AN EXACT GENERAL SOLUTION IN SPHERICAL HARMONICS OF THE BOLTZMANN EQUATION

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A solution of the one-velocity kinetic Boltzmann equation is obtained as a series of spherical harmonics. General expressions are obtained for the terms of the series, derived without any approximately valid assumptions. As particular cases of this solution, we obtain formulas for the known P_N-approximations for the spherical-harmonic method.

The exact general solution of the kinetic equation in the form of a series of spherical harmonics contains arbitrary functions which must depend on the formulation of boundary conditions. The general determination of the boundary conditions and the arbitrary functions is not considered. All the results of [4] remain valid for P_N-approximations.

The well-known and usually very effective spherical-harmonic method [1-6], by which the integro-differential Boltzmann equation is reduced to an infinite system of differential equations, is always applied in a P_N-approximation. In this approximation only the first N + 1 equations of the system are solved under the artificial condition that the spherical harmonic of number N + 1 be identically zero. In some cases, however, this condition is too rough, and for small values of N the spherical-harmonic method, in the form in which it is at present used, leads to large errors.

In the present work, we give an exact solution of the infinite system of equations obtained by the spherical-harmonic method; this solution is essentially equivalent to an exact solution of the Boltzmann equation. Our solution is general, i.e., it is not subject to any boundary conditions, and so it is determined with a precision up to certain arbitrary functions. We do not consider the formulation of boundary conditions and the solution of concrete physical problems in the present work. Different approximate approaches to this problem are possible, the use of which would obviously considerably broaden the range of applicability of the spherical-harmonic method. The ordinary P_N-approximations together with the appropriate boundary conditions are easily obtained as special cases of the general solution.

Harmonic Polynomials and Vector Operators

We seek the solution of the one-velocity kinetic Boltzmann equation in the form

$$F(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) Y_n(\mathbf{r}, \mathbf{\Omega}). \tag{1}$$

All the notation is the same as in [4] except when differences are specifically mentioned. Each spherical harmonic Y_n corresponds to a homogeneous harmonic field U_n:

$$U_n(\mathbf{r}, \boldsymbol{\mu}) = \mu^n Y_n(\mathbf{r}, \mathbf{\Omega}). \tag{2}$$

We note that U_n, being a polynomial in μ_x, μ_y and μ_z, satisfies in the space μ the Laplace equation

$$\dot{\nabla}^2 U_n = 0. \tag{3}$$

The coefficients of the polynomial are functions of r. We consider those polynomial satisfying the equation

$$\nabla^2 U_n = \varkappa^2 U_n, \tag{4}$$

in the space r, where $\varkappa^2$ is a constant, for the moment arbitrary.

Let the operator $(\mu\nabla)^l$ denote the l-tuple application of the operator $(\mu\nabla)$. Polynomials in $(\mu\nabla)$ introduced below will have their obvious meaning. We now demonstrate the theorem that enables us to obtain an exact expression for any term of the series (1).

Theorem. If f_k is a harmonic polynomial of degree k satisfying

$$(\nabla\dot{\nabla}) f_k = 0; \quad \nabla^2 f_k = \varkappa^2 f_k, \tag{5}$$

then the operator $\mu^{n-k} P_n^{(k)}\left(\frac{\mu\nabla}{\mu\varkappa}\right)$ converts it into a harmonic polynomial of degree n. Here and below $P_n^{(k)}(t) = d^k/dt^k P_n(t)$, where $P_n(t)$ is a Legendre polynomial.

Proof. The operator $\mu^{n-k} P_n^{(k)}\left(\frac{\mu\nabla}{\mu\varkappa}\right)$ is a homogeneous polynomial of degree n−k in μ_x, μ_y, and μ_z, since it is a linear combination of expressions of the type

$$\mu^{2q}\left(\frac{\mu\nabla}{\varkappa}\right)^{n-k-2q} \quad \text{or} \quad (\mu_x^2 + \mu_y^2 + \mu_z^2)^q \left(\frac{\mu\nabla}{\varkappa}\right)^{n-k-2q},$$

where $0 \le q \le n-k/2$. Hence $\mu^{n-k} P_n^{(k)}\left(\frac{\mu\nabla}{\mu\varkappa}\right) f_k$ is a homogeneous polynomial of degree n. It remains to prove that this polynomial satisfies the Laplace equation:

$$\dot{\nabla}^2 \mu^{n-k} P_n^{(k)}\left(\frac{\mu\nabla}{\mu\varkappa}\right) f_k = 0, \tag{6}$$

i.e., that it is harmonic.

This is proved by induction. We first show that Eq. (6), for a fixed k, is satisfied for any value of n if it holds for n−1 and n−2.

The operational relations needed for the proof are the following:

$$\dot{\nabla}^2 (\mu\nabla) = (\mu\nabla)\dot{\nabla}^2 + 2(\nabla\dot{\nabla}); \tag{7}$$

$$\dot{\nabla}^2 \mu^2 = \mu^2 \dot{\nabla}^2 + 4(\mu\dot{\nabla}) + 6; \tag{8}$$

$$(\nabla\dot{\nabla})\mu^{2k}(\mu\nabla)^l = \mu^{2k}(\mu\nabla)^l(\nabla\dot{\nabla}) + \frac{(\mu\nabla)}{\mu}\frac{\partial[\mu^{2k}(\mu\nabla)^l]}{\partial\mu} + \nabla^2\frac{\partial[\mu^{2k}(\mu\nabla)^l]}{\partial(\mu\nabla)}. \tag{9}$$

We also need the following recurrence relations for derivatives of the Legendre polynomials*:

$$(n-k)P_n^{(k)}(t) = (2n-1)tP_{n-1}^{(k)}(t) - (n+k-1)P_{n-2}^{(k)}(t); \tag{10}$$

$$(n-k-1)tP_{n-1}^{(k)}(t) + (1-t^2)P_{n-1}^{(k+1)}(t) = (n+k-1)P_{n-2}^{(k)}(t). \tag{11}$$

For brevity we use the notation

$$P_n^{(k)}\left(\frac{\mu\nabla}{\mu\varkappa}\right) \equiv P_n^{(k)}\left(\frac{\Omega\nabla}{\varkappa}\right) \equiv \hat{\mathbf{P}}_n^{(k)}. \tag{12}$$

Using (10), we write

$$\mu^{n-k}\hat{\mathbf{P}}_n^{(k)} = \frac{1}{n-k}\left[(2n-1)\frac{\mu\nabla}{\varkappa}\mu^{n-k-1}\hat{\mathbf{P}}_{n-1}^{(k)} - (n+k-1)\mu^2\mu^{n-k-2}\hat{\mathbf{P}}_{n-2}^{(k)}\right]. \tag{13}$$

*It is possible that (11) is not in the literature. It can be derived from other known recurrence relations.

Now using (7) and (8) and assuming that (6) holds for n − 1 and n − 2, we obtain

$$\dot{\nabla}^2 \mu^{n-k} \hat{\mathbf{P}}_n^{(k)} f_k = \frac{1}{n-k} \left\{ (2n-1)\, 2\, \frac{(\nabla\dot{\nabla})}{\varkappa}\, \mu^{n-k-1} \hat{\mathbf{P}}_{n-1}^{(k)} - (n+k-1)\,[4\,(\mu\dot{\nabla}) + 6]\, \mu^{n-k-2} \hat{\mathbf{P}}_{n-2}^{(k)} \right\} f_k. \tag{14}$$

To an operator of the form $\frac{(\nabla\dot{\nabla})}{\varkappa}\, \mu^{n-k-1} \hat{\mathbf{P}}_{n-1}^{(k)}$ we apply the formula (9) and, using (5), we have

$$\frac{(\nabla\dot{\nabla})}{\varkappa}\, \mu^{n-k-1} \hat{\mathbf{P}}_{n-1}^{(k)'} f_k = \mu^{n-k-2} \times \left\{ (n-k-1)\, \frac{(\mu\nabla)}{\mu\varkappa}\, \hat{\mathbf{P}}_{n-1}^{(k)} + \left[1 - \left(\frac{\mu\nabla}{\mu\varkappa} \right)^2 \right] \hat{\mathbf{P}}_{n-1}^{(k+1)} \right\} f_k \tag{15}$$

or, from (11),

$$\frac{(\nabla\dot{\nabla})}{\varkappa}\, \mu^{n-k-1} \hat{\mathbf{P}}_{n-1}^{(k)} f_k = (n+k-1)\, \mu^{n-k-2} \hat{\mathbf{P}}_{n-2}^{(k)} f_k. \tag{16}$$

We substitute the last expression in the right-hand side of (14) and eliminate the operator $(\mu\dot{\nabla})$ by using the Euler identity:

$$(\mu\dot{\nabla}) \mu^{n-k-2} \hat{\mathbf{P}}_{n-2}^{(k)} f_k = (n-2)\, \mu^{n-k-2} \hat{\mathbf{P}}_{n-2}^{(k)} f_k. \tag{17}$$

(17)

The relation (17) is a consequence of the fact that the expression $\mu^{n-k-2} \hat{\mathbf{P}}_{n-2}^{(k)} f_k$ is a homogeneous polynomial of degree n − 2.

Using (16) and (17), we find that

$$\dot{\nabla}^2 \mu^{n-k} \hat{\mathbf{P}}_n^{(k)} f_k = 0. \tag{18}$$

Hence, for the above assumptions a homogeneous polynomial of degree n of the form $\mu^{n-k} \hat{\mathbf{P}}_n^{(k)} f_k$ is harmonic.

We have $n \geq k$ in all expressions, and so for the completion of the proof, it is sufficient to prove that (18) is satisfied for n = k and n = k + 1. This is done directly:

for n = k

$$\dot{\nabla}^2 \hat{\mathbf{P}}_k^{(k)} f_k = \mathrm{const}\, \dot{\nabla}^2 f_k = 0;$$

for n = k + 1

$$\dot{\nabla}^2 \mu \hat{\mathbf{P}}_{k+1}^{(k)} f_k = \mathrm{const}\, \dot{\nabla}^2 (\mu\nabla)\, f_k = \mathrm{const}\, [(\mu\nabla)\, \dot{\nabla}^2 + 2\, (\nabla\dot{\nabla})]\, f_k = 0.$$

Consequences of the theorem. Every harmonic polynomial U_n, satisfying the equation

$$\nabla^2 U_n = \varkappa^2 U_n, \tag{19}$$

Can be written as a sum:

$$U_n = \sum_{k=0}^{n} \mu^{n-k} \hat{\mathbf{P}}_n^{(k)} f_k. \tag{20}$$

It is easily shown that the terms of this sum cannot be linearly dependent, and that the sum is a harmonic polynomial of general type.

Setting $\mu = 1$ in (20), we obtain an expression for the spherical harmonic Y_n:

$$Y_n = \sum_{k=0}^{n} \hat{\mathbf{P}}_n^{(k)} y_k, \tag{21}$$

where $y_k = [f_k]_{\mu=1}$ is a spherical harmonic of order k.

The Solution of the Kinetic Equation

The solution of the homogeneous equation is of special interest and so, because of the brevity of this article, we will consider the one-velocity kinetic Boltzmann equation only for a source-free medium:

$$(\Omega\nabla) F(\mathbf{r}, \Omega) + \Sigma F(\mathbf{r}, \Omega) - \Sigma_s \int\limits_{4\pi} F(\mathbf{r}, \Omega') W(\Omega', \Omega)\, d\Omega' = 0. \tag{22}$$

Substituting the function $F(\mathbf{r}, \Omega)$ in the form (1) into Eq. (22), we obtain

$$\sum_{n=0}^{\infty} (2n+1) [(\Omega\nabla) Y_n + \Sigma_n Y_n] = 0, \tag{23}$$

where $\Sigma_n = \Sigma - \Sigma_s c_n$ and the c_n are the coefficients in the expansion of the scattering indicatrix (for more details see [4])*.

We now express Y_n in the form (21) and use (10). In order not to mix functions y_k belonging to different harmonics Y_n, we use for them the notation y_{nk}, and remember in the following that the order of the spherical harmonics $y_{nk}(\mathbf{r}, \Omega)$ is determined only by the second subscript. Then (23) becomes

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \{\varkappa [(n-k+1) \hat{\mathbf{P}}^{(k)}_{n-1} + (n+k) \hat{\mathbf{P}}^{(k)}_{n+1}] y_{nk} + (2n+1) \Sigma_n \hat{\mathbf{P}}^{(k)}_n y_{nk}\} = 0. \tag{24}$$

If the linearly independent terms of the expression in (24) are equated to zero, we obtain

$$\varkappa [(n-k) y_{n-1,k} + (n+k+1)_{n+1,k}] + (2n+1) \Sigma_n y_{nk} = 0. \tag{25}$$

This relation enables us to express all the functions y_{nk} in terms of y_{kk}. We assume that

$$y_{nk} = (-1)^{n-k} M_{nk} y_{kk}, \tag{26}$$

where we obviously have $M_{kk} = 1$.

We now introduce the notation $\nu = \Sigma/\varkappa$ and $\varepsilon_n = \Sigma_n/\Sigma$. The relation (25) now yields the recurrence relation

$$(n+k+1) M_{n+1,k} = (2n+1) \nu\varepsilon_n M_{nk} - (n-k) M_{n-1,k}, \tag{27}$$

for the coefficients M_{nk}, and so these coefficients are polynomials of degree $n-k$ in ν, where each of these polynomials contain either purely even or purely odd powers of ν.

In (25) and (27) there are quantities with the same subscripts k, and so for each k there is an independent infinite sequence of functions y_{nk} beginning with the function y_{kk}. In the general solution the functions y_{kk} are arbitrary except for the fact that, from the assumption, they must satisfy the equations

$$(\nabla\dot{\nabla}) \mu^k y_{kk} = 0; \tag{28}$$

$$\nabla^2 y_{kk} = \left(\frac{\Sigma}{\nu}\right)^2 y_{kk}. \tag{29}$$

In concrete problems, the form of the functions y_{kk} is determined by the geometry of the medium and the boundary conditions. Substituting (26) in (21), we obtain

$$Y_n(\mathbf{r}, \Omega, \nu) = \sum_{k=0}^{n} (-1)^{n-k} M_{nk}(\nu) \hat{\mathbf{P}}^{(k)}_n y_{kk}(\mathbf{r}, \Omega, \nu). \tag{30}$$

* In [4] $\Sigma_n = (2n+1)(\Sigma - \Sigma_s c_n)$.

All real angular distributions are such that the functions Y_n in the expansion (1) cannot increase indefinitely with increasing n. We require that the coefficients $M_{nk}(\nu)$ also satisfy this condition. An investigation of the relation (27) shows that the functions $M_{nk}(\nu)$ are bounded for all real ν in the region $\nu| < 1$ and for certain values $\nu = \pm \nu_k$ in $|\nu| > 1$. Hence, the general expression for the spherical harmonic of order n must be

$$Y_n(\mathbf{r}, \mathbf{\Omega}) = \sum_{k=0}^{n} (-1)^{n-k} \left\{ M_{nk}(\nu_k)\,\hat{\mathbf{P}}_n^{(k)} y_{kk}(\mathbf{r}, \mathbf{\Omega}, \nu_k) + \int_0^1 M_{nk}(\nu)\,\hat{\mathbf{P}}_n^{(k)} y_{kk}(\mathbf{r}, \mathbf{\Omega}, \nu)\,d\nu \right\}. \qquad (31)$$

The integration here is only for positive values of ν, since the general form of the functions y_{kk} for the complete solution in the large is independent of the sign of ν. The numbers ν_k can always be given a plus sign. The values of the ν_k depend only on the physical properties of the medium. In each concrete case, the ν_k can in principle be calculated with any desired degree of accuracy.

P_N-Approximations of the Spherical-Harmonic Method

As a special case of (31), we can obtain formulas for the P_N-approximations of the spherical-harmonic method, if, without worrying about the convergence of the $M_{nk}(\nu)$ when $n \to \infty$, we specify the conditions

$$M_{N+1,\,k}(\nu) = 0, \quad 0 \leqslant k \leqslant N \qquad (32)$$

and $y_{N+1} \equiv 0$, which clearly agree with the assumption $Y_{N+1} \equiv 0$. Then the set of permissible values of ν reduces to the set of roots ν_{ki} of Eq. (32).

The roots ν_{0i} correspond to those numbers $\varkappa_{0i}$ which in [4] were called fundamental numbers, and the roots ν_{ki} for $k > 0$ correspond to the supplementary numbers $\varkappa_{ki}$. In one-dimensional, plane, or spherical geometry, the functions y_{kk} for $k > 0$ must be identically zero, since otherwise it is impossible to satisfy the condition (28), i.e., the supplementary roots take no part in the solution of such problems.

In the general case, among the roots ν_{ki} there will also be zero roots of the odd polynomials $M_{N+1,\,k}$. The corresponding functions $y_{kk}(\mathbf{r}, \Omega, 0)$, if they are defined to be

$$y_{kk}(\mathbf{r}, \mathbf{\Omega}, 0) = \lim_{\nu \to 0} y_{kk}(\mathbf{r}, \mathbf{\Omega}, \nu),$$

have an infinitely small relaxation distance, and their contribution to the general solution is in the form of finite discontinuities on the boundaries. It can be seen that for even N the discontinuities (or jumps) will occur in even harmonics, and for odd N in odd harmonics; the component of the first harmonic which can be interpreted as the normal component of the diffusion current, however, is always continuous.

The existence of zero roots was also noted in [3, 6], but these roots and the corresponding special solution were discarded. In [4], where the P_N-approximations of the spherical-harmonic method are described in their most general form, the zero roots are also not taken into account, but in this work the discontinuity of certain spherical harmonics follows from the boundary conditions. When zero roots are taken into account, the requirement that all the spherical harmonics be separately coupled becomes formally realizable. It can be shown that in the P_N-approximation, because of the above-mentioned properties of the functions $y_{kk}(\mathbf{r}, \mathbf{\Omega}, 0)$, the formal coupling of all harmonics is actually equivalent to the satisfaction of the boundary conditions given in [4].

CONCLUSIONS

The exact general expression for harmonics of any number on the one hand simplifies the analytical transition to P_N-approximations of the spherical-harmonic method. This simplification is particularly important in multidimensional problems. As is clear from the preceding, P_N-approximations are obtained by approximating the infinite set of characteristic numbers ν and functions $y_{kk}(\mathbf{r}, \Omega, \nu)$ by a finite set of numbers ν_{ki} and the corresponding functions $y_{kk}(\mathbf{r}, \Omega, \nu_{ki})$. The boundary conditions derived in [4] remain in force, but they have a new mathematical interpretation.

On the other hand the assumption $Y_{N+1} = 0$, which lies at the basis of the usually employed P_N-approximations, is not obligatory here. It is thus possible that new approximate forms of the spherical-harmonic method may be obtained, based on other, more precise assumptions. The problem is to choose the functions y_{kk} so that they will be in agreement with the boundary conditions. It is evidently also important to find a criterion for the character of the convergence of the approximate results towards the exact solution. In its general form this is at present a rather difficult problem.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.