2 \times C_8$
									$T_1 = \{\gamma_1 \rightarrow i\gamma_2, \gamma_2 \rightarrow \gamma_1\}$
									$T_2 = \{\gamma_1 \rightarrow i\gamma_1\gamma_2, \gamma_2 \rightarrow 1\}$
10	1	0	(16, 0)	1	1	2	4	8	C_{16}
11		2	(8, 0)	1	3	4	8		$C_2 \times C_8$
12			(8, 1)	1	1	2	4	8	C_{16}
13		4	(4, 1)	1	1	2	4	8	C_{16}
14			(4, 2)	1	3	4	8		$C_2 \times C_8$
15			(4, 0)	1	3	12			$C_4 \times C_4$
									$T = \{\gamma \rightarrow \gamma^3\}$
16		8	(2, 0)	1	3	4	8		$C_2 \times C_8$
17			(2, 1)	1	1	2	4	8	C_{16}
									$T = \{\gamma \rightarrow \omega\gamma\}$
									$\omega^8 = 1$
18	0	16		1	1	2	4	8	C_{16}

TABLE III. Degenerate cases of order 27. Three distinct groups

#	Generators	K	N	Elements of order 1	3	9	27	Transformations and comments
1	3	0	(3, 0, 3, 0, 3, 0)	1	26			$C_3 \times C_3 \times C_3$
2	2	0	(9, 0, 3, 0)	1	8	18		$C_3 \times C_9$
3			(3, 0, 3, 0)	1	26			$C_3 \times C_3 \times C_3$
4			(3, 0, 3, 1)	1	8	18		$C_3 \times C_9$ $T = \{\gamma_1 \to \gamma_1\gamma_2, \gamma_2 \to \gamma_2\}$
5	1	0	(27, 0)	1	2	6	18	C_{27}
6		3	(9, 0)	1	8	18		$C_3 \times C_9$
7			(9, 1)	1	2	6	18	C_{27} $T = \{\gamma^9 \to \omega \times \gamma^9\}$
8		9	(3, 0)	1	8	18		$C_3 \times C_9$
9			(3, 1)	1	2	6	18	C_{27} $T = \{\gamma^3 \to \omega\gamma^3\}$ ω = some ninth root of one
10	0	27		1	2	6	18	C_{27}

TABLE IV. Non-degenerate cases of order 8. Two distinct groups

#	Generators	K	W	N	Elements of order 1	2	4	Transformations and comments
1	2	2	(1)	(2, 1, 2, 1)	1	1	6	Quaternions
2				(2, 0, 2, 0)	1	5	6	$T = \{\gamma_1 \to \gamma_1\gamma_2, \gamma_2 \to \gamma_2\}$

TABLE V. Non-degenerate cases of order 16. Nine distinct groups

#	Gene-rators	K	W	N	Elements of order 1	2	4	8	Transformations and comments
1	3	2	(1, 1, 0)	(2, 1, 2, 1, 2, 1)	1	3	12		*Not* isomorphic to #8
2				(2, 1, 2, 0, 2, 0)	1	11	4		
3				(2, 0, 2, 0, 2, 0)	1	7	8		
									$T = \{\gamma_1 \to \gamma_1\gamma_3, \gamma_2 \to \gamma_2\gamma_3, \gamma_3 \to \gamma_3\}$
4			(1, 1, 0)	(2, 1, 2, 1, 2, 1)	1	5	10		
5				(2, 1, 2, 1, 2, 0)	1	7	8		
6				(2, 0, 2, 0, 2, 0)	1	11	4		$T_1 = \{\gamma_1 \to \gamma_1\gamma_3, \gamma_2 \to \gamma_2\gamma_3, \gamma_3 \to \gamma_3\}$
									$T_2 = \{\gamma_1 \to \gamma_1\gamma_2, \gamma_2 \to \gamma_2, \gamma_3 \to \gamma_3\}$
7	2	2	(1)	(2, 1, 4, 1)	1	3	4	8	*Not* isomorphic to #1
8				(2, 1, 4, 0)	1	3	12		
9				(2, 0, 4, 0)	1	7	8		
									$T = \{\gamma_1 \to \gamma_1\gamma_2^2, \gamma_2 \to \gamma_2\}$
10		4	(3)	(2, 1, 2, 1)	1	1	2	12	
11				(2, 1, 2, 2)	1	5	6	4	
12				(2, 2, 2, 2)	1	5	8	2	
									$T = \{\gamma_1 \to i\gamma_1, \gamma_2 \to i\gamma_2\}$
13			(2)	(2, 1, 2, 1)	1	3	4	8	
				(2, 2, 2, 2)	1	7	8		$T_1 = \{\gamma_1 \to \gamma_1, \gamma_2 \to \gamma_1\gamma_2\}$
									$T_2 = \{\gamma_1 \to \gamma_1, \gamma_2 \to i\gamma_2\}$

TABLE VI. Non-degenerate cases of order 27. Two distinct groups

#	Gene-rators	K	W	N	Elements of order 1	3	9	Transformations and comments
1	2	3	(1)	(3, 0, 3, 0)	1	26		
				(3, 0, 3, 1)	1	8	18	
								$T = \{\gamma_1 \to \gamma_1\gamma_2, \gamma_2 \to \gamma_2\}$

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Bibliography

[1] H. V. McIntosh, J. Mol. Spectr. **8,** 169–192 (1962).
[2] F. Ceconi, Ann. Univ. Toscane Ns **14** fasc. 2 1–49 (1931); review in Zentr. Mathematik Grenzgebiete **4,** 241 (1932).
[3] S. Cherubino, Atti Congr. Un. Mat. Ital. 342–345 (1938); review in Zentr. Mathematik Grenzgebiete **20,** 100 (1938).
[4] T. Kurosaki, Proc. Imp. Acad. Japan **17,** 24–28 (1941); reviews in Zentr. Mathematik Grenzgebiete **25,** 6 (1942), Math. Reviews **3,** 99 (1942) MF 4335.
[5] M. P. Drazin, Proc. Cambridge Phil. Soc. **47,** 7–10 (1951); Math. Reviews **12,** 582 (1951).
[6] P. A. M. Dirac, Proc. Roy. Soc. (London) Ser **A 117,** 610–624 (1928).
[7] G. Temple, Proc. Roy. Soc. (London) Ser **A 127,** 339–348 (1930).
[8] B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Julius Springer, Berlin, 1932), pp. 54–55.
[9] W. Pauli, Ann. Inst. Poincare **6,** 109–136 (1936).
[10] R. H. Good, Jr., Revs. Mod. Phys. **27,** 187–211 (1955).
[11] W. K. Clifford, Am. J. Math. **1,** 350–358 (1878).
[12] E. Witt, J. Reine Angew. Math. **176,** 31–44 (1937).
[13] E. Artin, *Geometric Algebra* (Interscience Publishers, New York, 1957), chapters 4 and 5.
[14] R. Brauer and H. Weyl, Am. J. Math. **57,** 425–449 (1935).
[15] H. C. Lee, J. London Math. Soc. **20,** 27–32 (1945), Ann. Math. **49,** 760–773 (1948).
[16] A. A. Albert, Math. Reviews **10,** 180 (1949), **7,** 361 (1946).
[17] A. S. Eddington, *Fundamental Theory,* (Cambridge University Press, 1953).
[18] D. E. Littlewood, J. London Math. Soc. **9,** 41–50 (1934).
[19] A. S. Eddington, J. London Math. Soc. **7,** 58–68 (1932).
[20] M. H. A. Newmann, J. London Math. Soc. **7,** 93–99s (1932).
[21] W. H. McCrea, Proc. Cambridge Phil. Soc. **35,** 123–125 (1939).
[22] A. S. Eddington, J. London Math. Soc. **8,** 142–152 (1933).
[23] Harish-Chandra, Proc. Indian Acad. Sci. Sect **A 22,** 30–41 (1945).
[24] B. S. Madhara Rao, Proc. Indian Acad. Sci. Sect **A 22,** 408–422 (1945).
[25] Harish-Chandra, Proc. Roy. Soc. (London) Ser **A 186,** 502–525 (1946), Proc. Cambridge Phil. Soc. **43,** 406–413 (1947).
[26] N. Kemmer, Proc. Cambridge Phil. Soc. **39,** 189–196 (1943). A. H. Wilson, Proc. Cambridge Phil. Soc. **36,** 363–380 (1940). D. E. Littlewood, Proc. Cambridge Phil. Soc. **43,** 406–413 (1947).
[27] N. Heerema, Duke Math. J. **22,** 423–443 (1954).
[28] L. J. Mordell, Quart. J. Math. **8,** 58–61 (1937).
[29] J. J. Sylvester, Johns Hopkins University Circulars **I** (1882) pp. 241, 242; **II** (1883) 46, reprinted in the Collected Mathematical Papers of James Joseph Sylvester (University Press Cambridge 1909), volume IV, pp. 647–650.
[30] J. Zak, Phys. Rev. **134,** A1602–A1606 (1964); A1607–A1611 (1964); **136,** A776–780 (1964); A1647–1649 (1964); **139,** A1159–1162 (1965).
[31] E. Brown, Phys. Rev. **133,** A1038–A1044 (1964).
[32] E. Daltabuit and H. V. McIntosh, Rev. Mex. Fis. **16,** 105–114 (1967).

[33] A. O. Barut and S. Komy, J. Math. Phys. **7,** 1903–1907 (1966).
[34] A. O. Barut, J. Math. Phys. **7,** 1908–1910 (1966).
[35] R. Frucht, J. Reine Angew. Math. **166,** 16–29 (1931).
[36] A. O. Morris, Quart. J. Math. **17,** 7–12 (1966); **19,** 289–299 (1968); references therein to K. Yamazaki, J. Fac. Sci. Univ. Tokyo **10,** 147–195 (1964); M. F. Atiyah, R. Roth and A. Shapiro, Topology **3,** 147–195 (1964).
[37] I. Schur, J. Reine Angew. Math. **132,** 85–137 (1907).
[38] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras,* Interscience Publishers, New York (1962).
[39] G. Frobenius, Sitzber. Preuss. Akad. Wiss. (1898) 501–515.
[40] G. W. Mackey, Ann. J. Math. **73,** 576–592 (1951).
[41] H. Boerner, *Darstellungen von Gruppen* (Springer Verlag, Berlin, 1955), chapters 8 and 2.

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