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Some results on the topology of varieties dominated by $\mathbf{C}^n$

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0 Introduction and statement of results

(0.0) Terminology and notation. Throughout this paper, the base field is $\mathbf{C}$ for all algebraic-geometric notions. For a variety X, X_{reg} denotes the smooth locus of X. If G is an algebraic group acting algebraically on an affine variety $X = \operatorname{Spec} A$, then $Q := X//G$ denotes $\operatorname{Spec} A^G$ whenever A^G is finitely generated. If X is affine space and the action of G is by linear transformations, then $Q = X//G$ will be called a linear quotient (variety).

(0.1) Wall's conjecture and related results. The main motivation for this work came from the following conjecture of Wall [14].

(0.1.1) Conjecture. *Let G be a reductive algebraic group acting linearly on $\mathbf{C}^n$. Suppose* $\dim C^n//G = 2$. *Then there exists a finite subgroup H of* $\mathrm{GL}(2, \mathbf{C})$ *such that* $\mathbf{C}^n//G$ *and* $\mathbf{C}^2/H$ *are isomorphic (as algebraic varieties).*

In [14], Wall verified this conjecture when G is an algebraic torus. In [8], Kempf proved that a two-dimensional linear quotient $Q = \mathbf{C}^n//G$ is actually smooth if G is connected and semi-simple; being a cone, Q must then be actually isomorphic to $\mathbf{C}^2$. More generally, Miyanishi [9] has proved the following result.

Let G be a connected algebraic group with no nontrivial characters, acting (not necessarily linearly) on $\mathbf{C}^n =: \operatorname{Spec} R$ *such that* $A := R^G$ *is finitely generated and two-dimensional. Then* $A \cong C[X, Y]$.

(0.2) Two questions on linear quotients. Easy examples (see §4) show that a statement analogous to Wall's conjecture is not valid if $\dim \mathbf{C}^n//G > 2$. On the other hand, the following two questions seem reasonable:

Let $Q = \mathbf{C}^n//G$ be any linear quotient, where G is reductive. Then

Question 1 Is $\pi_1(Q_{\text{reg}})$ a finite group?

Question 2 In $\bar{k}(Q_{\text{reg}}) = -\infty$?

Here, $\bar{k}(\cdot)$ denotes the logarithmic Kodaira dimension; for its definition and general properties, see [5, §11.2].

As we shall show in §1, an affirmative answer to either of these questions would imply that Wall's conjecture is true.

(0.3) Statement of the main results. The two main results of this paper, stated below, can be regarded as partial support for affirmative answers to the questions formulated above.

Theorem 1 *Let R be a normal affine subalgebra of the polynomial ring $\mathbf{C}[X_1, \ldots, X_n]$, and $V := \operatorname{Spec} R$. Then $H_1(V_{\mathrm{reg}}, \mathbf{Z})$ is a finite group. If further R is generated by monomials in $X_1, \ldots, X_n$, then $\pi_1(V_{\mathrm{reg}})$ is a finite abelian group.*

Actually the result we prove is stronger; see §2.3, Theorem 1, for details. The second part of Theorem 1 has been proved by Anderson [1] under the assumption $\dim R = 2$. Also, Brion has pointed out to us that the second part of Theorem 1 can also be proved by the methods of toric geometry.

Theorem 2 *Let $V = \mathbf{C}^n//G$, where G is either an algebraic torus, or is semi-simple. Then $\bar{k}(V_{\mathrm{reg}}) = -\infty$.*

The main technical result used in the proof of Theorem 1 is of independent interest, and we formulate it here separately:

Main Lemma. *Let X be an irreducible normal complex space, with a resolution of singularities $p: \tilde{X} \to X$. Assume that p is an isomorphism over X_{reg}, and that the homology groups $H_i(\tilde{X}, \mathbf{Z})$ are all finitely generated. Then the induced homomorphism $H_1(X_{\mathrm{reg}}, \mathbf{Z}) \to H_1(\tilde{X}, \mathbf{Z})$ (is surjective and) has finite kernel.*

1 Two results of two-dimensional linear quotients

In this section we shall show that an affirmative answer to either of the questions in (0.2) implies the truth of Walls's conjecture 0.1.1.

(1.1) Proposition. *Let V be a two-dimensional linear quotient (by a reductive group). Suppose $\pi_1(V_{\mathrm{reg}})$ is finite. Then V is the quotient of $\mathbf{C}^2$ by a finite subgroup of* $\mathrm{GL}(2, \mathbf{C})$.

Proof. Since V is a normal affine surface whose coordinate ring is a subring of a polynomial ring generated by homogeneous polynomials, V is quasi-homogeneous with a good $\mathbf{C}^*$-action. Hence V is topologically contractible (to its unique singularity), and the fundamental group at infinity of V is isomorphic to $\pi_1(V_{\mathrm{reg}})$. Since $\pi_1(V_{\mathrm{reg}})$ is finite by assumption, we can now use the following topological characterisation of finite quotients of $\mathbf{C}^2$ [3].

'Let W be a normal topologically contractible affine surface. Then W is (algebraically) isomorphic to $\mathbf{C}^2/H$ for a finite subgroup H of $\mathrm{GL}(2, \mathbf{C})$ iff the fundamental group at infinity of W is finite.'

Thus we see that Wall's conjecture is equivalent to the finiteness of $\pi_1(V_{\mathrm{reg}})$. □

(1.2) Proposition. *Let V be a two-dimensional linear quotient (by a reductive group). Suppose $\bar{k}(V_{\mathrm{reg}}) = -\infty$. Then V is isomorphic to a quotient of $\mathbf{C}^2$ by a finite subgroup.*

Proof. We have already observed that there is a good $\mathbf{C}^*$-action on V. Since V is obviously uni-rational, it is clear that the orbit space $V_{\mathrm{reg}}/\mathbf{C}^*$ is isomorphic to P^1.

Let $\varphi\colon V_{\mathrm{reg}} \to P^1$ be the orbit morphism. From the results of Pinkham [12], there exists a normal quasi-homogeneous surface W with a surjective $\mathbf{C}^*$-equivariant Galois morphism $\tilde{\sigma}\colon W \to V$ such that, in the commutative diagram

$$\begin{array}{ccc} W_{\mathrm{reg}} & \xrightarrow{\tilde{\sigma}} & V_{\mathrm{reg}} \\ {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ C & \xrightarrow{\sigma} & P^1 \end{array}$$

(where $\tilde{\varphi}\colon W_{\mathrm{reg}}/\mathbf{C}^* =: C$ is the orbit morphism), $\tilde{\sigma}$ is unramified and $\tilde{\varphi}$ is a locally trivial $\mathbf{C}^*$-fibration. By Theorem 11.10 of [5] $\bar{k}(W_{\mathrm{reg}}) = \bar{k}(V_{\mathrm{reg}})$. Hence $\bar{k}(W_{\mathrm{reg}}) = -\infty$. On the other hand, $\bar{k}(C) \leq \bar{k}(W_{\mathrm{reg}})$, by the inequality of Y. Kawamata [6] applied to the map $\tilde{\varphi}$. Hence C is isomorphic to P^1. Since the bundle $W_{\mathrm{reg}} \to P^1$ is clearly not trivial, it follows that $\pi_1(W_{\mathrm{reg}})$ is finite cyclic, and hence that $\pi_1(W_{\mathrm{reg}})$ is finite. We can now apply Proposition 1.1. □

2 Proof of Theorem 1

In this section, we show how Theorem 1 follows from the Main Lemma, and prove the Main Lemma. Let V be as in Theorem 1; thus V is a normal affine variety with a dominant morphism $\mathbf{C}^n \to V$. The idea of the proof is to observe that, if $\tilde{V} \to V$ is a resolution of singularities of V which is an isomorphism over V_{reg}, then $\pi_1(\tilde{V})$, and hence $H_1(\tilde{V}; \mathbf{Z})$, is a finite group. Hence the Main Lemma will imply that $H_1(V_{\mathrm{reg}}, \mathbf{Z})$ is a finite group. The proof of the Main Lemma can be reduced, by Poincaré duality, to the proof of the fact that the Chern classes of the irreducible exceptional divisors of the map $\tilde{V} \to V$ are linearly independent; this last fact is proved by reducing to the case of surfaces. The case when $V = \operatorname{Spec} R$ with R generated by monomials follows easily, since $\pi_1(V_{\mathrm{reg}})$ is abelian in this case. We now proceed to the details.

(2.1) Lemma. *Let V be a normal affine variety with a dominant morphism $\varphi\colon \mathbf{C}^n \to V$. Let $\tilde{V} \to V$ be a resolution of singularities of V. Then $\pi_1(\tilde{V})$ is finite.*

Proof. By Hironaka's theorem on removing the indeterminancies of a rational map, we can get a dominant morphism $\tilde{\varphi}\colon \tilde{\mathbf{C}}^n \to \tilde{V}$, where $\tilde{\mathbf{C}}^n$ is obtained from $\mathbf{C}^n$ by a finite succession of blow-ups with smooth centres. Thus $\tilde{\mathbf{C}}^n$ will still be simply connected. Now the finiteness of $\pi_1(\tilde{V})$ follows from Lemma 1.5 (B) of [10]: for any dominant morphism $f\colon X \to Y$ of smooth connected varieties, the image of $\pi_1(X)$ has finite index in $\pi_1(Y)$. □

(2.2) Lemma. *Let $V = \operatorname{Spec} R$ be a normal affine variety, where R is a subring of the polynomial ring generated by monomials. Then $\pi_1(V_{\mathrm{reg}})$ is abelian.*

Proof. By assumption, V is the Zariski closure of the image of a map $\mathbf{C}^n \to \mathbf{C}^k$ defined by k monomials. Hence V_{reg} contains the image T of $(\mathbf{C}^*)^n$, which is clearly a Zariski open subset of V isomorphic to a torus. Since $\pi_1(T) \to \pi_1(V_{\mathrm{reg}})$ is surjective, the lemma follows. □

(2.3) Proof of the Main Lemma. In view of Lemma 2.1 and 2.2, it is now clearly sufficient to prove the Main Lemma (see §0). Recall that the result to be proved can be formulated as follows.

Let U be a Zariski open subset of a connected complex manifold $\tilde{X}$ with finitely generated $\mathbf{Z}$-homology. Suppose there exists a proper morphism $\tilde{X} \to X$ which is an isomorphism on U and maps $E = \tilde{X} - U$ to a subvariety of codimension at least two. Then $H_1(U, \mathbf{Z}) \to H_1(\tilde{X}, \mathbf{Z})$ is surjective and has finite kernel.

In what follows, we shall work with homology and cohomology with complex coefficients. Thus we must prove that $H_1(U) \to H_1(\tilde{X})$ is injective. Note that the surjectivity even at the level of π_1 is obvious.

Now we observe that we have a commutative diagram

$$\begin{array}{ccc} H_c^{2n-1}(U) & \to & H_c^{2n-1}(\tilde{X}) \\ \cong\downarrow & & \downarrow\cong \\ H_1(U) & \to & H_1(\tilde{X}) \end{array}$$

where H_c^* denotes cohomology with compact support, and the vertical isomorphisms are provided by Poincaré duality ($n :=$ complex dimension of $\tilde{X}$). The cohomology exact sequence with compact support for the pair $(\tilde{X}, E)$ gives

$$\ldots \to H_c^{2n-2}(\tilde{X}) \to H_c^{2n-2}(E) \to H_c^{2n-1}(U) \to H_c^{2n-1}(\tilde{X}) \to 0\,,$$

hence our assertion is equivalent to the surjectivity of the map

$$H_c^{2n-2}(\tilde{X}) \to H_c^{2n-2}(E)\,.$$

By the same exact sequence for the pairs (E, E_{sing}) and (X, E_{sing}), we have the commutative diagram

$$\begin{array}{ccc} H_c^{2n-2}(E_{\text{reg}}) & \xrightarrow{\sim} & H_c^{2n-2}(E) \\ \rho\uparrow & & \uparrow \\ H_c^{2n-2}(\tilde{X} - E_{\text{sing}}) & \xrightarrow{\sim} & H_c^{2n-2}(\tilde{X}) \end{array}$$

since $\dim_{\mathbf{R}}(E_{\text{sing}}) \leqq 2n - 4$. Thus we must prove that ρ is surjective.

Now let $E = \bigcup E_i$ be the decomposition of E into irreducible components. We may clearly assume that all the E_i are of codimension one (lower dimensional components can be included in E_{sing} in the above reduction). It is clear that $\dim H_c^{2n-2}(E_{\text{reg}})$ is then equal to the number of irreducible components of E ($=$ number of connected components of E_{reg}). The first Chern classes $c_i := c_1(E_i)$ of the E_i are elements of $H^2(\tilde{X})$, hence can be regarded as elements of the dual of $H_c^{2n-2}(\tilde{X} - E_{\text{sing}})$ by Poincaré duality. In the de Rham description of cohomology, we have $c_i(\eta) = \int_{\tilde{X}} \eta \wedge c_i$ for any closed $(2n-2)$-form η with compact support on $\tilde{X} - E_{\text{sing}}$. By a standard property of the first Chern class (see e.g. [13, Chap. 1, §3]), $c_i(\eta) = \int_{E_i} \eta$, hence it is obvious that the c_i, as elements of the dual of $H_c^{2n-2}(\tilde{X} - E_{\text{sing}})$, vanish on $\ker \rho$. Thus, if we show that the c_i are linearly independent, the surjectivity of ρ will follow by dimension considerations.

To prove the linear independence of the c_i, suppose first that $\dim \tilde{X} = 2$. Then E_i are compact curves, and in the usual identification of $H^2(E_i, \mathbf{Z})$ with $\mathbf{Z}$, $c_i | E_j = (E_j \cdot E_j)$, the intersection number of E_i and E_j. Now any linear relation

$\sum \lambda_i c_i = 0$ would imply $(\sum \lambda_i c_i) | E_j = \sum_i \lambda_i (E_i \cdot E_j) = 0$ for all j. However, by a well-known theorem of Grauert, the matrix $((E_i \cdot E_j))$ is negative definite; in particular it is nonsingular. Hence we must have $\lambda_i = 0$ for all i.

The general case can be reduced to the two-dimensional case as follows. Suppose there is a relation $\sum \lambda_i c_i = 0$, with say $\lambda_1 \neq 0$. At a smooth point $\tilde{x}$ of E lying in E_1 choose a germ of a smooth curve C meeting E_1 transversally. If $\pi: \tilde{X} \to X$ is the proper morphism blowing down E, let $x := \pi(\tilde{x})$. Let Z be any germ of irreducible surface at $x \in X$ which contains $\pi(C)$ and meets X_{sing} only at x, and let Z' be the strict transform of Z in $\tilde{X}$. Then the $E_i \cap Z'$ are Cartier divisors on Z' whose Chern classes are $c_i | Z'$, hence there will be a non-trivial relation among the $c_i | Z'$ (note that $Z' \supset C$, hence $Z' \cap E_1$ is a non-empty divisor). Pulling this relation back to a desingularisation $\tilde{X}$ of Z', we will get a non-trivial relation among the Chern classes of the exceptional fibres of the composite map $\tilde{Z} \to Z$, contradicting what has already been proved for surfaces.

This completes the proof of the Main Lemma, hence also of Theorem 1. □

If we observe that Lemma 2.1 gives the analogous statement for $H_1(\tilde{V})$ whenever there is a dominant morphism φ: $Y \to V$ with Y smooth and $H_1(Y, \mathbf{Z})$ finite, we can strengthen Theorem 1 as follows:

Theorem 1' *Let Y be a non-singular algebraic variety with $H_1(Y, \mathbf{Z})$ finite. Suppose there is a dominant morphism φ: $Y \to X$ with X normal. Then $H_1(X_{\mathrm{reg}}, \mathbf{Z})$ is finite.*

3 Proof of Theorem 2

We recall the statement: if V is a linear quotient $\mathbf{C}^n//G$ with G either semi-simple or an algebraic torus, then $\bar{k}(V_{\mathrm{reg}}) = -\infty$.

(*3.1*) *Proof in the semi-simple case.* Let G be semi-simple, and φ: $\mathbf{C}^n \to V$ the quotient morphism. By Kempf [8], $\varphi^{-1}(V_{\mathrm{sing}})$ has codimension at least two. Let $d = \dim V$. Then, for a general linear subspace $\mathbf{C}^d$ of $\mathbf{C}^n$, $\psi := \varphi | \mathbf{C}^d$, $\mathbf{C}^d \to V$ is dominant, and the codimension of $\psi^{-1}(V_{\mathrm{sing}})$ in $\mathbf{C}^d$ is at least two. Hence $\bar{k}$ $(\mathbf{C}^d - \psi^{-1}(V_{\mathrm{sing}})) = -\infty$, which implies that $\bar{k}(V_{\mathrm{reg}}) = -\infty$. Note that the action of G need not be assumed to be linear in this case; the only assumption needed for applying [8] is that G should not admit non-trivial characters. □

(*3.2*) *Proof in the torus case.* Let now $V = \mathbf{C}^n//T$ be a linear quotient with T an algebraic torus. Thus V is an affine, normal toric variety. Thus, by Sumihiro's equivariant completion theorem and the equivariant resolution of singularities of a toric variety, we can imbed V_{reg} in a smooth projective toric variety W (see [7]). There exists a Zariski-open subset T' of V_{reg} which is a torus, and the action of T' on itself extends to an action of T' on W. Let $D_1, \ldots, D_r$ be the irreducible components of the divisor $W - T'$. By a result of Demazure [11, Proposition 6.6], the canonical divisor of W is given by

$$K_W = \mathcal{O}_W\left(-\sum_{i=1}^{r} D_i\right).$$

We may assume that $D := W - V_{\mathrm{reg}}$, is a union of some of the D_i. Also, $V_{\mathrm{reg}} - T'$ is a non-empty divisor in V_{reg} (since V_{reg} is not affine except in the trivial case

$V = V_{reg}$). Hence $H^0(W, n(K_W + D)) = \{0\}$ for all $n \geqq 1$. Hence $\bar{k}(V_{reg}) = -\infty$. □

(3.3) *Remarks.* (i) Consider the special case of Theorem 1 when Spec R is a linear quotient $\mathbf{C}^n//G$, with G semi-simple. For the natural map $\varphi: \mathbf{C}^n \to V$, we have seen above that $\varphi^{-1}(V_{sing})$ has codimension $\geqq 2$. It follows that $\mathbf{C}^n - \varphi^{-1}(V_{sing})$ is simply connected, and hence that $\pi_1(V_{reg})$ is finite. If G is connected, V_{reg} is in fact simply connected. This is proved by Kempf [8].

(ii) For certain unipotent groups U acting on $\mathbf{C}^n$, the ring of invariant polynomials is finitely generated; this is the case if e.g. U is the unipotent radical of a Borel subgroup of a semi-simple group acting on $\mathbf{C}^n$. In this case, we have again for $V = \mathbf{C}^n//G$: $\bar{k}(V_{reg}) = -\infty$ and $\pi_1(V_{reg})$ is finite. The proofs are identical with those in the semi-simple case.

(iii) If a torus acts algebraically on $\mathbf{C}^n$, it not known whether the action is linearisable. It is also not known whether, for the quotient $V = \mathbf{C}^n//T$, we must have $\bar{k}(V_{reg}) = -\infty$. Thus Theorems 1 and 2 can be regarded as necessary conditions for the linearisability of a torus action on affine space. Recent work of Kraft, Schwarz and Miyanishi shows that, for a $\mathbf{C}^*$-action on $\mathbf{C}^3$, the condition $\bar{k}(V_{reg}) = -\infty$ is also sufficient for linearisability.

(iv) The proof of the Main Lemma has the following corollary:

Let $D = \bigcup D_i$ be a compact divisor in a complex analytic manifold of dimension 2 with finitely generated homology and suppose that the intersection matrix $((D_i \cdot D_j))$ is non-singular. Then the map $H_1(S - D, \mathbf{R}) \to H_1(S, \mathbf{R})$ is an isomorphism.

4 Examples

(4.1) Let $\mathbf{C}^*$ act on $\mathbf{C}[X, Y, Z, W]$ by

$$\rho_t(X) = tX, \quad \rho_t(Y) = tY, \quad \rho_t(Z) = t^{-1}Z, \quad \rho_t(W) = t^{-1}W.$$

Then the ring of invariants is $R := \mathbf{C}[XZ, XW, YZ, YW]$. For $V :=$ Spec R, there is a unique singular point $\bar{0}$, the vertex of V. It can be shown that V_{reg} is simply connected. The height-one prime ideal (XZ, XW) in R has infinite order in the divisor-class group of R. Since the divisor-class group of the ring of invariants of any finite group of automorphisms of $\mathbf{C}[S, T, U]$ is always finite, it follows that R is not isomorphic to such a ring.

(4.2) Let A be an abelian surface, and $i: A \to A$ the involution $a \to -a$. Then $X := A/i$ has 16 ordinary double points, and the minimal resolution of singularities $\tilde{X}$ of X is a K-3 surface, hence simple-connected. By Theorem 1′, $H_1(X_{reg}, \mathbf{Z})$ is finite. But $\{A - 16 \text{ points}\} \to X_{reg}$ is an unramified two-sheeted covering, hence $\pi_1(X_{reg})$ contains $\mathbf{Z}^4$ as a subgroup of index two.

(4.3) The affine surface $S := \{X^{n+1} = Y^n + Z^n\}$ is parametrised by $X = U^n + V^n$, $Y = U(U^n + V^n)$, $Z = V(U^n + V^n)$. Hence the coordinate ring of S is isomorphic to the graded subring of $\mathbf{C}[U, V]$ generated by $U^n + V^n$, $U(U^n + V^n)$ and $V(U^n + V^n)$. Hence $H_1(S_{reg}, \mathbf{Z})$ is finite by Theorem 1. But $\pi_1(S_{reg})$ is infinite for $n \geqq 3$, since the singularity of S at its vertex is not even rational.

(*4.4*) There exist normal graded subrings R of $\mathbf{C}[X, Y]$ such that $V = \operatorname{Spec} R$ is a surface whose singularity is rational, but $\pi_1(V_{\mathrm{reg}})$ is infinite; see [4].

Appendix. The Main Lemma in the compact Kähler case

Since the Main Lemma was the crucial point in our proof of Theorem 1, we now give another proof for the lemma in the compact Kähler case, which does not depend on reducing to the two-dimensional case.

Lemma. *Let $\tilde{X}$ be a compact Kähler manifold, and $E = \bigcup E_i$ be a divisor on $\tilde{X}$. Suppose there exists a holomorphic map p: $\tilde{X} \to X$ with X a normal complex space, such that* (i) codim $p(E) \geqq 2$, and (ii) $\tilde{X} - E \to X - p(E)$ *is an isomorphism. Then the Chern classes $c_i = c_1(E_i)$ are linearly independent.*

Proof. Suppose there exists $a_i \in \mathbf{Z}$, with say $a_1 \neq 0$, such that $\sum a_i c_i = 0$. Then $c_1(\sum a_i E_i) = 0$. Then, by a theorem of Kodaira, there exists a meromorphic function f on the universal covering of $\tilde{X}$ such that, for each $\sigma \in \pi_1(\tilde{X})$, f^σ/f is a constant of absolute value 1, and the divisor of f on $\tilde{X}$ is $\sum a_i E_i$ (see [15, Chap. 5]). Thus $\log|f|$ is a pluri-subharmonic function on $\tilde{X} - E$ which is unbounded at smooth points of E_1. But $\log|f|$ can be regarded as a pluri-subharmonic function on $X - p(E)$; since codim $p(E) \geqq 2$, it extends as a pluri-subharmonic function to all of X, by a theorem of Grauert and Remmert [2]. In particular, $\log|f|$ cannot be unbounded. This contradiction proves the lemma. □

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