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Construction of exact solutions to the Jimbo–Miwa equation through Bäcklund transformation and symbolic computation

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ABSTRACT

Based on a simple transformation, and with the aid of symbolic computation, a Bäcklund transformation relating the Jimbo–Miwa equation and a system of linear partial differential equations is obtained, which enables us to construct exact solutions of the Jimbo–Miwa equation through the Wronskian determinants of independent solutions of the linear system. Particularly, explicit Wronskian form N-soliton solutions for the Jimbo–Miwa equation are presented. Moreover, the introduced transformation also helps to construct bi-soliton-like solutions of the Jimbo–Miwa equation. Due to the arbitrary functions they contain, the bi-soliton-like solutions can represent various waves such as classical cross-line bi-solitons, curved bi-solitons and bi-soliton-like breathers.

1. Introduction

Nonlinear evolution equations (NLEEs) arise in the study of many nonlinear problems in natural and applied sciences. For instance, the Korteweg–de Vries equation is established to model the surface water waves in long, narrow canals [1]. The Kadomtsev–Petviashvili (KP) equation describes the dynamics of small-amplitude, long wavelength solitary waves in two dimensions [2]. The Schrödinger equation is a model of the electromagnetic waves in optical fibers as well as surface waves in deep waters [3]. It is of significant importance to find ways solving the NLEEs explicitly, since the exact solutions of the NLEEs, if available, can provide better understanding of certain application problems such as optical communication. Although a general theory for solving NLEEs does not exist, there are some algorithms by which special types of exact solutions for some NLEEs can be constructed. Some of the most widely used NLEEs-solving algorithms include the bilinear approach [4], the Darboux transformation [5], the Tanh method and its extensions [6–9], the G'/G-expansion method and its extensions [10–13], and the multiple exp-function algorithm [14,15]. Particularly, the bilinear approach has been used to investigate a large number of NLEEs to get soliton and multi-soliton solutions. Moreover, based on the bilinear approach, Wronskian formulations of N-soliton solutions of some NLEEs, such as the KdV equation [16–19], the KP equation [16,17], the Boussinesq equation [17,20], the Toda type lattice [21] and the breaking soliton equation [22], have been obtained. The compact expression of the Wronskian makes it convenient to discuss the interactions among the N-solitons [23].

The $(3+1)$-dimensional Jimbo–Miwa equation

$$u_{xxxy} + 3\,u_x\,u_{xy} + 3\,u_y\,u_{xx} + 2\,u_{yt} - 3\,u_{xz} = 0 \tag{1}$$

is firstly introduced by Jimbo and Miwa as the second member of the entire Kadomtsev–Petviashvili hierarchy [24]. Different from the well known KP equation, this equation does not pass the Painlevé test of integrability [25]. It was shown that

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the symmetry algebra of Eq. (1) is infinite dimensional but has no Kac–Moody–Virasoro loop structure [26]. Symmetry reductions of the equation can be found in Refs. [26–28]. Although the Jimbo–Miwa equation (1) is non-integrable, different types of exact solutions of the equation were derived through various constructive algorithms [29–36]. Particularly, some Wronskian formulations of exact solutions to Eq. (1) are presented in Refs. [37,38]. As the high dimensional NLEEs often possess rich solution structures, new exact solutions of the equation with special properties might be expected.

In this paper, instead of using the bilinear approach, we find Wronskian formulations of exact solutions (hereinafter we call them Wronskian form solutions) for the Jimbo–Miwa equation (1) through two Bäcklund transformations. Particularly, we present explicit Wronskian form N-soliton solutions of Eq. (1). Still another Bäcklund transformation is used to construct bi-soliton-like solutions of the Jimbo–Miwa equation.

This paper is organized as follows. In Section 2, Wronskian form solutions of the Jimbo–Miwa equation is constructed and verified. As a result, explicit Wronskian form N-soliton solution is shown. In Section 3, we seek bi-soliton-like solutions of the Jimbo–Miwa equation. One will find that the obtained solutions contain arbitrary functions. Appropriate selections of the arbitrary functions lead to certain interesting bi-soliton-like waves, some of which are graphically revealed. Conclusion and discussion are given in Section 4.

2. Wronskian form solutions of the Jimbo–Miwa equation

The Wronskian form solutions of NLEEs often provide alternative expressions of the N-soliton solutions and beyond. Since the derivative of a Wronskian is a single determinant, and higher derivatives lead to sums of determinants with the length of the sums depending merely upon the number of differentiations, the Wronskian form solutions thus may be verified easily through direct substitution. This advantage is not available for the usual representations of N-soliton solution when $N \geq 2$.

To obtain exact solutions of the Jimbo–Miwa equation (1), we make the transformation

$$u = f + \frac{2\,\omega_x}{\omega}, \tag{2}$$

in which $f = f(x, y, z, t)$ and $\omega = \omega(x, y, z, t)$ are functions to be determined. This transformation can be obtained through the direct method presented in Ref. [39]. Substituting (2) into Eq. (1), and eliminating the coefficients of powers of $1/\omega$, yields the following partial differential equations involving f and ω.

$$\begin{aligned}
&f_{xxxy} + 3f_x f_{xy} + 3f_y f_{xx} + 2f_{yt} - 3f_{xz} = 0,\\
&\omega_x\,(3\,\omega_{xxy} - 3\,\omega_z + 3f_x\,\omega_y + 3f_y\,\omega_x) + \omega_y\,(\omega_{xxx} - \omega_t) + 3\,(\omega_y\,\omega_t - \omega_{xy}\,\omega_{xx}) = 0,\\
&3f_x\,\omega_{xxy} + 3f_{xy}\,\omega_{xx} + 3f_y\,\omega_{xxx} + 3f_{xx}\,\omega_{xy} + 2\,\omega_{txy} - 3\,\omega_{xxz} + \omega_{xxxxy} = 0,\\
&3\,\omega_x\,\omega_y f_{xx} + 9f_y\,\omega_x\,\omega_{xx} + 3f_x\,\omega_y\,\omega_{xx} + 6f_x\,\omega_{xy}\,\omega_x - 6\,\omega_x\,\omega_{xz} + 2\,\omega_{xy}\,\omega_t + 4\,\omega_{xxxy}\,\omega_x\\
&\quad - 2\,\omega_{xxx}\,\omega_{xy} + 2\,\omega_x\,\omega_{ty} + 2\,\omega_y\,\omega_{tx} + \omega_y\,\omega_{xxxx} - 3\,\omega_z\,\omega_{xx} + 3\,\omega_x^2 f_{xy} = 0.
\end{aligned} \tag{3}$$

If we can solve f and ω from the system (3), we can then get exact solutions of the Jimbo–Miwa equation (1) through (2).

The first equation in system (3) implies that f should be a solution of the Jimbo–Miwa equation (1). It is easy to test that if we set

$$f = \alpha(z, t) \tag{4}$$

and

$$\omega_y = \omega_{xx}, \qquad \omega_z = \omega_{xxxx} + 2\,\lambda\,\omega_{xx}, \qquad \omega_t = \omega_{xxx} + 3\,\lambda\,\omega_x, \tag{5}$$

with $\alpha(z, t)$ be arbitrary function of z and t, and λ be arbitrary constant, then the first two equations in (3) are satisfied identically. Moreover, we can also test with the aid of symbolic computation that (4) and (5) reduce every equation of (3) to the identity $0 = 0$. Note that (5) is a system of linear partial differential equations in the unknown function ω. Thus, we have the following proposition.

Proposition 1. *The Jimbo–Miwa equation* (1) *admits the solution*

$$u = \alpha(z, t) + \frac{2\,\omega_x}{\omega}, \tag{6}$$

where $\alpha(z, t)$ is an arbitrary analytical function of z and t, and ω satisfies (5).

For nonzero constants a_1, a_2 and a_3, we can test that

$$\begin{aligned}
\omega = {}& 1 + \exp\left[a_1\,x + a_1^2\,y + (a_1^4 + 2\,\lambda\,a_1^2)\,z + (a_1^3 + 3\,\lambda\,a_1)\,t\right]\\
&+ 2\,\exp\left[a_2\,x + a_2^2\,y + (a_2^4 + 2\,\lambda\,a_2^2)\,z + (a_2^3 + 3\,\lambda\,a_2)\,t\right]\\
&+ 3\,\exp\left[a_3\,x + a_3^2\,y + (a_3^4 + 2\,\lambda\,a_3^2)\,z + (a_3^3 + 3\,\lambda\,a_3)\,t\right]
\end{aligned} \tag{7}$$

is a solution of (5). Substituting (7) into (6) yields Y-junction bi-soliton solutions of the Jimbo–Miwa equation. Fig. 1 shows one of those solution waves.

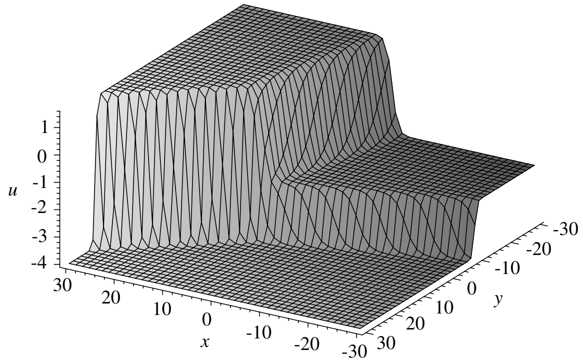

Fig. 1. A Y-junction bi-soliton in (x, y, u)-space determined by (6) and (7), with $\alpha(z, t) = 0$, $a_1 = -2, a_2 = -1, a_3 = 3/4, \lambda = -3$ and $z = 1$ at time $t = 1$.

The discussion above can also lead to the next result.

Proposition 2. *Let $\omega_1, \omega_2, \ldots, \omega_N$ be linearly independent functions satisfying (5), and*

$$W = \mathrm{Wr}(\omega_1, \omega_2, \ldots, \omega_N) = \begin{vmatrix} \omega_1 & \omega_1^{(1)} & \omega_1^{(2)} & \cdots & \omega_1^{(N-1)} \\ \omega_2 & \omega_2^{(1)} & \omega_2^{(2)} & \cdots & \omega_2^{(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_N & \omega_N^{(1)} & \omega_N^{(2)} & \cdots & \omega_N^{(N-1)} \end{vmatrix}, \tag{8}$$

in which $\omega_i^{(n)} := \partial_x^n \omega_i$ for $1 \le i \le N$, then for any analytical function $\alpha(z, t)$,

$$u = \alpha(z, t) + \frac{2\, W_x}{W} \tag{9}$$

is a solution of the Jimbo–Miwa equation (1).

Proposition 2 implies that (9) is a Bäcklund transformation relating the Jimbo–Miwa equation (1) and the linear system (5).

Before proving Proposition 2, we introduce some notations. We use $(\widehat{N-k})$ to denote the submatrix formed by the first $N-k$ columns of the $N \times M$ matrix

$$\Pi = \begin{pmatrix} \omega_1 & \omega_1^{(1)} & \omega_1^{(2)} & \cdots & \omega_1^{(M-1)} \\ \omega_2 & \omega_2^{(1)} & \omega_2^{(2)} & \cdots & \omega_2^{(M-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_N & \omega_N^{(1)} & \omega_N^{(2)} & \cdots & \omega_N^{(M-1)} \end{pmatrix}, \tag{10}$$

where $k < N < M$. Namely, we have

$$(\widehat{N-k}) = \begin{pmatrix} \omega_1 & \omega_1^{(1)} & \omega_1^{(2)} & \cdots & \omega_1^{(N-k-1)} \\ \omega_2 & \omega_2^{(1)} & \omega_2^{(2)} & \cdots & \omega_2^{(N-k-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_N & \omega_N^{(1)} & \omega_N^{(2)} & \cdots & \omega_N^{(N-k-1)} \end{pmatrix}. \tag{11}$$

We denote the kth column of the matrix Π as (k). Thus the Wronskian in (8) can be denoted as $|\widehat{N}|, |\widehat{N-1}, N|$ or $|1, 2, \ldots, N|$. Now we prove Proposition 2.

Proof. For any analytical functions $\alpha(z, t)$ and $\tau(x, y, z, t)$, substituting $u = \alpha(z, t) + 2\,\tau_x/\tau$ into the Jimbo–Miwa equation (1), integrating once with respect to x, and setting the integration constants to zero yields the following equation in τ:

$$\tau\, \tau_{xxxy} - \tau_y\, \tau_{xxx} - 3\, \tau_x\, \tau_{xxy} + 3\, \tau_{xx}\, \tau_{xy} + 2\, \tau\, \tau_{ty} - 2\, \tau_y\, \tau_t - 3\, \tau\, \tau_{xz} + 3\, \tau_x\, \tau_z = 0. \tag{12}$$

This implies that if τ satisfies (12), then $u = \alpha(z, t) + 2\,\tau_x/\tau$ is a solution of Eq. (1). So it is sufficient to show that $\tau = W = |\widehat{N}|$ satisfies (12).

If $\tau = W = |\widehat{N}|$, according to (5), we have

$$\begin{aligned}
&\tau_x = |\widehat{N-1}, N+1|,\\
&\tau_y = |\widehat{N-1}, N+2| - |\widehat{N-2}, N, N+1|,\\
&\tau_{xx} = |\widehat{N-1}, N+2| + |\widehat{N-2}, N, N+1|,\\
&\tau_{xy} = |\widehat{N-1}, N+3| - |\widehat{N-3}, N-1, N, N+1|,\\
&\tau_{xxx} = |\widehat{N-1}, N+3| + |\widehat{N-3}, N-1, N, N+1| + 2|\widehat{N-2}, N, N+2|,\\
&\tau_t = |\widehat{N-1}, N+3| + |\widehat{N-3}, N-1, N, N+1| - |\widehat{N-2}, N, N+2| + 3\lambda|\widehat{N-1}, N+1|,\\
&\tau_z = |\widehat{N-1}, N+4| - |\widehat{N-4}, N-2, N-1, N, N+1| - |\widehat{N-2}, N, N+3|\\
&\qquad + |\widehat{N-3}, N-1, N, N+2| + 2\lambda\, |\widehat{N-1}, N+2| - 2\lambda\, |\widehat{N-2}, N, N+1|,\\
&\tau_{xz} = |\widehat{N-1}, N+5| - |\widehat{N-2}, N+1, N+3| + |\widehat{N-3}, N-1, N+1, N+2|\\
&\qquad - |\widehat{N-5}, N-3, N-2, N-1, N, N+1| - 2\lambda\, |\widehat{N-3}, N-1, N, N+1| + 2\lambda\, |\widehat{N-1}, N+3|\\
&\tau_{yt} = |\widehat{N-4}, N-2, N-1, N, N+2| - |\widehat{N-5}, N-3, N-2, N-1, N, N+1|\\
&\qquad - |\widehat{N-2}, N, N+4| + |\widehat{N-2}, N+1, N+3| - |\widehat{N-3}, N-1, N+1, N+2|\\
&\qquad + |\widehat{N-1}, N+5| + 3\lambda\, |\widehat{N-1}, N+3| - 3\lambda\, |\widehat{N-3}, N-1, N, N+1|,\\
&\tau_{xxy} = |\widehat{N-1}, N+4| - |\widehat{N-4}, N-2, N-1, N, N+1| + |\widehat{N-2}, N, N+3| - |\widehat{N-3}, N-1, N, N+2|,\\
&\tau_{xxxy} = |\widehat{N-1}, N+5| + 2\, |\widehat{N-2}, N, N+4| - |\widehat{N-3}, N-1, N+1, N+2|\\
&\qquad - |\widehat{N-5}, N-3, N-2, N-1, N, N+1| - 2\, |\widehat{N-4}, N-2, N-1, N, N+2| + |\widehat{N-2}, N+1, N+3|.
\end{aligned} \tag{13}$$

Substituting Eq. (13) into the left hand side of Eq. (12) yields

$$\begin{aligned}
&6\,(|\,\widehat{N}\,|\,|\widehat{N-2}, N+1, N+3| - |\widehat{N-1}, N+1\,|\,|\widehat{N-2}, N, N+3| + |\widehat{N-1}, N+3\,|\,|\widehat{N-2}, N, N+1\,|)\\
&\quad - 6\,(|\,\widehat{N}\,|\,|\widehat{N-3}, N-1, N+1, N+2| - |\widehat{N-1}, N+1\,|\,|\widehat{N-3}, N-1, N, N+2|\\
&\quad + |\widehat{N-1}, N+2\,|\,|\widehat{N-3}, N-1, N, N+1\,|)\\
&= 6\,(-1)^N \begin{vmatrix} \widehat{N-1} & \widehat{N-2} & N & N+1 & N+3\\ O & \widehat{N-2} & N & N+1 & N+3 \end{vmatrix}\\
&\quad - 6\,(-1)^N \begin{vmatrix} \widehat{N-1} & \widehat{N-3} & N-1 & N & N+1 & N+2\\ O & \widehat{N-3} & N-1 & N & N+1 & N+2 \end{vmatrix} = 0.
\end{aligned} \tag{14}$$

This completes the proof. □

Obviously, the solution (6) can be viewed as a special case of the solution (9). According to Proposition 2, any linearly independent set of solutions of the linear system (5) will give rise to a solution of the Jimbo–Miwa equation (1). Hence it is possible to generate a large class of solutions in this way. Particularly, N-soliton solutions can be obtained from the choice

$$\omega_n = \sum_{m=1}^{M} a_{nm} e^{\theta_m}, \quad n = 1, 2, \ldots, N, \tag{15}$$

where $\theta_m = k_m\, x + k_m^2\, y + (k_m^4 + 2\,\lambda\, k_m^2)\, z + (k_m^3 + 3\,\lambda\, k_m)\, t + \theta_{0m}$ with distinct nonzero parameters: $k_1 < k_2 < \cdots < k_M$ and constants $\{\theta_{0m}\}_{m=1}^M$. The coefficients define an $N \times M$ constant matrix $A := (a_{nm})$ of rank N due to the linear independence of the functions $\{\omega_n\}_{n=1}^N$. The Wronskian (8) can then be expressed as

$$W = \sum_{1\le m_1\le\cdots\le m_N\le M} \left[A(m_1, \ldots, m_N) \exp[\theta(m_1, \ldots, m_N)] \prod_{1\le s<r\le N} (k_{m_r} - k_{m_s}) \right], \tag{16}$$

by expanding the determinant using the Binet–Cauchy formula [40]. In the formula (16), $\theta(m_1, \ldots, m_N) := \theta_{m_1} + \theta_{m_2} + \cdots + \theta_{m_n}$, and $A(m_1, \ldots, m_N)$ is the maximal minor, i.e. the determinant of the $N \times N$ sub-matrix of A obtained from columns $1 \le m_1 < \cdots < m_N \le M$. Regularity of the solutions in the entire (x, y, z)-space for all values of time t can be guaranteed, if all the $N \times N$ maximal minors of A are non-negative.

Fig. 2 shows a tri-soliton obtained from (15) by choosing $N = 3,\ M = 6$, and the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{17}$$

The Bäcklund transformations of Eq. (1) have also been considered by Ma et al. [32]. Here we mention that the Bäcklund transformation (3.24) in [32] can also be obtained through (2) and (3) by appropriately choosing f and ω. However, the

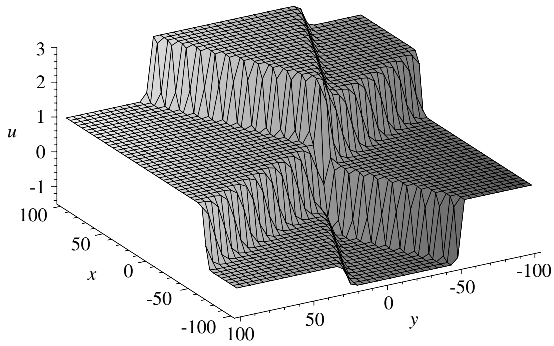

Fig. 2. Tri-soliton in (x, y, u)-space obtained from (9), (15) and (17), with $\alpha(z, t) = 0$, $k_1 = -3/2$, $k_2 = -3/5$, $k_3 = -1/5$, $k_4 = 2/3$, $k_5 = 1$, $k_6 = 4/3$, $\theta_{01} = -1$, $\theta_{03} = -1/3$, $\theta_{05} = 1/2$, $\theta_{02} = \theta_{04} = \theta_{06} = 0$, $\lambda = 4$ and $z = 1$ at time $t = 1$.

other two Bäcklund transformations, i.e. (3.16) and (3.25) in Ref. [32], cannot be obtained through (2) and (3) in this paper. There are also different N-soliton solutions of the Jimbo–Miwa equation (1) that cannot be obtained from (15) [36–38]. This reflects the special integral property of the equation.

3. Bi-soliton-like solutions of the Jimbo–Miwa equation

Following the proof of the Proposition 2, for any analytical functions $\alpha = \alpha(z, t)$ and $\tau = \tau(x, y, t)$, when $f = \alpha$ and $\omega = \tau$, the transformation (2) becomes a Bäcklund transformation relating the Jimbo–Miwa equation (1) and Eq. (12). This motivates us to seek exact solutions of the Jimbo–Miwa equation through solving (12). In order to get explicit bi-soliton-like solutions for Eq. (1), we assume that the solution of (12) is in the following form

$$\tau = 1 + e^{A} + e^{B} + C\, e^{A+B}, \tag{18}$$

where $A = A(x, y, z, t)$, $B = B(x, y, z, t)$ and $C = C(x, y, z, t)$ are functions to be determined. Substituting (18) into (12) yields an algebraic equation for e^A and e^B. Setting the coefficients of those powers of e^A and e^B to zero, we can get a set of partial differential equations in the unknown functions A, B and C. In what follows, we denote this system as *Eqs* (we do not list the system here due to its complexity). If we can solve A, B and C from the system *Eqs*, we can then get τ according to (18), and hence can obtain the solution of Eq. (1).

It is difficult for us to solve the system *Eqs* in general. However, some nontrivial exact solutions of the system can be obtained by making special assumptions on A, B and C.

Assumption 1. Let $A = a\,x + f(y, z) + g(t)$, $B = b\,x + p(y, z) + q(t)$, $C = l(z)$, for a and b be nonzero constants.

Under this assumption, and with the aid of symbolic computation, we can solve the system *Eqs* to obtain

$$\begin{aligned} f(y, z) &= \Phi\left(z + \frac{6\,b\,y}{4\,c - b^3 + 3\,a^2\,b}\right), \qquad g(t) = \frac{a\,(a^2\,b - b^3 + 4\,c)t}{4\,b}, \\ p(y, z) &= \Psi\left(z + \frac{3\,b\,y}{2\,c + b^3}\right), \qquad q(t) = c\,t, \qquad l(z) = \frac{(a-b)^2}{(a+b)^2}, \end{aligned} \tag{19}$$

where $\Phi(\cdot)$, $\Psi(\cdot)$ are arbitrary functions, and c is arbitrary constant.

Thus we have the following proposition.

Proposition 3. *For arbitrary analytical function $\alpha(z, t)$, the Jimbo–Miwa equation (1) possesses the bi-soliton-like solution*

$$u = \alpha(z, t) + \frac{2\,\tau_x}{\tau}, \tag{20}$$

where

$$\tau = 1 + e^{a\,x + f(y,z) + g(t)} + e^{b\,x + p(y,z) + q(t)} + l(z)\, e^{(a+b)\,x + f(y,z) + p(y,z) + g(t) + q(t)}, \tag{21}$$

and the functions $f(y, z)$, $g(t)$, $p(y, z)$, $q(t)$ and $l(z)$ are given by (19).

The solution (20) can model a wide categories of bi-soliton-like waves due to the arbitrary functions contained in the solution. For example, if setting

$$\begin{aligned} &a = 2, \qquad b = -3, \qquad c = -1, \qquad \alpha(z, t) = 0, \\ &\Phi(X) = 9\,\arctan X + 50, \qquad \Psi(X) = -18\,\sin\frac{3X}{4} + 50, \end{aligned} \tag{22}$$

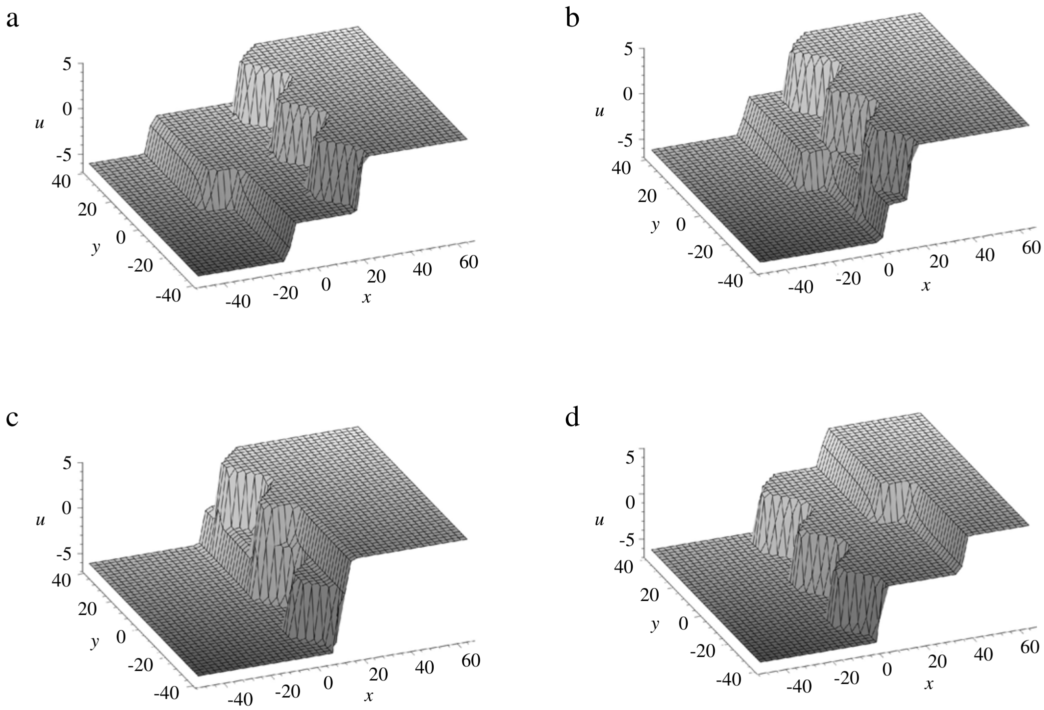

Fig. 3. Collision between the two curved solitons determined by solution (20), with the parameter choice (22) and $z = 1$ at times (a). $t = 10$; (b). $t = 25$; (c). $t = 30$; (d). $t = 60$.

we obtain curved bi-soliton. Fig. 3 illustrates the evolution of this kind of bi-soliton. As we can see, both waves keep their original shapes and speeds during the propagation, just like the usual line-shaped bi-soliton.

Assumption 2. $A = a\,x + \xi(z, t) + \eta(y)$, $B = b\,x + \theta(z, t) + \phi(y)$, $C = \nu(z)$, in which a and b are nonzero constants.

Under this assumption, the system *Eqs* can be solved to obtain

$$\begin{aligned}
&\xi(z,t) = \frac{c\,a^2\,z}{3} + F\left(t + \frac{2\,c\,z}{3\,a}\right), \qquad \eta(y) = c\,y, \\
&\theta(z,t) = -\frac{c\,b^3\,z}{3\,a} + G\left(t - \frac{2\,c\,z}{3\,a}\right), \qquad \phi(y) = -\frac{c\,b\,y}{a}, \qquad \nu(z) = 1,
\end{aligned} \tag{23}$$

where $F(\cdot)$, $G(\cdot)$ are arbitrary functions and c is arbitrary nonzero constant.

Hence we also have the following proposition.

Proposition 4. *Let $\alpha(z, t)$ be arbitrary analytical function, and*

$$\tau = 1 + e^{a\,x+\xi(z,t)+\eta(y)} + e^{b\,x+\theta(z,t)+\phi(y)} + \nu(z)\,e^{(a+b)\,x+\xi(z,t)+\theta(z,t)+\eta(y)+\phi(y)}, \tag{24}$$

with the functions $\xi(z, t)$, $\eta(y)$, $\theta(z, t)$, $\phi(y)$ and $\nu(z)$ be given by (23), then

$$u = \alpha(z,t) + \frac{2\,\tau_x}{\tau} \tag{25}$$

is a solution of the Jimbo–Miwa equation (1).

Similar to the upper case, the solution (25) may also model different types of bi-soliton-like waves under different choices of the functions F and G. A special bi-soliton-like wave and a typical bi-soliton modeled by solution (25) are shown in Fig. 4. The parameters and functions corresponding to Fig. 4 are as follows

$$\text{(a)}: \ a = \frac{3}{2}, \qquad b = -2, \qquad c = 1, \qquad \alpha(z,t) = 0, \qquad F(X) = -2\,\sin X, \qquad G(X) = 6\,X. \tag{26}$$

$$\text{(b)}: \ a = 3, \qquad b = -2, \qquad c = 2, \qquad \alpha(z,t) = 0, \qquad F(X) = -3\,X - 15, \qquad G(X) = 5\,X - 2. \tag{27}$$

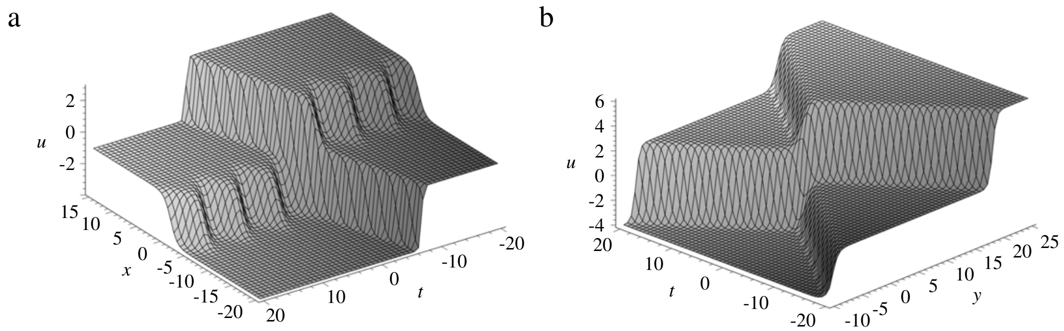

Fig. 4. Solution waves determined by (25). (a) A bi-soliton-like wave corresponds to the parameter choice (26) at $z = 1, y = 3$; (b) A bi-soliton with the parameter choice (27) at $z = 1,\ t = 2$.

Fig. 5 shows the time-evolution of a bi-soliton-like wave determined by the solution (25). We can see that one wave (left) is a traditional line soliton and another (right) is a soliton-like breather. Both waves keep their shapes during the propagation, but the breather changes its speed periodically.

The parameter and function choices in Fig. 5 are

$$a = \frac{3}{2}, \qquad b = -1, \qquad c = -\frac{3}{4}, \qquad \alpha(z,t) = 0, \qquad F(X) = -8\,\sin X, \qquad G(X) = 3\,X. \tag{28}$$

For both solutions (20) and (25), it is easy to prove that

$$|u(x,y,z,t)| < |\alpha(z,t)| + 4\,(|a| + |b|), \quad \forall\,(x,y,z,t) \in R^4. \tag{29}$$

Therefore, in either case, u is a bounded solution of the Jimbo–Miwa equation (1) once α is bounded. Moreover, if $\alpha(z,t) = u_0$, with u_0 be real constant, we have

$$\lim_{x\to+\infty} u = \begin{cases} u_0 + 2\,(a+b), & a > 0,\ b > 0 \\ u_0, & a < 0,\ b < 0 \\ u_0 + 2\max\{a,b\}, & a\,b < 0, \end{cases} \tag{30}$$

and

$$\lim_{x\to-\infty} u = \begin{cases} u_0, & a > 0,\ b > 0 \\ u_0 + 2\,(a+b), & a < 0,\ b < 0 \\ u_0 + 2\,\min\{a,b\}, & a\,b < 0. \end{cases} \tag{31}$$

One may also make other assumptions on the functions A, B and C in (18), so as to obtain new bi-soliton-like solutions for the Jimbo–Miwa equation.

4. Conclusion and discussion

The Wronskian form solutions of the NLEEs are often constructed by means of the bilinear approach. In this paper, we obtained such solutions of the $(3+1)$-dimensional Jimbo–Miwa equation in a different way. When compared with those presented in Refs. [37,38], our solution has less restrictions on the entries of the Wronskian.

Besides the Wronskian form solutions, we also got bi-soliton-like solutions of the Jimbo–Miwa equation through Bäcklund transformation and symbolic computation. Based on those solutions, we demonstrated the dynamic behaviors of some solution waves, and revealed different phenomena such as curved bi-soliton and bi-soliton-like breather. Due to the arbitrary functions they contain, the bi-soliton-like solutions can model a wide categories of solution waves, and hence will be useful for the analyzing of nonlinear phenomena arising in related models.

In order to get some nontrivial solutions of the Jimbo–Miwa equation (1), we made certain restrictions on the functions involved in the solving procedure. Different restrictions on those functions may lead to new solutions of the equation. For example, consider Eq. (3) in Section 2. If we make the following assumptions other than (4) and (5):

$$f = \alpha(t)\,x + \mu\,y + \beta(z,t), \tag{32}$$

and

$$\omega_y = \omega_{xx}, \qquad \omega_z = \omega_{xxxx} + [2\,\lambda + \alpha(t)]\,\omega_{xx} + \mu\,\omega_x, \qquad \omega_t = \omega_{xxx} + 3\,\lambda\,\omega_x, \tag{33}$$

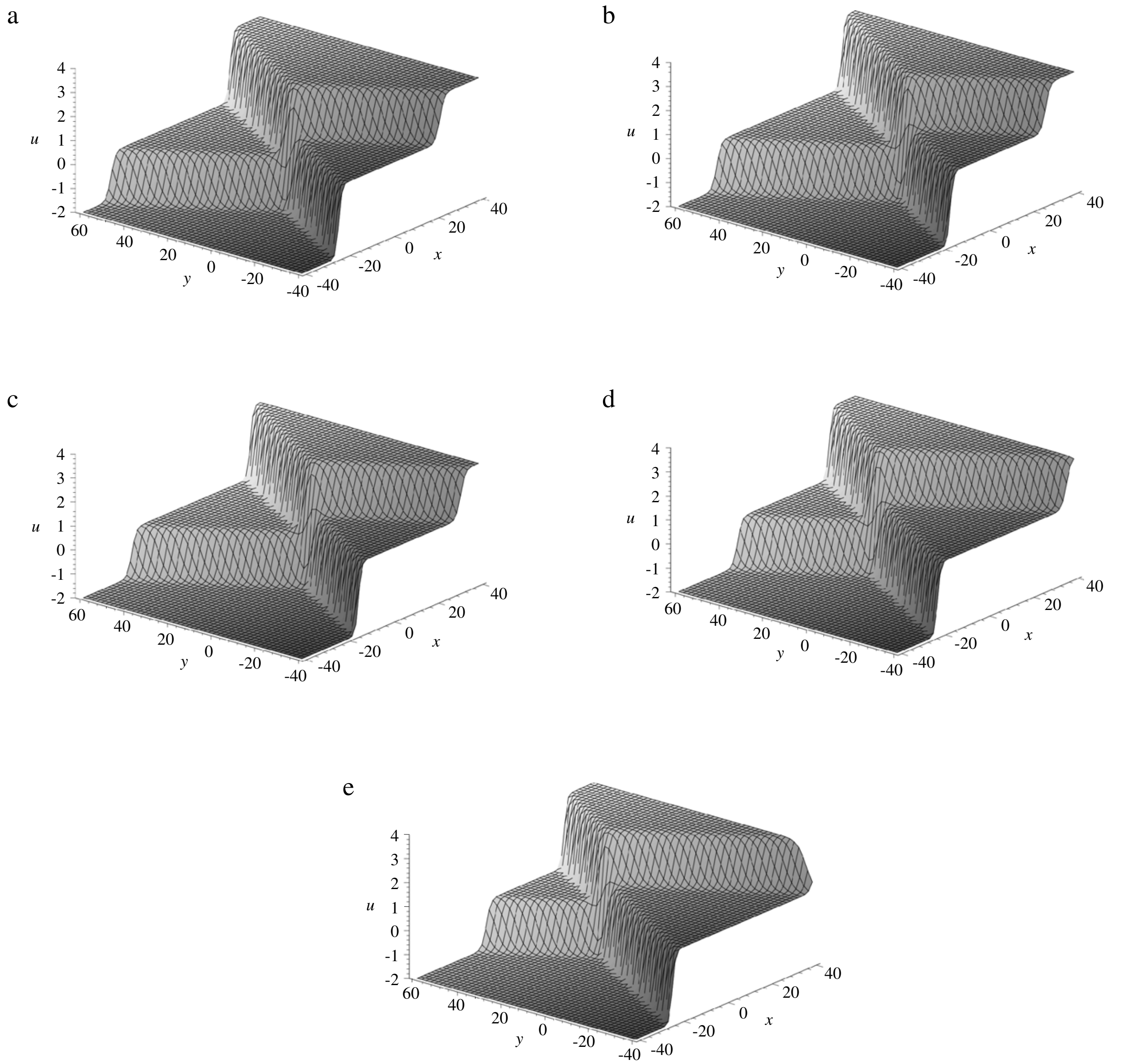

Fig. 5. Propagation of a bi-soliton-like breather determined by the solution (25), with the parameter choice (28) and $z = 3$ at times (a). $t = 0$; (b). $t = \pi/2$; (c). $t = \pi$; (d). $t = 3\pi/2$; (e). $t = 2\pi$.

with $\alpha(t)$, $\beta(z, t)$ be arbitrary analytical functions, and λ, μ be arbitrary constants, then we can prove that (32) and (33) also reduce each equation of (3) to the identity $0 = 0$. Thus, we expect that the Jimbo–Miwa equation (1) also possess the following Wronskian form solution

$$u = \alpha(t)\,x + \mu\,y + \beta(z, t) + \frac{2\,W_x}{W}, \tag{34}$$

in which

$$W = \mathrm{Wr}(\omega_1, \omega_2, \ldots, \omega_N), \tag{35}$$

and $\omega_1, \omega_2, \ldots, \omega_N$ are linearly independent solutions of the linear system (33). In fact, correctness of the solution (34) can be proved in a similar way to the proof of Proposition 2. Here we omit it.

References

[1] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, London, 1991.
[2] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl. 15 (1970) 539–541.

[3] V.E. Zakharov, A.B. Shabat, Exact theory of two-dimentional self-focusing and one-dimentional self-modulation of waves in nonlinear media, Sov. Phys. JETP 34 (1972) 62–69.
[4] R. Hirota, Exact solution of the KdV equation for multiple collisions of solitons, Phys. Rev. Lett. 27 (1971) 1192–1194.
[5] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer-Verlag, Heidelberg, 1991.
[6] H.B. Lan, K.L. Wang, Exact solutions for two nonlinear equations: I, J. Phys. A: Math. Gen. 23 (1990) 3923–3928.
[7] W.X. Ma, B. Fuchssteiner, Explicit and exact solutions to a Kolmogorov–Petrovskii–Piskunov equation, Internat. J. Non-Linear Mech. 31 (1996) 329–338.
[8] Z.Y. Yan, H.Q. Zhang, New explicit solitary wave solutions and periodic wave solutions for Whitham–Broer–Kaup equation in shallow water, Phys. Lett. A 285 (2001) 355–362.
[9] Z.S. Lü, H.Q. Zhang, On a further extended tanh method, Phys. Lett. A 307 (2003) 269–273.
[10] M.L. Wang, X.Z. Li, J.L. Zhang, The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A 372 (2008) 417–423.
[11] İ. Aslan, The discrete (G'/G)-expansion method applied to the differential–difference Burgers equation and the relativistic Toda lattice system, Numer. Methods Partial Differential Equations 28 (2012) 127–137.
[12] İ. Aslan, Exact and explicit solutions to the discrete nonlinear Schrodinger equation with a saturable nonlinearity, Phys. Lett. A 375 (2011) 4214–4217.
[13] İ. Aslan, Some exact solutions for Toda type lattice differential equations using the improved (G'/G)-expansion method, Math. Methods Appl. Sci. 35 (2012) 474–481.
[14] W.X. Ma, T.W. Huang, Y. Zhang, A multiple exp-function method for nonlinear differential equations and its application, Phys. Scr. 82 (2010) 065003.
[15] W.X. Ma, Solving the $(3+1)$-dimensional generalized KP and BKP equations by the multiple exp-function algorithm, Appl. Math. Comput. 218 (2012) 11871–11879.
[16] N.C. Freeman, J.J.C. Nimmo, Soliton solutions of the Korteweg–de Vries and Kadomtsev–Petviashvili equations: the Wronskian technique, Phys. Lett. A 95 (1983) 1–3.
[17] J.J.C. Nimmo, N.C. Freeman, The use of Bäcklund transformations in obtaining N-soliton solutions in Wronskian form, J. Phys. A: Math. Gen. 17 (1984) 1415–1424.
[18] W.X. Ma, Complexiton solutions to the Korteweg–de Vries equation, Phys. Lett. A 301 (2002) 35–44.
[19] W.X. Ma, Y.C. You, Solving the Korteweg–de Vries equation by its bilinear form: Wronskian solutions, Trans. Amer. Math. Soc. 357 (2005) 1753–1778.
[20] W.X. Ma, C.X. Li, J.S. He, A second Wronskian formulation of the Boussinesq equation, Nonlinear Anal. TMA 70 (2009) 4245–4258.
[21] W.X. Ma, Combined Wronskian solutions to the 2D Toda molecule equation, Phys. Lett. A 375 (2011) 3931–3935.
[22] Y.Q. Yao, D.Y. Chen, Y.B. Zeng, N-soliton solutions for a $(3+1)$-dimensional breaking soliton equation with self-consistent sources, Nonlinear Anal. TMA 72 (2010) 57–64.
[23] S. Chakravarty, Y. Kodama, Soliton solutions of the KP equation and application to shallow water waves, Stud. Appl. Math. 123 (2009) 83–151.
[24] M. Jimbo, T. Miwa, Solitons and infinite dimensional Lie algebras, Publ. Res. Inst. Math. Sci. 19 (1983) 943–1001.
[25] B. Dorrizzi, B. Grammaticos, A. Ramani, P. Winternitz, Are all the equations of the KP hierarchy integrable? J. Math. Phys. 27 (1986) 2848–2852.
[26] J. Rubin, P. Winternitz, Point symmetries of conditionally integrable nonlinear evolution equations, J. Math. Phys. 31 (1990) 2085–2090.
[27] X.Y. Tang, J. Lin, Conditional similarity reductions of Jimbo–Miwa equation via the classical Lie group approach, Commun. Theor. Phys. 39 (2003) 6–8.
[28] H.C. Ma, A simple method to generate Lie point symmetry groups of the $(3+1)$-dimensional Jimbo–Miwa equation, Chin. Phys. Lett. 22 (2005) 554–557.
[29] B. Tian, Y.T. Gao, W. Hong, The solitonic features of a nonintegrable $(3+1)$-dimensional Jimbo–Miwa equation, Comput. Math. Appl. 44 (2002) 525–528.
[30] T. Özis, I. Aslan, Exact and explicit solutions to the $(3+1)$-dimensional Jimbo–Miwa equation via the exp-function method, Phys. Lett. A 372 (2008) 7011–7015.
[31] X.Y. Tang, Z.F. Liang, Variable separation solutions for the $(3+1)$-dimensional Jimbo–Miwa equation, Phys. Lett. A 351 (2006) 398–402.
[32] W.X. Ma, J.H. Lee, A transformed rational function method and exact solutions to the $3+1$ dimensional Jimbo–Miwa equation, Chaos Solitons Fractals 42 (2009) 1356–1363.
[33] Z.D. Dai, J. Liu, X.P. Zeng, Z.J. Liu, Periodic kink-wave and kinky periodic-wave solutions for the Jimbo–Miwa equation, Phys. Lett. A 372 (2008) 5984–5986.
[34] G.Q. Xu, The soliton solutions, dromions of the Kadomtsev–Petviashvili and Jimbo–Miwa equations in $(3+1)$-dimensions, Chaos Solitons Fractals 30 (2006) 71–76.
[35] Zhaqilao, Z.B. Li, Multiple periodic-soliton solutions for $(3+1)$-dimensional Jimbo–Miwa equation, Commun. Theor. Phys. 50 (2008) 1036–1040.
[36] W.X. Ma, E.G. Fan, Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl. 61 (2011) 950–959.
[37] Y.N. Tang, W.X. Ma, W. Xu, L. Gao, Wronskian determinant solutions of the $(3+1)$-dimensional Jimbo–Miwa equation, Appl. Math. Comput. 217 (2011) 8722–8730.
[38] Y.N. Tang, J.Y. Tu, W.X. Ma, Two new Wronskian conditions for the $(3+1)$-dimensional Jimbo–Miwa equation, Appl. Math. Comput. 218 (2012) 10050–10055.
[39] Z.S. Lü, H.Q. Zhang, Soliton-like and period form solutions for high dimensional nonlinear evolution equations, Chaos Solitons Fractals 17 (2003) 669–673.
[40] F.R. Gantmacher, The Theory of Matrices, Chelsea Publishing Company, New York, 1959.