Optical and Quantum Electronics **24** *(1992) S1021–S1026*

Third- and fourth-order parametric characterization of partially coherent beams propagating through *ABCD* optical systems

R. MARTÍNEZ-HERRERO, P. M. MEJÍAS,
Universidad Complutense, Departamento de Optica, Facultad de Ciencias Físicas, 28040 Madrid, Spain
M. SÁNCHEZ, J. L. H. NEIRA
Empresa Nacional de Optica (ENOSA), R & D Department, Spain

Received 16 January; accepted 2 April 1992

Within the formalism of the Wigner distribution function, third- and fourth-order measurable beam parameters are proposed and their physical meaning is investigated. The general propagation law of such characteristic quantities in *ABCD* optical systems is derived and a number of parameters are shown to be invariant under propagation in *ABCD* devices.

1. Introduction

Beams generated by gas flow lasers, employing either stable or unstable resonator cavities, frequently present multimode profiles. In fact, beams emerging from transversal flow amplifiers usually have astigmatic or asymmetrical structure. Moreover, their spatial properties will be modified when they propagate through optical systems. Consequently, to obtain optimum control in industrial processes a sufficiently precise description of laser beams is required, along with a parametric characterization of their behaviour under propagation through *ABCD* systems. Up to now, the most frequently used measurable parameters for such a description have been [1–8] the output power, beam position, propagation direction, beam width and far-field divergence, closely related to the zero-, first- and second-order moments of both the intensity and the radiant intensity of the field. Recently [9], higher-order moments have been considered, providing information about other beam characteristics such as beam symmetry and sharpness, and their propagation laws through free space have also been reported.

In the present work attention will be focused on the third- and fourth-order moments of the above intensity distributions. In particular, the general propagation law of these measureable beam parameters through *ABCD* systems will be derived, and their physical meaning will be investigated. Also, a number of parameters will be given which are invariant under propagation through *ABCD* optical devices. The resulting conservation laws will reveal competition mechanisms which can be of use in the design of optical systems for the spatial control of laser beams.

2. Formalism and physical meaning of third- and fourth-order moments

To begin, let us introduce the Wigner distribution function h, defined in terms of the cross-spectral density function G which characterizes a partially coherent beam, as follows

$$h(\boldsymbol{x}, u, z) = \int \exp(-\mathrm{i}kus) G(\boldsymbol{x} + s/2, \boldsymbol{x} - s/2, z)\, \mathrm{d}s \tag{1}$$

where the coordinate z denotes the direction of the beam propagation, $\boldsymbol{x}$ the transversal variable (for simplicity, we will use a 2-D picture) and $ku = k_x$ is the wave vector component along $\boldsymbol{x}$. Hence u represents an angle of propagation (without taking the evanescent waves into account). In Equation 1 normalization of h has been imposed and, consequently, division by the total power is implicit. As usual, averages (moments) of function h (denoted by sharp brackets) are defined in the form

$$\langle \boldsymbol{x}^m u^n \rangle = \int \mathrm{d}x\, \mathrm{d}u\, h(\boldsymbol{x}, u, z)\, \boldsymbol{x}^m u^n \tag{2}$$

where m and n are non-negative integers. For the sake of brevity, in what follows it will be assumed that $\langle x \rangle = \langle u \rangle = 0$. This is not a restriction, since it is simply equivalent to a shift of the coordinate system [2]. Concentrating our attention on third- and fourth-order averages, it might appear, at the first sight, that to characterize any beam, up to fourth order, it would suffice to know its intensity moments $\langle \boldsymbol{x}^3 \rangle$ and $\langle \boldsymbol{x}^4 \rangle$, which account for the beam symmetry and sharpness, respectively [9], along with the radiant intensity moments $\langle u^3 \rangle$ and $\langle u^4 \rangle$, closely connected with the above properties but in the far-field. It is easy to see, however, that, when one propagates these characteristic beam parameters, a number of crossed averages arise, namely, $\langle \boldsymbol{x}^2 u \rangle$, $\langle \boldsymbol{x} u^2 \rangle$, $\langle \boldsymbol{x}^2 u^2 \rangle$, $\langle \boldsymbol{x}^3 u \rangle$ and $\langle \boldsymbol{x} u^3 \rangle$. To find their physical meaning will be the purpose of the remainder of this section.

Note that, at any plane z,

$$\partial(\langle \boldsymbol{x}^3 \rangle)/\partial z = 3\langle \boldsymbol{x}^2 u \rangle \tag{3}$$

and

$$\partial(\langle \boldsymbol{x}^4 \rangle/\partial z = 4\langle \boldsymbol{x}^3 u \rangle \tag{4}$$

Accordingly, the crossed moments $\langle \boldsymbol{x}^2 u \rangle$ and $\langle \boldsymbol{x}^3 u \rangle$ can be considered as a measure of the spatial variation range of the beam symmetry and sharpness, respectively (in this sense, recall the meaning of the second-order parameter $\langle \boldsymbol{x} u \rangle$). By spatial variation range of a moment, we understand the distance (along z) the beam should travel (in the neighbourhood of the plane we are considering) to appreciably modify the value of such a moment.

Let us now consider the so-called Fourier transform optical system, in which the beam amplitude in the back focal plane of a converging lens is imaged onto the front focal plane of the same lens. For such a system it is not difficult to show that the crossed averages $\langle \boldsymbol{x} u^2 \rangle$ and $\langle \boldsymbol{x} u^3 \rangle$ transform (proportionality factors apart) in $\langle \boldsymbol{x}^2 u \rangle$ and $\langle \boldsymbol{x}^3 u \rangle$, respectively. Consequently, the moments $\langle \boldsymbol{x} u^2 \rangle$ and $\langle \boldsymbol{x} u^3 \rangle$ will constitute a measure of the spatial variation range of symmetry and sharpness beam properties over the Fourier plane of a lens.

Finally, to gain physical insight about the remaining parameter $\langle \boldsymbol{x}^2 u^2 \rangle$ let us now consider a certain class of fields, namely, the quasihomogeneous (QH) fields, whose spectral density function at some plane z takes the form [10]

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2, z_0) = A((\boldsymbol{x}_1 + \boldsymbol{x}_2)/2, z_0)\, B(\boldsymbol{x}_2 - \boldsymbol{x}_1, z_0) \tag{5}$$

Significant examples of QH fields are Gaussian and Gauss Schell model beams. It is immediate to prove that, for QH fields, the Wigner distribution function factorizes at z in the form

$$h(\boldsymbol{x}, u, z) = A(\boldsymbol{x}, z_0)\, T(u, z_0) \tag{6}$$

where

$$T(u, z_0) = \int \exp(-\mathrm{i}kus)\, B(s, z_0)\, \mathrm{d}s \tag{7}$$

From Equation 6 it follows at once that

$$\langle \boldsymbol{x}^m u^n \rangle_{z_0} = \langle \boldsymbol{x}^m \rangle_{z_0} \langle u^n \rangle_{z_0} \qquad \text{for any } m, n = 1, 2, \ldots \tag{8}$$

Furthermore, if, as assumed, $\langle \boldsymbol{x} \rangle = \langle u \rangle = 0$, then we have

$$\langle \boldsymbol{x} u^n \rangle_{z_0} = \langle \boldsymbol{x}^n u \rangle_{z_0} = 0 \qquad \text{for any } n = 1, 2, \ldots \tag{9}$$

In particular $\langle \boldsymbol{x}u \rangle_{z_0} = 0$, so that the waist of a beam that behaves as a QH field at z_0 is placed just over this plane z_0. Returning to the parameter $\langle \boldsymbol{x}^2 u^2 \rangle$, Equation 8 implies that, for QH fields, the ratio $\langle \boldsymbol{x}^2 u^2 \rangle / \langle \boldsymbol{x}^2 \rangle \langle u^2 \rangle$ equals unity. Therefore, for a general beam, the above ratio expresses, in a sense, its degree of similarity with respect to QH fields whose waists are placed at the plane we are considering. It then appears to be physically reasonable to relate the fourth-order average $\langle \boldsymbol{x}^2 u^2 \rangle$ of a general beam with its degree of quasihomogeneity. Incidentally, note that, for any beam, the product $\langle \boldsymbol{x}^2 \rangle \langle u^2 \rangle$ reaches its minimum value at the waist plane and provides (over this plane) the value of the so-called beam quality parameter [3–7].

3. Propagation through *ABCD* optical systems

To derive the propagation law of beam moments (up to fourth order) through *ABCD* optical systems it will be useful to characterize a beam by the following 5×5 matrix

$$\boldsymbol{M} = \left(\begin{array}{c|c} \boldsymbol{M}_1 & \boldsymbol{M}_2 \\ \hline \boldsymbol{M}_3 & \boldsymbol{M}_4 \end{array}\right) = \left(\begin{array}{cc|ccc} \langle \boldsymbol{x}^2 \rangle & \langle \boldsymbol{x}u \rangle & \langle \boldsymbol{x}^3 \rangle & \langle \boldsymbol{x}u^2 \rangle & \langle \boldsymbol{x}^2 u \rangle \\ \langle \boldsymbol{x}u \rangle & \langle u^2 \rangle & \langle \boldsymbol{x}^2 u \rangle & \langle u^3 \rangle & \langle \boldsymbol{x}u^2 \rangle \\ \hline \langle \boldsymbol{x}^3 \rangle & \langle \boldsymbol{x}^2 u \rangle & \langle \boldsymbol{x}^4 \rangle & \langle \boldsymbol{x}^2 u^2 \rangle & \langle \boldsymbol{x}^3 u \rangle \\ \langle \boldsymbol{x}u^2 \rangle & \langle u^3 \rangle & \langle \boldsymbol{x}^2 u^2 \rangle & \langle u^4 \rangle & \langle \boldsymbol{x}u^3 \rangle \\ \langle \boldsymbol{x}^2 u \rangle & \langle \boldsymbol{x}u^2 \rangle & \langle \boldsymbol{x}^3 u \rangle & \langle \boldsymbol{x}u^3 \rangle & \langle \boldsymbol{x}^2 u^2 \rangle \end{array}\right) \tag{10}$$

defined at each plane z. We refer to matrix $\boldsymbol{M}$ as the moment matrix of the beam. This matrix can formally be obtained from the relation

$$\boldsymbol{M} = \langle \boldsymbol{R}\boldsymbol{R}^{\mathrm{t}} \rangle \tag{11}$$

where

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{x} \\ u \\ \boldsymbol{x}^2 \\ u^2 \\ \boldsymbol{x}u \end{pmatrix} \tag{12}$$

Also note that $\boldsymbol{M}_1$ and $\boldsymbol{M}_4$ are symmetric and $\boldsymbol{M}_2 = \boldsymbol{M}_3^t$, the superscript t denoting the transposed matrix. Inspection of Equation 10 reveals that the moment matrix contains only 12 different beam parameters.

On the other hand, the propagation of function h through $ABCD$ optical systems characterized by the 2×2 matrix

$$\boldsymbol{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{13}$$

is described by

$$h(\boldsymbol{x}_o, u_o, z) = h(A\boldsymbol{x}_i + Bu_i, C\boldsymbol{x}_i + Du_i, z) \tag{14}$$

the subscripts i and o denoting, respectively, the corresponding values at the input and output planes of our $ABCD$ system.

Accordingly, taking Equations 10 and 14 into account, it can be shown that the moment matrix at the output, $\boldsymbol{M}_o$, is given in terms of its value, $\boldsymbol{M}_i$, at the entrance plane of the system as follows:

$$\boldsymbol{M}_o = \boldsymbol{L}\boldsymbol{M}_i\boldsymbol{L}^t = \left(\begin{array}{c|c} \boldsymbol{S}(\boldsymbol{M}_1)_i\boldsymbol{S}^t & \boldsymbol{S}(\boldsymbol{M}_2)_i\boldsymbol{T}^t \\ \hline \boldsymbol{T}(\boldsymbol{M}_2^t)_i\boldsymbol{S}^t & \boldsymbol{T}(\boldsymbol{M}_4)_i\boldsymbol{T}^t \end{array}\right) \tag{15}$$

where

$$\boldsymbol{L} = \left(\begin{array}{c|c} \boldsymbol{S} & 0 \\ \hline 0 & \boldsymbol{T} \end{array}\right) \tag{16}$$

and

$$\boldsymbol{T} = \begin{pmatrix} A^2 & B^2 & 2AB \\ C^2 & D^2 & 2CD \\ AC & BD & AD + BC \end{pmatrix} \tag{17}$$

Equation 15 thus constitutes the propagation law we are looking for. This expression makes it possible to find the values of the moments (which globally characterize the beam) at any plane (for example, in inaccessible regions) by measuring them over another (more suitable) plane. Also note that the relation $(\boldsymbol{M}_1)_o = \boldsymbol{S}(\boldsymbol{M}_1)_i\boldsymbol{S}^t$ contained in Equation 15 is the well-known formula for the propagation of the second-order beam moments.

4. Invariant parameters

The above propagation law also enable us to derive a number of quantities which remain invariant under propagation through $ABCD$ systems. They are

(1) $\det \boldsymbol{M}_1$
(2) $\det (\boldsymbol{M}_2^t\boldsymbol{K}\boldsymbol{M}_2)$
(3) $\det \boldsymbol{M}_4$
(4) $\det \boldsymbol{M}_2\boldsymbol{M}_4^{-1}\boldsymbol{M}_2^t(\boldsymbol{M}_1^{-1})^t$
(5) $\operatorname{tr} \boldsymbol{M}_2\boldsymbol{M}_4^{-1}\boldsymbol{M}_2^t(\boldsymbol{M}_1^{-1})^t$

where

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{18}$$

and the symbol tr denotes the trace of the matrix.

Proof. (1) and (3) are immediate, since det S = det T = 1. To show that $\det(M_2^t K M_2)$ does not change under propagation through *ABCD* optical systems note that, taking Equation 15 into account, we have

$$(M_2^t)_o = T(M_2^t)_i S^t \tag{19}$$

which implies

$$(M_2^t)_o K = T(M_2^t)_i S^t K \tag{20}$$

and then

$$(M_2^t)_o K(M_2)_o = T(M_2^t)_i S^t KS(M_2)_i T^t \tag{21}$$

But $S^t KS = K$, so that

$$(M_2^t)_o K(M_2)_o = T(M_2^t)_i K(M_2)_i T^t. \tag{22}$$

Finally, recalling that det $T = 1$, we find at once the invariance of (2).

To prove the invariance of expressions (4) and (5) note that

$$(M_1^{-1})_o = (S^t)^{-1}(M_1^{-1})_i S^{-1} \tag{23}$$

and, therefore,

$$(M_1^{-1})_o(M_2)_o = (S^t)(M_1^{-1})_i(M_2)_i T^t \tag{24}$$

In a similar way it can be shown that

$$(M_2)_o(M_4^{-1})_o = S(M_2)_i(M_4^{-1})_i T^{-1} \tag{25}$$

Then, taking Equations 24 and 25 into account, we obtain

$$(M_2)_o(M_4^{-1})_o(M_2^t)_o((M_1^{-1})^t)_o = S(M_2)_i(M_4^{-1})_i(M_2^t)_i((M_1^{-1})^t)_i S^{-1} \tag{26}$$

from which the invariant quantities (4) and (5) follow. □

Note that the first invariant, namely, det M_1, is the beam quality parameter we have referred to before. On the other hand, invariance of the third-order expression (2) implies that the symmetry parameter of both the intensity and the radiant intensity cannot be increased while maintaining constant their spatial variation ranges.

The fourth-order invariant parameter (3) indicates that the sharpness of both the intensity and the radiant intensity of a beam cannot be changed without modifying its degree of quasihomogeneity and its corresponding spatial variation range.

Finally, the invariants (4) and (5) show that second-, third- and fourth-order averages are closely connected with each other, and no change can be introduced in any of the beam moments without altering the others.

All these facts reveal, in conclusion, competition mechanisms between the beam moments we have considered, and, consequently, any modification we try to introduce to any of these parameters should be carefully evaluated for each particular case.

Note

The definition of the generalized moments in this paper given by Equation 2 differs from that used by H. Weber, in another paper of this volume. As it will be shown in a future paper, both definitions are related by simple expressions (at least up to fourth-order), but the differences, however, affect the corresponding *ABCD* propagation laws for $n \geqslant 4$.

Acknowledgements

The research work leading to this paper was supported by Empresa Nacional de Optica, S.A. (ENOSA) within the framework of EUREKA Project EU-180, and by the Comisión Interministerial de Ciencia y Tecnología of Spain (CICYT) under Project ROB90-539.

References

1. M. J. BASTIAANS, *J. Opt. Soc. Am.* **69** (1979) 1710.
2. M. J. BASTIAANS, *J. Opt. Soc. Am. A* **3** (1986) 1227.
3. S. LAVI, R. PROCHASKA and E. KEREN, *Appl. Opt.* **27** (1988) 3696.
4. R. SIMON, N. MUKUNDA and E. C. G. SUDARSHAN, *Opt. Commun.* **65** (1988) 322.
5. J. SERNA, R. MARTÍNEZ-HERRERO and P. M. MEJÍAS, *J. Opt. Soc. Am. A* **8** (1991) 1094.
6. R. MARTÍNEZ-HERRERO and P. M. MEJÍAS, *Opt. Commun.* **85** (1991) 162.
7. J. SERNA, P. M. MEJÍAS and R. MARTÍNEZ-HERRERO, *Appl. Opt.* **31** (1992) 7730.
8. J. SERNA, P. M. MEJÍAS and R. MARTÍNEZ-HERRERO, *J. Modern Opt.* **39** (1992) 625.
9. M. SÁNCHEZ, J. L. H. NEIRA, J. DELGADO and G. CALVO, *Proc. SPIE* **1397** (1990) 635.
10. W. H. CARTER, *J. Opt. Soc. Am. A* **1** (1984) 716.