## The Numerical Range of a Continuous Mapping of a Normed Space†

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Let $X$ be a normed linear space, let $X'$ denote its dual space, and let $P=\{(x,f)\in X\times X': \|x\|=\|f\|=f(x)=1\}$. Given a continuous mapping $T$ of the unit sphere $S(X)=\{x\in X: \|x\|=1\}$ into $X$, the *numerical range* $V(T)$ of $T$ is defined by

$$V(T)=\{f(Tx): (x,f)\in P\}.$$

If the unit ball of $X$ is smooth, so that there is a unique semi-inner-product on $X$ giving the norm of $X$, then $V(T)$ coincides with the numerical range $W(T)$ in the sense of G. LUMER [*Semi-Inner-Product Spaces*, Trans. Amer. Math. Soc. *100*, 29–43 (1961)]. It is also easy to see that in general $V(T)$ is the union of the numerical ranges $W(T)$ corresponding to all choices of semi-inner-product that give the norm of $X$.

We prove that $V(T)$ is connected, except perhaps when $X$ is one-dimensional, in which case the result fails for non-linear mappings $T$. It is classical that if $X$ is a Hilbert space and $T$ is linear, then $V(T)$ (which coincides with $W(T)$ in this case) is convex. Examples of linear mappings $T$ on normed linear spaces of finite dimension are known in which, respectively, $V(T)$ is not convex and $W(T)$ is not connected.

We give two proofs of the connectedness of $V(T)$. One depends on the connectedness of $P$ as a subset of $X\times X'$ with the product of the norm topology on $X$ and the weak$*$ topology on $X'$. The other proof uses the fact that the mapping $x\to D(x)$ (where $D(x)=\{f:(x,f)\in P\}$) is an upper semi-continuous mapping of $S(X)$ into the subsets of $X'$ with respect to the norm topology on $S(X)$ and the weak$*$ topology on $X'$.

A final section is concerned with the upper semi-continuity of the mapping $x\to D(x)$ with respect to the norm topologies in both spaces. This holds in particular for the space $c_0$ but not for the space $c$.

## Eigenvectors Obtained from the Adjoint Matrix††

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Let $A$ be a square matrix with complex entries and $\lambda_1$ an eigenvalue of $A$. If $x\neq 0, y, z, \dots$ are vectors such that

$$Ax=\lambda_1 x,\quad Ay=\lambda_1 y+x,\quad Az=\lambda_1 z+y, \dots$$

then we say that $y, z, \dots$ are generalized eigenvectors of $A$ associated to the proper eigenvector $x$. The vectors $y, z, \dots$ are not uniquely determined by $x$. If $B(\lambda)$ is the adjoint of $\lambda I-A$ then it is easy to see that nonzero columns of $B(\lambda_1)$ are eigenvectors

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