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Isometric reflection vectors in Banach spaces

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Abstract

The aim of this paper is to study the set I_X^r of isometric reflection vectors of a real Banach space X. We deal with geometry of isometric reflection vectors and parallelogram identity vectors, and we prove that a real Banach space is a Hilbert space if the set of parallelogram identity vectors has nonempty interior. It is also shown that every real Banach space can be decomposed as an I^r-sum of a Hilbert space and a Banach space with some points which are not isometric reflection vectors. Finally, we give a new proof of the Becerra–Rodríguez result: a real Banach space X is a Hilbert space if and only if I_X^r is not rare.

1. Introduction

Let X be a real Banach space and M a closed linear subspace. M is said to be an *isometric reflection subspace* of X if there exists a linear subspace N of X such that $X = M \oplus N$ and $\|m+n\| = \|m-n\|$ for every $m \in M,\ n \in N$. This determines two mappings: A linear surjective isometry $T_M : X \to X$ defined by $T_M(m+n) = m-n$, which is called the *isometric reflection* of M in X; and a linear projection $P_M : X \to M$ defined by $P_M(m+n) = m$, which is called the *isometric reflection projection* of X onto M. Note that $\|m\| \leqslant \frac{1}{2}(\|m+n\| + \|m-n\|) = \|m+n\|$, so $\|P_M\| = 1$.

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Obviously $T_M(x) = 2P_M(x) - x$ for every $x \in X$, and $\{0\}$ and X are trivial isometric reflection subspaces. A 2-dimensional Banach space in which those are the two only isometric reflection subspaces appears in [7].

The following proposition can be easily proved and provides a simple characterization of the linear isometries which are isometric reflections.

Proposition 1.1. *Let X be a real Banach space and $T : X \to X$ a linear isometry. T is an isometric reflection if and only if $T^2 = \mathrm{Id}$. In such case, $M = \{x \in X\colon\ Tx = x\}$ is an isometric reflection subspace, with $P_M = \frac{1}{2}(\mathrm{Id} + T)$ and thus $T_M = T$.*

Whenever A is a subset of X, we use the notation $\mathcal{L}(A)$ for the linear span of A and $\overline{A}$ for the closure of A. If $M = \mathcal{L}(u)$ is an isometric reflection subspace for some $u \in X$ then u is called *isometric reflection vector*. If $u \neq 0$ then the expression $P_{\mathcal{L}(u)}x = f_u(x)u$ determines a functional $f_u \in X^*$, which is called *the isometric reflection functional* of u, with $f_u(u) = 1$ and $\|f_u\| = \|u\|^{-1}$. We will usually write T_u and P_u instead of $T_{\mathcal{L}(u)}$ and $P_{\mathcal{L}(u)}$.

Let us recall two relevant results about isometric reflection vectors:

(a) In [8], A. Skorik and M. Zaidenberg proved that a real Banach space is a Hilbert space if and only if it is almost transitive and its unit sphere has an isometric reflection vector.
(b) In [4], J. Becerra and A. Rodríguez proved that a real Banach space is a Hilbert space if and only if the subset of its unit sphere whose points are isometric reflection vectors is not rare in the unit sphere.

Note that the "only if" part in both results is trivial: In a Hilbert space, orthogonal projections are isometric reflection projections.

The set of isometric reflection vectors of X will be denoted by I_X^r. It is well known (see [3, p. 43]) that this set is closed in X.

2. Geometry of parallelogram identity vectors

Definition 2.1. Let X be a real Banach space. Consider $u \in X$. We will say that u is a parallelogram identity vector if

$$\frac{\|u + x\|^2 + \|u - x\|^2}{2} = \|u\|^2 + \|x\|^2$$

for every $x \in X$.

Proposition 2.2. *Let X be a real Banach space. If $u \in X$ is a parallelogram identity vector then $\mathcal{L}(u, x)$ is a Hilbert space for every $x \in X$.*

Proof. Consider $Y = \mathcal{L}(u, x)$. We may assume that $u \in S_Y$ and $\dim Y = 2$. The function $f : S_Y \to \mathbb{R}$ defined by $f(z) = \|u + z\|^2$ is continuous and verifies $f(u) = 4$, $f(-u) = 0$. Since S_Y is connected there exists $x \in S_Y$ such that $f(x) = 2$. Being u a parallelogram identity vector, necessarily $\|u+x\|^2 = \|u-x\|^2 = 2$. Note that if Y were a Hilbert space, x would be orthogonal to u.

Take $m, n \in \mathbb{Z}$, we will prove that $\|mu + nx\|^2 = m^2 + n^2$. It is obviously true for $m, n \in \{-1, 0, 1\}$.

Let us denote $a_k = \|ku + nx\|^2$, where $k \in \{1, \dots, n-1\}$. Since u is a parallelogram identity vector, we obtain the following equations:

$$
\begin{aligned}
&(a_2 + n^2)/2 = 1 + a_1,\\
&(a_3 + a_1)/2 = 1 + a_2,\\
&(a_4 + a_2)/2 = 1 + a_3,\\
&\quad\vdots\\
&(a_{n-1} + a_{n-3})/2 = 1 + a_{n-2},\\
&(2n^2 + a_{n-2})/2 = 1 + a_{n-1}.
\end{aligned}
$$

This system has $n-1$ equations and $n-1$ variables; it is compatible determined and allows us to obtain the value of $\|mu + nx\|^2$ for $n \geqslant 2$, $1 \leqslant m < n$. Analogously we would work with $a_k = \|ku - nx\|^2$ to obtain the value of $\|mu - nx\|^2$ for $n \geqslant 2$ and $1 \leqslant m < n$; the resulting system of equations is the same.

Let us consider $n \geqslant 2$ and $m > n$, and suppose that we know the value of $\|(m-1)u + nx\|^2$ and $\|(m-2)u + nx\|^2$. Then u provides the equation $(\|mu + nx\|^2 + \|(m-2)u + nx\|^2)/2 = 1 + \|(m-1)u + nx\|^2$. Inductively we can determine the value of $\|mu + nx\|^2$ for $n \geqslant 2$, $m > n$. We would proceed analogously with $\|mu - nx\|^2$.

We have considered every possible case for m and n. Observe also that the coefficients of all the equations we have used do not depend on what concrete space Y is; therefore these coefficients will be the same in the Euclidean-plane case. Hence, necessarily we will obtain the value $\|mu + nx\|^2 = m^2 + n^2$ for every $m, n \in \mathbb{Z}$.

By the standard argument we can prove $\|pu + qx\|^2 = p^2 + q^2$ for every $p, q \in \mathbb{Q}$ and by density $\|\alpha u + \beta x\|^2 = \alpha^2 + \beta^2$ for every $\alpha, \beta \in \mathbb{R}$. We conclude that Y is a Hilbert space. □

The next result was proved in [1].

Proposition 2.3. *Let X be a real Banach space and $u \in X$. The following assertions are equivalent*:

(1) *u is an L^2-summand vector.*
(2) *u is a parallelogram identity vector and an isometric reflection vector.*
(3) *$\mathcal{L}(u, x)$ is a Hilbert space for every $x \in X$.*

And, in such case, u is strongly smooth and locally uniformly rotund.

Taking into account Proposition 2.2 we deduce immediately the following corollary.

Corollary 2.4. *Let X be a real Banach space. $u \in X$ is a parallelogram identity vector if and only if it is an L^2-summand vector. In such case, u is strongly smooth, locally uniformly rotund and an isometric reflection vector.*

The lemma and theorem that follow could be deduced from the previous corollary and the results in [1]. Here we give a direct proof.

Lemma 2.5. *Let X be a real Banach space. If u, v and $u+v$ are parallelogram identity vectors of X then $u-v$ is also a parallelogram identity vector.*

Proof. Let x be in X. On the one hand, we have

$$\begin{aligned}
&\frac{\|u+v+x\|^2+\|u+v-x\|^2}{2}\\
&\quad=\frac{2\|u\|^2+2\|v+x\|^2-\|u-v-x\|^2+2\|u\|^2+2\|v-x\|^2-\|u-v+x\|^2}{2}\\
&\quad=2\|u\|^2+2\|v\|^2+2\|x\|^2-\frac{\|u-v-x\|^2+\|u-v+x\|^2}{2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\frac{\|u+v+x\|^2+\|u+v-x\|^2}{2}\\
&\quad=\|u+v\|^2+\|x\|^2=2\|u\|^2+2\|v\|^2-\|u-v\|^2+\|x\|^2.
\end{aligned}$$

We deduce

$$\frac{\|u-v-x\|^2+\|u-v+x\|^2}{2}=\|u-v\|^2+\|x\|^2,$$

which concludes the proof. □

Example 2.6. It is worth mentioning that the previous lemma fails for isometric reflection vectors. The 2-dimensional real Banach space whose unit ball is the convex closure of the set $\{(2,1),(2,-3),(0,-4),(-2,-1),(-2,3),(0,4)\}$ verifies that $(2,1)$, $(-2,1)$ and $(0,2)$ are isometric reflection vectors but $(4,0)$ is not an isometric reflection vector. The latter fact can be proved as a consequence of Proposition 3.3 on this paper.

Theorem 2.7. *Let X be a real Banach space. If the set of parallelogram identity vectors of X has nonempty interior then X is a Hilbert space.*

Proof. Let P be the set of parallelogram identity vectors of X. Take $u \in P$ and $r > 0$ such that $B(u,r) \subseteq P$. Consider $x \in B(0,r)$. We have that $x+u \in B(u,r) \subseteq P$. In addition, $(x+u)+u = x+2u \in B(2u,r) \subseteq B(2u,2r) = 2B(u,r) \subseteq P$. By Lemma 2.5, $x = (x+u)-u \in P$. We have proved that $B(0,r) \subseteq P$, which concludes the proof. □

3. Geometry of isometric reflection vectors

The notions of exposed and rotund point can be found in [6] and [5], respectively.

Theorem 3.1. *Let X be a real Banach space. Let $u \in S_X$ be an isometric reflection vector. Then*

(1) *u is an extreme point of B_X if and only if it is an exposed point of B_X.*
(2) *f_u is an isometric reflection vector of X^* and its isometric reflection functional is $\hat{u}$.*
(3) *If v is an isometric reflection vector of X such that $f_u = f_v$ then $u = v$.*

Proof. (1) We only have to prove one of the implications. Assume that u is an extreme point of B_X. We know that $f_u(u) = 1$; if $y \in S_X$ and $f_u(y) = 1$ then $T_u(y) = 2f_u(y)u - y = 2u - y$ and therefore $\|2u - y\| = \|y\| = 1$. Since $u = \frac{1}{2}y + \frac{1}{2}(2u - y)$, we deduce that $u = y$. Thus u is an exposed point of B_X.

(2) Since $f_u(u) = 1$, we obtain $X^* = \ker(\hat{u}) \oplus \mathcal{L}(f_u)$. For every $g \in \ker \hat{u}$, $m \in \ker f_u$ and $\alpha, \lambda \in \mathbb{R}$, we have $(g + \lambda f_u)(m + \alpha u) = g(m) + \lambda\alpha = (g - \lambda f_u)(m - \alpha u) = (g - \lambda f_u)(-T_u)(m + \alpha u)$, therefore $\|g + \lambda f_u\| = \|(g - \lambda f_u)(-T_u)\| = \|g - \lambda f_u\|$.

(3) If $f_u = f_v$ then $\hat{u} = f_{f_u} = f_{f_v} = \hat{v}$, therefore $u = v$. $\square$

If we consider $(\mathbb{R}^2, \|\cdot\|_\infty)$, every vertex of its unit ball is an extreme point and an isometric reflection vector, whereas not a rotund point; so (1) in Theorem 3.1 cannot be extended that way.

Now we give two propositions that try to clarify the relation between extreme points and isometric reflection points in the 2-dimensional case.

Proposition 3.2. *Let X be a 2-dimensional, real Banach space. If u and v are extreme points of B_X such that $u + v \in I_X^r$ then $u - v \in I_X^r$.*

Proof. Since $T = T_{u+v}$ verifies $T(u + v) = u + v$, there exists $x \in X$ such that $Tu = v + x$ and $Tv = u - x$. We also know that $u = T^2u = T(v + x) = u - x + Tx$, this implies $x = Tx$ and so there exists $\gamma \in \mathbb{R}$ such that $x = \gamma(u + v)$. By symmetry, we can suppose without loss of generality that $\gamma \geqslant 0$. The extreme points v, $-u = v - (u + v)$ and $Tu = v + \gamma(u + v)$ lie on a straight line, thus $\gamma = 0$.

Then we have $Tu = v$, $Tv = u$ and $\|\alpha(u + v) + \beta(u - v)\| = \|T(\alpha(u + v) + \beta(u - v))\| = \|\alpha(u + v) - \beta(u - v)\|$ for every $\alpha, \beta \in \mathbb{R}$. Therefore $u - v \in I_X^r$. $\square$

Proposition 3.3. *Let X be a 2-dimensional, real Banach space. Suppose that $x, y \in B_X$ are extreme points such that the segment $[x, y] = \{tx + (1 - t)y\colon\ t \in [0, 1]\}$ is contained in S_X. If $u \in I_X^r \cap ([x, y] \setminus \{x, y\})$ then $u = \frac{1}{2}(x + y)$.*

Proof. Suppose that $[x, y]$ and $[T_u x, T_u y]$ do not lie in the same straight line. Then $[x, y] \cap [T_u x, T_u y] = \{u\}$. For every $z \in [x, y]$ we take the semistraight line $r_z = \{tz\colon\ t \in \mathbb{R}^+\}$. Let us define $A = \bigcup\{r_z\colon\ z \in [x, y]\}$.

It is easy to prove that $A = \{\alpha x + \beta y\colon\ \alpha \geqslant 0,\ \beta \geqslant 0\}$. Consider the continuous mapping $f : A \to \mathbb{R}^+ \times \mathbb{R}^+$ defined by $f(\alpha x + \beta y) = (\alpha, \beta)$; since $u \in [x, y] \setminus \{x, y\}$, we deduce that $u \in \operatorname{int}(A)$. There exist $\varepsilon > 0$ such that $B(u, \varepsilon) \subset A$ and $a \in [T_u x, T_u y]$ such that $0 < \|u - a\| < \varepsilon$. As $a \in A$, there exist $z \in [x, y]$ and $t \geqslant 0$ such that $a = tz$, but $\|a\| =$

$\|z\| = 1$, hence $t = 1$ and $a = z$. This implies that $a \in [x, y] \cap [T_u x, T_u y]$ and thus $a = u$, contradicting the choice of a.

Therefore $[x, y]$ and $[T_u x, T_u y]$ lie in the same straight line. Since x, y, $T_u x$ and $T_u y$ are extreme points, necessarily $[x, y] = [T_u x, T_u y]$ and, as T_u cannot be the identity mapping, $x = T_u y$ and $y = T_u x$. Then $\|u - x\| = \|T_u u - T_u x\| = \|u - y\|$, which implies $u = \frac{1}{2}(x + y)$. $\square$

4. Isometric reflection vectors and Hilbert spaces.

The following result is well known and can be found in [2].

Theorem 4.1. *Let X be a real Banach space. If $I_X^r = X$ then X is a Hilbert space.*

Now we need some preliminary lemmas.

Lemma 4.2. *Let $M \subseteq Y \subseteq X$ be real Banach spaces. If M is an isometric reflection subspace of X then M is an isometric reflection subspace of Y.*

Proof. Obviously $Y = M \oplus (\ker P_M \cap Y)$. Besides, for every $m \in M$ and $n \in \ker P_M \cap Y$ we have $\|m + n\| = \|m - n\|$. $\square$

Corollary 4.3. *Let $Y \subseteq X$ be real Banach spaces. If $Y \subseteq I_X^r$ then Y is a Hilbert space.*

Lemma 4.4. *Let $M \subseteq Y \subseteq X$ be real Banach spaces. If M and Y are isometric reflection subspaces of X then* $\ker P_Y \subseteq \ker P_M$.

Proof. If $x \in \ker P_Y$ then $x = m + n$ with $m \in M$ and $n \in \ker P_M$. So

$$\begin{aligned}\|x\| &= \|m + n\| = \|m - n\| = \|2m - x\| = \|2m + x\| = \|3m + n\| \\ &= \|3m - n\| = \|4m - x\| = \|4m + x\| = \|5m + n\| = \cdots.\end{aligned}$$

We have $\|x\| = \|(2k - 1)m + n\|$ for each $k \in \mathbb{N}$, which implies $m = 0$. $\square$

Corollary 4.5. *Let X be a real Banach space. If $M \subseteq X$ is an isometric reflection subspace then there exists a unique closed linear subspace $N \subseteq X$ such that $M \oplus N = X$ and $\|m + n\| = \|m - n\|$ for every $m \in M$, $n \in N$.*

The space N in Corollary 4.5 will be called the *isometric reflection complement* of M in X. In addition, we obtain that P_M and T_M are also unique.

Proposition 4.6. *Let X be a real Banach space, M an isometric reflection subspace and u an isometric reflection vector. Then*

(1) $\mathcal{L}(M \cup \{u\}) \oplus (\ker P_M \cap \ker P_u) = X$.
(2) *If $u \in \ker P_M$ then $\mathcal{L}(M \cup \{u\})$ is an isometric reflection subspace.*

Proof. (1) First, we will see that $\mathcal{L}(M \cup \{u\}) \cap (\ker P_M \cap \ker P_u) = \{0\}$. Take $x \in \mathcal{L}(M \cup \{u\}) \cap (\ker P_M \cap \ker P_u)$. We can write $x = m + \alpha u$ with $m \in M$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \|m\| &= \|x - \alpha u\| = \|x + \alpha u\| = \|2x - m\| = \|2x + m\| = \|3x - \alpha u\| \\ &= \|3x + \alpha u\| = \|4x - m\| = \|4x + m\| = \|5x - \alpha u\| = \cdots. \end{aligned}$$

We deduce $\|m\| = \|(2k-1)x - \alpha u\|$ for every $k \in \mathbb{N}$, which implies $x = 0$.

Now, we will see that $\mathcal{L}(M \cup \{u\}) + (\ker P_M \cap \ker P_u) = X$.

If $u - P_M(u) \in \ker P_u$ then $u - P_M(u) \in \mathcal{L}(M \cup \{u\}) \cap (\ker P_M \cap \ker P_u) = \{0\}$ and thus $u = P_M(u) \in M$. By Lemma 4.4 we deduce the desired equality.

If $u - P_M(u) \notin \ker P_u$ then $\mathcal{L}(u - P_M(u)) + (\ker P_M \cap \ker P_u) = \ker P_M$ and therefore $M + \mathcal{L}(u - P_M(u)) + (\ker P_M \cap \ker P_u) = X$. Since $\mathcal{L}(M \cup \{u\}) = M + \mathcal{L}(u - P_M(u))$ we obtain the result in this case as well.

(2) By Lemma 4.4, $M \subseteq \ker P_u$. For every $m \in M$, $\alpha \in \mathbb{R}$ and $x \in \ker P_M \cap \ker P_u$ we have

$$\begin{aligned} \big\|(m + \alpha u) + x\big\| &= \big\|(m + x) + \alpha u\big\| = \big\|(m + x) - \alpha u\big\| = \big\|m + (x - \alpha u)\big\| \\ &= \big\|m - (x - \alpha u)\big\| = \big\|(m + \alpha u) - x\big\|, \end{aligned}$$

which concludes the proof. □

Theorem 4.7. *Let X be a real Banach space. There exists a closed linear subspace H of X such that*

(1) *H is a Hilbert space.*
(2) *H is an isometric reflection subspace of X.*
(3) *Every element of H is an isometric reflection vector of X.*
(4) *H is maximal for the inclusion with respect to Properties* 2 *and* 3.

Proof. Consider the set

$$\mathcal{S} = \big\{M \subseteq I_X^r \colon\ M \text{ is an isometric reflection subspace of } X\big\}$$

ordered by inclusion.

Let I be a completely ordered set, and let $(M_\alpha)_{\alpha \in I}$ be a chain of $\mathcal{S}$. Define

$$M = \bigcup_{\alpha \in I} M_\alpha \quad \text{and} \quad L = \bigcap_{\alpha \in I} \ker(P_{M_\alpha}).$$

Since I_X^r is closed, we have $\overline{M} \subseteq I_X^r$. By Corollary 4.3, $\overline{M}$ is a Hilbert space. Since M and L are topological complements, so are $\overline{M}$ and L. It is also clear, by density, that $\|m + n\| = \|m - n\|$ for every $m \in \overline{M}$ and $n \in L$. Therefore $\overline{M}$ is an isometric reflection subspace of $F = \overline{M} \oplus L$.

Now we will prove that $F = X$. Take $x \in X$ with $\|x\| = 1$. For every $\alpha \in I$ there exist $x_\alpha \in M_\alpha$ and $z_\alpha \in \ker P_{M_\alpha}$ such that $x = x_\alpha + z_\alpha$. Since $\overline{M}$ is a Hilbert space and $(x_\alpha)_{\alpha \in I}$ is a net in $\overline{M}$ such that $\|x_\alpha\| \leqslant 1$ for every $\alpha \in I$, we deduce that there exists a subnet $(x_\beta)_{\beta \in J}$ of $(x_\alpha)_{\alpha \in I}$ which is ω-convergent to some $x_0 \in \overline{M}$. If we fix $\gamma \in I$ then for every

$\beta \geqslant \gamma$ we have $x - x_\beta = z_\beta \in \ker P_{M_\beta} \subseteq \ker P_{M_\gamma}$. Hence, $x - x_0 \in \ker P_{M_\gamma}$ and, since γ is arbitrary, we deduce that $x - x_0 \in \bigcap\{\ker P_{M_\alpha}\colon \alpha \in I\} = L$. Thus $F = X$.

We have proved that $\overline{M}$ is an upper bound for the chain $(M_\alpha)_{\alpha\in I}$ and belongs to $\mathcal{S}$. By Zorn's lemma $\mathcal{S}$ has a maximal element H, which automatically verifies conditions (2)–(4). Condition (1) is deduced from condition (3) and Corollary 4.3. $\square$

It is very tempting to affirm that in Theorem 4.7 no point of the isometric reflection complement of H in X is an isometric reflection vector. However $(\mathbb{R}^2, \|\cdot\|_\infty)$ is a trivial counterexample to this assertion.

Finally, we provide the announced proof of the Becerra–Rodríguez result.

Theorem 4.8. *Let X be a real Banach space. If $I_X^r \cap S_X$ has nonempty interior in S_X then X is a Hilbert space.*

Proof. Let u be an interior point of $I_X^r \cap S_X$ in S_X. Given $x \in X$, take $Y = \mathcal{L}(u, x)$. We can suppose $\dim Y = 2$, otherwise Y would be a Hilbert space. By Lemma 4.2 we have $I_X^r \cap Y \subset I_Y^r$, therefore $I_Y^r \cap S_Y$ has nonempty interior in S_Y; in other words, there exists a connected component C of $I_Y^r \cap S_Y$ which is not unitary. C is closed in S_Y since $I_Y^r \cap S_Y$ is closed in S_Y. Take $z \in C$ and consider the isometric reflection $T_z : Y \to Y$ and the open halfplanes A and B determined by the straight line $R = \{tz\colon t \in \mathbb{R}\}$. Note that $T_z(A) = B$ and $T_z(B) = A$.

Take $a \in A$ and $b = T_z a \in B$. For every $s \in [0,1]$ consider the sets

$$A_s = \left\{\frac{ta + (1-t)z}{\|ta + (1-t)z\|}\colon t \in [0,s]\right\}$$

and

$$B_s = \left\{\frac{tb + (1-t)z}{\|tb + (1-t)z\|}\colon t \in [0,s]\right\}.$$

Since C is connected and nonunitary, there exists $s_0 \in [0,1]$ such that either $A_{s_0} \subseteq C$ or $B_{s_0} \subseteq C$. But $T_z(A_{s_0}) = B_{s_0}$ and $T_z(B_{s_0}) = A_{s_0}$, so we deduce that $A_{s_0} \cup B_{s_0} \subseteq C$ and thus z is an interior point of C in S_Y.

Therefore C is closed and open in S_Y, so $C = S_Y$ and every point of S_Y is an isometric reflection vector. This implies $I_Y^r = Y$ and, by Theorem 4.1, Y is a Hilbert space.

Since x was arbitrary we deduce that u is a parallelogram identity vector. Consequently, the set of parallelogram identity vectors of X has nonempty interior, which implies that X is a Hilbert space. $\square$

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