CANONICAL QUANTIZATION ON THE LIGHT CONE

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A procedure of canonical quantization on the light cone in the model of a self-interacting scalar field and in the Yukawa model is constructed.

Introduction

Knowledge of the commutators or anticommutators of interacting fields on the light cone, i.e., quantities of the type

$$[\Phi(|\mathbf{x}|,\mathbf{x}),\quad \Phi(|\mathbf{y}|,\mathbf{y})]_-, \tag{1}$$

is important in many applications of quantum field theory, especially in high energy physics. However, the methods used at present to study quantities of the type (1) do not by any means give exhaustive information about them. If not impossible, it appears at least very difficult to find a complete solution to the problem of calculating the commutators or anticommutators of interacting fields on the light cone on the basis of traditional field theory quantized by means of equal-time commutation relations or on the basis of perturbation theory. It therefore appears sensible to proceed from a theory in which the quantization is done on the light cone, and not on a surface $X^0 = 0$, as is customary. Such an approach is used in the present paper, and it makes it possible, at least in some models, to calculate the commutators or anticommutators of interacting fields on the light cone.

There are several methods of quantization on the light cone (see [1-3]). However, in all these papers the quantization is done, not by means of the canonical procedure, but by certain other methods, and the treatment is limited to the case of free fields, so that the constructed theory does not actually give any new information (in the case of flat space–time; we mention in this connection that the methods proposed in [2,3] were created for quantization in curved space–time). We also mention [4], which discusses quantization on the light cone of a massless scalar field in two-dimensional space–time.

In contrast to [1-3], to quantize fields we shall, as in traditional field theory, use the canonical method. However, canonical quantization on the light cone has a number of important differences from quantization on spacelike surfaces. These differences are mainly related to the fact that the light cone is a characteristic surface of the field equations.

In the present paper, we consider the problem of quantization on the light cone in the model of a self-interacting scalar field (Sec. 1) and in the Yukawa model (Sec. 2).

In the Appendix, we derive some expressions used in the main text.

We use the following notation. Lower case Latin indices j, i, ... take the values 0, 1, 2, 3. The Greek indices α, β, ... take the values 1, 2, 3. Summation over repeated indices is understood. Rectangular coordinates in Minkowski space are denoted by upper case Latin letters. Coordinates on the light cone are denoted by lower case Latin letters.

1. Scalar Field

A. Hamiltonian Formulation of the Goursat Problem for the Klein–Gordon Equation. To construct a canonical quantization procedure, it is necessary to give the Hamiltonian formulation of the original classical field theory. In other words, it is necessary to introduce a phase space M, a Hamilton function $\mathcal{H}$, and a symplectic structure ω on M such that the Hamilton equations corresponding to the differential form ω and the function $\mathcal{H}$ are identical to the field equations. Moreover, if the initial data for the solution of the field equations are given on some surface Σ, then functions on this surface are points of M.

In our case, the field equation is the Klein–Gordon equation with arbitrary self-interaction:

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$$\Box\Phi(X)=-V'(\Phi(X)). \tag{2}$$

We introduce the new coordinates (u, x^α):

$$u=X^0-\sqrt{X^\alpha X^\alpha}, \quad x^\alpha=X^\alpha.$$

In the variables (u, x^α), Eq. (2) takes the form

$$2\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial\Phi}{\partial u}=\Delta\Phi-V'(\Phi), \tag{3}$$

where $r=\sqrt{x^\alpha x^\alpha}$, $\Delta=\partial^2/\partial x^\alpha\partial x^\alpha$.

We consider the problem of solving Eq. (2) with initial data on the forward light cone. This problem is obviously equivalent to the problem of solving Eq. (3) with initial data on the surface $u = 0$.

Equation (3) is of first order in the variable u. Therefore, to solve this equation, we need to specify only one function on the surface $u = 0$ (in the class of "sufficiently good functions," in which there is an inverse of the operator $\partial/\partial r+1/r$). Therefore, in accordance with the assertion made at the beginning of this section, the phase space M of the Hamiltonian system defined by Eq. (3) consists of the functions $\Phi(x)\equiv\Phi(u, x)|_{u=0}$. We define the norm in this space as follows:

$$\|\Phi\|=\int d^3x\Phi^2(x). \tag{4}$$

We introduce the 2-form ω on M:

$$\omega(\Phi_1, \Phi_2)=C\int d^3x\Phi_1(x)\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\Phi_2(x), \tag{5}$$

where C is some normalization factor. It is easy to see that the 2-form ω is closed and nondegenerate. Therefore, this 2-form defines a symplectic structure on M.

We define the Hamilton function H by

$$H=-\frac{C}{2}\int d^3x\left(\frac{1}{2}\left(\frac{\partial\Phi}{\partial x^\alpha}\right)^2+V(\Phi)\right). \tag{6}$$

It is readily verified that Eq. (3) is equivalent to the Hamilton equations corresponding to the symplectic structure ω and the Hamiltonian H (see (4)-(6)).

We can now calculate the Poisson brackets of the functions $F(\Phi)$ and $G(\Phi)$ using the standard definition

$$\{F, G\}=dF(IG), \tag{7}$$

$$\{F, G\}=-\{G, F\}, \tag{8}$$

where IG is the vector field defined by

$$\omega(\xi, IG)=dG(\xi) \text{ for } \forall\xi\in M. \tag{9}$$

Let $F_1(\Phi)$ and $F_2(\Phi)$ be functions on M. Using the definitions (7) and (9), we can readily show that the Poisson brackets of these functions corresponding to the 2-form ω are defined by

$$\{F_1, F_2\}=\int d^3xd^3y\frac{\delta F_1}{\delta\Phi(x)}\frac{\mathrm{sgn}(|x|-|y|)}{2C|x||y|}\delta(\Omega_x-\Omega_y)\frac{\delta F_2}{\delta\Phi(y)}, \tag{10}$$

where $|x|=r$, $|y|=\sqrt{y^\alpha y^\alpha}$, $\delta(\Omega_x-\Omega_y)$ is the δ function on the unit sphere:

$$\delta(\Omega_x-\Omega_y)=\frac{1}{\sin\theta_x}\delta(\theta_x-\theta_y)\delta(\varphi_x-\varphi_y),$$

where θ_x, φ_x and θ_y, φ_y are the polar angles of the vectors x and y, respectively. For details of the calculations, see §1 of the Appendix.

We set

$$F_1(\Phi)=\Phi(x), \quad F_2(\Phi)=\Phi(y), \tag{11}$$

where x and y are fixed. Substituting (11) in (10), we obtain

$$\{\Phi(x), \Phi(y)\} = \frac{1}{2C|x||y|} \operatorname{sgn}(|x|-|y|)\,\delta(\Omega_x-\Omega_y). \tag{12}$$

We now consider the determination of the coefficient C in Eq. (6), i.e., the normalization of the Hamiltonian H.

In ordinary theory, the Hamiltonian $\mathscr{H}$ is defined by

$$\mathscr{H} = \int d^3X T_{00}, \tag{13}$$

where T_{00} is the component of the tensor

$$T_{mn} = \frac{\partial\Phi}{\partial X^m}\frac{\partial\Phi}{\partial X^n} - g_{mn}\left(\frac{1}{2}\frac{\partial\Phi}{\partial X^k}\frac{\partial\Phi}{\partial X_k} - V(\Phi)\right). \tag{14}$$

In quantum theory, the operator $\mathscr{H}$ is the generator of translations along the time axis. Since $\partial/\partial u = \partial/\partial X^0$, the operator H (see (6)) is also a generator of translations along the time axis. Therefore,

$$H \sim \mathscr{H}. \tag{15}$$

The coefficient of proportionality in (15) depends on the normalization factor C (see (5), (6)). We require

$$H = \mathscr{H}. \tag{16}$$

Equation (16) enables us to calculate C. To do this calculation, we express $\mathscr{H}$ in terms of the value of $\Phi(X)$ on the forward light cone. By Noether's theorem,

$$\frac{\partial T_{mn}}{\partial X_n} = 0. \tag{17}$$

We integrate both sides of Eq. (17) over the region V of Minkowski space bounded by the forward light cone and the plane $X^0 = 0$. By Gauss's theorem

$$0 = \int_V d^4X \frac{\partial T_{mn}}{\partial X_n} = \int_{X^0=0} T_{mn}\,d\Sigma_1{}^n - \int_{X^0=R} T_{mn}\,d\Sigma_2{}^n, \tag{18}$$

where

$$d\Sigma_1{}^n = n_1{}^n d^3X, \quad n_1{}^n = (1, 0, 0, 0), \quad d\Sigma_2{}^n = n_2{}^n \sqrt{2}\, d^3x, \quad n_2{}^n = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\frac{x^\alpha}{r}\right). \tag{19}$$

Substituting (19) in (18), and using (13), we obtain

$$\mathscr{H} = \int d^3x \left(T_{00} + \frac{x^\alpha}{r} T_{0\alpha}\right). \tag{20}$$

We substitute (14) in (20) and go over to the coordinates (u, x^α):

$$\mathscr{H} = \int d^3x \left(\frac{1}{2}\left(\frac{\partial\Phi}{\partial x^\alpha}\right)^2 + V(\Phi)\right). \tag{21}$$

Comparing (21) with (6) and (16), we find that

$$C = -2. \tag{22}$$

Substituting (22) in (12), we finally obtain

$$\{\Phi(x), \Phi(y)\} = -\frac{1}{4|x||y|} \operatorname{sgn}(|x|-|y|)\,\delta(\Omega_x-\Omega_y). \tag{23}$$

To conclude this section, we note that using the method employed in the derivation of Eq. (21) we can express all the generators of the Poincaré group in terms of the values of the field $\Phi(X)$ on the forward light cone.

B. Canonical Quantization. In the considered case, the procedure of canonical quantization consists of replacing the function $\Phi(u, x)$ by an operator that in accordance with (23) satisfies the commutation relations

$$[\Phi(u,x), \Phi(u,y)]_- = -\frac{i}{4|x||y|} \operatorname{sgn}(|x|-|y|)\,\delta(\Omega_x-\Omega_y), \tag{24}$$

with

$$\frac{\partial\Phi(u,x)}{\partial u}=i[H,\Phi(u,x)]_-,$$

and

$$H=\int d^3x\left(\frac{1}{2}:\left(\frac{\partial\Phi}{\partial x^\alpha}\right)^2:+:V(\Phi):\right).$$

Note that in the case of a free field quantization on the light cone is equivalent to quantization on the surface $X^0 = 0$. Indeed, it is easy to show (see, for example, [5]) that

$$[\Phi(x),\Phi(y)]_-=iG(x,y)=iD(X-Y)\,|_{X^2=Y^2=0}, \tag{25}$$

where $D(X)$ is the Pauli–Jordan commutation function.

2. Spinor Field

We now consider the problem of quantization on the light cone in the Yukawa model. As in the case of the scalar field, to construct the procedure of canonical quantization on the light cone we must give a Hamiltonian formulation of the Goursat problem for Eqs. (26) and (27), i.e., the problem of the solution of these equations with initial data on the forward light cone. The question of the initial data necessary in the solution of the Goursat problem in the considered case is investigated in Sec. 2A. In Sec. 2B we give the Hamiltonian formulation of the Goursat problem for Eqs. (26) and (27) and construct the procedure of canonical quantization on the light cone.

A. Characteristic Form of the Dirac Equation. The field equations in the considered model have the form

$$i\gamma^n\frac{\partial\Psi}{\partial X^n}-m\Psi-g\Phi\Psi=0, \tag{26}$$

$$\Box\Phi+V'(\Phi)+g\overline{\Psi}\Psi=0. \tag{27}$$

We shall use the following representation of the γ matrices:

$$\gamma^0=\beta=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \gamma^\alpha=\begin{pmatrix}0 & -\sigma^{(\alpha)}\\ \sigma^{(\alpha)} & 0\end{pmatrix}, \tag{28}$$

where $\sigma^{(\alpha)}$ are the Pauli matrices. In the variables (u, x^α) introduced in Sec. 1, Eq. (26) is

$$i\left(1+\frac{x_\alpha}{r}\rho^\alpha\right)\frac{\partial\Psi}{\partial u}=-i\rho^\alpha\frac{\partial\Psi}{\partial x^\alpha}+m\beta\Psi+g\beta\Phi\Psi, \tag{29}$$

where $x_\alpha=-x^\alpha$, $\rho^\alpha=\gamma^0\gamma^\alpha$. We introduce the matrices

$$\Pi(x)=\frac{1}{2}\left(1+\frac{x_\alpha}{r}\rho^\alpha\right), \tag{30}$$

$$Q(x)=\frac{1}{2}\left(1-\frac{x_\alpha}{r}\rho^\alpha\right). \tag{31}$$

Using the definitions (30) and (31) and the properties of the ρ matrices, we can readily show that $\Pi^2(x) = \Pi(x)$, $Q^2(x) = Q(x)$. Therefore, $\Pi(x)$ and $Q(x)$ are projection matrices, and

$$\Pi(x)+Q(x)=1, \tag{32}$$

$$\Pi(x)Q(x)=Q(x)\Pi(x)=0. \tag{33}$$

We define the functions $A(x)$ and $B(x)$:

$$A(x)=Q(x)\Psi(x),\quad B(x)=\Pi(x)\Psi(x). \tag{34}$$

By virtue of (32) $A(x)+B(x)=\Psi(x)$.

The components of the bispinors $A(x)$ and $B(x)$ are not independent, for by virtue of (33) and (34)

$$Q(x)B(x)=0,\quad \Pi(x)A(x)=0. \tag{35}$$

Using the expressions (34) and (35), we can readily transform Eqs. (29) to

$$\frac{\partial B}{\partial u}=-\frac{1}{2}\rho^{\alpha}\frac{\partial A}{\partial x^{\alpha}}+\frac{1}{2}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)(A+B)-\frac{i}{2}m\beta A-\frac{i}{2}g\beta\Phi A, \tag{36}$$

$$\rho^{\alpha}\frac{\partial B}{\partial x^{\alpha}}+\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)(A+B)+im\beta B+ig\beta\Phi B=0, \tag{37}$$

and Eq. (27) to

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial\Phi}{\partial u}=\frac{1}{2}\Delta\Phi-\frac{1}{2}V'(\Phi)-\frac{1}{2}(B^{+}\beta A+A^{+}\beta B). \tag{38}$$

Note that Eqs. (36) and (37) do not contain the derivatives $\partial A/\partial u$. Therefore, to solve the Goursat problem we must specify only the functions $B(x)$ on the surface $u = 0$. At the same time, Eq. (35) must be satisfied. The functions A are determined from Eq. (37) for every u.

Equations (36) and (37) are the Dirac equations in characteristic form.

B. Hamiltonian Formulation of the Goursat Problem and Canonical Quantization. The variables $B(x)$ are not independent dynamical variables, being related by Eq. (35). Therefore, we cannot give a Hamiltonian formulation of the Goursat problem for Eqs. (26) and (27) in terms of the variables $B(x)$. Therefore, it is necessary to "solve" the constraint equations (35).

We consider in more detail the structure of the matrix $\Pi(x)$. Using the representation (29) of the γ matrices, we can readily show that

$$\Pi(x)=\frac{1}{\sqrt{2}r}\begin{pmatrix}(X_{AA'})^{T} & 0\\ 0 & (X^{AA'})\end{pmatrix}, \tag{39}$$

where $X^{AA'}$ is the spinor form of expression of the 4-vector $X=(|\mathbf{x}|, x^{\alpha})$ ($A, A'=0, 1$ are, respectively, undotted and dotted indices). It is clear that

$$X^{2}=0,\quad X^{0}=\mathbf{x}>0. \tag{40}$$

By virtue of (40), the matrix $X^{AA'}$ can be represented in the form

$$X^{AA'}=\xi^{A}(x)\bar{\xi}^{A'}(x). \tag{41}$$

Suppose the spinor $\eta_{A}(x)$ is such that

$$\eta^{A}(x)\xi_{A}(x)=1. \tag{42}$$

Then the spinors $\xi^{A}(x)$ and $\eta^{A}(x)$ for every x form a basis in the space of undotted two-component spinors, and the spinors $\bar{\xi}_{A'}(x)$ and $\bar{\eta}_{A'}(x)$ form a basis in the space of dotted two-component spinors. In the case when the representation (28) is chosen for the γ matrices, the first two components of the bispinor $\Psi(x)$ form an undotted spinor, and the second two components a dotted spinor, i.e.,

$$\Psi(x)=\begin{pmatrix}\varphi^{A}(x)\\ \chi_{A'}(x)\end{pmatrix}. \tag{43}$$

We decompose the spinors $\varphi^{A}(x)$ and $\chi_{A'}(x)$ with respect to the basis introduced above:

$$\varphi^{A}(x)=a_{1}(x)\xi^{A}(x)+b_{1}(x)\eta^{A}(x),\quad \chi_{A'}(x)=a_{2}(x)\bar{\xi}_{A'}(x)+b_{2}(x)\bar{\eta}_{A'}(x). \tag{44}$$

Using (39), (42), and (44), we find that

$$B(x)=\Pi(x)\Psi(x)=\frac{1}{\sqrt{2}r}\begin{pmatrix}\bar{\xi}_{A'}(x)b_{1}(x)\\ -\xi^{A}(x)b_{2}(x)\end{pmatrix}, \tag{45}$$

and

$$b_{1}(x)=\varphi^{A}(x)\xi_{A}(x),\quad b_{2}(x)=-\chi_{A'}(x)\bar{\xi}^{A'}(x). \tag{46}$$

The complex-valued functions $b_1(x)$ and $b_2(x)$ are the required independent dynamical variables in terms of which we can give a Hamiltonian formulation of the Goursat problem for Eqs. (26) and (27) (see below).

Exactly as we did in Sec. 1A for the scalar field, we can show that the Hamiltonian H, expressed in terms of the values of the fields Φ and Ψ on the forward light cone, has the form

$$H=i\int d^{3}x\left(\bar{b}_{1}\frac{\partial b_{1}}{\partial u}+\bar{b}_{2}\frac{\partial b_{2}}{\partial u}\right)\frac{\sqrt{2}}{r}+H_{(\mathrm{sc})}, \tag{47}$$

where

$$H_{(sc)}=\int d^3x\left(\frac{1}{2}\left(\frac{\partial\Phi}{\partial x^\alpha}\right)^2+V(\Phi)\right).$$

Using (45) and (47), we find that

$$H=2i\int d^3xB^+\frac{\partial B}{\partial u}+H_{(sc)}, \tag{48}$$

where $\partial B/\partial u$ is determined by Eqs. (36) and (37). We introduce the space M as the space of pentuplets of functions $(b_1, \bar{b}_1, b_2, \bar{b}_2, \Phi)$ defined on the forward light cone with norm

$$\|(b_1,\bar{b}_1,b_2,\bar{b}_2,\Phi)\|^2=\int d^3x(\bar{b}_1b_1+\bar{b}_2b_2+\Phi^2). \tag{49}$$

We define the 2-form ω on M:

$$\omega=i\int d^3x\frac{1}{\sqrt{2}r}(d\bar{b}_1(x)\wedge db_1(x)+d\bar{b}_2(x)\wedge db_2(x))+\omega_{(sc)}, \tag{50}$$

where $\omega_{(sc)}$ is the form (5). It is readily verified that Eq. (50) defines a symplectic structure on M.

Proceeding from the expressions (49) and (50), we can calculate the Poisson brackets for any two functions on M:

$$\{F_1,F_2\}=i\int d^3x\sqrt{2}r\left(\frac{\delta F_1}{\delta\bar{b}_1(x)}\frac{\delta F_2}{\delta b_1(x)}-\frac{\delta F_1}{\delta b_1(x)}\frac{\delta F_2}{\delta\bar{b}_1(x)}+\frac{\delta F_1}{\delta\bar{b}_2(x)}\frac{\delta F_2}{\delta b_2(x)}-\frac{\delta F_1}{\delta b_2(x)}\frac{\delta F_2}{\delta\bar{b}_2(x)}\right)+$$
$$\int d^3xd^3y\frac{\delta F_1}{\delta\Phi(x)}G(x,y)\frac{\delta F_2}{\delta\Phi(y)},\quad G(x,y)=-\frac{1}{4|x||y|}\mathrm{sgn}(|x|-|y|)\delta(\Omega_x-\Omega_y). \tag{51}$$

We can show that the Hamiltonian equations corresponding to the Hamiltonian H and the symplectic structure ω are identical to the field equations (26) and (27). For details, see §2 of the Appendix.

We can now consider the quantization of the classical theory. In accordance with the canonical procedure, the classical fields $B(u, x)$, $B^+(u, x)$, and $\Phi(u, x)$ are replaced by operators, and by virtue of (51),

$$\begin{gathered}[B(u,x),B^+(u,y)]_+={}^1/_2\Pi(x)\delta(x-y),\quad [B(u,x),B(u,y)]_+=[B^+(u,x),B^+(u,y)]_+=0,\\ [B(u,x),\Phi(u,y)]_-=[B^+(u,x),\Phi(u,y)]_-=0,\quad [\Phi(u,x),\Phi(u,y)]_-=iG(x,y).\end{gathered} \tag{52}$$

At the first glance it appears that by using (37) to express $\Psi(x)$ in terms of $B(x)$ and $\Phi(x)$ we can, using (52), calculate $[\Psi(x), \Psi^+(y)]_+$. But, as is shown in the Appendix, even in the theory of free fields the anticommutator $[\psi(x), \psi^+(y)]_+$ is not uniquely defined. In the theory of free fields, we can nevertheless calculate

$$\left[\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\psi(u,x),\left(\frac{\partial}{\partial r'}+\frac{1}{r'}\right)\psi^+(u,y)\right]_+, \tag{53}$$

where $r' = |y|$ (see Eq. (A.11)). We show that in the case of interacting fields, we can also calculate the anticommutator (53).

Using (37) and (51), we obtain (after simple but rather lengthy calculations)

$$i\left\{\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\Psi(u,x),\left(\frac{\partial}{\partial r'}+\frac{1}{r'}\right)\Psi^+(u,y)\right\}=\left[-\frac{1}{2}Q(x)\Delta_x\delta(x-y)+\frac{1}{2}m^2Q(x)\delta(x-y)+\right.$$
$$\left.\frac{1}{2}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(\rho^\beta\frac{\partial}{\partial x^\beta}+im\beta\right)\delta(x-y)\right]+g\left[\frac{i}{2}\beta\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\delta(x-y)\Phi(u\ y)+\frac{i}{2}gQ(x)\beta\rho^\alpha\frac{\partial\Phi(u,x)}{\partial x^\alpha}\times\right.$$
$$\left.\delta(x-y)+mQ(x)\Phi(u,x)\delta(x-y)\right]+g^2[iG(x,y)\beta B(u,x)B^+(u,y)\beta+{}^1/_2\Phi^2(u,x)Q(x)\delta(x-y)]. \tag{54}$$

The third square brackets on the right-hand side of Eq. (54) contains operators quadratic in the fields. Therefore, on the transition to the quantum theory the problem of operator ordering arises. Using the Källén–Lehmann representation, we can show that the operators $B(u, x)B^+(u, y)$ and $\Phi^2(x)$ on the right-hand side of (54) must be expressed in normal form, i.e., for example, $B(u,x)B^+(u,y)\to{:}B(u,x)B^+(u,y){:}=B(u,x)B^+(u,y)-\langle 0|B(u,x)B^+(u,y)|0\rangle$ (for more detail, see [6]).

Bearing this remark in mind and applying the canonical method, we obtain

$$\left[\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\Psi(u,x),\left(\frac{\partial}{\partial r'}+\frac{1}{r}\right)\Psi^{+}(u,y)\right]_{+}=\Big[-{}^{1}/{}_{2}Q(x)\Delta_{x}\delta(x-y)+{}^{1}/{}_{2}m^{2}Q(x)\delta(x-y)+$$
$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(\rho^{\beta}\frac{\partial}{\partial x^{\beta}}+im\beta\right)\delta(x-y)\Big]+g\Big[\frac{i}{2}\beta\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\delta(x-y)\Phi(u,y)+\frac{i}{2}gQ(x)\times$$
$$\beta\rho^{\alpha}\frac{\partial\Phi}{\partial x^{\alpha}}(u,x)\delta(x-y)+mQ(x)\Phi(u,x)\delta(x-y)\Big]+g^{2}[iG(x,y)\beta:B(u,x)B^{+}(u,y):\beta]+{}^{1}/{}_{2}g^{2}:\Phi^{2}(u,x):Q(x)\delta(x-y). \tag{55}$$

Similarly, we can show that

$$\left[\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\Psi_{a}(u,x),\left(\frac{\partial}{\partial r'}+\frac{1}{r'}\right)\Psi_{b}(u,y)\right]_{+}=-ig^{2}G(x,y)\beta_{ac}\beta_{bd}:B_{c}(u,x)B_{d}(u,y):, \tag{56}$$

where $a, b, c, d=1, 2, 3, 4$. Equations (52), (55), and (56) solve the problem of quantization on the light cone in the Yukawa model.

Note that in the case of a free spinor field (i.e., for $g = 0$) quantization on the light cone is equivalent to quantization on the surface $X^0 = 0$. Indeed, it is easy to show that for $g = 0$ Eq. (52) is identical to Eq. (A.11), which solves the problem of quantization on the light cone in the case of the ordinary (quantized by means of equal-time commutation relations) theory of the free spinor field.

Conclusions

We have considered the problem of calculating the commutators and anticommutators of interacting fields on the light cone in the model of a self-interacting scalar field and in the Yukawa model. To solve this problem, we have constructed a procedure of canonical quantization on the light cone. We have obtained the following results:

1) we have calculated the commutators of the scalar fields in the considered models (see Eqs. (24) and (51)). The commutators have been found to be equal to the restriction to the light cone of the ordinary commutator of the free scalar field;

2) we have shown that the anticommutator of spinor fields on the light cone is not defined uniquely even in the case of free fields (see the Appendix). However, it is possible to calculate anticommutators of the type

$$\left[\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\Psi(x),\left(\frac{\partial}{\partial r'}+\frac{1}{r'}\right)\Psi^{+}(y)\right]_{+}$$

(see Eqs. (55) and (56)). In contrast to the scalar case, the anticommutator of spinor fields is an operator and not a c number;

3) we have shown that quantization on the light cone and quantization on the surface $X^0 = 0$ are equivalent in the case of free fields.

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Appendix

1. In this section, we give a detailed deriviation of Eq. (10). We write Eq. (9) in the form

$$C\int d^{3}x\,\xi(x)\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)IG(x)=\int d^{3}x\frac{\delta G}{\delta\Phi(x)}\xi(x). \tag{A.1}$$

Because ξ is arbitrary, it follows from (A.1) that

$$IG(x)=\int dy\,K(x,y)\frac{\delta G}{\delta\Phi(y)}, \tag{A.2}$$

where

$$C\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)K(x,y)=\delta(x-y). \tag{A.3}$$

From (A.2) and (7) we find that

$$\{F, G\} = \int dx\, dy \frac{\delta F}{\delta \Phi(x)} K(x, y) \frac{\delta G}{\delta \Phi(y)}. \tag{A.4}$$

To satisfy (8), we must require

$$K(x, y) = -K(y, x). \tag{A.5}$$

Substituting the solution of Eqs. (A.3) and (A.5) in (A.4), we obtain Eq. (10).

2. Here, we show that the Hamilton equations corresponding to the Hamiltonian H and the symplectic structure ω (defined by Eqs. (48) and (50), respectively) are identical to the field equations (26) and (27).

We note that by virtue of (52) and (48)

$$\left\{ B(y), \frac{\partial B}{\partial u}(x) \right\} = 0. \tag{A.6}$$

In this last equation, $\partial B/\partial u$ is defined in accordance with Eqs. (36) and (37). Using (A.6) and (52), we obtain

$$\{B(x), H\} = 2i \int d^3y \{B(x), B^+(y)\} \frac{\partial B}{\partial u} = \frac{\partial B}{\partial u}. \tag{A.7}$$

Therefore, Eqs. (36) and (37) are equivalent to the Hamilton equations (A.7). By direct calculation of the Poisson brackets we can also establish the equivalence of Eq. (38) and the Hamilton equation

$$\frac{\partial \Phi}{\partial u} = \{\Phi, H\}. \tag{A.8}$$

But Eqs. (36)-(38) are equivalent to Eqs. (26) and (27). Therefore, we conclude that the Hamilton equations (A.7) and (A.8) are equivalent to Eqs. (26) and (27).

3. In this section, we study the behavior of the commutation function $S(X - Y)$ of a spinor field when both arguments of this function are on the light cone.

The function $S(X - Y)$ is given by

$$S(X-Y) = -\left(i\gamma^n \frac{\partial}{\partial X^n} + m \right) D(X-Y). \tag{A.9}$$

Going over here to the coordinates (u, x) and using the fact that the function $D(X - Y)$ satisfies the Klein–Gordon equation, we obtain

$$-i\beta \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) S(X-Y) = \left[-Q(x)(\Delta_x - m^2) + \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \left(\rho^\alpha \frac{\partial}{\partial x^\alpha} + im\beta \right) \right] D(X-Y). \tag{A.10}$$

We denote $u' = Y^0 - \sqrt{Y^\alpha Y^\alpha}$. In Eq. (A.10), we can set $u = u' = 0$. Using (25), we obtain

$$\left[\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \psi(x), \left(\frac{\partial}{\partial r'} + \frac{1}{r'} \right) \psi^+(y) \right]_+ = -i\beta \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \left(\frac{\partial}{\partial r'} + \frac{1}{r'} \right) S(X-Y) =$$

$$\left[-\frac{1}{2} Q(x)(\Delta_x - m^2) + \frac{1}{2} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \left(\rho^\alpha \frac{\partial}{\partial x^\alpha} + im\beta \right) \right] \delta(x-y). \tag{A.11}$$

From this expression, we cannot obtain the anticommutator $[\psi(x), \psi^+(y)]_+$. Indeed, if we attempt to "invert" the operators $(\partial/\partial r + 1/r)$ and $(\partial/\partial r' + 1/r')$, then the calculation of $[\psi(x), \psi^+(y)]_+$ gives rise to divergent integrals of the type

$$\int d^3y G(x, y) G(y, z).$$

Therefore, the anticommutator $[\psi(x), \psi^+(y)]_+$ is not uniquely defined.

LITERATURE CITED

1. R. Sachs, Phys. Rev., **128**, 2851 (1962).
2. A. Komar, Phys. Rev. B, **134**, 1430 (1964).
3. I. V. Volovich, V. A. Zagrebnov, and V. P. Frolov, Fiz. Elem Chastits At. Yadra, **9**, 10 (1978).
4. A. I. Oksak, Teor. Mat. Fiz., **48**, 297 (1981).
5. F. A. Lunev, "Canonical quantization on the light cone. I. Scalar field," Preprint 81-74 [in Russian], Institute of High Energy Physics, Serpukhov (1981).

6. F. A. Lunev, "Canonical quantization on the light cone. II. Spinor field," Preprint 81-75 [in Russian], Institute of High Energy Physics, Serpukhov (1981).

ON QUANTUM AND THERMODYNAMIC FLUCTUATIONS OF FERMION FIELDS IN A FINITE VOLUME

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The effective potential (V_{eff}) of fermion fields enclosed in a finite region of space is calculated at nonzero temperatures. In a one-dimensional model, an exact expression for V_{eff} is obtained in terms of Jacobi theta functions as a function of the temperature (T) and length (L) of the space. In four-dimensional space, the low- and high-temperature behaviors, and also the limit $T \to \hbar c/2L$ are studied. It is shown that the thermal corrections ($TL \ll 1$) to the vacuum energy density and the dimensional corrections ($TL \gg 1$) to the fermion free energy density are exponentially small.

1. Introduction

There has recently been much interest in the investigation of quantum effects in an external field specified by boundary conditions (see, for example, [1]). The study of problems of this kind is stimulated in particular by questions in the theory of elementary particles. The energy of the zero-point vibrations of color fields confined in a bounded region of space differs from the energy of the zero-point vibrations of the vacuum in Minkowski space and may play an important part in determining the hadron mass spectrum (in this connection, see [2]).

In recent years, much attention has also been devoted to the construction of a quantum theory of electroweak and strong interactions at the high temperatures that characterized the early stage in the evolution of the Universe and also multiparticle production processes associated with the interaction of high-energy hadrons (see the reviews [3]).

In problems of the quantization of fields in the presence of nontrivial boundary counditions, finite-temperature corrections have been considered only for the Casimir effect [4, 5]. In the light of what we have said above, it is of interest to study the thermodynamics of fermion fields in finite volume.

We first consider massless fermions in two-dimensional space–time. In this case, the problem can be analytically solved to the end and a compact expression can be found for the regularized effective potential of the system as a function of the temperature and volume (length L) of the space.

For four-dimensional space, one can, even specifying boundary conditions for only one of the coordinates, obtain explicit expressions for the effective potential in the special cases of low ($T \ll T_K$, $T_K = \hbar c/2L$) and high ($T \gg T_K$) temperatures, and also the limit $T = T_K$. In contrast to calculations of the fluctuations of the electromagnetic field (the Casimir effect), in which the temperature corrections for $T \ll T_K$ are suppressed by power-law smallness [4, 5], for fermions the analogous terms are exponentially small. The expressions are symmetric under the substitution $T \rightleftharpoons T_K$, and for $T \sim T_K$ the effective potential is approximately two times less than its limiting values.

For comparison, we also calculate the effective potential of a massless scalar field at $T \neq 0$ with Dirichlet boundary conditions for one of the coordinates.

2. Fermions in the Space $S^1 \times E^1$

We consider a gas of one-dimensional fermions confined to a finite interval $[0, L]$. The corresponding conditions of confinement in this case have the standard form [2] (n_μ is a unit spacelike vector):

$$n_\mu \bar{\psi} \gamma_\mu \psi |_{0, L} = 0. \tag{1}$$

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