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A note on convective effects in elastic contact problems for dissimilar materials

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Abstract – In this paper we discuss the effect of neglecting relative tangential surface displacements in forming the boundary conditions of elastic contact problems between dissimilar materials. This is one of the known approximations made by Hertz in his original theory. Attempts have been made only recently to build up procedures to take this 'convective' effect into account, for simple plane problems (Soldatenkov, 1996). However, before questioning all the existing solutions for elastically dissimilar contact problems, it is considered important to estimate quantitatively the order of the possible correction. Here a simple iterative procedure is set up to solve frictionless plane contact problems taking into account the 'convective effect'. Attention is focused on the problem of wedge indentation, as this provides a reasonably tractable problem, and on the parabolic indenter, to discuss the Hertzian case. The correction introduced is shown not to be negligible, but is of practical significance only in extreme conditions, viz. frictionless contact and large Dundurs' constant, β. In these extreme cases, the maximum correction to the contact are dimension may be of the order of an *increase of* 10% for the contact area dimension. The effect tends to be more significant for Hertzian indenter and higher order profiles. © Elsevier, Paris

convective effect / contact / dissimilar materials / elastic mismatch / wedge

1. Introduction

One of the original hypotheses of Hertz's theory is that, in any contact between elastic bodies, surface particles within the contact zone must move *normal* to the free surface by an amount equal to the original gap: they are not precluded from moving laterally, but this is not taken into account in solving the normal contact problem. This is equivalent to requiring that normal displacements have a rigid body component together with a contribution equal to the original gap, at any particular point. However, this ignores the possibility of a 'convective effect', i.e. that surface particles which eventually come into contact are not the ones that were originally opposite each other in the undeformed configuration. There are also other idealisations implicit in Hertz' original contact theory. The stipulation that the contact patch be small by comparison with the characteristic radii of the contacting bodies has two consequences in simplifying the mathematics of the problem. These are

(1) The local shape of the bodies may be reduced to a simple second order surface — in the case of a plane problem, the circular form of the cylinders may be replaced by parabolae.

(2) The problem may be formulated using elasticity procedures appropriate to half spaces, or half-planes in the case of the two dimensional problem.

The first effect may be viewed separately — we could, for example, treat the case where the contacting solids really were parabolic in form precisely — but the influence of the second assumption is difficult to separate out, and in some of the results to be presented later, we will be careful to use geometries where careful numerical work using the finite element method has been used to show that the half-plane idealisation is a good one.

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It has long been noted (Love, 1927) that Hertz' calculation is precise only for elastically similar materials, where both components of surface displacement under the pressure are assumed *equal* in the two bodies. This is *approximate* in the general case. The error is usually *assumed* to be negligible, but very few studies have quantified this effect, as it requires a sophisticated treatment, even for simple plane problems (Soldatenkov, 1996).

It is believed that the error induced in neglecting the 'convective' effect may be important in an engineering context for real material combinations, and this will be discussed in the present paper with a simple improvement of the classical solution for frictionless plane contact problems. An analogous analytical treatment of more general cases, such as axisymmetric or general 3-D configurations, would seem to be prohibitively complicated, as the correction is likely to change the contact area *shape*. For example, in Hertzian (second order profile) contacts the contact area will no longer be elliptical. The present treatment is useful for estimating the error in these cases, but for a better comparison, and to take into account other idealisations at the same time, it would seem necessary to employ the Finite Element Method. The difficulty here will be that an extremely dense mesh with great refinement will be needed to eliminate the possible influence of convergence difficulties, which may mask the effect under consideration.

Although the formulation presented in the next section may, in principle, be applied to indenters having many profiles, here the results found will be applied first to a *wedge* shaped indenter, as this provides the possibility of a closed form first-order correction, and permits a better discussion of the influence of elastic mismatch and geometry. For comparison, some calculations are carried out for a parabolic indenter, in order to discuss the implications for the Hertzian geometry.

2. Formulation

We will assume frictionless contact as this does not require an incremental model, although the formulation may readily be modified to include the effects of friction. Indeed, the 'convective effect' has a maximum effect in the case of frictionless contact, as the action of friction is always to resist relative tangential motion. As the shear tractions are assumed zero, the equation governing the normal contact, within the framework of infinitesimal elasticity, and under the Hertz hypothesis of no 'convective effect' is

$$\frac{1}{A}h'(x) = \frac{1}{\pi}\int_{-b}^{b}\frac{p(\xi)\,\mathrm{d}\xi}{x-\xi}, \tag{1}$$

where $h(x)$ is the profile of the indenter, and A is a measure of the composite compliance of the bodies, defined under plane strain conditions by

$$A = 2\left(\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}\right). \tag{2}$$

Here E_i is Young's modulus and ν_i is the Poisson's ratio of body i, b is the semi-dimension of the contact area, to be determined from a side condition expressing equilibrium between applied load and pressure distribution

$$P = -\int_{-b}^{b} p(x)\,\mathrm{d}x. \tag{3}$$

The shape of the indenter, $h(x)$, is assumed to be symmetrical with respect to the application point of P, so that the problem itself is symmetrical. The function $h(x)$ may also be viewed as the amount of overlap if the bodies could freely interprenetate each other, and is defined from the geometry of the undeformed bodies $y = f_1(x)$ and $y = f_2(x)$ as

$$h(x) = C - \left[f_1(x) - f_2(x)\right], \tag{4}$$

where C is the approach of two remote points.

Now, in an 'exact' formulation, allowing for the convection phenomenon, particles in the two bodies come into contact at a position different from their locations in the unloaded bodies. It is clear that we do not know *a priori* the exact value of this displacement, as it is dependent on the pressure distribution. However, the relative tangential surface displacement, which, within the half-plane idealisation is the same as the relative x-direction displacement, $g(x)$, is given by

$$\frac{1}{A}\frac{\partial g(x)}{\partial x} = \beta p(x), \tag{5}$$

where

$$\beta = \frac{\frac{\mu_2}{\mu_1}(\kappa_1 - 1) - (\kappa_2 - 1)}{\frac{\mu_2}{\mu_1}(\kappa_1 + 1) + (\kappa_2 + 1)} \tag{6}$$

is a Dundurs' constant, which gives a measure of the 'elastic mismatch', and κ is Kolosov's constant, given by $\kappa = (3 - 4\nu)$ under plane strain conditions, and μ_i is the modulus of rigidity of body i. Note that if body 1 is rigid, and body 2 has Poisson's ratio $\nu_2 = 0$, the maximum absolute value of β is reached, given by

$$\beta = \frac{-(\kappa_2 - 1)}{(\kappa_2 + 1)} = \frac{-2}{4} = -\frac{1}{2}. \tag{7}$$

Now, from symmetry, $g(0) = 0$, and suppose body 1 is the more rigid, so that $\beta < 0$. But integrating Eq. (5), the formulation that takes into account the 'convective effect' is seen to be

$$\frac{1}{A}h'\bigl(x - \operatorname{sign}(x)g(x)\bigr) = \frac{1}{\pi}\int_{-b}^{b}\frac{p(\xi)\,\mathrm{d}(\xi)}{x - \xi}, \quad -b \leqslant x \leqslant b \tag{8}$$

together with

$$g(x) = \beta A\int_0^x p(\xi)\,\mathrm{d}\xi, \quad -b \leqslant x \leqslant b. \tag{9}$$

The way we propose to solve the problem is to set up an iterative procedure:

– start by neglecting the 'convective effect', i.e. use Eq. (1), to solve for the pressure, $p_0(x)$,
– (i) estimate the relative tangential displacement of the surface, $g(x)$, from Eq. (9),
– (ii) re-calculate the pressure from (8), and repeat (i).

It is clear that, as normal surface displacements due to a concentrated normal force P grow as $\ln|x|$ in the neighbourhood of the point of application, whereas tangential displacements due to P are *constant* and directed towards the point of application of the force, and moreover the relative displacements are due only to the *difference* in elastic constants, the 'convective effect' cannot result in more than a small correction to the pressure distribution. Therefore, a first order correction is probably sufficient in most cases, and there is no need to employ a full series solution as proposed in (Soldatenkov, 1996) as will be discussed below.

Both $p(b)$ and $p(-b)$ are bounded (the so-called *incomplete contact conditions*), and it may be proved that $p(\pm b) = 0$ must hold, so that the general solution for a contact over the range $-b \leqslant x \leqslant b$ is given by the following implicit formula

$$p(x) = \frac{1}{\pi A}\sqrt{b^2 - x^2}\int_{-b}^{b} \frac{h'(t - \mathrm{sign}(t) g(t))}{\sqrt{b^2 - t^2}(t - x)}\,\mathrm{d}t, \tag{10}$$

where the conventional solution, $p_0(x)$, is obtained by neglecting, as is usual, convection, by approximating $h'(t - \mathrm{sign}(t) g(t)) \simeq h'(t)$. Equilibrium between the applied load and the pressure distribution gives, according to (Shtayermann, 1949)

$$P = -\frac{1}{A}\int_{-b}^{b} \frac{h'(t - \mathrm{sign}(t) g(t)) t\,\mathrm{d}t}{\sqrt{b^2 - t^2}}, \tag{11}$$

where convection is correctly allowed for. Now, on substituting $g(x)$ form Eq. (9), one has, from the chain rule

$$h'\big(x - \mathrm{sign}(x) g(x)\big) = h'\left(x - \beta A\int_0^x p(\xi)\,\mathrm{d}\xi\right)\big(1 - \mathrm{sign}(x)\beta A p(x)\big). \tag{12}$$

3. Wedge-shaped punch

For a wedge in contact with a half-plane, the function $h(x)$ is

$$h'(x) = \begin{cases} \theta, & -b \leqslant x \leqslant 0, \\ -\theta, & 0 \leqslant x \leqslant +b, \end{cases} \tag{13}$$

where θ is the external angle of the wedge, which must remain small, (i) to maintain the realism of the half-plane idealization, and (ii) for the strains in the vicinity of the apex to be within the definition of linear elasticity theory.

This geometry is particulary interesting as the derivative of the profile is constant, i.e.

$$\begin{aligned} h'\big(x - \mathrm{sign}(x) g(x)\big) &= -\theta\big(1 - \mathrm{sign}(x)\beta A p(x)\big) \\ &= h'(x) + \theta\,\mathrm{sign}(x)\beta A p(x), \quad 0 \leqslant x \leqslant +b \end{aligned} \tag{14}$$

and so

$$\begin{aligned} p_0(x) &= -\frac{\theta}{\pi A}\sqrt{b^2 - x^2}\int_{-b}^{b} \frac{\mathrm{sign}(t)\,\mathrm{d}t}{\sqrt{b^2 - t^2}(t - x)} \\ &= -\frac{2\theta}{\pi A}\cosh^{-1}\left|\frac{b}{x}\right|. \end{aligned} \tag{15}$$

Then, considering the first correction,

$$p_1(x) = \frac{1}{\pi A}\sqrt{b^2 - x^2}\int_{-b}^{b} \frac{h'(t - \mathrm{sign}(t) g(t))}{\sqrt{b^2 - t^2}(t - x)}\,\mathrm{d}t \tag{16}$$

$$= p_0(x) + \frac{\theta\beta}{\pi}\sqrt{b^2 - x^2}\int_{-b}^{b} \frac{\mathrm{sign}(t) p_0(t)}{\sqrt{b^2 - t^2}(t - x)}\,\mathrm{d}t. \tag{17}$$

Now, substituting the result for $p_0(x)$,

$$p_1(x) = p_0(x) - \frac{2\theta}{\pi A}\frac{\theta\beta}{\pi}\sqrt{b^2 - x^2}\int_{-b}^{b} \frac{\mathrm{sign}(t)\cosh^{-1}|\frac{b}{t}|}{\sqrt{b^2 - t^2}(t - x)}\,\mathrm{d}t. \tag{18}$$

By defining the function

$$I_1(s) = \sqrt{1-s^2}\int_{-1}^{1} \frac{\mathrm{sign}(t)\cosh^{-1}|\frac{1}{t}|}{\sqrt{1-s^2}(t-s)}\,\mathrm{d}t \tag{19}$$

we can write the correction in dimensionless form as

$$p_1(s) = p_0(s) - \frac{2\theta}{\pi A}\frac{\theta\beta}{\pi}I_1(s) = -\frac{2\theta}{\pi A}\left[\cosh^{-1}\left|\frac{1}{s}\right| + \frac{\theta\beta}{\pi}I_1(s)\right]. \tag{20}$$

We now calculate the corrected contact area dimension, from Eq. (11). This gives

$$\begin{aligned} P &= -\frac{1}{A}\int_{-b}^{b} \frac{h'(t-\mathrm{sign}(t)g(t))t\,\mathrm{d}t}{\sqrt{b^2-t^2}} = -\frac{1}{A}\int_{-b}^{b}\frac{[h'(t)+\mathrm{sign}(t)\theta\beta A p(t)]t\,\mathrm{d}t}{\sqrt{b^2-t^2}} \\ &= 2\frac{\theta}{A}\int_0^b \frac{[1-\beta A p(t)]t\,\mathrm{d}t}{\sqrt{b^2-t^2}} = 2\frac{\theta}{A}\frac{\pi}{2}b - 2\theta\beta\int_0^b \frac{p(t)t\,\mathrm{d}t}{\sqrt{b^2-t^2}} \\ &= \frac{\theta}{A}\pi b + \frac{2\theta}{\pi A}2\theta\beta b\int_0^1 \frac{\cosh^{-1}|\frac{1}{t}|t\,\mathrm{d}t}{\sqrt{1-t^2}} = \frac{\theta}{A}b\left(\pi + 0.693147\frac{4}{\pi}\beta\theta\right). \end{aligned} \tag{21}$$

Therefore, as $b_0 = P/(\frac{\theta}{A}\pi)$, and $b = P/[\frac{\theta}{A}\pi(1+0.693147\frac{4}{\pi^2}\beta\theta)]$,

$$\frac{b}{b_0} = \frac{\frac{\theta}{A}\pi}{\frac{\theta}{A}\pi(1-0.693147\frac{4}{\pi^2}\beta\theta)} = \frac{1}{1-0.693147\frac{4}{\pi^2}\beta\theta}.$$

It was stated at the beginning of this section that the external wedge angle must be small for the half-plane idealisation to be appropriate. Detailed numerical calculations using ABAQUS have shown that, providing $\theta < \pi/5$, the internal stress state for the elastically similar problem does not differ by more than 0.5% from the exact solution, and so, in order to reduce this influence to a small fraction of the effect under consideration, we will choose this angle as a representative geometry. Also, we include the maximum amount of elastic mismatch by taking $\beta = -0.5$, giving

$$\frac{b}{b_0} = \frac{1}{1-0.693147\frac{2}{5\pi}} \simeq 1.097,$$

i.e. the maximum correction correction is just less than 10%. *Figure 1* shows the comparison between the pressure $p_o(x)$ obtained neglecting convective effects, and the pressure $p_1(x)$ on considering first order correction, with the constants indicated. It is clear that the correction is surprisingly significant at the center of the contact, where the singularity state of pressure is in effect released.

We can draw the following conclusions

- if $\beta \neq 0$, for a 'convex' indenter, the correction will *decrease* the pressure, and therefore, the contact area dimension must increase;
- the correction is linearly proportional to β and, for the wedge, also to the external wedge angle, θ;
- the correction produces a change in pressure that is not definable in a pointwise sense: indeed, there are areas of contact where contact was not predicted, as well as a large reduction of the pressure at the center of the contact area; it may be appreciated that the most significant differences occur at the edge of the contact area, where pressure was not predicted, and at the centre, where the pressure concentration (specifically a logarithmic singularity of the stresses) is greatly reduced. Therefore the classical solution, on taking into account the 'convective effect', is conservative with respect to the strength of the contact.

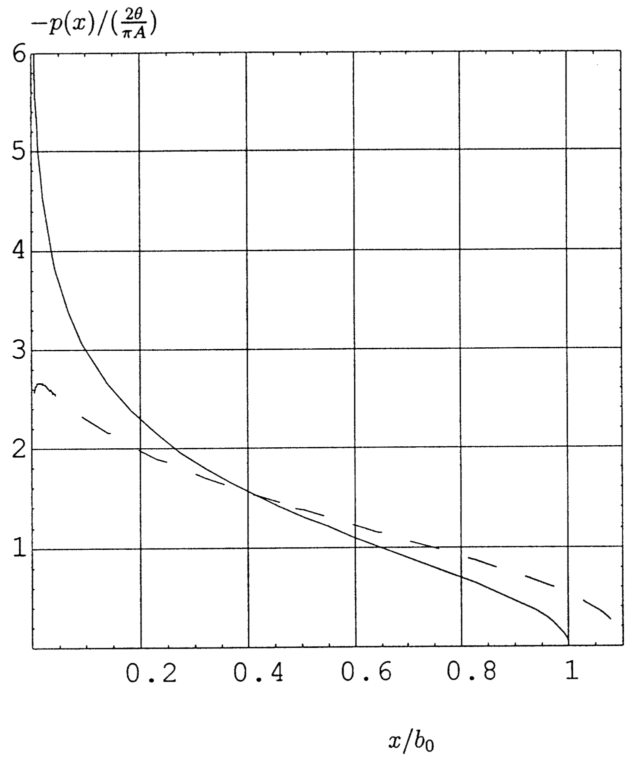

Figure 1. Pressure distribution for a wedge punch, without 'convective effect' $p_0(x)$ (solid line), and with first order correction $p_1(x)$ (dashed line). $\theta = \pi/5$, $\beta = -0.5$.

- the second correction, for the wedge case, is

$$p_2(s) = p_0(s) + \frac{\theta\beta}{\pi}\sqrt{1-s^2}\int_{-1}^{1}\frac{\operatorname{sign}(t)p_1(t)}{\sqrt{1-t^2}(t-s)}\,\mathrm{d}t \tag{22}$$

$$= p_1(s) - \left(\frac{\theta\beta}{\pi}\right)^2\frac{2\theta}{\pi A}\sqrt{1-s^2}\int_{-1}^{1}\frac{\operatorname{sign}(t)I_1(t)}{\sqrt{1-t^2}(t-s)}\,\mathrm{d}t. \tag{23}$$

  It is clear that the correction has a series representation growing as a power series of $\theta\beta$. As the first product, $\theta\beta$, is already small, there is, in practice, no need for more than one correction, within the context of linear elasticity theory.
- convection produces the effect of making the profile shallower.

4. Parabolic punch

Some consideration has been given to carrying out the above calculation for a parabolic indenter, in order to investigate the Hertz problem itself. However, although this is certainly possible, we report only the results found for the contact area — load relationship.

The function $h'(t)$, with R representing the radius of curvature at the origin, is

$$\begin{aligned} h(x) &= -\frac{1}{2R}x^2, \\ h'(x) &= -\frac{1}{R}x. \end{aligned} \tag{24}$$

Now, as the profile is not linear, the profile derivative depends on the point of evaluation, so that the correction for the convective effect is not as simple as in the wedge case. Note that, for the parabola to represent adequately a circle of radius R, or the half-plane assumption to be satisfactory, the ratio b/R should not be greater than, say, 0.3–0.4 (Fessler and Ollerton, 1957); let us then calculate

$$h'\big(x - \mathrm{sign}(x)g(x)\big) = -\frac{1}{R}\left(x - \beta A \int_0^x p(\xi)\,\mathrm{d}\xi\right)\big(1 - \mathrm{sign}(x)\beta A p(x)\big) \tag{25}$$

and so

$$p_0(x) = -\frac{1}{AR}\sqrt{b^2 - x^2}. \tag{26}$$

In order to calculate the increase of contact area, we take

$$\begin{aligned} P &= -\frac{1}{A}\int_{-b}^{b} \frac{h'(t - \mathrm{sign}(t)g(t))t\,\mathrm{d}t}{\sqrt{b^2 - t^2}} \\ &= \frac{2}{AR}\int_0^b \frac{(t - \beta A\int_0^t p(\xi)\,\mathrm{d}\xi)(1 - \beta A p(t))t\,\mathrm{d}t}{\sqrt{b^2 - t^2}} \\ &= \frac{2}{AR}\int_0^b \frac{(t - \beta A\int_0^t p(\xi)\,\mathrm{d}\xi)t\,\mathrm{d}t}{\sqrt{b^2 - t^2}} - \frac{2\beta}{R}\int_0^b \frac{p(t)t^2\,\mathrm{d}t}{\sqrt{b^2 - t^2}} + \frac{2\beta^2 A}{R}\int_0^b \frac{p(t)\int_0^t p(\xi)\,\mathrm{d}\xi t\,\mathrm{d}t}{\sqrt{b^2 - t^2}}. \end{aligned} \tag{27}$$

Then, on neglecting the last term as it involves β^2,

$$\begin{aligned} P &\simeq \frac{2}{AR}\int_0^b \frac{t^2\,\mathrm{d}t}{\sqrt{b^2 - t^2}} - \frac{2\beta}{R}\int_0^b \frac{\int_0^t p(\xi)\,\mathrm{d}\xi t\,\mathrm{d}t}{\sqrt{b^2 - t^2}} - \frac{2\beta}{R}\int_0^b \frac{p(t)t^2\,\mathrm{d}t}{\sqrt{b^2 - t^2}} \\ &= \frac{2}{AR}b^2\frac{\pi}{4} + \frac{1}{AR^2}\beta b^3\int_0^1 \frac{(t\sqrt{1 - t^2} + \arcsin t)t\,\mathrm{d}t}{\sqrt{1 - t^2}} - \frac{2\beta}{R}\int_0^b \frac{p(t)t^2\,\mathrm{d}t}{\sqrt{b^2 - t^2}} \\ &= \frac{b^2}{AR}\frac{\pi}{2} + \frac{1}{AR^2}\beta b^3\left(\frac{4}{3} + \frac{2}{3}\right) = \frac{b^2}{AR}\frac{\pi}{2}\left(1 + \frac{4}{\pi}\frac{\beta b}{R}\right). \end{aligned}$$

Thus, since $b_0^2 = P/[\frac{\pi}{2}\frac{1}{AR}]$, and $b^2 = P/[\frac{\pi}{2}\frac{1}{AR}(1 + \frac{4}{\pi}\frac{\beta}{R})]$, then the following approximate expression is obtained

$$\left(\frac{b}{b_0}\right)^2 = \frac{\frac{\pi}{2}\frac{1}{AR}}{\frac{\pi}{2}\frac{1}{AR}(1 + \frac{4}{\pi}\frac{\beta b}{R})} = \frac{1}{1 + \frac{4}{\pi}\frac{\beta b}{R}}.$$

If $\beta = -0.5$, and we put a value of b/R such that the slope of the profile at the edge of the contact area is the same as the slope of the wedge we considered in the example of the previous paragraph (i.e. 36 degrees), $b/R = 0.611$, then

$$\frac{b}{b_0} = \sqrt{\frac{1}{1 - \frac{2}{\pi}\frac{b}{R}}} \simeq 1.28,$$

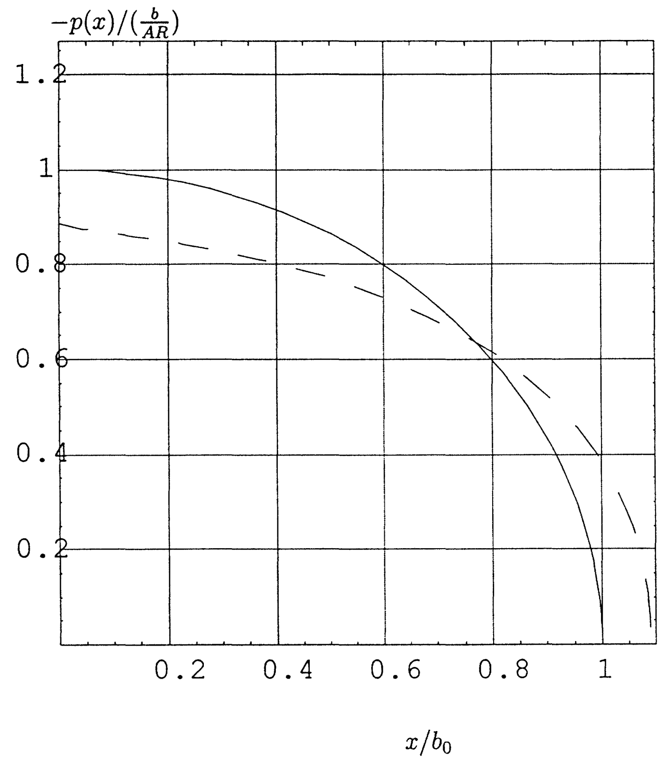

Figure 2. Pressure distribution for a parabolic punch, without 'convective effect' $p_0(x)$ (solid line), and with first order correction $p_1(x)$ (dashed line). $b/R = 1/4$, $\beta = -0.5$.

i.e. the correction is about 16%, which is significant indeed, although obtained under extreme conditions. Note in fact that in the Hertzian case, for the parabola to be a proper approximation of a circle, and for the approximations of linear half-plane elasticity to be satisfactory, b/R should not be greater than, say, $1/4$, in which case

$$\frac{b}{b_0} = \sqrt{\frac{1}{1-\frac{1}{2\pi}}} \simeq 1.09$$

and in practical cases this considered to be the upper limit.

Details of the first iteration for determination of the pressure distribution are given in Appendix I, where it is shown that

$$p_1(x) \simeq p_0(x)\left(1 + \frac{\beta b}{2\pi R} J_1(x)\right), \tag{28}$$

where

$$J_1(x) = \int_{-1}^{1} \frac{t + 2|t| + \arctan \frac{t}{\sqrt{1-t^2}} / \sqrt{1-t^2}}{t - x}\, \mathrm{d}t. \tag{29}$$

Figure 2 shows the comparison between $p_0(x)$ and $p_1(x)$, where the latter includes first order approximation of the convective effect, with $b/R = 1/4$ and $\beta = -0.5$.

5. Conclusion

It is widely thought that convection produces only a tiny correction to problems solved within the framework of the linear theory of elasticity. Further, not only are the absolute value of the corrections small, but for contact problems they are tacitly thought of as appropriate only when other refinements are made, and these, in turn, are often of questionable value within linear elasticity. We have shown here that, in fact, convection *is* a problem for strongly dissimilar contacts under light friction, and that the correction to the size of the contact is significant. Interestingly, the effect is to cause a reduction in the gradient of the profile, so that the contact pressure is *lower* than implied by a ‘no-convection’ solution, and as a consequence the extent of the contact patch is greater. The effect is shown to be likely to be more significant for higher order contacts, although it is usually significant when other assumptions in the solution of the problem can be questioned as well.

Appendix I: Contact law for parabolic punch

For the parabolic punch, as the profile is not linear, the correction for convective effect is not as simple as in the wedge case, as the profile derivative *does* depend on the point of evaluation, and the magnitude of the correction will depend on the extent of the contact, as there is a scale factor, R;

$$h'\big(x - \operatorname{sign}(x)g(x)\big) = -\frac{1}{R}\left(x - \beta A\int_0^x p(\xi)\,\mathrm{d}\xi\right)\big(1 - \operatorname{sign}(x)\beta A p(x)\big) \tag{30}$$

and so

$$\begin{aligned} p_0(x) &= -\frac{1}{\pi AR}\sqrt{b^2-x^2}\int_{-b}^{b}\frac{t\,\mathrm{d}t}{\sqrt{b^2-t^2}(t-x)} \\ &= -\frac{1}{\pi AR}\sqrt{b^2-x^2}\left[\int_{-b}^{b}\frac{\mathrm{d}t}{\sqrt{b^2-t^2}} + x\int_{-b}^{b}\frac{\mathrm{d}t}{\sqrt{b^2-t^2}(t-x)}\right] \\ &= -\frac{1}{AR}\sqrt{b^2-x^2}. \end{aligned} \tag{31}$$

Then, on considering the first correction

$$\begin{aligned} p_1(x) &= \frac{1}{\pi A}\sqrt{b^2-x^2}\int_{-b}^{b}\frac{h'(t-\operatorname{sign}(t)g(t))}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t \\ &= -\frac{\sqrt{b^2-x^2}}{\pi AR}\int_{-b}^{b}\frac{t-\beta A\int_0^t p_0(\xi)\,\mathrm{d}\xi - \beta A\operatorname{sign}(t)p_0(t)(t-\beta A\int_0^t p_0(\xi)\,\mathrm{d}\xi)}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t \\ &= p_0(x) + \frac{\sqrt{b^2-x^2}}{\pi AR}\int_{-b}^{b}\frac{\beta A\int_0^t p_0(\xi)\,\mathrm{d}\xi}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t \\ &\quad + \frac{\sqrt{b^2-x^2}}{\pi AR}\int_{-b}^{b}\frac{\beta A\operatorname{sign}(t)p_0(t)(t-\beta A\int_0^t p_0(\xi)\,\mathrm{d}\xi)}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t \\ &= p_0(x) + \frac{\beta\sqrt{b^2-x^2}}{\pi R}\int_{-b}^{b}\frac{\int_0^t p_0(\xi)\,\mathrm{d}\xi}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t \\ &\quad + \frac{\beta\sqrt{b^2-x^2}}{\pi R}\int_{-b}^{b}\frac{\operatorname{sign}(t)p_0(t)(t-\beta A\int_0^t p_0(\xi)\,\mathrm{d}\xi)}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t \end{aligned}$$

$$= p_0(x) + \frac{\beta\sqrt{b^2-x^2}}{\pi R}\int_{-b}^{b}\frac{\int_0^t p_0(\xi)\,\mathrm{d}\xi}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t$$
$$+ \frac{\beta\sqrt{b^2-x^2}}{\pi R}\int_{-b}^{b}\frac{\mathrm{sign}(t)\,p_0(t)\,t}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t - \frac{\beta^2 A\sqrt{b^2-x^2}}{\pi R}\int_{-b}^{b}\frac{\mathrm{sign}(t)\,p_0(t)\int_0^t p_0(\xi)\,\mathrm{d}\xi}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t. \quad (32)$$

On substituting the value of $p_0(x)$, and neglecting the last term, as it depends on β^2

$$p_1(x) \simeq p_0(x) - \frac{1}{\pi AR}\frac{\beta}{R}\sqrt{b^2-x^2}\int_{-b}^{b}\frac{\sqrt{b^2-\xi^2}\,\mathrm{d}\xi}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t - \frac{1}{AR}\frac{\beta\sqrt{b^2-x^2}}{\pi R}\int_{-b}^{b}\frac{\mathrm{sign}(t)t}{(t-x)}\,\mathrm{d}t$$

or

$$p(x) = p_0(x) - \frac{1}{\pi A}\frac{\beta}{R^2}\sqrt{b^2-x^2}\left[\int_{-b}^{b}\frac{\int_0^t\sqrt{b^2-\xi^2}\,\mathrm{d}\xi}{\sqrt{b^2-t^2}(t-x)}\,\mathrm{d}t + \int_{-b}^{b}\frac{\mathrm{sign}(t)t}{(t-x)}\,\mathrm{d}t\right] \quad (33)$$

and, performing the integration

$$\int_0^t\sqrt{b^2-\xi^2}\,\mathrm{d}\xi = \frac{t}{2}\sqrt{b^2-t^2} + \frac{b^2}{2}\arctan\frac{t}{\sqrt{b^2-t^2}}$$

yields formula (28) and (29).

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