Fuzzy Optimization Problems Based on the Embedding Theorem and Possibility and Necessity Measures

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Abstract—The embedding theorem shows that the set of all fuzzy numbers can be embedded into a Banach space. Inspired by this embedding theorem, we propose a solution concept of fuzzy optimization problem based on the possibility and necessity measures by solving a biobjective optimization problem. This biobjective optimization problem is obtained by applying the embedding function to the original fuzzy optimization problem. We then also consider the fuzzy optimization problem with fuzzy coefficients (i.e., the coefficients are assumed to be fuzzy numbers). Under a setting of core value of fuzzy number, we show that the optimal solution of its corresponding crisp optimization problem (the usual optimization problem) is also a 1-optimal solution of the original fuzzy optimization problem. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [1]. Since then, many applications of fuzzy sets have been widely developed. One of them is the fuzzy optimization in operations research. The randomness occurring in the optimization problems is categorized as the stochastic optimization problems [2–5]. However, the imprecision (fuzziness) occurring in the optimization problems is categorized as the fuzzy optimization problems. Inuiguchi and Ramík [6] give a brief review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problem.

Bellman and Zadeh [7] inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there came a lot of articles dealing with the fuzzy optimization problems. Zimmermann [8–10] applied fuzzy sets theory to the linear programming problems and linear multiobjective programming problems. The collection of papers on fuzzy optimization edited by Slowinski [11] and Delgado *et al.* [12] gives the main stream of this topic. Lai and Hwang [13,14] also give an insightful survey.

Puri and Ralescu [15] and Kaleva [16] have proven that the set of all fuzzy numbers can be embedded into a Banach space isometrically and isomorphically. Wu and Ma [17] provide a

specific Banach space that the set of all fuzzy numbers can be embedded into the Banach space $\bar{C}[0,1] \times \bar{C}[0,1]$ (its norm $\|\cdot\|$ will be seen in the context), where $\bar{C}[0,1]$ is the set of all real-valued bounded functions f on $[0,1]$ such that f is left-continuous for any $t \in (0,1]$ and right-continuous at 0, and f has a right limit for any $t \in [0,1)$. Inspired by this specific Banach space, we can transform the fuzzy optimization problem into a biobjective optimization problem using this embedding theorem.

In Section 2, we introduce some basic properties of fuzzy numbers and provide the embedding theorem. In Section 3, we introduce a ranking method on the set of all fuzzy numbers using the notion of possibility and necessity measures. In Section 4, we formulate the fuzzy optimization problems using the embedding theorem. We also introduce a solution concept of fuzzy optimization problem, and then we can obtain its optimal solutions by solving its corresponding biobjective optimization problem. In Section 5, we solve the fuzzy optimization problem with fuzzy coefficients. Under a setting of core values of fuzzy numbers, the optimal solution of its corresponding crisp optimization problem (the usual optimization problem) is also a 1-optimal solution of the original fuzzy optimization problem.

2. EMBEDDING THEOREM

We denote by $\mathcal{F}_{cc}(\mathbb{R})$ the set of all fuzzy numbers $\tilde{a}$ with membership function $\xi_{\tilde{a}} : \mathbb{R} \to [0,1]$ satisfying the following conditions:

   (i) $\tilde{a}$ is normal, i.e., there exists an $x \in \mathbb{R}$, such that $\xi_{\tilde{a}}(x) = 1$;
   (ii) $\xi_{\tilde{a}}$ is quasi-concave, i.e., $\xi_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$, for all $\lambda \in [0,1]$;
   (iii) $\xi_{\tilde{a}}$ is upper semicontinuous, i.e., $\{x : \xi_{\tilde{a}}(x) \geq \alpha\}$ is a closed set, for all $\alpha \in [0,1]$;
   (iv) the closure of the set $\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) > 0\}$ is compact (i.e., a closed and bounded set).

We see that if $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$, then the α-level set of $\tilde{a}$, defined and denoted by $\tilde{a}_\alpha = \{x : \xi_{\tilde{a}}(x) \geq \alpha\}$, is a compact (by Conditions (iii) and (iv)) and convex (by Condition (ii)) set, for all $\alpha \in [0,1]$, i.e., $\tilde{a}_\alpha$ is a closed interval. Therefore, we also write $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$.

Let $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\odot$ be any binary operations $\oplus$, $\ominus$, or $\otimes$ between $\tilde{a}$ and $\tilde{b}$. The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\xi_{\tilde{a} \odot \tilde{b}}(z) = \sup_{x \cdot y = z} \min\left\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\right\},$$

where $\odot = \oplus$, $\ominus$, or $\otimes$ and $\cdot = +$, $-$, or $\times$, according to the extension principle proposed by Zadeh [18-20]. Then we have the following well-known result.

PROPOSITION 2.1. Let $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$. Then we have

   (i) $\tilde{a} \oplus \tilde{b}, \tilde{a} \ominus \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$ and $(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U]$, $(\tilde{a} \ominus \tilde{b})_\alpha = [\tilde{a}_\alpha^L - \tilde{b}_\alpha^U, \tilde{a}_\alpha^U - \tilde{b}_\alpha^L]$;
   (ii) $\tilde{a} \otimes \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$ and $(\tilde{a} \otimes \tilde{b})_\alpha = [\min\{\tilde{a}_\alpha^L \tilde{b}_\alpha^L, \tilde{a}_\alpha^L \tilde{b}_\alpha^U, \tilde{a}_\alpha^U \tilde{b}_\alpha^L, \tilde{a}_\alpha^U \tilde{b}_\alpha^U\}, \max\{\tilde{a}_\alpha^L \tilde{b}_\alpha^L, \tilde{a}_\alpha^L \tilde{b}_\alpha^U, \tilde{a}_\alpha^U \tilde{b}_\alpha^L, \tilde{a}_\alpha^U \tilde{b}_\alpha^U\}]$.

We say that $\tilde{a}$ is a crisp number with value m if its membership function is

$$\xi_{\tilde{a}}(r) = \begin{cases} 1, & \text{if } r = m, \\ 0, & \text{otherwise.} \end{cases}$$

We also use the notation $\tilde{1}_{\{m\}}$ to represent the crisp number with value m.

Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$. The *Hausdorff* metric is defined by

$$d_H(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\right\}.$$

According to Puri and Ralescu [21], we define the metric $d_{\mathcal{F}}$ in $\mathcal{F}_{cc}(\mathbb{R})$ as

$$d_{\mathcal{F}}\left(\tilde{a}, \tilde{b}\right) = \sup_{0 \leq \alpha \leq 1} d_H\left(\tilde{a}_\alpha, \tilde{b}_\alpha\right).$$

For $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, we have

$$d_H\left(\tilde{a}_\alpha, \tilde{b}_\alpha\right) = \max\left\{\left|\tilde{a}_\alpha^L - \tilde{b}_\alpha^L\right|, \left|\tilde{a}_\alpha^U - \tilde{b}_\alpha^U\right|\right\}.$$

The space $\bar{C}[0,1]$ is the set of all real-valued bounded functions f on $[0,1]$, such that f is left-continuous for any $t \in (0,1]$ and right-continuous at 0, and f has a right limit for any $t \in [0,1)$. Then $(\bar{C}[0,1], \|\cdot\|)$ is a Banach space with the norm defined by $\|f\| = \sup_{t \in [0,1]} |f(t)|$. Furthermore, $(\bar{C}[0,1] \times \bar{C}[0,1], \|\cdot\|)$ is also a Banach space with the norm defined by

$$\|(f,g)\| = \max\{\|f\|, \|g\|\},$$

where $(f,g) \in \bar{C}[0,1] \times \bar{C}[0,1]$ (see [17]). Let $\tilde{a}$ be a fuzzy number in $\mathcal{F}_{cc}(\mathbb{R})$. We write $a^L(\alpha) = \tilde{a}_\alpha^L$ and $a^U(\alpha) = \tilde{a}_\alpha^U$. Then the function $\pi : \mathcal{F}_{cc}(\mathbb{R}) \to \bar{C}[0,1] \times \bar{C}[0,1]$ defined by $\pi(\tilde{a}) = (a^L(\alpha), a^U(\alpha))$ is injective. Wu and Ma [17] proved the following embedding theorem.

THEOREM 2.1. EMBEDDING THEOREM. *(See [17].) The function $\pi : \mathcal{F}_{cc}(\mathbb{R}) \to \bar{C}[0,1] \times \bar{C}[0,1]$ is defined by $\pi(\tilde{a}) = (a^L(\alpha), a^U(\alpha))$. Then the following properties hold true.*

(i) *π is injective.*
(ii) *$\pi((\tilde{1}_{\{s\}} \otimes \tilde{a}) \oplus (\tilde{1}_{\{t\}} \otimes \tilde{b})) = s\pi(\tilde{a}) + t\pi(\tilde{b})$, for all $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$, $s \geq 0$, and $t \geq 0$.*
(iii) *$d_{\mathcal{F}}(\tilde{a}, \tilde{b}) = \|\pi(\tilde{a}) - \pi(\tilde{b})\|$.*

That is to say, $\mathcal{F}_{cc}(\mathbb{R})$ can be embedded into $\bar{C}[0,1] \times \bar{C}[0,1]$ isometrically and isomorphically.

The above theorem says that each element in $\mathcal{F}_{cc}(\mathbb{R})$ can be regarded as an element in $\bar{C}[0,1] \times \bar{C}[0,1]$. Or, equivalently, each element in $\mathcal{F}_{cc}(\mathbb{R})$ can be identified with an element in $\bar{C}[0,1] \times \bar{C}[0,1]$. More precisely, each element $\tilde{a}$ in $\mathcal{F}_{cc}(\mathbb{R})$ can be identified with an element $(a^L(\alpha), a^U(\alpha))$ in $\bar{C}[0,1] \times \bar{C}[0,1]$, where $a^L(\alpha) = \tilde{a}_\alpha^L$ and $a^U(\alpha) = \tilde{a}_\alpha^U$, and this identification is isometric and isomorphic.

3. RANKING OF FUZZY NUMBERS

Now we are going to discuss the ranking concept of fuzzy numbers using the possibility and necessity indices proposed by Dubois and Prade [22].

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Dubois and Prade [22] defined the following four indices:

$$\text{Poss}\left(\tilde{a} \succeq \tilde{b}\right) = \sup_{u \geq v} \min\left\{\xi_{\tilde{a}}(u), \xi_{\tilde{b}}(v)\right\},$$

$$\text{Poss}\left(\tilde{a} \succ \tilde{b}\right) = \sup_u \inf_{\{v: u \leq v\}} \min\left\{\xi_{\tilde{a}}(u), 1 - \xi_{\tilde{b}}(v)\right\},$$

$$\text{Nece}\left(\tilde{a} \succeq \tilde{b}\right) = \inf_u \sup_{\{v: u \geq v\}} \max\left\{1 - \xi_{\tilde{a}}(u), \xi_{\tilde{b}}(v)\right\},$$

$$\text{Nece}\left(\tilde{a} \succ \tilde{b}\right) = \inf_{u \leq v} \max\left\{1 - \xi_{\tilde{a}}(u), 1 - \xi_{\tilde{b}}(v)\right\} = 1 - \sup_{u \leq v} \max\left\{\xi_{\tilde{a}}(u), \xi_{\tilde{b}}(v)\right\}.$$

The following results can also be obtained from Sakawa and Yano [23] when the membership functions are assumed to be upper semicontinuous.

PROPOSITION 3.1. *(See [23].) If $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers, then we have:*

(i) *$\text{Poss}(\tilde{a} \succeq \tilde{b}) \geq \alpha$ if and only if $\tilde{a}_\alpha^U \geq \tilde{b}_\alpha^L$;*
(ii) *$\text{Poss}(\tilde{a} \succ \tilde{b}) \geq \alpha$ if and only if $\tilde{a}_\alpha^U \geq \tilde{b}_{1-\alpha}^U$;*
(iii) *$\text{Nece}(\tilde{a} \succeq \tilde{b}) \geq \alpha$ if and only if $\tilde{a}_{1-\alpha}^L \geq \tilde{b}_\alpha^L$;*
(iv) *$\text{Nece}(\tilde{a} \succ \tilde{b}) \geq \alpha$ if and only if $\tilde{a}_{1-\alpha}^L \geq \tilde{b}_{1-\alpha}^U$.*

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then we write $\tilde{a} \succeq_\alpha \tilde{b}$ if and only if $\text{Poss}(\tilde{a} \succ \tilde{b}) \geq \alpha$ and $\text{Nece}(\tilde{a} \succeq \tilde{b}) \geq \alpha$ for any $\alpha \in [0,1]$. We also write $\tilde{b} \preceq_\alpha \tilde{a}$ if and only if $\tilde{a} \succeq_\alpha \tilde{b}$.

PROPOSITION 3.2. *We have $\tilde{a} \succeq_\alpha \tilde{b}$ if and only if $\tilde{a}_\alpha^L \geq \tilde{b}_\alpha^L$ and $\tilde{a}_\alpha^U \geq \tilde{b}_\alpha^U$ for each $\alpha \geq 0.5$.*

PROOF. It is easy to see that if $\tilde{a}$ is a fuzzy number, then $\tilde{a}_{\alpha_1}^L \leq \tilde{a}_{\alpha_1}^L$ and $\tilde{a}_{\alpha_1}^U \geq \tilde{a}_{\alpha_1}^U$ for $\alpha_1 \leq \alpha_2$. From Proposition 3.1, we have $\text{Poss}(\tilde{a} \succ \tilde{b}) \geq \alpha$ if and only if $\tilde{a}_\alpha^U \geq \tilde{b}_{1-\alpha}^U$. Since $\alpha \geq 1 - \alpha$ for $\alpha \geq 0.5$, we have $\tilde{b}_{1-\alpha}^U \geq \tilde{b}_\alpha^U$. Therefore, $\tilde{a}_\alpha^U \geq \tilde{b}_\alpha^U$ for $\alpha \geq 0.5$. On the other hand, $\text{Nece}(\tilde{a} \succeq \tilde{b}) \geq \alpha$ if and only if $\tilde{a}_{1-\alpha}^L \geq \tilde{b}_\alpha^L$. Since $\alpha \geq 1 - \alpha$ for $\alpha \geq 0.5$, we have $\tilde{a}_\alpha^L \geq \tilde{a}_{1-\alpha}^L$. Therefore, $\tilde{a}_\alpha^L \geq \tilde{b}_\alpha^L$ for $\alpha \geq 0.5$. This completes the proof. ∎

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. We say that $\tilde{a} \prec_\alpha \tilde{b}$ if and only if

$$
\begin{array}{ccccc}
\tilde{a}_\alpha^L < \tilde{b}_\alpha^L, & & \tilde{a}_\alpha^L \leq \tilde{b}_\alpha^L, & & \tilde{a}_\alpha^L < \tilde{b}_\alpha^L, \\
 & \text{or} & & \text{or} & \\
\tilde{a}_\alpha^U \leq \tilde{b}_\alpha^U, & & \tilde{a}_\alpha^U < \tilde{b}_\alpha^U, & & \tilde{a}_\alpha^U < \tilde{b}_\alpha^U.
\end{array}
\tag{1}
$$

We also write $\tilde{a} \succ_\alpha \tilde{b}$ if and only if $\tilde{b} \prec_\alpha \tilde{a}$.

4. FORMULATION OF THE FUZZY OPTIMIZATION PROBLEMS

Let f_1, f_2, g_1, and g_2 be real-valued functions defined on the same real vector space V. We say that, for any fixed $x_0 \in V$, $(f_1, g_1) \leq_{x_0} (f_2, g_2)$ if and only if $f_1(x_0) \leq f_2(x_0)$ and $g_1(x_0) \leq g_2(x_0)$. We say that $(f_1, g_1) <_{x_0} (f_2, g_2)$ if and only if

$$
\begin{array}{ccccc}
f_1(x_0) < f_2(x_0), & & f_1(x_0) \leq f_2(x_0), & & f_1(x_0) < f_2(x_0), \\
 & \text{or} & & \text{or} & \\
g_1(x_0) \leq g_2(x_0), & & g_1(x_0) < g_2(x_0), & & g_1(x_0) < g_2(x_0).
\end{array}
\tag{2}
$$

PROPOSITION 4.1. ORDER PRESERVING. *Let $\tilde{a}, \tilde{b} \in \mathcal{F}_{cc}(\mathbb{R})$ and π be the embedding function defined in Theorem 2.1. Then $\tilde{a} \preceq_\alpha \tilde{b}$ if and only if $\pi(\tilde{a}) \leq_\alpha \pi(\tilde{b})$ for $\alpha \geq 0.5$. We also have $\tilde{a} \prec_\alpha \tilde{b}$ if and only if $\pi(\tilde{a}) <_\alpha \pi(\tilde{b})$ for $\alpha \geq 0.5$.*

PROOF. We see that $\tilde{a} \preceq_\alpha \tilde{b}$ if and only if $a^L(\alpha) = \tilde{a}_\alpha^L \leq \tilde{b}_\alpha^L = b^L(\alpha)$ and $a^U(\alpha) = \tilde{a}_\alpha^U \leq \tilde{b}_\alpha^U = b^U(\alpha)$ for $\alpha \geq 0.5$. Then $\pi(\tilde{a}) = (a^L(\alpha), a^U(\alpha)) \leq_\alpha \pi(\tilde{b}) = (b^L(\alpha), b^U(\alpha))$ for $\alpha \geq 0.5$. Similarly, from (1) and (2), we see that $\tilde{a} \prec_\alpha \tilde{b}$ if and only if $\pi(\tilde{a}) <_\alpha \pi(\tilde{b})$ for $\alpha \geq 0.5$. This completes the proof. ∎

Let $\tilde{f}$ be a function defined by $\tilde{f} : V \to \mathcal{F}_{cc}(\mathbb{R})$, where V is a real vector space. Then $\tilde{f}$ is called a fuzzy-valued function (defined on a real vector space V).

We consider a fuzzy optimization problem as follows.

$$
\begin{aligned}
\min \quad & \tilde{f}(x), \\
\text{subject to} \quad & g_i(x) \leq 0, \qquad i = 1, \ldots, m, \\
& x \in X,
\end{aligned}
\tag{FOP1}
$$

where $\tilde{f}$ is a fuzzy-valued function defined on a real vector space V, $g_i(x)$ are real-valued functions defined on X for $i = 1, \ldots, m$, and X is a subspace of the real vector space V. We say that x^* is an α-optimal solution of problem (FOP1) if there exists no x ($\neq x^*$), such that $\tilde{f}(x) \prec_\alpha \tilde{f}(x^*)$. Let π be the function defined in Theorem 2.1. Then, we consider the following optimization problem (OP1):

$$
\begin{aligned}
\min \quad & \pi \circ \tilde{f}(x) = \pi \left(\tilde{f}(x) \right), \\
\text{subject to} \quad & g_i(x) \leq 0, \qquad i = 1, \ldots, m, \\
& x \in X,
\end{aligned}
\tag{OP1}
$$

by applying the embedding function π to problem (FOP1). We say that x^* is an α-optimal solution of problem (OP1) if there exists no x ($\neq x^*$), such that $\pi(\tilde{f}(x)) <_\alpha \pi(\tilde{f}(x^*))$. From the embedding Theorem 2.1, we have

$$
\pi \circ \tilde{f}(x) = \pi \left(\tilde{f}(x) \right) = \left(\left(\tilde{f}(x) \right)^L (\alpha), \left(\tilde{f}(x) \right)^U (\alpha) \right) \stackrel{\text{def}}{=} \left(f^L(x, \alpha), f^U(x, \alpha) \right),
\tag{3}
$$

for any fixed $x \in X$. Therefore x^* is an α-optimal solution of problem (OP1) if there exists no x $(\neq x^*)$, such that $(f^L(x, \alpha), f^U(x, \alpha)) <_\alpha (f^L(x^*, \alpha), f^U(x^*, \alpha))$, i.e.,

$$f^L(x, \alpha) < f^L(x^*, \alpha), \qquad f^L(x, \alpha) \leq f^L(x^*, \alpha), \qquad f^L(x, \alpha) < f^L(x^*, \alpha),$$
$$f^U(x, \alpha) \leq f^U(x^*, \alpha), \quad \text{or} \quad f^U(x, \alpha) < f^U(x^*, \alpha), \quad \text{or} \quad f^U(x, \alpha) < f^U(x^*, \alpha).$$

PROPOSITION 4.2. x^* *is an α-optimal solution of problem (FOP1) if and only if x^* is an α-optimal solution of problem (OP1) for $\alpha \geq 0.5$.*

PROOF. From Proposition 4.1, we have $\tilde{f}(x) \prec_\alpha \tilde{f}(x^*)$ if and only if $\pi(\tilde{f}(x)) <_\alpha \pi(\tilde{f}(x^*))$, for $\alpha \geq 0.5$. This completes the proof. ∎

From equation (3) and problem (OP1), we consider the following biobjective optimization problem (BOP1):

$$\min \quad \left(f^L(x, \alpha), f^U(x, \alpha) \right),$$
$$\text{subject to} \quad g_i(x) \leq 0, \qquad i = 1, \ldots, m, \qquad \text{(BOP1)}$$
$$x \in X, \qquad 0 \leq \alpha \leq 1.$$

We adopt the notation $\vec{x} = (x, \alpha)$. Then we have the following result.

PROPOSITION 4.3. *If $\vec{x}^* = (x^*, \alpha^*)$ is a Pareto optimal solution of the biobjective optimization problem (BOP1) for some $\alpha^* \in [0, 1]$, then x^* is an α^*-optimal solution of problem (OP1).*

PROOF. Suppose that x^* is not an α^*-optimal solution of problem (OP1). Then there exists an x $(\neq x^*)$, such that

$$f^L(x, \alpha^*) < f^L(x^*, \alpha^*), \qquad f^L(x, \alpha^*) \leq f^L(x^*, \alpha^*), \qquad f^L(x, \alpha^*) < f^L(x^*, \alpha^*),$$
$$f^U(x, \alpha^*) \leq f^U(x^*, \alpha^*), \quad \text{or} \quad f^U(x, \alpha^*) < f^U(x^*, \alpha^*), \quad \text{or} \quad f^U(x, \alpha^*) < f^U(x^*, \alpha^*). \qquad (4)$$

Let $\vec{x} = (x, \alpha^*)$. Then $\vec{x} \neq \vec{x}^*$ since $x \neq x^*$. This means that there exists an $\vec{x}$ $(\neq \vec{x}^*)$, such that equation (4) holds. Therefore, $\vec{x}^*$ is not a Pareto optimal solution of problem (BOP1), which contradicts the hypothesis. This completes the proof. ∎

THEOREM 4.1. *If (x^*, α^*) is a Pareto optimal solution of the biobjective optimization problem (BOP1) for some $\alpha^* \geq 0.5$, then x^* is an α^*-optimal solution of the fuzzy optimization problem (FOP1).*

PROOF. The result follows from Propositions 4.2 and 4.3. ∎

REMARK 4.1. From the proof of Proposition 4.3, we see that if x^* is a Pareto optimal solution of the following biobjective optimization problem:

$$\min \quad \left(f^L(x, \alpha^*), f^U(x, \alpha^*) \right),$$
$$\text{subject to} \quad g_i(x) \leq 0, \qquad i = 1, \ldots, m,$$
$$x \in X,$$

for some $\alpha^* \geq 0.5$, then x^* is also an α^*-optimal solution of problem (FOP1).

The scalarization methods proposed in the multiobjective optimization problems are really useful for obtaining the Pareto optimal solutions. For instance, the weighting problem of (BOP1) defined by

$$\min \quad w_1 f^L(x, \alpha) + w_2 f^U(x, \alpha),$$
$$\text{subject to} \quad g_i(x) \leq 0, \qquad i = 1, \ldots, m, \qquad \text{(OP2)}$$
$$x \in X, \qquad 0 \leq \alpha \leq 1,$$

where w_1 and w_2 are positive real numbers (w_1 and w_2 are not necessarily to be normalized as $0 \leq w_1, w_2 \leq 1$), provides the Pareto optimal solutions of problem (BOP1). If (x^*, α^*) is an optimal solution of problem (OP2), then (x^*, α^*) is a Pareto optimal solution of problem (BOP1) (see [24]); that is, x^* is an α^*-optimal solution of problem (FOP1) from Theorem 4.1 for $\alpha^* \geq 0.5$.

5. FUZZY OPTIMIZATION PROBLEMS WITH FUZZY COEFFICIENTS

In this section, we consider another kind of fuzzy optimization problems, i.e., the fuzzy optimization problems with fuzzy coefficients. For example, the fuzzy objective function will look like

$$\tilde{5}x_1^3 \oplus \tilde{8}x_2x_4^7 \oplus \tilde{7}x_3^2 \oplus \tilde{4}x_1x_4 \oplus \tilde{6}x_3x_5, \tag{5}$$

where the decision variables x_1, x_2, x_3, x_4, x_5 take values on $\mathbb{R}$. Therefore, x_1^3, $x_2x_4^7$, x_3^2, x_1x_4, and x_3x_5 can be regarded as the crisp numbers $\tilde{1}_{\{x_1^3\}}$, $\tilde{1}_{\{x_2x_4^7\}}$, $\tilde{1}_{\{x_3^2\}}$, $\tilde{1}_{\{x_1x_4\}}$, and $\tilde{1}_{\{x_3x_5\}}$ with values x_1^3, $x_2x_4^7$, x_3^2, x_1x_4, and x_3x_5, respectively. Then equation (5) will be rewritten as follows:

$$\left(\tilde{5} \otimes \tilde{1}_{\{x_1^3\}}\right) \oplus \left(\tilde{8} \otimes \tilde{1}_{\{x_2x_4^7\}}\right) \oplus \left(\tilde{7} \otimes \tilde{1}_{\{x_3^2\}}\right) \oplus \left(\tilde{4} \otimes \tilde{1}_{\{x_1x_4\}}\right) \oplus \left(\tilde{6} \otimes \tilde{1}_{\{x_3x_5\}}\right) \tag{6}$$

in order to use Proposition 2.1.

Let $\tilde{a} \in \mathcal{F}_{cc}(\mathbb{R})$. We say that $\tilde{a}$ is a fuzzy real number if its 1-level set $\tilde{a}_1$ is a singleton set. We see that if $\tilde{a}$ is a fuzzy real number, then we have

$$\tilde{a}_\alpha^L < \tilde{a}_1^L = \tilde{a}_1^U \stackrel{\text{def}}{=} a < \tilde{a}_\alpha^U, \qquad \text{for } 0 \le \alpha < 1. \tag{7}$$

The real number a is called the core value of $\tilde{a}$ and is denoted by $a = \text{core}(\tilde{a})$. For example, $\text{core}(\tilde{6}) = 6$. If $\tilde{1}_{\{m\}}$ is a crisp number with value m, then $\text{core}(\tilde{1}_{\{m\}}) = m$. The following proposition follows immediately from Proposition 2.1.

PROPOSITION 5.1. Let $\tilde{a}$, $\tilde{b}$ be fuzzy real numbers and $\tilde{1}_{\{m\}}$, $\tilde{1}_{\{m_1\}}$, $\tilde{1}_{\{m_2\}}$ be crisp numbers with values m, m_1, and m_2, respectively. Then the following properties hold true.

(i) $\text{core}(\tilde{a} \oplus \tilde{b}) = \text{core}(\tilde{a}) + \text{core}(\tilde{b}) = a + b$.

(ii) $\text{core}(\tilde{a} \otimes \tilde{1}_{\{m\}}) = m \cdot \text{core}(\tilde{a}) = m \cdot a$.

(iii) $\text{core}((\tilde{a} \otimes \tilde{1}_{\{m_1\}}) \oplus (\tilde{b} \otimes \tilde{1}_{\{m_2\}})) = m_1 \cdot \text{core}(\tilde{a}) + m_2 \cdot \text{core}(\tilde{b}) = m_1 \cdot a + m_2 \cdot b$.

Using Proposition 5.1, the core value of equation (6) is given by

$$5x_1^3 + 8x_2x_4^7 + 7x_3^2 + 4x_1x_4 + 6x_3x_5.$$

Therefore, the core value of fuzzy objective function is a real-valued function.

We now consider a fuzzy optimization problem with fuzzy coefficients that are assumed to be fuzzy real numbers as follows:

$$\begin{aligned} \min \quad & \tilde{f}(\mathbf{x}), \\ \text{subject to} \quad & g_i(\mathbf{x}) \le 0, \qquad i = 1, 2, \ldots, m, \\ & \mathbf{x} \in \mathbb{R}_+^n. \end{aligned} \tag{FOP2}$$

Then we have the corresponding crisp optimization problem

$$\begin{aligned} \min \quad & f(\mathbf{x}), \\ \text{subject to} \quad & g_i(\mathbf{x}) \le 0, \qquad i = 1, 2, \ldots, m, \\ & \mathbf{x} \in \mathbb{R}_+^n, \end{aligned} \tag{OP3}$$

by taking the core value of fuzzy objective function in problem (FOP2) using Proposition 5.1.

By considering problem (FOP2), we also have the corresponding biobjective optimization problem (BOP2) as follows.

$$\begin{aligned} \min \quad & \left(f^L(\mathbf{x}, \alpha), f^U(\mathbf{x}, \alpha)\right), \\ \text{subject to} \quad & g_i(\mathbf{x}) \le 0, \qquad i = 1, 2, \ldots, m \\ & \mathbf{x} \in \mathbb{R}_+^n, \qquad 0 \le \alpha \le 1. \end{aligned} \tag{BOP2}$$

Then the following result follows from Theorem 4.1 immediately.

THEOREM 5.1. *If (x^*, α^*) is a Pareto optimal solution of problem (BOP2) for some $\alpha^* \geq 0.5$, then x^* is an α^*-optimal solution of problem (FOP2).*

THEOREM 5.2. *If $\mathbf{x}^*$ is an optimal solution of problem (OP3), then $\mathbf{x}^*$ is a 1-optimal solution of problem (FOP2).*

PROOF. From Theorem 5.1, we just need to show that $(\mathbf{x}, \alpha) = (\mathbf{x}^*, 1)$ is a Pareto optimal solution of problem (BOP2). It is easy to see that $(\mathbf{x}^*, 1)$ is a feasible solution of problem (BOP2). From equation (7), we have, for any fixed $\mathbf{x}$,

$$f^L(\mathbf{x}, \alpha) < f^L(\mathbf{x}, 1) = f(\mathbf{x}) = f^U(\mathbf{x}, 1) < f^U(\mathbf{x}, \alpha), \qquad \text{for } 0 \leq \alpha < 1. \tag{8}$$

Suppose that $(\mathbf{x}^*, 1)$ is not a Pareto optimal solution of problem (BOP2), then there exists an $(\mathbf{x}, \alpha) \in \mathbb{R}_+^n \times [0, 1]$, such that

$$
\begin{aligned}
&f^L(\mathbf{x}, \alpha) < f^L(\mathbf{x}^*, 1) = f(\mathbf{x}^*), && f^L(\mathbf{x}, \alpha) \leq f^L(\mathbf{x}^*, 1) = f(\mathbf{x}^*), \\
&f^U(\mathbf{x}, \alpha) \leq f^U(\mathbf{x}^*, 1) = f(\mathbf{x}^*), && \text{or} && f^U(\mathbf{x}, \alpha) < f^U(\mathbf{x}^*, 1) = f(\mathbf{x}^*), \\
&&& f^L(\mathbf{x}, \alpha) < f^L(\mathbf{x}^*, 1) = f(\mathbf{x}^*), \\
&\text{or} && f^U(\mathbf{x}, \alpha) < f^U(\mathbf{x}^*, 1) = f(\mathbf{x}^*).
\end{aligned}
\tag{9}
$$

Since $\mathbf{x}^*$ is an optimal solution of problem (OP3) by hypothesis, using equation (8), we have

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) = f^U(\mathbf{x}, 1) < f^U(\mathbf{x}, \alpha),$$

which contradicts equation (9). Therefore, we conclude that $(\mathbf{x}^*, 1)$ is a Pareto optimal solution of problem (BOP2). This completes the proof. ∎

6. CONCLUSIONS

The notions of possibility and necessity measures were used to introduce the ranking method on the set of all fuzzy numbers. Under this setting, the α-optimal solution concept of fuzzy optimization problem was provided by applying the embedding theorem. If we consider the fuzzy optimization problem (FOP1), then x^* is an α-optimal solution of (FOP1) when (x^*, α^*) is a Pareto optimal solution of problem (BOP1) for some $\alpha^* \geq 0.5$. However, if we consider the fuzzy optimization problem (FOP2) (with fuzzy coefficients), then, from Theorem 5.2, we further have that $\mathbf{x}^*$ is a 1-optimal solution of problem (FOP2) when $\mathbf{x}^*$ is an optimal solution of the crisp optimization problem (OP3).

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