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THE INTUITION OF INFINITY

The Problem

It is surprising that psychology has done so little in exploring the fascinating concept of infinity, whose importance for science, mathematics and philosophy is undeniable. Even Piaget, who is an . . . infinite source of new ideas and new outlooks concerning a variety of fundamental scientific concepts, has made a very limited contribution in this direction.

What is the real psychological background of that concept? It is, of course, a pure construct: no direct experience may be invoked in order to support it. It is not even an hypothesis. No conceivable test is able to support or reject infinity.

On the other hand, we have to agree that infinity is a meaningful, mathematical (ideal) concept: if we are able to prove – such as Cantor did – that infinity is a non-contradictory concept, consistent with the totality of the other mathematical concepts, we may accept its mathematical reality.

But there are here two fundamentally different aspects, which have to be taken into account (as always in mathematical thinking). There is the ideal, pure, final mathematical structure which is unquestionable as a logical construct. And there is the psychological reality of the same concept which may remain complex, contradictory, strongly related to intuitive difficulties.

That is exactly the case with the concept of infinity. Accepting definitions, theorems and logical proofs is one thing. Using the concept of infinity in various real, psychological contexts in the process of thinking and interpreting, is another.

It is fair to suppose that the main source of difficulties which accompany the concept of infinity is the deep contradiction between this concept and our intellectual schemes. Genuinely built on our practical, real life experience, these schemes are naturally adapted to finite objects and events.

The first solution to that contradiction was to admit, as Aristotle did, that infinity expresses, in fact, only a pure potentiality, i.e., the non-limited *possibility* to increase an interval or to divide it. It was that interpretation of infinity as a potentiality, which dominated mathematics until the Cantorian revolution.

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The first aspect of that revolution was to prove the mathematical meaningfulness of the concept of actual infinity. This fact required first of all the elaboration of new logical schemes which were partially in contradiction with our usual mental schemes. For instance, in admitting infinity as actually existing we have to admit the strange proposition that the whole may be equivalent to some of its parts.

Kant's first two antinomies (concerning the infinity of space and the infinite divisibility of matter) are, in our opinion, not pure philosophical constructs, but rather an expression of the psychologically real, profound conflict which is characteristic of the intuition of infinity.

The conflict is probably so deep that, in fact, we cannot completely avoid it even with the most sophisticated mathematical tools. And with this we arrive at the second aspect of the Cantorian revolution. Cantor proved – and this is again in direct conflict with our intuitive understanding of infinity – that more than one kind of infinity exists. It may be proved, for instance, that the (infinite) cardinality of the points of a segment is larger than the (infinite) cardinality of the natural numbers. Instead of one concept of infinity corresponding to our intuitive understanding of endlessness, Cantor proved the existence of an infinite possible world of infinities, hierarchially organized – the world of transfinite cardinals.

Now, when considering the infinite set of transfinite cardinals, a remarkable fact may be noticed. The world of $\aleph_0, \aleph_1, \aleph_2, \ldots$ composed of actual infinities represents a potential, not an actual form of infinity. As O. Becker writes, "The potential nature of infinity, which seemed to have been completely abandoned as a result of Cantor's theory, reappeared on a superior level" (Becker, 1959, p. 125). The contradictory nature of infinity can be pushed to higher levels but cannot be completely eliminated.

The main objective of our present research was to investigate this contradictory nature of infinity – as considered in the actual dynamics of the process of thinking. Our intention was to determine the impact of various figural contexts on solutions to problems where the concept of infinity intervenes. More precisely, we wanted to determine the dependence of that impact on such variables as age, mathematical knowledge and general school achievement. As was pointed out above, psychology has contributed very little to elucidate these problems.

Piaget and Inhelder (1948, pp. 152–179) studied the concepts of points and continuum in their book devoted to the representation of space in children. Their questions referred to the following aspects: (1) The subjects had to draw the "smallest" and the "biggest" square on a sheet of paper. (2) The successive division of a geometrical figure (for instance by two). The subjects had to predict what would happen if the process of division were continued mentally. (3) The form of the final element of such a division. (4) The reconstruction of the original figure starting from the final element. Piaget and Inhelder's results showed the following: up to the age of 8, the child considers only a small number of points as final elements, which, in addition, preserve the form of the original figure. During the concrete operational stage the child indicates a great number of elements as a result of continuous divisions, but he is not able to understand the infinite nature of the process. These last elements, though not necessarily isomorphic to the original figure, nevertheless preserve a certain magnitude and form. At the formal operational stage, the children are able to conceive of the infinite divisibility of a figure, to consider the final elements as points, i.e., without form or dimensionality.

Fischbein (1963) replicated some of the Piagetian techniques and his results practically confirmed those obtained by Piaget and Inhelder. For instance, concerning the number of points which can be constructed on a segment, the following results were obtained (percentages of students indicating a finite number of points): (the Roman figures stand for school grades) II, 42.0; III, 28.5; IB, 23.5; V, 26.3; and VI, 0.8 (Fischbein, 1963, p. 245). Subjects considering an infinite number of points – Grades II and III, no answer of this type, IV, 3.8; V, 8.8; and VI 39.5. There were also other types of answers, such as, "We can put as many points as we want", etc. Altogether the answers which expressed the idea that the process is endless represented 49.5% at Grade VI (Fischbein, 1963, p. 245). Analyzing Piaget and Inhelder's findings and taking our previous results into account, the following aspects have to be mentioned. Firstly, Piaget and Inhelder give no details concerning the frequencies of the different types of answers. In fact, as our own data have shown, only half of the subjects of age 11–12 consider the process as being infinite. What happens at higher ages? To that question neither Piaget and Inhelder nor our previous research offer any answer. Secondly, Piaget and Inhelder refer to the concept of continuum, but *operationally* they do not differentiate between the power of a denumerably infinite set of elements (such as the number of points obtained as the result of successive divisions) and the "power of continuum". Thus, only one nonspecified type of infinity is taken into account. It is true that Piaget and Inhelder have pointed out that, "what is lacking for our subjects to attain the full concept of continuum is the concept of irrational numbers" (1948, pp. 178–179). Nevertheless, their research is built, and the main conclusions are drawn, as if no essential differences exist between $\aleph_0$ and $\aleph_1$.

Our present research is focused on what may be called *the intuition of infinity*. We use the term *intuition* for direct, global, self-evident forms of knowledge. The statement: "If $A > B$ and $B > C$, then $A > C$" is an example of such an intuitively accepted truth. Of course, not all intuitively accepted statements are really self-evident or correct.

In our present research, we have investigated higher ages than those studied by Piaget and Inhelder (1948) and by Fischbein (1963). Moreover, we have attempted to separate the two levels of infinity which correspond to the power of the denumerably infinite sets and the power of the continuum. Furthermore, we wanted to determine the resistance of the intuition of infinity through age and teaching influences. It is characteristic of an intuitive line of interpretation to remain very stable in spite of instructional-school influences. Finally, we wanted to determine the relation between the intuitive interpretation of the infinite and the school achievement level of the subjects.

Hypotheses

As we mentioned above, it was supposed that the concept of infinity (and specifically of infinite divisibility) is intuitively contradictory. Consequently, it was hypothesized that the answers would fall into two opposite categories, one supporting the idea of infinite divisibility of an interval and the other rejecting it.

Secondly, it was supposed that neither age nor the teaching process would significantly influence the nature of genuinely intuitive answers. Consequently, we hypothesized that the frequencies of the main categories of answers would remain relatively stable across age and grades.

We had no definite expectations concerning the influence of the intellectual level of the subjects on their intuitive interpretations.

Although we had no specific hypothesis concerning the concept of limit, we thought that investigating this aspect in conjunction with the other aspects under review, would shed further light on understanding intuitive notions about infinity.

METHOD

Subjects

Four hundred and seventy primary and junior high school pupils were tested. In the first category there were 46 fifth and 58 sixth graders, while the second category contained 152 seventh, 104 eighth, and 110 ninth graders. In order to obtain a relatively representative sample of the population, two categories of pupils at each grade level were tested, one category belonging to a higher

SES and a second category corresponding to a lower SES. Starting with junior high school, each grade in Israel has three levels, representing high, medium and low general school achievement. Consequently, *for the last three grades* the general achievement level was taken into account as a separate independent variable. It must be mentioned that only after this study was in progress did we learn of the fact that some of the sixth graders underwent supplementary training in modern mathematics, in addition to what is normally taught in school. From one of the classes, 6 of the 28 children studied in a course for gifted children, while from the other class 13 of the 30 children participate in a course at Tel-Aviv University. The school prinicipal informed us that the latter class *in toto* is considered as being very good in mathematics.

Procedure

One female experimenter administered the test to all subjects. The test consisted of 10 items in questionnaire form, which each class received in one session. The items referred to the divisibility of segments, transfinite cardinals and limits.

Items

(1) We divide the segment AB into two equal parts (Figure 1). Point H is the midpoint of the segment. Now we divide AH and HB. Points P and Q represent the midpoints of the segments AH and HB, respectively. We continue dividing in the same manner. With each division, the fragments become smaller and smaller. Question: Will we arrive at a situation such that the fragments will be so small that we will be unable to divide further? Explain your answer.

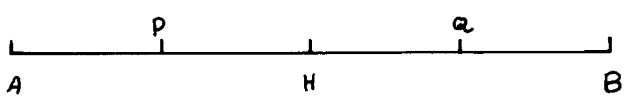

Fig. 1.

(2) Consider again segment AB of question 1. This time, instead of dividing the segment into two equal parts we will divide it into three equal parts (Figure 2). Will we arrive at a situation such that the segments will be so small that we will be unable to continue dividing? Explain your answer.

Fig. 2.

(3a) If the answer to question 2 is "yes", will we arrive at this situation sooner than in the first question (where we divided the segment into two equal parts)? Explain your answer.

(3b) AB and CD are segments of different lengths, such that AB is longer than CD (Figure 3). Divide segment AB and call the midpoint H_1. Divide segment CD and call its midpoint H_2. Let us term H_1 and H_2 *correspondent points*. Let us continue to divide the resultant segments. We will term the midpoints of AH_1 and H_1B respectively P_1 and P_3 and the midpoints of CH_2 and H_2D respectively P_2 and P_4. P_1 corresponds to P_2, P_3 corresponds to P_4. Let us continue to perform this process. Question: Will we arrive at a situation such that for a division point on segment AB there will be no more corresponding points on segment CD? Explain your answer.

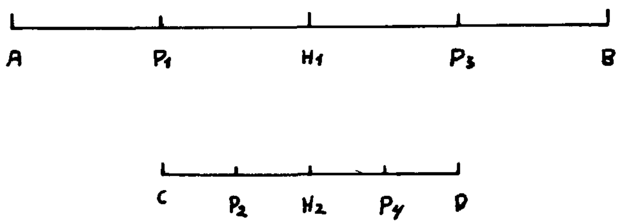

Fig. 3.

(4) Consider the set of natural numbers N = {1, 2, 3, 4, . . .} and the set of even numbers D = {2, 4, 6, 8, . . .}. Question: Which of the two sets contains more elements? Explain your answer.

(5) C is an arbitrary point somewhere on segment AB (Figure 4). We divide and subdivide segment AB as we did in question 1. Question: Will we arrive at a situation such that one of the points of division will coincide with point C? Explain your answer.

Fig. 4.

(6) Let us consider a segment AB whose length is 1 cm. and a square whose side is 1 cm. (Figure 5). Question: Is it possible to find a point of correspondence on the segment for each point on the square? Explain your answer.

(7) Consider a square and a cube (Figure 6). Question: Is it possible to find a point of correspondence on the square for each point on the cube? Explain your answer. What concerns question 6 and 7, some oral and graphical

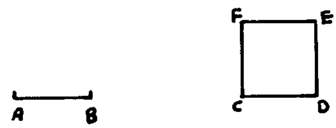

Fig. 5.

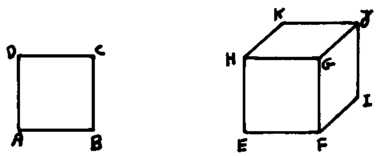

Fig. 6.

explanations were added in order to make clear that the questions were referring to a one-to-one correspondence between the two sets.

(8a) Construct a semicircle with segment AB as a diameter. Divide AB into two equal parts, AC and CB, and construct two semicircles on AC and CB as in Figure 7. Continue dividing and constructing semicircles (see Figure 7). Question: What will happen to the length of the wavy line as we shorten the length of each sub-segment? Explain your answer.

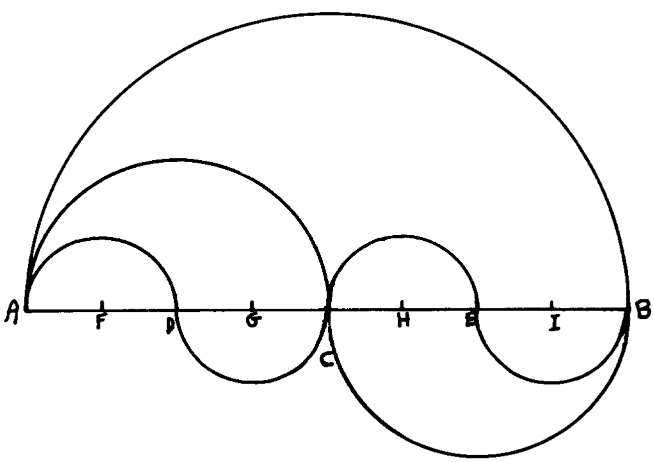

Fig. 7.

(8b) What will happen to the *sum of the areas* determined by the semicircles as we shorten the length of each sub-segment? Explain your answer.

(9) Let us consider the rectangle ABCD (Figure 8). Construct new rectangles by increasing its length and decreasing its width in such a way that the

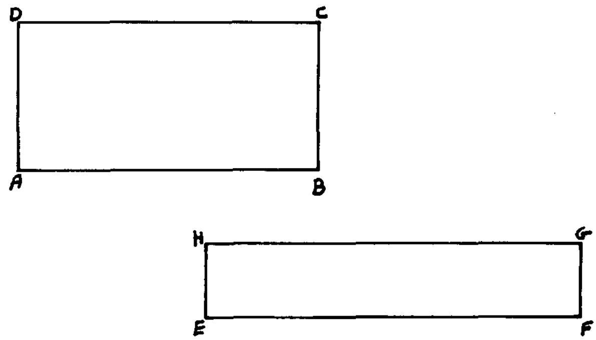

Fig. 8.

perimeter remains constant. What happens to the areas of the rectangles as the process continues?

(10a) ABC is an equilateral triangle. Divide each side into 2 equal parts and label the midpoints A_1, B_1 and C_1. Let us consider the triangle A_1, B_1, C_1, and let us label the midpoints of its sides A_2, B_2, and C_2 (Figure 9). Continue the process in the same manner. Question: Does this process come to an end?

(10b) What will be the area of the final Figure? See Note 1.

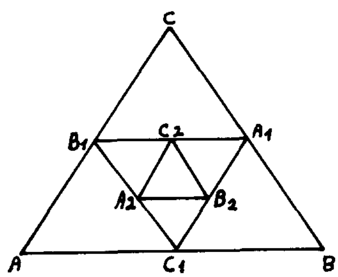

Fig. 9.

RESULTS

Each answer was cross tabulated separately with five independent variables. These were: (1) grade (fifth through ninth grade), (2) type of school (primary versus junior high), (3) the three grades within junior high, (4) the three achievement levels within junior high, and (5) sex. Chi-square tests of independence were performed on each of the above. All tables contain the relevant number of categories in which the answers could be classified (for all answers $N = 470$). However, the chi-square tests were usually not performed on all categories, because this was not always meaningful. In each instance, the categories which actually entered the analysis will be specified.

For the sake of brevity it will be stated now that none of the chi-square tests of the answers by sex were significant. Therefore, we shall not repeat this when presenting the results for each question.

Infinite divisibility. Let us first consider the answers to *question 1.* Table I distributes the answers into four categories: (1) actually the process comes to an end, but theoretically it is infinite; (2) the process is infinite; (3) the process comes to an end; and (4) no answer. We combined type 1 with type 2 in order to have a more complete picture of "infinitists" answers. As can be seen in Table I, the answers are split into two main categories, namely, "the process is infinite" and "the process comes to an end." With the exception of grade six (the subjects who received a supplementary course in modern mathematics), the majority of respondents at each grade level supported the idea that the process comes to an end. (If we dichotomize the answers as belonging to "infinitist" and "finitist" solutions (i.e., categories 1 + 2 versus 3) the majority of all subjects tested thought the process to be finite (55.3%). When subjecting this dichotomy across grades to a chi-square test, the obtained value of 12.25 (df = 4) is significant at the .02 level. *The same analysis for junior high school subjects alone and the primary versus junior high school subjects yielded nonsignificant chi-square values.*

The last three columns of Table I show the percentages of solutions for the three achievement levels of the junior high school subjects. The highest achievement level has twice as many "infinitist" as "finitist" subjects, the intermediate level is fairly evenly split, and the lowest level strongly favors a "finitist" conception ($X^2_{(2)} = 64.79, P < .001$).

It is important to mention that for both types of answers ("infinitists" and "finitists") there are "abstract" and "concrete" forms of explanation.

The following categories of relevant explanations have been found:

The process is infinite because: (1) In every segment there is an infinite number of points, 5.5%. (2) We can always go on in dividing a segment, 13.2%. (3) With perfect instruments we can always go on, 5.1%. (4) A line has no area, 4.0%.

The process is finite because: (1) The segment is limited, 47.2%. (2) Our instruments are limited, 4.7%. (3) Everything has an end, 0.6%.

Because not all subjects have explained their answers by summing up the percentages we do not get 100%. The same situation will be encountered in the case of the other questions too.

TABLE I

The successive divisions of a segment by two (in %)

	Categories of answers		Grade					Sex		Levels of School Achievement		
	N =	Total 470	5 46	6 58	7 152	8 104	9 110	M 218	F 252	High 122	Medium 141	Low 103
1.	Practically the process comes to an end. Theoretically the process is infinite.	12.2	6.5	8.6	11.2	16.3	13.8	12.4	11.9	24.5	12.9	1.0
2.	The process in infinite	29.0	17.4	44.8	31.6	27.9	22.9	30.0	28.2	39.8	30.7	9.8
1 +* 2		41.2	23.9	53.4	42.8	44.2	36.7	42.4	40.1	64.3	43.6	10.8
3.	The process comes to an end.	55.4	76.1	44.8	51.3	51.0	62.4	53.5	57.1	33.3	52.1	83.3
4.	No answer	3.4	–	1.7	5.9	4.8	0.9	4.1	2.8	2.4	4.3	5.9

* The first two categories have been combined to indicate both types of "infinitist" answers.

TABLE II

The successive divisions of a segment by three (in %)

Categories of answers			Grade					Sex		Levels of School Achievement		
	N =	Total 470	5 46	6 58	7 152	8 104	9 110	M 218	F 252	High 122	Medium 141	Low 103
1.	Practically the process comes to an end. Theoretically the process is infinite.	11.3	8.7	8.6	9.2	11.5	16.5	10.1	12.3	20.3	12.1	2.0
2.	The process is infinite.	26.7	17.4	39.7	29.6	26.0	20.2	30.0	23.8	43.9	24.3	5.9
1 + 2*		38.0	26.1	48.3	38.8	37.5	36.7	40.1	36.1	64.3	36.4	7.9
3.	The process comes to an end.	59.9	73.9	50.0	59.2	57.7	62.4	57.1	62.3	34.1	60.0	90.1
4.	No answer	2.1	–	1.7	2.0	4.8	0.9	2.8	1.6	1.6	3.6	2.0

* The first two categories have been combined to indicate both types of "infinitist" answers.

The answers to question 2 (the successive division of an interval by three) are presented in Table II. The answers to the second question closely parallel those of the first. Here too, the analysis was made on categories 1 + 2 versus category 3. On this question the sixth graders were like the other subjects in their answers, consequently the chi-square test did not yield significance. Once again the three achievement levels differ ($X^2_{(2)} = 75.79$, $P < .001$), while neither the type of school nor within junior high school tests were significant. The explanations to the answers also paralleled those of the first question.

The process in infinite because: (1) In every segment there is an infinite number of points, 4.7%. (2) We can always go on dividing a segment, 11.7%. (3) With perfect instruments we can always go on, 5.5%. (4) A line has no area, 0.2%.

The process is finite because: (1) The segment is limited, 52.3%. (2) Our instruments are limited, 4.3%. (3) Everything has an end, 0.5%.

Question 3a required the subjects to compare the processes referred to in the first two questions (successive divisions by two versus three) and asked if one of these would finish sooner. The answers (Table IIIa) are again split into two main categories. The sixth graders are markedly different from the other subjects with 84.5% maintaining "equally infinite" solutions. A chi-square test across all grades was highly significant ($X^2_{(4)} = 33.76, p < .001$). However, this is attributable to the performance of the sixth graders. The answers within junior high school (grades 7, 8, and 9) were quite stable ($X^2_{(2)} = 1.77$, N.S.). When they were compared to the primary school children the difference was significant ($X^2_{(1)} = 7.92, p < .005$). In comparing the three achievement levels within junior high school, the high proportion of "infinitist" answers by the high achievers resulted in a significant chi-square value of 41.44 (df = 2), $p < .0001$.

It will be observed in Table IIIa that about one seventh of all subjects did not answer the question. We therefore included "no answer" as a type of answer and recalculated the chi-square tests of independence with an additional category. In brief, the addition of "no answer" did not alter the previous findings in any way.

Question 3b concerns the number of points in two segments of different lengths. Table IIIb shows the distribution of answers to this question. Here too the results closely resemble those obtained for the previous questions regarding infinite divisibility. The vast majority of all answers fell into one of two categories supporting either the idea of an "infinity of points" or the idea of "less numerous points in the shorter segment". Over all subjects, the

TABLE IIIa

Comparison of the two successive divisions of a segment (by two and by three – in %)

Categories of answers	Total	Grade					Sex		Levels of School Achievement		
		5	6	7	8	9	M	F	High	Medium	Low
N =	470	46	58	152	104	110	218	252	122	141	103
1. In both cases the process is infinite	45.7	28.3	84.5	42.8	37.5	44.5	48.2	43.7	67.7	32.9	22.5
2. The division by three will finish sooner	38.5	50.0	10.3	39.5	48.1	39.1	35.3	41.7	26.6	46.4	53.9
3. Both process will finish at the same time	2.3	–	–	0.8	1.9	7.3	1.8	2.8	–	3.6	5.9
4. No answer	13.2	21.7	5.2	17.1	12.5	9.1	14.7	11.9	5.6	17.1	17.6

TABLE IIIb

Corresponding points in two segments of different lengths (in %)

Categories of answers		Grade						Sex		Levels of School Achievement		
	N =	Total 470	5 46	6 58	7 152	8 104	9 110	M 218	F 252	High 122	Medium 141	Low 103
1.	Practically the process of collecting corresponding points will end sooner in the shorter segment, though theoretically the process is infinite	5.9	4.5	5.5	7.0	4.1	6.9	6.7	5.1	10.8	6.0	1.0
2.	The process is infinite	32.0	27.3	49.1	33.6	24.5	29.7	33.4	30.2	46.8	24.6	17.5
1 + 2*		37.9	31.8	54.6	40.6	28.6	36.6	40.1	35.3	57.6	30.6	18.5
3.	The process of collecting corresponding points will finish sooner in the shorter interval	56.9	59.1	43.6	53.8	64.3	60.4	51.5	61.7	38.8	64.9	73.3
4.	No answer	5.2	9.1	1.8	5.6	7.1	3.0	7.8	3.0	3.6	4.5	8.2

* The first two categories have been combined to indicate both types of "infinitist" answers.

majority favored the latter view. This was also true of each grade level except for grade 6, which contributed towards a significant chi-square value (9.80 (df = 4), $p < .05$). Without grade 6 and without grades 5 and 6 the chi-square test no longer revealed differences, nor did the comparison of grades 5 and 6 to grades 7, 8 and 9. The achievement level effect was again significant with similar trends ($X^2_{(2)} = 35.49, p < .001$).

At question 3b there are no additional relevant explanations other than those expressed in the solutions themselves, except the following statement: "There are more points in the longer segment", 10.0%.

In order to get a correct image with respect to the role of age, the sixth graders must be omitted. Consequently we compared also 5 versus 7, and 5 versus 7 + 8 + 9 graders. (For questions 1, 2, 3a, and 3b). As can be seen when inspecting Tables I, II, IIIa and IIIb a clear tendency can be detected, i.e., there are constantly more "infinitist" answers for grades 7, 8, 9 than for grade 5. Despite this, only for question 1 is the "leap" statistically significant (grade 5 versus grade 7, and grade 5 versus grades 7 + 8 + 9, $p < 0.02$).

Transfinite cardinals. Question 4 concerns the comparison of the set of natural numbers with the set of even numbers. Table IV reveals that most of the subjects (70%) support the idea that there are more natural numbers than even numbers. Their main argument is that the set of natural numbers *contains* the set of even numbers. Only one tenth of the subjects maintain that the two sets are "equally numerous". Their reasoning was that both sets contain an infinite number of elements. Another tenth of the subjects could not decide.

The following explanations may be mentioned:

The two sets are "equally numerous": (1) In both sets there is an infinite number of elements (0.2%); (2) If we consider only a limited set, N is bigger, if not, the sets are "equally numerous" or infinite; (0.4%). (3) We can find a one-to-one correspondence between the two sets (0.4%). N is larger because: (1) In both groups there is an infinite number of elements but N contains D (5.3%); (2) N contains D (37.4%); (3) There are more elements in N, because N is an infinite set (8.9%).

The intuition on which the answers to this question are based is very stable over age. When the answers to this question were dichotomized (a: "In both sets there is an infinity of numbers", and b: " The sets are not "equally numerous" "), none of the chi-square values obtained for any of the five independent variables was significant. When the answers were trichotomized (a: both infinite; b: the set of the natural numbers is "more numerous"; c: the set of the even numbers is "more numerous") then the achievement levels of the junior high school children yielded significance ($X^2_{(4)} = 34.14$,

TABLE IV

Comparison between the set of natural numbers and the set of positive even numbers (in %)

	Categories of answers		Grade					Sex		Levels of School Achievement		
	N =	Total 470	5 46	6 58	7 152	8 104	9 110	M 218	F 252	High 122	Medium 141	Low 103
1.	There is an infinite number of elements in both sets	4.5	2.2	5.2	5.3	3.8	4.6	5.0	4.4	5.7	4.3	4.9
2.	There is an infinite number of elements in both sets, but I don't know which set is "bigger"	5.1	10.9	5.2	5.3	2.9	4.6	6.0	4.4	5.7	3.5	3.9
1 +* 2		9.6	13.1	10.4	10.6	6.7	9.2	11.0	8.8	11.4	7.8	8.8
3.	The set of the natural numbers is "bigger"	71.0	60.9	77.6	63.2	80.8	73.4	67.9	73.4	81.3	78.0	49.0
4.	The set of the positive even numbers is "bigger"	8.3	10.9	6.9	10.5	8.7	4.6	11.0	6.0	2.4	5.0	19.6
5.	No answer	11.1	15.1	5.1	15.7	3.8	12.8	10.1	11.8	4.9	9.2	22.5

* The first two categories have been combined to indicate both types of "infinitist" answers.

TABLE V
The coincidence of a point of division of a segment with any arbitrarily given point (in %)

Categories of answers		Grade						Sex		Levels of School Achievement		
	N =	Total 470	5 46	6 58	7 152	8 104	9 110	M 218	F 252	High 122	Medium 141	Low 103
1.	If the point corresponds to an "even point" we shall reach it	1.9	–	1.7	2.6	3.8	–	2.8	1.2	4.9	0.7	1.0
2.	It depends on the point's place	8.3	6.5	5.2	9.9	8.7	8.3	9.6	7.1	11.4	10.6	3.9
1 +* 2		10.2	6.5	6.9	12.5	12.5	8.3	12.4	8.3	16.3	11.3	4.9
3.	We shall reach the point	81.2	82.6	91.4	81.6	67.3	88.1	75.7	86.1	74.8	78.0	87.3
4.	We shall not reach the point	6.0	6.5	1.7	2.6	16.3	2.8	7.3	4.8	5.7	6.4	7.8
5.	No answer	2.6	4.2	–	3.3	3.8	0.8	4.5	0.8	3.2	4.3	–

* The first two categories have been combined to indicate the total of correct answers.

$p < .001$). This effect is due to the relatively large proportion of high achievers who maintained that the set of natural numbers is "more numerous".

Question 5: If successively dividing a segment AB by two, will any one of the points of division coincide precisely with a given point C of the segment? The answers to this question (Table V) were analyzed by an overall chi-square test (categories 1 and 2 versus 3). The only significant result was for achievement level ($X^2_{(2)} = 7.45$, $p < .05$). In fact, even in the high achievers, the proportion of wrong answers is still very high (79.8%). In the sixth graders, the proportion of wrong answers is even higher (91.4%).

The following additional explanations offered by subjects have to be noted:

The point will be reached because: (1) The division is infinite (7.5%). (2) The segment is limited (58.1%). (3) A segment contains a finite number of points (0.4%). Comparisons between grades 5 and 7; and 5 versus 7 + 8 + 9 yield no significance for both questions 4 and 5.

Question 6: The comparison between the points of a segment AB and the points of a square. Table VI reveals that over 65% of the subjects at each grade claim that it is not possible to map all the points of a square into the points of a segment. A chi-square test across grades was significant ($X^2_{(4)} = 20.26$, $p < .001$). Though there is a visible improvement from the fifth grade to the higher grades, at least two thirds of the subjects at each grade still give an incorrect answer to this question. This finding of change from the fifth to the higher grades might be sufficient in causing the chi-square value to be significant, but what is interesting here is the unexpected drop of the ninth graders, with over 80% believing that the points will not fit. Consequently, the within junior high school analysis yielded a significant chi-square value (8.32, df = 2, $p < .02$). Neither the primary/junior high school comparison, nor the achievement levels resulted in significant differences. In spite of this, both comparisons: 5 versus 7, and 5 versus 7 + 8 + 9 yielded significant differences ($p < 0.001$ and $p < 0.01$, respectively).

Subjects' explanations to their answers were rather diverse, hence only the two most common types of explanations are presented for each position. For those subjects who maintained that the points will always fit, 10% said that there are an infinite number of points in the square and in the segment; 14% offered the explanation that the side of the square and the segment have the same length. For those who maintained that the points will not fit, 38% reasoned that the square contains the segment; and 8% that the segment's area is smaller than the square's area.

TABLE VI

The correspondence (one-to-one) between the points of a segment and the points of a square (in %)

Categories of answers		Grade						Sex		Levels of School Achievement		
	Total	5	6	7	8	9		M	F	High	Medium	Low
N =	470	46	58	152	104	110		218	252	122	141	103
1. The points will fit	26.7	6.5	34.5	32.9	31.7	17.6		28.0	25.5	33.6	23.4	27.5
2. The points will not fit	73.3	93.5	65.5	67.1	68.3	82.4		72.0	74.5	66.4	76.6	72.5

TABLE VII

The correspondence (one-to-one) between the points of a square and the points of a cube (in %)

Categories of answers		Grade						Sex		Levels of School Achievement		
	Total	5	6	7	8	9		M	F	High	Medium	Low
N =	470	46	58	152	104	110		218	252	122	141	103
1. The points will fit	25.9	13.0	29.3	30.3	27.9	21.3		28.9	23.1	33.6	22.0	25.5
2. The points will not fit	74.1	87.0	70.7	69.7	72.1	78.7		71.1	76.9	66.4	78.0	74.5

Question 7: The correspondence between the points of a square and the points of a cube. The distribution of answers to this question appears in Table VII. At least 70% of the subjects at each grade level affirm that the points of the cube and those of the square will not fit. The comparisons between primary versus junior high school and between 7, 8 and 9 graders are not significant. On the other hand, there are significantly more infinitists in grade 7 than in grade 5 ($X^2_{(1)} = 4.55, p < .05$).

Due to the diversity in the type of explanations offered, only a few types are presented. For those maintaining that the *points will fit*, the most frequent responses were: (1) there is an infinite number of points in each (10.6%); (2) the points have no area, so we can always continue (2.1%); (3) the edge of the cube and the side of the square have the same length (10.3%). For those maintaining *that the points will not fit*, the most frequent responses were: (1) the square is a part of the cube (the cube contains the square – 37.9%); (2) the square is two dimensional and the cube is three dimensional (12.1%).

Limits. Question 8a: The length of the curved line. The answers to this question (see Table VIIIa) were analyzed once with three categories (the first three) and once with two categories (first category versus second plus third). The two sets of results were identical. The results reported here are from the dichotomized analysis. Over age, the chi-square value of 36.75, df = 4, $p < .0001$ was highly significant. A brief glance at Table VIIIa shows that this effect is mainly due to the very high proportion of sixth graders affirming constancy (93%). A within junior high school analysis yielded a non significant chi-square value ($X^2_{(2)} = 1.34$, N.S.).

The difference between achievement levels was significant ($X^2_{(2)} = 26.21$, $p < .001$). About 60% of the medium and high achievers believe that the length of the wavy line remains constant, while only one third of the low achievers agree with them.

Question 8b enquired what would happen to the sum of the areas determined by the semi-circles. The answers to this question (Table VIIIb) are again distributed over four categories. However, for the analyses the second and third categories (i.e., the area changes) were jointly compared to the first category (the area remains constant). This question significantly differentiated between grades ($X^2_{(4)} = 17.92$, $p < .002$), between primary and junior high school subjects ($X^2_{(1)} = 10.05$, $p < .002$), within junior high school ($X^2_{(2)} =$ 6.82, $p < .05$), and between achievement groups ($X^2_{(2)} = 13.42$, $p < .001$). Once again, the sixth graders are largely responsible for the strong effects both across all grades and between primary and junior high schools. *The most*

TABLE VIIIa

The semicircles – the length of the curved line (in %)

	Categories of answers	Grade						Sex		Levels of School Achievement		
		Total	5	6	7	8	9	M	F	High	Medium	Low
	N =	470	46	58	152	104	110	218	252	122	141	103
1.	The length remains the same	56.3	37.0	93.1	50.0	51.9	57.8	54.8	57.5	60.2	62.1	31.4
2.	The length decreases	23.9	30.4	3.4	27.0	26.0	25.7	25.3	22.6	19.5	19.3	44.1
3.	The length increases	14.5	15.2	1.7	17.1	19.2	12.8	13.4	15.5	17.9	13.6	18.6
4.	No answer	5.3	17.4	1.8	5.9	2.9	3.7	6.5	4.4	2.4	5.0	5.9

TABLE VIIIb

The semicircles – the area (in %)

	Categories of answers		Grade					Sex		Levels of School Achievement		
	N =	Total 470	5 46	6 58	7 152	8 104	9 110	M 218	F 252	High 122	Medium 141	Low 103
1.	The sum of the areas is constant	51.7	37.0	70.7	46.7	44.2	61.8	45.0	57.5	53.2	58.6	36.3
2.	The sum of the areas decreases	26.6	13.0	17.2	29.6	34.6	25.5	30.7	23.0	31.5	19.3	42.2
3.	The sum of the areas increases	6.8	–	1.7	8.6	11.5	5.5	7.8	6.0	10.5	7.9	6.9
4.	No answer	14.9	50.0	10.4	15.1	9.7	7.2	16.5	13.5	4.8	14.2	14.6

striking finding was that the percentages of subjects affirming that the sum of the areas remains constant (wrong answer) are higher in grades 7, 8 and 9 than in grade 5; in grade 6 than in the other grades; and in high achievers than in low achievers.

The following explanations offered by subjects could be mentioned:

The area decreases (correct answer) because: (1) At the end we shall have a straight line which has no area (3.6%). (2) The area tends to zero (1.5%). (3) The pieces become smaller (4.0%). (4) The length of the semi-circles decreases (1.1%). (5) The radius decreases (0.4%).

The area remains the same because: (1) Compensation between the increasing number of semi-circles and decreasing length of each semi-circle (25.9%). (2) (Incorrect) geometric proofs (1.1%).

It is of interest to note whether those subjects who maintain that the area under the curved line remains constant are the same as those who believe that the length of the curved line remains constant. Of the 388 subjects who could be classified in this way three quarters gave consistent answers to the two questions with 68% of the subjects maintaining constancy in both cases. The phi coefficient of 0.44 is highly significant, $p < .001$.

Question 9: The variation of the area of a rectangle with constant perimeter. Table IX gives the response distribution for five categories. The majority of all subjects at each grade (except the eighth) falsely believe that the area remains constant along with the perimeter, despite changes in length and width.

The analyses were performed once for category 1 (the area decreases) versus category 2 (the area remains constant), and once for constancy versus change (categories 1 + 3 + 4). Both sets of analyses yielded significant differences for all grades, primary/junior high schools, and achievement levels.

Constancy versus change (whether smaller, bigger or uncertain) differentiated the three grades significantly ($X^2_{(2)} = 7.04$, $p < .05$), however, "constancy" versus "smaller" did not significantly differentiate between grades 7, 8 and 9.

The results reported below are based on the comparison between the first two categories (the area becomes smaller versus the area remains the same). The significant chi-square effect for this question by all grades ($X^2_{(4)} = 18.94$, $p < .001$) is clearly due to the low proportion of primary school children giving the correct answer. This is reflected by the highly significant chi-square value of 14.31 ($\mathrm{df} = 1$, $p < .001$) obtained for the primary versus junior high school comparison. Within junior high the distribution of answers is fairly stable (nonsignificant chi-square) between grades, but not between achievement levels ($X^2_{(2)} = 27.43$, $p < .001$). *Note the higher proportion of non-correct answers for the high achievement groups, and for the sixth graders.*

TABLE IX

The variation of the area of a rectangle with a constant perimeter (in %)

	Categories of answers		Grade					Sex		Levels of School Achievement		
	N =	Total 470	5 46	6 58	7 152	8 104	9 110	M 218	F 252	High 122	Medium 141	Low 103
1.	The area decreases	17.2	6.5	5.2	18.4	23.1	20.9	17.4	17.1	17.7	14.3	32.4
2.	The area remains the same	58.1	67.4	81.0	61.2	44.2	50.9	61.0	55.6	65.3	61.4	27.5
3.	The area increases	2.6	2.2	3.4	2.6	2.9	1.8	2.3	2.8	0.8	5.0	1.0
4.	The area changes but not sure of direction	2.6	–	–	0.7	8.7	1.8	1.4	3.5	1.6	3.6	4.9
5.	No answer	19.5	23.9	10.4	17.1	21.1	24.6	17.9	21.0	14.6	15.7	34.2

The above analyses were also calculated with the inclusion of "no answer" as a category (in addition to "smaller" and "same"). None of the chi-square values became significant or nonsignificant as a result of this change.

It is difficult to ascribe any steadfast developmental trends from the data. However, if we compare the primary school children (grades 5 and 6) to the junior high school (grades 7, 8 and 9) it may be observed that the percentage of correct answers increases with age. On the other hand, *even for the oldest children (grade 9) the percentage of subjects maintaining invariance of the area (a wrong answer) is significantly higher than those who conclude that the area diminishes.*

The following relevant justifications have to be mentioned:

The area decreases because: (1) The area tends to zero (5.1%); (2) The width decreases (0.2%); (3) Mathematical proof (7.9%).

The area remains constant because: (1) The perimeter does not change (20.3%); (2) Compensation between the length and the width (18.1%).

Question 10a: The equilateral triangles. Will the process of decreasing the sizes eventually terminate?

Over three-quarters of the subjects at each grade agree that the process is infinite (see Table Xa). Although the percentage of subjects supporting the idea that the process is infinite decreases after the seventh grade, an overall chi-square test did not reveal significant differences ($X^2_{(4)} = 8.49$, N.S.).

The following types of justifications have been found.

The process is infinite because: (1) There is an infinite number of points in every segment (12.5%); (2) We can always divide every side of the triangle (0.2%); (3) The triangles become smaller but the process will never finish (3.4%).

The process is finite because: (1) The area decreases and there will be no area left (3.0%); (2) There will be no place to move points (3.0%).

The primary to junior high school comparison was also not significant, although within junior high school the increasing proportion of older subjects maintaining that the process will come to an end resulted in a significant chi-square value of 6.42 (df = 2, $p < .05$). The larger proportion of high achievers who thought that the process is infinite were responsible for a significant difference when this variable was examined ($X^2_{(2)} = 6.26, p < .05$).

TABLE Xa

The equilateral triangles – will the process come to an end? (in %)

Categories of answers	Total	Grade					Sex		Levels of School Achievement		
	Total	5	6	7	8	9	M	F	High	Medium	Low
N =	470	46	58	152	104	110	218	252	122	141	103
1. The process is infinite	85.5	89.1	89.7	89.5	83.7	78.0	82.6	88.1	91.1	81.6	80.4
2. The process will come to an end	14.5	10.9	10.3	10.5	16.3	22.0	17.4	11.9	8.9	18.4	19.6

TABLE Xb

The equilateral triangles – the area of the final figure (in %)

Categories of answers	Total	Grade					Sex		Levels of School Achievement		
	Total	5	6	7	8	9	M	F	High	Medium	Low
N =	470	46	58	152	104	110	218	252	122	141	103
1. A small triangle	16.2	6.5	3.4	15.8	26.0	18.3	16.1	16.3	20.2	13.6	26.5
2. A point	18.9	15.2	1.7	19.7	24.0	23.6	15.1	22.3	12.9	26.4	27.5
3. The area will be zero	2.3	2.2	5.2	0.7	1.9	3.6	1.8	2.8	–	2.9	2.9
4. A very small area	34.3	47.8	48.3	33.6	32.7	23.6	38.5	30.6	53.2	23.6	11.8
5. No answer	28.4	28.3	41.4	30.3	15.4	30.9	28.4	28.2	13.7	33.6	31.4

Question 10b: The equilateral triangles – the area of the final figure. The distribution of answers to this question appear in Table Xb. No clear-cut conclusions regarding an age effect can be drawn from the data. There is one type of answer, "a very small area", which becomes less frequent as the subjects get older: from almost 50% at the 5th grade to under a quarter of the 9th graders.

In analyzing the data, the two categories of answers: "a point" and "the area will be zero", were collapsed. The chi-square tests of independence were performed once for all categories and once without the "no answer" category. Once again the inclusion or deletion of "no answer" as a type of answer did not effect the significance levels of the results. Below, we give the results based on the analysis of three categories ("a little triangle", "a point or no area" and "a very small area").

These categories differentiated significantly between grades ($X^2_{(8)} = 31.24$, $p < .001$). This was due to the younger children's more frequent responses of "a very small area", whereas the older children were more inclined to reply "a small triangle". Hence, the primary to junior high school comparison was significant ($X^2_{(2)} = 25.03$, $p < .001$), while within junior high school the stability of responses was reflected by the nonsignificant difference.

The chi-square test for achievement level was highly significant ($X^2_{(8)} = 44.96$, $p < .001$). However, the interpretation of this result is neither simple nor obvious. The only category which distinguished sharply between the achievement levels was "a very small area". This response was given by 12% of the low achievers, by twice as many medium achievers, and twice again as many high achievers.

DISCUSSION

1. *The contradictory nature of the intuition of infinite divisibility*

It has been argued that the idea of the infinite divisibility of an interval is in itself contradictory. Consequently, we anticipated that this contradiction would be manifested by having the answers split into two main opposite categories: Those accepting the infinite divisibility and those rejecting it. Our findings have confirmed that hypothesis. At all age levels, for all questions concerning the successive divisions of an interval, the answers were oriented into these two opposite directions. Although the "finitist" answers generally prevailed, the "infinitist" answers followed closely with 40% of all answers.

Let us consider some alternative interpretations to the above finding. Firstly, it might be argued that the subjects do not possess the concept of infinity (and specifically, the concept of infinite divisibility) and consequently, their answers are in fact random – the infinite division is accepted or rejected.

However, this argument will be discarded on the following grounds. The concept of infinity clearly *is* understood by most subjects, as demonstrated by the nature of their answers to some of the other questions. These questions are: (a) "By successively dividing a segment into two parts will an arbitrarily given point on the segment, be reached by any one of the points of division?" It has been shown that the majority of subjects (wrongly) support the idea that such a coincidence will (necessarily) take place (see Table V). Only by accepting (at least implicitly) that the process of division is endless can it be argued that *every* point on the segment will be reached, sooner or later, by one of the points of division. (b) The construction of equilateral triangles. At *all age levels* most of the subjects (78–90%) affirm that the process is infinite (see Table Xa).

Consequently, we must admit that the subjects have an intuitive idea of the concept of infinity and that their answers are generally not random answers.

A second plausible explanation of the duality of the answers may be the following: there are pupils who reproduce what they have been taught (for instance "a segment contains an infinity of points.") and there are pupils who do not.

Our data really support the conclusion that the subjects' reactions are, in fact, influenced by their mathematical knowledge. As can be noted by inspecting the data, the 6th graders (well trained in mathematics) and the high achievers distinguish themselves from the other subjects. But the influence of mathematical education cannot by itself explain the following findings: (1) At least from grade 7 there is no increase in the percentage of "infinitist" answers. Conversely, in all cases referring to the concept of infinity (except question 3a), the percentage of "infinitist" answers decreases from grade 7 to grade 9. (2) For some of the questions, there are very high percentages of "finitist" answers even for the high achievers (questions 4, 6, 7, 8). Consequently, even if we accept that the teaching process influenced the answers, a second factor has to be taken into account as well.

A third plausible explanation of our findings may be the following: there are two opposite categories of answers, because the questions themselves are in fact interpretable in two different ways. For instance, the question referring to the successive divisions of a segment may be interpreted either as concerning an ideal, mathematical segment divided by ideal non-dimensional points, or as a material segment (i.e., a drawing) divided materially by small concrete figures. In the first case the correct answer would be "the process of successive division is infinite." In the second case an acceptable answer would be "the process is finite".

Although this "dualistic" interpretation sounds plausible at first, scrutiny of subjects' argumentations permit us to reject it. Not all "finitist" answers are accompanied by "concretist" argumentations, and not all "infinitist" answers are accompanied by "purist" argumentations. Generally the subjects who claim that the process will come to an end support their answers by the argument: "the interval is limited" (i.e., an "abstract" argumentation). Conversely, there are subjects who accept the division of an interval as being infinite though the process is considered to be a concrete one, "if we use perfect instruments we can always go on" (a "concrete" argumentation).

Taking the above-mentioned facts into account we support the conclusion that the subjects, at least starting from age 11, have a certain intuition of infinity. Being contradictory in itself that intuition is very labile, i.e., very sensitive to the conceptual and figural context of the problem.

The split into two opposite categories of answers cannot completely be explained as an effect of teaching or as a result of personal interpretation. An additional, more fundamental, factor must also be taken into account, and this is the contradiction between the "finitist" character of our intellectual schemes and the concept of infinity itself. Nearly 50% of all subjects argue that a segment cannot be endlessly divided because, "the segment is limited". Eighty-seven percent of subjects claim that the set of points of a cube and the set of points of a square "cannot fit". About 60% of them express their "finitist" arguments: "The square is a part of the cube", and "the square is two dimensional while the cube is three dimensional".

An analogous situation happens when asking the subjects to compare the set of natural numbers with the set of even (positive) numbers. Seventy-one percent claim that the "set of natural numbers is larger". The main argument: "The set of natural numbers *contains* the set of even numbers".

These findings seem rather trivial, because they are easily predictable and they are easily predictable, because they express very general, elementary intuitions. But they are *not trivial*. In fact, *in each case we have asked the subjects to compare two infinite sets*. Logically, the natural answers should have been, "both sets are constituted of an infinity of elements" (again, supposing that the subjects have no idea of the different powers of transfinite cardinals, but have an elementary intuition of infinity). In this case too, we have a conflictual situation. The compared sets, though infinite, are in fact represented by objects having a finite magnitude. The "finitising" effect of such a comparison is much stronger than the "finitising" effect of a single object (e.g., a segment).

The proportion of "infinitist" answers varies greatly over some of the different questions. In the question on whether the arbitrarily given point of the segment will be reached, 81.2% of the subjects are "infinitists". The two questions concerned with successive divisions of a segment (by two or by three) resulted in about 40% of the subjects giving infinitist answers. When comparing the number of points of a segment and a square, 26.7% of the subjects are infinitists. The comparison of the sets of natural numbers and positive, even numbers resulted in about 10% infinitists answers.

These huge discrepancies prove, as has been said, that the natural intuition of infinity is highly labile, depending on conjectural and contextual influences. *The lability of the intuition of infinity can be explained if admitting its intrinsic contradictory nature as a psychological reality.*

2. *The stability of intuitive interpretations*

A major hypothesis in the present study was that intuitions and, particularly, the intuitions of infinity are very resistant to the effect of age. Our findings partially confirmed this prediction. *Starting with grade 7 (12–13 years), the percentage of the main categories of answers (especially "finitists" and "infinitists") is generally stable across ages with a slight tendency of domination of the "infinitist" answers toward the higher grades).*

It may be supposed that up to the age of 12 (which corresponds roughly with the beginning of the formal-operational stage) the intuitive interpretations concerning the concept of infinity are still in a process of formation. This raises the more general question of the development of intuitive interpretations. It may be hypothesized that the development of intuitive (natural) interpretations takes place in connection with the general intellectual development. But what defines their specificity is their stability, their resistance (starting with a certain age level) to age and even to teaching influences.

It is difficult to give a general rule which would enable us to predict periods of formation and those of stability of intuitions. These periods probably depend on the specific features of the domain. On the other hand, it may be supposed that many aspects of the development of intuitions can be explained in reference to the piagetian theory of stages. For instance, it may be supposed that the relative stability of the frequency of answers, starting at age 12, may be explained by the strong connection between the *intuition* of infinity and the general features of the formal operational period. Indeed, infinity must necessarily exceed direct, tangible, concrete information and must, of necessity, be built on mental experiences resorting to the propositional system.

So, what is the *intuition* of infinity as distinct from the *concept* of infinity?

Let us recall that the main characteristics of an intuition are its syncretic nature and its self-evidence (its spontaneous acceptance). In the piagetian theory, such modalities of knowledge are incompatible with operations and operational schemata, which, by definition, have an analytical structure. But, in fact, such syncretic, self-evident interpretations and representations exist at all ages, not only at the preoperational period, and at all phases of the construction of some systems of knowledge.

We believe that the intuition of infinity may be (and indeed is) affected by the basic schemes of intelligence at each stage of intellectual development. However, this does not cancel the specificity of the intuition as such. The intuition of infinity means what we *really feel* as being true or self-evident concerning the magnitude (the numerosity, the power) of infinite sets, and not what we accept as being true as a consequence of a logical, explicit analysis. *This means that the concept of infinity may develop itself by the instructional process, while the intuitions of infinity may remain unchanged, starting with age 12.*

3. *Limits and the problem of "formal order" conservation*

In connection with the problem of the semi-circles – two questions were asked: (a) What would happen to the length of the curved line as the diameters of the semi-circles decrease, and (b) What would happen to the sum of the areas under the same condition?

A first remark: nearly all subjects (except a few who did not answer at all) understood the question, and none raised any objection concerning the possibility of continuing the process indefinitely. Not one of the 470 subjects asked himself, "What would happen if the semi-circles would become so small that any continuation would be practically impossible?" This means that such a concrete interpretation of geometrical problems is not natural, at least not at the ages we have considered, if the question itself does not suggest such a possible interpretation. As was pointed out earlier, the "finitist" solutions, when they appear, are not necessarily imposed by "concretist" interpretations.

A second remark concerns the internal dynamics of the intuitions themselves. The proportion of correct answers for the length of the line (the length is constant) increases with age (except grade 6), but does not exceed 57.8% – at grade 9. The rest of the subjects are divided into two groups; about 24% think that the length of the line decreases and about 14% think that it increases.

If we pay attention to the arguments we get some hints concerning the manner in which intuition acts in that case. Those who claim that the length of the line increases refer mainly to the *number* of semi-circles. Those who conclude that the length of the line decreases focus especially on the *magnitude of the diameters*. We may therefore suppose that the correct answer is an effect of compensation (in Piaget's sense): the increasing number of semi-circles is compensated by the decreasing magnitude of the diameters. Subjects who fail to give correct solutions are those who do not succeed in that compensatory activity. The attention of some of them is centered on the number of semi-circles only, while the others are only considering magnitude of the diameters.

It may be supposed that the problem we are facing here is related to the piagetian domain of "conservations". As is well known, problems of this kind refer to situations in which the child has to grasp an invariant feature of an object or of a set of objects (length, number, weight, volume, etc.) in spite of certain qualitative transformations (e.g., shape). In Piaget's view the conservation effect is produced by inner equilibration, i.e., by compensating and coordinating two opposite transformations. It is worthwhile remembering that the equilibration process is not an automatic effect of interacting perceptual properties. In fact, conservation is the result of complex mental activities. Different properties of objects (such as quantity, weight, volume, etc.) are conserved starting with different ages.

The curved line problem does not belong to the same class of concrete-operational conservation problems. It is not an object which changes its shape, but keeps its weight or volume (ultimately, because it keeps its identity). What the subjects have in our curved line problem is an *infinite succession* of different curved lines, with different shapes (the construction of which follows of course, a unique rule). Consequently, the problem is not a concrete operational one – with compensations which take place in the realm of concrete operations. Rather, we are facing here a typical formal operational problem. The reasons for this statement are the following: (1) our subjects were asked to grasp the conservation of a certain property through an infinite number of transformations; (2) the objects considered (the curved lines) are not *given*, they are *possible, constructable ideal* objects; (3) the reasoning used is of the type of hypothetico-deductive reasoning: What will happen to the length of the line *if* the number of semi-circles increases? In short, we may consider that we have here a special type of conservation problem in which formal operational modalities of reasoning are implicated (See: Inhelder and Piaget, 1958, pp. 328–9).

It may be supposed that the correct answers are simply an effect of higher mathematical knowledge (without connections to the conservation-equilibration process). But when inspecting the data concerning the sum of the areas determined by the curved line, we find a similar distribution of percentages across ages and achievement levels, though in this case the *"conservation" answers are wrong answers*. The number of subjects affirming constancy ("conservation") of area increases from 37% in grade 5 to 62% in grade 9. *The high achievers support the same wrong solution more often than the low achievers.* So we are led to believe that beyond the mathematical knowledge of the pupils there is, in this case, a specific intuitive mechanism (similar to the piagetian conservation mechanism), which is responsible for their solutions. We would like to emphasize again, that in our case the "conservation" phenomenon deals with processes which are specific to the formal operational stage.

The results obtained from question 9, the variation of the area of a rectangle with constant perimeter, supplement the picture. The percentage of correct answers increases with age, and it may be supposed that this is due to the effect of increasing mathematical knowledge. But, at the same time, it is reasonable to suppose that this effect is counter-balanced by the same conservation mechanism described above. *Even half of the 15–16 year olds do not solve this question correctly* (i.e., affirm that the area decreases). It seems reasonable to suppose that the subjects have to face two contradictory intuitive effects. One effect is determined by the real decreasing tendency of the areas – with the feeling that the areas tend to zero. The other effect is that of the general conservation mechanism – the length and width of the rectangles vary conversely. At grade 9 the two effects are balanced; at younger ages the conservation effect seems to be stronger.

Up to now it has generally been accepted that conservation, expressing the mechanism of mental equilibration, represents a positive attribute of intelligent behavior. We have seen in the above that conservation (acting as an autonomous mechanism) may, in some instances, *increase the chance for incorrect solutions.*

The final question concerns the equilateral triangles. *A very high percentage of all subjects considers the process of constructing the successively inscribed equilateral triangles as being endless.*

This finding supports our previously expressed idea that "finitist" and "infinitist" positions are not directly tied to what may be called "abstract" and "concretist" orientations. In fact the process of building successively smaller triangles is much more complicated than dividing a segment, if it is considered as a practical activity. Consequently, it is fair to suppose that such a process is much closer to a "finitist" interpretation than is the process of dividing a segment. Despite this, the percentage of "infinitists" in the triangle problem is about twice that in the division problems. Generally the higher-achievers present higher percentages of infinitist interpretations. But for this question even 80% of the lower-achievers offer "infinitist" solutions.

The contradictory structure of the intuition of infinity is clearly revealed by the answers concerning the area of the final figure. These answers are divided into three categories: no area, a small area, a little triangle, which occur with comparable frequency in grades 8 and 9. Let us bear in mind that many of those voting for some non-zero image ("a little triangle" or "a small area") have adopted the "infinitist" solution. It is also interesting to note that the percentages of the "concretist" solution ("a very small triangle") increase with age with a maximum of 26% at grade 8. Such an evolution could not be predicted adopting the hypothesis of structural "concretist" and "abstract" interpretations. In fact from an intuitive point of view, the solution "a very small triangle" is no less acceptable as the limit of the process, than the solution "the limit is a point", because in the latter case one may ask the question: "What will be the figure just before the final one?" This question is stupid from the mathematical point of view, but is very legitimate from an intuitive standpoint.

4. *The effect of teaching*

It has been supposed that intuitions remain unchanged under the influence of teaching. Our findings present a much more complex image. In some cases a positive effect of teaching could be detected, while in other cases the effect is zero or even negative.

Relevant information on this aspect can be drawn while inspecting the results obtained by the 6th graders. As was mentioned above, a part of these pupils get special training in mathematics and most of them have been estimated as being very good in mathematics. The beneficial effect of their high mathematical standard could be identified in all the questions concerning the division of a segment. On these questions they scored the highest percentages of "infinitist" answers in comparison with the other grades.

The *same* subjects present high percentages of *wrong* answers to some other questions. Question 4: "There are 'more' natural than even numbers" (77.6%). Question 5: "The point C will be reached by continuing divisions of the segment" (91.4%). Question 6: "There are more points in a square than in a segment" (65.5%), and Question 7: "There are more points in a cube than in a square" (70.7%). Question 8b: "The sum of the areas under the curved line remains constant" (70.7%). Question 9: "The area of a rectangle with constant perimeter remains constant" (63.3%).

Briefly speaking, it may be safely supposed that the only categories of answers which revealed a positive teaching effect were those which were directly based on the notions which have been taught during class activities. Starting with the fifth grade, pupils learn about straight lines, segments; about the infinity of straight lines and about the fact that a segment is composed of an infinity of points. For instance: "A segment contains an infinity of points". This correct knowledge resisted even when two segments of different lengths are compared. But when comparing uni-dimensional with bi-dimensional or bi-dimensional with three-dimensional figures, the concept of infinity was not more active. There were the "finitist" schemes which prevailed. What was still more fascinating was the fact that *exactly for the same question the majority of the high achievers also presented very high percentages of wrong answers*, nearly always more than 60%. For instance, for question 4 the answer, "There are more natural than positive even numbers": high achievement level subjects: 81.3%, medium achievers: 78%, low achievers: 49%.

In connection with the concept of infinity, mathematical training has, then, two divergent (and apparently contradictory effects). On some concepts the influence is highly positive (the division of a segment and the related problems). On other concepts the influence is negative or absent (questions 4, 5, 6, 7).

This finding can be explained if accepting our above interpretation. What explains the contradictory behavior of the intuition of infinity is the fact that we tend to think on infinite sets of resorting to our usual logical schemes, which are naturally adapted to finite realities.

Usual mathematical training overcomes that difficulty only in what concerns concepts which are directly taught in school. *The intuition of infinity is not basically affected by this training.* On the other hand, mathematics education, in general, strengthens logical thinking, i.e., the finitist schemes of our natural manner of thinking. *For non-standard questions concerning infinite sets, or for questions for which the pupil did not get specific information, we must expect high percentages of "finitist" (wrong) reactions even in spite of his more advanced general mathematical training (and sometimes as an indirect effect of just this mathematical training).*

SUMMARY AND CONCLUSIONS

The contradictory nature of infinity. Infinity appears to our intuition as being contradictory. The explanation of the fact is that the logical schemes are naturally adapted to finite realities. To overcome that genuine contradiction, the intelligence has invented the *potential infinity* or the *improper infinity* in Cantor's terminology. This contradiction is expressed in our data by the fact that the subject's reactions are generally split into two main categories: "finitists" and "non-finitists". Finitists interpretations tend to prevail. Age and teaching effects could explain only a part of that dichotomy. There are paradoxical results which could be interpreted only as an effect of that genuine contradiction. There are subjects who give concrete explanations for "infinitist" answers and *vice versa.* There are mathematically well trained pupils who present a higher percentage of wrong "finitist" answers, than less trained pupils. Higher mathematical training determines more systematic use of formal logical schemes which are naturally adapted to finite objects and events.

The effect of age. Generally speaking, the intuition of infinity appears to be relatively stable starting approximately with grades 7 (age 12–13). There are the following types of situations: (a) In some cases there is a visible progress from grade 5 to grade 7, after which the frequencies tend to become stable (questions related to the division of a segment, i.e., questions 1, 2, 3a, 3b; the correspondence between the points of a segment and of a square; between the points of a square and of a cube). For the last two questions, the proportion of "finitist" interpretations remains above 60% at all ages. The progress from grade 5 to 7 is probably due to age (emergence of formal operations) and to the effect of teaching. But teaching cannot explain, by itself, these results because no improvement could be generally detected from grade 7 to grade 9. (b) In a second category of cases there are no significant changes with age; changes present rather an oscillatory character. In these cases, the finitist or other non-correct answers show higher constant percentages in all ages investigated by us (the comparison of the set of natural numbers with the set of positive even numbers; the coincidence of a point of division with an arbitrary given point of a segment; the sum of the areas of the semicircles; the area of the rectangle with a constant perimeter). (c) With regard to the building of equilateral triangles we found 70–80% of infinitist answers at all ages. Briefly speaking, concerning the intuition of infinity, progress could be detected only from age 11 to age 12 and only for part of the questions.

The effect of teaching. The effect of teaching has been differentiated from the effect of age by taking into account the following data: (a) results obtained by the 6th graders who received special mathematical training and; (b) the results presented by the classes of high achievers. Two divergent effects could be identified: (a) A positive influence on reactions related to the division of a segment; (b) A negative (or no) effect on reactions to other categories of questions. This finding has been interpreted in the following manner. Regular mathematical training affects only the formal, superficial understanding of the concept of infinity. Intuitions remain unaffected. Instead, mathematical training systematically strengthens the current logical schemes which are genuinely finitist. When a conflictual situation is generated (an infinite set has to be considered equivalent with some of its sub-sets) *the finitist (wrong) interpretations tend to prevail even (and sometimes mainly) in well trained pupils.* The better results obtained by the high achievers in connection with the division of segments are interpreted as directly (and superficially) reflecting what has been taught in classes.

Limits. Two questions were asked concerning the evolution of the area of figures having a constant length of their border line. In both cases the majority of subjects affirm that the area is also constant though, in fact, the area tends to zero. What is still more interesting is the fact that in these cases too "the best trained pupils" (the 6th graders and the high achievers) present the highest percentages of wrong answers. This finding supports, though indirectly, our above interpretations: Mathematics teaches the pupils to think consistently. *Without adequate intuitions and without resorting to an adequate mathematical control, this consistency may degenerate into blind rigidity.*

On the other hand, we interpret the high percentage of answers supporting the idea of constancy of the area as being due to a conservation mechanism – similar to that described by Piaget, acting on the formal level and implying specific, formal operational schemes. It is that compensation-conservation mechanism, which in this case, explains the wrong intuitions, i.e., constancy of areas.

It may be supposed that both factors – the conservation effect and the logical consistency – cooperate in making the pupils' reactions deteriorate.

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NOTE

[1] The following concepts related to our questionnaire are initially introduced to the school children as listed: Grade 5 – a line segment is composed of an infinity of points. Grade 7 – the general concept of finite and infinite sets. Grade 8 – one-to-one correspondence, rational and irrational numbers. Our subjects were never taught about comparisons between infinite cardinals.

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