A CONVERGENCE THEOREM ON A KIND OF BAND-LIMITED FUNCTIONS WITH APPLICATIONS*

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Abstract

A theorem on the convergence of a particular sequence of bandlimited functions is proved. As its applications, the convergence of a speed up error energy reduction algorithm for extrapolating bandlimited functions in noiseless cases and the convergence of an iterative algorithm to obtain estimations of bandlimited functions in noise cases are derived. Both algorithms are the improved versions of the Papoulis-Gercheberg algorithm.

§ 1. Introduction

$f(t)$ is called a σ-bandlimited function if $F(\omega)=0$ for $|\omega|>\sigma$ and

$$\int_{-\infty}^{+\infty}|f(t)|^2dt<\infty$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

The σ-bandlimited functions form a Hilbert space with norm

$$\|f\|^2=\int_{-\infty}^{+\infty}|f(t)|^2dt \tag{1}$$

which is denoted by L_2^σ. Obviously, it is a subspace of L_2. All the quadric integrable functions on $[-T, T]$ form another Hilbert space L_2^T with norm

$$\|f\|_T^2=\int_{-T}^{T}|f(t)|^2dt. \tag{2}$$

By the elementary properties of Hilbert space which is isomorphic to L_2[1], $\|f_n-f\|\to 0$ in L_2^σ if and only if f_n converges to f in the weak sense, i.e., $f_n\xrightarrow{w}f$, and $\|f_n\|\to\|f\|$. Furthermore, $f_n\xrightarrow{w}f$ is equivalent to that $\{\|f_n\|,\ n=0,\ 1,\ \cdots\}$ are bounded and f_n converges to f in coordinates. Because, when f_n, $f\in L_2^\sigma$, they can be expanded as

$$f_n(t)=\sum_{k=0}^{\infty}a_k^{(n)}\phi_k(t) \tag{3}$$

and

$$f(t)=\sum_{k=0}^{\infty}a_k\phi_k(t) \tag{4}$$

where $\phi_k(t)$ are prolate spheroidal functions (PSF) (see [2] or [3]), convergence in coordinates means $a_k^{(n)}\to a_k$, $k=0,\ 1,\ \cdots$. Moreover, for any $g\in L_2^T$, it yields that

* Received June 16, 1983.

$$g(t)=\sum_{k=0}^{\infty} b_k\phi_k(t), \quad t\in[-T, T] \tag{5}$$

where $\sum_{k=0}^{\infty}\lambda_k\, b_k^2<\infty$ and λ_k are eigenvalues corresponding to $\phi_k(t)$.

For any $g\in L_2^T$, let

$$w_n(t)=\begin{cases} g(t), & |t|\leqslant T, \\ A_n f_n(t), & |t|>T, \end{cases} \tag{6}$$

and

$$\begin{cases} f_{n+1}(t)=w_n(t)*\dfrac{\sin\sigma t}{\pi t}, \\ f_0(t)=0 \end{cases} \tag{7}$$

where A_n, $n=0, 1, \cdots$, are a sequence of nonnegative constants. It is easy to verify f_n are σ-bandlimited functions. In next section it will be shown that the necessary and sufficient conditions for $\|f_n-f\|\to 0$ are $A_n\to\alpha$ and $\|f_n\|\to\|f\|$ where $0<\alpha\leqslant 1$ and

$$f(t)=\sum_{k=0}^{\infty} a_k\phi_k(t)=\sum_{k=0}^{\infty}\frac{\lambda_k b_k}{1-\alpha+\alpha\lambda_k}\phi_k(t)\in L_2^\sigma. \tag{8}$$

Obviously, if and only if $g(t)=f(t)$, α equals 1. In this case, i.e., $g(t)$ is the segment of a σ-bandlimited function, let $A_n\equiv 1$; then (6), (7) become the iteration algorithem proposed by Papoulis[4] and Gercheberg[5] for extrapolating bandlimited function. When A_n in (6) is properly chosen the speed up of convergence of original algorithm can be achieved as long as the conditions for convergence mentioned above are satisfied. In Section 3, on choosing A_n such that $\|f_{n+1}-f\|^2$ is minimized a speed up error energy reduction algorithm for extrapolating bandlimited functions is proved. By the convergence theorem in Section 2, the proposed algorithm is convergent. In addition, an iterative formula for evaluating A_n which is concerned only with data on finite interval is presented. On the other hand, it is well known that Papoulis–Gercheberg algorithm is only suitable to noiseless case. But, if $A_n\to\alpha$ and $0<\alpha<1$, (6), (7) can be used in noise case to obtain estimations of bandlimited function. This result is shown in Section 4.

§ 2. A Convergence Theorem

$f_n(t)$, $f(t)$, $g(t)$ are defined as in (7), (8) and (5) respectively.

Theorem 1. $\|f_n-f\|\to 0$ if and only if $A_n\to\alpha$ and $\|f_n\|\to\|f\|$.

Proof. Sufficiency.

Expanding $f_n(t)$ in PSF we obtain

$$f_n(t)=\sum_{k=0}^{\infty} a_k^{(n)}\phi_k(t).$$

Since

$$f_{n+1}(t)=[g(t)p_T(t)+A_n f_n(t)(1-p_T(t))]*\frac{\sin\sigma t}{\pi t} \tag{9}$$

where

$$p_T(t)=\begin{cases} 1, & |t|\leqslant T, \\ 0, & |t|>T, \end{cases} \tag{10}$$

it can be derived that

$$\begin{aligned} &a_k^{(n+1)} = \lambda_k b_k + A_n a_k^{(n)}(1-\lambda_k), \\ &a_k^{(0)} = 0, \qquad k = 0, 1, \cdots. \end{aligned} \tag{11}$$

For $k \in \{k: b_k = 0\}$, obviously, $a_k^{(n)} = a_k = 0$. For $k \in \{k: b_k \neq 0\}$, let $c_k^{(n)} = a_k^{(n)}/b_k$; then

$$c_k^{(n+1)} = \lambda_k + c_k^{(n)} A_n (1-\lambda_k). \tag{12}$$

Because $c_k^{(0)} = 0$ and A_n are nonnegative we claim

$$c_k^{(n)} > 0, \quad n = 1, 2, \cdots.$$

Furthermore,

$$\|f_n\|^2 = \sum_{\{k: b_k \neq 0\}} (c_k^{(n)})^2 b_k^2; \tag{13}$$

therefore

$$c_k^{(n)} < \frac{\|f_n\|}{|b_k|}. \tag{14}$$

From $\|f_n\| \to \|f\|$, $\{\|f_n\|, n=0, 1, \cdots\}$ are bounded, so are $\{c_k^{(n)}, n=0, 1, \cdots\}$ because of (14). Hence,

$$\overline{\lim_n}\, c_k^{(n+1)} \leqslant \alpha \overline{\lim_n}\, c_k^{(n)} - \alpha\lambda_k \overline{\lim_n}\, c_k^{(n)} + \lambda_k \tag{15}$$

i.e.,

$$\overline{\lim_n}\, c_k^{(n)} \leqslant \frac{\lambda_k}{1-\alpha+\alpha\lambda_k}. \tag{16}$$

Similarly, it can be derived that

$$\underline{\lim_n}\, c_k^{(n)} \geqslant \frac{\lambda_k}{1-\alpha+\alpha\lambda_k}. \tag{17}$$

Thus,

$$\lim_n c_k^{(n)} = \frac{\lambda_k}{1-\alpha+\alpha\lambda_k} \tag{18}$$

i.e.,

$$\lim_n a_k^{(n)} = a_k. \tag{19}$$

Now, we assert $f_n(t)$ converges to $f(t)$ in coordinates. Under the condition $\|f_n\| \to \|f\|$ the conclusion of the theorem is obtained.

Necessity.

Because $\|f_n - f\| \to 0$, it yields that $\|f_n\| \to \|f\|$ and $f_n \xrightarrow{w} f$, i.e.,

$$\lim_n a_k^{(n)} = a_k. \tag{20}$$

$g(t)$ is a nonzero function, so at least there is one b_k such that $b_k \neq 0$. For this b_k, from (20) we maintain

$$\lim_n A_n = \lim_n \frac{a_k^{(n+1)} - \lambda_k b_k}{a_k^{(n)}(1-\lambda_k)} = \alpha. \tag{21}$$ ∎

§ 3. A Speedup Error Energy Reduction Algorithm Extrapolating Bandlimited Functions

In this section, suppose $g(t)$, $t \in [-T, T]$ in (5) is the segment of a bandlimited function.

It is well known that one of the problems associated with Papoulis–Gercheberg iteration algorithm is the speed of convergence. Chamzas and Xu[6] proposed an improved version which is equivalent to choosing A_n such that it minimizes

$$I_n=\int_{-\infty}^{+\infty}|f(t)-A_nf_n(t)|^2dt \tag{22}$$

or

$$I_n=\int_{-\infty}^{+\infty}|f(t)-A_nf_n(t)|^2(1-p_T(t))dt. \tag{23}$$

Here, more exactly, the constant A_n is chosen to minimize

$$E^2_{n+1}=\|f_{n+1}(t)-f(t)\|^2. \tag{24}$$

For simplicity, hereafter we assume $f(t)$ is real. Because

$$\|f_{n+1}(t)-f(t)\|^2=\sum_{k=0}^{\infty}(A_na_k^{(n)}-a_k)^2(1-\lambda_k), \tag{25}$$

minimizing E^2_{n+1} gives

$$A_n=\frac{\sum\limits_{k=0}^{\infty}a_ka_k^{(n)}(1-\lambda_k)^2}{\sum\limits_{k=0}^{\infty}(a_k^{(n)})^2(1-\lambda_k)^2}$$

$$=\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(s)f_n(\rho)[1-p_T(s)][1-p_T(\rho)]\dfrac{\sin\sigma(\rho-s)}{\pi(\rho-s)}d\rho\,ds}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(s)f(\rho)[1-p_T(s)][1-p_T(\rho)]\dfrac{\sin\sigma(\rho-s)}{\pi(\rho-s)}d\rho\,ds}. \tag{26}$$

From (11) and (26) it is easy to find $A_n>0$ and

$$\sum_{k=0}^{\infty}(a_k^{(n)})^2(1-\lambda_k)^2\geqslant\sum_{k=0}^{\infty}a_k^2\lambda_k^2(1-\lambda_k)^2. \tag{27}$$

Theorem 2. If A_n is defined by (26) then $\|f_n-f\|\to 0$.

Proof.

$$E^2_n=\|f_n-f\|^2=E^2_{n+1}+(1-A_n)^2\sum_{k=0}^{\infty}(a_k^{(n)})^2(1-\lambda_k)^2$$

$$+\sum_{k=0}^{\infty}(a_k^{(n)}-a_k)^2\lambda_k+\sum_{k=0}^{\infty}(a_k^{(n)}-a_k)^2\lambda_k(1-\lambda_k). \tag{28}$$

Here, the orthogonal principle is used. Because $0<\lambda_k<1$, all terms in the last equation are nonnegative (28) means $E^2_n>E^2_{n+1}$, i.e., E^2_n is reduced in each iteration. Therefore the limit of E^2_n exists and

$$\lim_n(1-A_n)^2\sum_{k=0}^{\infty}(a_k^{(n)})^2(1-\lambda_k)^2=0, \tag{29}$$

$$\lim_n\sum_{k=0}^{\infty}(a_k^{(n)}-a_k)^2\lambda_k=0, \tag{30}$$

$$\lim_n\sum_{k=0}^{\infty}(a_k^{(n)}-a_k)^2\lambda_k(1-\lambda_k)=0. \tag{31}$$

From (29) and (27) it yields that

$$\lim_n A_n=1. \tag{32}$$

Since all the sequences $(d_0, d_1, \cdots, d_k, \cdots)$ where $\sum_{k=0}^{\infty} d_k^2 < \infty$ can be regarded as elements of the Hilbert space with norm$\left(\sum_{k=0}^{\infty} d_k^2 \lambda_k\right)^{1/2}$ and the convergence in norm sense implies convergence in weak sense, by (30) and (31) it is not difficult to prove

$$\int_{-\infty}^{+\infty} [f(t) - A_n f_n(t)] f_n(t)\, dt \to 0. \tag{33}$$

Furthermore, from (7) and (28) it can be asserted that $f_n(t)$ converges to $f(t)$ in weak sense in L_2^g. Therefore,

$$\lim_n \int_{-\infty}^{+\infty} f_n(t) f(t)\, dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt.$$

Finally

$$\lim_n [\|f\|^2 - \|f_n\|^2] = \lim_n \int_{-\infty}^{+\infty} [f(t) - A_n f_n(t)] f_n(t)\, dt = 0. \tag{34}$$

According to Theorem 1, (32), (34) imply $\|f_n - f\| \to 0$. ∎

Although $f(t) = g(t)$, $t \in [-T, T]$, it seems impossible to evaluate the A_n in (26) because it contains the value of $f(t)$ outside interval $[-T, T]$. In fact we can find

$$A_n = \frac{l_n - 2m_n + h_n}{p_n}, \quad n = 1, 2, \cdots \tag{35}$$

where

$$l_n = \int_{-T}^{T} |f(t)|^2 dt + A_{n-1}[l_{n-1} - m_{n-1}] \tag{36}$$

$$\begin{aligned} p_n = & \int_{-T}^{T}\int_{-T}^{T} f(s) f(\rho) \frac{\sin\sigma(s-\rho)}{\pi(s-\rho)}\, ds\, d\rho - 2\int_{-T}^{T} |f_n(t)|^2 dt \\ & + \int_{-T}^{T}\int_{-T}^{T} f_n(s) f_n(\rho) \frac{\sin\sigma(s-\rho)}{\pi(s-\rho)}\, ds\, d\rho \\ & + 2A_{n-1}(m_{n-1} - h_{n-1}) + A_{n-1}^2 p_{n-1}, \quad n = 1, 2, \cdots \end{aligned} \tag{37}$$

$$m_n = \int_{-T}^{T} f_n(t) f(t)\, dt, \quad n = 0, 1, \cdots \tag{38}$$

$$h_n = \int_{-T}^{T}\int_{-T}^{T} f_n(\rho) f(s) \frac{\sin\sigma(\rho - s)}{\pi(\rho - s)}\, d\rho\, ds, \quad n = 0, 1, \cdots \tag{39}$$

with $A_0 = 1$ and $l_0 = p_0 = 0$. It means A_n can be evaluated iteratively on the basis of the data of $f_n(t)$, $f(t)$ in the interval $[-T, T]$.

It should be emphasized that the algorithm proved here only in part improves the speed of convergence of Papoulis–Gercheberg algorithm.

§ 4. Estimation of Bandlimited Function in Noise Case

Because the problem of extrapolating bandlimited function is ill-posed, the result of Papoulis–Gercheberg algorithm is divergent when the given segment is not of a bandlimited function. In noise case we can only discuss some kind of estimation. In order to obtain a unique estimation certain additive information should be given.

The following theorem points out that with slight revision Papoulis-Gercheberg algorithm can be used in noise case.

Theorem 3. If A_n in (6) satisfy $\lim\limits_n A_n=\alpha$ where $0<\alpha<1$, then $\|f_n-f\|\to 0$.

Proof. Since

$$\|f_n\|^2=\sum_{k=0}^{\infty}(a_k^{(n)})^2=\sum_{k=0}^{\infty}[\lambda_k b_k+\alpha a_k^{(n-1)}(1-\lambda_k)]^2$$
$$=\sum_{k=0}^{\infty}(\lambda_k b_k)^2[1+A_{n-1}(1-\lambda_k)+\cdots+A_{n-1}A_{n-2}\cdots A_1(1-\lambda_k)^{n-1}], \tag{40}$$

then, for any sufficiently small $\varepsilon>0$ such that $\alpha-\varepsilon>0$ and $\alpha+\varepsilon<1$, it can be proved that

$$\sum_{k=0}^{\infty}(\lambda_k b_k)^2\left[\frac{1}{1-(\alpha-\varepsilon)(1-\lambda_k)}\right]^2\leqslant\varliminf_n\|f_n\|^2\leqslant\varlimsup_n\|f_n\|^2$$
$$\leqslant\sum_{k=0}^{\infty}(\lambda_k b_k)^2\left[\frac{1}{1-(\alpha+\varepsilon)(1-\lambda_k)}\right]^2 \tag{41}$$

which implies

$$\lim_n\|f_n\|^2=\sum_{k=0}^{\infty}\left(\frac{\lambda_k b_k}{1-\alpha+\alpha\lambda_k}\right)^2=\|f\|^2. \tag{42}$$

According to Theorem 1, we maintain $\|f_n-f\|\to 0$. ∎

Of course, the value of α should be determinated on the basis of a additional information.

§ 5. Conclusion

Two improved versions of Papoulis-Gercheberg algorithm are proposed. They can be used to speed up convergence and to suit the noise case respectively. Their convergences are dealt with in a unified manner based on a more general convergence theorem.

References

[1] Linsternik, L. A. and Sobolev, V. J., Elements of Functional Analysis, Frederick Ungar Publishing Co., New York, 1961.

[2] Slepian, D., Pollak, H. O. and Landau, H. J., Prolate Spheroidal Wave Functions I, II, *Bell Syst. Tech. J.*, **40** (1961), 43—84.

[3] Papoulis, A., Signal Analysis, McGraw-Hill, New York, 1977.

[4] Papoulis, A., A New Algorithm in Spectral Analysis and Bandlimited Extrapolation, *IEEE Tran. Circuits Syst.*, CAS-**22**: 9 (1975), 735—742.

[5] Gercheberg, R. W., Super-Resolution through Error Energy Reduction, *Optica Acta*, **21**:9 (1974). 709—720.

[6] Chamzas, C. and Xu, W., An Improved Version of Papoulis-Gercheberg Algorithm on Bandlimited Extrapolation, *IEEE Trans. Acoust., Speech, Signal Processing*, ASSP-**32**: 2 (1984), 437—440.