REPRESENTATIONS OF THE SUPERGROUP U(p, q)

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1. Let $\mathfrak{G}$ be a Grassman algebra over a field C with a denumerable set of generators and an involution $f \to f^*$. Let L(p, q) denote the algebra of matrices of order p + q, of the form $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are, respectively, (p × p) and (q × q) matrices composed of even elements of $\mathfrak{G}$, and B and C are (p × q) and (q × p) matrices composed of odd elements of $\mathfrak{G}$. There is an involution $A \to A^*$ defined in L(p, q) as follows: if $A = \|\alpha_{ik}\|$, then $A^* = \|\alpha^*_{ki}\|$, where $\alpha_{ki} \to \alpha^*_{ki}$ is the involution in $\mathfrak{G}$. Then the supergroup U(p, q) is the group consisting of the elements of L(p, q) which satisfy the condition $A^* = A^{-1}$. (This definition is based on the general definition of a supergroup, equivalent to, but in a different form than that given in [1] and [2].) A representation of U(p, q) is a homomorphism from U(p, q) into U(p', q'). The representation T_U is irreducible if linear combinations with complex coefficients of the matrices T_U exhaust the algebra L(p, q). The supertrace of the matrix $A \in L(p, q)$ is sp' A = spA − spD. The character of the representation T_U is sp'T_U.

2. The characteristic set (c. s.) of the group U(p, q) is the set U_0 of elements of the form $g = u\theta$, where

$$u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \qquad \theta = \begin{pmatrix} (1+zz^*)^{-1/2} & (1+zz^*)^{-1/2} z \\ -(1+z^*z)^{-1/2} z^* & (1+z^*z)^{-1/2} \end{pmatrix},$$

where $u_1 \in U(p)$, $u_2 \in U(q)$, and z and z* are, respectively, (p × q) and (q × p) matrices, whose elements z_{ij}, z^*_{ji} are generators of $\mathfrak{G}$. The characteristic set acts as a system of generators of U(p, q). A function on U(p, q) is a map $U(p, q) \to \mathfrak{G}$. There exists an invariant integral on U(p, q)

$$\int f(gg_0)\, d\mu(g_0) = \int f(g_0 g)\, d\mu(g_0) = \int f(g_0)\, d\mu(g_0),$$

where $g \in U(p, q)$, $g_0 = u\theta \in U_0$, $d\mu(g_0) = d\mu(u) \prod dz_{ij}\, dz^*_{ij}$, $d\mu(u)$ is an invariant measure on U(p) × U(q), and the integral with respect to $dz_{ij} dz^*_{ij}$ is understood in the sense of [3]. Note that if $f(g) \equiv 1$, then $\int f(g_0)\, d\mu(g_0) = 0$. In view of this fact, the theory of representations of the group U(p, q) is closer to the theory of noncompact, as opposed to compact, Lie groups.

3. Let $f(g_1 g g_1^{-1}) = f(g)$ $\forall g, g_1 \in U(p, q)$. Denote the set of such functions by L. Consider the group H consisting of diagonal matrices with eigenvalues $\theta_k = \exp(i\varphi_k)$, $\sigma_{k'} = \exp(i\psi_{k'})$ $(1 \leqslant k \leqslant p,\ 1 \leqslant k' \leqslant q)$, where φ_k and $\psi_{k'}$ are real numbers. If the function $f(g) \in L$ is continuous (in the natural sense), it is completely defined by its restriction to H. Denote this by $\tilde{f}(\varphi, \psi)$. The function $\tilde{f}$ is an arbitrary periodic function of all its arguments, which is invariant with respect to independent permutations of φ_k and $\psi_{k'}$.

Let $g = u\theta \in U_0$ and let the eigenvalues of the matrix u be distinct. Then $g = g_0 hk g_0^{-1}$, where $g_0 \in U$, $h \in H$, and k is a diagonal matrix with eigenvalues $k_i = 1 + r_i$, where r_i is an even nilpotent element of $\mathfrak{G}$, and

$$d\mu(g) = j^2(\varphi, \psi) \prod d\varphi_k d\psi_{k'} d\sigma(g_0), \quad j = \prod_{i<j} \sin\frac{\varphi_i - \varphi_j}{2} \prod_{i<j} \sin\frac{\psi_i - \psi_j}{2} \left(\prod_{i,j} \sin\frac{\varphi_i - \psi_j}{2}\right)^{-1}.$$

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4. In the space of smooth functions on U(p, q), we may define the Laplace operators Δ_k in the usual way. The restriction of Δ_k to the space L, described in Para. 3, is called the radial part of Δ_k and is denoted by $\overset{\circ}{\Delta}_k$. By the method proposed in [4], operators $\overset{\circ}{\Delta}_k$ may be found such that

$$\overset{\circ}{\Delta}_k = (j(\varphi, \psi))^{-1}\left(\sum \frac{\partial^k}{\partial \varphi_i^k} + (-1)^k \sum \frac{\partial^k}{\partial \psi_i^k}\right) j(\varphi, \psi), \qquad k = 1, 2, \ldots$$

The operators Δ_k are independent for $k \leqslant p+q$. The characters of irreducible representations satisfy equations $\overset{\circ}{\Delta}_k \chi = \lambda_k \chi$. The point $\lambda = (\lambda_1, \ldots, \lambda_{p+q}) \in \mathrm{R}^{p+q}$ is called admissible if this equation is satisfied by a character of at least one irreducible representation, and is called nonsingular if the equation is satisfied by a character of precisely one. In this case the representation is also called nonsingular. The point λ is admissible if and only if there exist integers k_i and l_j which satisfy the equations $i^k(\Sigma k_j^s + (-1)^s \Sigma l_j^s) = \lambda_s$, $i = \sqrt{-1}$. The characters of nonsingular representations for $g \in H$ are of the form

$$\chi_g = \chi(\varphi, \psi) = 2^{pq} \chi_1(\varphi) \chi_2(\psi) \prod_{i,j} \sin \frac{\varphi_i - \psi_j}{2},$$

where $\chi_1(\varphi)$ and $\chi_2(\psi)$ are characters of irreducible representations of the groups U(p) and U(q), respectively.

Let $t \in H$ have eigenvalues $\theta_k = 1$, $\sigma_{k'} = -1$. Let T_g be a nonsingular representation of U(p, q). In this case $\chi_{tg} = \mathrm{sp}T_g$. For $g \in H$ $\chi_{tg} = \chi(\varphi_1, \ldots, \varphi_p, \psi_1 + \pi, \ldots, \psi_q + \pi)$. It follows from these results that a nonsingular representation is a homomorphism from U(p, q) into U(p', p'), $p' = 2^{pq-1}N_1N_2$, where $N_1 = \chi_1(0)$, $N_2 = \chi_2(0)$.

5. Let $p \geqslant q$, and let λ_k be the eigenvalues of the Laplace operators, and let $\varphi(z) = \exp\left(\sum_1^\infty \frac{(-1)^k}{k} \lambda_k z^k\right) = \sum \varphi_k(\lambda) z^k$, $z \in \mathrm{C}$. A representation is nonsingular if and only if $\det\|a_{ij}\| \neq 0$, where $a_{ij} = \varphi_{p+i-j}$, $i, j = 1, \ldots, q$.

6. Each irreducible nonsingular representation is contained in the decomposition of the left (and right) regular representation, in factors whose shortness is equal to its dimension. The character of the left (and right) regular representation is equal to zero.

7. The matrix elements of nonsingular irreducible representations satisfy the orthogonality relations

$$\int a_{ij}^T(g) (a_{i'j'}^{T'}(g))^* d\mu(g) = \delta_{TT'} \delta_{ii'} \delta_{jj'} N^{-1}(k) N^{-1}(l) R(k, l).$$

where $k = (k_1, \ldots, k_p)$, $l = (l_1, \ldots, l_q)$, $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_p$, $l_1 \geqslant \ldots \geqslant l_q$ are integers connected with λ_s by the relations shown in Para. 4, N(k) and N(l) are the dimensions of irreducible representations of the groups U(p) and U(q) with leading weights k, l respectively, and

$$R(k, l) = \prod_{i,j} \left(\sum_0^i k_{p-s} - \sum_0^j l_{q-s} - i + j\right), \qquad 0 \leqslant i \leqslant p-1, \quad 0 \leqslant j \leqslant q-1.$$

Note that this formula cannot be obtained by the traditional method based on Schur's lemma.

8. U(p, q) acts on the set $\mathfrak{M}$, consisting of (p × q) matrices $z = \|z_{ij}\|$, where z_{ij} are odd generators of $\mathfrak{G}$, according to the formula $z \to gz = (Az + B)(Cz + D)^{-1}$. Each irreducible nonsingular representation of the group U(p, q) is realized in the space of vector functions on $\mathfrak{M}$ with components belonging to $\mathfrak{G}$, according to the formula $(T_g f)(z) = A(g, z) f(g^{-1}z)$. The results obtained above relate to supergroups with semisimple even part, which do not have zero odd roots.

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LITERATURE CITED

1. F. A. Berezin and G. I. Kats, Matem. Sb., 82, 343-359 (1970).
2. F. A. Berezin and D. A. Leites, Dokl. Akad. Nauk., 224, No. 3, 505-508 (1975).
3. F. A. Berezin, Dokl. Akad. Nauk., 137, No. 2, 311-314 (1961).
4. F. A. Berezin, Trudy Mosk. Matem. Ob-va, 6, 371-463 (1957).

THE NUMBER OF INTEGRAL POINTS IN INTEGRAL POLYHEDRA

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An integral polyhedron is the convex hull in R^k of a finite subset of the lattice Z^k. The set $A+B=\{x+y \mid x\in A,\ y\in B\}$ is called the sum of the two sets A and B. Denote by N(A) the number of integral points in the polyhedron A, and by $N_i(A)$ the number of integral points in the union of all the i-faces of A.

THEOREM. Let $A_1, \ldots, A_q$ be integral polyhedra. Then

a) $N(n_1A_1 + \ldots + n_qA_q)$ is a polynomial in $n_1, \ldots, n_q$ for integral $n_1 \geqslant 0, \ldots, n_q \geqslant 0$

b) for any i, $N_i(n_1A_1 + \ldots + n_qA_q)$ is a polynomial in $n_1, \ldots, n_q$ for integral $n_1 \geqslant 1, \ldots n_q \geqslant 1$.

Proposition a) was proved for k = 2 by A. G. Kushnirenko, who also formulated it as a hypothesis for k > 2, in an attempt to prove the formula

$$k!\,V(A_1,\ldots,A_k) = (-1)^k\Big(1-\sum N(A_i)+\sum N(A_i+A_j)\ldots+(-1)^k N(A_1+\ldots+A_k)\Big), \tag{1}$$

where $V(A_1, \ldots, A_k)$ is the compound volume of the polyhedra $A_1, \ldots, A_k$. In particular, this formula follows from the formulation of the theorem. The following special case of formula (1): $k!\,V(A)=\Sigma\,(-1)^{k-n}C_k^nN(nA)$, where V(A) is the volume of the polyhedron A, together with the polynomiality of N(nA) for $n \geqslant 0$, was known before (see [1]). A. G. Kushnirenko approached this combinatorial formula from several algebraic-geometric standpoints.

We note that Proposition b) ceases to be true if any of the n_j are equal to zero. For example, the polynomial $N_i(nA)$ under the substitution n = 0 gives the Euler characteristic of the i-skeleton of the polyhedron A, and this may differ from 1.

To prove the theorem it is sufficient to prove polynomiality in each variable.

LEMMA 1. N(A + nB) is a polynomial for integral $n \geqslant 0$.

LEMMA 2. $N_i(A + nB)$ is a polynomial for integral $n \geqslant 1$.

The theorem follows from these lemmas and from the trivial estimates of the degrees of the polynomials $N(A + nB) = O(n^k)$.

In the proof we shall introduce several auxiliary operations on convex compacta. If α is a covector and A a compactum, we set $h(A,\alpha)=\max\{(\alpha,x)\mid x\in A\}$, $A(\alpha)=\{x\in A\mid(\alpha,x)=h(A,\alpha)\}$, i.e., $A(\alpha)$ is the trace excised on A by the hyperplane of support in the direction of α. If $\Omega = (\alpha, \ldots, \beta)$ is a collection of covectors, we set $A(\Omega)=A(\alpha)\cap\ldots\cap A(\beta)$. A linear operator P defines a map on the set of convex compacta; $A \to PA$. It is easily seen that the operations PA and $A(\Omega)$ are additive with respect to A, and are linearly homogeneous under multiplication of A by a positive number. Under multiplication by 0, homogeneity may break down in the unique case when $0A(\Omega)=\phi$, $(0A)(\Omega)=0$.

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