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The induced linguistic continuous ordered weighted geometric operator and its application to group decision making ☆

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ABSTRACT

In this paper, based on the induced linguistic ordered weighted geometric (ILOWG) operator and the linguistic continuous ordered weighted geometric (LCOWG) operator, we develop the induced linguistic continuous ordered weighted geometric (ILCOWG) operator, which is very suitable for group decision making (GDM) problems taking the form of uncertain multiplicative linguistic preference relations. We also present the consistency of uncertain multiplicative linguistic preference relation and study some properties of the ILCOWG operator. Then we propose the relative consensus degree ILCOWG (RCD-ILCOWG) operator, which can be used as the order-inducing variable to induce the ordering of the arguments before aggregation. In order to determine the weights of experts in group decision making (GDM), we define a new distance measure based on the LCOWG operator and develop a nonlinear model on the basis of the criterion of minimizing the distance of the uncertain multiplicative linguistic preference relations. Finally, we analyze the applicability of the new approach in a financial GDM problem concerning the selection of investments.

1. Introduction

Group decision making (GDM) consists of finding the best alternative(s) from a feasible set. To do this, experts have to express their preferences by means of a set of alternatives over a set of alternatives and find the best alternative(s) by using a proper aggregation technique (Alonso, Cabrerizo, Chiclana, Herrera, & Herrera-Viedma, 2009). Various types of preference relations have been investigated in the literature, including the multiplicative preference relation (Herrera, Herrera-Viedma, & Chiclana, 2001; Saaty, 1980; Xu, 2000), the fuzzy preference relation (Herrera-Viedma, Herrera, Chiclana, & Luque, 2004; Wu & Cao, 2011), the multiplicative linguistic preference relation (Dong, Xu, & Yu, 2009; Wu, Cao, & Zhang, 2011; Xu, 2004c), the fuzzy linguistic preference relation (Alonso et al., 2009; Chen, Wang, & Wu, 2011; Dong, Xu, & Li, 2008; Xu, 2005) and the intuitionistic fuzzy preference relation (Gong, Li, Zhou, & Yao, 2009; Xu, 2007b, 2008). However, due to time pressure, lack of knowledge and limited expertise related to the problem domain, experts are willing to provide uncertain information. For example, experts may be possible to give the uncertain fuzzy preference relations (Chen & Zhou, 2011; Xu, 2006b, 2011b), the uncertain multiplicative preference relation (Chen & Zhou, 2012; Wang & Elhag, 2007; Wang, Yang, & Xu, 2005; Wu, Cao, & Zhang, 2010; Wu, Li, Li, & Duan, 2009; Yager & Xu, 2006), the uncertain multiplicative linguistic preference relation (Xu, 2006a), the uncertain fuzzy linguistic preference relation (Chen, Zhou, & Han, 2011; Wang & Chen, 2010; Xu, 2004a) and the interval-valued intuitionistic fuzzy preference relation (Xu, 2010).

Consistency is a very important problem in GDM using preference relations (Xu, 2007a). The lack of consistency can lead to inconsistent conclusions in decision making with preference relations. There have been a lot of discussions on consistency of preference relations (Dong et al., 2008; Genc, Boran, Akay, & Xu, 2010; Herrera-Viedma et al., 2004; Ma, Fan, & Jiang, 2006; Saaty, 1980; Wang & Chen, 2008; Xia & Xu, 2011b; Xu, 2011b). For example, Herrera-Viedma et al. (2004) proposed the additive transitivity property of the fuzzy preference relations, which can be used to construct consistent fuzzy preference relations. Xia and Xu (2011b) developed a new method to avoid inconsistency based on the multiplicative consistency of the fuzzy preference relation. Dong et al. (2008) presented a consistency index of linguistic preference relations and developed a consistency measure method for linguistic preference relations. Wang and Chen (2008) applied fuzzy linguistic preference relations to construct a pairwise comparison matrix with additive reciprocal property and consistency.

Another crucial issue of GDM is to find the proper way to aggregate experts' preferences. The ordered weighted averaging (OWA) operator introduced by Yager (1988) is a useful tool for aggregating

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the exact arguments that lie between the max and the min operators. Since it has appeared, the OWA operator has been studied in a wide range of applications and extensions (Calvo, Mayor, & Mesiar, 2002; Chen, Lee, Liu, & Yang, 2012; Li, 2011; Liu, 2011; Merigó & Gil-Lafuente, 2009, 2010, 2011b; Merigó, Gil-Lafuente, Zhou, & Chen, 2011; Wei, 2010b; Wei & Zhao, 2012b; Wu & Cao, 2013; Xia & Xu, 2011a; Xia, Xu, & Zhu, 2012; Xu, 2004b, 2006c, 2007c, 2008, 2011a; Xu & Chen, 2008; Xu & Da, 2002a, 2002b; Xu & Xia, 2011a, 2011b; Yager, 2004b; Yager & Kacprzyk, 1997; Yager, Kacprzyk, & Beliakov, 2011; Yang & Chen, 2012; Yu, Wu, & Lu, 2012; Yu & Xu, 2013; Yue, 2011; Zeng & Su, 2011; Zhang, Jiang, Jia, & Luo, 2011; Zhao, Xu, Ni, & Liu, 2010; Zhou & Chen, 2010, 2012; Zhou, Chen, & Liu, 2012, 2013; Zhou, Chen, Merigó, & Gil-Lafuente, 2012). A very practical extension of the OWA operator is the continuous ordered weighted averaging (COWA) operator (Yager, 2004a) in which the argument to be aggregated is interval number (Sengupta & Pal, 2009). In the last few years, many researchers have paid attention to the different extensions and applications of the COWA operator (Chen, Liu, & Wang, 2008; Chen & Zhou, 2011, 2012; Wu et al., 2010; Wu et al., 2009; Xu, 2006b; Yager & Xu, 2006; Zhang & Xu, 2005; Zhou & Chen, 2011; Zhou, Chen, Wang, & Ding, 2010). For example, Yager and Xu (2006) extended the COWA operator and obtained the continuous ordered weighted geometric (COWG) operator. Zhang and Xu (2005) extended the COWG operator to the linguistic environment and obtained the linguistic COWG (LCOWG) operator. Another useful extension of the OWA operator is the induced ordered weighted averaging (IOWA) operator (Yager & Filev, 1999), which uses order-inducing variables in the reordering of the arguments. Recently, several authors have developed different extensions and applications of the IOWA operator (Chiclana, Herrera, Herrera-Viedma, Herrera, & Alonso, 2004; Merigó, 2011; Merigó & Casanovas, 2011a, 2011b, 2011c; Merigó & Gil-Lafuente, 2009, 2011a; Merigó, Gil-Lafuente, Zhou, & Chen, 2012; Su, Xia, Chen, & Wang, 2012; Wei, 2010a; Wei & Zhao, 2012a; Xu, 2006d; Xu & Wang, 2012; Xu & Xia, 2011a). For example, Chiclana et al. (2004) presented the induced ordered weighted geometric (IOWG) operator, which is very suitable for aggregating multiplicative preference relations. In order to aggregate the uncertain multiplicative preference relations, Wu et al. (2009) developed the induced continuous ordered weighted geometric (ICOWG) operator. Xu (2006) proposed the induced linguistic ordered weighted geometric (ILOWG) operator, which can be used to aggregate the multiplicative linguistic preference relations.

The aim of this paper is to develop the ILCOWG operator which is very suitable for GDM problems taking the form of uncertain multiplicative linguistic preference relations. Firstly, we present the ILCOWG operator based on ILOWG operator and the LCOWG operator. Then, we propose the consistency of uncertain multiplicative linguistic preference relation and study some properties of the ILCOWG operator. In particular, we develop the relative consensus degree ILCOWG (RCD-ILCOWG) operator, which applies the ordering of the arguments based on the relative consensus degree of the preference information. Furthermore, we define a new distance measure based on the LCOWG operator and develop a nonlinear model to determine experts' weights based on the criterion of minimizing the distance. Finally, the ILCOWG operator is applied to group decision making with uncertain multiplicative linguistic preference relations.

In order to do so, this paper is organized as follows. In Section 2, we briefly review some basic concepts. Section 3 presents ILCOWG operator and investigates some properties of the ILCOWG operator. In Section 4, we present the RCD-ILCOWG operator and build up the optimal model to determine the optimal experts' weights in GDM. Section 5 proposes a new approach for uncertain multiplicative linguistic preference relations. In Section 6, we develop an illustrative example of the new approach focusing on the selection of investment. Finally, in Section 7 we summarize the main conclusions of the paper.

2. Preliminaries

In this section, we briefly review the uncertain multiplicative linguistic variable, the uncertain multiplicative linguistic preference relation, the OWA operator, the COWG operator, the ILOWG operator, the ICOWG operator and the LCOWG operator.

2.1. Uncertain multiplicative linguistic variable and operational laws

Let $S = \{s_\alpha | \alpha = 1/t, \ldots, 1/2, 1, 2, \ldots, t\}$ be a multiplicative linguistic label set with odd cardinality, which requires that the multiplicative linguistic label set should satisfy the following characteristics (Herrera & Herrera-Viedma, 2000; Xu, 2004d):

(1) The set S is ordered: if $s_\alpha, s_\beta \in S$ and $\alpha > \beta$, then $s_\alpha > s_\beta$.
(2) There exists the reciprocal operator: $rec(s_\alpha) = s_\beta$ such that $\alpha\beta = 1$,

where s_α and s_β represent possible values for the linguistic variables and t is a positive integer.

The multiplicative linguistic label set S is called the multiplicative linguistic scale. For example, a set of nine labels S can be defined as:

$$S = \{s_{1/5} = EL, s_{1/4} = VL, s_{1/3} = L, s_{1/2} = SL, s_1 = M, s_2 = SH, s_3 = H, s_4 = VH, s_5 = EH\}.$$

Note that *EL* = *Extremely low*, *VL* = *Very low*, *L* = *Low*, *SL* = *Slightly low*, *M* = *Medium*, *SH* = *Slightly high*, *H* = *High*, *VH* = *Very high*, *EH* = *Extremely high*.

To preserve all the given information, we can extend the discrete linguistic term set S to a continuous linguistic term set $\widetilde{S} = \{s_\alpha | \alpha \in [1/q, q]\}$, where $q(q > t)$ is a sufficiently large positive integer (Xu, 2006a). If $s_\alpha \in S$, we call s_α the original multiplicative linguistic term, which is provided to evaluate alternatives by the decision makers, otherwise, we call s_α the virtual multiplicative linguistic term, which can only appear in operations.

Definition 1 Xu, 2006a. Let $\tilde{s} = [s_\alpha, s_\beta] = \{x | s_\alpha \leqslant x \leqslant s_\beta\}$, then $\tilde{s}$ is called the uncertain multiplicative linguistic variable, where $s_\alpha, s_\beta \in \widetilde{S}, s_\alpha, s_\beta$ are the lower and upper limits, respectively. Especially, $\tilde{s}$ is called the multiplicative linguistic variable if $s_\alpha = s_\beta$.

Suppose that $\tilde{s} = [s_\alpha, s_\beta], \tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$ are any three uncertain multiplicative linguistic variables, and μ, μ_1, $\mu_2 \in [0, 1]$. Xu (2006a) defined some operational laws as follows:

(1) $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \otimes [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \otimes s_{\alpha_2}, s_{\beta_1} \otimes s_{\beta_2}] = [s_{\alpha_1\alpha_2}, s_{\beta_1\beta_2}]$.
(2) $\tilde{s}_1 \otimes \tilde{s}_2 = \tilde{s}_2 \otimes \tilde{s}_1$.
(3) $\tilde{s}^\mu = [s_\alpha, s_\beta]^\mu = \left[s_\alpha^\mu, s_\beta^\mu\right] = \left[s_{\alpha^\mu}, s_{\beta^\mu}\right]$.
(4) $\tilde{s}^{\mu_1} \otimes \tilde{s}^{\mu_2} = \tilde{s}^{\mu_1+\mu_2}$.
(5) $(\tilde{s}_1 \otimes \tilde{s}_2)^\mu = \tilde{s}_1^\mu \otimes \tilde{s}_2^\mu$.

2.2. Uncertain multiplicative linguistic preference relation

In a GDM problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives. When an expert makes pairwise comparisons using the multiplicative linguistic term set S, he/she can express his/her own opinions by a multiplicative linguistic preference relation on X. The multiplicative linguistic preference relation can be defined as follows:

Definition 2 Xu, 2004d. A multiplicative linguistic preference relation $\widehat{A}=(\hat{a}_{ij})_{n\times n}$ on the set X is denoted by a linguistic decision matrix $\widehat{A}=(\hat{a}_{ij})_{n\times n}\subset X\times X$, such that

$$\hat{a}_{ij}\in S,\quad \hat{a}_{ij}\otimes\hat{a}_{ji}=s_1,\quad \hat{a}_{ii}=s_1,\quad \forall i,j=1,2,\ldots,n, \tag{1}$$

where $\hat{a}_{ij}$ represents the preference degree of the alternative x_i over x_j. Especially, $\hat{a}_{ij}=s_1$ indicates that x_i is equivalent to x_j, $\hat{a}_{ij}>s_1$ indicates that x_i is preferred to x_j, and $\hat{a}_{ij}<s_1$ indicates that x_j is preferred to x_i.

A very crucial property of multiplicative linguistic preference relation is the consistency. It can be defined as follows:

Definition 3 Xu, 2004d. Let $\widehat{A}=(\hat{a}_{ij})_{n\times n}$ be a multiplicative linguistic preference relation, then $\widehat{A}$ is called a consistent linguistic preference relation if there exists $\hat{a}_{ik}\otimes\hat{a}_{kj}=\hat{a}_{ij}$ for i, j, k = 1, 2, ..., n.

However, experts may be possible to provide only uncertain multiplicative linguistic preference relations because of time pressure, lack of knowledge or data and their limited expertise related to the problem domain. The uncertain multiplicative linguistic preference relation can be defined as follows.

Definition 4 Xu, 2006a. An uncertain multiplicative linguistic preference relation on the set X is defined as matrix $\widetilde{A}=(\tilde{a}_{ij})_{n\times n}\subset X\times X$ satisfying

$$\tilde{a}^U_{ij}\otimes\tilde{a}^L_{ji}=s_1,\quad \tilde{a}^L_{ij}\otimes\tilde{a}^U_{ji}=s_1,\quad \tilde{a}^U_{ij}=\tilde{a}^L_{ij}=s_1,\quad \forall i,j=1,2,\ldots,n, \tag{2}$$

where $\tilde{a}_{ij}=\left[\tilde{a}^L_{ij},\tilde{a}^U_{ij}\right]$ indicates the multiplicative linguistic preference relation degree of the alternative x_i over x_j, $\tilde{a}^L_{ij},\tilde{a}^U_{ij}\in\widetilde{S}$, $\tilde{a}^U_{ij}\geqslant\tilde{a}^L_{ij}$, $\tilde{a}^L_{ij}$ and $\tilde{a}^U_{ij}$ are the lower and upper bounds of uncertain multiplicative linguistic variables $\tilde{a}_{ij}$, respectively.

Note that throughout this paper, let M_n be the set of all $n\times n$ uncertain multiplicative linguistic preference relations. For convenience, supposing that $s_\alpha\in\widetilde{S}$, $I(s_\alpha)$ denotes the lower index of multiplicative linguistic term s_α, then we have $I(s_\alpha)=\alpha>0$.

2.3. The OWG operator, the IOWG operator and the ILOWG operator

The OWA operator (Yager, 1988) is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum, which can be defined as follows:

Definition 5 Yager, 1988. An OWA operator of dimension n is a mapping OWA: $R^n\to R$ that has an associated weighting vector w with $\sum_{j=1}^n w_j=1$ and $w_j\in[0,1]$, such that

$$OWA(a_1,a_2,\ldots,a_n)=\sum_{j=1}^n w_j b_j, \tag{3}$$

where b_j is the jth largest of the arguments a_1, a_2, ..., a_n.

The OWA operator is monotonic, commutative, bounded and idempotent. Other properties could be studied such as different families of the OWA operators (Calvo et al., 2002; Fodor, Marichal, & Roubens, 1995; Yager & Kacprzyk, 1997; Yager et al., 2011). Moreover, the OWA operator is suitable for aggregating the fuzzy preference relations. Motivated by Yager (1988), Chiclana, Herrera and Herrera-Viedma (2000); Xu and Da (2002b) developed the ordered weighted geometric (OWG) operator, which is based on the OWA operator and the geometric mean. It can be defined as follows:

Definition 6 (*Chiclana, Herrera, and Herrera-Viedma, 2000; Xu and Da, 2002b*). An OWG operator of dimension n is a mapping OWG: $(R^+)^n\to R^+$ that has an associated weighting vector w with $\sum_{j=1}^n w_j=1$ and $w_j\in[0,1]$, such that

$$OWG(a_1,a_2,\ldots,a_n)=\prod_{j=1}^n b_j^{w_j}, \tag{4}$$

where b_j is the jth largest of the arguments a_1, a_2, ..., a_n.

The OWG operator can be used to deal with the multiplicative decision making models (Herrera, Herrera-Viedma, & Chiclana, 2003). Based on the OWG operator and the induced ordered weighted averaging (IOWA) operator, Chen and Sheng (2005) introduced the induced ordered weighted geometric averaging (IOWG) operator, which is a more general type of the OWG operator. It can be defined as follows:

Definition 7 Chen and Sheng, 2005. An IOWG operator of dimension n is a mapping $IOWG$: $(R^+)^n\to R^+$ that has an associated exponential weighting vector w with $\sum_{j=1}^n w_j=1$ and $w_j\in[0,1]$, such that

$$IOWG(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\ldots,\langle u_n,a_n\rangle)=\prod_{j=1}^n a_{\sigma(j)}^{w_j}, \tag{5}$$

where $a_{\sigma(j)}$ is the a_i value of the IOWG pair $\langle u_i,a_i\rangle$ having the jth largest u_i. Especially, if $u_i=a_i$ for all i, then the IOWG operator is reduced to the OWG operator.

In (2006c), Xu extended the IOWG operator to linguistic environment and obtained the ILOWG operator.

Definition 8 Xu, 2006c. An ILOWG operator can be defined as follows:

$$ILOWG(\langle u_1,s_{\alpha_1}\rangle,\ldots,\langle u_n,s_{\alpha_n}\rangle)=\otimes_{j=1}^n(s_{\gamma_j})^{w_j}, \tag{6}$$

where $w=(w_1,w_2,\ldots,w_n)^T$ is the exponential weighting vector of the s_{γ_j} satisfying $w_j\in[0,1]$ and $\sum_{j=1}^n w_j=1$, s_{γ_j} is the value of the ILOWG pair $\langle u_i,s_{\alpha_i}\rangle$ having the jth largest u_i, and u_i in $\langle u_i,s_{\alpha_i}\rangle$ is referred to as the order-inducing variable and s_{γ_j} as the multiplicative linguistic variable.

2.4. The COWG operator and the ICOWG operator

The COWG operator was developed by Yager and Xu (2006), which is based on the continuous ordered weighted averaging (COWA) operator (Yager, 2004a). It can be defined as follows:

Definition 9 Yager and Xu, 2006. A COWG operator is a mapping G: $\Sigma^+\to R^+$ associated with a basic unit interval monotonic (BUM) function Q, such that

$$G_Q(a)=G_Q([a^L,a^U])=a^U\left(\frac{a^L}{a^U}\right)^{\int_0^1\frac{dQ(y)}{dy}y\,dy}, \tag{7}$$

where Σ^+ is the set of closed intervals, in which the lower limits of all closed intervals are positive, R^+ is the set of positive real numbers, the BUM function Q: $[0,1]\to[0,1]$ is monotonic, and $Q(0)=0$, $Q(1)=1$.

If $\lambda=\int_0^1 Q(y)dy$ is the attitudinal character of Q, then a general formulation of $G_Q(a)$ can be obtained as follows:

$$G_Q(a)=G_Q([a^L,a^U])=(a^U)^\lambda(a^L)^{1-\lambda}. \tag{8}$$

As we can see, the COWG operator $G_Q(a)$ is always the weighted geometric mean of end points based on the attitudinal character. That is to say, the interval number a can be replaced by $G_Q(a)$.

In (2009), Wu et al. presented the induced continuous ordered weighted geometric (ICOWG) operator based on the COWG operator and the IOWG operator, which can be defined as follows:

Definition 10 Wu et al., 2009. An ICOWG operator is a mapping $ICOWG$: $(\Sigma^+)^n \to R^+$ that has an associated exponential weighting vector w with $\sum_{j=1}^n w_j = 1$ and $w_j \in [0, 1]$, such that

$$ICOWG(\langle u_1, [a_1, b_1]\rangle, \ldots, \langle u_n, [a_n, b_n]\rangle) = \prod_{j=1}^{n} (G_Q([a_{\sigma(j)}, b_{\sigma(j)}]))^{w_j}, \quad (9)$$

with σ: {1, …, n} → {1, …, n} being a permutation such that $u_{\sigma(j)} \geqslant u_{\sigma(j+1)}$ for $j = 1, 2, \ldots, n-1$, $\langle u_{\sigma(j)}, G_Q([a_{\sigma(j)}, b_{\sigma(j)}])\rangle$ is the two tuples with $u_{\sigma(j)}$ the ith largest value in the set $\{u_1, \ldots, u_n\}$ and $G_Q([a_{\sigma(j)}, b_{\sigma(j)}])$ is determined by the COWG operator.

2.5. Linguistic COWG operator

Zhang and Xu (2005) extended the COWG operator to linguistic environment and obtained the linguistic COWG operator, which can be defined as follows.

Definition 11 Zhang and Xu, 2005. Let $\tilde{s} = [s_\alpha, s_\beta]$ be an uncertain multiplicative linguistic variable. If

$$g_Q(\tilde{s}) = g_Q([s_\alpha, s_\beta]) = s_\gamma, \quad (10)$$

and

$$\gamma = G_Q([I(s_\alpha), I(s_\beta)]) = G_Q([\alpha, \beta]), \quad (11)$$

then g is called the linguistic COWG (LCOWG) operator, where $s_\alpha, s_\beta \in \widetilde{S}$ and Q is the BUM function.

If $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of Q, then the LCOWG operator can be written as follows:

$$g_Q(\tilde{s}) = g_Q([s_\alpha, s_\beta]) = s_{\beta^\lambda \times \alpha^{1-\lambda}} = (s_\beta)^\lambda \otimes (s_\alpha)^{1-\lambda}. \quad (12)$$

From Eq. (11), we can see that the LCOWG operator may be determined by the attitudinal character λ. For convenience, in this paper, we assume that $g_\lambda(\tilde{s})$ denotes $g_Q(\tilde{s})$, i.e.,

$$g_\lambda(\tilde{s}) = g_Q(\tilde{s}) = (s_\beta)^\lambda \otimes (s_\alpha)^{1-\lambda} = s_{\beta^\lambda \times \alpha^{1-\lambda}}. \quad (13)$$

As we can see, the LCOWG operator is able to deal with individual uncertain multiplicative linguistic preference relation, but it is unsuitable for aggregating individual preference relations into group preference relation. In the following section, we shall develop the induced linguistic continuous ordered weighted geometric (ILCOWG) operator to deal with the uncertain multiplicative linguistic preference relations in GDM.

3. The ILCOWG operator

For convenience, in this paper, we assume that Ω is a set of uncertain multiplicative linguistic variables.

Definition 12. An ILCOWG operator is a mapping $ILCOWG : \Omega^n \to \widetilde{S}$, which has an associated exponential weighting vector w with $\sum_{j=1}^n w_j = 1$ and $w_j \in [0, 1]$, and is defined to aggregate the set of the second arguments of list of n tuples $\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots, \langle u_n, [s_{\alpha_n}, s_{\beta_n}]\rangle$ according to the following formula:

$$\begin{aligned} &ILCOWG(\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots, \langle u_n, [s_{\alpha_n}, s_{\beta_n}]\rangle) \\ &= IOWG(\langle u_1, g_\lambda([s_{\alpha_1}, s_{\beta_1}])\rangle, \ldots, \langle u_n, g_\lambda([s_{\alpha_n}, s_{\beta_n}])\rangle) \\ &= \otimes_{j=1}^n (g_\lambda([s_{\alpha_{\sigma(j)}}, s_{\beta_{\sigma(j)}}]))^{w_j}, \end{aligned} \quad (14)$$

with σ: {1, …, n} → {1, …, n} being a permutation such that $u_{\sigma(j)} \geqslant u_{\sigma(j+1)}$ for $j = 1, 2, \ldots, n-1$, $\langle u_{\sigma(j)}, g_\lambda([s_{\alpha_{\sigma(j)}}, s_{\beta_{\sigma(j)}}])\rangle$ is the two tuples with $u_{\sigma(j)}$ the ith largest value in the set $\{u_1, \ldots, u_n\}$ and $g_\lambda([s_{\alpha_{\sigma(j)}}, s_{\beta_{\sigma(j)}}])$ is determined by the LCOWG operator.

Specially, if $s_{\alpha_i} = s_{\beta_i}$ for all i, then the ILCOWG operator reduces to the ILOWG operator (Xu, 2006c).

Example 1. Assume that there are four two-tuples $\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots, \langle u_4, [s_{\alpha_4}, s_{\beta_4}]\rangle$, where $(u_1, u_2, u_3, u_4) = (0.75, 0.82, 0.68, 0.85)$, $[s_{\alpha_1}, s_{\beta_1}] = [s_1, s_3]$, $[s_{\alpha_2}, s_{\beta_2}] = [s_2, s_4]$, $[s_{\alpha_3}, s_{\beta_3}] = [s_3, s_5]$, $[s_{\alpha_4}, s_{\beta_4}] = [s_2, s_5]$, the exponential weighting vector $w = (0.3, 0.2, 0.4, 0.1)^T$ and the BUM function $Q(y) = \sqrt{y}$, then $\lambda = 2/3$. By Eq. (13), we have that

$$g_\lambda([s_{\alpha_1}, s_{\beta_1}]) = s_{3^{2/3} \times 1^{1/3}} = s_{2.0801}, \quad g_\lambda([s_{\alpha_2}, s_{\beta_2}]) = s_{4^{2/3} \times 2^{1/3}} = s_{3.1748},$$
$$g_\lambda([s_{\alpha_3}, s_{\beta_3}]) = s_{5^{2/3} \times 3^{1/3}} = s_{4.2172}, \quad g_\lambda([s_{\alpha_4}, s_{\beta_4}]) = s_{5^{2/3} \times 2^{1/3}} = s_{3.6840}.$$

Thus,

$$g_\lambda([s_{\alpha_{\sigma(1)}}, s_{\beta_{\sigma(1)}}]) = s_{3.6840}, \quad g_\lambda([s_{\alpha_{\sigma(2)}}, s_{\beta_{\sigma(2)}}]) = s_{3.1748},$$
$$g_\lambda([s_{\alpha_{\sigma(3)}}, s_{\beta_{\sigma(3)}}]) = s_{2.0801}, \quad g_\lambda([s_{\alpha_{\sigma(4)}}, s_{\beta_{\sigma(4)}}]) = s_{4.2172}.$$

Therefore,

$$\begin{aligned} &ILCOWG(\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots, \langle u_4, [s_{\alpha_4}, s_{\beta_4}]\rangle) = \otimes_{j=1}^4 (g_\lambda([s_{\alpha_{\sigma(j)}}, s_{\beta_{\sigma(j)}}]))^{w_j} \\ &= (s_{3.6840})^{0.3} \otimes (s_{3.1748})^{0.2} \otimes (s_{2.0801})^{0.4} \otimes (s_{4.2172})^{0.1} = s_{2.8839}. \end{aligned}$$

The ILCOWG operator is monotonic, idempotent, bounded and commutative. These properties can be expressed by the following theorems.

Theorem 1 (*Monotonicity*). *Let f be the ILCOWG operator. If* $s_{\alpha_i} \geqslant s_{\alpha'_i}$ *and* $s_{\beta_i} \geqslant s_{\beta'_i}$ *for all i, then*

$$\begin{aligned} &f(\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots, \langle u_n, [s_{\alpha_n}, s_{\beta_n}]\rangle) \\ &\geqslant f\left(\left\langle u_1, \left[s_{\alpha'_1}, s_{\beta'_1}\right]\right\rangle, \ldots, \left\langle u_n, \left[s_{\alpha'_n}, s_{\beta'_n}\right]\right\rangle\right). \end{aligned} \quad (15)$$

Theorem 2 (*Idempotency*). *Let f be the ILCOWG operator. If* $[s_{\alpha_i}, s_{\beta_i}] = [s_\alpha, s_\beta]$ *for all i, then*

$$f(\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots, \langle u_n, [s_{\alpha_n}, s_{\beta_n}]\rangle) = g_\lambda([s_\alpha, s_\beta]). \quad (16)$$

Theorem 3 (*Boundedness*). *Let f be the ILCOWG operator. Then*

$$\min_i s_{\alpha_i} \leqslant f(\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots, \langle u_n, [s_{\alpha_n}, s_{\beta_n}]\rangle) \leqslant \max_i s_{\beta_i}. \quad (17)$$

Theorem 4 (*Commutativity*). *Let f be the ILCOWG operator. Then*

$$\begin{aligned} &f(\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots, \langle u_n, [s_{\alpha_n}, s_{\beta_n}]\rangle) \\ &= f\left(\left\langle u'_1, \left[s_{\alpha'_1}, s_{\beta'_1}\right]\right\rangle, \ldots, \left\langle u'_n, \left[s_{\alpha'_n}, s_{\beta'_n}\right]\right\rangle\right), \end{aligned} \quad (18)$$

where $\left\langle u'_1, \left[s_{\alpha'_1}, s_{\beta'_1}\right]\right\rangle, \ldots, \left\langle u'_n, \left[s_{\alpha'_n}, s_{\beta'_n}\right]\right\rangle$ *is any permutation of* $\langle u_1, [s_{\alpha_1}, s_{\beta_1}]\rangle, \ldots,$
$\langle u_n, [s_{\alpha_n}, s_{\beta_n}]\rangle$.

The consistency measure is a very important problem in decision making with all kinds of preference relations. The lack of consistency in decision making with preference relations can lead to inconsistent conclusions. Similar to (Dong et al., 2008), we can define the consistency of uncertain multiplicative linguistic preference relation as follows:

Definition 13. Let $\widetilde{A} = (\tilde{a}_{ij})_{n \times n} \in M_n$, Then $\widetilde{A} = (\tilde{a}_{ij})_{n \times n}$ is called a consistent uncertain multiplicative linguistic preference relation if there exists

$$\tilde{a}^L_{ik} \otimes \tilde{a}^L_{kj} = \tilde{a}^L_{ij}, \quad \tilde{a}^U_{ik} \otimes \tilde{a}^U_{kj} = \tilde{a}^U_{ij}, \quad \text{for } i, j, k = 1, 2, \ldots, n. \quad (19)$$

Definition 14. Let $\widetilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$, then we call $g_\lambda(\widetilde{A}) = (g_\lambda(\tilde{a}_{ij}))_{n\times n}$ the expected multiplicative linguistic preference relation corresponding to $\widetilde{A}$, where $g_\lambda(\tilde{a}_{ij})$ is obtained by the LCOWG operator:

$$g_\lambda(\tilde{a}_{ij}) = g_\lambda\left(\left[\tilde{a}_{ij}^L, \tilde{a}_{ij}^U\right]\right) = \left(\tilde{a}_{ij}^U\right)^\lambda \otimes \left(\tilde{a}_{ij}^L\right)^{1-\lambda},$$
$$g_\lambda(\tilde{a}_{ij}) = rec(g_\lambda(\tilde{a}_{ji})), \quad \text{for } i \leqslant j, \tag{20}$$

where λ is the attitudinal parameter of the BUM function Q.

By Eq. (20), we can obtain the following theorem:

Theorem 5. *Let $\widetilde{A} = (\tilde{a}_{ij})_{n\times n} \in M_n$ and $g_\lambda(\widetilde{A}) = (g_\lambda(\tilde{a}_{ij}))_{n\times n}$ be the expected multiplicative linguistic preference relation corresponding to $\widetilde{A}$. If $\widetilde{A}$ is consistent, then $g_\lambda(\widetilde{A})$ is also consistent.*

Proof. Since $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}$ is consistent, by Definition 13, we have
$\tilde{a}_{ik}^L \otimes \tilde{a}_{kj}^L = \tilde{a}_{ij}^L, \quad \tilde{a}_{ik}^U \otimes \tilde{a}_{kj}^U = \tilde{a}_{ij}^U, \quad \text{for } i,j,k = 1,2,\ldots,n.$

Thus,

$$\begin{aligned} g_\lambda(\tilde{a}_{ik}) \otimes g_\lambda(\tilde{a}_{kj}) &= \left(\left(\tilde{a}_{ik}^U\right)^\lambda \otimes \left(\tilde{a}_{ik}^L\right)^{1-\lambda}\right) \otimes \left(\left(\tilde{a}_{kj}^U\right)^\lambda \otimes \left(\tilde{a}_{kj}^L\right)^{1-\lambda}\right) \\ &= \left(\tilde{a}_{ik}^U \otimes \tilde{a}_{kj}^U\right)^\lambda \otimes \left(\tilde{a}_{ik}^L \otimes \tilde{a}_{kj}^L\right)^{1-\lambda} = \left(\tilde{a}_{ij}^U\right)^\lambda \otimes \left(\tilde{a}_{ij}^L\right)^{1-\lambda} \\ &= g_\lambda(\tilde{a}_{ij}). \end{aligned}$$

By Definition 4, we get that $g_\lambda(\widetilde{A})$ is consistent. □

Let $E = \{e_1, e_2, \ldots, e_m\}$ be a finite set of experts and $\widetilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n$ be the uncertain multiplicative linguistic preference relation provided by expert e_k, $k = 1, 2, \ldots, m$, then we can get the synthetic linguistic preference relation of $\widetilde{A}^{(1)}, \widetilde{A}^{(2)}, \ldots, \widetilde{A}^{(m)}$ as follows:

Definition 15. Let $\widetilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n$ for $k = 1, 2, \ldots, m$. If

$$\begin{aligned} \tilde{a}_{ij} &= ILCOWG\left(\left\langle u_1, a_{ij}^{(1)}\right\rangle, \ldots, \left\langle u_m, a_{ij}^{(m)}\right\rangle\right) \\ &= \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{ij}^{(k)}\right)\right)^{\omega_k}, \quad i,j = 1,2,\ldots,n \end{aligned} \tag{21}$$

then the matrix $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}$ is called the synthetic linguistic preference relation of all experts, where $\tilde{b}_{ij}^{(k)}$ is the $\tilde{a}_{ij}^{(l)}$ value of the ILCOWG pair $\left\langle u_l, \tilde{a}_{ij}^{(l)}\right\rangle$ having the kth largest u_l, $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ is the weighting vector of experts, which satisfies that $\omega_k \geqslant 0$ for all $k = 1, 2, \ldots, m$ and $\sum_{k=1}^m \omega_k = 1$.

Theorem 6. *Let $\widetilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n, k = 1,2,\ldots,m$. If $g_\lambda(\widetilde{A}^{(k)}) = \left(g_\lambda\left(\tilde{a}_{ij}^{(k)}\right)\right)_{n\times n}(k = 1,2,\ldots,m)$ are the expected multiplicative linguistic preference relations corresponding to $\widetilde{A}^{(k)}$, then the synthetic linguistic preference relation $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}$ is also the expected multiplicative linguistic preference relation.*

Proof. Since $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}$ is the synthetic linguistic preference relation, by Definition 15, we have

$$\tilde{a}_{ij} = ILCOWG\left(\left\langle u_1, a_{ij}^{(1)}\right\rangle, \ldots, \left\langle u_m, a_{ij}^{(m)}\right\rangle\right) = \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{ij}^{(k)}\right)\right)^{\omega_k},$$
$$i,j = 1,2,\ldots,n.$$

If $g_\lambda(\widetilde{A}^{(k)}) = \left(g_\lambda\left(\tilde{a}_{ij}^{(k)}\right)\right)_{n\times n}$ are the expected multiplicative linguistic preference relations corresponding to $\widetilde{A}^{(k)}$, respectively, we get that

$$g_\lambda\left(\tilde{b}_{ij}^{(k)}\right) = g_\lambda\left(\left[\tilde{b}_{ij}^{(k)L}, \tilde{b}_{ij}^{(k)U}\right]\right) = \left(\tilde{b}_{ij}^{(k)U}\right)^\lambda \otimes \left(\tilde{b}_{ij}^{(k)L}\right)^{1-\lambda},$$
$$g_\lambda\left(\tilde{b}_{ij}^{(k)}\right) = rec\left(g_\lambda\left(\tilde{b}_{ji}^{(k)}\right)\right), \quad \text{for } i \leqslant j.$$

Thus,

$$\begin{aligned} \tilde{a}_{ij} \otimes \tilde{a}_{ji} &= \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{ij}^{(k)}\right)\right)^{\omega_k} \otimes \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{ji}^{(k)}\right)\right)^{\omega_k} \\ &= \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{ij}^{(k)}\right) \otimes g_\lambda\left(\tilde{b}_{ji}^{(k)}\right)\right)^{\omega_k} = \otimes_{k=1}^m (s_1)^{\omega_k} = s_1. \end{aligned}$$

Because $\widetilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n$, we obtain $\tilde{a}_{ii}^{(k)U} = \tilde{a}_{ii}^{(k)L} = s_1$ for all i, which means that $\tilde{b}_{ii}^{(k)U} = \tilde{b}_{ii}^{(k)L} = s_1$. Then

$$\tilde{a}_{ii} = \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{ii}^{(k)}\right)\right)^{\omega_k} = \otimes_{k=1}^m \left(\left(\tilde{b}_{ii}^{(k)U}\right)^\lambda \otimes \left(\tilde{b}_{ii}^{(k)L}\right)^{1-\lambda}\right)^{\omega_k} = s_1,$$

which completes the proof of Theorem. □

Theorem 7. *Let $\widetilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}(k = 1,2,\ldots,m)$ be the uncertain multiplicative linguistic preference relations provided by m experts. Assume that $g_\lambda(\widetilde{A}^{(k)}) = \left(g_\lambda\left(\tilde{a}_{ij}^{(k)}\right)\right)_{n\times n}(k = 1,2,\ldots,m)$ are the expected multiplicative linguistic preference relations corresponding to $\widetilde{A}^{(k)}$ and $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}$ is the synthetic linguistic preference relation of all experts. If $\widetilde{A}^{(k)}$ are consistent for all k, then $\widetilde{A}$ is also consistent.*

Proof. If $\widetilde{A}^{(k)}$ are consistent for all k and $\widetilde{A}^{(k)} = \left(\tilde{a}_{ij}^{(k)}\right)_{n\times n} \in M_n$, then by Theorem 5, $g_\lambda(\widetilde{A}^{(1)}), g_\lambda(\widetilde{A}^{(2)}), \ldots, g_\lambda(\widetilde{A}^{(m)})$ are consistent, i.e.,

$$g_\lambda\left(\tilde{a}_{it}^{(k)}\right) \otimes g_\lambda\left(\tilde{a}_{tj}^{(k)}\right) = g_\lambda\left(\tilde{a}_{ij}^{(k)}\right), \quad i,j,t = 1,2,\ldots,n, \quad k = 1,2,\ldots,m.$$

Thus,

$$\begin{aligned} \tilde{a}_{it} \otimes \tilde{a}_{tj} &= \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{it}^{(k)}\right)\right)^{\omega_k} \otimes \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{tj}^{(k)}\right)\right)^{\omega_k} \\ &= \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{it}^{(k)}\right) \otimes g_\lambda\left(\tilde{b}_{tj}^{(k)}\right)\right)^{\omega_k} = \otimes_{k=1}^m \left(g_\lambda\left(\tilde{b}_{ij}^{(k)}\right)\right)^{\omega_k} \\ &= \tilde{a}_{ij}, \quad i,j,t = 1,2,\ldots,n, \end{aligned}$$

which completes the proof of Theorem. □

4. An approach for GDM problem with uncertain multiplicative linguistic preference relations

4.1. The relative consensus degree ILCOWG (RCD-ILCOWG) operator

In group decision making problem, each expert has an important degree associated with them, which can be used as the order-inducing variable to induce the ordering of the arguments before aggregation. Wu et al. (2009) presented a relative consensus degree ICOWG (RCD-ICOWG) operator based on the concept of the relative consensus degree (RCD) variable.

Definition 16. The matrix $A = (a_{ij})_{n\times n}$ is an antisymmetric matrix, if

$$a_{ij} = -a_{ji}, \quad i,j = 1,2,\ldots,n. \tag{22}$$

And if

$$a_{ij} = -a_{ji}, \quad a_{ij} = a_{ik} + a_{kj}, \quad i,j,k = 1,2,\ldots,n, \tag{23}$$

then A is called the transfer matrix.

Definition 17. Let $A = (a_{ij})_{n\times n}$ be an antisymmetric matrix. If there exists matrix $B = (b_{ij})_{n\times n}$ and the distance $d = \|B - A\| = \sum_{i=1}^n \sum_{j=1}^n (b_{ij} - a_{ij})^2$ arrives the minimum, then we call B the optimal transfer matrix to A.

Based on Definitions 16 and 17, we can obtain the following theorem (Wu et al., 2009):

Theorem 8. *Let $A^{(1)}$, $A^{(2)}$, …, $A^{(m)}$ be the multiplicative preference relations, and $B^{(1)}$, $B^{(2)}$, …, $B^{(m)}$ be the corresponding antisymmetric matrices, where $B^{(l)} = \ln A^{(l)}$. If $C^* = \left(c^*_{ij}\right)_{n\times n}$ is the optimal transfer matrix to $B^{(1)}$, $B^{(2)}$, …, $B^{(m)}$ satisfying*

$$c^*_{ij} = \frac{1}{mn}\sum_{l=1}^{m}\sum_{k=1}^{n}\left(b^{(l)}_{ik} - b^{(l)}_{jk}\right), \tag{24}$$

then the matrix $A^* = e^{C^*}$ is consistent.

Lemma 1. *If $s_\alpha, s_\beta \in \widetilde{S}$, then $I(s_\alpha \otimes s_\beta) = I(s_\alpha)I(s_\beta)$.*

Proof. $I(s_\alpha \otimes s_\beta) = I(s_{\alpha\beta}) = \alpha\,\beta = I(s_\alpha)I(s_\beta)$. □

Theorem 9. *If $\widehat{A} = (\hat{a}_{ij})_{n\times n}$ is the multiplicative linguistic preference relation, then $I(\widehat{A}) = (I(\hat{a}_{ij}))_{n\times n}$ is the multiplicative preference relation.*

Proof. If $\widehat{A} = (\hat{a}_{ij})_{n\times n}$ is the multiplicative linguistic preference relation, then we have

$$\hat{a}_{ij} \otimes \hat{a}_{ji} = s_1, \quad \hat{a}_{ii} = s_1, \quad i,j = 1,2,\ldots,n.$$

By Lemma 1, we get $I(\hat{a}_{ij} \otimes \hat{a}_{ji}) = I(\hat{a}_{ij})I(\hat{a}_{ji}) = 1$ and $I(\hat{a}_{ii}) = I(s_1) = 1$. Thus, $I(\widehat{A}) = (I(\hat{a}_{ij}))_{n\times n}$ is the multiplicative preference relation. □

Based on Definitions 14, 16 and 17 and Theorems 8 and 9, we can define the relative consensus degree of uncertain multiplicative linguistic preference relation as follows:

Definition 18. Let $\widetilde{A}^{(k)} = \left(\tilde{a}^{(k)}_{ij}\right)_{n\times n}(k = 1,2,\ldots,m)$ be the uncertain (k = 1, 2, …, m) and $g_\lambda(\widetilde{A}^{(k)}) = \left(g_\lambda\left(\tilde{a}^{(k)}_{ij}\right)\right)_{n\times n}(k = 1,2,\ldots,m)$ be the expected multiplicative linguistic preference relations corresponding to $\widetilde{A}^{(k)}$. Assume that $B^{(1)}$, $B^{(2)}$, …, $B^{(m)}$ are the corresponding antisymmetric matrices to $I(g_\lambda(\widetilde{A}^{(1)})), I(g_\lambda(\widetilde{A}^{(2)})), \ldots, I(g_\lambda(\widetilde{A}^{(m)}))$ and $C^* = \left(c^*_{ij}\right)_{n\times n}$ is the optimal transfer matrix to $B^{(1)}$, $B^{(2)}$, …, $B^{(m)}$, where $B^{(k)} = \ln I(g_\lambda(\widetilde{A}^{(k)}))$ for k = 1, 2, …, m and c^*_{ij} is determined by Eq. (24). If $A^* = e^{C^*}$, then we call

$$RCD^{(k)} = \frac{\left\langle vec(I(g_\lambda(\widetilde{A}^{(k)}))), vec(A^*)\right\rangle}{\|vec(I(g_\lambda(\widetilde{A}^{(k)})))\| \cdot \|vec(A^*)\|}, \tag{25}$$

the relative consensus degree of e_k, where $vec(I(g_\lambda(\widetilde{A}^{(k)}))) = \left(I\left(g_\lambda\left(\tilde{a}^{(1)}_{11}\right)\right), I\left(g_\lambda\left(\tilde{a}^{(1)}_{21}\right)\right), \ldots, I(g_\lambda\left(\tilde{a}^{(1)}_{n1}\right)); \ldots; I\left(g_\lambda\left(\tilde{a}^{(1)}_{1n}\right)\right), I\left(g_\lambda\left(\tilde{a}^{(1)}_{2n}\right)\right), \ldots, I\left(g_\lambda\left(\tilde{a}^{(1)}_{nn}\right)\right)\right)^T$, $vec(A^*) = (a^*_{11}, a^*_{21}, \ldots, a^*_{n1}; \ldots; a^*_{1n}, a^*_{2n}, \ldots, a^*_{nn})^T$.

As we can see, the greater $RCD^{(k)}$ is, the closer $\widetilde{A}^{(k)}$ to consistency, then more importance should be placed on the preference relation $\widetilde{A}^{(k)}$, which means that the $RCD^{(k)}$ can be considered as the induced variable of expert e_k in the ILCOWG operator. Then we obtain the RCD-ILCOWG operator, which can be defined as follows:

Definition 19. Let $\widetilde{A}^{(k)} = \left(\tilde{a}^{(k)}_{ij}\right)_{n\times n} \in M_n$ for k = 1, 2, …, m provided by expert d_k and $g_\lambda(\widetilde{A}^{(k)}) = \left(g_\lambda\left(\tilde{a}^{(k)}_{ij}\right)\right)_{n\times n}(k = 1,2,\ldots,m)$ be the expected multiplicative linguistic preference relations corresponding to $\widetilde{A}^{(k)}$. Then we can obtain the synthetic linguistic preference relation $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}$ based on the RCD-ILCOWG operator according to the formula:

$$\begin{aligned}\tilde{a}_{ij} &= RCD-ILCOWG\left(\left\langle RCD^{(1)}, \tilde{a}^{(1)}_{ij}\right\rangle, \ldots, \left\langle RCD^{(m)}, \tilde{a}^{(m)}_{ij}\right\rangle\right) \\ &= \otimes_{k=1}^{m}\left(g_\lambda\left(\tilde{b}^{(k)}_{ij}\right)\right)^{\omega_k},\end{aligned} \tag{26}$$

where $RCD^{(k)}$ (k = 1, 2, …, m) are determined by Eq. (25), $\tilde{b}^{(k)}_{ij}$ is the $\tilde{a}^{(l)}_{ij}$ value of the ILCOWG pair $\left\langle RCD^{(l)}, \tilde{a}^{(l)}_{ij}\right\rangle$ having the kth largest $RCD^{(l)}$, $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ is the weighting vector of experts, which satisfies that $\omega_k \geqslant 0$ for all k = 1, 2, …, m and $\sum_{k=1}^{m}\omega_k = 1$.

Example 2. Let $\widetilde{A}^{(1)}, \widetilde{A}^{(2)}, \widetilde{A}^{(3)}$ be three uncertain multiplicative linguistic preference relations, whose weighting vector is $\omega = (0.2, 0.6, 0.2)^T$. They are listed as follows:

$$\widetilde{A}^{(1)} = \begin{pmatrix} [s_1, s_1] & [s_3, s_4] & [s_{1/4}, s_{1/2}] & [s_4, s_5] & [s_{1/3}, s_{1/2}] \\ [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_3, s_4] & [s_3, s_4] & [s_{1/4}, s_{1/2}] \\ [s_2, s_4] & [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_3, s_4] & [s_4, s_5] \\ [s_{1/5}, s_{1/4}] & [s_{1/4}, s_{1/3}] & [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_3, s_4] \\ [s_2, s_3] & [s_2, s_4] & [s_{1/5}, s_{1/4}] & [s_{1/4}, s_{1/3}] & [s_1, s_1] \end{pmatrix},$$

$$\widetilde{A}^{(2)} = \begin{pmatrix} [s_1, s_1] & [s_3, s_4] & [s_{1/2}, s_1] & [s_4, s_5] & [s_{1/2}, s_1] \\ [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_3, s_4] & [s_3, s_4] & [s_{1/4}, s_{1/3}] \\ [s_1, s_2] & [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_2, s_4] & [s_4, s_5] \\ [s_{1/5}, s_{1/4}] & [s_{1/4}, s_{1/3}] & [s_3, s_4] & [s_1, s_1] & [s_3, s_4] \\ [s_1, s_2] & [s_3, s_4] & [s_{1/5}, s_{1/4}] & [s_{1/4}, s_{1/3}] & [s_1, s_1] \end{pmatrix},$$

$$\widetilde{A}^{(3)} = \begin{pmatrix} [s_1, s_1] & [s_3, s_4] & [s_{1/2}, s_1] & [s_2, s_3] & [s_{1/5}, s_{1/4}] \\ [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_3, s_5] & [s_3, s_4] & [s_{1/4}, s_{1/3}] \\ [s_1, s_2] & [s_{1/5}, s_{1/3}] & [s_1, s_1] & [s_4, s_5] & [s_3, s_4] \\ [s_{1/3}, s_{1/2}] & [s_{1/4}, s_{1/3}] & [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_4, s_5] \\ [s_4, s_5] & [s_3, s_4] & [s_{1/4}, s_{1/3}] & [s_{1/5}, s_{1/4}] & [s_1, s_1] \end{pmatrix}.$$

Assume that $Q(y) = \sqrt{y}$, then $\lambda = \int_0^1 Q(y)dy = 2/3$. Based on Eq. (12), we obtain

$$g_\lambda(\widetilde{A}^{(1)}) = \begin{pmatrix} s_1 & s_{3.6342} & s_{0.3969} & s_{4.6416} & s_{0.4368} \\ s_{0.2752} & s_1 & s_{3.6342} & s_{3.6342} & s_{0.3969} \\ s_{2.5198} & s_{0.2752} & s_1 & s_{3.6342} & s_{4.6416} \\ s_{0.2154} & s_{0.2752} & s_{0.2752} & s_1 & s_{3.6342} \\ s_{2.2895} & s_{2.5198} & s_{0.2154} & s_{0.2752} & s_1 \end{pmatrix},$$

$$g_\lambda(\widetilde{A}^{(2)}) = \begin{pmatrix} s_1 & s_{3.6342} & s_{0.7937} & s_{4.6416} & s_{0.7937} \\ s_{0.2752} & s_1 & s_{3.6342} & s_{3.6342} & s_{0.3028} \\ s_{1.2599} & s_{0.2752} & s_1 & s_{3.1748} & s_{4.6416} \\ s_{0.2154} & s_{0.2752} & s_{0.3150} & s_1 & s_{3.6342} \\ s_{1.2599} & s_{3.3021} & s_{0.2154} & s_{0.2752} & s_1 \end{pmatrix},$$

$$g_\lambda(\widetilde{A}^{(3)}) = \begin{pmatrix} s_1 & s_{3.6342} & s_{0.7937} & s_{2.6207} & s_{0.2321} \\ s_{0.2752} & s_1 & s_{4.2172} & s_{3.6342} & s_{0.3028} \\ s_{1.2599} & s_{0.2371} & s_1 & s_{4.6416} & s_{3.6342} \\ s_{0.3816} & s_{0.2752} & s_{0.2154} & s_1 & s_{4.6416} \\ s_{4.3089} & s_{3.3021} & s_{0.2752} & s_{0.2154} & s_1 \end{pmatrix}.$$

Then the corresponding antisymmetric matrices to $I(g_\lambda(\widetilde{A}^{(1)})), I(g_\lambda(\widetilde{A}^{(2)})), I(g_\lambda(\widetilde{A}^{(3)}))$ are $B^{(1)}$, $B^{(2)}$, $B^{(3)}$, respectively, where

$$B^{(1)} = \begin{pmatrix} 0 & 1.2904 & -0.9242 & 1.5351 & -0.8283 \\ -1.2904 & 0 & 1.2904 & 1.2904 & -0.9242 \\ 0.9242 & -1.2904 & 0 & 1.2904 & 1.5351 \\ -1.5351 & -1.2904 & -1.2904 & 0 & 1.2904 \\ 0.8283 & 0.9242 & -1.5351 & -1.2904 & 0 \end{pmatrix},$$

$$B^{(2)}=\begin{pmatrix} 0 & 1.2904 & -0.2310 & 1.5351 & -0.2310 \\ -1.2904 & 0 & 1.2904 & 1.2904 & -1.1946 \\ 0.2310 & -1.2904 & 0 & 1.1552 & 1.5351 \\ -1.5351 & -1.2904 & -1.1552 & 0 & 1.2904 \\ 0.2310 & 1.1946 & -1.5351 & -1.2904 & 0 \end{pmatrix},$$

$$B^{(3)}=\begin{pmatrix} 0 & 1.2904 & -0.2310 & 0.9635 & -1.4607 \\ -1.2904 & 0 & 1.4392 & 1.2904 & -1.1946 \\ 0.2310 & -1.4392 & 0 & 1.5351 & 1.2904 \\ -0.9635 & -1.2904 & -1.5351 & 0 & 1.5351 \\ 1.4607 & 1.1946 & -1.2904 & -1.5351 & 0 \end{pmatrix}.$$

According to Eq. (24), we get the optimal transfer matrix to $B^{(1)}$, $B^{(2)}$, $B^{(3)}$:

$$C^*=\left(c_{ij}^*\right)_{5\times 5}=\begin{pmatrix} 0 & 0.1358 & -0.2628 & 0.7933 & 0.5199 \\ -0.1358 & 0 & -0.3986 & 0.6575 & 0.3841 \\ 0.2628 & 0.3986 & 0 & 1.0561 & 0.7827 \\ -0.7933 & -0.6575 & -1.0561 & 0 & -0.2734 \\ -0.5199 & -0.3841 & -0.7827 & 0.2734 & 0 \end{pmatrix}.$$

It follows that

$$A^*=\left(a_{ij}^*\right)_{5\times 5}=\begin{pmatrix} 1 & 1.1455 & 0.7689 & 2.2107 & 1.6819 \\ 0.8730 & 1 & 0.6713 & 1.9300 & 1.4684 \\ 1.3005 & 1.4897 & 1 & 2.8752 & 2.1874 \\ 0.4523 & 0.5181 & 0.3478 & 1 & 0.7608 \\ 0.5946 & 0.6810 & 0.4572 & 1.3144 & 1 \end{pmatrix}.$$

By Eq. (25), we have

$$RCD^{(1)}=0.8191,\quad RCD^{(2)}=0.8079,\quad RCD^{(3)}=0.7503.$$

Thus, based on Eq. (26), we obtain the synthetic linguistic preference relation as follows:

$$\widetilde{A}=(\tilde{a}_{ij})_{5\times 5}=\begin{pmatrix} s_1 & s_{3.6342} & s_{0.6910} & s_{4.1402} & s_{0.5508} \\ s_{0.2752} & s_1 & s_{3.7440} & s_{3.6342} & s_{0.3197} \\ s_{1.4473} & s_{0.2671} & s_1 & s_{3.5192} & s_{4.4199} \\ s_{0.2415} & s_{0.2752} & s_{0.2842} & s_1 & s_{3.8165} \\ s_{1.8156} & s_{3.1283} & s_{0.2262} & s_{0.2620} & s_1 \end{pmatrix}.$$

4.2. The optimal model for weighting experts based on the distance measure of uncertain multiplicative linguistic preference relations

In order to determine the weights of experts in GDM, in the following, we shall define a new distance measure based on the LCOWG operator.

Definition 20. Let $\tilde{s}_1=[s_{\alpha_1},s_{\beta_1}]\in\Omega$ and $\tilde{s}_2=[s_{\alpha_2},s_{\beta_2}]\in\Omega$. If

$$d_\lambda(\tilde{s}_1,\tilde{s}_2)=\frac{1}{2t}|I(g_\lambda(\tilde{s}_1))-I(g_\lambda(\tilde{s}_2))|, \tag{27}$$

then $d_\lambda(\tilde{s}_1,\tilde{s}_2)$ is called the distance $\tilde{s}_1$ and $\tilde{s}_2$ based on the LCOWG operator, where $g_\lambda(\tilde{s}_1)$ and $g_\lambda(\tilde{s}_2)$ are determined by Eqs. (10) and (11).

Specially, if $\tilde{s}_1$ and $\tilde{s}_2$ are simple linguistic variables, then the $d_\lambda(\tilde{s}_1,\tilde{s}_2)$ reduces the deviation measure proposed in (Xu, 2005).

It can be seen easily that the smaller the value of $d_\lambda(\tilde{s}_1,\tilde{s}_2)$, the nearer the uncertain linguistic variables $\tilde{s}_1$ and $\tilde{s}_2$ will be. With Eqs. (10) and (11), $d_\lambda(\tilde{s}_1,\tilde{s}_2)$ can be expressed as follows:

$$d_\lambda(\tilde{s}_1,\tilde{s}_2)=\frac{1}{2t}\left|I\left(s_{\beta_1^\lambda\alpha_1^{1-\lambda}}\right)-I\left(s_{\beta_2^\lambda\alpha_2^{1-\lambda}}\right)\right|=\frac{1}{2t}\left|\beta_1^\lambda\alpha_1^{1-\lambda}-\beta_2^\lambda\alpha_2^{1-\lambda}\right|, \tag{28}$$

where $\lambda=\int_0^1 Q(y)dy$ is the attitudinal character of Q.

From Definition 20, we can get the following theorem easily:

Theorem 10. *Let $\tilde{s}_1=[s_{\alpha_1},s_{\beta_1}]\in\Omega,\tilde{s}_2=[s_{\alpha_2},s_{\beta_2}]\in\Omega$, and $\tilde{s}_3=[s_{\alpha_3},s_{\beta_3}]\in\Omega$, then*

(1) *Nonnegativity:* $d_\lambda(\tilde{s}_1,\tilde{s}_2)\geqslant 0$.
(2) *Reflexivity:* $d_\lambda(\tilde{s}_1,\tilde{s}_1)=0$.
(3) *Commutativity:* $d_\lambda(\tilde{s}_1,\tilde{s}_2)=d_\lambda(\tilde{s}_2,\tilde{s}_1)$.
(4) *Transitivity: If* $d_\lambda(\tilde{s}_1,\tilde{s}_2)=0$ *and* $d_\lambda(\tilde{s}_2,\tilde{s}_3)=0$, *then* $d_\lambda(\tilde{s}_1,\tilde{s}_3)=0$.
(5) *Triangle inequality:* $d_\lambda(\tilde{s}_1,\tilde{s}_3)\leqslant d_\lambda(\tilde{s}_1,\tilde{s}_2)+d_\lambda(\tilde{s}_2,\tilde{s}_3)$.

Definition 21. Let $\widetilde{A}=(\tilde{a}_{ij})_{n\times n}\in M_n$ and $\widetilde{B}=(\tilde{b}_{ij})_{n\times n}\in M_n$ be two uncertain multiplicative linguistic preference relations, then

$$d_\lambda(\widetilde{A},\widetilde{B})=\frac{1}{2tn^2}\sum_{i=1}^n\sum_{j=1}^n|I(g_\lambda(\tilde{a}_{ij}))-I(g_\lambda(\tilde{b}_{ij}))|, \tag{29}$$

is called the distance of $\widetilde{A}$ and $\widetilde{B}$ based on the LCOWG operator, where $g_\lambda(\tilde{a}_{ij}),g_\lambda(\tilde{b}_{ij})$ are determined by Eqs. (10) and (11), and $\tilde{a}_{ij}=\left[\tilde{a}_{ij}^L,\tilde{a}_{ij}^U\right],\tilde{b}_{ij}=\left[\tilde{b}_{ij}^L,\tilde{b}_{ij}^U\right]$, for all i, j = 1, 2, ..., n.

Table 1
Experts' optimal weights with different attitudinal character λ.

ω_i^*	λ										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
ω_1^*	0.4836	0.4455	0.3061	0.2700	0.2481	0.2319	0.2189	0.2081	0.1988	0.1907	0.1836
ω_2^*	0.2965	0.2973	0.2999	0.2984	0.2966	0.2946	0.2925	0.2902	0.2877	0.2849	0.2818
ω_3^*	0.2199	0.2572	0.3940	0.4316	0.4553	0.4735	0.4886	0.5017	0.5135	0.5244	0.5346

Table 2
Aggregation results with different attitudinal character λ.

	λ										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\tilde{a}_1^*$	0.8756	0.9204	0.9817	1.0320	1.0827	1.1350	1.1893	1.2457	1.3045	1.3656	1.4293
$\tilde{a}_2^*$	1.4684	1.4748	1.3667	1.3789	1.4077	1.4441	1.4855	1.5309	1.5800	1.6325	1.6881
$\tilde{a}_3^*$	0.8225	0.8418	0.8363	0.8559	0.8797	0.9056	0.9332	0.9623	0.9927	1.0245	1.0576
$\tilde{a}_4^*$	0.4277	0.4416	0.4511	0.4654	0.4806	0.4966	0.5133	0.5305	0.5484	0.5669	0.5860
$\tilde{a}_5^*$	0.8728	0.9426	1.1296	1.2190	1.2967	1.3715	1.4458	1.5210	1.5975	1.6759	1.7565

Table 3
Ordering of the companies.

λ	Ordering	λ	Ordering
0	$x_2 \succ x_1 \succ x_5 \succ x_3 \succ x_4$	0.6	$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4$
0.1	$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4$	0.7	$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4$
0.2	$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4$	0.8	$x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$
0.3	$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4$	0.9	$x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$
0.4	$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4$	1	$x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$
0.5	$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4$	2/3	$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4$

As we can see, the distance $d_\lambda(\widetilde{A},\widetilde{B})$ based on the LCOWG operator reflects the total reciprocal difference between the uncertain multiplicative linguistic preference relations $\widetilde{A}$ and $\widetilde{B}$, in which all the corresponding elements are aggregated by the LCOWG operator.

Based on Definition 21 and Theorem 10, we can obtain the following theorem:

Theorem 11. *Let* $\widetilde{A}=(\tilde{a}_{ij})_{n\times n}\in M_n, \widetilde{B}=(\tilde{b}_{ij})_{n\times n}\in M_n$ *and* $\widetilde{C}=(\tilde{c}_{ij})_{n\times n}\in M_n$*, then*

(1) *Nonnegativity*: $d_\lambda(\widetilde{A},\widetilde{B})\geqslant 0$.
(2) *Reflexivity*: $d_\lambda(\widetilde{A},\widetilde{A})=0$.
(3) *Commutativity*: $d_\lambda(\widetilde{A},\widetilde{B})=d_\lambda(\widetilde{B},\widetilde{A})$.
(4) *Transitivity: If* $d_\lambda(\widetilde{A},\widetilde{B})=0$ *and* $d_\lambda(\widetilde{B},\widetilde{C})=0$*, then* $d_\lambda(\widetilde{A},\widetilde{C})=0$.
(5) *Triangle inequality:* $d_\lambda(\widetilde{A},\widetilde{C})\leqslant d_\lambda(\widetilde{A},\widetilde{B})+d_\lambda(\widetilde{B},\widetilde{C})$.

Let $\widetilde{A}^{(k)}=\left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}\in M_n$ be the uncertain multiplicative linguistic preference relation provided by expert e_k (k = 1, 2, ..., m) and $\widetilde{A}=(\tilde{a}_{ij})_{n\times n}$ be the synthetic linguistic preference relation of all experts determined by Eq. (26). Then the less distance of uncertain multiplicative linguistic preference relations provided by expert e_k, the more reliable information given by e_k. Therefore, the aggregation weight of e_k maybe depends on the distance of uncertain multiplicative linguistic preference relations. In order to determine the weights of experts, we can construct the distance of $\widetilde{A}^{(k)}$ and the synthetic linguistic preference relation $\widetilde{A}$, i.e.,

$$d_\lambda(\widetilde{A}^{(k)},\widetilde{A})=\frac{1}{2tn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|I\left(g_\lambda\left(\tilde{a}_{ij}^{(k)}\right)\right)-I(g_\lambda(\tilde{a}_{ij}))\right|, \tag{30}$$

where $\widetilde{A}=(\tilde{a}_{ij})_{n\times n}$ is determined by Eq. (26).

Then we obtain the optimal model as follows:

$$\min J=\sum_{k=1}^{m}d_\lambda(\widetilde{A}^{(k)},\widetilde{A})s.t.\quad \begin{cases}\sum_{k=1}^{m}\omega_k=1,\\ \omega_k\geqslant 0, k=1,2,\ldots,m.\end{cases} \tag{31}$$

Note that the model (31) is nonlinear and can be solved by using Matlab or LINGO software package.

Example 3. Take the same information in Example 2, then by the model (31), we have the weighting vector of $\widetilde{A}^{(1)},\widetilde{A}^{(2)},\widetilde{A}^{(3)}$ as follows:

$\omega=(0.4435,0.2175,0.3390)^T$.

5. The application of the ILCOWG operator to group decision making with uncertain multiplicative linguistic preference relations

In this section, we shall present a new approach based on the ILCOWG operator to group decision making with uncertain multiplicative linguistic preference relations.

Consider a GDM problem. Let $X=\{x_1, x_2, \ldots, x_n\}$ be a set of finite alternatives and $E=\{e_1, e_2, \ldots, e_m\}$ be a finite set of experts. Each expert provides his/her own decision matrix $\widetilde{A}^{(k)}=\left(\tilde{a}_{ij}^{(k)}\right)_{n\times n}$, which are uncertain multiplicative linguistic preference relation given by the expert $e_k\in E$. The process of new approach can be summarized as follows:

Step 1: Utilize the Eq. (25) to determine the relative consensus degree of e_k.

Step 2: Utilize the model (31) to determine the optimal weights of experts:

$$\omega^*=(\omega_1^*,\omega_2^*,\ldots,\omega_m^*)^T.$$

Step 3: Utilize Eq. (26) to obtain the synthetic linguistic preference relation $\widetilde{A}^*=\left(\tilde{a}_{ij}^*\right)_{n\times n}$ based on the optimal weights of experts.

Step 4: Calculate the expected value $\tilde{a}_i^*$ of preference degree of the alternative x_i to all the alternative by the following formula:

$$\tilde{a}_i^*=\left(\otimes_{j=1}^{n}\tilde{a}_{ij}^*\right)^{1/n},\quad i=1,2,\ldots,n. \tag{32}$$

Step 5: Rank the expected value $\tilde{a}_i^*(i=1,2,\ldots,n)$ in descending order.

Step 6: Rank all the alternatives x_i (i = 1, 2, ..., n) and select the best one(s) in accordance with the expected value $\tilde{a}_i^*(i=1,2,\ldots,n)$.

Step 7: End.

6. Illustrative example

In this section, we regard the use of the ILCOWG operator with the uncertain multiplicative linguistic preference relations in a GDM problem. Supplier selection is a very important strategic decision involving decisions balancing a number of conflicting criteria. With the increase in outsourcing, offshore sourcing and various electronic businesses, supplier's service performance is becoming ever more complex in the global market. The International Logistics Company ChaoYang established in Hefei wanted to select potential partners for a collaborative project. In order to select an ideal supplier, the company formed a term of three experts e_k (k = 1,2,3) including the Finance Manager, the Technology Manager and the Quality Manager, to evaluate five potential supplier partners x_i (i = 1,2,3,4,5). Three experts are invited to compare these five suppliers with respective to the main criterion service performance by using the multiplicative linguistic scale:

$$S=\{s_{1/5}=EL,s_{1/4}=VL,s_{1/3}=L,s_{1/2}=SL,s_1=M,s_2=SH,s_3=H,s_4=VH,s_5=EH\}.$$

Note that *EL* = *Extremely low*, *VL* = *Very low*, *L* = *Low*, *SL* = *Slightly low*, *M* = *Medium*, *SH* = *Slightly high*, *H* = *High*, *VH* = *Very high*, *EH* = *Extremely high*.

Experts constructed the uncertain multiplicative linguistic preference relations $\widetilde{A}^{(k)}(k=1,2,3)$, respectively, which are listed as follows:

$$\widetilde{A}^{(1)}=\begin{pmatrix}[s_1,s_1] & [s_2,s_3] & [s_{1/4},s_{1/3}] & [s_4,s_5] & [s_{1/2},s_1]\\ [s_{1/3},s_{1/2}] & [s_1,s_1] & [s_4,s_5] & [s_2,s_3] & [s_{1/5},s_{1/4}]\\ [s_3,s_4] & [s_{1/5},s_{1/4}] & [s_1,s_1] & [s_3,s_4] & [s_{1/5},s_{1/4}]\\ [s_{1/5},s_{1/4}] & [s_{1/3},s_{1/2}] & [s_{1/4},s_{1/3}] & [s_1,s_1] & [s_5,s_6]\\ [s_1,s_2] & [s_4,s_5] & [s_4,s_5] & [s_{1/6},s_{1/5}] & [s_1,s_1]\end{pmatrix};$$

$$\widetilde{A}^{(2)} = \begin{pmatrix} [s_1, s_1] & [s_3, s_4] & [s_{1/3}, s_{1/2}] & [s_3, s_4] & [s_{1/4}, s_{1/3}] \\ [s_{1/4}, s_{1/3}] & [s_1, s_1] & [s_5, s_6] & [s_3, s_4] & [s_{1/3}, s_{1/2}] \\ [s_2, s_3] & [s_{1/6}, s_{1/5}] & [s_1, s_1] & [s_4, s_5] & [s_{1/4}, s_{1/3}] \\ [s_{1/4}, s_{1/3}] & [s_{1/4}, s_{1/3}] & [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_5, s_6] \\ [s_3, s_4] & [s_2, s_3] & [s_3, s_4] & [s_{1/6}, s_{1/5}] & [s_1, s_1] \end{pmatrix};$$

$$\widetilde{A}^{(3)} = \begin{pmatrix} [s_1, s_1] & [s_4, s_5] & [s_{1/4}, s_{1/3}] & [s_2, s_3] & [s_{1/4}, s_{1/3}] \\ [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_4, s_5] & [s_3, s_4] & [s_6, s_7] \\ [s_3, s_4] & [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_4, s_5] & [s_{1/3}, s_{1/2}] \\ [s_{1/3}, s_{1/2}] & [s_{1/4}, s_{1/3}] & [s_{1/5}, s_{1/4}] & [s_1, s_1] & [s_5, s_6] \\ [s_3, s_4] & [s_{1/7}, s_{1/6}] & [s_2, s_3] & [s_{1/6}, s_{1/5}] & [s_1, s_1] \end{pmatrix}.$$

With this information, we can use the proposed decision making method to get the ranking of the suppliers. Letting $Q(y) = \sqrt{y}$, then $\lambda = \int_0^1 Q(y)dy = 2/3$. The following steps are involved:

Step 1: Utilize the Eq. (25) to determine the relative consensus degree of e_k and obtain

$$RCD^{(1)} = 0.7301, \quad RCD^{(2)} = 0.7491, \quad RCD^{(3)} = 0.7240.$$

Step 2: Utilize the model (31) to determine the optimal weights of experts:

$$\omega_1^* = 0.2115, \quad \omega_2^* = 0.2910, \quad \omega_3^* = 0.4975.$$

Step 3: Utilize Eq. (26) to obtain the synthetic linguistic preference relation $\widetilde{A}^* = \left(\tilde{a}_{ij}^*\right)_{n\times n}$ based on the optimal weights of experts, where

$$\widetilde{A}^* = \begin{pmatrix} s_1 & s_{3.4799} & s_{0.3634} & s_{3.6417} & s_{0.4008} \\ s_{0.3185} & s_1 & s_{5.1168} & s_{3.3044} & s_{0.6463} \\ s_{3.0887} & s_{0.2091} & s_1 & s_{4.3226} & s_{0.3095} \\ s_{0.3028} & s_{0.3369} & s_{0.2508} & s_1 & s_{5.6462} \\ s_{2.8558} & s_{1.7097} & s_{3.6417} & s_{0.1882} & s_1 \end{pmatrix}.$$

Step 4: Calculate the expected value $\tilde{a}_i^*$ of preference degree of the alternative x_i to all the alternative by Eq. (32):

$$\tilde{a}_1^* = s_{1.2267}, \quad \tilde{a}_2^* = s_{1.5154}, \quad \tilde{a}_3^* = s_{0.9524}, \quad \tilde{a}_4^* = s_{0.5247}, \quad \tilde{a}_5^* = s_{1.4958}.$$

Step 5: Rank the expected value $\tilde{a}_i^* (i = 1, 2, \ldots, 5)$ in descending order:

$$\tilde{a}_2^* > \tilde{a}_5^* > \tilde{a}_1^* > \tilde{a}_3^* > \tilde{a}_4^*.$$

Step 6: Rank all the alternatives x_i (i = 1, 2, …, 5) in accordance with the expected value $\tilde{a}_i^* (i = 1, 2, \ldots, 5)$:

$$x_2 \succ x_5 \succ x_1 \succ x_3 \succ x_4.$$

The best alternative is the second partner.

Moreover, according to Eq. (30), we get the distance of $\widetilde{A}^{(k)} (k = 1, 2, 3)$ and $\widetilde{A}^*$:

$$d_\lambda(\widetilde{A}^{(1)}, \widetilde{A}^*) = \frac{1}{2 \times 5 \times 5^2} \sum_{i=1}^{5} \sum_{j=1}^{5} \left| I\left(g_\lambda\left(\tilde{a}_{ij}^{(1)}\right)\right) - I\left(g_\lambda\left(\tilde{a}_{ij}^*\right)\right) \right| = 0.0591,$$

$$d_\lambda(\widetilde{A}^{(2)}, \widetilde{A}^*) = \frac{1}{2 \times 5 \times 5^2} \sum_{i=1}^{5} \sum_{j=1}^{5} \left| I\left(g_\lambda\left(\tilde{a}_{ij}^{(2)}\right)\right) - I\left(g_\lambda\left(\tilde{a}_{ij}^*\right)\right) \right| = 0.0089,$$

$$d_\lambda(\widetilde{A}^{(3)}, \widetilde{A}^*) = \frac{1}{2 \times 5 \times 5^2} \sum_{i=1}^{5} \sum_{j=1}^{5} \left| I\left(g_\lambda\left(\tilde{a}_{ij}^{(3)}\right)\right) - I\left(g_\lambda\left(\tilde{a}_{ij}^*\right)\right) \right| = 0.1731.$$

Furthermore, it is possible to analyze how the different attitudinal character λ plays a role in the aggregation results, in this case, we consider different value of λ: 0, 0.1, …, 0.9, 1, which are provided by experts. The results of experts' weights by model (31) are shown in Table 1 and the results of $\tilde{a}_i^*$ $(i = 1, 2, 3, 4, 5)$ are shown in Table 2.

It is observed from Table 1 that ω_1^* decreases as λ increases, while ω_3^* increases as λ increases, and ω_2^* increases first and then decreases as λ increases. Moreover, from Table 2 we can see that $\tilde{a}_1^*$, $\tilde{a}_3^*$, $\tilde{a}_4^*$ and $\tilde{a}_5^*$ increase as λ increases, while $\tilde{a}_2^*$ is irregular as λ increases.

We can establish an ordering of the companies for each value of λ. The results are shown in Table 3. Note that "$\succ$" means "preferred to".

As we can see, depending on the particular cases of the attitudinal character λ used, the ordering of the companies is different, thus leading to different decisions. However, it seems that x_2 is the best choice when $\lambda \leqslant 0.7$, and x_5 sometimes is also the best one.

Furthermore, in order to analyze how other aggregation operators have affection for the aggregation results, in this example, we consider the induced uncertain linguistic ordered weighted geometric (IULOWG) operator and the uncertain linguistic ordered weighted geometric (ULOWG) operator introduced in (Xu, 2006a). In this case, we assume that the weighting vector of experts $\omega = (0.2, 0.5, 0.3)^T$. Firstly, we can utilize the IULOWG operator to aggregate all the uncertain multiplicative linguistic preference relations $\widetilde{A}^{(k)} (k = 1, 2, 3)$ into the collective uncertain multiplicative linguistic preference relation $\widetilde{A} = (\tilde{a}_{ij})_{5\times 5}$:

$$\widetilde{A} = \begin{pmatrix} [s_1, s_1] & [s_{3.2875}, s_{4.3174}] & [s_{0.2648}, s_{0.3615}] & [s_{2.4915}, s_{3.5195}] & [s_{0.2872}, s_{0.4152}] \\ [s_{0.2316}, s_{0.3042}] & [s_1, s_1] & [s_{4.1826}, s_{5.1857}] & [s_{2.7663}, s_{3.7764}] & [s_{1.7048}, s_{2.1205}] \\ [s_{2.7663}, s_{3.7764}] & [s_{0.1928}, s_{0.2391}] & [s_1, s_1] & [s_{3.7764}, s_{4.7818}] & [s_{0.2971}, s_{0.4014}] \\ [s_{0.2841}, s_{0.4014}] & [s_{0.2648}, s_{0.3615}] & [s_{0.2091}, s_{0.2648}] & [s_1, s_1] & [s_5, s_6] \\ [s_{2.4082}, s_{3.4822}] & [s_{0.4717}, s_{0.5867}] & [s_{2.4915}, s_{3.5195}] & [s_{0.1667}, s_{0.2}] & [s_1, s_1] \end{pmatrix}.$$

Secondly, we utilize the ULOWG operator (let its weighting vector be $\omega' = (0.15, 0.2, 0.3, 0.2, 0.15)^T$) to aggregate $\tilde{a}_{ij} (j = 1, 2, 3, 4, 5)$ corresponding to the alternative x_i, and then get the collective uncertain linguistic preference degree $\tilde{a}_i$ of the ith alternative over all the other alternatives: $\tilde{a}_1 = [s_{0.8477}, s_{1.1130}], \tilde{a}_2 = [s_{1.6379}, s_{2.0012}], \tilde{a}_3 = [s_{0.9113}, s_{1.0933}], \tilde{a}_4 = [s_{0.5053}, s_{0.6248}], \tilde{a}_5 = [s_{0.9022}, s_{1.1458}]$. Thirdly, we compare each $\tilde{a}_i$ by using the degree of possibility of uncertain linguistic variables, and develop a fuzzy preference relation:

$$P = \begin{pmatrix} 0.5 & 0 & 0.4510 & 1 & 0.4143 \\ 1 & 0.5 & 1 & 1 & 1 \\ 0.5490 & 0 & 0.5 & 1 & 0.4490 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0.5857 & 0 & 0.5510 & 1 & 0.5 \end{pmatrix}.$$

Then by summing all elements in each line of matrix P, we have $p_1 = 2.3653$, $p_2 = 4.5$, $p_3 = 2.4980$, $p_4 = 0.5$, $p_5 = 2.6367$. Finally, we can rank $\tilde{a}_i$ in descending order in accordance with the values of p_i: $\tilde{a}_2 > \tilde{a}_5 > \tilde{a}_3 > \tilde{a}_1 > \tilde{a}_4$, which means that $x_2 \succ x_5 \succ x_3 \succ x_1 \succ x_4$.

As we can see, depending on the other aggregation operators and the ILCOWG operator used, the ordering of the companies is different, but leading to the same decisions as $\lambda \leqslant 0.7$, because the best alternative is the same. However, according to the aggregation process, the induced variables in the IULOWG operator and the weighting vector of experts are all given by decision maker in advance, but the induced variables in the ILCOWG operator are determined by the relative consensus degree and the weighting vector of experts is dominated by the model (31). It is obvious that the approach developed in this paper is more objective than the method using the IULOWG operator. Moreover, the decision maker can make decision by choosing different value of the parameter λ with his/her attitude, which is superior to the traditional operators.

7. Conclusions

In this paper, we developed the ILCOWG operator, which is an extension of the ILOWG operator. We have studied some desirable properties of the ILCOWG operator, which can be applied to GDM with uncertain multiplicative linguistic preference relations based on the concept of the consistency. Particularly, we have defined the RCD-ILCOWG operator. Moreover, we have presented the distance measure of uncertain multiplicative linguistic preference relations based on the LCOWG operator and developed an optimal model to determine experts' weights, which are the scientific bases of using the uncertain multiplicative linguistic preference relations in the GDM. Furthermore, we have applied the developed aggregation operators to solving the group decision making problem with uncertain multiplicative linguistic preference relations. The prominent characteristic of the new approach is that it can measure the reliability of information given by different experts based on the criterion of minimizing the collective distance index of uncertain multiplicative linguistic preference relations, which is more objective and effective than other existing ones.

In future, we expect to develop more linguistic aggregation operators by adding new characteristic for aggregating the uncertain linguistic preference relations, such as uncertain additive linguistic preference relations. We will also consider other decision making problems with linguistic preference relations, such as strategic decision making and product management.

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