ON THE QUESTION OF THE EXTREMAL PROPERTIES OF QUADRATIC DIFFERENTIALS WITH END DOMAINS IN THE STRUCTURE OF THE TRAJECTORIES

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One considers the possible definitions of the reduced modules of domains of a special form with respect to corresponding families of curves in these domains. One investigates the role of quadratic differentials, having poles of order ≥ 3, in the problems of the extremal partition of the z-sphere, formulated in terms of the introduced definitions.

The problems of extremal partition play a considerable role in the geometric theory of functions. These problems include questions regarding the maximum of a linear combination of functions of domains — conformal invariants or various reduced modules — in the family of systems of nonoverlapping domains, associated with some family of homotopic classes of curves. A general principle, establishing the equivalence of the module problem for families of classes of curves of a definite type and the problems of extremal partition, as well as the essential role of the quadratic differentials in each of these extremal questions, have been proved for the first time by J. A. Jenkins [1]. The extension of this principle to the extremal metric problem, connected with the concept of the reduced module of a simply connected domain with respect to an interior point of it, has been done in [2] and has obtained a series of applications (some of them are mentioned in the survey [3]). K. Strebel's monograph [4] is devoted to the extremal properties of quadratic differentials, related to questions of the indicated character.

In recent investigations of the author [5] and of E. G. Emel'yanov [6] one has introduced the concept of the reduced module of a biangle with respect to its distinguished boundary elements and one has established the extremal properties of quadratic differentials with poles of the first and second order in a larger sphere of problems of extremal partition. We mention that in [7] there are given interesting applications of the results of [6] to covering problems. Regarding the application of the symmetrization method to these questions, see V. N. Dubinin's paper [8] in this volume.

This paper is devoted to the investigation of the role of quadratic differentials, having poles of order ≥ 3, in extremal partition problems. Here first of all there arises the question of the definition of the reduced modules of domains, similar in the small to strip and end domains of such differentials, with respect to a corresponding family of curves. In Sec. 1 we consider some of the possible definitions of the mentioned reduced modules. Section 2 is devoted to the investigation of the extremal properties of quadratic differentials in problems extremal partition problems, formulated in terms of the introduced concepts.

The present paper is a preliminary investigation, devoted to the given sphere of problems, and does not touch upon some of its aspects.

1. Reduced Modules of Domains, Similar in the Small to Strip-Shaped and End Domains of Quadratic Differentials

1°. Let D be a simply connected domain with two distinguished boundary elements $\tilde{c}_1$, $\tilde{c}_2$ with supports at the (distinct or coinciding) points c_1, c_2 (for the sake of convenience, we assume that the points c_1, c_2 are finite). We shall assume that the boundary arcs of the domain D are analytic in the neighborhoods of the boundary elements $\tilde{c}_k$ and are asymptotically similar to the boundary arcs of the strip-shaped domain of a quadratic differential, having at the points c_k poles of order $\mu_k \geq 3$; condition (*), formulated below, holds. Let $\zeta = g(z)$ be the conformal homeomorphism of the domain D onto the band $\Pi = \{\zeta : -h/2 < \operatorname{Im} \zeta < h/2\}$ such that $g(\tilde{c}_1) = -\infty$, $g(\tilde{c}_2) = +\infty$. Then in the neighborhood of each of the boundary elements $\tilde{c}_k$, $k = 1, 2$, we have one of the equalities:

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$$g(z) = (-1)^{K-1}\{-\beta_K[e^{-i\gamma_K}(z-c_K)]^{-m_K} + \\ +\sum_{j=1}^{m_K-1} \beta_K^{(j)}[e^{-i\gamma_K}(z-c_K)]^{-m_K+j} + B_K \log[e^{-i\gamma_K}(z-c_K)] + (z-c_K)g_K(z)\} + C_K \tag{1}$$

or

$$g(z) = (-1)^{K-1}\{-\beta_K[e^{-i\gamma_K}(z-c_K)]^{-m_K+1/2} + \\ + \sum_{j=1}^{m_K-1} \beta_K^{(j)}[e^{-i\gamma_K}(z-c_K)]^{-m_K+1/2+j} + (z-c_K)^{1/2}g_K(z)\} + C_K . \tag{2}$$

Here $m_k \geq 1$ is a natural number, $\beta_k > 0$, γ_k is real, $\beta_K^{(j)}, B_K$, and C_k are complex constants, $g_k(z)$ is a function, regular in the neighborhood of the point c_k, in (1) one considers the appropriate branch of the logarithm, while in (2) the appropriate branch of the root. Clearly, γ_k is the argument of the tangent to the boundary arc of the domain D at the boundary element $\tilde{c}_k$. In the sequel the expressions (1) and (2) will be written in the form

$$g(z) = (-1)^{K-1}\left[G_K(z) + (z-c_K)^{\omega_K} g_K(z)\right] + C_K , \tag{3}$$

where $\omega_k = 1$ or $\omega_k = 1/2$, respectively.

Condition (*) allows us to investigate the behavior of the function $g(z)$ on certain sections of the domain D. Assume that for $\varepsilon > 0$ we have

$$\mathcal{E}_K(\varepsilon) = \{z : [e^{-i\gamma_K}(z-c_K)]^{r_K} = 2\varepsilon e^{i\varphi}\cos\varphi, -\pi/2 \leq \varphi \leq \pi/2\},$$

here and below $r_k = m_k$ or $r_k = m_k - 1/2$, depending on which of the expansions (1) or (2) holds for the function $g(z)$ in the neighborhood of $\tilde{c}_k$; for $r_k = 1$, $\mathcal{E}_K$ is a circumference with diameter $[c_K, c_K + 2\varepsilon e^{i\gamma_K}]$.

For sufficiently small $\varepsilon > 0$ the set

$$S_K(\varepsilon) = D \cap \mathcal{E}_K(\varepsilon)$$

consists of a unique arc. Assume that in the neighborhood of the boundary element $\tilde{c}_k$ we have the representation (1). Then for $z \in S_k(\varepsilon)$ we have

$$G_K(z) = -\frac{\beta_K}{2\varepsilon}(1 - i\,\mathrm{tg}\,\varphi) + \sum_{j=1}^{m_K-1}\left(\frac{1}{2\varepsilon}\right)^{1-j/m_K} A_K^{(j)} + \frac{B_K}{m_K}\log(2\varepsilon e^{i\varphi}\cos\varphi), \tag{4}$$

where

$$A_K^{(j)} = \beta_K^{(j)}(1 - i\,\mathrm{tg}\,\varphi)^{1-j/m_K} .$$

Let Γ_1 and Γ_2 be the sides of the biangle D, which under the mapping $\zeta = g(z)$ go into the lines $\mathrm{Im}\,\zeta = -h/2$ and $\mathrm{Im}\,\zeta = h/2$, respectively. We denote by $\varphi_K^-(\varepsilon)$ and $\varphi_K^+(\varepsilon)$ the values of the parameter φ, corresponding to the endpoints of the arc $S_k(\varepsilon)$ on the sides Γ_1 and Γ_2. From the expansions (3) and (4) we find

$$\varphi_K^-(\varepsilon) = -\frac{h}{\beta_K}\varepsilon + o(\varepsilon), \quad \varphi_K^+(\varepsilon) = \frac{h}{\beta_K}\varepsilon + o(\varepsilon) . \tag{5}$$

From (4) and (5) we obtain that for $z \in S_k(\varepsilon)$ one has

$$\operatorname{Re} G_k(z) = -\frac{\beta_k}{2\varepsilon} + \sum_{j=1}^{m_k-1} \frac{\operatorname{Re} \beta_k^{(j)}}{(2\varepsilon)^{1-j/m_k}} + O(\varepsilon^{1/m_k}) . \tag{6}$$

Assume now that in the neighborhood of the boundary element $\tilde{c}_k$ for $g(z)$ we have the representation (2). Then, in a manner similar to the previous case, we find that for $z \in S_k(\varepsilon)$ one has

$$\operatorname{Re} G_k(z) = -\frac{\beta_k}{2\varepsilon} + \sum_{j=1}^{m_k-1} \frac{\operatorname{Re} \beta_k^{(j)}}{(2\varepsilon)^{1-j/(m_k-1/2)}} + O(\varepsilon^{1/(2m_k-1)}) . \tag{7}$$

The conditions (6) and (7), determining the values of the function $G_k(z)$ for $z \in S_k(\varepsilon)$, will be written in the form

$$\operatorname{Re} G_k(z) = F_k(\varepsilon) + O(\varepsilon^{\omega_k/\tau_k}), \quad k=1,2 . \tag{8}$$

Then from (3) and (8) we obtain that for $z \in S_k(\varepsilon)$ we have

$$\operatorname{Re} g(z) = (-1)^{k-1} F_k(\varepsilon) + \operatorname{Re} C_k + O(\varepsilon^{\omega_k/\tau_k}) .$$

Assume now that ε_1 and ε_2 are sufficiently small positive numbers. By $D(\varepsilon_1, \varepsilon_2)$ we denote the quadrangle lying in D and having as their opposite sides the arcs $S_k(\varepsilon_k)$, $k=1,2$. Let $\Gamma^{(1)}$ be the family of arcs in D, having c_1, c_2 as their limiting endpoints and separating the sides Γ_1 and Γ_2 of the biangle D, let $\Gamma^{(2)}$ be the family of arcs in D, joining the sides of this biangle. By $\Gamma^{(j)}(\varepsilon_1, \varepsilon_2)$, $j = 1, 2$, we denote the family of arcs in $D(\varepsilon_1, \varepsilon_2)$, joining and separating, respectively, its sides $S_1(\varepsilon_1)$ and $S_2(\varepsilon_2)$, and by $\operatorname{mod}^{(j)}(\varepsilon_1, \varepsilon_2)$ the modulus of the quadrangle $D(\varepsilon_1, \varepsilon_2)$ with respect to the family $\Gamma^{(j)}(\varepsilon_1, \varepsilon_2)$.

Let $\Pi(\varepsilon_1, \varepsilon_2)$ be the image of $D(\varepsilon_1, \varepsilon_2)$ under the mapping $\zeta = g(z)$. Let $\pi_k(\varepsilon_k)$, $k=1,2$ be the opposite sides of quadrangle $\Pi(\varepsilon_1, \varepsilon_2)$, which are the g-images of the arcs $S_k(\varepsilon_k)$. By what has been proved above, for $\zeta \in \pi_k(\varepsilon_k)$ we have

$$\operatorname{Re} \zeta = (-1)^{k-1} F_k(\varepsilon_k) + \operatorname{Re} C_k + O(\varepsilon^{\omega_k/\tau_k}) . \tag{9}$$

From (9) we obtain

$$\operatorname{mod}^{(2)} D(\varepsilon_1, \varepsilon_2) = \frac{1}{h} \{ -(F_2(\varepsilon_2) + F_1(\varepsilon_1)) + \operatorname{Re}(C_2 - C_1) \} + O(\varepsilon^{\omega_1/\tau_1}) + O(\varepsilon_2^{\omega_2/\tau_2}).$$

Consequently, there exists the limit

$$\mathcal{M}^{(2)}(D) = \mathcal{M}^{(2)}(D; \tilde{c}_1, \tilde{c}_2) = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \{ \operatorname{mod}^{(2)} D(\varepsilon_1, \varepsilon_2) + \frac{1}{h}(F_2(\varepsilon_2) + F_1(\varepsilon_1)) \} = \frac{1}{h} \operatorname{Re}(C_2 - C_1) . \tag{10}$$

The limit (10) is called the reduces module-2 of biangle D with respect to boundary elements $\tilde{c}_1$, $\tilde{c}_2$. If $\mathcal{M}^{(2)}(D) \neq 0$, then the reciprocal quantity $\mathcal{M}^{(1)}(D) = 1/\mathcal{M}^{(2)}(D)$ is called the reduces module-1 of the biangle D.

It is easy to indicate extremal metric problems for the classes $\Gamma^{(1)}$ and $\Gamma^{(2)}$ of curves in D, separating and joining, respectively, the boundary sides of D, for each of which the module is the quantity $\mathcal{M}^{(2)}(D)$. In both cases the extremal metric is $\rho(z)|dz| = h^{-1}|g'(z)||dz|$, where $g(z)$ is the above defined mapping of D onto the strip, and this metric is unique: every other extremal metric differs from $\rho(z)|dz|$ perhaps only on a set of zero measure. The proof of this assertion is completely analogous to the proof of Lemmas 2, 3 in [5].

2°. One can give other definitions to the reduced module of a biangle D, satisfying condition (*). Let $n \geq 0$. We write the decomposition (3) for the function $g(z)$ in the form

$$g(z) = (-1)^{k-1} [G_k^{[n]}(z) + (z - c_k)^{n+\omega_k} g_k^{[n]}(z)] + C_k . \tag{11}$$

Depending on which of the expansions (1) or (2) holds for $g(z)$, in $G_k^{[n]}(z)$ one has the initial terms of the expansion (1), containing the powers $(z - c_k)^j$ for $j \le n$, and the logarithmic term or the initial terms of the expansion (2), containing the powers $(z - c_k)^{j+1/2}$ for $j \le n - 1$; $g_k^{[n]}(z)$ is a function that is regular in the neighborhood of the point c_k. Let ε_1 and ε_2 be sufficiently small positive numbers and let

$$S_k^{[n]}(\varepsilon_k) = \{ z : z \in D, \operatorname{Re} G_k^{[n]}(z) = -\frac{1}{2\varepsilon_k} \}, \quad k=1,2 .$$

It is easy to see that for $z \in S_k^{[n]}(\varepsilon_k)$ we have

$$|z - c_k| = O(\varepsilon_k^{(n+\omega_k)/\tau_k}) .$$

Let $D^{[n]}(\varepsilon_1, \varepsilon_2)$ be a quadrangle, lying in D and having as opposite sides the arcs $S_k^{[n]}(\varepsilon_k)$, $k=1,2$. By $\operatorname{mod}^{(j)} D^{[n]}(\varepsilon_1, \varepsilon_2)$, $j = 1, 2$, we denote the modulus of the quadrangle $D^{[n]}(\varepsilon_1, \varepsilon_2)$ relative to the family of arcs in this quadrangle, joining and separating, respectively, its sides $S_1^{[n]}(\varepsilon_1)$ and $S_2^{[n]}(\varepsilon_2)$.

Let $\Pi^{[n]}(\varepsilon_1, \varepsilon_2)$ be the image of $D^{[n]}(\varepsilon_1, \varepsilon_2)$ under the mapping $\zeta = g(z)$, let $\pi_k^{[n]}(\varepsilon_k)$, $k = 1, 2$, be the opposite sides of $\Pi^{[n]}(\varepsilon_1, \varepsilon_2)$, which are g-images of arcs $S_k^{[n]}(\varepsilon_k)$. By virtue of what has been said above, for $\zeta \in \pi_k^{[n]}(\varepsilon_k)$ we have

$$\operatorname{Re} \zeta = (-1)^k \frac{1}{2\varepsilon_k} + \operatorname{Re} C_k + O(\varepsilon_k^{(n+\omega_k)/\tau_k}).$$

Consequently,

$$\operatorname{mod}^{(2)} D^{[n]}(\varepsilon_1, \varepsilon_2) = \frac{1}{h}\left\{\frac{1}{2\varepsilon_1} + \frac{1}{2\varepsilon_2} + \operatorname{Re}(C_2 - C_1)\right\} + O(\varepsilon_1^{(n+\omega_1)/\tau_1}) + O(\varepsilon_2^{(n+\omega_2)/\tau_2}) \tag{12}$$

and, obviously, there exists the limit

$$\mathcal{M}^{(2)}(D) = \mathcal{M}^{(2)}(D; \tilde{c}_1, \tilde{c}_2) = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \left\{\operatorname{mod}^{(2)} D^{[n]}(\varepsilon_1, \varepsilon_2) - \frac{1}{h}\left(\frac{1}{2\varepsilon_1} + \frac{1}{2\varepsilon_2}\right)\right\} = \frac{1}{h}\operatorname{Re}(C_2 - C_1). \tag{13}$$

The limit (13) will be also called the reduced module-2 of the domain D with respect to the boundary elements $\tilde{c}_1$, $\tilde{c}_2$.

3°. Assume now that D is a simply connected domain in $\overline{\mathbb{C}}$ and $c \in \mathbb{C}$ is a boundary point of D. We shall assume that in the neighborhood of the point c the boundary arcs of the domain D are asymptotically similar to the boundary arcs of an end domain of the quadratic differential, having at the point c a pole of order $\mu \ge 3$: the boundary arcs of the domain D are analytic in the neighborhood of the point c and the interior angle of the domain D at the point c is equal to $2\pi/(\mu - 2)$. Namely, we assume that the following condition (**) holds. Let $\zeta = g(z)$ be the conformal homeomorphism of the domain D onto the upper half-plane, normalized by the condition $g(c) = \infty$. Then, depending whether μ is an even or an odd number, for the function $g(z)$ in the neighborhood of the point c we have one of the following expansions:

$$g(z) = -\beta\, [e^{-i\gamma}(z-c)]^{-m} + \sum_{j=1}^{m-1} \beta^{(j)} [e^{-i\gamma}(z-c)]^{-m+j} + B \log [e^{-i\gamma}(z-c)] + (z-c)\, g_1(z) + C \tag{14}$$

if $\mu = 2m + 2 \ge 4$, and

$$g(z) = -\beta\, [e^{-i\gamma}(z-c)]^{-m+1/2} + \sum_{j=1}^{m-1} \beta^{(j)} [e^{-i\gamma}(z-c)]^{-m+1/2+j} + (z-c)\, g_1(z) + C , \tag{15}$$

if $\mu = 2m + 1 \geq 3$. Here γ and $\gamma + 2\pi/(\mu - 2)$ are the arguments of the tangent and boundary arcs of the domain D at the point c, $\beta > 0$, $\beta^{(j)}$, B, and C are complex constants, $g_1(z)$ is a function, regular in the neighborhood of the point c, and in (15) we consider the suitable branch of the root.

Let $n \geq 0$. Then the expansions (14) and (15) can be written, respectively, in the form

$$g(z) = G^{[n]}(z) + (z-c)^{n+1} g^{[n]}(z) + C, \tag{16}$$

where $G^{[n]}(z)$ contains the $m + n$ initial terms of the expansion (14), containing the powers $(z - c)^j$ for $j \leq n$, and the logarithmic term, and

$$g(z) = G^{[n]}(z) + (z-c)^{n+1/2} g^{[n]}(z) + C, \tag{17}$$

where $G^{[n]}(z)$ contains the $m + n + 1$ initial terms of the expansion (15), containing $(z - c)^{j+1/2}$ for $j \leq n - 1$; here $g^{[n]}(z)$ is a function, regular in the neighborhood of the point c.

Let δ and ε be sufficiently small positive numbers. We consider the arcs ($n \geq 0$, $n' \geq 0$)

$$\mathcal{C}^{[n]}(\delta) = \left\{ z : z \in D,\ \operatorname{Im} G^{[n]}(z) = \frac{1}{2\delta} \right\},$$

$$\mathcal{C}_K^{[n']}(\varepsilon) = \left\{ z : z \in D,\ \operatorname{Re} G^{[n']}(z) = (-1)^K \frac{1}{2\varepsilon} \right\}.$$

By $U^{[n, n']}(\delta, \varepsilon)$ we denote the neighborhood of the point c in the domain D, bounded by arcs belonging to $\mathcal{C}^{[n]}(\delta)$ and $\mathcal{C}_K^{[n']}(\varepsilon)$, , $K = 1, 2$. Let

$$D^{[n,n']}(\delta,\varepsilon) = D \setminus \overline{U^{[n,n']}(\delta,\varepsilon)},$$

where $\overline{U^{[n,n']}(\delta,\varepsilon)}$ is the closure of $U^{[n, n']}(\delta, \varepsilon)$. Then $D^{[n, n']}(\delta, \varepsilon)$ is a rectangle, lying in D. The sides of this rectangle, belonging to the arcs $\mathcal{C}^{[n]}(\delta)$ and $\mathcal{C}_K^{[n']}(\varepsilon)$, will be denoted by $T^{[n]}(\delta)$ and $S_K^{[n']}(\varepsilon)$, respectively.

Let $\Pi^{[n, n']}(\delta, \varepsilon)$ be the rectangle, corresponding to $D^{[n, n']}(\delta, \varepsilon)$ under the mapping $\zeta = g(z)$ and let $t^{[n]}(\delta)$ and $s_K^{[n']}(\varepsilon)$ be the g-images of the arcs $T^{[n]}(\delta)$ and $S_K^{[n']}(\varepsilon)$. By virtue of the construction of the neighborhoods $U^{[n, n']}(\delta, \varepsilon)$, for $\zeta \in t^{[n]}(\delta)$ we have

$$\operatorname{Im} \zeta = \frac{1}{2\delta} + \operatorname{Im} C + O(\delta^{\rho(m,n)}), \tag{18}$$

and for $\zeta \in s_K^{[n']}(\varepsilon)$, $K = 1, 2$,

$$\operatorname{Re} \zeta = (-1) \frac{1}{2\varepsilon} + \operatorname{Re} C + O(\varepsilon^{\rho(m,n')}); \tag{19}$$

here $\rho(m,n) = \frac{n+1}{m}$ or $\rho(m,n) = \frac{n+1/2}{m-1/2}$, depending on whether (16) or (17) holds.

Let $\operatorname{mod}^{(1)} D^{[n, n']}(\delta, \varepsilon)$ be the module of the rectangle $D^{[n, n']}(\delta, \varepsilon)$ with respect to the family of curves joining the sides $S_1^{[n']}(\varepsilon)$ and $S_2^{[n']}(\varepsilon)$ and let $\operatorname{mod}^{(1)} \Pi^{[n,n']}(\delta,\varepsilon)$ be the module of $\Pi^{[n,n']}(\delta,\varepsilon)$ with respect to the family of curves joining $s_1^{[n']}(\varepsilon)$ and $s_2^{[n']}(\varepsilon)$. From (18), (19) we obtain

$$\begin{aligned} \operatorname{mod}^{(1)} D^{[n,n']}(\delta,\varepsilon) &= \operatorname{mod}^{(1)} \Pi^{[n,n']}(\delta,\varepsilon) = \\ &= \frac{\frac{1}{2\delta} + \operatorname{Im} C + O(\delta^{\rho(m,n)})}{\frac{1}{\varepsilon} + O(\varepsilon^{\rho(m,n')})} = \frac{\varepsilon}{2\delta} \, \frac{1 + 2\delta \operatorname{Im} C + \delta\, O(\delta^{\rho(m,n)})}{1 + \varepsilon\, O(\varepsilon^{\rho(m,n')})}. \end{aligned} \tag{20}$$

Considering the area of the quadrangle $D^{[n,n']}(\delta, \varepsilon)$ in the metric $\rho(z)|dz| = |g'(z)||dz|$, i.e., the area of $\Pi^{[n,n']}(\delta, \varepsilon)$ in the Euclidean metric $\rho_1(\zeta)|d\zeta| = |d\zeta|$ on the ζ-sphere, for the later area we obtain the expression

$$\text{area } \Pi^{[n,n']}(\delta,\varepsilon) = \left[\frac{1}{\varepsilon} + O(\varepsilon^{p(m,n')})\right]\left[\frac{1}{2\delta} + \operatorname{Im} C + O(\delta^{p(m,n)})\right] =$$
$$= \frac{1}{2\delta\varepsilon} + \frac{1}{\varepsilon}\operatorname{Im} C + \frac{1}{\delta} O(\varepsilon^{p(m,n')}) + \frac{1}{\varepsilon} O(\delta^{p(m,n)}). \tag{21}$$

This equality shows the possibility of selecting the parameters n, n', δ, ε, occurring in the definition of the neighborhood $U^{[n,n']}(\delta, \varepsilon)$, in such a manner that the remainders in (21) should tend to zero when $\delta \to 0$, $\varepsilon \to 0$. Below we restrict ourselves to the consideration of the simplest case $n = n' = m$, $\delta = \varepsilon$. In this case (21) assumes the form

$$\text{area } \Pi^{[m,m]}(\varepsilon,\varepsilon) = \frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\operatorname{Im} C + O(\varepsilon^{1/r(m)}), \tag{22}$$

where $r(m) = m$ or $r(m) = m - 1/2$. Consequently, there exists the limit

$$\lim_{\varepsilon \to 0}\left\{\varepsilon\left[\text{area } \Pi^{[m,m]}(\varepsilon,\varepsilon) - \frac{1}{2\varepsilon^2}\right]\right\} = \operatorname{Im} C. \tag{23}$$

The limit (23) will be called the reduced area of the domain D with respect to the point $c \in \partial D$ and we shall denote it by $\sigma(D, c)$. With the indicated selection of the parameters, from (20) we obtain

$$\operatorname{mod}^{(1)} D^{[m,m]}(\delta,\varepsilon) = \frac{1}{2} + \varepsilon \operatorname{Im} C + O(\varepsilon^{2+1/r(m)}). \tag{24}$$

2. Extremal Properties of Quadratic Differentials

1°. Let $Q(z)dz^2$ be a quadratic differential, regular in $\overline{\mathbb{C}}$ with the exception of simple poles at the points a_n, $n=1,\dots,N_1'$, second-order poles at the points b_j, $j=1,\dots,N_2$ (these poles may be absent), and poles at points c_k, $k = 1, \dots, N$, of order $\mu_k \geq 3$. Assume that the interior closure Φ of the set Φ for the differential $Q(z)dz^2$ is empty; consequently, in the structure of the trajectories there are no dense domains. If the points b_j exist, then, for the simplicity of the formulation, we shall assume that each of the points b_j is contained in a circular domain for the differential $Q(z)dz^2$. In addition, we shall assume that all the points of the sets $\{c_k\}$ and $\{b_j\}$ are finite.

Let $D_i^{(1)}$, $i=1,\dots,N_1$, be the ring domains for the differential $Q(z)dz^2$ and let $D_j^{(2)}$, $j=1,\dots,N_2$, be the circular domains for this differential, $b_j \in D_j^{(2)}$ (domains of one or both of these types may be absent). By $\alpha_i^{(1)}$ and $\alpha_j^{(2)}$ we shall denote the Q-length of the trajectories of differential $Q(z)dz^2$ in the domains $D_i^{(1)}$ and $D_j^{(2)}$, respectively.

Assume that in the neighborhoods of the points c_k the function $Q(z)$ has the expansion

$$Q(z) = A_k e^{i\gamma_k(\mu_k-2)}(z-c_k)^{-\mu_k} + A_{k-1}(z-c_k)^{-\mu_k+1} + \dots,$$

where $A_k > 0$ and γ_k is real. Then

$$\gamma_{k,\ell} = \gamma_k + \frac{(\ell-1)2\pi}{\mu_k-2}, \quad \ell=1,\dots,\mu_k-2,$$

are the values of $\arg(z-c_k)$, defining those limiting directions along which the arcs of the trajectories of the differential $Q(z)dz^2$, lying in a sufficiently small neighborhood of the point c_k, tend to this point.

Let $D_{k,\ell}$ $\ell = 1, \dots, \mu_k - 2$, be the ring domains of the differential $Q(z)dz^2$, enumerated in the natural order: $\gamma_{k,\ell}$ and $\gamma_{k,\ell+1}$ are the limiting directions of tangents to the boundary of domain $D_{k,\ell}$ at the point c_k. Let $\zeta = g_{k,\ell}(z)$ be the conformal homeomorphism of the domain $D_{k,\ell}$ onto the upper half-plane $\operatorname{Im}\zeta > 0$, $g_{k,\ell}(c_k) = \infty$. We have

$$g_{k,\ell}(z) = \int \sqrt{Q(z)}\,dz + C_{k,\ell},$$

where the branches of the root are selected in the appropriate manner and $C_{k,l}$ is a constant. We denote by $g_k(z)$ the formal expansion of the function $\int\sqrt{Q(z)}dz$ in the neighborhood of the point c_k (without the constant term):

$$g_\kappa(z) = -\beta_\kappa\left[e^{-i\gamma_\kappa}(z-c_\kappa)\right]^{-m_\kappa} + \sum_{j=1}^{m_\kappa-1}\beta_\kappa^{(j)}\left[e^{-i\gamma_\kappa}(z-c_\kappa)\right]^{-m_\kappa+j} + B_\kappa \log\left[e^{-i\gamma_\kappa}(z-c_\kappa)\right] + \ldots \tag{25}$$

if $\mu_k = 2(m_k + 2)$ is an even number and

$$g_\kappa(z) = e^{-i\gamma_\kappa/2}(z-c_\kappa)^{1/2}\left\{-\beta_\kappa\left[e^{-i\gamma_\kappa}(z-c_\kappa)\right]^{-m_\kappa} + \sum_{j=1}^{m_\kappa-1}\beta_\kappa^{(j)}\left[e^{-i\gamma_\kappa}(z-c_\kappa)\right]^{-m_\kappa+j} + \ldots\right\}, \tag{26}$$

if $\mu_k = 2m_k + 1$ is an odd number; $\beta_\kappa = \sqrt{A_\kappa}\cdot 2/(\mu_\kappa-2) > 0$. In the sequel, for the simplicity of the formulation, we shall assume that the logarithmic term in the expansion (25) is missing. Then, in the case of an even μ_k, the function $g_k(z)$, defined by this expansion, is univalent in the neighborhood of the point c_k. If μ_k is odd, then in the neighborhood of the point c_k we consider a cut along an arc with endpoint at this point, for example, along an arc of the boundary of the domain $D_{k,1}$, and we isolate a univalent branch of the function $g_k(z)$, defining in (26) the branch of $(z - c_k)^{1/2}$ by the condition $\arg(z - c_k)^{1/2} = i\gamma_k/2$ for $z - c_k = |z - c_k| e^{i\gamma_k}$. The function $g_k(z)$ will be also written in the form (compare with (3))

$$g_\kappa(z) = G_\kappa^{[n]}(z) + (z-c_\kappa)^{n+\omega_\kappa} g_\kappa^{[n]}(z), \tag{27}$$

where $n \geq 0$, $\omega_k = 1$ or $\omega_k = 1/2$, $g_\kappa^{[n]}(z)$ is a function that is regular in the neighborhood of the point c_k. For the above given definition of the function $g_k(z)$ we have

$$g_{\kappa,\ell}(z) = (-1)^{\ell-1} g_\kappa(z) + C_{\kappa,\ell}, \quad \ell = 1, \ldots, \mu_\kappa - 2,$$

where $C_{k,\ell}$ is a constant.

The strip domains for the differential $Q(z)dz^2$ have points from the set $\{c_k\}$ as supports of their distinguished boundary elements. Those strip domains for which the point c_k is a support for at least one boundary element, while the quantity $\gamma_{k,\ell}$ is the argument of the tangent to the boundary of the domain at the boundary element $\tilde{c}$, will be denoted by $D_{k,\ell,p}$; here $p = 1, \ldots, p(k, \ell)$, where $p(k, \ell)$ is the number of strip domains possessing the mentioned property. If the biangle $D_{k,\ell,p}$ has the point c_k as the support of both of its distinguished boundary elements $\tilde{c}_k$ and $\tilde{c}_\kappa'$, then $D_{k,\ell,p}$ will be denoted also by $D_{\kappa,\ell',p'}$; for this domain the values $\gamma_{\kappa,\ell}$ and $\gamma_{\kappa,\ell'}$ are the arguments of the tangents to the boundary of the domain at the boundary elements $\tilde{c}_k$ and $\tilde{c}_\kappa'$. If the biangle $D_{k,\ell,p}$ has boundary elements $\tilde{c}_k$ and $\tilde{c}_\nu$, whose supports are the distinct points c_k and c_ν, while $\gamma_{k,\ell}$ and $\gamma_{k,\ell'}$ are the arguments of the tangents to the boundary of this domain at the boundary elements $\tilde{c}_k$ and $\tilde{c}_\nu$, then $D_{k,\ell,p}$ will be denoted also by $D_{\nu,\ell',p'}$. For the mapping $\zeta = g_{\kappa,\ell,p}(z) = g_{\nu,\ell',p'}(z)$ of the domain $D_{k,\ell,p}$ onto the strip $-h_{k,\ell,p}/2 < \Im\zeta < h_{k,\ell,p}/2$, in the neighborhoods of $\tilde{c}_k$ and $\tilde{c}_\nu$ we have the representations

$$g_{\kappa\ell p}(z) = (-1)^{\ell-1} g_\kappa(z) + C_{\kappa,\ell,p}$$

and

$$g_{\nu,\ell',p'}(z) = (-1)^{\ell'-1} g_\nu(z) + C_{\nu,\ell',p'},$$

respectively, where $C_{\kappa,\ell,p}$ and $C_{\nu,\ell',p'}$ are constants; $\nu = k$ or $\nu \neq k$.

Let $\varepsilon = \{\varepsilon_j\}_{j=1}^{N_2}$ and $\delta = \{\delta_\kappa\}_{\kappa=1}^{N}$ be systems of sufficiently small positive numbers. By $U(b_j, \varepsilon_j)$ we denote the circular ε_j-neighborhood of the point b_j : $U(b_j,\varepsilon_j) = \{z : |z-b_j| < \varepsilon_j\}$. Let $T_{\kappa,\ell}(\delta_\kappa)$ and $S_{\kappa,\ell}(\delta_\kappa)$

$\ell = 1, \ldots, \mu_\kappa - 2$ be the trajectories and the orthogonal trajectories, respectively, of the differential $Q(z)dz^2$, having c_k as limit endpoint in both directions and defined by the conditions: for $z \in T_{\kappa,\ell}(\delta_\kappa)$

$$\operatorname{Im}\{(-1)^{\ell-1} q_\kappa(z)\} = \frac{1}{2\delta_\kappa},$$

while for $z \in S_{\kappa,\ell}(\delta_\kappa)$

$$\operatorname{Re}\{(-1)^{\ell-1} q_\kappa(z)\} = \frac{1}{2\delta_\kappa}; \qquad \ell = 1, \ldots, \mu_\kappa - 2.$$

By $U_\kappa(c_\kappa, \delta_\kappa)$ we denote neighborhood of point c_k, being the union of interiors of closed curves $T_{\kappa,\ell}(\delta_\kappa) \cup \{c_\kappa\}$ and $S_{\kappa,\ell}(\delta_\kappa) \cup \{c_\kappa\}$, $\ell = 1, \ldots, \mu_\kappa - 2$. Let $\overline{\mathbb{C}}(\delta, \varepsilon) - (N_1 + N_2)$ be the connected domain obtained from $\overline{\mathbb{C}}$ by removing the closures of the neighborhoods $U(b_j, \varepsilon_j)$, $j = 1, \ldots, N_1$ and $U(c_\kappa, \delta_\kappa)$, $\kappa = 1, \ldots, N$.

The intersections of the domains $D_{k,\ell}$ and $D_{\kappa,\ell,p} = D_{\nu,\ell',p'}$ with $\overline{\mathbb{C}}(\delta, \varepsilon)$ will be denoted by $D_{k,\ell}(\delta_k)$ and $D_{\kappa,\ell,p}(\delta_\kappa, \delta_\nu)$. Let $t_{k,\ell}(\delta_k)$ and $s_{k,\ell}(\delta_k)$ be the Q-lengths of the arcs of the trajectories and of the orthogonal trajectories of the differential $Q(z)dz^2$ in the quadrangle $D_{k,\ell}(\delta_k)$. Let $t_{\kappa,\ell,p}(\delta_\kappa, \delta_\nu) = t_{\nu,\ell',p'}(\delta_\nu, \delta_\kappa)$ be the Q-length of the trajectory of the differential $Q(z)dz^2$ in the quadrangle $D_{\kappa,\ell,p}(\delta_\kappa, \delta_\nu) = D_{\nu,\ell',p'}(\delta_\nu, \delta_\kappa)$ and let $h_{k,\ell,p} = h_{\nu,\ell',p'}$ be the Q-length of the orthogonal trajectory in the indicated quadrangle.

2°. We obtain an expression for the area of the domain in $\overline{\mathbb{C}}(\varepsilon, \delta)$ in the Q-metric. In the sequel $\mathcal{M}(D_i^{(1)})$ is the module of the doubly connected domain $D_i^{(1)}$ with respect to the family of curves separating its boundary components, $\mathcal{M}(D_j^{(2)}, b_j)$ is the reduced module of the domain $D_j^{(2)}$ with respect to the points $b_j \in D_j^{(2)}$. For the mentioned area we have the equality

$$\operatorname{area}_Q \overline{\mathbb{C}}(\varepsilon, \delta) + \sum_{j=1}^{N_2} \frac{1}{2\pi} \alpha_j^{(2)2} \log \varepsilon_j = \sum\nolimits_1 + \sum\nolimits_2 + O(\|\varepsilon\|^\lambda), \ \lambda > 0, \tag{28}$$

where

$$\sum\nolimits_1 = \sum_{i=1}^{N_1} \alpha_i^{(1)2} \mathcal{M}(D_i^{(1)}) + \sum_{j=1}^{N_2} \alpha_j^{(2)2} \mathcal{M}(D_j^{(2)}, b_j), \tag{29}$$

$$\sum\nolimits_2 = \sum_{\kappa=1}^{N} \sum_{\ell=1}^{\mu_\kappa - 2} t_{\kappa,\ell}^2(\delta_\kappa) \cdot \operatorname{mod}^{(1)} D_{\kappa,\ell}(\delta_\kappa) + \frac{1}{2} \sum_{\kappa=1}^{N} \sum_{\ell=1}^{\mu_\kappa - 2} \sum_{p=1}^{p(\kappa,\ell)} t_{\kappa,\ell,p}^2(\delta_\kappa, \delta_\nu) \cdot \operatorname{mod}^{(1)} D_{\kappa,\ell,p}(\delta_\kappa, \delta_\nu). \tag{30}$$

(the factor 1/2 in front of the triple sum in (30) is due to the fact that the module of each of the strip-shaped domains occurs twice in this sum). We write equality (30) in the form

$$\begin{aligned} 2\sum\nolimits_2 = {} & \sum_{\kappa=1}^{N} \sum_{\ell=1}^{\mu_\kappa - 2} \left[t_{\kappa,\ell}(\delta_\kappa) s_{\kappa,\ell}(\delta_\kappa) + t_{\kappa,\ell+1}(\delta_\kappa) s_{\kappa,\ell+1}(\delta_\kappa) \right] + \\ & + \sum_{\kappa=1}^{N} \sum_{\ell=1}^{\mu_\kappa - 2} \sum_{p=1}^{p(\kappa,\ell)} t_{\kappa,\ell,p}(\delta_\kappa, \delta_\nu)\, h_{\kappa,\ell,p}. \end{aligned} \tag{31}$$

Due to the absence of a logarithmic term in the expressions for $G_k(z)$, the Q-lengths of the arcs of the trajectories in the domains $D_{\kappa,\ell}(\delta_\kappa)$, $\ell = 1, \ldots, \mu_\kappa - 2$, are equal among themselves:

$$t_{\kappa,1}(\delta_\kappa) = \ldots = t_{\kappa,\mu_\kappa - 2}(\delta_\kappa) = \frac{1}{\delta_\kappa}. \tag{32}$$

Further, since $D_{\kappa,\ell,p}(\delta_\kappa,\delta_\nu)=D_{\nu,\ell',p'}(\delta_\nu,\delta_\kappa)$, where $\nu = k$ or $\nu \neq k$, we have

$$\begin{aligned} t_{\kappa,\ell,p}(\delta_\kappa,\delta_\nu) &= \frac{1}{2\delta_\kappa}+\frac{1}{2\delta_\nu}+h_{\kappa,\ell,p}\,\mathcal{M}^{(2)}_{\kappa,\ell,p} = \\ &= \frac{1}{2}\left[\frac{1}{\delta_\kappa}+\frac{1}{\delta_\nu}+h_{\kappa,\ell,p}\mathcal{M}^{(2)}_{\kappa,\ell,p}+h_{\nu,\ell',p'}\mathcal{M}^{(2)}_{\nu,\ell',p'}\right], \end{aligned} \tag{33}$$

where $\mathcal{M}^{(2)}_{\kappa,\ell,p}=\mathcal{M}^{(2)}(D_{\kappa,\ell,p})$ is the reduced module-2 of the strip-shaped domain $D_{k,\ell,p}$ (we make use of the definitions of Sec. 1). Therefore, from (31) we obtain

$$\begin{aligned} 2\sum\nolimits_2 = \sum_{\kappa=1}^{N}\sum_{\ell=1}^{\mu_\kappa-2}\Big\{ &\frac{1}{\delta_\kappa}\left[s_{\kappa,\ell}(\delta_\kappa)+s_{\kappa,\ell+1}(\delta_\kappa)+\sum_{p=1}^{p(\kappa,\ell)}h_{\kappa,\ell,p}\right]+ \\ &+\sum_{p=1}^{p(\kappa,\ell)}h^2_{\kappa,\ell,p}\,\mathcal{M}^{(2)}_{\kappa,\ell,p}\Big\}. \end{aligned} \tag{34}$$

The quantity occurring in square brackets in the right-hand side of (34), represents the Q-length of arc $S_{\kappa,\ell}(\delta_\kappa)$. Taking into account that

$$s_{\kappa,\ell}(\delta_\kappa) = \frac{1}{2\delta_\kappa} + \sigma_{\kappa,\ell}\ ,$$

where $\sigma_{\kappa,\ell}=\sigma(D_{\kappa,\ell},c_\kappa)=\operatorname{Im} C_{\kappa,\ell}$ is the reduced area of the domain $D_{k,\ell}$ (we make use of the definitions of Sec. 1, 3°), we find

$$\begin{aligned} &\sum_{\ell=1}^{\mu_\kappa-2}\left[s_{\kappa,\ell}(\delta_\kappa)+s_{\kappa,\ell+1}(\delta_\kappa)+\sum_{p=1}^{p(\kappa,\ell)}h_{\kappa,\ell,p}\right]= \\ &=\sum_{\ell=1}^{\mu_\kappa-2}\left[\frac{1}{\delta_\kappa}+\sigma_{\kappa,\ell}+\sigma_{\kappa,\ell+1}+\sum_{p=1}^{p(\kappa,\ell)}h_{\kappa,\ell,p}\right]= \\ &=(\mu_\kappa-2)\frac{1}{\delta_\kappa}+\sum_{\ell=1}^{\mu_\kappa-2}\left[2\sigma_{\kappa,\ell}+\sum_{p=1}^{p(\kappa,\ell)}h_{\kappa,\ell,p}\right]. \end{aligned}$$

Consequently, (34) can be written in the form

$$\begin{aligned} 2\sum\nolimits_2 = &\sum_{\kappa=1}^{N}(\mu_\kappa-2)\frac{1}{\delta_\kappa^2}+\sum_{\kappa=1}^{N}\frac{1}{\delta_\kappa}\sum_{\ell=1}^{\mu_\kappa-2}\left[2\sigma_{\kappa,\ell}+\sum_{p=1}^{p(\kappa,\ell)}h_{\kappa,\ell,p}\right]+ \\ &+\sum_{\kappa=1}^{N}\sum_{\ell=1}^{\mu_\kappa-2}\sum_{p=1}^{p(\kappa,\ell)}h^2_{\kappa,\ell,p}\,\mathcal{M}^{(2)}_{\kappa,\ell,p}\ . \end{aligned} \tag{35}$$

3°. By an admissible system of domains $\tilde{D}^{(1)}_i, \tilde{D}^{(2)}_j, \tilde{D}_{\kappa,\ell}, \tilde{D}_{\kappa,\ell,p}$, we shall understand a system of nonoverlapping domains, satisfying the following conditions. 1) $\tilde{D}^{(1)}_i$ is a doubly connected domain in $\overline{\mathbb{C}}'$, where $\overline{\mathbb{C}}'$ is obtained by removing from $\overline{\mathbb{C}}$ all distinguished points a_n, b_j, c_k. The curves, separating boundary components of $\tilde{D}^{(1)}_i$, are homotopic on $\overline{\mathbb{C}}'$ to the curves, separating the boundary components of $D^{(1)}_i$ (without loss of generality we can assume that the number of domains $\tilde{D}^{(1)}_i$ is equal to the number of domains $D^{(1)}_i$, since, otherwise, by some of the domains $D^{(1)}_i$ we shall mean the degenerate doubly connected domains and then it is clear what has to be meant by the previous condition). 2) $\tilde{D}^{(2)}_j$ is a simply connected domain in $\overline{\mathbb{C}}'\cup\{b_j\}$, $b_j\in\tilde{D}^{(2)}_j$. 3) $\tilde{D}_{\kappa,\ell}$ is a simply connected domain in $\overline{\mathbb{C}}'$, $c_\kappa\in\partial\tilde{D}_{\kappa,\ell}$. For the function $\tilde{g}_{k,\ell}(z)$, mapping $\tilde{D}_{k,\ell}$ onto the upper half-plane, in the neighborhood of the point c_k we have the representation (see (27))

$$\tilde{g}_{\kappa,\ell}(z) = (-1)^{\ell-1}\left[G^{[m_\kappa]}_\kappa(z)+(z-c_\kappa)^{m_\kappa+\omega_\kappa}\hat{g}_{\kappa,\ell}(z)\right]+\tilde{C}_{\kappa,\ell}, \tag{36}$$

where $\hat{g}_{k,\ell}(z)$ is regular in the neighborhood of c_k and $\tilde{C}_{k,\ell}$ is a constant. 4) $\tilde{D}_{\kappa,\ell,p}=\tilde{D}_{\kappa',\ell',p'}$ is a biangle in $\overline{\mathbb{C}}'$ with distinguished boundary elements $\tilde{c}_\kappa$, $\tilde{c}_{\kappa'}$ with supports at the points c_k, $c_{k'}$, where $k = k'$ or $k \neq k'$. The arcs in

$\widetilde{D}_{\kappa,\ell,p}$, separating the sides of $\widetilde{D}_{\kappa,\ell,p}$, are homotopic in $\overline{\mathbb{C}}'$ to the arcs in the biangle $D_{\kappa,\ell,p}$, separating the sides of $D_{\kappa,\ell,p}$ (we can assume that the number of domains $\widetilde{D}_{\kappa,\ell,p}$ is equal to number of domains $D_{\kappa,\ell,p}$). For the functions $\widetilde{g}_{\kappa,\ell,p}(z)$, mapping $\widetilde{D}_{\kappa,\ell,p}$ onto the strip $\{\zeta: -\widetilde{h}_{\kappa,\ell,p}/2 < \operatorname{Im}\zeta < \widetilde{h}_{\kappa,\ell,p}/2\}$, in the neighborhood of the boundary element $\tilde{c}_k$ we have the representation

$$\widetilde{g}_{\kappa,\ell,p}(z) = (-1)^{\ell-1}\left[G_{\kappa}^{[m_\kappa]}(z) + (z-c_\kappa)^{m_\kappa+\omega_\kappa}\hat{g}_{\kappa,\ell,p}(z)\right] + \widetilde{C}_{\kappa,\ell,p}, \tag{37}$$

where $\hat{g}_{\kappa,\ell,p}(z)$ is regular in the neighborhood of the point c_κ, $\widetilde{C}_{\kappa,\ell,p}$ is a constant, and an analogous representations holds in the neighborhood of the boundary element $\tilde{c}_{k'}$.

By virtue of the above mentioned properties of the Q-metric, we have the inequalities

$$\iint_{\widetilde{D}_i^{(1)}} |Q(z)|\,dx\,dy \geqslant \alpha_i^{(1)2}\mathcal{M}(\widetilde{D}_i^{(1)}),$$

$$\lim_{\varepsilon_j\to 0}\left\{\iint_{\widetilde{D}_j^{(2)}\setminus \overline{U(b_j,\varepsilon_j)}} |Q(z)|\,dx\,dy + \frac{\alpha_j^{(2)2}}{2\pi}\log\varepsilon_j\right\} \geqslant \alpha_j^{(2)2}\mathcal{M}(\widetilde{D}_j^{(2)}, b_j).$$

Consequently,

$$\sum_{i=1}^{N_1}\iint_{\widetilde{D}_i^{(1)}} + \lim_{\|\varepsilon\|\to 0}\sum_{j=1}^{N_2}\left\{\iint_{D_j^{(2)}\setminus\overline{U(b_j,\varepsilon_j)}} + \frac{\alpha_j^{(2)2}}{2\pi}\log\varepsilon_j\right\} \geqslant \widetilde{\sum}_1, \tag{38}$$

where

$$\widetilde{\sum}_1 = \sum_{i=1}^{N_1}\alpha_i^{(1)2}\mathcal{M}(\widetilde{D}_i^{(1)}) + \sum_{j=1}^{N_2}\alpha_j^{(2)2}\mathcal{M}(\widetilde{D}_j^{(2)}, b_j).$$

Now we have

$$\iint_{\widetilde{D}_{\kappa,\ell}(\delta_\kappa)} |Q(z)|\,dx\,dy \geqslant t_{\kappa,\ell}^2(\delta_\kappa)\ \mathrm{mod}^{(1)}\widetilde{D}_{\kappa,\ell}(\delta_\kappa), \tag{39}$$

$$\iint_{\widetilde{D}_{\kappa,\ell,p}(\delta_\kappa,\delta_\nu)} |Q(z)|\,dx\,dy \geqslant t_{\kappa,\ell,p}^2(\delta_\kappa,\delta_\nu)\ \mathrm{mod}^{(1)}\widetilde{D}_{\kappa,\ell,p}(\delta_\kappa,\delta_\nu); \tag{40}$$

here $\widetilde{D}_{\kappa,\ell,p}(\delta_\kappa,\delta_\nu) = \widetilde{D}_{\nu,\ell',p'}(\delta_\kappa,\delta_\nu)$, where $\nu = k$ or $\nu \neq k$.

By virtue of the conditions (36) and (37), imposed on an admissible family of domains, we have the equalities (see equalities (24) and (12) of Sec. 1)

$$\mathrm{mod}^{(1)}\widetilde{D}_{\kappa,\ell}(\delta_\kappa) = \frac{1}{2} + \delta_\kappa\widetilde{\sigma}_{\kappa,\ell} + O(\delta_\kappa^{2+1/r_\kappa}), \tag{41}$$

$$\mathrm{mod}^{(1)}\widetilde{D}_{\kappa,\ell,p}(\delta_\kappa,\delta_\nu) = \widetilde{h}_{\kappa,\ell,p}\left[\frac{1}{2\delta_\kappa} + \frac{1}{2\delta_\nu} + \widetilde{h}_{\kappa,\ell,p}\widetilde{\mathcal{M}}^{(2)}_{\kappa,\ell,p} + O(\delta_\kappa^{1+1/r_\kappa}) + O(\delta_\nu^{1+1/r_\nu})\right]^{-1}, \tag{42}$$

where $r_k = m_k$ or $r_k = m_k - 1/2$. Here $\widetilde{\sigma}_{\kappa,\ell} = \sigma(\widetilde{D}_{\kappa,\ell}, c_\kappa)$ is reduced area of domain $\widetilde{D}_{\kappa,\ell}$, $\widetilde{\mathcal{M}}^{(2)}_{\kappa,\ell,p} = \mathcal{M}^{(2)}(\widetilde{D}_{\kappa,\ell,p})$ is the reduced module-2 of the biangle $\widetilde{D}_{\kappa,\ell,p} = \widetilde{D}_{\nu,\ell',p'}$. From (39)-(42) and the expressions for $t_{\kappa,\ell}(\delta_\kappa)$ and $t_{\kappa,\ell,p}(\delta_\kappa,\delta_\nu)$ (see (32) and (33)) we find $(\lambda' = \min\{1/r_1,\ldots,1/r_N, 1\} > 0)$

$$2\sum_{\kappa=1}^{N}\sum_{\ell=1}^{\mu_\kappa-2}\iint_{\widetilde{D}_{\kappa,\ell}(\delta_\kappa)} + \sum_{\kappa=1}^{N}\sum_{\ell=1}^{\mu_\kappa-2}\sum_{p=1}^{p(\kappa,\ell)}\iint_{\widetilde{D}_{\kappa,\ell,p}(\delta_\kappa,\delta_\nu)} \geqslant 2\widetilde{\sum}_2 + O(\|\delta\|^{\lambda'}), \tag{43}$$

where

$$2\tilde{\sum}_2 = \sum_{\kappa=1}^{N} (\mu_\kappa - 2)\frac{1}{\delta_\kappa^2} + \sum_{\kappa=1}^{N} \frac{1}{\delta_\kappa} \sum_{\ell=1}^{\mu_\kappa-2} \Big[2\tilde{\sigma}_{\kappa,\ell} + \sum_{p=1}^{p(\kappa,\ell)} \tilde{h}_{\kappa,\ell,p}\Big] + $$
$$+ \sum_{\kappa=1}^{N} \sum_{\ell=1}^{\mu_\kappa-2} \sum_{p=1}^{p(\kappa,\ell)} \Big[2 h_{\kappa,\ell,p} \tilde{h}_{\kappa,\ell,p} \mathcal{M}^{(2)}_{\kappa,\ell,p} - \tilde{h}^2_{\kappa,\ell,p} \tilde{\mathcal{M}}^{(2)}_{\kappa,\ell,p}\Big].$$

From (28), (38), and (43) we obtain

$$\sum_1 + \sum_2 \geqslant \tilde{\sum}_1 + \tilde{\sum}_2 + O(\|\delta\|^{\lambda'}) . \tag{44}$$

It is natural to assume that for the system of domains of an admissible family the following condition holds:

$$\sum_{\ell=1}^{\mu_\kappa-2} \Big[2\sigma_{\kappa,\ell} + \sum_{p=1}^{p(\kappa,\ell)} h_{\kappa,\ell,p}\Big] = \sum_{\ell=1}^{\mu_\kappa-2} \Big[2\tilde{\sigma}_{\kappa,\ell} + \sum_{p=1}^{p(\kappa,\ell)} \tilde{h}_{\kappa,\ell,p}\Big], \quad \kappa = 1, \dots, N . \tag{45}$$

As it follows from what has been said above, this condition means that the closures of the domains of an admissible system fill out completely the neighborhoods of the points c_k. Then from (44) we obtain the inequality

$$\sum_{i=1}^{N_1} \alpha_i^{(1)2} [\mathcal{M}(D_i^{(1)}) - \mathcal{M}(\tilde{D}_i^{(1)})] + \sum_{j=1}^{N_2} \alpha_j^{(2)2} [\mathcal{M}(D_j^{(2)}, b_j) - \mathcal{M}(\tilde{D}_j^{(2)}, b_j)] +$$
$$+ \frac{1}{2} \sum_{\kappa=1}^{N} \sum_{\ell=1}^{\mu_\kappa-2} \sum_{p=1}^{p(\kappa,\ell)} \Big[h^2_{\kappa,\ell,p} \mathcal{M}^{(2)}(D_{\kappa,\ell,p}) - 2 h_{\kappa,\ell,p} \tilde{h}_{\kappa,\ell,p} \mathcal{M}^{(2)}(D_{\kappa,\ell,p}) +$$
$$+ \tilde{h}^2_{\kappa,\ell,p} \mathcal{M}^{(2)}(\tilde{D}_{\kappa,\ell,p})\Big] \geqslant 0 . \tag{46}$$

From the known uniqueness results of the method of extremal metrics and from the statements regarding the uniqueness of the extremal metric in module problems, defining the reduced area $\sigma(D_{\kappa,\ell}; c_\kappa)$ and the reduced module-2 $\mathcal{M}^{(2)}(D_{\kappa,\ell,p})$, there follows that equality holds in (46) only in the case when the considered systems of domains coincide. Thus, we have

THEOREM. Let $Q(z)dz^2$ be the quadratic differential defined in Subsection 1° and assume that in the formal expansions of the function $\int^z \sqrt{Q(t)}\,dt$ in the neighborhoods of the points c_k the logarithmic terms are missing. Let $\mathbf{D} = \{ D_i^{(1)}, D_j^{(2)}, D_{\kappa,\ell}, D_{\kappa,\ell,p} \}$ be a system of annular, circular, end, and strip-shaped domains for the differential $Q(z)dz^2$. Let $\alpha_i^{(1)}$ and $\alpha_j^{(2)}$ be the Q-lengths of the trajectories of the differential $Q(z)dz^2$ in the domains $D_i^{(1)}$ and $D_j^{(2)}$, let $h_{k,l,p}$ be the Q-lengths of arcs of its orthogonal trajectories in domains $D_{k,l,p}$. Let $\tilde{\mathbf{D}} = \{\tilde{D}_i^{(1)}, \tilde{D}_j^{(2)}, \tilde{D}_{\kappa,\ell}, \tilde{D}_{\kappa,\ell,p}\}$ be the admissible system of domains, defined by the conditions 1)-4) of Subsection 3°, let $\tilde{h}_{k,l,p}$ be the parameters of the domains $\tilde{D}_{k,l,p}$, occurring in condition 4), and assume that condition (45) is satisfied. Then, with the above introduced notations of the functions and the domains, the inequality (46) holds and equality prevails in (46) only in the case when the systems $\tilde{\mathbf{D}}$ and $\mathbf{D}$ coincide.

4°. The obtained theorem is only the simplest result in the given circle of problems. The restrictive conditions of this theorem, imposed on the admissible family of systems of domains, are connected with the use in the proof of the (m, m)-definition of the reduced area $\sigma(D)$ of the domain D, satisfying the condition (**), given in Sec. 1, Subsection 3°. Making use of other definitions of the reduced area of a domain D (and, respectively, of the reduced module of a biangle, satisfying condition (*) (see Sec. 1, Subsection 1°)), one can obtain extremal partition theorems under other conditions on the admissible systems of domains.

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GROUPS OF CLASSES OF PSEUDOHOMOTOPIC SINGULAR LINKS. I

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By a singular link of type (p_1, p_2) in S^n we mean a pair of continuous mappings $S^{p_1} \to S^n$, $S^{p_2} \to S^n$ with disjoint images. In the paper the concept of the pseudohomotopy of singular links is defined, similar to the concept of concordance of classical links, and it is proved that for $n > p_2 + 2$ the set of the classes of pseudohomotopic singular links of type (p_1, p_2) in S^n forms an Abelian group with respect to a componentwise connected summation. This group has been obtained in case $n \geq 2p_2 + 1 - \max\{n - p_1 - 2, 0\}$.

The primary purpose of this paper is to carry over the concept of concordance from classical links to singular links, to construct groups for singular links, similar to Haefliger's groups for classical links [1], and to obtain these groups in the simplest cases.

1. A Historical Survey

Let n, p_1, p_2 be natural numbers. By a *singular link of type* (p_1, p_2) in S^n we mean an ordered pair of continuous mappings $f_1 : S^{p_1} \to S^n$, $f_2 : S^{p_2} \to S^n$, with disjoint images. If f_1, f_2 are smooth imbeddings, then a singular link is called also a *classical link*. By a *suspension over a singular link* $(f_1 : S^{p_1} \to S^n, f_2 : S^{p_2} \to S^n)$ we mean a singular link $(in \circ f_1, \sigma f_2)$, where in is the inclusion $S^n \subset S^{n+1}$ and σ is a suspension. It is easy to see that, in general, a suspension over a classical link is not a classical link*; this has prompted the author to define singular links and to start their investigation. We note that the singular links are not new objects; thus, one-dimensional singular links in $\mathbb{R}^3$ have been investigated by Milnor [5], singular links with $n - p_i > 2$ $(i = 1, 2)$ by Scott [6].

*If f_2 is the inclusion $S^{p_2} \subset S^n$, then a suspension is a classical link. This observation has allowed us to introduce a suspension in the theory of classical links (see [2], [3], [4]).

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