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VECTOR FIELDS AND DIFFERENTIAL EQUATIONS ON SUPERMANIFOLDS

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In the latest papers in physics devoted to supergravitation, the generalized Yang–Mills equations, etc., an important role is played by odd vector fields of the form $\partial/\partial\xi + \xi\partial/\partial u$, considered as "square roots" of the shift generator $\partial/\partial u$ (cf. [1]). In this note a theorem on rectifiable vector fields is proved, which shows that the field $\partial/\partial\xi + \xi\partial/\partial u$ has a simple invariant characterization; on the basis of it differential equations are defined on supermanifolds for which an existence and uniqueness theorem for the solutions is proved. All the preliminary information is contained in [2, 3].

1. Let $\mathcal{M}$ be a superdomain of dimension (p, q). In coordinates $x = (u, \xi)$ on $\mathcal{M}$, each vector field D, obviously can be described in the form $D = \Sigma D(u_i)\partial/\partial u_i + \Sigma D(\xi_j)\partial/\partial\xi_j$.

We call the field D weakly nondegenerate at the point $m \in \mathcal{M}$, if not all the coefficients of D vanish at the point m, and nondegenerate if $D|_{\mathcal{U}}: C^\infty(\mathcal{U}) \to C^\infty(\mathcal{U})$ is an epimorphism for some neighborhood $\mathcal{U}$ of the point m.

THEOREM 1. Let the field D be nondegenerate at the point $m \in \mathcal{M}$. Then there exists a coordinate system $x = (u, \xi)$ in a neighborhood of the point m, in which $D = \partial/\partial u_1$, where D is even and $D = \partial/\partial\xi_1 + \xi_1\partial/\partial u_1$ if D is odd.

Proposition 1. Let the field D be weakly nondegenerate at the point $m \in \mathcal{M}$. Then if D is even, then D is nondegenerate, and if D is odd, then in some neighborhood $\mathcal{U}$ of the point m there exists a coordinate system in $D|_{\mathcal{U}} = \partial/\partial\xi_1 + \xi_1 L$, where L is an even field on $\mathcal{U}$.

2. Proof. We denote by J the ideal in $C^\infty(\mathcal{M})$ generated by all odd functions. Induction on k shows that an even weakly nondegenerate field can be reduced to the form $\partial/\partial u_1 (\mathrm{mod}\ J^k)$. Theorem 1 and Proposition 1 follows from the fact that $J^{q+1} = 0$, and from the fact that $D^2 = (1/2)[D, D]$ is an even nondegenerate vector field.

3. It is known (cf. [4]) that by an ordinary differential equation it is convenient to understand a vector field on a manifold M, depending on time, a one-dimensional manifold T; by a solution of this equation is meant a T-family of diffeomorphisms of the manifold M (translations along integral curves).

Analogously, let $(\mathcal{T} \subset \mathcal{T}_0, x, D_{\mathcal{T}})$, where $\mathcal{T}$ is a subsuperdomain in the superspace $\mathcal{T}_0$, and x and $D_{\mathcal{T}}$ are coordinates and a vector field on $\mathcal{T}$, and one has one of the two quadruples:

a) $(I^{1,0} \subset \mathcal{R}^{1,0}, t, \partial/\partial t)$, where $I^{1,0}$ is an interval containing 0;

b) $(I^{1,1} \subset \mathcal{R}^{1,1}, (t, \tau), \partial/\partial\tau + \tau\partial/\partial t)$, where $I^{1,1} = I^{1,0} \times \mathcal{R}^{0,1}$.

We shall call $\mathcal{T}$ time and $D_{\mathcal{T}}$ differentiation with respect to time.

Let $\mathcal{M}$ be a supermanifold, $\mathcal{U}$ be an open subsupermanifold in $\mathcal{M}$, and $\mathcal{T}$ be time. We call $\mathcal{U}\times\mathcal{T}$ a cylinder over $\mathcal{M}$, and an open subsupermanifold $\mathcal{W} \subset \mathcal{M} \times \mathcal{T}$, representable as a union of cylinders and containing $\mathcal{M} \times \{0\}$ a pseudocylinder. (If $\mathcal{M}$ is compact, then below pseudocylinder can be replaced by cylinder.) Let $\pi_{\mathcal{M}}, \pi_{\mathcal{T}}$ be the projections of $\mathcal{M}\times\mathcal{T}$ onto $\mathcal{M}$ and $\mathcal{T}$, respectively, $i: \mathcal{M} \to \mathcal{M} \times \mathcal{T}$ be the inclusion defined by the projection $i^*: C^\infty(\mathcal{M} \times \mathcal{T}) \to C^\infty(\mathcal{M} \times \mathcal{T})/\{f\colon f|_{\mathcal{M}\times\{0\}} = 0\} \simeq C_\infty(\mathcal{M})$. By $\tilde{D}_{\mathcal{T}}$ we denote the field on $\mathcal{M} \times \mathcal{T}$, uniquely defined by the conditions $\pi^*_{\mathcal{T}} \circ D_{\mathcal{T}} = \tilde{D}_{\mathcal{T}} \circ \pi^*_{\mathcal{T}},\ \tilde{D}_{\mathcal{T}} \circ \pi^*_{\mathcal{M}} = 0$.

By a differential equation on $\mathcal{M}$ with time $\mathcal{T}$ we shall mean a field D on a pseudocylinder over $\mathcal{M}$, such that $D \circ \pi^*_{\mathcal{T}} = \pi^*_{\mathcal{T}} \circ D_{\mathcal{T}}$. By a $\mathcal{T}$-family of diffeomorphisms of the supermanifold $\mathcal{M}$ we shall mean a diffeo-

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morphism $\varphi: \mathcal{W}_1 \to \mathcal{W}_2$ of pseudocylinders over $\mathcal{M}$, such that $\pi_{\mathcal{T}} = \pi_{\mathcal{T}} \circ \varphi$. By a solution of the differential equation D, defined on the pseudocylinder $\mathcal{W}$, we shall mean a $\mathcal{T}$-family of diffeomorphisms $\varphi: \mathcal{W}_1 \to \mathcal{W}_2$, where $\mathcal{W}_1 \subset \mathcal{W}$, such that $\varphi \circ i = i$ and $D \circ \varphi^* = \varphi^* \circ \tilde{D}_{\mathcal{T}}$.

4. THEOREM 2. Any differential equation has a solution, which is unique in the sense that two solutions coincide on their common domain of definition.

Sketch of the Proof. For an even vector field X on a supermanifold $\mathcal{N}$ one can construct uniquely a vector field πX on the underlying manifold N, applying at each point of the manifold N the same value as X does. Hence to the equation D corresponds an equation on the underlying manifold, defined by the field πD, if D is even, and by the field πD^2 if D is odd.

LEMMA. If $\varphi: \mathcal{U} \times \mathcal{T} \to \mathcal{U} \times \mathcal{T}$ is a diffeomorphism of a cylinder, while $\varphi^* \circ \tilde{D}_{\mathcal{T}} = \tilde{D}_{\mathcal{T}} \circ \varphi^*$, then there exists a unique diffeomorphism $\psi: \mathcal{U} \to \mathcal{U}$, such that $\varphi = \psi \times \mathrm{id}_{\mathcal{T}}$.

The solution of the equation on the underlying manifold, which exists and is unique according to [5], is used as the foundation on which, using Theorem 1 and the Lemma, one erects a solution on the supermanifold.

5. We associate with the field D on a superdomain $\mathcal{M}$ the differential equation defined by a field D' on $\mathcal{M} \times \mathcal{T}$, where $p(D_{\mathcal{T}}) = p(D)$, so $D' \circ \pi^*_{\mathcal{M}} = \pi^*_{\mathcal{M}} \circ D$ and $D' \circ \pi^*_{\mathcal{T}} = \pi^*_{\mathcal{T}} \circ D_{\mathcal{T}}$.

Let $p(D) = \bar{1}$, $\dim \mathcal{T} = (1,1)$, $D' = D'_0 + \tau D'_1$ and $f = f_0 + \tau f_1 \in C^\infty(\mathcal{M} \times \mathcal{T})$. The equation $D'f = 0$ is obviously equivalent with the system

$$\partial/\partial t\,(f_0) = -(D'_0 + D'^2_1) f_0, \quad f_1 = -D'_1 f_0.$$

Examples. Let $\mathcal{M}$ be a superspace of dimension (2n, q) (respectively (m, m)). We define a Lie superalgebra structure in $C^\infty(\mathcal{M})$ (respectively $P(C^\infty(\mathcal{M}))$), setting (cf. [6], respectively [7])

$$\{f, g\}_{P.B.} = \sum \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) + (-1)^{p(f)} \sum \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \xi_j}$$

(respectively, $\{f, g\}_{L.B.} = \sum \left((-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial u_i} + \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial \xi_i} \right)$).

Setting $D(f) = \{f, H\}$, where $f, H \in C^\infty(\mathcal{M})$, we get the dynamics on $\mathcal{M}$. One can show that equations of the type considered in [6], where the Hamiltonian and brackets are even, are among the equations obtained with (1, 1)-dimensional time, $p(H) = \bar{1}$.

6. The problem of local classification of weakly nondegenerate fields is meaningless: it contains the problem of local classification of all fields on $\mathbf{R}^n$ (it suffices to associate with the field $D \in \mathrm{Vect}\,\mathbf{R}^n$ the field $\partial/\partial\xi + \xi D \in \mathrm{Vect}\,\mathcal{R}^{n,1}$). We single out the weakly nondegenerate fields the class of automatically interesting ones, which, apparently one can succeed in classifying. We call an odd vector field D homological, if $D^2 = 0$. Considering D as an operator on functions, we set $H_D = \mathrm{Ker}\, D/\mathrm{Im}\, D$.

Example. Let M be a manifold. Then on the supermanifold $\hat{M} = (M, \Omega(M))$ (cf. [3]) the differential d is a homological vector field, where $\dim H_d = (1, 0)$.

If $H_D \equiv \{0\}$, then one can succeed in describing the homological fields.

THEOREM 3. Let D be a homological field on the superdomain $\mathcal{M}$. Then the following conditions are equivalent:

1) the field D is weakly nondegenerate at all points of $\mathcal{M}$;

2) $H_D = \{0\}$;

3) there exists a coordinate system in which $D = \partial/\partial\xi_1$.

7. By analogy with paragraph 2, we mean by odd time on a supermanifold $\mathcal{T} = \mathcal{R}^{0,1}$ with coordinate τ, and by time differentiation the field $\partial/\partial\tau$.

Proposition 2. If $\mathcal{T}$ is an odd time, then only those differential equations with time $\mathcal{T}$, which are defined by homological vector fields on $\mathcal{M} \times \mathcal{T}$, are integrable. Such a field can be described uniquely in the form $D - \tau D^2 + \partial/\partial\tau$, where D is an odd field on $\mathcal{M}$. A solution of such an equation is a diffeomorphism $\varphi: \mathcal{M} \times \mathcal{T} \to \mathcal{M} \times \mathcal{T}$, where $\varphi^* = e^{-\tau D} = 1 - \tau D$.

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FIXED POINTS OF LINEAR-FRACTIONAL TRANSFORMATIONS

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1. Let K be the unit ball in Hilbert space H (real or complex). A transformation φ:K → K is called linear-fractional for short (l.f.t.) if $\varphi(x) = [(Ax + x_0)/(l(x) + \alpha_0)]$, where $A \in B(H)$, $x_0 \in H$, $l \in H^*$, $\alpha_0 \notin -l(K)$. As proved by Krein [1], the problems of existence of invariant nonnegative subspaces for families of J-unitary operators in the space Π_1 reduce to finding fixed points of families of l.f.t. If H is real, then all l.f.t. are quasiaffine (preserve the class of convex sets), and this in a series of cases allows one to prove the existence of fixed points (cf. [1, 2]). In complex space l.f.t. are not quasiaffine. It turns out, however, that invertible l.f.t. are quasiaffine relative to a certain "non-Euclidean" convexity and this makes it possible to establish the existence of common fixed points of equicontinuous groups of l.f.t.

2. We set for any $z \in \mathbf{C}$ (Re z < 1)

$$T_z = \left\{\frac{1-t}{1-tz}:\ 0 \leqslant t \leqslant 1\right\}$$

and we mean by the h-segment joining the vectors $x \in K$, $y \in K$, the set

$$h(x, y) = \{\alpha x + (1-\alpha)y:\ \alpha \in T_{(x, y)}\}.$$

A set $S \subset K$ is called h-convex if from $x \in S$, $y \in S$ follows $h(x, y) \subset S$.

Obviously in the case of real H, h-convexity coincides with ordinary convexity. In the one-dimensional case H = C, an h-segment is a segment in the Poincaré model of Lobachevskian geometry so that an h-convex set is a set which is Lobachevskii convex. The following two lemmas establish respectively the h-convexity and uniform h-convexity of a ball with center at the point 0. For short, by the length of an h-segment we shall mean the distance between its ends.

LEMMA 1. Any ball αK $(0 < \alpha < 1)$ is an h-convex set.

LEMMA 2. For any α $(0 < \alpha < 1)$ and $\varepsilon > 0$ there exists a β $(0 < \beta < \alpha)$ such that any segment contained in $\alpha K \setminus \beta K$ has length less than ε.

A transformation φ: K → K will be called h-quasiaffine if it is weakly continuous and carries h-convex sets into h-convex ones. A set $S \subset K$ will be called isolated from the boundary if $S \subset \alpha K$ for some $\alpha < 1$.

LEMMA 3. If G is an equicontinuous group of h-quasiaffine transformations, then the G-orbit of any interior point of K is isolated from the boundary.

THEOREM 1. Any equicontinuous group G of h-quasiaffine transformations has a fixed point in int K.

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