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# Two-Vortex Interactions and Elastic Constants in Type II Superconductors

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*The elastic energy of a distorted flux-line lattice is calculated on the basis of a two-vortex interaction. Such a description is completely sufficient throughout the whole induction range between the upper and lower critical fields* $H_{c1}$ *and* $H_{c2}$. *Therefore it is possible to calculate all elastic moduli from a common potential consisting of two parts, one of a combined "electromagnetic London type," the other based on the core overlap of the flux lines. The results are highly nonlocal and are in agreement with previous calculations of Brandt, but are modified near* $H_{c1}$ *for small* $\kappa$ *(the ratio between the penetration depth and the coherence length).*

## 1. INTRODUCTION

The maximum critical current density $\mathbf{j}_c$ supportable by a type II superconductor in a magnetic field without energy loss is determined by the ability of defects in the material to hold the flux-line lattice against the force produced by the transport current. To explain the high values of $\mathbf{j}_c$ obtained in experiment, an extended examination of the pinning forces is needed (see, e.g., Ref. 10). In addition, the smallest force that allows a pinning of the vortex lattice is essentially influenced by its elasticity, as first successfully investigated by Labusch.[1]

Brandt[2,3] presented a nonlocal theory of elasticity concerning the flux-line lattice of type II superconductors, based in Ref. 2 on the linearized Ginzburg–Landau functional and in Ref. 3 on a two-body interaction consisting of the magnetic field energy and the kinetic energy of the supercurrents, following London. The nonlocality is necessary since the interaction between the vortices is of extremely long range in comparison with their distance. The ansatz for the interaction in Ref. 3, however, turns out to be insufficient, since Brandt himself shows[3] that in his model it is impossible to find a potential producing all the elastic constants of Ref. 2.



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*Note**: A system is *exactly* described by two-body forces *alone* when the interaction field obeys a *linear* differential equation (as, e.g., in case of a London superconductor), whereas many-body forces can only appear in connection with nonlinear equations. In general their neglect is a good approximation if the distances are large. Due to this, even in a Ginzburg–Landau description the assumption of a two-body interaction is sufficient near the lower critical field $H_{c1}$. Near the upper critical field $H_{c2}$, however, where, because of the high vortex density the nonlinear term of the order parameter should lead to "many-vortex effects," this term is so small that it can be neglected compared to the linear term.

It must therefore be possible to calculate *all* the elastic moduli computed in Ref. 2 also using only a two-string interaction. It is the aim of this analysis to give one: This is done by supplementing the combined electromagnetic London term ("emL term") by an additional part, which only considers the lengths of the vortex sections but not their orientation. This contribution corresponds to the energy change due to the overlap of the normal-conducting vortex cores.

Furthermore, a more elaborate ansatz is made for the total vortex interaction, based on the use of the Clem model for curved flux lines,[4,5] which leads to a modification of the results for small $\kappa = \lambda/\xi$.

The basis of the calculation is a deformed flux-line lattice with vortex cores along the curves

$$\mathbf{r}_\mu(z) = \mathbf{R}_\mu(z) + \mathbf{s}_\mu(z) \tag{1}$$

where $\mathbf{R}_\mu(z) = (X_\mu, Y_\mu, z)$ signifies the flux lines in the regular, undeformed lattice situated parallel to the $z$ direction, and $\mathbf{s}_\mu$ represents the two-dimensional displacement vector located in the $xy$ plane. We limit ourselves to deformations with radius of curvature amounting to at least several penetration depths $\lambda$. The density of the vortex points in the $xy$ plane shall be $n$.

## 2. THE INTERACTION

In the London theory the magnetic field of an isolated, curved vortex (see, e.g., Ref. 6) with center along $\mathbf{r}_1$ is determined by the inhomogeneous London equation:

$$\mathbf{B}_1(\mathbf{r}) + \lambda^2 \nabla \times [\nabla \times \mathbf{B}_1(\mathbf{r})] = \Phi_0 \int d\mathbf{r}_1 \, \delta(\mathbf{r} - \mathbf{r}_1) \tag{2}$$

*For help on these points I am very grateful to W. Macke.

The solution of (2) is as follows:

$$\mathbf{B}_1(\mathbf{r}) = \frac{\Phi_0}{4\pi\lambda^2} \int d\mathbf{r}_1 \frac{\exp(-|\mathbf{r}-\mathbf{r}_1|/\lambda)}{|\mathbf{r}-\mathbf{r}_1|} \tag{3}$$

In the case of several flux lines along $\mathbf{r}_\mu$ we get the magnetic field by linear superposition of the fields of the single vortices:

$$\mathbf{B}(\mathbf{r}) = \sum_\mu \mathbf{B}_\mu(\mathbf{r}) \tag{4}$$

The result for the London free energy

$$F^{\mathrm{emL}} = \frac{1}{8\pi} \int d^3\mathbf{r} \, [\mathbf{B}^2 + \lambda^2 (\nabla \times \mathbf{B})^2]$$

takes the following form, after inserting (4), using partial integration, and using (2):

$$F^{\mathrm{emL}} = \frac{1}{2} \sum_{\mu,\nu} \int d\mathbf{r}_\mu \int d\mathbf{r}_\nu \frac{\Phi_0^2}{(4\pi)^2\lambda} \frac{\exp(-|\mathbf{r}_\mu - \mathbf{r}_\nu|/\lambda)}{|\mathbf{r}_\mu - \mathbf{r}_\nu|} \tag{5}$$

Kramer[7] calculated the interaction between two parallel, straight flux lines on the basis of the Ginzburg–Landau theory under the assumption of well-separated vortices. He obtains an approximate expression for the Ginzburg–Landau two-vortex interaction minus the combined electromagnetic London part. This approximate contribution to the energy is of an attractive nature and can be interpreted as the interaction caused by the overlap of the normal conducting vortex cores. We denote it therefore by $F^{\mathrm{c}}$. The relaxation length of this contribution has to be chosen in a way that ensures the disappearance of the total interaction for parallel, straight flux lines for $\kappa = 1/\sqrt{2}$ (see Ref. 7).

Generalizing this "core interaction" to curved geometries, it must be considered* that it can only depend on the mutual distance and on the lengths of the vortex sections but not on their orientation toward each other. At least it is transmitted by a scalar—namely by the absolute magnitude of the order parameter. (An influence of the phase of the order parameter would mean a speed-dependent interaction.)

Kramer's result, transferred to curved geometries, modifies the London free energy, valid only in the limit of infinitely thin vortex lines, by the

*I thank again W. Macke for help on this point.

following additional term:

$$F = F^{\mathrm{emL}} + F^{\mathrm{c}}$$
$$= F^{\mathrm{emL}} - \frac{1}{2} \sum_{\mu \neq \nu} \int |d\mathbf{r}_\mu|$$
$$\times \int |d\mathbf{r}_\nu| \frac{\Phi_0^2}{(4\pi)^2 \lambda} \frac{\exp(-|\mathbf{r}_{\mu\nu}|\sqrt{2}\kappa/\lambda)}{|\mathbf{r}_{\mu\nu}|} \tag{6}$$

with $\mathbf{r}_{\mu\nu} = \mathbf{r}_\mu - \mathbf{r}_\nu$.

Clem[5] suggested a model for the description of isolated flux lines that avoids the divergences of London's theory, and Wagenleithner generalized it to several curved vortices.[4] The "emL contribution" of the energy changes to

$$F^{\mathrm{emL}} = \frac{1}{2} \sum_{\mu,\nu} \int d\mathbf{r}_\mu \int d\mathbf{r}_\nu \frac{\Phi_0^2}{(4\pi)^2 \lambda} \frac{\lambda/\xi_v}{K_1(\xi_v/\lambda)}$$
$$\times \frac{\exp[(|\mathbf{r}_{\mu\nu}|^2 + \xi_v^2)^{1/2}/\lambda]}{(|\mathbf{r}_{\mu\nu}|^2 + \xi_v^2)^{1/2}} \tag{7}$$

This generalization is justified by the fact that on the one hand the expression (7) for large $\kappa$ or for large vortex distances turns into the London result (5), and on the other hand the electromagnetic contribution to the self-energy calculated by Clem is reproduced in the case of a straight, isolated vortex.

Generalizing the core interaction in the same way, namely by replacing every

$$|\mathbf{r}_{\mu\nu}| \to (|\mathbf{r}_{\mu\nu}|^2 + \xi_v^2)^{1/2}$$

(and adding a factor accordingly to $F^{\mathrm{emL}}$), one has to realize that this procedure does not provide us with the correct[5] vortex self-energy. Nevertheless, we use it for the interaction due to the core overlap of two *different* vortices. The remaining part of the self-energy shall be estimated in the same manner as in Ref. 4 by means of the corresponding expression of Clem's result for the straight vortex. This means for the correction terms of (7)

$$F = F^{\mathrm{emL}} + \sum_{\mu} \int |d\mathbf{r}_\mu| \frac{\Phi_0^2}{4(4\pi)^2 \lambda} \left( \frac{\xi_v^2}{\xi^2} + 1 \right)$$
$$- \frac{1}{2} \sum_{\mu,\nu} \int |d\mathbf{r}_\mu| \int |d\mathbf{r}_\nu| (1 - \delta_{\mu,\nu}) \frac{\Phi_0^2}{(4\pi)^2 \lambda} \frac{\lambda/\xi_v}{K_1(\xi_v/\lambda)}$$
$$\times \frac{\exp[(|\mathbf{r}_{\mu\nu}|^2 + \xi_v^2)^{1/2} \sqrt{2}\kappa/\lambda]}{(|\mathbf{r}_{\mu\nu}|^2 + \xi_v^2)^{1/2}} \tag{8}$$

Expression (5) together with (6) [respectively (7) together with (8)] form the expressions used for the interaction energy. They can be written in the following way:

$$F = \frac{1}{2} \sum_{\mu,\nu} \int d\mathbf{r}_\mu \int d\mathbf{r}_\nu \, V^{\mathrm{emL}}(|\mathbf{r}_{\mu\nu}|) + \sum_{\mu} \int |d\mathbf{r}_\mu| E_1 + \frac{1}{2} \sum_{\mu,\nu} \int |d\mathbf{r}_\mu| \int |d\mathbf{r}_\nu| \, (1-\delta_{\mu,\nu}) \, V^{\mathrm{c}}(|\mathbf{r}_{\mu\nu}|) \tag{9}$$

(*Remark*: The diverging contributions to the self-energy in the London model resulting from the vanishing extension of the vortex cores are given in the first term of this formula if $\mu$ is equal to $\nu$. Here $E_1$ denotes exclusively those parts of the self-energy estimated by the Ginzburg–Landau theory, minus the electromagnetic London contribution. Choosing the London model therefore means that $E_1$ is equated to zero.)

## 3. ELASTIC MATRIX

On the basis of a two-body interaction according to (9) we are now going to calculate the change of energy up to the second order of the displacements. We get from (1)

$$\begin{aligned} d\mathbf{r}_\mu(z) \cdot d\mathbf{r}_\nu(z') &= dz\, dz' \left(1 + \frac{d\mathbf{s}_\mu(z)}{dz} \frac{d\mathbf{s}_\nu(z')}{dz'}\right) \\ |d\mathbf{r}_\mu(z)| \cdot |d\mathbf{r}_\nu(z')| &\simeq dz\, dz' \left[1 + \frac{1}{2}\left(\frac{d\mathbf{s}_\mu(z)}{dz}\right)^2 + \frac{1}{2}\left(\frac{d\mathbf{s}_\nu(z')}{dz'}\right)^2 + \cdots\right] \\ V(\mathbf{r}_{\mu\nu}) &\simeq V(\mathbf{R}_{\mu\nu}) - \frac{1}{2} s^i_{\mu\nu} \frac{\partial}{\partial u_i} s^j_{\mu\nu} \frac{\partial}{\partial u_j} V(\mathbf{u})\Big|_{\mathbf{u}=\mathbf{R}_{\mu\nu}} + \cdots \end{aligned} \tag{10}$$

By doing this we obtain for the emL part of (9)

$$\begin{aligned} \Delta F^{\mathrm{emL}} &\equiv F[\mathbf{s}] - F[\mathbf{0}] \\ &\simeq \frac{1}{2} \sum_{\mu,\nu} \int dz \int dz' \left[\frac{d\mathbf{s}_\mu}{dz} \frac{d\mathbf{s}_\nu}{dz'} V^{\mathrm{emL}}(\mathbf{R}_{\mu\nu}) \right. \\ &\quad + \frac{1}{2}\left(s^i_\mu \frac{\partial}{\partial u_i}\right)\left(s^j_\nu \frac{\partial}{\partial u_j}\right) V^{\mathrm{emL}}(\mathbf{u})\Big|_{\mathbf{u}=\mathbf{R}_{\mu\nu}} \\ &\quad \left. + \frac{1}{2}\left(s^i_\mu \frac{\partial}{\partial u_i}\right)\left(s^j_\mu \frac{\partial}{\partial u_j}\right) V^{\mathrm{emL}}(\mathbf{u})\Big|_{\mathbf{u}=\mathbf{R}_{\mu\nu}} + \cdots \right. \end{aligned} \tag{11}$$

Integrating by parts over $z$ and $z'$ in the first term and substituting in the second term

$$s_\mu^j = s_\mu^j(z) \to \sum_\lambda \int dz''\, s_\lambda^j(z'') \delta_{\mu,\lambda} \delta(z - z'')$$

one recognizes that it is possible to express $\Delta F^{\mathrm{emL}}$ as

$$\Delta F^{\mathrm{emL}} = \text{“}(\tfrac{1}{2}\mathbf{s}\mathbf{M}^{\mathrm{emL}}\mathbf{s})\text{”} \equiv \tfrac{1}{2} s_\mu^i(z) [\mathbf{M}_{\mu\nu}^{\mathrm{emL}}(z - z')]^{ij} s_\nu^j(z') \tag{12}$$

In the sense of a generalized Einstein convention one has to sum (respectively, to integrate) over repeated vector components $i, j, \ldots$, string numerations $\mu, \nu, \ldots$, and variables $z, z'$ on the right-hand side of (12). The electromagnetic London part of all the $2 \times 2$ tensors $\mathbf{M}_{\mu\nu}(z - z')$ forming together the so-called “elastic matrix” $\mathbf{M}$ is found to be

$$\begin{aligned} -[\mathbf{M}_{\mu\nu}^{\mathrm{emL}}(z - z')]^{ij} &= \left[ \delta^{ij} \frac{\partial^2}{\partial z^2} V^{\mathrm{emL}} + \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} V^{\mathrm{emL}}(u) \right] \Bigg|_{\mathbf{u} = \mathbf{R}_{\mu\nu}} \\ &\quad - \delta_{\mu,\nu} \delta(z - z') \sum_\lambda \int dz'' \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} V^{\mathrm{emL}}(u) \Bigg|_{\mathbf{u} = \mathbf{R}_{\mu\lambda}} \end{aligned} \tag{13}$$

Substituting in this expression the interaction potential by its Fourier transform

$$V(\mathbf{r}) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} V(\mathbf{q}) \exp(i\mathbf{q}\mathbf{r}) \tag{14a}$$

and calculating with it the Fourier transform of the elastic matrix

$$\tilde{\mathbf{M}}(\mathbf{k}) = \sum_\mu \int dz\, \mathbf{M}_{\mu 0}(z) \exp(-i\mathbf{k}\mathbf{R}_\mu) \tag{14b}$$

we have

$$\begin{aligned} &[\tilde{\mathbf{M}}^{\mathrm{emL}}(\mathbf{k})]^{ij} \\ &\quad = n \sum_{\mathbf{g}} \{ [\delta^{ij} k_z^2 + (g + k)^i (g + k)^j] V^{\mathrm{emL}}(\mathbf{g} + \mathbf{k}) - g^i g^j V^{\mathrm{emL}}(\mathbf{g}) \} \end{aligned} \tag{15}$$

where the sum runs over all vectors of the reciprocal lattice.

This result arises *here* from *direct* calculation using the representation of $\mathbf{M}_{\mu\nu}$ in real space (13), whereas Brandt could also obtain it from an expression differing from (13) (see Ref. 3, Appendix A) but only by the additional demand

$$\tilde{\mathbf{M}}(\mathbf{k} = \mathbf{0}) \overset{!}{=} \mathbf{O}$$

The Fourier transform of (13) satisfies this relation automatically and it does not have to be postulated separately.

The elastic matrix due to core overlap is calculated in the same way. The expansion of both the line elements and of $V^c(\mathbf{r}_{\mu\nu})$ up to the second order in the displacements yields (we again neglect surface effects)

$$
\begin{aligned}
&-[\mathsf{M}^c_{\mu\nu}(z-z')]^{ij} \\
&\quad = \delta^{ij}\frac{\partial^2}{\partial z^2}\,\delta_{\mu,\nu}\delta(z-z') \\
&\qquad \times\left[E_1-\sum_\lambda\int dz''\,(1-\delta_{\mu,\lambda})\,V^c(u)\right]\Bigg|_{\mathbf{u}=\mathbf{R}_{\mu\lambda}} \\
&\qquad +\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}(1-\delta_{\mu,\nu})\,V^c(u)\Bigg|_{\mathbf{u}=\mathbf{R}_{\mu\nu}} \\
&\qquad -\delta_{\mu,\nu}\delta(z-z')\sum_\lambda\int dz''\,(1-\delta_{\mu,\lambda})\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}\,V^c(u)\Bigg|_{\mathbf{u}=\mathbf{R}_{\mu\lambda}}
\end{aligned}
\tag{16}
$$

This part is Fourier-transformed in the same manner as (13). If we then transform the lattice sums into integrals, the second and fourth terms of (16) vanish. They are therefore smaller than the others by one order of magnitude in $1/n$. We can neglect them in the following, which means the neglect of some anisotropy effects. Precisely, these terms result from the change of the core energy by elongation of the vortices, an effect surely small in comparison with the change of the self-energy or the emL energy caused by different orientation of the flux lines.

The remaining core part of $\mathbf{M}$ is

$$
\begin{aligned}
[\tilde{\mathsf{M}}^c(\mathbf{k})]^{ij} &= \delta^{ij}k_z^2E_1+n\sum_{\mathbf{g}}[(g+k)^i(g+k)^jV^c(\mathbf{g}+\mathbf{k})] \\
&\quad -\int\frac{d^2\mathbf{q}}{(2\pi)^2}\,q^iq^jV^c(q_x,q_y,k_z)
\end{aligned}
\tag{17}
$$

Together (15) and (17) form the complete elastic matrix. The potentials in these formulas in the London limit are as follows:

$$
\begin{aligned}
E_1 &= 0 \\
V_{\mathrm{L}}^{\mathrm{emL}}(k) &= \frac{\Phi_0^2/4\pi}{1+\lambda^2k^2} \\
V_{\mathrm{L}}^{c}(k) &= -\frac{\Phi_0^2/4\pi}{2\kappa^2+\lambda^2k^2}
\end{aligned}
\tag{18}
$$

Using Clem's model (7) and (8), we obtain[8] for the Fourier transform of the potential, additionally putting $\xi_v \approx \xi$ (see Ref. 5),

$$E_1 = \frac{\Phi_0^2}{32\pi^2\lambda^2}$$

$$V_{\mathrm{Cl}}^{\mathrm{emL}}(k) = \frac{\Phi_0^2/4\pi}{K_1(1/\kappa)} \frac{K_1(\kappa^{-1}(1+\lambda^2k^2)^{1/2})}{(1+\lambda^2k^2)^{1/2}} \tag{19}$$

$$V_{\mathrm{Cl}}^{\mathrm{c}}(k) = -\frac{\Phi_0^2/4\pi}{K_1(1/\kappa)} \frac{K_1(\kappa^{-1}(2\kappa^2+\lambda^2k^2)^{1/2})}{(2\kappa^2+\lambda^2k^2)^{1/2}}$$

The use of (19) instead of (18) is more complicated, but offers the advantage of the nondivergence of the lattice sums in (15) and (17). Since the potentials (18) and (19) decline sufficiently rapidly, the isotropic approximation suggested by Brandt[3] shall be presented now: This means that we only take into account the term $\mathbf{g} = \mathbf{0}$ in our lattice sums. In addition, we neglect the integral term in (17), since errors of this order of magnitude have already been made. The complete elastic matrix is simplified by this to

$$[\tilde{\mathbf{M}}(\mathbf{k})]^{ij} = \delta^{ij}k_z^2[nV^{\mathrm{emL}}(k) + E_1] + nk^ik^j[V^{\mathrm{emL}}(k) + V^{\mathrm{c}}(k)] \tag{20}$$

## 4. CALCULATION OF THE ELASTIC CONSTANTS

In the elasticity of continuous solids the expansion of the free energy with respect to the strains defines the fourth-order tensor of the elastic constants $\mathbf{C}$:

$$\Delta F = \delta^2 F = \tfrac{1}{2}\varepsilon_{ik}(\mathbf{r})\mathsf{C}_{iklm}(\mathbf{r}, \mathbf{r}')\varepsilon_{lm}(\mathbf{r}') \tag{21}$$

($\varepsilon_{ik}$ denotes the strain tensor formed by symmetric combination of the derivatives of the displacement vector,[9] since homogeneous displacements or rotations do not affect the energy. We integrate over repeated variables $\mathbf{r}, \mathbf{r}'$.) In homogeneous systems, due to translational invariance $\mathbf{C}$ depends only on the difference vector $\mathbf{r}-\mathbf{r}'$, in a local theory in the form of $\mathbf{C} \cdot \delta(\mathbf{r}-\mathbf{r}')$. In a nonlocal theory it contains characteristic lengths of range that essentially correspond to the lengths of interaction.

Considering all symmetries of $\mathbf{C}$ (see, e.g., Ref. 9) and integrating by parts, we can write (21) in the form

$$\Delta F = \text{“}(\tfrac{1}{2}\mathbf{s}\vec{\nabla}\mathbf{C}\vec{\nabla}'\mathbf{s})\text{”} = \tfrac{1}{2}s^i(\mathbf{r})\nabla_{\mathbf{r}}^k\mathsf{C}_{iklm}(\mathbf{r}-\mathbf{r}')\nabla_{\mathbf{r}'}^l s^m(\mathbf{r}') \tag{22}$$

Expressions (22) and (12) connect the elastic constants with the elastic

matrix **M**. The Fourier transform of this relation is

$$[\tilde{\mathbf{M}}(\mathbf{k})]_{ij} = \frac{1}{n} k_l \mathbf{C}_{ilmj} k_m \tag{23}$$

This allows us to calculate the components of **C** using (23), (15), and (17). Hexagonal string symmetry[1] leaves only three independent elastic moduli. In the approximation (20) these are

$$c_{11}(\mathbf{k}) \simeq c_{11}(k_x, k_y) = n^2 V^{\mathrm{emL+c}}(k_x^2 + k_y^2)$$
$$c_{44}(\mathbf{k}) \simeq c_{44}(k_z) = n^2 V^{\mathrm{emL}}(k_z^2) + nE_1$$
$$c_{66}(k) = 0$$

Following Ref. 3, we correct $c_{66}$ and $c_{44}$ in such a way that their local part is the same as the result that Labusch[1] obtained by simple considerations. The errors of this simplification are not larger[3] than those already made in (20). The final result is therefore

$$c_{11}(\mathbf{k}) \simeq n^2 V^{\mathrm{emL+c}}(k_x^2 + k_y^2)$$
$$c_{44}(\mathbf{k}) \simeq n^2 [V^{\mathrm{emL}}(k_z^2) - V^{\mathrm{emL}}(0)] + BH \tag{24}$$
$$c_{66}(\mathbf{k}) \simeq \frac{H_{c2} B}{4\pi} \frac{1}{8\kappa^2}$$

Using (18) in (24), we have precisely the elastic constants calculated by Brandt in Ref. 2.

## 5. DISCUSSION

In contrast to the assertion in Ref. 3 that there is no potential producing both elastic moduli $c_{11}$ and $c_{44}$ as obtained by the linearized Ginzburg–Landau theory in Ref. 2, it was possible to give one here: Such a potential is found by adding an interaction independent of the mutual orientations of the vortices. (In principle it must be possible to describe a type II superconductor on the basis of a two-string interaction, as was pointed out in the introduction.)

A numerical evaluation of the lattice sums in (15) and (17) would be worth further investigation, especially to compare the more exact results found for the elastic constants with those of the isotropic approximation. A comparison with the numerical results of Brandt in Ref. 3 would also be of considerable interest.

In spite of its simplicity, the crude calculation neglecting all terms of the lattice sums except $\mathbf{g} = \mathbf{0}$ provides us with highly nonlocal expressions for $c_{11}$ and $c_{44}$ (here as in Ref. 3). The lengths of nonlocality are, as expected,

the lengths of the interaction potential, $\lambda$ and $\xi$. The lattice shows the softening against point forces (that is, short-wavelength perturbations) found by Brandt in Ref. 11. The core interaction added in this paper influences mainly the compression modulus $c_{11}$; because of its independence of orientation, however, it affects the tilting modulus $c_{44}$ only in more exact calculations. A description of $c_{11}$ and $c_{44}$ by only one common potential is therefore evidently possible.

Near the lower critical field $H_{c1}$ in the case of small $\kappa$ the London model should be substituted by that of Clem, and as a consequence the results of Brandt should be substituted by (24), simultaneously using (19). If we extrapolate this modified result up to $H_{c2}$ by introducing a field-dependent penetration depth, as did Brandt,[3] we notice a too rapid decline of the elastic moduli with increasing average induction in the sample. This is due to the fact that in the case of high inductions the ansatz suggested for the order parameter becomes of poorer quality. In the range of low inductions, however, (19) is to be preferred to (18).

The choice of the corrections to the self-energy caused by core overlap is of minor importance, as it turned out in this analysis: In the isotropic approximation these corrections do not affect anything, since the local part of $c_{44}$ is chosen to be that of Labusch. In a more exact theory considering all lattice contributions, it would be interesting, however, to use an improved model of the core interaction: This could provide us with more elaborate information on the nonlocal elastic constants already found by the simple interaction ansatz in the "isotropic approximation" and would therefore, in addition to the calculations mentioned above, be another important extension of this work.

## NOTE ADDED

While this paper was in preparation Brandt[12] published a very interesting paper on computer simulations of flux pinning. As the basis for the simulations he suggests that (besides an adequate expression for the vortex–pin interaction) the same potential as used here in (5) and (6) is the best starting point for the subsequent calculations. (However, for simplicity, the results given in Ref. 12 were then obtained by taking a Gaussian-like function for all types of interaction.)

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