Nonlinear Differ. Equ. Appl. 20 (2013), 1411–1452

1021-9722/13/041411-42
published online January 26, 2013
DOI 10.1007/s00030-012-0216-8

Nonlinear Differential Equations and Applications NoDEA

The Aubry set for a version of the Vlasov equation

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Abstract. We check that several properties of the Aubry set, first proven for finite-dimensional Lagrangians by Mather and Fathi, continue to hold in the case of the infinitely many interacting particles of the Vlasov equation on the circle.

Mathematics Subject Classification. Primary 37J50; Secondary 35Q83.

Keywords. Vlasov equation, Aubry-Mather theory.

0. Introduction

The Vlasov equation on the circle governs the motion of a group of particles on $S^1 := \frac{\mathbf{R}}{\mathbf{Z}}$ under the action of an external potential $V(t, x)$ and a mutual interaction W. More precisely, we let $I = [0, 1)$, we lift our particles to $\mathbf{R}$, and we parametrize them at time t by a function $\sigma_t \in L^2(I, \mathbf{R})$; we require that σ_t satisfies the differential equation in $L^2(I, \mathbf{R})$

$$\ddot{\sigma}_t z = -V'(t, \sigma_t z) - \int_I W'(\sigma_t z - \sigma_t \bar{z}) \mathrm{d}\bar{z}. \qquad (ODE)_{Lag}$$

Our standing hypotheses on the potentials V and W are

• $V \in C^2(S^1 \times S^1)$, $W \in C^2(S^1)$; moreover W, seen as a function on $\mathbf{R}$, is even; up to adding a constant, we can suppose that $W(0) = 0$.

There is an element of arbitrariness in choosing the lift of the particles to $\mathbf{R}$ and in parametrizing them; that's why we are less interested in the evolution of the labelling σ_t than in the evolution of the measure it induces. In other words, we want to study the measures on $S^1 \times \mathbf{R}$ given by $\mu_t := (\pi \circ \sigma_t, \dot{\sigma}_t)_\sharp \nu_0$, where ν_0 denotes the Lebesgue measure on I, $\pi \colon \mathbf{R} \to S^1$ is the natural projection and $(\cdot)_\sharp$ denotes the push-forward. A standard calculation shows that, if σ_t satisfies $(ODE)_{Lag}$, then μ_t satisfies, in the weak sense,

$$\partial_t \mu_t + v \partial_x \mu_t = \partial_v (\mu_t \partial_x P_t) \qquad (ODE)_{meas}$$

This work was partially supported by the PRIN2009 grant “Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations.

where

$$P_t(x) = V(t,x) + \int_{S^1\times\mathbf{R}} W(x-\bar{x})\mathrm{d}\mu_t(\bar{x},v) = V(t,x) + \int_I W(x-\sigma_t\bar{z})\mathrm{d}\bar{z}.$$

Problem $(ODE)_{meas}$ (see [9,10]) is Lagrangian; actually, many results of Aubry–Mather theory can be extended to curves of measures which are "minimal" in a suitable way. Here, however, we follow the approach of [8]: we are going to work with $(ODE)_{Lag}$, keeping track of its symmetries. Quotienting $(ODE)_{Lag}$ by its symmetry group, we shall get a problem equivalent to $(ODE)_{meas}$. Though in this paper we restrict ourselves to the one-dimensional situation, we recall that Nurbekian [14] has extended the results of [8] about minimal parametrizations to tori of any dimension.

The aim of this paper is to do a few simple checks, showing that many features of Aubry–Mather theory persist in this setting; actually, we shall check that the main theorems of [7] continue to hold. In Sect. 1, we recall the main results of [8] on $(ODE)_{Lag}$ and its symmetries; in Sect. 2, following [7,8,10], we define the Hopf–Lax semigroup and we show that it has fixed points. We also show that the value function satisfies the Hamilton–Jacobi equation on L^2. In Sect. 3, we show that $(ODE)_{Lag}$ admits invariant measures minimal in the sense of Mather; as a consequence, we can define Mather's conjugate actions α and β. In Sect. 4, we recall two different definitions of the Aubry set, one of Mather's and the other of Fathi's; we show that, also in this case, the two definitions coincide. In Sect. 5, we shall show that $(ODE)_{meas}$ admits a solution μ_t which is periodic (i.e. $\mu_0=\mu_1$) and has irrational rotation number. We shall see that, as a consequence of the KAM theorem, if the rotation number ω is sufficiently irrational, and V and W are sufficiently regular and small (depending on ω), then μ_t has a smooth density.

1. Notation and preliminaries

Since V and W are periodic, we have that $(ODE)_{Lag}$ is invariant by the action of $L^2_{\mathbf{Z}} := L^2(I,\mathbf{Z})$; in other words, if σ_t is a solution and $h\in L^2_{\mathbf{Z}}$, then σ_t+h is a solution too. Moreover, $(ODE)_{Lag}$ is invariant by the group G of the measure-preserving transformations of I into itself; indeed, such maps do not change the value of the integral defining $P_t(x)$. An idea of [8] is to quotient $L^2(I)$ by these two groups; we recall from [8] some facts about this quotient.

We shall denote by $\|\cdot\|$ the norm on $L^2(I)$, $\langle\cdot,\cdot\rangle$ the internal product. We set

$$\mathbf{T}\colon = \frac{L^2(I)}{L^2_{\mathbf{Z}}(I)}.$$

The space $\mathbf{T}$ is metric, with distance between the equivalence classes $[M]$ and $[\bar{M}]$ given by

$$dist_{\mathbf{Z}}([M],[\bar{M}]) = \inf_{Z\in L^2_{\mathbf{Z}}} ||M-\bar{M}-Z|| = |||M-\bar{M}|_{S^1}||$$

where

$$|m|_{S^1} := \min_{k\in\mathbf{Z}} |m+k|.$$

We note that, for each $x \in I$, we can measurably choose $Zx \in \mathbf{Z}$ such that $|Mx - \bar{M}x - Zx| = |Mx - \bar{M}x|_{S^1}$; as a consequence, we get that the inf in the definition of $dist_{\mathbf{Z}}$ is a minimum; we also get the second equality above.

Let Group denote the group of the measure-preserving transformations of I with measurable inverse; for $M, \bar{M} \in L^2(I)$ we set

$$dist_{weak}(M, \bar{M}) = \inf_{G \in \mathrm{Group}} dist_{\mathbf{Z}}(M \circ G, \bar{M}).$$

This yields that M and $M \circ G$, which we would like to consider equivalent, have zero distance; however, if we say that $M \simeq \bar{M}$ when $\bar{M} = M \circ G$ for some $G \in$ Group, then the equivalence classes are not closed in $\mathbf{T}$, essentially because the inf in the definition of $dist_{weak}$ is not a minimum: it is possible (see [8]) that $dist(M, \bar{M}) = 0$ even if M and $\bar{M}$ are not equivalent. But we can consider their closure if we look at the equivalence relation from the right point of view, i.e. that of the measure induced by M.

We denote by Meas the space of Borel probability measures on S^1, and we let $\pi\colon \mathbf{R} \to S^1$ be the natural projection. We introduce the map

$$\Phi\colon L^2(I) \to \mathrm{Meas}, \qquad \Phi\colon M \to (\pi \circ M)_\sharp \nu_0$$

where $(\cdot)_\sharp$ denotes push-forward and ν_0 is the Lebesgue measure on I. We note that Φ is invariant under the action of $L^2_{\mathbf{Z}}$ and Group; in other words, if $Z \in L^2_{\mathbf{Z}}$ and $G \in$ Group, then $\Phi(u) = \Phi((u + Z) \circ G)$. We say that $M \simeq \bar{M}$ if $\Phi(M) = \Phi(\bar{M})$. We set $\mathbf{S} := \frac{L^2(I)}{\simeq}$; on this space, we consider the metric

$$dist_{\mathbf{S}}([M], [\bar{M}]) = \inf\{||M^* - \bar{M}^*|| \ : \ M^* \in [M], \quad \bar{M}^* \in [\bar{M}]\}.$$

The infimum above is a minimum: one can always find a minimal couple $(M^*, \bar{M}^*)$ with M monotone and taking values in $[0, 1]$, and $\bar{M}$ monotone and taking values in $[-\frac{3}{2}, \frac{3}{2}]$. Now $\mathbf{S}$ is isometric to the space of Borel probability measures on S^1 with the 2-Wasserstein distance; in particular, it is a compact space.

Another fact proven in [8] is that $dist_{weak}(M, \bar{M}) = dist_{\mathbf{S}}([M], [\bar{M}])$.

By Proposition 2.9 of [8], which we copy below, the $L^2_{\mathbf{Z}}$-equivariant (or $L^2_{\mathbf{Z}}$-equivariant and Group-equivariant) closed forms on $L^2(I)$ have a particularly simple structure: the first equivariant cohomology group of $L^2(I)$ is $\mathbf{R}$.

Proposition 1.1. *Let $S\colon L^2(I) \to \mathbf{R}$ be C^1.*

1. *If $\mathrm{d}S$ is $L^2_{\mathbf{Z}}$-periodic in the sense that $\mathrm{d}_{M+Z}S = \mathrm{d}_M S$ for all $Z \in L^2_{\mathbf{Z}}(I)$, then there is a unique $C \in L^2(I)$ and a function $s\colon L^2(I) \to \mathbf{R}$, of class C^1 and $L^2_{\mathbf{Z}}$-periodic, such that*

$$S(M) = s(M) + \langle C, M \rangle.$$

2. *If, in addition, $: M \to \mathrm{d}_M S$ is rearrangement-invariant (i.e. if $\mathrm{d}_M S = \mathrm{d}_{M \circ G} S$ for all $G \in$ Group), then C is constant and s is rearrangement-invariant.*

In view of the lemma above, for $c\in\mathbf{R}$ we define the Lagrangian $\mathcal{L}_c$ as

$$\mathcal{L}_c\colon S^1\times L^2(I)\times L^2(I)\to\mathbf{R},\quad \mathcal{L}_c(t,M,N)=\frac{1}{2}||N||^2-\langle c,N\rangle-\mathcal{V}(t,M)\\ -\mathcal{W}(M)$$

where

$$\mathcal{V}(t,M)=\int_I V(t,Mx)\mathrm{d}x,\quad\text{and}\quad \mathcal{W}(M)=\frac{1}{2}\int_{I\times I}W(Mx-Mx')\mathrm{d}x\mathrm{d}x'.$$

In order to define the c-minimal orbits of $\mathcal{L}$, we let $K\subset\mathbf{R}$ be an interval; following [1], we say that $u\in L^1(K,L^2(I))$ is absolutely continuous if there is $\dot u\in L^1(K,L^2(I))$ such that, for any $\phi\in C^1_0(K,\mathbf{R})$, we have that

$$\int_K u_t(x)\dot\phi(t)\mathrm{d}t=-\int_K \dot u_t(x)\phi(t)\mathrm{d}t. \tag{1.1}$$

The equality above is an equality in $L^2(I)$, i.e. it holds for a.e. $x\in I$; however, it is easy to see that the exceptional set does not depend on ϕ, and thus that, for a.e. x, the map $: t\to u_t(x)$ is A. C. with derivative $\dot u_t(x)$. We shall denote by $AC(K,L^2(I))$ the class of A. C. functions from K to $L^2(I)$.

Let $c\in\mathbf{R}$; we say that $\sigma\in AC(K,L^2(I))$ is c-minimal for $\mathcal{L}$ if, for any interval $[t_0,t_1]\subset K$ and any $\tilde\sigma\in AC((t_0,t_1),L^2(I))$ satisfying

$$\tilde\sigma_{t_1}-\sigma_{t_1}\in L^2_{\mathbf{Z}}(I)\quad\text{and}\quad \tilde\sigma_{t_2}-\sigma_{t_2}\in L^2_{\mathbf{Z}}(I),$$

we have that

$$\int_{t_0}^{t_1}\mathcal{L}_c(t,\sigma_t,\dot\sigma_t)\mathrm{d}t\le\int_{t_0}^{t_1}\mathcal{L}_c(t,\tilde\sigma_t,\dot{\tilde\sigma}_t)\mathrm{d}t.$$

We forego the standard proof that c-minimal orbits solve $(ODE)_{Lag}$.

Let now $n\in\mathbf{N}$, and let $\mathcal{A}_n$ be the σ-algebra on I generated by the intervals $[\frac{i}{n},\frac{i+1}{n})$ with $i\in(0,\dots,n-1)$; we call $\mathcal{C}_n$ the closed subspace of the $\mathcal{A}_n$-measurable functions of $L^2(I)$, and we denote by $P_n\colon L^2(I)\to\mathcal{C}_n$ the orthogonal projection. We have a bijection

$$D_n\colon\mathbf{R}^n\to\mathcal{C}_n,\qquad D_n\colon(q_1,\dots,q_n)\to\sum_{i=0}^{n-1}q_i1_{[\frac{i}{n},\frac{i+1}{n})}(x).$$

We also note that the space $S^1\times\mathcal{C}_n\times\mathcal{C}_n$ is invariant by the Euler–Lagrange flow of $(ODE)_{Lag}$.

2. The Hopf–Lax semigroup

Definition. Let us denote by $C_{\text{Group}}(\mathbf{T})$ the set of functions $U\in C(L^2(I),\mathbf{R})$ which are $L^2_{\mathbf{Z}}$ and Group equivariant. It is standard (Proposition 2.8 of [8]) each $U\in C_{\text{Group}}(\mathbf{T})$ quotients to a continuous function on the compact space $\mathbf{S}$; in particular, it is bounded.

Given $M\in L^2(I)$, $U\in C_{\text{Group}}(\mathbf{T})$ and $t>0$, we define

$$(A_c^t U)(M) = \inf\left\{\int_0^t \mathcal{L}_c(s,\sigma_s,\dot\sigma_s)\mathrm{d}s + U(\sigma_0)\ :\ \sigma\in AC([0,t],L^2(I)),\right.$$
$$\left.\sigma_t = M\right\}. \tag{2.1}$$

We shall denote by Mon the space of the maps $\sigma\colon I\to\mathbf{R}$ which are monotone increasing and satisfy $\sigma(1-)\le\sigma(0)+1$. We endow Mon with the topology it inherits from $L^2(I)$, which turns it into a locally compact space.

We group together the statements of a few lemmas of [8] and [10]; for a slightly different proof, point 1 is lemma 2.1 of [5], point 2 is lemma 2.8, point 4 Proposition 2.2.

Proposition 2.1. *Let $U\in C_{\mathrm{Group}}(\mathbf{T})$, let $t>0$ and let $A_c^tU\colon L^2(I)\to\mathbf{R}$ be defined by (2.1); then, the following statements hold.*

1. *A_c^tU is $L^2_{\mathbf{Z}}$ and* Group-*equivariant.*
2. *A_c^tU is $L(t)$-Lipschitz for $dist_{weak}$ (or for $dist_{\mathbf{S}}$, since we have seen that the two distances coincide). The constant $L(t)$ does not depend on U. Moreover, $L(t)\le L$ for $t\ge 1$.*
3. *As a consequence of 1 and 2, $A_c^tU\in C_{\mathrm{Group}}(\mathbf{T})$.*
4. *Let $M\in Mon$; then, the* inf *in (2.1) is a minimum; more precisely, there is $\sigma\in AC([0,t],L^2(I))$ with $\sigma_t=M$, $\sigma_s\in Mon$ for $s\in[0,t]$ and such that*
$$(A_c^tU)(M)=\int_0^t\mathcal{L}_c(s,\sigma_s,\dot\sigma_s)\mathrm{d}s+U(\sigma_0).$$
The function σ is c-minimal on $(0,t)$ and solves $(ODE)_{Lag}$.
5. *Since $\mathcal{L}_c$ is one-periodic in time, A_c^t has the semigroup property on the integers: in other words, if $t>0$ and $s\in\mathbf{N}$, then*
$$A_c^{t+s}U = A_c^t(A_c^sU).$$

Let $l\in\mathbf{R}$; by point 3 of the last lemma, we can define a map
$$\Lambda_{c,\lambda}\colon C_{\mathrm{Group}}(\mathbf{T})\to C_{\mathrm{Group}}(\mathbf{T})$$
$$\Lambda_{c,\lambda}\colon U\to(A_c^1U)(\cdot)+\lambda.$$
It follows immediately from the definition of A_c^1U that

- $\Lambda_{c,\lambda}$ is monotone, i.e., if $U_1\le U_2$, then $\Lambda_{c,\lambda}U_1\le\Lambda_{c,\lambda}U_1$.
- If $a\in\mathbf{R}$, then $\Lambda_{c,\lambda}(U+a)=\Lambda_{c,\lambda}U+a$. These two facts easily imply that
- $\Lambda_{c,\lambda}$ is continuous (actually, 1-Lipschitz) from $C_{\mathrm{Group}}(\mathbf{T})$ to itself, if we put on $C_{\mathrm{Group}}(\mathbf{T})$ the sup norm.

Again, we refer the reader to [8,10] (or to [7], since the finite dimensional proof is the same) for the next lemma; in [5], point 1) is Proposition 2.11. Point 2 follows in a standard way by point 1 and the semigroup property.

Proposition 2.2. 1. *There is a unique $\lambda\in\mathbf{R}$ (which we shall call $\alpha(c)$) such that $\Lambda_{c,\lambda}$ has a fixed point in $C_{\mathrm{Group}}(\mathbf{T})$. By point 2 of Proposition 2.1, any fixed point is L-Lipschitz.*

2. *Let U be a fixed point of $\Lambda_{c,\lambda}$, and let $M \in Mon$. Then, there is $\sigma \in AC_{loc}((-\infty, 0], L^2(I))$ such that $\sigma_t \in Mon$ for $t \in (-\infty, 0)$, $\sigma_0 = M$ and, for all $k \in \mathbf{N}$,*
$$U(M) = \int_{-k}^{0} [\mathcal{L}_c(t, \sigma_t, \dot{\sigma}_t) + \alpha(c)]\, dt + U(\sigma_{-k}).$$
The function σ is c-minimal on $(-\infty, 0)$ and solves $(ODE)_{Lag}$.

Now we introduce the notation of [7] for the Hopf–Lax semigroups, forward (T_t^-) and backward (T_{-t}^+) in time. The signs $+$ and $-$ point, apparently, in the wrong direction; a possible justification is that, when the semigroup goes forward in time, the characteristics go backward, and vice-versa.

Definition. Let $U \in C_{\text{Group}}(\mathbf{T})$, let $M \in L^2(I)$ and let $\alpha(c)$ be as in Proposition 2.2; for $t \geq 0$, we define
$$(T_t^- U)(M) = \inf \left\{ U(\gamma_0) + \int_0^t [\mathcal{L}_c(s, \gamma_s, \dot{\gamma}_s) + \alpha(c)] \mathrm{d}s \,:\, \gamma_t = M \right\}$$
and
$$(T_{-t}^+ U)(M) = \sup \left\{ U(\gamma_0) - \int_{-t}^0 [\mathcal{L}_c(s, \gamma_s, \dot{\gamma}_s) + \alpha(c)] \mathrm{d}s \,:\, \gamma_{-t} = M \right\}.$$

We note that, by Proposition 2.1, $T_t^- U$ and $T_{-t}^+ U$ belong to $C_{\text{Group}}(\mathbf{T})$. By Proposition 2.2, $T_1^- = \Lambda_{c,\alpha(c)}$ has a fixed point; we cannot say the same for T_{-1}^+ because the choice $\lambda = \alpha(c)$, which yields a fixed point of T_1^-, may not yield a fixed point of T_{-1}^+; we shall have to wait until Theorem 4.2 below to see that this is actually the case, and that both operators have fixed points.

By point 5 of Proposition 2.1, if U is a fixed point of T_1^-, then, for $t \geq 0$, $T_{t+1}^- U = T_t^- U$; in other words, the function $(T_t^- U)(M)$ defined on $[0, +\infty) \times L^2(I)$ can be extended by periodicity to $\mathbf{R} \times L^2(I)$. As a final remark, if $M \in Mon$, it follows by Proposition 2.1 that $(T_t^- U)(M)$ and $(T_{-t}^+ U)(M)$ are a minimum and a maximum respectively.

Definition. We shall say that a function $U \in C_{\text{Group}}(\mathbf{T})$ is c-dominated if, for every $m < n \in \mathbf{Z}$ and every $\sigma \in AC([m, n], L^2(I))$, we have that
$$U(\sigma_n) - U(\sigma_m) \leq \int_m^n [\mathcal{L}_c(t, \sigma_t, \dot{\sigma}_t) + \alpha(c)] \mathrm{d}t.$$

We note that there are c-dominated functions: for instance, the fixed points of T_1^-, given by Proposition 2.2, are c-dominated by formula 2.1.

Definition. If $\sigma \in AC([a, b], L^2(I))$ and $\sigma_t \in Mon$ for $t \in [a, b]$, we shall say that $\sigma \in AC_{mon}([a, b])$. By point 4) of Proposition 2.1, if $M \in Mon$ there is $\sigma \in AC_{mon}$ minimal (or maximal) in the definition of $T_t^- U(M)$ (or of $T_{-t}^+ U(M)$.).

Definition. Let $U \in C_{\text{Group}}(\mathbf{T})$ be c-dominated and let $a < b \in \mathbf{Z} \cup \{\pm\infty\}$; we say that $\gamma \in AC_{mon}([a, b])$ is calibrating if, for any $[m, n] \subset [a, b]$ with m and n integers, we have

$$U(\gamma_n) - U(\gamma_m) = \int_m^n [\mathcal{L}_c(t, \gamma_t, \dot{\gamma}_t) + \alpha(c)]\mathrm{d}t.$$

It follows from (2.1) that a calibrating function γ is c-minimal on $[a, b]$, and thus it satisfies $(ODE)_{Lag}$.

We state at once a relation between these definitions; it comes, naturally, from [7].

Lemma 2.3. 1. *Let $U \in C_{\mathrm{Group}}(\mathbf{T})$; then U is c-dominated iff $U \le T_n^- U$ (or iff $T_{-n}^+ U \le U$) for all $n \ge 0$.*

2. *Moreover, $T_n^-(U) = U$ (or $T_{-n}^+ U = U$) for all $n \in \mathbf{N}$ iff U is c-dominated and, for each $M \in Mon$, there is a calibrating curve $\gamma \in AC_{mon}((-\infty, 0])$ (or $\gamma \in AC_{mon}([0, +\infty))$ with $\gamma_0 = M$.*

Proof. Point 1 is a rewording of the definition of c-dominated. We prove point 2; if $T_n^- U = U$, then U is c-dominated by point 1; the existence of a calibrating curve γ follows from point 2 of Proposition 2.2. To prove the converse, let $M \in Mon$ and let γ be calibrating on $(-\infty, 0]$ with $\gamma_0 = M$; then,

$$U(M) - U(\gamma_{-1}) = U(\gamma_0) - U(\gamma_{-1}) = \int_{-1}^0 [\mathcal{L}_c(t, \gamma_t, \dot{\gamma}_t) + \alpha(c)]\mathrm{d}t.$$

By the definition of T_1^-, this means that $(T_1^- U)(M) \le U(M)$; since the opposite inequality holds by point 1, we have that $(T_1^- U)(M) = U(M)$ for all $M \in Mon$. Since U is continuous and equivariant, and since by [8] any $N \in L^2(I)$ can be approximated by $M \circ G_n + Z_n$ with $M \in Mon$, $G_n \in$ Group and $Z_n \in L^2_{\mathbf{Z}}$, we have that $(T_1^- U)(N) = U(N)$ for all $N \in L^2(I)$, and we are done.

The Lagrangian $\mathcal{L}_c$ has a Legendre transform $\mathcal{H}_c$; an easy calculation shows that

$$\mathcal{H}_c \colon S^1 \times L^2(I) \times L^2(I) \to \mathbf{R}$$

$$\mathcal{H}_c(t, \sigma, p) = \frac{1}{2}||c + p||^2_{L^2(I)} + \mathcal{V}(t, \sigma) + \mathcal{W}(\sigma).$$

What we really need are subsolutions of Hamilton–Jacobi; that's why we give the following definition.

Definition. We define Mon_3 as the set of monotone functions γ on I such that $\gamma(1-) \le \gamma(0) + 3$. Let $U \colon \mathbf{R} \times L^2(I) \to \mathbf{R}$, and let $M \in Mon$. We say that $(a, \xi) \in \mathbf{R} \times L^2(I)$ is the "lazy differential" of U at (t, M) if there is $K > 0$ such that

i) $U(t + h, M + N) - U(t, M) \le ah + \langle \xi, N \rangle + K(|h|^2 + ||N||^2)$
$\forall (h, N) \in \mathbf{R} \times L^2(I)$

ii) $U(t + h, M + N) - U(t, M) \ge ah + \langle \xi, N \rangle + o(|h| + ||N||)$

for all $N \in L^2(I)$ such that $M + N \in Mon_3$.

We set $(\partial_t U(t, M), \partial_M U(t, M)) := (a, \xi)$.

Lemma 2.4. *If U is lazily differentiable at (t, M), then the lazy differential (a, ξ) is unique.*

Proof. Let (a', ξ') be another lazy differential; if we set $N = 0$ in $i)$, $ii)$, we get that $a = a'$.

If we set $h = 0$, $N = \epsilon\bar{N}$ for $\epsilon > 0$, and we subtract $ii)$

$$U(t, M + \epsilon\bar{N}) - U(t, M) \geq \epsilon\langle\xi', \bar{N}\rangle + o(\epsilon)$$

from $i)$

$$U(t, M + \epsilon\bar{N}) - U(t, M) \leq \epsilon\langle\xi, \bar{N}\rangle + K\epsilon^2,$$

we get that

$$\langle\xi - \xi', \bar{N}\rangle \geq 0$$

for all $\bar{N}$ such that $M + \epsilon\bar{N} \in Mon_3$ for ϵ positive and small. Exchanging the rôles of ξ and ξ', we get that

$$\langle\xi - \xi', \bar{N}\rangle = 0$$

for all $\bar{N}$ such that $M + \epsilon\bar{N} \in Mon_3$ for ϵ positive and small. In particular, the formula above holds for $\bar{N} = 1_{[c,1]}$ and $\bar{N}_1 = 1_{[d,1]}$; subtracting, we get that

$$\int_c^d (\xi(x) - \xi'(x))\mathrm{d}\nu_0(x) = 0 \qquad \forall 0 \leq c < d \leq 1.$$

Thus, $\xi = \xi'$, as we wanted.

Proposition 2.5. *Let $U \in C_{\mathrm{Group}}(\mathbf{T})$. For $t > 0$, let us set $\hat{U}(t, M) = (T_t^- U)(M)$. For $(t, M) \in (0, +\infty) \times Mon$, let us suppose that there is a unique curve σ such that $\sigma_t = M$ and*

$$\hat{U}(t, \sigma_t) - U(\sigma_0) = \int_0^t [\mathcal{L}_c(s, \sigma_s, \dot{\sigma}_s) + \alpha(c)]\mathrm{d}s. \tag{2.2}$$

Then, $\hat{U}$ is lazily differentiable at (t, M) and

$$\partial_t\hat{U}(t, M) + \mathcal{H}_0(t, M, c + \partial_M\hat{U}(t, M)) = \alpha(c). \tag{2.3}$$

As a partial converse, if $\hat{U}$ is Fréchet differentiable at $(t, M) \in (0, +\infty) \times Mon$, then there is a unique σ minimal in (2.2), which satisfies (2.3) by the statement above.

Proof. The proof is identical to the finite-dimensional one. We begin with the converse.

Let $\hat{U}$ be Fréchet differentiable at $(t, M) \in (0, +\infty) \times Mon$; by Proposition 2.1, there is a curve σ such that (2.2) holds; we want to prove that it is unique. For $N \in L^2(I)$, let us set

$$\tilde{\sigma}_s = \sigma_s + (N - M)\frac{s}{t}.$$

Since $\tilde\sigma_t = N$, $\tilde\sigma_0 = \sigma_0$ and σ is minimal, the definition of $\hat U$ implies the first inequality below.

$$\begin{aligned}
&\hat U(t,N) - \hat U(t,M)\\
&\quad\le \int_0^t [\mathcal{L}_c(s,\tilde\sigma_s,\dot{\tilde\sigma}_s) - \mathcal{L}(s,\sigma_s,\dot\sigma_s)]\mathrm{d}s \le \int_0^t \left[\left\langle \dot\sigma_s - c, \frac{N-M}{t}\right\rangle\right.\\
&\qquad\left. -\left\langle \mathcal{V}'(s,\sigma_s) + \mathcal{W}'(\sigma_s), \frac{(N-M)s}{t}\right\rangle\right]\mathrm{d}s + K||N-M||^2\\
&\quad= \langle \dot\sigma_t - c, N-M\rangle + K||N-M||^2.
\end{aligned}\tag{2.4}$$

The second inequality above comes from a Taylor development of $\mathcal{L}_c$, and from the fact that the second derivatives of V and W are bounded; the equality comes from an integration by parts and the fact that σ, by point 4) of Proposition 2.1, solves $(ODE)_{Lag}$.

If $\hat U$ is Fréchet differentiable at (t,M), the last formula implies that

$$\partial_M \hat U(t,M) = (\dot\sigma_t - c).\tag{2.5}$$

Since σ satisfies (2.2), it is calibrating, and thus it solves $(ODE)_{Lag}$; we have just seen that its final speed at t satisfies the formula above; since the existence and uniqueness theorem holds for $(ODE)_{Lag}$, we get that the minimizer at (t,M) is unique. It remains to prove that (2.3) holds; since we have just shown that the minimizer σ is unique, this follows from the direct statement, which we presently prove.

Let us suppose that $(t,M) \in (0,+\infty)\times Mon$, and let the minimum in (2.2) be attained on a unique σ. We want to prove that $\hat U$ is lazily differentiable and satisfies (2.3) at (t,M). For $h \in \mathbf{R}$ and $N \in L^2(I)$, we set

$$\tilde\sigma_s = \sigma_s + \frac{N-M}{t+h}s - (\sigma_{t+h} - \sigma_t)\frac{s}{t+h}$$

and we see that $\tilde\sigma_{t+h} = N$ while $\tilde\sigma_0 = \sigma_0$. We get as above that

$$\begin{aligned}
&\hat U(t+h,N) - \hat U(t,M)\\
&\quad\le \int_0^{t+h} [\mathcal{L}_c(s,\tilde\sigma_s,\dot{\tilde\sigma}_s) + \alpha(c)]\mathrm{d}s - \int_0^t [\mathcal{L}_c(s,\sigma_s,\dot\sigma_s) + \alpha(c)]\mathrm{d}s\\
&\quad= \int_t^{t+h} [\mathcal{L}_c(s,\sigma_s,\dot\sigma_s) + \alpha(c)]\mathrm{d}s + \int_0^{t+h} [\mathcal{L}_c(s,\tilde\sigma_s,\dot{\tilde\sigma}_s) - \mathcal{L}_c(s,\sigma_s,\dot\sigma_s)]\mathrm{d}s.
\end{aligned}$$

We also note that, since $||V'||_\infty + ||W'||_\infty \le K$, we have $||\mathcal{V}'|| + ||\mathcal{W}'|| \le K$; we recall that $||\cdot||$ denotes the norm on $L^2(I)$. Since σ_t solves $(ODE)_{Lag}$, this yields that $||\ddot\sigma_t|| \le K$; by a Taylor development, this implies that

$$||\frac{\sigma_{t+h} - \sigma_t}{h} - \dot\sigma_t|| \le K|h|.\tag{2.6}$$

The last two formulas and a Taylor development imply the first inequality below; the equality comes from an integration by parts; the last inequality comes again from (2.6).

$$\begin{aligned}
\hat{U}(t+h,N)-\hat{U}(t,M) &\le h[\mathcal{L}_c(t,\sigma_t,\dot\sigma_t)+\alpha(c)]\\
&\quad+\int_0^{t+h}\left[\left\langle \dot\sigma_s-c,\frac{(N-M)-(\sigma_{t+h}-\sigma_t)}{t+h}\right\rangle\right.\\
&\quad\left.-\left\langle \mathcal{V}'(s,\sigma_s)+\mathcal{W}'(\sigma_s)\frac{[(N-M)-(\sigma_{t+h}-\sigma_t)]s}{t+h}\right\rangle\right]\mathrm{d}s\\
&\quad+K(h^2+||N-M||^2)=h[\mathcal{L}_c(t,\sigma_t,\dot\sigma_t)+\alpha(c)]\\
&\quad+\langle\dot\sigma_{t+h}-c,-\sigma_{t+h}+\sigma_t+(N-M)\rangle+K(h^2+||N-M||^2)\\
&\le h[\mathcal{L}_c(t,\sigma_t,\dot\sigma_t)+\alpha(c)]\\
&\quad-h\langle\dot\sigma_t-c,\dot\sigma_t\rangle+\langle\dot\sigma_t-c,N-M\rangle+2K(h^2+||N-M||^2).
\end{aligned}$$

Since

$$-\mathcal{H}_0(t,\sigma_t,\dot\sigma_t)=\mathcal{L}_c(t,\sigma_t,\dot\sigma_t)-\langle\dot\sigma_t-c,\dot\sigma_t\rangle,$$

the last formula implies that

$$\begin{aligned}
\hat{U}(t+h,N)-\hat{U}(t,M)&\le -h[\mathcal{H}_0(t,\sigma_t,\dot\sigma_t)-\alpha(c)]\\
&\quad+\langle\dot\sigma_t-c,N-M\rangle+2K(h^2+||N-M||^2).
\end{aligned}\tag{2.7}$$

To prove differentiability and (2.3), we need an inequality opposite to (2.7). We let σ be as above, the minimizing curve for $\hat{U}(t,M)$; by hypothesis, σ is unique. We note that point 4) of Proposition 2.1 holds for $\gamma(0)\in Mon_3$, with the same proof. In other words, if $N\in Mon_3$ we can find $\sigma^{h,N}$ minimal for $\hat{U}(t+h,N)$ such that $\sigma^{h,N}_{t+h}=N$; moreover, $\sigma^{h,N}_s\in Mon_3$ for $0\le s\le t+h$. We set

$$\hat\sigma_s=\sigma^{h,N}_s+\frac{M-N}{t}s+\frac{\sigma^{h,N}_{t+h}-\sigma^{h,N}_t}{t}s$$

and we see that $\hat\sigma_t=M$, $\hat\sigma_0=\sigma^{h,N}_0$. With the same calculations of (2.7), we get that

$$\begin{aligned}
\hat{U}(t,M)-\hat{U}(t+h,N)&\le\int_0^t[\mathcal{L}_c(s,\hat\sigma_s,\dot{\hat\sigma}_s)+\alpha(c)]\mathrm{d}s\\
&\quad-\int_0^{t+h}[\mathcal{L}_c(s,\sigma^{h,N}_s,\dot\sigma^{h,N}_s)+\alpha(c)]\mathrm{d}s\le-\left\langle\dot\sigma^{h,N}_t-c,N-M\right\rangle\\
&\quad+h[\mathcal{H}_0(t,N,\dot\sigma^{h,N}_{t+h})-\alpha(c)]+K(h^2+||M-N||^2).
\end{aligned}\tag{2.8}$$

We forego the easy proof ([5]) that, if $(t+h,N)$ belongs to a ball centered in (t,M), we have a uniform bound

$$\int_0^t\|\dot\sigma^{h,N}_s\|^2\mathrm{d}s\le C_1.\tag{2.9}$$

In particular, $\sigma^{h,N}_s$ is uniformly $\frac{1}{2}$-Hölder for $|h|\le 1$ if $N\in Mon_3$ and $\|M-N\|\le 1$.

We assert that the uniform Holderianity of $\sigma^{h,N}_s$ implies the following: if $N\in Mon_3$ and $|h|+||M-N||<\delta$, then

$$||\sigma^{h,N}-\sigma||_{C^0([0,t],Mon_3)}<\epsilon(\delta)\quad\text{with }\epsilon(\delta)\to 0\quad\text{as}\quad\delta\to 0.$$

It suffices to show that, if $(h_k, N_k) \to (0, M)$ in $\mathbf{R} \times Mon_3$, then, up to subsequences, $\sigma^{h_k,N_k} \to \sigma$ in $C^0([0,t], Mon_3)$. We show this fact.

Since $\sigma^{h_k,N_k} \colon [0,t] \to Mon_3$ and Mon_3 is locally compact, we can use (2.9) and Ascoli–Arzelà as in [5] to get that, up to subsequences, $\sigma^{h_k,N_k} \to \sigma^1$ in $C^0([0,t], Mon_3)$. Since σ^{h_k,N_k} minimizes in (2.1), we easily see [5] that σ^1 minimizes (2.1) at (t, M). By our hypotheses, σ is the only minimizer; this yields that $\sigma^1 = \sigma$. In other words, $\sigma^{h,N} \to \sigma$ in $C^0([0,t], Mon_3)$ as $(t+h, N) \to (t, M)$; since $\sigma^{h,N}$ satisfies $(ODE)_{Lag}$, it follows that $\sigma^{h,N} \to \sigma$ in $C^2([0,t], Mon_3)$. This fact and (2.8) imply that, for $N \in Mon_3$,

$$\begin{aligned} \hat{U}(t,M) - \hat{U}(t+h,N) \le & -\langle \dot{\sigma}_t - c, N - M\rangle + h[\mathcal{H}_0(t, M, \dot{\sigma}_t) - \alpha(c)] \\ & + \epsilon(\|M-N\| + |h|) \cdot (\|M-N\| + |h|) \end{aligned}$$

where $\epsilon(\gamma) \to 0$ as $\gamma \to 0$.

The last formula, together with (2.7), implies that $\hat{U}$ is lazily differentiable and that

$$\partial_M \hat{U}(t,M) = (\dot{\sigma}_t - c), \qquad \partial_t \hat{U}(t,M) = -\mathcal{H}_0(t, \sigma_t, \dot{\sigma}_t) + \alpha(c).$$

Since $\sigma_t = M$, (2.3) holds.

Lemma 2.6. *There is $K \ge 0$ such that, for any $U \in C_{\mathrm{Group}}(\mathbf{T})$, the function $T_1^- U$ is K-quasiconcave. In other words, there is $K \ge 0$ such that the map Φ_K*

$$\Phi_K \colon L^2(I) \to \mathbf{R}, \qquad \Phi_K \colon M \to (T_1^- U)(M) - \frac{K}{2}\|M\|^2$$

is concave.

Proof. We define a Lagrangian on $S^1 \times (S^1)^n \times \mathbf{R}^n$ by

$$L_{n,c}(t,q,\dot{q}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}|\dot{q}_i|^2 - c\dot{q}_i\right) - \frac{1}{n}\sum_{i=1}^{n} V(t, q_i) - \frac{1}{2n^2}\sum_{i,j=1}^{n} W(q_i - q_j)$$

where $q = (q_1, \dots, q_n)$. This is the Lagrangian for the Vlasov equation with n particles, each of mass $\frac{1}{n}$; its value function is

$$\begin{aligned} \hat{u}_n(x) = \min \Bigg\{ & \int_0^1 [L_{n,c}(t,q,\dot{q}) + \alpha(c)]\mathrm{d}t + U(D_n q(0)) \;:\; q \in AC([0,1], \mathbf{R}^n), \\ & q(1) = x \Bigg\} \end{aligned} \tag{2.10}$$

where the operator D_n has been defined at the end of Sect. 1, and $x = (x_1, \dots, x_n)$.

Since $L_{n,c}$ is a finite-dimensional Lagrangian, the minimum above is attained by Tonelli's theorem. Let q be minimal in the definition of $\hat{u}_n(x)$; for $h \in \mathbf{R}^n$, we set

$$q^{\pm h} = q(t) \pm ht.$$

Formula (2.10) implies the first inequality below.

$$\hat{u}_n(x+h)+\hat{u}_n(x-h)-2\hat{u}_n(x)\leq\int_0^1[L_{n,c}(t,q^h,\dot{q}^h)+L_{n,c}(t,q^{-h},\dot{q}^{-h})$$
$$-2L_{n,c}(t,q,\dot{q})]\mathrm{d}t=\int_0^1\left\{\frac{1}{n}|h|^2-\frac{1}{2n}\sum_{i=1}^n[V''(t,q_i(t)+\theta_i^+h_i(t-1))\right.$$
$$+V''(t,q_i(t)-\theta_i^-h_i(t-1))]h_i^2-\frac{1}{4n^2}\sum_{i,j=1}^n[W''(q_i(t)-q_j(t)$$
$$+\theta_{i,j}^+(h_i-h_j)(t-1))+W''(q_i(t)-q_j(t)-\theta_{i,j}^-(h_i-h_j)(t-1))](h_i-h_j)^2.$$

The equality above comes from a second order Taylor development (the constants $\theta_i^\pm$ and $\theta_{i,j}^\pm$ belong to $(0,1)$ and depend on t); since

$$|V''(t,x)|\leq C_1,\quad |W''(x)|\leq C_1,\quad\text{and}\quad (h_i-h_j)^2\leq 2h_i^2+2h_j^2,$$

we get that

$$\hat{u}_n(x+h)+\hat{u}_n(x-h)-2\hat{u}_n(x)$$
$$\leq\frac{1}{n}|h|^2+\frac{1}{2n}\sum_{i=1}^n 2C_1(h_i)^2+\frac{1}{4n^2}\sum_{i,j=1}^n 4C_1(h_i^2+h_j^2)=\frac{K}{n}|h|^2$$

where we have denoted by $|\cdot|$ the euclidean norm in $\mathbf{R}^n$. It is well-known that the formula above implies that the function from $\mathbf{R}^n$ to $\mathbf{R}$

$$:x\to\hat{u}_n(x)-\frac{1}{2}\frac{K}{n}|x|^2$$

is concave. By the definition of the operator $D_n\colon\mathbf{R}^n\to L^2(I)$, we have that $\frac{1}{\sqrt{n}}|q|=||D_nq||$; thus, the formula above says that the function from $L^2(I)$ to $\mathbf{R}$

$$:M\to\hat{u}_n(P_nM)-\frac{K}{2}||P_nM||^2$$

is concave. The thesis follows from this and from the fact, proven in [5], that, if $M\in L^2(I)$, then

$$\hat{u}_n(P_nM)\to(T_1^-U)(M)\quad\text{as}\quad n\to+\infty.$$

Definition. Let $U\in C_{\text{Group}}(\mathbf{T})$ be c-dominated; we define A_U as the set of the $M\in Mon$ for which there is $\gamma\in AC_{mon}([-1,1])$ with $\gamma_0=M$ and

$$U(\gamma_1)-U(\gamma_{-1})=\int_{-1}^1[\mathcal{L}_c(s,\gamma_s,\dot{\gamma}_s)+\alpha(c)]\mathrm{d}s.\tag{2.11}$$

Theorem 2.7. *There is a constant $A>0$ such that, if U is c-dominated, then the following holds.*

1) *A_U is closed in Mon.*
2) *If $M\in A_U$, then $U(M)=(T_1^-U)(M)=(T_{-1}^+U)(M)$.*
3) *If γ is as in (2.11) and $\gamma_0=M\in A_U$, then $\gamma|_{[-1,0]}$ is the unique curve on which the* inf *in the definition of $(T_1^-U)(M)$ is attained; analogously, $\gamma|_{[0,1]}$ is the unique curve on which the* sup *in the definition of $(T_{-1}^+U)(M)$ is attained.*

4) *Let us call* γ^M *the curve which satisfies* (2.11) *and* $\gamma_0^M = M$*; we recall that, by point 3),* γ^M *is unique. Then, the map* $: M \to \dot\gamma_0^M$ *is continuous.*
5) U *is Fréchet differentiable at* $M \in A_U$*, and* $d_M U = \dot\gamma_0^M - c$*. Moreover, the map*

$$: A_U \to L^2(I) \times L^2(I), \qquad : M \to (M, d_M U)$$

is Lipschitz with Lipschitz inverse.

Proof. We only sketch the proof, which is identical to theorem 4.5.5 of [7]. We begin with point 1). Let $M^n \in A_U$ and let $M^n \to M$ in $L^2(I)$; let $\gamma^n \in AC_{mon}([-1,1])$ be a curve satisfying (2.11) with $\gamma_0^n = M^n$. We shall prove that γ^n converges to a curve γ which satisfies (2.11) and such that $\gamma_0 = M$.

Since γ^n is calibrating, it is c-minimal; this implies in a standard way (see Lemma 3.4 below for a proof) that there is $C_1 > 0$ such that

$$\sup_{t\in(-1,1)} ||\dot\gamma_t^n|| \le C_1 \qquad \forall n \in \mathbf{N}.$$

As a consequence, $\gamma^n \colon [-1,1] \to Mon$ is equilipschitz and, since $\gamma_0^n = M^n$ is bounded, γ^n is equibounded too. Since Mon is a locally compact subset of $L^2(I)$, we get by Ascoli-Arzelà that, up to subsequences, $\gamma^n \to \gamma$ in $C^0([-1,1], L^2(I))$, and that $\gamma_0 = M$. Using the fact that γ_n solves $(ODE)_{Lag}$, we see that $\ddot\gamma_n \to \ddot\gamma$ in $C^0([-1,1], L^2(I))$; by the interpolation inequalities, $\gamma^n \to \gamma$ in $C^2([-1,1], L^2(I))$. Now, the action functional is continuous under convergence in C^1; together with the fact that U is continuous and that γ^n satisfies (2.11), this implies that

$$U(\gamma_1) - U(\gamma_{-1}) = \int_{-1}^{1} [\mathcal{L}_c(t, \gamma_t, \dot\gamma_t) + \alpha(c)]\mathrm{d}t,$$

proving point 1). We prove point 2). Since U is c-dominated, point 1) of Lemma 2.3 implies that

$$U(M) \le (T_1^- U)(M) \quad \text{and} \quad (T_{-1}^+ U)(M) \le U(M) \quad \text{for all} \quad M \in L^2(I). \tag{2.12}$$

Let now $M \in A_U$ and let γ with $\gamma_0 = M$ satisfy (2.11). We re-write (2.11) as

$$U(\gamma_1) - U(M) + U(M) - U(\gamma_{-1}) = \int_{-1}^{0} [\mathcal{L}_c(s, \gamma_s, \dot\gamma_s) + \alpha(c)]\mathrm{d}s + \int_0^1 [\mathcal{L}_c(s, \gamma_s, \dot\gamma_s) + \alpha(c)]\mathrm{d}s.$$

Since U is c-dominated, we also have that

$$\begin{cases} U(\gamma_1) - U(M) = U(\gamma_1) - U(\gamma_0) \le \int_0^1 [\mathcal{L}_c(s, \gamma_s, \dot\gamma_s) + \alpha(c)]\mathrm{d}s \\ U(M) - U(\gamma_{-1}) = U(\gamma_0) - U(\gamma_{-1}) \le \int_{-1}^{0} [\mathcal{L}_c(s, \gamma_s, \dot\gamma_s) + \alpha(c)]\mathrm{d}s. \end{cases}$$

From the last two formulas, we get that

$$\begin{aligned} U(M) - U(\gamma_{-1}) &= \int_{-1}^{0} [\mathcal{L}_c(s, \gamma_s, \dot{\gamma}_s) + \alpha(c)] \mathrm{d}s, \\ U(\gamma_1) - U(M) &= \int_{0}^{1} [\mathcal{L}_c(s, \gamma_s, \dot{\gamma}_s) + \alpha(c)] \mathrm{d}s. \end{aligned} \tag{2.13}$$

By the definitions of $(T_1^- U)(M)$ and $(T_{-1}^+ U)(M)$, the two formulas above imply respectively that, if $M \in A_U$,

$$(T_1^- U)(M) \le U(M) \quad \text{and} \quad U(M) \le (T_{-1}^+ U)(M).$$

This and (2.12) prove point 2).

We prove point 3). By point 2), if $M \in A_U$, then $(T_1^- U)(M) = U(M)$; thus, it suffices to prove that any curve $\tilde{\gamma}$ with $\tilde{\gamma}_0 = M$ and

$$(T_1^- U)(\tilde{\gamma}_0) - U(\tilde{\gamma}_{-1}) = U(\tilde{\gamma}_0) - U(\tilde{\gamma}_{-1}) = \int_{-1}^{0} [\mathcal{L}_c(t, \tilde{\gamma}_t, \dot{\tilde{\gamma}}_t)] \mathrm{d}t \tag{2.14}$$

coincides with γ.

Let us suppose by contradiction that $\tilde{\gamma} \ne \gamma$ on $[-1, 0]$; we define

$$\hat{\gamma}_t = \begin{cases} \tilde{\gamma}_t & t \in [-1, 0] \\ \gamma_t & t \in [0, 1]. \end{cases}$$

By (2.14) and the second formula of (2.13), it follows easily that (2.11) holds for $\hat{\gamma}$. We have seen that this implies that $\hat{\gamma}$ is c-minimal on $[-1, 1]$; in particular, it satisfies $(ODE)_{Lag}$. Now γ satisfies $(ODE)_{Lag}$ for the same reason; since $\hat{\gamma} = \gamma$ on $[0, 1]$, we have a contradiction with the existence and uniqueness theorem.

We prove point 4). Let $M_n \in A_U$, and let $M_n \to M$ in $L^2(I)$; point 1) implies that $M \in A_U$. Let γ^{M_n} satisfy (2.11) with $\gamma_0^{M_n} = M_n$; we see as in the proof of point 1) that the sequence $\gamma^{M_n} \in AC_{mon}(-1, 1)$ has a subsequence converging to a limit γ in $C^2([-1, 1], L^2(I))$. As a consequence, γ satisfies (2.11) and $\gamma_0 = M$. By the uniqueness of point 3), this implies that $\gamma^{M_n} \to \gamma^M$ in $C^2((-1, 1), L^2(I))$, proving 4).

We prove point 5). Let $M \in A_U$; the inequality below is point 1) Lemma 2.3; the equality, point 2) of the present theorem.

$$U(N) \le (T_1^- U)(N) \quad \forall N \in L^2(I) \quad \text{and} \quad U(M) = (T_1^- U)(M).$$

This implies the first inequality and the equality below; in the proof of Proposition 2.4, we got (2.4), i.e. the second inequality below.

$$\begin{aligned} U(N) &\le (T_1^- U)(N) \le (T_1^- U)(M) + \left\langle \dot{\gamma}_0^M - c, N - M \right\rangle + K\|N - M\|^2 \\ &= U(M) + \left\langle \dot{\gamma}_0^M - c, N - M \right\rangle + K\|N - M\|^2 \quad \forall N \in L^2(I). \end{aligned}$$

Applying the same argument to T_{-1}^+ with time reversed, we get that

$$U(N) \ge U(M) + \left\langle \dot{\gamma}_0^M - c, N - M \right\rangle - K\|N - M\|^2 \quad \forall N \in L^2(I).$$

Now a general fact (Proposition 4.5.3 of [7]) implies that, if the two inequalities above hold, then U is Fréchet differentiable at any point of A_U, with $d_M U = \dot{\gamma}_0^M - c$. Moreover, the map : $M \to d_M U$ is Lipschitz.

3. The minimal measures

Definition. Let $M, N \in Mon$; we say that $M \simeq N$ if $M - N \equiv z \in \mathbf{Z}$. We denote by $Mon_{\mathbf{Z}}$ the space of equivalence classes; it is easy to see that $Mon_{\mathbf{Z}}$ is compact for the topology it inherits from Mon (or from $L^2(I)$, which is the same.) We shall denote by $[[M]]$ the equivalence class of M in $Mon_{\mathbf{Z}}$: we use the double brackets to avoid confusion with the equivalence class of M in $\mathbf{S}$, which we denoted by [M]. We denote by Π the natural projection of Mon into $Mon_{\mathbf{Z}}$. In the following, we shall work mostly on Mon, though we shall turn to $Mon_{\mathbf{Z}}$ in all situations in which we need compactness.

We let B_R be the closed ball of radius R in $L^2(I)$, with the weak topology; we endow

$$F_R := S^1 \times Mon_{\mathbf{Z}} \times B_R$$

with the product topology. We see that F_R, being the product of compact sets, is compact; moreover, it is a metric space.

Let $\psi_s(t, M, v)$ be the flow of $(ODE)_{Lag}$; in other words,

$$\psi_s\colon \mathbf{R} \times L^2(I) \times L^2(I) \to \mathbf{R} \times L^2(I) \times L^2(I), \quad \psi_s(t, M, v) = (t + s, \gamma_{t+s}, \dot\gamma_{t+s})$$

where γ_τ solves

$$\begin{cases} \ddot\gamma_\tau = -\mathcal{V}'(\tau, \gamma_\tau) - \mathcal{W}'(\gamma_\tau) \quad \text{for} \quad \tau \in \mathbf{R} \\ \gamma_t = M \\ \dot\gamma_t = v. \end{cases}$$

We want to restrict this flow to a compact subset K_R of F_R.

Definition. We define the set $K_R \subset F_R$ in the following way. Let $(t, [[M]], v) \in F_R$ and let $\psi_s(t, M, v) = (t + s, \gamma_{t+s}, \dot\gamma_{t+s})$; if $\gamma_\tau \in Mon$ and $||\dot\gamma_\tau|| \le R$ for all $\tau \in \mathbf{R}$, we say that $(t, [[M]], v) \in K_R$. Note that, since $\psi_s(t, M + k, v) = \psi_s(t, M, v) + (0, k, 0)$, the condition just stated does not depend on the choice of the representative M.

We are going to see below that, for R large, $K_R \neq \emptyset$; meanwhile, we prove the following.

Lemma 3.1. *K_R is compact in F_R.*

Proof. Since we saw above that F_R is compact, it suffices to prove that K_R is closed in F_R. Thus, let $(t_n, [[M_n]], v_n) \in K_R$, and let $(t_n, [[M_n]], v_n) \to (t, [[M]], v)$; we must prove that $(t, [[M]], v) \in K_R$.

First of all, up to adding integers, we can suppose that $M_n \to M$ in Mon. By the definition of K_R, we can find a solution γ^n of $(ODE)_{Lag}$ such that $(\gamma^n_{t_n}, \dot\gamma^n_{t_n}) = (M_n, v_n)$ and $\gamma^n_s \in Mon$, $\|\dot\gamma^n_s\| \le R$ for all $s \in \mathbf{R}$. As a consequence, $\gamma^n\colon \mathbf{R} \to Mon$ is R-Lipschitz for all n; since $(t_n, \gamma^n_{t_n}) \to (t, M)$, we get that γ_n is locally bounded. Now, bounded sets of Mon are relatively compact, and we can apply Ascoli-Arzelà and get that, up to subsequences, $\gamma^n \to \gamma$ in $C^0_{loc}(\mathbf{R}, Mon)$.

Clearly, $\gamma_s \in Mon$ for all s; indeed, it is the $L^2(I)$-limit of $\gamma^n_s \in Mon$.

We note that $\sup_{s\in\mathbf{R}}\|\dot\gamma_s\|$ is l. s. c. for the $C^0_{loc}(\mathbf{R},Mon)$ topology; indeed, if $\|\psi\|\le 1$, we have that

$$|\langle\dot\gamma_s,\psi\rangle|=\left|\frac{\mathrm{d}}{\mathrm{d}s}\langle\gamma_s,\psi\rangle\right|\le\sup_s\left|\frac{\mathrm{d}}{\mathrm{d}s}\langle\gamma_s,\psi\rangle\right|\le\liminf_{n\to+\infty}\sup_{\tau\in\mathbf{R}}|\frac{\mathrm{d}}{\mathrm{d}\tau}\langle\gamma^n_\tau,\psi\rangle|\le R.$$

The second inequality above comes from the well-known fact that, in dimension 1, the sup norm of the derivative is l. s. c. for uniform convergence; the third one comes from the fact that $(\tau,\gamma^n_\tau,\dot\gamma^n_\tau)\in K_R$. Since ψ is arbitrary in B_1 and s is arbitrary in $\mathbf{R}$, the formula above implies

$$\sup_{s\in\mathbf{R}}\|\dot\gamma_s\|\le R.$$

We prove that γ is an orbit with $\gamma_t=M$ and $\dot\gamma_t=v$. Since $\gamma^n\to\gamma$ in $C^0_{loc}(\mathbf{R},L^2(I))$ and γ^n satisfies $(ODE)_{Lag}$, it follows that $\ddot\gamma^n\to\ddot\gamma$ in $C^0_{loc}(\mathbf{R},L^2(I))$. By the usual interpolation inequalities, we get that $\dot\gamma^n\to\dot\gamma$ in $C^0_{loc}(\mathbf{R},L^2(I))$, which implies that $\dot\gamma_t=v$. Finally, since $\gamma_n\to\gamma$ in C^2, taking limits in $(ODE)_{Lag}$, we get that γ solves this equation.

Definition. We define $\mathcal{M}^R_1$ as the set of the probability measures on K_R, invariant by the Euler–Lagrange flow of $\mathcal{L}$. We note that $\mathcal{M}^R_1$ is not empty, if R is large enough; to show this, we recall that $W'(0)=0$, because W is even; thus, if $q(t)$ is an orbit of the one-particle Lagrangian

$$L\colon S^1\times S^1\times\mathbf{R}\to\mathbf{R},\qquad L(t,q,\dot q)=\frac{1}{2}|\dot q|^2-V(t,q),$$

then $D_1q(t)$ is an orbit of $\mathcal{L}$; the operator D_1 has been defined at the end of Sect. 1. As a consequence, if R is large enough, $\mathcal{M}^R_1$ contains the measures induced by $(t,[[D_1q(t)]],\dot q(t))$, where q is a periodic orbit of L.

We endow $\mathcal{M}^R_1$ with the weak$*$ topology; since K_R is a compact metric space, we get that $\mathcal{M}^R_1$ is a compact metric space too.

We also define

$$I_c\colon\mathcal{M}^R_1\to\mathbf{R},\qquad I_c(\mu)=\int_{K_R}\mathcal{L}_c(t,\sigma,v)\mathrm{d}\mu(t,[[\sigma]],v).$$

Lemma 3.2. *The functional I_c on $\mathcal{M}^R_1$ is lower semicontinuous.*

Proof. We note that

$$\mathcal{L}_c(t,\sigma,v)=Cin(v)-Hom(v)-P(t,\sigma)$$

where

$$Cin(v)=\frac{1}{2}\|v\|^2,\qquad Hom(v)=\langle c,v\rangle,\qquad P(t,\sigma)=\mathcal{V}(t,\sigma)+\mathcal{W}(t,\sigma).$$

Since V and W are Lipschitz, and the topology on $Mon_{\mathbf{Z}}$ is the one induced by $L^2(I)$, it is immediate that

$$P\colon K_R\to\mathbf{R}$$

is continuous. By the definition of the weak$*$ topology on $\mathcal{M}^R_1$, this implies that the map

$$:\mu\to\int_{K_R}P(t,\sigma)\mathrm{d}\mu(t,[[\sigma]],v)$$

is continuous. Since we have endowed B_R with the weak topology, the map $Hom\colon K_R \to \mathbf{R}$ is continuous; as a consequence, the map

$$: \mu \to \int_{K_R} Hom(v)\mathrm{d}\mu(t,[[\sigma]],v)$$

is continuous too. Let us prove that the map

$$: \mu \to \int_{K_R} \frac{1}{2}\|v\|^2\mathrm{d}\mu(t,[[\sigma]],v) \tag{3.1}$$

is l. s. c.. To do this, we let $\{\psi_n\}_{n\geq 1}$ be a sequence dense in B_R for the strong topology of $L^2(I)$, and we define

$$g_n\colon L^2(I) \to \mathbf{R}$$

$$g_n(v) = \sup\left\{\langle v,\psi_i\rangle - \frac{1}{2}\|\psi_i\|^2 \;:\; i \in (1,\dots,n)\right\}. \tag{3.2}$$

It is a standard fact that, if $\|v\| \leq R$, then

$$\frac{1}{2}\|v\|^2 = \sup_{\|\psi\|\leq R}\left\{\langle v,\psi\rangle - \frac{1}{2}\|\psi\|^2\right\}.$$

Since $\{\psi_i\}$ is dense in B_R, the last formula implies that, if $v \in B_R$, then

$$g_n(v) \nearrow \frac{1}{2}\|v\|^2 \quad \forall v \in B_R.$$

Formula (3.2) and Cauchy–Schwarz imply the first inequality below; since v and ψ_1 are in B_R, also the second one follows.

$$g_n(v) \geq -||v||\cdot||\psi_1|| - \frac{1}{2}||\psi_1||^2 \geq -\frac{3}{2}R^2 \quad \forall v \in B_R.$$

Since $-\frac{3R^2}{2} \in L^1(K_R,\mu)$, we can apply monotone convergence and get that

$$\int_{K_R}\frac{1}{2}\|v\|^2\mathrm{d}\mu(t,[[\sigma]],v) = \sup_{n\geq 1}\int_{K_R} g_n(v)\mathrm{d}\mu(t,[[\sigma]],v).$$

Thus, the lower semicontinuity of (3.1) follows, if we prove that each map

$$: \mu \to \int_{K_R} g_n(v)\mathrm{d}\mu(t,[[\sigma]],v)$$

is continuous. By the definition of the weak$*$ topology, it suffices to prove that each function $g_n\colon K_R \to \mathbf{R}$ is continuous. But this is true because, by (3.2), g_n is the sup of a finite family of maps, each of which is continuous on K_R.

The next corollary follows at once from the last two lemmas.

Corollary 3.3. *If $c \in \mathbf{R}$ and $R > 0$ is so large that K_R is not empty, then there is $\bar\mu \in \mathcal{M}_1^R$ such that*

$$I_c(\bar\mu) = \inf_{\mu\in\mathcal{M}_1^R} I_c(\mu).$$

We call c-minimal the measures which satisfy the formula above. We want to prove, following [12] and [7], that

1. • for R large, the set of the c-minimal measures does not depend on R;

2. • the orbits in the support of a c-minimal measure are c-minimal.

We need a lemma.

Lemma 3.4. *There is a function* $R\colon \mathbf{R} \to (0,+\infty)$*, bounded on bounded sets, such that, for any* c*-minimal* $\sigma \in AC_{mon}([0,1])$*, we have that*

$$\sup_{t\in[0,1]} \|\dot\sigma_t\| \le R(c).$$

Proof. Since $\sigma_0, \sigma_1 \in Mon$, we have $\sigma_0(1-) \le \sigma_0(0)+1$ and $\sigma_1(1-) \le \sigma_1(0)+1$; thus, we can find $z_0, z_1 \in \mathbf{Z}$ such that $\sigma_0 + z_0$ and $\sigma_1 + z_1$ have range in $[-1,1]$. We set $\bar A = \sigma_0 + z_0$, $\bar B = \sigma_1 + z_1$ and

$$\tilde\sigma_t = (1-t)\bar A + t\bar B.$$

We denote, as usual, by $||\cdot||_{C^0}$ the sup norm; an easy calculation shows that

$$C_1 := \|V\|_{C^0(S^1\times S^1)} + \|W\|_{C^0(S^1)} \ge \|\mathcal{V}\|_{C^0(S^1\times L^2)} + \|\mathcal{W}\|_{C^0(L^2)}. \tag{3.3}$$

This implies the first inequality below.

$$\begin{aligned}\int_0^1 \mathcal{L}_c(t,\tilde\sigma_t,\dot{\tilde\sigma}_t)\mathrm{d}t &= \int_0^1 \left[\frac{1}{2}\|\bar B-\bar A\|^2 - \langle c, \bar B - \bar A\rangle - \mathcal{V}(t,\tilde\sigma_t) - \mathcal{W}(\tilde\sigma_t)\right]\mathrm{d}t\\ &\le \frac{1}{2}\|\bar B-\bar A\|^2 - \langle c, \bar B-\bar A\rangle + C_1 \le \frac{4}{2} + 2|c| + C_1 = C_2.\end{aligned}$$

The last inequality above follows from the fact that $\bar A$ and $\bar B$ have range in $[-1,1]$. Since σ is c-minimal, and $\sigma_i - \tilde\sigma_i \in L^2_{\mathbf{Z}}$ for $i = 0,1$, we get that

$$\int_0^1 \mathcal{L}_c(t,\sigma_t,\dot\sigma_t)\mathrm{d}t \le \int_0^1 \mathcal{L}_c(t,\tilde\sigma_t,\dot{\tilde\sigma}_t)\mathrm{d}t \le C_2.$$

The first inequality below follows from Cauchy–Schwarz; the second one from the fact that $\frac{1}{4}x^2 - c^2 \le \frac{1}{2}x^2 - cx$; the third one, from (3.3) and the last one from the formula above.

$$\begin{aligned}\frac{1}{4}\left(\int_0^1 \|\dot\sigma_t\|\mathrm{d}t\right)^2 - c^2 &\le \frac{1}{4}\int_0^1 \|\dot\sigma_t\|^2\mathrm{d}t - c^2 \le \int_0^1\left[\frac{1}{2}\|\dot\sigma_t\|^2 - \langle c,\dot\sigma_t\rangle\right]\mathrm{d}t\\ &\le \int_0^1 \mathcal{L}_c(t,\sigma_t,\dot\sigma_t)\mathrm{d}t + C_1 \le C_2 + C_1.\end{aligned}$$

From this it follows that

$$\int_0^1 \|\dot\sigma_t\|\mathrm{d}t \le C_4. \tag{3.4}$$

We get as in (3.3) that

$$\|\mathcal{V}'\|_{C^0(S^1\times L^2)} + \|\mathcal{W}'\|_{C^0(L^2)} \le \|V'\|_{C^0(S^1\times S^1)} + \|W'\|_{C^0(S^1)}.$$

Now σ, being minimal, satisfies $(ODE)_{Lag}$; by the last formula, this implies that

$$\|\ddot\sigma_t\| \le C_5 \quad \forall t\in[0,1].$$

The last formula and (3.4) imply the thesis.

Lemma 3.5. *Let $R(c) > 0$ be as in Lemma 3.4 and let $R \geq R(c)$. Let μ minimize I_c in $\mathcal{M}_1^R$. Then, the following three points hold.*

1) *μ a.e. $(t, [[M]], v) \in K_R$ is the initial condition of a c-minimal orbit. Moreover, μ induces, in a natural way, a measure γ_0 on $K_R \cap \{t = 0\}$ such that the following happens. If U is c-dominated, for γ_0 a.e. $(0, [[M]], v)$, we have that the curve $\sigma_s = \pi_{mon} \circ \psi_s(0, M, v)$ is calibrating for U.*
2) *μ is supported in $K_{R(c)}$.*
3) *For the function α defined in Sect. 2, we have that*

$$-\alpha(c) = I_c(\mu) = \min_{\nu \in \mathcal{M}_1^R} I_c(\nu).$$

Proof. We begin to note that point 2) follows from point 1) and Lemma 3.4; before proving point 1), we sketch the proof of 3) given in [7].

We have to prove only the first equality, since the fact that that μ is minimal in $\mathcal{M}_1^R$ is one of the hypotheses. Let us consider the projections

$$\pi_{mon} \colon S^1 \times Mon_{\mathbf{Z}} \times L^2(I) \to Mon_{\mathbf{Z}}, \quad \pi_{L^2} \colon S^1 \times Mon_{\mathbf{Z}} \times L^2(I) \to L^2(I),$$
$$\pi_{mon \times L^2} \colon S^1 \times Mon_{\mathbf{Z}} \times L^2(I) \to Mon_{\mathbf{Z}} \times L^2(I),$$
$$\pi_{time} \colon S^1 \times Mon_{\mathbf{Z}} \times L^2(I) \to S^1.$$

Let us consider $(\pi_{time})_\sharp \mu$, the marginal of μ on S^1; since μ is invariant by ψ_s, it is easy to see that $(\pi_{time})_\sharp \mu$ is translation-invariant, and thus it must coincide with the Lebesgue measure on S^1. As a consequence, we can disintegrate μ as $\mu = \mathcal{L}^1 \otimes \gamma_t$, where $\mathcal{L}^1$ is the Lebesgue measure on S^1 and γ_t is a probability measure on $Mon_{\mathbf{Z}} \times B(0, R)$. Using again the fact that μ is invariant by the flow ψ_s, we easily see that $\gamma_t = (\pi_{mon \times L^2} \circ \psi_t(0, \cdot, \cdot))_\sharp \gamma_0$; as a consequence, γ_0 is invariant by the time-one map $\Psi \colon (M, v) \to \pi_{mon \times L^2} \circ \psi_1(0, M, v)$.

Let now $U \colon L^2(I) \to \mathbf{R}$ be a fixed point of T_1^-; we have seen in Proposition 2.2 that such a function exists. By Lemma 2.3, U is c-dominated, and thus, for $k \in \mathbf{N}$, we have

$$U \circ \pi_{mon} \circ \Psi^k(M, v) - U \circ \pi_{mon}(M, v) \leq \int_0^k [\mathcal{L}_c(t, \sigma_t, \dot{\sigma}_t) + \alpha(c)] \mathrm{d}t \quad (3.5)$$

for every $\sigma \in AC_{mon}([0, k])$ with $\sigma_k = \pi_{mon} \circ \Psi^k(M, v)$ and $\sigma_0 = M$. We let $\sigma_t^{M,v} = \pi_{mon} \circ \psi_t(0, M, v)$; we consider (3.5) for $k \geq 1$ and $\sigma = \sigma^{M,v}$; we integrate it under γ_0 and we get the inequality below.

$$\begin{aligned} 0 &= \int_{Mon_{\mathbf{Z}} \times B_R} [U \circ \pi_{mon} \circ \Psi^k(M, v) - U \circ \pi_{mon}(M, v)] \mathrm{d}\gamma_0([[M]], v) \\ &\leq \int_{Mon_{\mathbf{Z}} \times B_R} \mathrm{d}\gamma_0([[M]], v) \int_0^k [\mathcal{L}_c(\psi_t(0, M, v)) + \alpha(c)] \mathrm{d}t \\ &= \int_0^k \mathrm{d}t \int_{Mon_{\mathbf{Z}} \times B_R} [\mathcal{L}_c(t, M, v) + \alpha(c)] \mathrm{d}\gamma_t([[M]], v) \\ &= k \int_{K_R} [\mathcal{L}_c(t, M, v) + \alpha(c)] \mathrm{d}\mu(t, M, v). \end{aligned} \quad (3.6)$$

The first equality above follows because γ_0 is invariant by the time-one map Ψ, the second one because $\gamma_t = \psi_t(0,\cdot,\cdot)_\sharp\gamma_0$, and the third one because $\mu = \mathcal{L}^1 \otimes \gamma_t$.

Now (3.6) implies that, for μ c-minimal,

$$I_c(\mu) \geq -\alpha(c). \tag{3.7}$$

We want to prove the opposite inequality. Let $M \in Mon$, we recall from Proposition 2.2 that there is $\sigma \in AC_{mon}((-\infty,0])$ with $\sigma_0 = M$ such that, for any $k \in \mathbf{N}$,

$$U(M) - U(\sigma_{-2k}) = \int_{-2k}^{0} [\mathcal{L}_c(t,\sigma_t,\dot\sigma_t) + \alpha(c)]\mathrm{d}t. \tag{3.8}$$

Now we use the Krylov-Bogoljubov argument: we consider the map

$$\Phi_k\colon [-k,k] \to F_R, \qquad \Phi_k\colon t \to (t \mod 1, \sigma_{t-k} \mod 1, \dot\sigma_{t-k})$$

and the probability measure $\mu_k = (\Phi_k)_\sharp\nu_k$, where ν_k is the Lebesgue measure on $[-k,k]$ normalized to 1.

Since σ is c-minimal on $(-\infty,0]$, Lemma 3.4 implies that $\Phi_k([-k,k]) \in F_{R(c)} \subset F_R$ for $k \geq 1$.

This implies that μ_k is supported in the compact set F_R; thus, up to subsequences, μ_k converges weak$*$ to a probability measure $\bar\mu$ on F_R. We assert that $\bar\mu \in \mathcal{M}_1^R$, i.e. that $\bar\mu$ is invariant and supported on K_R. The Kryolov-Bogolyubov construction implies in a standard way that $\bar\mu$ is invariant; moreover, $\bar\mu$ is supported on the limits of the orbits σ_{t-k}; but $\sigma_{t-k} \in Mon$ for $t \in (-\infty,k]$, and thus any of its limits $\tilde\sigma_t$ belongs to Mon for all $t \in \mathbf{R}$.

This and Lemma 3.2 imply the inequality below.

$$\begin{aligned} I_c(\bar\mu) &\leq \liminf_{k\to+\infty} I_c(\mu_k) = \liminf_{k\to+\infty} \frac{1}{2k}\int_{-2k}^{0} \mathcal{L}_c(t,\sigma_t,\dot\sigma_t)\mathrm{d}t \\ &= \liminf_{k\to+\infty} \frac{1}{2k}[U(M) - U(\sigma_{-2k}) - 2k\alpha(c)] = -\alpha(c). \end{aligned}$$

The first equality above comes from the definition of μ_k, the second one comes from (3.8) and the third one from the fact, which we saw at the beginning of Sect. 2, that U is bounded. Since $\bar\mu$ is an invariant probability measure on K_R, the last formula and (3.7) imply point 3).

By point 3), for $k \in \mathbf{N}$ formula (3.6) collapses to

$$\begin{aligned} 0 &= \int_{K_R\cap\{t=0\}} [U\circ\pi_{mon}\circ\Psi^k(M,v) - U\circ\pi_{mon}(M,v)]\mathrm{d}\gamma_0([[M]],v) \\ &= \int_{K_R\cap\{t=0\}} \mathrm{d}\gamma_0([[M]],v)\int_0^k [\mathcal{L}_c(t,\sigma_t^{M,v},\dot\sigma_t^{M,v}) + \alpha(c)]\mathrm{d}t. \end{aligned}$$

This and (3.5) imply that, for all $k \in \mathbf{N}$ and γ_0 a.e. $([[M]],v)$,

$$U\circ\pi_{mon}\circ\Psi^k(0,M,v) - U(M) = \int_0^k [\mathcal{L}_c(t,\sigma_t^{M,v},\dot\sigma_t^{M,v}) + \alpha(c)]\mathrm{d}t. \tag{3.9}$$

We have seen that, since U is c-dominated, this implies that $\sigma^{M,v}$ is c-minimal for a.e. $([[M]],v)$; but this is point 1).

Now we briefly define, following [12], the two "conjugate mean actions" α and β.

For starters, we define the rotation number of $\mu \in \mathcal{M}_1^R$ in the standard way, by duality with the equivariant homology of $L^2(I)$. We recall from Proposition 1.1 that, if $S \in C^1(L^2(I))$ and $\mathrm{d}S$ is $L^2_{\mathbf{Z}}$ and Group-equivariant, then

$$\mathrm{d}S = c + \mathrm{d}s$$

with $c \in \mathbf{R}$; the function s, which belongs to $C^1(L^2(I))$, is $L^2_{\mathbf{Z}}$ and is Group-equivariant. Let $\mu \in \mathcal{M}_1^R$; as in [12], the ergodic theorem implies the first equality below.

$$\begin{aligned}
&\int_{K_R} \langle \mathrm{d}_M s, v\rangle \mathrm{d}\mu(t,[[M]],v) \\
&\quad = \int_{K_R} \mathrm{d}\mu(t,[[M]],v) \lim_{n\to+\infty} \frac{1}{n}\int_0^n \frac{\mathrm{d}}{\mathrm{d}\tau} s(\pi_{mon}\circ\psi_\tau(t,M,v))\mathrm{d}\tau \\
&\quad = \int_{K_R} \lim_{n\to+\infty} \frac{1}{n}[s\circ\pi_{mon}\circ\psi_n(t,M,v) - s\circ\pi_{mon}(t,M,v)]\mathrm{d}\mu(t,[[M]],v) = 0.
\end{aligned}$$

The last equality above comes from the fact that any $s \in C_{\mathrm{Group}}(\mathbf{T})$ is bounded; we saw this right at the beginning of Sect. 2. As a consequence,

$$\int_{K_R} \langle c + \mathrm{d}_M s, v\rangle \mathrm{d}\mu(t,[[M]],v)$$

depends only on $c \in \mathbf{R}$. If we define $\rho(\mu)$ as

$$\rho(\mu) = \int_{K_R} \langle 1, v\rangle \mathrm{d}\mu(t,[[M]],v), \tag{3.10}$$

we see by the formula above that

$$\int_{K_R} \langle c + \mathrm{d}_M s, v\rangle \mathrm{d}\mu(t,[[M]],v) = c\cdot\rho(\mu)$$

for all $c \in \mathbf{R}$ and $s \in C^1(L^2(I))$, $L^2_{\mathbf{Z}}$ and Group-equivariant. One can look on $\rho(\mu)$ as on the "mean number of turns of all the particles around S^1"; indeed, by the ergodic theorem, (3.10) implies the first equality below.

$$\begin{aligned}
\rho(\mu) &= \int_{K_R} \mathrm{d}\mu(t,[[M]],v) \lim_{n\to+\infty} \frac{1}{n}\int_0^n \frac{\mathrm{d}}{\mathrm{d}\tau}\langle 1, \pi_{mon}\circ\psi_\tau(t,M,v)\rangle \mathrm{d}\tau \\
&= \int_{K_R} \mathrm{d}\mu(t,[[M]],v) \lim_{n\to+\infty} \frac{1}{n}\langle 1, [\pi_{mon}\circ\psi_n(t,M,v) - M]\rangle.
\end{aligned}$$

Now, since $: x \to \pi_{mon}\circ\psi_n(t,M,v)(x)$ belongs to Mon, it is easy to see that

$$\lim_{n\to+\infty} \frac{1}{n}[\pi_{mon}\circ\psi_n(t,M,v)(x) - M(x)]$$

does not depend on $x \in I$; actually, it is equal to

$$\lim_{n\to+\infty} \frac{1}{n}\langle 1, [\pi_{mon}\circ\psi_n(t,M,v) - M]\rangle,$$

yielding that

$$\rho(\mu) = \lim_{n\to+\infty} \frac{1}{n}[\pi_{mon} \circ \psi_n(t, M, v)(x) - M(x)]$$

for all $x \in I$.

Let the space $\mathcal{C}_1$ be as in the end of Sect. 1; it is a standard fact (see [12]) that $S^1 \times \mathcal{C}_1 \times \mathcal{C}_1$ (the phase space of a single particle), which is invariant by the Euler–Lagrange flow of $\mathcal{L}$, contains measures of any rotation number $\rho \in \mathbf{R}$; as a consequence, if $\rho \in \mathbf{R}$ is given and $R > 0$ is large enough, $\mathcal{M}_1^R$ contains measures of rotation number ρ.

We define

$$\beta^R(\rho) = \min\left\{\int_{K_R} \mathcal{L}_0(t, M, v)\mathrm{d}\mu(t, [[M]], v) \,:\, \mu \in \mathcal{M}_1^R \quad \text{and} \quad \rho(\mu) = \rho\right\}$$

and

$$-\alpha^R(c) = \min\left\{\int_{K_R} \mathcal{L}_c(t, M, v)\mathrm{d}\mu(t, [[M]], v) \,:\, \mu \in \mathcal{M}_1^R\right\}.$$

The second minimum is attained by corollary 3.3; by Lemma 3.2, to prove that the first minimum is attained, it suffices to prove that the set

$$\{\mu \in \mathcal{M}_1^R \,:\, \rho(\mu) = \rho\}$$

is compact. Since $\mathcal{M}_1^R$ is compact, it suffices to prove that : $\mu \to \rho(\mu)$ is continuous for the weak$*$ topology on $\mathcal{M}_1^R$; this in turn follows from the fact that the integral on the right hand side of (3.10) is a continuous function of μ; we saw this in the proof of Lemma 3.2, where we called it $Hom(v)$.

By point 3) of Lemma 3.5, we get that, for $R \geq R(c)$, $\alpha^R(c) = \alpha(c)$. By point 2) of the same lemma, the c-minimal measures are supported in $K_{R(c)}$. By definition, β^R is decreasing in R; we set

$$\beta(\rho) = \inf_{R>0} \beta^R(\rho) = \lim_{R\to+\infty} \beta^R(\rho).$$

It is easy to see that α and β are convex; we recall the proof, which is identical to [12], that each of them is the Legendre transform of the other one. Indeed,

$$\begin{aligned}
\beta^*(c) &= \sup_{\rho}\{\rho \cdot c - \beta(\rho)\} = \sup_{\rho,R}\{\rho \cdot c - \beta_R(\rho)\} \\
&= \sup_{\rho,R} \sup \left\{\rho \cdot c - \int_{K_R} \mathcal{L}\mathrm{d}\mu \,:\, \rho(\mu) = \rho, \quad \mu \in \mathcal{M}_1^R\right\} \\
&= \sup_{R} \sup_{\mu\in\mathcal{M}_1^R} \left(-\int_{K_R} \mathcal{L}_c \mathrm{d}\mu\right) = \alpha(c)
\end{aligned}$$

where the last but one equality comes from (3.10) and the last one the fact that $\alpha^R(c) = \alpha(c)$ for R large enough. The proof that β is the Legendre transform of α is analogous.

The fact that α and β are each the Legendre transform of the other, implies that both have superlinear growth. Since α is the Legendre transform of β, we have that $\beta(\rho) = c \cdot \rho - \alpha(c)$ for any $c \in \partial\beta(\rho)$; as a consequence,

$\beta(\rho)$ is attained exactly on the c-minimal measures, for $c \in \partial\beta(\rho)$; since β is superlinear, $\partial\beta(\rho)$ is compact, and thus

$$R_\rho := \sup_{c\in\partial\beta(\rho)} R(c)$$

is finite. In other words, for $R \geq R_\rho$, the set of measures μ on K_R such that $\rho(\mu) = \rho$ and $I_c(\mu) = \beta^R(\rho)$ does not depend on R; or $\beta^R(\rho) = \beta(\rho)$ for R large enough.

We define $\hat{Mat}_c$ as the closure of the union of the supports of all the c-minimal measures; we define

$$\tilde{Mat}_c := \ (\Pi, id)^{-1}\{\hat{Mat}_c \cap \{t = 0\}\} \subset \ Mon \times L^2(I),$$
$$Mat_c\colon \ = \pi_{mon}(\tilde{Mat}_c) \subset Mon$$

where the projection Π was defined at the beginning of this section. In other words, $\tilde{Mat}_c$ is the set of all initial conditions (M, v) in $Mon \times L^2(I)$ such that $(\Pi, id) \circ \psi_s(0, M, v)$ lies in the support of a c-minimal measure; Mat_c is what we get from this set forgetting the velocity variable.

4. The Aubry set

In this section, we define the Aubry set in terms of the operators T_1^- and T_{-1}^+; we shall check that the arguments of [7] continue to work.

Lemma 4.1. *If U is c-dominated, if $M \in Mat_c$ and $n \in \mathbf{N}$, then*

$$(T_n^- U)(M) = (T_{-n}^+ U)(M) = U(M).$$

Proof. Since U is c-dominated, $U(M) \geq (T_{-n}^+ U)(M)$ by point 1) of Lemma 2.3. Since $M \in Mat_c$, we have that formula (3.9) holds; now (3.9) immediately implies that $(T_{-n}^+ U)(M) \geq U(M)$, and we are done.

Theorem 4.2. *If $U \in C_{\mathrm{Group}}(\mathbf{T})$ is c-dominated, there are a fixed point U^- of T_1^- and a fixed point U^+ of T_{-1}^+ which satisfy the following points.*

1) *$U(M) = U^-(M) = U^+(M)$ if $M \in Mat_c$.*
2) *$U^+(M) \leq U(M) \leq U^-(M)$ for all $M \in L^2(I)$.*
3) *U^- is the smallest of the fixed points of T_1^- which are larger than U, and U^+ is the largest of the fixed points of T_{-1}^+ which are smaller than U. In other words,*
   (a) • *if U_1^- is a fixed point of T_1^- such that $U \leq U_1^-$, then $U^- \leq U_1^-$, and*
   (b) • *if U_1^+ is a fixed point of T_{-1}^+ such that $U \geq U_1^+$, then $U^+ \geq U_1^+$.*
4) *The sequences $T_n^- U$ and $T_{-n}^+ U$ converge to U^- and U^+ respectively, uniformly on $L^2(I)$.*
5) *If U^- is a fixed point of T_1^-, then there is a fixed point U^+ of T_{-1}^+ such that $U^- = U^+$ on Mat_c; moreover, $U^+ \leq U^-$ on Mon.*

Proof. We only sketch the proof, since it is identical to [7].

We note that

$$T^-_{n+1}U = T^-_n \circ T^-_1 U \geq T^-_n U,$$

where the equality comes from the semigroup property, and the inequality from point 1) of Lemma 2.3 and the fact, which we saw at the beginning of Sect. 2, that T^-_n is monotone. Thus, $T^-_n U$ is an increasing sequence. Moreover, by point 2) of Proposition 2.1, $T^-_n U$ is L-Lipschitz for $dist_{\mathbf{S}}$, for some $L > 0$ independent on n. Thus, $T^-_n U$ quotients on the compact set $\mathbf{S}$ as an increasing sequence of L-Lipschitz functions. By Lemma 4.1, $T^-_n U = U$ on Mat_c; since $\mathbf{S}$ is compact, and $T^-_n U$ is uniformly Lipschitz, the sequence $T^-_n U$ is bounded in the sup norm. Thus, $T^-_n U$ quotients to an increasing, bounded, uniformly Lipschitz sequence of functions on $\mathbf{S}$; as a result, $T^-_n U$ converges uniformly to a L-Lipschitz function U^- on $\mathbf{S}$. We go back to $L^2(I)$; what we just said implies that $T^-_n U$ converges uniformly to U^- in $L^2(I)$; since $dist_{\mathbf{S}}([u],[v]) = dist_{weak}(u,v) \leq ||u - v||$, we get that U^- is L-Lipschitz on $L^2(I)$. Since $T^-_n U \geq U$, we get that $U^- \geq U$. Since $T^-_n U = U$ on Mat_c, we get that $U^- = U$ on Mat_c. Thus, U^- (and U^+, with the same proof) satisfies points 1), 2) and 4).

We saw right after the definition of $\Lambda_{c,l}$ that the map $: U \to T^-_1 U$ is continuous for the sup norm; this implies the second equality below, while the first and last one follow by point 4).

$$T^-_1 U^- = T^-_1 \left(\lim_{n\to+\infty} T^-_n U \right) = \lim_{n\to+\infty} T^-_{(n+1)} U = U^-.$$

This proves that U^- is a fixed point of T^-_1.

We prove 3); let U^-_1 be as in this point. The first equality below is point 4); the inequality is the fact, which we saw before Proposition 2.2, that T^-_n is monotone: $T^-_n(V_1) \leq T^-_n(V_2)$ if $V_1 \leq V_2$.

$$U^-(M) = \lim_{n\to+\infty} T^-_n U(M) \leq \lim_{n\to+\infty} T^-_n U^-_1(M) = U^-_1(M).$$

The last equality above follows because U^-_1 is a fixed point of T^-_1.

We prove 5). Let U^- be a fixed point of T^-_{-1}; by point 4), we can build U^+ as the limit of $T^+_{-n}(U^-)$ as $n \to +\infty$; by point 1), $U^- = U^+$ on Mat_c; applying point 2) with $U = U^-$, we get that $U^+ \leq U^-$.

Lemma 4.3. *Let $\mathcal{U}$ be an open neighbourhood of $\hat{Mat}_c$. Then, there is $t(\mathcal{U}) > 0$ with the following property. If $t \geq t(\mathcal{U})$ and $\gamma \in AC_{mon}([0,t])$ is c-minimal, then there is $s \in [0,t] \cap \mathbf{N}$ with $(s, \gamma_s, \dot\gamma_s) \in \mathcal{U}$.*

Proof. The proof of this lemma is identical to [7]; essentially, it follows from the fact that, as $k \to +\infty$, the push-forward of the normalized Lebesgue measure on $[0,k]$ by the map $: s \to (s, \gamma_s, \dot\gamma_s)$ accumulates on a c-minimal measure. We used this fact in proving point 3) of Lemma 3.5.

Proposition 4.4. *Let $U \in C_{\text{Group}}(\mathbf{T})$ be c-dominated. Then, there is a unique couple (U^-, U^+) such that U^- is a fixed point of T^-_1, U^+ is a fixed point of T^+_{-1} and $U^- = U^+ = U$ on Mat_c. Moreover, $U^+ \leq U^-$.*

Proof. Existence of the couple (U^-, U^+) follows from Theorem 4.2. We prove uniqueness. Let $(\tilde U^-, \tilde U^+)$ be another such couple and let $M \in Mon$; since $\tilde U^-$ is a fixed point of T_1^-, by point 2) of Proposition 2.2 there is $\sigma \in AC_{loc}((-\infty, 0])$ such that $\sigma_0 = M$ and, for all $k \in \mathbf{N}$,

$$\tilde U^-(M) - \tilde U^-(\sigma_{-k}) = \int_{-k}^{0} [\mathcal{L}_c(t, \sigma_t, \dot\sigma_t) + \alpha(c)]\mathrm{d}t.$$

By Lemma 4.3, there is a sequence $k_j \to +\infty$ such that $\sigma_{-k_j} \to N \in Mat_c$. Since $\tilde U^-$ is continuous, the formula above implies that, in the formula below, the limit on the right exists and it is equal to the expression on the left.

$$\tilde U^-(M) - \tilde U^-(N) = \lim_{j \to +\infty} \int_{-k_j}^{0} [\mathcal{L}_c(t, \sigma_t, \dot\sigma_t) + \alpha(c)]\mathrm{d}t.$$

Using the fact that U^- is c-dominated, we get that

$$U^-(M) - U^-(N) \le \lim_{j \to +\infty} \int_{-k_j}^{0} [\mathcal{L}_c(t, \sigma_t, \dot\sigma_t) + \alpha(c)]\mathrm{d}t.$$

Since $N \in Mat_c$, we have that $U^-(N) = U(N) = \tilde U^-(N)$; from this and the last two formulas we get that

$$U^-(M) \le \tilde U^-(M) \qquad \forall M \in Mon$$

which implies in the usual way that

$$U^-(M) \le \tilde U^-(M) \qquad \forall M \in L^2(I).$$

Exchanging the rôles of $\tilde U^-$ and U^-, we get the opposite inequality; this proves the first assertion of the lemma.

The last assertion, i.e. that $U^+ \le U^-$, follows, in the obvious way, from uniqueness and point 2) of Theorem 4.2.

Definition. A pair of functions $U^-, U^+ \in C_{\mathrm{Group}}(\mathbf{T})$ is said to be conjugate if U^- is a fixed point of T_1^-, U^+ is a fixed point of T_{-1}^+ and $U^+ = U^-$ on Mat_c. We denote by $\mathcal{D}$ the set of the couples (U^-, U^+) of conjugate functions. By Proposition 2.2, there is a c-dominated function U; thus, by Proposition 4.4, $\mathcal{D}$ is not empty.

Always by Proposition 4.4, if $(U^-, U^+) \in \mathcal{D}$, then $U^+ \le U^-$.

We forego the easy proof that $\mathcal{D}$ is closed in $C(L^2(I), \mathbf{R}) \times C(L^2(I), \mathbf{R})$.

Definition. For $(U^-, U^+) \in \mathcal{D}$, we set

$$\mathcal{I}_{(U^-,U^+)} = \{M \in Mon : \ U^-(M) = U^+(M)\}.$$

Let $(U^-, U^+) \in \mathcal{D}$; then, by definition of conjugate couple,

$$Mat_c \subset \mathcal{I}_{(U^-,U^+)}.$$

We note that $\Pi(\mathcal{I}_{(U^-,U^+)})$ is a compact set of $Mon_{\mathbf{Z}}$; indeed, we have already seen that $Mon_{\mathbf{Z}}$ is compact; since the functions $U^\pm$ are continuous, $\mathcal{I}_{(U^-,U^+)}$ is a closed set of Mon, implying that $\Pi(\mathcal{I}_{(U^-,U^+)})$ is a closed set of $Mon_{\mathbf{Z}}$.

Theorem 4.5. *Let $(U^-, U^+) \in \mathcal{D}$ and let $M \in \mathcal{I}_{(U^-,U^+)}$. Then, there is a unique c-minimal curve $\gamma \in AC_{mon}(\mathbf{R})$ such that $\gamma_0 = M$ and, for all $m \le n \in \mathbf{Z}$,*

$$U^{\pm}(\gamma_n) - U^{\pm}(\gamma_m) = \int_m^n [\mathcal{L}_c(t, \gamma_t, \dot{\gamma}_t) + \alpha(c)] dt. \tag{4.1}$$

In other words, γ is calibrating both for U^- and for U^+. Moreover, $U^{\pm}$ is Fréchet differentiable at M and

$$\mathrm{d}_M U^+ = \mathrm{d}_M U^- = -c + \dot{\gamma}_0. \tag{4.2}$$

Proof. Let $M \in Mon$; since U^- and U^+ are fixed points of T^-_{-1} and T^+_1 respectively, we can apply point 2) of Proposition 2.2 and get that there are two minimal curves, $\gamma^- \in AC_{loc}((-\infty, 0])$ and $\gamma^+ \in AC_{loc}([0, +\infty))$, such that

$$\gamma^-_0 = \gamma^+_0 = M$$

and, for any $n \in \mathbf{N}$ and $-m \in \mathbf{N}$,

$$\begin{cases} U^-(\gamma^-_0) - U^-(\gamma^-_m) = \displaystyle\int_m^0 [\mathcal{L}_c(t, \gamma^-_t, \dot{\gamma}^-_t) + \alpha(c)]\mathrm{d}t \\ U^+(\gamma^+_n) - U^+(\gamma^+_0) = \displaystyle\int_0^n [\mathcal{L}_c(t, \gamma^+_t, \dot{\gamma}^+_t) + \alpha(c)]\mathrm{d}t. \end{cases} \tag{4.3}$$

We define

$$\gamma_t = \begin{cases} \gamma^-_t & t \le 0 \\ \gamma^+_t & t \ge 0 \end{cases}$$

and we get, by (4.3), that

$$\int_m^n [\mathcal{L}_c(t, \gamma_t, \dot{\gamma}_t) + \alpha(c)]\mathrm{d}t = U^+(\gamma_n) - U^-(\gamma_m) + [U^-(\gamma_0) - U^+(\gamma_0)]. \tag{4.4}$$

We prove that, if $M \in \mathcal{I}_{(U^-,U^+)}$, then γ satisfies (4.1); clearly, up to integer translations, we can always suppose that $m < 0 < n$. The first inequality below comes from the fact that $U^- \ge U^+$; the first equality comes from the fact that $\gamma_0 = M \in \mathcal{I}_{(U^-,U^+)}$; the second one comes from (4.4). The last inequality comes from the fact that U^- is c-dominated.

$$\begin{aligned} U^-(\gamma_n) - U^-(\gamma_m) &\ge U^+(\gamma_n) - U^-(\gamma_m) = U^+(\gamma_n) - U^-(\gamma_m) + [U^-(\gamma_0) - U^+(\gamma_0)] \\ &= \int_m^n [\mathcal{L}_c(t, \gamma_t, \dot{\gamma}_t) + \alpha(c)]\mathrm{d}t \ge U^-(\gamma_n) - U^-(\gamma_m). \end{aligned}$$

This formula implies (4.1) for U^-; the proof for U^+ is analogous.

We saw above that, if U is c-dominated and γ satisfies (4.1), i.e. it is calibrating, then γ is c-minimal. This gives existence.

We prove uniqueness. Let $\tilde{\gamma}$ be any curve such that $\tilde{\gamma}_0 = M$ and such that (4.1) holds. If we define

$$\hat{\gamma}_t = \begin{cases} \gamma_t & t \le 0 \\ \tilde{\gamma}_t & t \ge 0 \end{cases}$$

we see as above that $\hat{\gamma}$ satisfies (4.1) and thus it is c-minimal; since c-minimal curves are C^2, we get that $\dot{\gamma}_0 = \dot{\tilde{\gamma}}_0$; since both curves satisfy $(ODE)_{Lag}$, we get that $\tilde{\gamma} = \gamma$.

Formula (4.2) comes from (4.1) and point 5) of Theorem 2.7.

Definition. Let $(U^-,U^+)\in\mathcal{D}$; in view of Theorem 4.5, we can define

$$\tilde{\mathcal{I}}_{(U^-,U^+)}=\{(M,c+\mathrm{d}_M U^-)\ :\ M\in\mathcal{I}_{(U^-,U^+)}\}$$
$$=\{(M,c+\mathrm{d}_M U^+)\ :\ M\in\mathcal{I}_{(U^-,U^+)}\}$$

where the derivatives are in the Fréchet sense.

Theorem 4.6. 1. *Let $(U^-,U^+)\in\mathcal{D}$. Then, the projection*

$$\pi_{mon}\colon\tilde{\mathcal{I}}_{(U^-,U^+)}\to\mathcal{I}_{(U^-,U^+)}$$

is bi-Lipschitz.

2. *The set $\tilde{\mathcal{I}}_{(U^-,U^+)}$ is invariant by the time-one map Ψ of the Euler–Lagrange flow of $\mathcal{L}$, and it contains the set $\tilde{Mat}_c$ defined at the end of Sect. 3. Moreover, $(\Pi\times id)(\tilde{\mathcal{I}}_{(U^-,U^+)})$ is compact in $Mon_{\mathbf{Z}}\times L^2(I)$; we recall that $\Pi\colon Mon\to Mon_{\mathbf{Z}}$ is the projection.*
3. *If $(M,v)\in\tilde{\mathcal{I}}_{(U^-,U^+)}$, and if $\gamma_t=\pi_{mon}\circ\psi_t(0,M,v)$, then, for $m\le n\in\mathbf{Z}$,*

$$U^{\pm}(\gamma_n)-U^{\pm}(\gamma_m)=\int_m^n[\mathcal{L}_c(t,\gamma_t,\dot\gamma_t)+\alpha(c)]dt.$$

Proof. Let $(M,v)\in\tilde{\mathcal{I}}_{(U^-,U^+)}$ and let $\gamma_t=\pi_{mon}\circ\psi_t(0,M,v)$; this mean that $\dot\gamma_0$ satisfies formula (4.2); by the uniqueness part of Theorem 4.5, it satisfies (4.1) too, and this yields point 3).

Since $\gamma_0=M$, setting $m=-1$ and $n=1$ in point 3) of the present theorem, we see that $M\in A_{U^-}$; the set A_{U^-} has been defined before theorem 2.7. Since M is arbitrary in $\mathcal{I}_{(U^-,U^+)}$, we get that $\mathcal{I}_{(U^-,U^+)}\subset A_{U^-}$. Point 1) follows by this and point 5) of theorem 2.7.

Point 2): the fact that $(\Pi\times id)(\tilde{\mathcal{I}}_{(U^-,U^+)}$ is compact follows from point 1) and the fact that $\Pi(\mathcal{I}_{(U^-,U^+)})$ is compact, which we proved just before theorem 4.5.

We prove that $\tilde{\mathcal{I}}_{(U^-,U^+)}$ is invariant by Ψ. Let γ be as in point 3); we have that

$$\begin{cases} U^-(\gamma_1)-U^-(\gamma_{-n})=\int_{-n}^1[\mathcal{L}_c(t,\gamma_t,\dot\gamma_t)+\alpha(c)]\mathrm{d}t\\ U^+(\gamma_n)-U^+(\gamma_1)=\int_1^n[\mathcal{L}_c(t,\gamma_t,\dot\gamma_t)+\alpha(c)]\mathrm{d}t.\end{cases}$$

Let us suppose by contradiction that $\gamma_1\not\in\mathcal{I}_{(U^-,U^+)}$, i.e. that $U^-(\gamma_1)-U^+(\gamma_1)>0$; summing the two formulas above, this implies that

$$U^+(\gamma_n)-U^-(\gamma_{-n})<\int_{-n}^n[\mathcal{L}_c(t,\gamma_t,\dot\gamma_t)+\alpha(c)]\mathrm{d}t.$$

On the other side, since $\gamma_0\in\mathcal{I}_{(U^-,U^+)}$, we have that $U^-(\gamma_0)-U^+(\gamma_0)=0$; arguing as above, this implies that

$$U^+(\gamma_n)-U^-(\gamma_{-n})=\int_{-n}^n[\mathcal{L}_c(t,\gamma_t,\dot\gamma_t)+\alpha(c)]\mathrm{d}t.$$

This contradiction proves that $\gamma_1\in\mathcal{I}_{(U^-,U^+)}$; since

$$-c+\dot\gamma_1=\mathrm{d}_{\gamma_1}U^+=\mathrm{d}_{\gamma_1}U^-$$

by (4.2), we get that that $\tilde{\mathcal{I}}_{(U^-,U^+)}$ is invariant by Ψ.

The fact that $Mat_c\subset\mathcal{I}_{(U^-,U^+)}$ follows from the definition of conjugate pair; to prove that $\tilde{Mat}_c\subset\tilde{\mathcal{I}}_{(U^-,U^+)}$, we recall that, in formula (3.9, we have shown that, if $(M,v)\in\tilde{Mat}_c$ and $(\gamma_0,\dot\gamma_0)=(M,v)$, then γ satisfies (4.1); by the uniqueness of theorem 4.5, we get that $\dot\gamma_0=c+\mathrm{d}_M U^-(\gamma_0)$, i.e. that $\tilde{Mat}_c\subset\tilde{\mathcal{I}}_{(U^-,U^+)}$.

Definition. We define the Aubry set $\mathcal{A}_c$ and the Mañe set $\mathcal{MN}_c$ in the following way.

$$\mathcal{A}_c=\bigcap_{(U^-,U^+)\in\mathcal{D}}\mathcal{I}_{(U^-,U^+)},\qquad \tilde{\mathcal{A}}_c=\bigcap_{(U^-,U^+)\in\mathcal{D}}\tilde{\mathcal{I}}_{(U^-,U^+)}$$

$$\mathcal{MN}_c=\bigcup_{(U^-,U^+)\in\mathcal{D}}\mathcal{I}_{(U^-,U^+)},\qquad \tilde{\mathcal{MN}}_c=\bigcup_{(U^-,U^+)\in\mathcal{D}}\tilde{\mathcal{I}}_{(U^-,U^+)}.$$

Theorem 4.7. 1) *The quotiented Aubry sets $\Pi(\mathcal{A}_c)$ and $(\Pi\times id)(\tilde{\mathcal{A}}_c)$ are compact; we have that $Mat_c\subset\mathcal{A}_c$ and $\tilde{Mat}_c\subset\tilde{\mathcal{A}}_c$. Moreover, $\tilde{\mathcal{A}}_c$ is invariant by the time-one map Ψ.*

2) *There is a pair $(U^-,U^+)\in\mathcal{D}$ such that $\mathcal{A}_c=\mathcal{I}_{(U^-,U^+)}$.*

3) *The map $\pi_{mon}\colon\tilde{\mathcal{A}}_c\to\mathcal{A}_c$ is bi-Lipschitz.*

Proof. By definition, each $\Pi(\mathcal{I}_{(U^-,U^+)})$ is compact and contains $\Pi(Mat_c)$; moreover, by point 2) of theorem 4.6, each $(\Pi\times id)(\tilde{\mathcal{I}}_{(U^-,U^+)})$ is compact, invariant by Ψ, and contains $(\Pi\times id)(\tilde{Mat}_c)$; this implies point 1).

We note that point 2) and theorem 4.6 imply point 3); actually, point 3) is also implied directly by theorem 4.6, because the restriction of a Lipschitz map to a smaller set is Lipschitz.

We prove point 2). First of all, we restrict our conjugate couples to Mon, and quotient them on $Mon_{\mathbf{Z}}$; in other words, we look at them as functions in $C(Mon_{\mathbf{Z}},\mathbf{R})$. This is justified by the fact, which we saw in Sect. 1, that any $U\in C(Mon_{\mathbf{Z}},\mathbf{R})$ can be uniquely extended to a function in $C_{\text{Group}}(\mathbf{T})$.

Since $Mon_{\mathbf{Z}}$ is a compact metric space, $C(Mon_{\mathbf{Z}},\mathbf{R})$ is separable; since $\mathcal{D}$ is a closed set of $C(Mon_{\mathbf{Z}},\mathbf{R})\times C(Mon_{\mathbf{Z}},\mathbf{R})$, we can find a dense sequence $\{(U_n^+,U_n^-)\}_{n\ge1}\subset\mathcal{D}$. Since $\{U_n^\pm\}$ is a sequence of fixed points of $T^\pm_{\pm1}$, it is equilipschitz by Proposition 2.2. We note that, if $(U_n^+,U_n^-)\in\mathcal{D}$ and $a_n\in\mathbf{R}$, then $(U_n^++a_n,U_n^-+a_n)$ is a conjugate pair too; since $Mon_{\mathbf{Z}}$ has finite diameter, since $U_n^\pm$ is equilipschitz and $U_n^+=U_n^-$ on Mat_c, we can choose a_n in such a way that $U_n^\pm+a_n$ is equibounded. Setting $\tilde U_n^\pm=U_n^\pm+a_n$, we get that the two series below converge uniformly to two Lipschitz functions on $Mon_{\mathbf{Z}}$, which we call $\tilde U^-$ and $\tilde U^+$ respectively.

$$\tilde U^-=\sum_{n\ge1}\frac{1}{2^n}\tilde U_n^-,\qquad \tilde U^+=\sum_{n\ge1}\frac{1}{2^n}\tilde U_n^+. \tag{4.5}$$

Since $\tilde U_n^-=\tilde U_n^+$ on Mat_c, we get that $\tilde U^-=\tilde U^+$ on Mat_c. Since $\tilde U^-$ and $\tilde U^+$ are convex combinations of c-dominated functions, it follows easily that they are c-dominated; by points 1) and 2) of theorem 4.2, we can find U^-, a fixed point of T^-_{-1}, satisfying $U^- \geq \tilde U^-$, and $U^- = \tilde U^-$ on Mat_c. Analogously, there is U^+, a fixed point of T^+_1, satisfying $U^+ \leq \tilde U^+$, with equality on Mat_c. Since $\tilde U^- = \tilde U^+$ on Mat_c, we have that $U^- = U^+$ on Mat_c, and thus $(U^-, U^+) \in \mathcal{D}$. As a consequence,

$$\mathcal{A}_c \subset \mathcal{I}_{(U^-,U^+)}. \tag{4.6}$$

On the other side, since $\{(U^-_n, U^+_n)\}_{n\geq 1}$ is dense in $\mathcal{D}$, we see that, if $M \notin \mathcal{A}_c$, then

$$U^+_n(M) < U^-_n(M)$$

for at least one n; this implies that $\tilde U^+_n(M) < \tilde U^-_n(M)$ for at least one n. On the other side, $\tilde U^+_n \leq \tilde U^-_n$ for all n, since $(\tilde U^-_n, \tilde U^+_n)$ is a conjugate pair; by (4.5), this implies that

$$\tilde U^+(M) < \tilde U^-(M).$$

We saw above that $U^+ \leq \tilde U^+$ and $\tilde U^- \leq U^-$; thus, if $M \notin \mathcal{A}_c$,

$$U^+(M) < U^-(M).$$

Together with (4.6), this implies point 2).

Definition. Given $[[M]], [[N]] \in Mon_{\mathbf{Z}}$ and $n \in \mathbf{N}$, we define as in [13]

$$h_n([[M]],[[N]]) = \min\left\{\int_0^n [\mathcal{L}_c(t,\gamma_t,\dot\gamma_t)+\alpha(c)]\mathrm{d}t \,:\, \sigma_0 \in [[M]], \quad \sigma_n \in [[N]],\right\}$$

and

$$h_\infty([[M]],[[N]]) = \liminf_{n\to+\infty} h_n([[M]],[[N]]).$$

The minimum in the definition of h_n is attained by an argument similar to that of point 4) of Proposition 2.1. Naturally, we have to prove that h_∞ is finite; for this, we refer the reader to [13], since the proof is identical.

Lemma 4.8. *If $(U^-, U^+) \in \mathcal{D}$, then*

$$\forall M_-, M_+ \in Mon, \quad U^-(M_-) - U^+(M_+) \leq h_\infty([[M_-]],[[M_+]]).$$

Proof. We recall the proof of [7]. By the definition of h_∞, we can find a sequence of integers $n_k \to +\infty$ and a minimal $\gamma_k \in AC_{mon}([0, n_k])$ such that

$$\begin{cases} h_\infty([[M_-]],[[M_+]]) = \lim_{k\to+\infty}\int_0^{n_k}[\mathcal{L}_c(t,\gamma^k_t,\dot\gamma^k_t)+\alpha(c)]\mathrm{d}t \\ \gamma^k_0 \in [[M_-]], \quad \gamma^k_{n_k} \in [[M_+]]. \end{cases} \tag{4.7}$$

By Lemma 4.3 and the fact that $\Pi(Mat_c)$ is compact in $Mon_{\mathbf{Z}}$, there are two integers $n'_k \in [0, n_k]$ and $a_k \in \mathbf{Z}$ such that $\gamma^k_{n'_k} - a_k \to N \in Mat_c$. Since U^- and U^+ are c-dominated, we have that

$$U^+(\gamma^k_{n'_k}) - U^+(M_-) \leq \int_0^{n'_k}[\mathcal{L}_c(t,\gamma^k_t,\dot\gamma^k_t)+\alpha(c)]\mathrm{d}t$$

$$U^-(M_+) - U^-(\gamma^n_{n'_k}) \leq \int_{n'_k}^{n_k}[\mathcal{L}_c(t,\gamma^k_t,\dot\gamma^k_t)+\alpha(c)]\mathrm{d}t.$$

We recall that U^- and U^+ are $L^2_{\mathbf{Z}}$-invariant; adding the inequalities above, and letting $k \to +\infty$, we get by (4.7) that

$$U^-(M_+) - U^-(N) + U^+(N) - U^+(M_-) \le h_\infty([[M_-]], [[M_+]]).$$

Since $N \in Mat_c$, the definition of $\mathcal{D}$ implies that $U^-(N) = U^+(N)$, and the thesis follows.

Theorem 4.9. *For $M_-, M_+ \in Mon$, we have that*

$$h_\infty([[M_-]], [[M_+]]) = \sup_{(U^-,U^+)\in\mathcal{D}} [U^-(M_-) - U^+(M_+)].$$

Proof. By Lemma 4.8, we know that

$$h_\infty([[M_-]], [[M_+]]) \ge \sup_{(U^-,U^+)\in\mathcal{D}} [U^-(M_-) - U^+(M_+)]. \tag{4.8}$$

To prove the opposite inequality, we see as in theorem 5.3.6 of [7] that, for all $M_+ \in Mon$, the function

$$U^-_{M_-} : M_+ \to h_\infty([[M_-]], [[M_+]])$$

is a fixed point of T^-_1, while for all $M_- \in Mon$, the function

$$U^+_{M_+} : M_- \to h_\infty([[M_-]], [[M_+]])$$

is a fixed point of T^+_{-1}. The reason for this is essentially the following: it is not hard to see that $Q\colon M_+ \to h_\infty([[M_-]], [[M_+]])$ is c-dominated; moreover, the curves γ^n which minimize in the definition of $h_n([[M_-]], [[M_+]])$ converge, up to subsequences, to a curve γ calibrating for Q on $(-\infty, 0]$; now the assertion follows by point 2) of Lemma 2.3.

Moreover, we can prove as in [7] that the conjugate function U^+ of $U^-_{M_-}$ vanishes at M_-, while the conjugate function U^- of $U^+_{M_+}$ vanishes at M_+. Indeed, since $U^-_{M_-}$ is c-dominated, we can apply point 4) of theorem 4.2 and get that

$$\begin{aligned} U^+(M_-) &= \lim_{n\to+\infty} (T^+_{-n} U^-_{M_-})(M_-) \\ &= \lim_{n\to+\infty} \max \left\{ h_\infty([[M_-]], [[\gamma_n]]) \right. \\ &\quad \left. - \int_0^n [\mathcal{L}_c(t, \gamma, \dot\gamma) + \alpha(c)]\mathrm{d}t \,:\, \gamma_0 = M_- \right\}. \end{aligned}$$

Let $\bar\gamma^n$ maximize in the formula above. For each n we choose γ^n minimal in the definition of $h_n([[M]], [[\bar\gamma^n_n]])$; by compactness, there is $n_k \to +\infty$ such that $[[\gamma^{n_k}_{n_k}]] \to [[N]]$. By an argument like that of point 2) of Proposition 2.1, the functions h_n can be shown to be L-Lipschitz in both variables, with the constant L independent on n; this implies that h_∞ is Lipschitz too. This, and the fact that $\gamma^{n_k}_{n_k} \to N$, imply the first and third equalities below; the last one follows by our choice of γ^n.

$$\lim_{k\to+\infty} h_\infty([[M_-]],[[\gamma^{n_k}_{n_k}]]) = h_\infty([[M_-]],[[N]]) \le \liminf_{k\to+\infty} h_{n_k}([[M_-]],[[N]])$$
$$= \liminf_{k\to+\infty} h_{n_k}([[M_-]],[[\gamma^{n_k}_{n_k}]])$$
$$= \liminf_{k\to+\infty} \int_0^{n_k} [\mathcal{L}_c(t,\gamma^{n_k},\dot\gamma^{n_k}) + \alpha(c)]\mathrm{d}t.$$

The last two formulas imply that $U^+(M_-) \le 0$. Since $(U^-_{M_-}, U^+) \in \mathcal{D}$, Lemma 4.8 implies the inequality below; the equality is the definition of $U^-_{M_-}$.

$$h_\infty([[M_-]],[[M_+]]) \ge U^-_{M_-}(M_+) - U^+(M_-) = h_\infty([[M_-]],[[M_+]]) - U^+(M_-).$$

This implies that $U^+(M_-) \ge 0$, ending the proof that $U^+(M_-) = 0$.

Since $U^+(M_-) = 0$, we get the second equality below; the first one is the definition of $U^-_{M_+}$.

$$h_\infty([[M_-]],[[M_+]]) = U^-_{M_-}(M_+) = U^-_{M_-}(M_+) - U^+(M_-).$$

Since $(U^-_{M_+}, U^+) \in \mathcal{D}$, this yields the inequality opposite to (4.8).

As an immediate consequence, we can reunite Mather's definition in [13] with Fathi's definition, which we gave before theorem 4.7.

Theorem 4.10. *$M \in \mathcal{A}_c$ iff $h_\infty(M,M) = 0$.*

We forego another check, i.e. that the Mañe set $\mathcal{M}N_c$ is the set of the c-minimal orbits.

5. Fixed points and KAM

Now we want to to look at the minimal orbits of $\mathcal{L}_c$ from another point of view, that of fixed point theory.

Definition. Let $\tilde\mu_{-1}$, $\tilde\mu_1$ be two Borel probability measures on $\mathbf{R}$, which we shall always suppose to be compactly supported. Actually, we shall only consider $\tilde\mu_{\pm1}$ of the form $\tilde\mu_{\pm1} = (\sigma_{\pm1})_\sharp\nu_0$, with $\sigma_{\pm1} \in Mon$, implying that $\tilde\mu_{\pm1}$ is supported in an interval of length 1.

We denote by $\mathcal{M}_1(\tilde\mu_1,\tilde\mu_2)$ the space of the Borel probability measures on $[-1,1]\times\mathbf{R}\times\mathbf{R}$ which satisfy the following three points.

i)

$$\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}} (1+|v|)\mathrm{d}\mu(t,q,v) < +\infty.$$

ii) Let $\pi\colon (t,q,v) \to t$. We ask that, if $\mu \in \mathcal{M}_1(\tilde\mu_{-1},\tilde\mu_1)$, then $\pi_\sharp\mu = \frac{1}{2}\mathcal{L}^1$, where $\mathcal{L}^1$ denotes the Lebesgue measure on $[-1,1]$. In particular, μ is a probability measure, and can be disintegrated as $\mu = \frac{1}{2}\mathcal{L}^1 \otimes \mu_t$, with μ_t a measure on $\mathbf{R}\times\mathbf{R}$.

iii) We also ask that the elements μ of $\mathcal{M}_1(\tilde{\mu}_{-1},\tilde{\mu}_1)$ are closed, i.e. for any $\phi\in C_0^1([-1,1]\times\mathbf{R})$, we have that

$$\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}\mathrm{d}\phi(t,q)\cdot(1,v)\mathrm{d}\mu(t,q,v)=\frac{1}{2}\int_{\mathbf{R}}\phi(1,q)\mathrm{d}\tilde{\mu}_1(q)-\frac{1}{2}\int_{\mathbf{R}}\phi(-1,q)\mathrm{d}\tilde{\mu}_{-1}(q). \quad (5.1)$$

In [4], the elements of $\mathcal{M}_1(\tilde{\mu}_{-1},\tilde{\mu}_1)$ are called the transport measures.

Point i) above essentially says that the integral on the left of (5.1) converges; point iii) says that μ has "boundary values" $\tilde{\mu}_{-1}$ at $t=-1$, and $\tilde{\mu}_1$ at $t=1$. As an example, consider $\sigma\in C^1([-1,1],L^2(I))$; if we define $\tilde{\mu}_{\pm1}=(\sigma_{\pm1})_\sharp\nu_0$, $\mu_t=(\sigma_t,\dot{\sigma}_t)_\sharp\nu_0$ and $\mu=\frac{1}{2}\mathcal{L}^1\otimes\mu_t$, then it is easy to check that $\mu\in\mathcal{M}_1(\tilde{\mu}_{-1},\tilde{\mu}_1)$. We saw above that $\tilde{\mu}_{\pm1}$ are supported in an interval of length 1.

It is well-known ([4]) that we can endow $\mathcal{M}_1(\tilde{\mu}_{-1},\tilde{\mu}_1)$ with a distance d (called a Kantorovich-Rubinstein distance) with the following property: $d(\mu_n,\mu)\to 0$ if, for any $\phi\in C([-1,1]\times\mathbf{R}\times\mathbf{R})$ such that

$$\sup_{(t,q,v)}\frac{|\phi(t,q,v)|}{1+|v|}<+\infty,$$

we have that

$$\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}\phi(t,q,v)\mathrm{d}\mu_n(t,q,v)\to\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}\phi(t,q,v)\mathrm{d}\mu(t,q,v).$$

By [4], d turns $\mathcal{M}_1(\tilde{\mu}_1,\tilde{\mu}_2)$ into a complete metric space.

It is a standard consequence of i), ii), and iii) above (the proof is akin to Lemma 8.1.2 of [2]) that, for any choice of the C^1 function ϕ, the function

$$: t\to\int_{\mathbf{R}\times\mathbf{R}}\phi(q)\mathrm{d}\mu_t(q,v)$$

is absolutely continuous. In particular, the function

$$W_\mu(t,x)=\int_{\mathbf{R}\times\mathbf{R}}W(x-y)\mathrm{d}\mu_t(y,v)$$

is continuous in t. Since we are supposing that $W\in C^2(S^1)$, differentiating under the integral sign we get that $W_\mu\in C([-1,1],C^2(S^1))$; actually, we get that $||W_\mu||_{C([-1,1],C^2(S^1))}$ is bounded by a constant independent on μ. This prompts us to define, for $\mu\in\mathcal{M}_1(\tilde{\mu}_{-1},\tilde{\mu}_1)$,

$$L_{\mu,c}\colon[-1,1]\times S^1\times\mathbf{R}\to\mathbf{R}$$

by

$$L_{\mu,c}(t,x,\dot{x})=\frac{1}{2}|\dot{x}|^2-c\dot{x}-V(t,x)-W_\mu(t,x).$$

An important case is that in which $\mu_t=(\sigma_t,\dot{\sigma}_t)_\sharp\nu_0$, with σ c-minimal; we saw in Sect. 3 that, in this case, $\sigma\in C^2(\mathbf{R},L^2(I))$; actually, there is $C_1>0$ such that, for any c-minimal σ, $||\sigma||_{C^2(\mathbf{R},L^2(I))}\le C_1$; as a consequence, $||W_\mu||_{C^2([-1,1]\times S^1)}\le C_2$, with C_2 not depending on the c-minimal σ.

To avoid proving theorems about compactness, a small haircut on transfer measures is necessary.

Definition. We define $A(\tilde{\mu}_{-1},\tilde{\mu}_1)$ as the smallest R for which $B_R:=[-R,R]$ contains the supports of both $\tilde{\mu}_{-1}$ and $\tilde{\mu}_1$.

For $R\geq A(\tilde{\mu}_{-1},\tilde{\mu}_1)$, let us call $\mathcal{M}_1^R(\tilde{\mu}_{-1},\tilde{\mu}_1)$ the set of the elements of $\mathcal{M}_1(\tilde{\mu}_{-1},\tilde{\mu}_1)$ which are supported in $[-1,1]\times B_R\times B_R$. Note that $\mathcal{M}_1^R(\tilde{\mu}_{-1},\tilde{\mu}_1)$ is a compact subset of $\mathcal{M}_1(\tilde{\mu}_{-1},\tilde{\mu}_1)$. It follows from [3] that, for R large enough, $\mathcal{M}_1^R(\tilde{\mu}_{-1},\tilde{\mu}_1)$ is not empty.

Lemma 5.1. *Let $\delta\in\mathcal{M}_1(\tilde{\mu}_{-1},\tilde{\mu}_1)$, and let K be so large that $\mathcal{M}_1^K(\tilde{\mu}_{-1},\tilde{\mu}_1)$ is not empty.*

1) *Then, there is $\bar{\mu}\in\mathcal{M}_1^K(\tilde{\mu}_{-1},\tilde{\mu}_1)$ such that*
$$\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta,c}(t,q,v)\mathrm{d}\bar{\mu}(t,q,v)$$
$$=\inf\left\{\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta,c}(t,q,v)\mathrm{d}\mu(t,q,v)\ :\ \mu\in\mathcal{M}_1^K(\tilde{\mu}_{-1},\tilde{\mu}_1)\right\}.$$
2) *The set of all the measures $\bar{\mu}$ which satisfy the formula above is a compact, convex set C_δ of $\mathcal{M}_1^K(\tilde{\mu}_{-1},\tilde{\mu}_1)$.*
3) *There is $R>0$, depending on $A(\tilde{\mu}_{-1},\tilde{\mu}_1)$ but not on δ, such that, for $K\geq R$, C_δ does not depend on K.*

Proof. We only sketch the standard proof of this lemma. We saw above that $\mathcal{M}_1^K(\tilde{\mu}_{-1},\tilde{\mu}_1)$ is compact; thus, point 1) is a standard consequence of the fact that the functional
$$:\mu\rightarrow\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta,c}(t,q,v)\mathrm{d}\mu(t,q,v)$$
is l. s. c. (see for instance [4]). We prove point 2); since $\mathcal{M}_1^K(\tilde{\mu}_{-1},\tilde{\mu}_1)$ is compact, it suffices to prove that C_δ is convex and closed; this is again a consequence of the fact that the map displayed above is linear and l.s.c..

As for point 3), we recall the fact, proven in [3], that any minimal $\bar{\mu}$ is supported in a set of orbits q minimal for $L_{\delta,c}$; thus, the thesis follows if we prove that there is $R>0$, independent on δ, such that any minimal q, connecting a point in the support of $\tilde{\mu}_{-1}$ with another in the support of $\tilde{\mu}_1$, satisfies $(q(t),\dot{q}(t))\in B_R\times B_R$. Since $q(\pm1)$ lie in the supports of $\tilde{\mu}_{\pm1}$, i.e. in the interval $B_{A(\tilde{\mu}_{-1},\tilde{\mu}_1)}$, it suffices a bound on $\dot{q}$: we shall prove that q satisfies $|\dot{q}(t)|\leq C$ for a constant C depending only on $A(\tilde{\mu}_{-1},\tilde{\mu}_1)$.

Actually, with the same argument of Lemma 3.4, we can prove that there is $C>0$ such that, if q is minimal for $L_{\delta,c}$ and connects two points in $B_{A(\tilde{\mu}_{-1},\tilde{\mu}_1)}$, then $|\dot{q}|\leq C$. The constant C depends only on $||V+W_\delta||_{C([-1,1],C^2(\mathbf{T}^p))}$ (which we know to be bounded independently on δ) and on $A(\tilde{\mu}_{-1},\tilde{\mu}_1)$ (the maximal distance of the points to be connected), ending the proof.

Definition. We settle a bit of notation: from now on, R will be the constant of point 3) of the lemma above.

If $\mu\in\mathcal{M}_1^R(\tilde\mu_{-1},\tilde\mu_1)$ is minimal in point 1) of Lemma 5.1, we call it a minimal transfer measure for $L_{\delta,c}$.

Let $\mathcal{C}$ denote the class of all closed, convex subsets of $\mathcal{M}_1^R(\tilde\mu_{-1},\tilde\mu_1)$; by point 2) of Lemma 5.1, we have a map

$$\Phi\colon \mathcal{M}_1^R(\tilde\mu_{-1},\tilde\mu_1)\rightarrow\mathcal{C}$$

which brings δ into the set C_δ of minimal transfer measures for $L_{\delta,c}$.

We assert that the set valued map Φ is upper semicontinuous, i.e. that, if $\delta_n\rightarrow\delta$, if μ_n is minimal for $\mathcal{L}_{\delta_n,c}$ and $\mu_n\rightarrow\mu$, then μ is minimal for $\mathcal{L}_{\delta,c}$. We sketch the standard proof of this; for starters, since $\mathcal{M}_1^R(\tilde\mu_{-1},\tilde\mu_1)$ is closed in $\mathcal{M}_1(\tilde\mu_{-1},\tilde\mu_1)$, we get that $\mu\in\mathcal{M}_1^R(\tilde\mu_{-1},\tilde\mu_1)$. It is proven in [3] that the function

$$:\mu\rightarrow\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}[\frac{1}{2}|v|^2-c\cdot v-V(t,q)]\mathrm{d}\mu(t,q,v)$$

is l. s. c.. Moreover, since $\delta_n\rightarrow\delta$, we have that $W_{\delta_n}\rightarrow W_\delta$ uniformly; these two facts imply that

$$\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta,c}(t,q,v)\mathrm{d}\mu(t,q,v)\le\liminf_{n\rightarrow+\infty}\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta_n,c}(t,q,v)\mathrm{d}\mu_n(t,q,v).$$

Let us suppose by contradiction that μ is not a minimal transfer measure for $L_{\delta,c}$; by the formula above, this means that there is $\bar\mu\in\mathcal{M}_1^R(\tilde\mu_{-1},\tilde\mu_1)$ such that

$$\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta,c}(t,q,v)\mathrm{d}\bar\mu(t,q,v)<\liminf_{n\rightarrow+\infty}\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta_n,c}(t,q,v)\mathrm{d}\mu_n(t,q,v).$$

Since $W_{\delta_n}\rightarrow W_\delta$ uniformly, the formula above implies that, for n large enough, the inequality below holds.

$$\begin{aligned}&\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta_n,c}(t,q,v)\mathrm{d}\bar\mu(t,q,v)\\&\quad=\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta,c}(t,q,v)\mathrm{d}\bar\mu(t,q,v)-\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}[W_{\delta_n}-W_\delta]\mathrm{d}\bar\mu(t,q,v)\\&\quad<\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\delta_n,c}(t,q,v)\mathrm{d}\mu_n(t,q,v).\end{aligned}$$

This contradicts the fact that μ_n is minimal for $L_{\delta_n,c}$, i.e. that $\mu_n\in\Phi(\delta_n)$.

Since $\mathcal{M}_1^R(\tilde\mu_1,\tilde\mu_2)$ is compact and the map Φ is upper semicontinuous, we can apply the Ky Fan theorem [11] and find μ such that $\mu\in\Phi(\mu)$; let us gather in a set S the measures μ for which $\mu\in\Phi(\mu)$. Again from the fact that Φ is u. s. c., it follows that S is a closed set of $\mathcal{M}_1^R(\tilde\mu_{-1},\tilde\mu_1)$; thus, it is compact, and we can find $\bar\mu\in S$ such that

$$\begin{aligned}a(\tilde\mu_1,\tilde\mu_2)\colon&=\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\frac{1}{2}\bar\mu,c}(t,x,v)\mathrm{d}\bar\mu(t,x,v)\\&=\inf_{\mu\in S}\int_{[-1,1]\times\mathbf{R}\times\mathbf{R}}L_{\frac{1}{2}\mu,c}(t,x,v)\mathrm{d}\mu(t,x,v).\end{aligned}$$

We need a definition.

Definition. We shall say that σ is minimal for $\mathcal{L}_c$ if it is minimal among all A. C. curves γ with $\gamma_{\pm 1} = \sigma_{\pm 1}$. This is a weaker notion that the c-minimality of Sect. 1, where we only required that $\gamma_{\pm 1} - \sigma_{\pm 1} \in L^2_{\mathbf{Z}}$. In other words, now we are considering particles on $\mathbf{R}$, not on S^1.

The next lemma gives us the relation between the minimal transfer measures μ and the minimal paths σ; it can be seen as a different proof of formula (12) of [8]. Note the quirk of notation: in the definition of $a(\tilde{\mu}_1, \tilde{\mu}_2)$ we are minimizing the integral of $L_{\frac{1}{2}\mu,c}$, but over all the minimal transfer measures μ for $L_{\mu,c}$. We shall see the reasons for this factor $\frac{1}{2}$ in the proof below.

Lemma 5.2. 1) *Let $\bar{\sigma} \in AC_{mon}(-1,1)$ be minimal for $\mathcal{L}_c$, and let us consider the two measures on $\mathbf{R}$ $\tilde{\mu}_{-1} = (\bar{\sigma}_{-1})_\sharp \nu_0$, $\tilde{\mu}_1 = (\bar{\sigma}_1)_\sharp \nu_0$. Then,*

$$a(\tilde{\mu}_{-1}, \tilde{\mu}_1) = \int_{-1}^{1} \mathcal{L}_c(t, \bar{\sigma}_t, \dot{\bar{\sigma}}_t) \mathrm{d}t \tag{5.2}$$

2) *Moreover, if $a(\tilde{\mu}_{-1}, \tilde{\mu}_1)$ is attained on μ, then μ is induced by a minimal parametrization σ_t; vice-versa, if σ_t is a minimal parametrization, then $a(\tilde{\mu}_{-1}, \tilde{\mu}_1)$ is attained on the measure induced by σ_t.*

Proof. We begin with point 1). For $M_{-1}, M_1 \in Mon$, we define

$$b(M_{-1}, M_1) = \min \left\{ \int_{-1}^{1} \mathcal{L}_c(t, \sigma_t, \dot{\sigma}_t) \mathrm{d}t \, : \, \sigma_{-1} = M_{-1}, \quad \sigma_1 = M_1 \right\}.$$

Thus, we have to prove that

$$b(\sigma_{-1}, \sigma_1) = a(\tilde{\mu}_1, \tilde{\mu}_2).$$

We begin to show that

$$b(\bar{\sigma}_{-1}, \bar{\sigma}_1) \leq a(\tilde{\mu}_1, \tilde{\mu}_2). \tag{5.3}$$

Let μ minimize in the definition of $a(\tilde{\mu}_{-1}, \tilde{\mu}_1)$; then, $\mu \in S$, which implies that μ is a minimal transfer measure for $L_{\mu,c}$.

We assert that there is a parametrization $\sigma \in AC_{mon}(-1,1)$ such that $\mu = \frac{1}{2}\mathcal{L}^1 \otimes (\sigma_t, \dot{\sigma}_t)_\sharp \nu_0$; note that this implies, by the definition of push-forward, that

$$\int_{-1}^{1} \mathcal{L}_c(t, \sigma_t, \dot{\sigma}_t) \mathrm{d}t = \int_{[-1,1]\times\mathbf{R}\times\mathbf{R}} L_{\frac{1}{2}\mu,c}(t,x,v) \mathrm{d}\mu(t,x,v)$$

from which (5.3) follows. Once we shall have proven that $b(\sigma_{-1}, \sigma_1) = a(\tilde{\mu}_{-1}, \tilde{\mu}_1)$, the formula above will yield part of point 2), i.e. that the minimal measure μ is induced by a minimal parametrization σ.

The proof of the assertion is essentially contained in Sect. 4.2 of [3], which says that, if μ is a minimal transfer measure for a Lagrangian, say $L_{\mu,c}$, then μ is supported on a bunch of minimal orbits of $L_{\mu,c}$, which can be easily parametrized.

More precisely, let ψ^t_s be the Euler–Lagrange flow of $L_{\mu,c}$: ψ^t_s brings an initial condition (x,v) at time s into its evolution at time t. By Sect. 4.2 of [3], there is a probability measure $\tilde{\mu}_0$ on $\mathbf{R}$ and a Lipschitz function $v \colon \mathbf{R} \to \mathbf{R}$ such that, setting

$$\mu_t = (\psi_0^t)_\sharp (id, v)_\sharp \tilde{\mu}_0,$$

then $\mu = \frac{1}{2}\mathcal{L}^1 \otimes \mu_t$. Take the monotone map σ_0 which brings ν_0, the Lebesgue measure on the parameter space $[0,1]$, into μ_0 and set

$$(\sigma_t, \dot\sigma_t) = \psi_0^t \circ (id, v) \circ \sigma_0.$$

By the two formulas above, it is immediate that $\mu_t = (\sigma_t, \dot\sigma_t)_\sharp \nu_0$, and this proves the assertion.

We prove the inequality opposite to (5.3). Let $b(M_{-1}, M_1)$ be attained on σ; let $\tilde\mu_{\pm 1} = (\sigma_{\pm 1})_\sharp \nu_0$, $\mu_t = (\sigma_t, \dot\sigma_t)_\sharp \nu_0$ and $\mu = \frac{1}{2}\mathcal{L}^1 \otimes \mu_t$.

We begin to prove that, for ν_0 a.e. x, the orbit $: t \to \sigma_t x$ is minimal, for fixed endpoints. To show this, let x_0 be a Lebesgue point for both maps $: x \to (\sigma_{\pm 1} x, \dot\sigma_{\pm 1} x)$, i.e. let

$$\lim_{\epsilon \to 0+} \frac{1}{2\epsilon} \int_{x_0-\epsilon}^{x_0+\epsilon} (\sigma_{\pm 1} x, \dot\sigma_{\pm 1} x) \mathrm{d}x = (\sigma_{\pm 1} x_0, \dot\sigma_{\pm 1} x_0).$$

We write

$$\begin{aligned}
&\int_{-1}^{1} \mathrm{d}t \left[\int_I \frac{1}{2} |\dot\sigma_t x|^2 \mathrm{d}x - \int_I V(t, \sigma_t x) \mathrm{d}x - \frac{1}{2} \int_{I \times I} W(\sigma_t x - \sigma_t y) \mathrm{d}x \mathrm{d}y \right] \\
&\quad = \int_{-1}^{1} \mathrm{d}t \left[\int_{I \setminus [x_0-\epsilon, x_0+\epsilon]} \frac{1}{2} |\dot\sigma_t x|^2 \mathrm{d}x - \int_{I \setminus [x_0-\epsilon, x_0+\epsilon]} V(t, \sigma_t x) \mathrm{d}x \right. \\
&\qquad \left. - \frac{1}{2} \int_{(I \setminus [x_0-\epsilon, x_0+\epsilon])^2} W(\sigma_t x - \sigma_t y) \mathrm{d}x \mathrm{d}y \right] \qquad (5.4a) \\
&\qquad + \int_{-1}^{1} \mathrm{d}t \left[\int_{[x_0-\epsilon, x_0+\epsilon]} \frac{1}{2} |\dot\sigma_t x|^2 \mathrm{d}x - \int_{[x_0-\epsilon, x_0+\epsilon]} V(t, \sigma_t x) \mathrm{d}x \right. \\
&\qquad - \int_{[x_0-\epsilon, x_0+\epsilon] \times (I \setminus [x_0-\epsilon, x_0+\epsilon])} W(\sigma_t x - \sigma_t y) \mathrm{d}x \mathrm{d}y \\
&\qquad \left. + \frac{1}{2} \int_{[x_0-\epsilon, x_0+\epsilon]^2} W(\sigma_t x - \sigma_t y) \mathrm{d}x \mathrm{d}y \right]. \qquad (5.4b)
\end{aligned}$$

Note that, if $x \in [x_0 - \epsilon, x_0 + \epsilon]$, the trajectory $: t \to \sigma_t x$ doesn't appear in (5.4a) of the formula above, but only in (5.4b); thus, the curve parametrized by $\sigma_t|_{[x_0-\epsilon, x_0+\epsilon]}$ minimizes the integral in (5.4b) for fixed boundary conditions. We assert that this implies that $: t \to \sigma_t x_0$ is minimal for $L_{\mu,c}$, endpoints fixed.

Indeed, let us suppose by contradiction that there is $: t \to q(t)$, with the same extrema, such that

$$\int_{-1}^{1} L_{\mu,c}(t, q(t), \dot q(t)) \mathrm{d}t < \int_{-1}^{1} L_{c,\mu}(t, \sigma_t x_0, \dot\sigma_t x_0) \mathrm{d}t. \qquad (5.5)$$

For $\epsilon > 0$, define $\gamma = \gamma(\epsilon)$ as the largest one among $\sigma_1(x_0 + \epsilon) - \sigma_1(x_0)$, $\sigma_1(x_0) - \sigma_1(x_0 - \epsilon)$, $\sigma_{-1}(x_0 + \epsilon) - \sigma_1(x_0)$ and $\sigma_{-1}(x_0) - \sigma_1(x_0 - \epsilon)$; since x_0 is a Lebesgue point of $\sigma_{\pm 1}$, we have that $\gamma \to 0$ as $\epsilon \to 0$.

For $x \in [x_0-\epsilon, x_0+\epsilon]$, we define

$$\tilde\sigma_t x = \begin{cases} \frac{t+1-\gamma}{-\gamma}\sigma_{-1}x + \frac{t+1}{\gamma}q(t) & -1 \le t \le -1+\gamma \\ q(t) & -1+\gamma \le t \le 1-\gamma \\ \frac{t-1}{\gamma}q(t) + \frac{t-1+\gamma}{\gamma}\sigma_1 x & 1-\gamma \le t \le 1. \end{cases}$$

If $x \notin [x_0-\epsilon, x_0+\epsilon]$, we set $\tilde\sigma_t x = \sigma_t x$. Since σ_t solves $(ODE)_{Lag}$, for ν_0 a.e. x $: t \to \sigma_t x$ is an orbit of $L_{\mu,c}$; in particular, it is C^2 and depends continuously, in the C^2 topology, from the initial condition $(\sigma_0 x, \dot\sigma_0 x)$. Using this, the fact that x_0 is a Lebesgue point of $: x \to (\sigma_t x, \dot\sigma_t x)$ and formula (5.5), it is easy to see that

$$\begin{aligned} &\int_{-1}^{1}\mathrm{d}t\Bigg[\int_{[x_0-\epsilon,x_0+\epsilon]}\frac{1}{2}|\dot{\tilde\sigma}_t x|^2\mathrm{d}x - \int_{[x_0-\epsilon,x_0+\epsilon]}V(t,\tilde\sigma_t x)\mathrm{d}x \\ &\quad -\int_{[x_0-\epsilon,x_0+\epsilon]\times(I\setminus[x_0-\epsilon,x_0+\epsilon])}W(\tilde\sigma_t x-\tilde\sigma_t y)\mathrm{d}x\mathrm{d}y \\ &\quad +\frac{1}{2}\int_{[x_0-\epsilon,x_0+\epsilon]^2}W(\tilde\sigma_t x-\tilde\sigma_t y)\mathrm{d}x\mathrm{d}y\Bigg] \\ &< \int_{-1}^{1}\mathrm{d}t\Bigg[\int_{[x_0-\epsilon,x_0+\epsilon]}\frac{1}{2}|\dot\sigma_t x|^2\mathrm{d}x - \int_{[x_0-\epsilon,x_0+\epsilon]}V(t,\sigma_t x)\mathrm{d}x \\ &\quad -\int_{[x_0-\epsilon,x_0+\epsilon]\times(I\setminus[x_0-\epsilon,x_0+\epsilon])}W(\sigma_t x-\sigma_t y)\mathrm{d}x\mathrm{d}y \\ &\quad +\frac{1}{2}\int_{[x_0-\epsilon,x_0+\epsilon]^2}W(\sigma_t x-\sigma_t y)\mathrm{d}x\mathrm{d}y\Bigg] \end{aligned}$$

contradicting the fact that $\sigma_t|_{[x_0-\epsilon,x_0+\epsilon]}$ minimizes the integral in (5.4b).

Thus, for ν_0 a.e. x the orbit $: t \to \sigma_t x$ is minimal for $L_{\mu,c}$; note that we have lost the factor $\frac{1}{2}$ in the potential of $\mathcal{L}_c$ (the reason for this in in formula (5.4a)), and this explains the quirk of notation mentioned above.

The fact that, for ν_0 a.e. x the orbit $: t \to \sigma_t x$ is minimal for $L_{\mu,c}$, together with the fact that the map $: \sigma_{-1}x \to \sigma_1 x$ brings $\tilde\mu_{-1}$ into $\tilde\mu_1$, implies by [3] that $\mu = \frac{1}{2}\mathcal{L}^1 \otimes \mu_t$ is a minimal transfer measure. Actually, this is a standard fact of transport theory: in dimension one, if minimal characteristics cannot intersect, then the unique monotone map bringing $\tilde\mu_{-1}$ into $\tilde\mu_1$ is a minimal transfer map. As a consequence, we get the other half of point 2): if σ is minimal, then the measure induced by σ is a minimal transfer measure.

Proposition 5.3. *Let $\sigma \in AC_{mon}(\mathbf{R})$ be c-minimal. Let us consider the set*

$$A = \{(t,\sigma_t x) :\ t \in \mathbf{R},\quad x \in I\} \subset \mathbf{R}\times S^1.$$

Then, the map

$$\Gamma\colon A \to \mathbf{R},\qquad \Gamma\colon (t,\sigma_t x) \to \dot\sigma_t x$$

is Lipschitz.

Proof. Since σ is c-minimal, it is also minimal for $\mathcal{L}_c$ in the weaker sense defined above; in particular, Lemma 5.2 holds.

Let $\tilde{\mu}_t = (\sigma_t)_\sharp \nu_0$, and let $\mu = \frac{1}{2}\mathcal{L}^1 \otimes (\sigma_t, \dot{\sigma}_t)_\sharp \nu_0$; by Lemma 5.2, μ is a minimal transport measure for $L_{\mu,c}$ between $\tilde{\mu}_{k-1}$ and $\tilde{\mu}_{k+1}$, $k \in \mathbf{Z}$. Now we can apply the addendum in Section 1.3 of [3], which says that Γ is L-Lipschitz for $t \in [k - \frac{1}{2}, k + \frac{1}{2}]$; the Lipschitz constant L depends only on the C^2 norm of V and W_μ, and on the radius of the smallest ball containing the supports of $\tilde{\mu}_{k-1}$ and $\tilde{\mu}_{k+1}$. Since we are free to translate by an integer, L depends on the diameter of the union of the supports of $\tilde{\mu}_{k-1}$ and $\tilde{\mu}_{k+1}$. We note that V is fixed, while $||W_\mu||_{C^2(\mathbf{R}\times\mathbf{R})}$ is bounded, since $W \in C^2(S^1)$ and the C^2 norm of σ_t, which solves $(ODE)_{Lag}$, is bounded (we saw in Lemma 3.4 that $\sup_{t\in\mathbf{R}} \|\dot{\sigma}_t\|$ is bounded). Since σ_t belongs to Mon and has bounded speed in $L^2(I)$, the diameter of the union of the supports of $\tilde{\mu}_{k-1}$ and $\tilde{\mu}_{k+1}$ is bounded, uniformly in k. Thus, the Lipschitz constant L of Γ on $[k - \frac{1}{2}, k + \frac{1}{2}]$ does not depend on k, and the thesis follows.

We want to study the regularity of periodic minimal measures of irrational rotation number. We consider the Lagrangian

$$\mathcal{L}^\epsilon(t, \sigma, v) = \frac{1}{2}\|v\|^2 - \epsilon\mathcal{V}(t, \sigma) - \epsilon\mathcal{W}(\sigma)$$

where $\mathcal{V}$ and $\mathcal{W}$ are defined as in Sect. 1; the only difference is that we ask that the potentials V and W are C^k for some large k which we shall determine in the following.

We want to study

$$\min\left\{\int_0^1 \mathcal{L}^\epsilon(t, \sigma_t, \dot{\sigma}_t)\mathrm{d}t \;:\; \sigma \in \mathcal{P}_\omega\right\} \tag{5.6}$$

where by $\mathcal{P}_\omega$ we denote the set of those $\sigma \in AC_{mon}([0, 1])$ such that

- $\sigma_0 \simeq \sigma_1$ in the sense of Sect. 1, or σ projects to a periodic curve on $\mathbf{S}$; in other words,

$$(\sigma_0)_\sharp \nu_0 = (\sigma_1)_\sharp \nu_0. \tag{5.7}$$

- Moreover, we ask that the rotation number of σ is ω; in other words,

$$\int_0^1 \langle 1, \dot{\sigma}_t\rangle \mathrm{d}t = \omega. \tag{5.8}$$

We note that $\mathcal{P}_\omega$ is closed in $AC_{mon}([0, 1])$: we forego the easy proof that (5.7) and (5.8) are closed under uniform convergence. Moreover, $\mathcal{P}_\omega$ is not empty, since it is easy to see that $\sigma_t x = x + \omega t$ is periodic in $\mathbf{S}$ (actually, it is constant) and has rotation number ω. As a consequence, we were justified in writing min in (5.6).

Let $\gamma, \tau > 0$; we say that $\omega \in \mathbf{R}$ is (γ, τ)-diophantine if

$$|\omega q - p| \geq \frac{\gamma}{q^\tau} \quad \text{if} \quad (q, p) \in (\mathbf{N} \setminus \{0\}) \times \mathbf{Z}.$$

We want to prove the following.

Theorem 5.4. *Let ω be (γ,τ)-diophantine, and let σ minimize in (5.6). Then, there is $k_0(\gamma,\tau)>0$ such that, if V and W are C^k with $k\ge k_0(\gamma,\tau)$ and ϵ is small enough, the measure on $[0,1]\times S^1\times\mathbf{R}$ given by $\mu:=\mathcal{L}^1\otimes(\sigma_t,\dot\sigma_t)_\sharp\nu_0$ is the push-forward of the Lebesgue measure on $S^1\times S^1$ by a C^1 map $\Phi\colon(t,x)\to(t,\phi_1(t,x),\phi_2(t,x))$.*

Proof. Let σ be minimal in (5.6), and let $\mu=\mathcal{L}^1\otimes(\sigma_t,\dot\sigma_t)_\sharp\nu_0$; let the potential $W_\mu(x,t)$ and the Lagrangian $L_{\mu,0}(t,x,\dot x)$ be defined at the beginning of this section. Since $\sigma_0\simeq\sigma_1$ by (5.7), the definition of W_μ implies that $W_\mu(1,x)=W_\mu(0,x)$; thus $L_{\mu,0}$ is 1-periodic in time.

Since σ is minimal, it is a periodic solution of $(ODE)_{Lag}$; since the potentials V and W are C^k, we get that $\sigma\in C^{k+1}(S^1,L^2(I))$; as a consequence, $W_\mu\in C^k(S^1\times S^1)$, while $V\in C^k(S^1\times S^1)$ by hypothesis. Using again the fact that σ is minimal, we see as in Lemma 3.4 that $||\dot\sigma||_{C^0(\mathbf{R},L^2(I))}$ is bounded, independently on $\epsilon\in[0,1]$; differentiating in $(ODE)_{Lag}$, we get that the higher derivatives are bounded too. Thus, $||\sigma||_{C^k(\mathbf{R},L^2(I))}$ is bounded independently on $\epsilon\in[0,1]$; as a consequence, $||W_\mu||_{C^k(\mathbf{R}\times S^1)}$ is bounded independently on ϵ. In particular, $||\epsilon V+\epsilon W_\mu||_{C^k(S^1\times S^1)}$ tends to zero as $\epsilon\to 0$; thus, by [15], for ϵ small and k large enough, $L_{\mu,0}$ has a KAM torus of rotation number ω.

We are supposing that σ is minimal in (5.6); by periodicity, this implies that σ is minimal, with fixed boundary conditions, on each interval $[t_0,t_0+1]$. From Lemma 5.2 we gather that, for a.e. $x\in I$, $\sigma_t x$ is minimal for $L_{\mu,0}$ on each interval $[t_0,t_0+1]$ for fixed boundary conditions; in particular, $:t\to\sigma_t x$ is an orbit of $L_{\mu,0}$. This immediately implies that μ is invariant by the Euler–Lagrange flow of $L_{\mu,0}$. Moreover, (5.8), and the fact that $\sigma_t\in Mon$, imply as in Sect. 3 that

$$\lim_{t\to+\infty}\frac{\sigma_t x-\sigma_0 x}{t}=\omega \qquad \forall x\in I. \tag{5.9}$$

We saw above that there is $k_0(\tau,\gamma)$ such that, if $k\ge k_0(\tau,\gamma)$ and ϵ is small enough, then $L_{\mu,0}$ has a KAM torus of frequency ω. In other words, there is a C^1 map $\Phi\colon S^1\times S^1\to S^1\times S^1\times\mathbf{R}$ such that, denoting as usual by ψ_s the Euler–Lagrange flow of $L_{\mu,0}$,

$$\psi_s\circ\Phi(t,x)=\Phi(t+s,x+\omega s),$$

or Φ conjugates the rotation on $S^1\times S^1$ given by $:(t,x)\to(t+s,x+\omega s)$ to the Euler–Lagrange flow on the image of Φ. We have to show that μ is the push-forward by Φ of the Lebesgue measure on $S^1\times S^1$. Since the KAM torus is conjugate to an irrational rotation, it supports just one invariant measure, i.e. the push-forward of Lebesgue. Thus, it suffices to prove that μ, which we proved to be invariant, is supported on the KAM torus; equivalently, that, for each $x\in I$, the orbit $(t,\sigma_t x,\dot\sigma_t x)$ lies on the KAM torus. This is a consequence of (5.9) and of the fact that $:t\to\sigma_t x$ is an orbit. We explain why.

By [12] and [13], we know that the KAM torus is a graph; in other words, there is a Lipschitz map

$$v\colon S^1\times S^1\to\mathbf{R}$$

such that the image of Φ coincides with the graph of v. Moreover, the two sets

$$A_- = \{(t, q, \dot{q}) : \dot{q} < v(t, q)\}, \qquad A_+ = \{(t, q, \dot{q}) : \dot{q} > v(t, q)\}$$

are invariant by the flow ϕ_s.

Let us call $\mathcal{T}$ the KAM torus of frequency ω; it is standard that both A_- and A_+ contain sequences $\mathcal{T}_-^n$ and $\mathcal{T}_+^n$ respectively of KAM tori which, as $n \to +\infty$, converge to $\mathcal{T}$. Since no orbit can cross a KAM torus, we get that, for any $z \in I$, the closure C of

$$\{(t, \sigma_t z, \dot{\sigma}_t z)\}_{t \in \mathbf{R}}$$

either is contained in $\mathcal{T}$, or in one ot the two invariant sets $A_\pm$, and at a finite distance from $\mathcal{T}$. Let us suppose by contradiction that, for some $z \in I$, C is not contained in $\mathcal{T}$; to fix ideas, let $C \subset A_+$.

Let us denote by $q_{x,t}(s)$ the orbit on the KAM torus such that $q_{x,t}(t) = x$. We assert two facts:

1) if $x' > x$, then there is a positive number $\delta(x' - x)$, only depending on $x' - x$, such that

$$q_{x',t}(s) \geq q_{x,t}(s) + \delta(x' - x) \qquad \forall s \in \mathbf{R}.$$

2) There is $\epsilon > 0$, independent on $t \in \mathbf{R}$, such that, if $\sigma_t z = x$, and if $q_{x,t}(s) = q_{x,t}(t) + 1 = x + 1$, then $\sigma_s z \geq q_{x,t}(s) + 1 + \epsilon$.

Before proving 1) and 2), we show how they imply the thesis. Let $z \in I$ and C be as above; we set $\sigma_0 z = x$; by 2), we see that, if $q_{x,0}(s) = x + 1$, then $\sigma_s z \geq x + 1 + \epsilon$. Now we set $x' = \sigma_s z$; applying again point 2), we have that, if $q_{x',s}(s_1) = x' + 1$, then $\sigma_{s+s_1} z \geq x' + 1 + \epsilon$. By point 1), this means that $\sigma_{s+s_1} z \geq q_{x,0}(s + s_1) + \delta(\epsilon) + \epsilon$. Iterating, we have that

$$\sigma_{s+s_1+\cdots+s_n} z \geq q_{x,0}(s + s_1 + \cdots + s_n) + (n-1)\delta(\epsilon) + \epsilon.$$

This fact implies the inequality below; the equality comes from the fact that the KAM torus has rotation number ω.

$$\lim_{n \to +\infty} \frac{\sigma_{s+s_1+\cdots+s_n} z}{s + s_1 + \cdots + s_n} \geq \lim_{n \to +\infty} \frac{q_{x,0}(s + s_1 + \cdots + s_n)}{s + s_1 + \cdots + s_n} + \delta(\epsilon) = \omega + \delta(\epsilon).$$

We have reached a contradiction with (5.9).

We prove the two assertions above. To prove point 1), we begin to note that

$$\Phi(t, x) = (t, \Phi_x(t, x), \Phi_v(t, x)).$$

Now point 1) is true for the rotation : $(t, x) \to (t + s, x + \omega s)$, with $\delta(x' - x) = x' - x$; since Φ is a conjugation, we have that $q_{x,t}(s) = \Phi_x(t + s, x + \omega s)$; thus, it suffices to show that the map : $x \to \Phi_x(t, x)$ is strictly monotone for all t. This follows since, by the KAM theorem, the map : $(t, x) \to (t, \Phi_x(t, x))$ is close to the identity.

Since $C \subset A_+$ is at finite distance from $\mathcal{T}$, we get from the definition of A_+ that there is $a > 0$ such that

$$\dot{\sigma}_t z \geq v(\sigma_t z) + a \qquad \forall x \in I, \quad \forall t \in \mathbf{R}. \tag{5.10}$$

Let now $x \in I$, $t \in \mathbf{R}$ and let $q_{x,t}(s)$ be the orbit of the KAM torus with initial conditions $q_{x,t}(t) = \sigma_t z = x$ and $\dot{q}_{x,t}(t) = v(t,x)$. By (5.10), $\sigma_s z > q_{x,t}(s)$ for $s-t$ positive and small; let (t,T) be the largest interval on which $\sigma_s z > q_{x,t}(s)$. We assert that $T = +\infty$. Indeed, if T were finite, then we would have $\sigma_T z = q_{x,t}(T)$; together with $\sigma_s z > q_{x,t}(s)$ for $s \in (0,T)$, this implies that $\dot{\sigma}_T z \le \dot{q}_{x,t}(T)$, contradicting (5.10). As a consequence, if s is such that $q_{x,t}(s) = q_{x,t}(t) + 1 = \sigma_t z + 1$, then $\sigma_s z \ge \sigma_t z + 1 + \epsilon$, for some $\epsilon > 0$ independent on $t \in \mathbf{R}$.

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Received: 13 April 2012.
Accepted: 11 December 2012.