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Multigrid method and multilevel additive preconditioner for mixed element method for non-self-adjoint and indefinite problems ☆

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Abstract

In this paper, a V-cycle multigrid method is proposed for mixed element method for non-self-adjoint and indefinite second-order elliptic problems and the uniform convergence of the V-cycle multigrid method is proven under minimal regularity assumption. Meanwhile, a multilevel additive preconditioner is given for these problems and an optimal convergence rate for preconditioned GMRES method is obtained under minimal regularity assumption. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Mixed element methods have been increasingly used in application, in particular, for such problems where instead of the primal variable, its gradient is of major interest. Many authors have studied the solving methods for mixed element methods for second-order elliptic problems, especially for self-adjoint and positive definite problems (see [1,2,7,8,10], and references therein). Among

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these methods, multigrid methods and multilevel additive preconditioning methods are very efficient.

For self-adjoint and positive definite second-order elliptic problems, Brenner [2] proposed a multigrid algorithm for the lowest-order Raviart–Thomas mixed triangular element method. In Refs. [13,14], Mathew studied Schwarz methods for the lowest-order Raviart–Thomas mixed rectangular element method. Recently, Chen [5] proposed so-called modified mixed element methods and showed that the modified mixed element formulation can be algebraically condensed to a symmetric and positive definite system for Lagrange multipliers. Based on this idea, authors in Refs. [5,7] discussed multigrid methods and domain decomposition methods for modified mixed element methods.

For non-self-adjoint and indefinite second-order elliptic problems, Chen, Kwak and Yon [8] studied V-cycle multigrid algorithms for modified mixed element methods under minimal regularity assumption. Chen and Li [10] proposed a simpler modified mixed element method and discussed its domain decomposition method under minimal regularity assumption. For standard mixed element method for non-self-adjoint and indefinite problems, Arbogast and Chen [1] established the equivalence between mixed element methods and so-called projection nonconforming element methods. But they demanded that the original problems are coercive, and only gave a W-cycle multigrid for self-adjoint positive definite problems under H^2 regular assumption. Recently, Chen and Li [9] proved the equivalence between the lowest-order Raviart–Thomas triangular mixed element method and projection P_1 nonconforming element method without coercive hypothesis and gave a V-cycle multigrid preconditioning method based on self-adjoint positive definite part under $H^{1+\alpha}$ $(\alpha \in (1/2, 1])$ regularity assumption. In a later paper [11], Chen and Li obtained the equivalence between the lowest-order Raviart–Thomas triangular mixed element method and projection P_1 nonconforming element method only under minimal regularity assumption. Meanwhile, they got the uniform convergence of the solution of the mixed element method and discussed preconditioning methods by using generalized Goldstein preconditioning framework [12].

In this paper, we discuss V-cycle multigrid method and multilevel additive preconditioning method for the lowest-order Raviart–Thomas triangular mixed element method for non-self-adjoint and indefinite second-order elliptic problems under minimal regularity assumption. Our discussion is based on the equivalence between mixed element method and projection nonconforming element method. If the coarest grid size is small enough, we prove the uniform convergence of V-cycle multigrid method. Also we give a multilevel additive preconditioner and obtain the optimal convergence of preconditioned GMRES method [16].

It should be pointed out that in order to deal with nonconforming element case for non-self-adjoint and indefinite problems, authors [8,10] usually assumed that the coefficient of the first-order term is continuous differentiable and piecewise C^2 with the second-order derivative over pieces being bounded. In this paper, we only need that the coefficient of the first-order term belongs to W^1_∞.

The remainder of this paper is outlined as follows. In Section 2 we give the continuous non-self-adjoint and indefinite second-order elliptic problem, its mixed and projection nonconforming element methods. In Section 3 some lemmas are given. In Section 4 a V-cycle multigrid method is given and the uniform convergence of the V-cycle multigrid method is proven. In Section 5, a multilevel additive preconditioner is constructed and optimal convergence for preconditioned GMRES method is obtained. In this paper, C (with or without subscripts) will denote a generic positive constant with possibly different values in different contexts. For any subdomain $D \subset \Omega$, we use usual L^2 inner product $(\cdot,\cdot)_D$, Sobolev space $H^s(D)$ with usual Sobolev norm $\|\cdot\|_{H^s(D)}$ and seminorm $|\cdot|_{H^s(D)}$. If $D=\Omega$, we denote the usual L^2 inner product by $(\cdot,\cdot)$, the Sobolev norm by $\|\cdot\|_s$ and seminorm by $|\cdot|_s$, where s may be fractional (for details see [18]).

2. Preliminaries

We consider the following elliptic problem on a bounded polygonal domain $\Omega \subset \mathscr{R}^2$ without crack and with boundary $\Gamma = \partial\Omega$

$$-\nabla\cdot(a\nabla u + bu) + du = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma, \tag{2.1}$$

where $a(x) = (a_{ij}) \in W^1_\infty(\Omega)$ is a uniformly symmetric positive definite tensor on $\bar{\Omega}$, $b(x) \in W^1_\infty(\Omega)$, $d(x) \in L^\infty(\Omega)$, and $f(x) \in L^2(\Omega)$.

Eq. (2.1) can be written in the form of the first-order system

$$\nabla\cdot\sigma + du = f \quad \text{in } \Omega, \tag{2.2}$$

$$\sigma = -(a\nabla u + bu) \quad \text{in } \Omega, \tag{2.3}$$

$$u = 0 \quad \text{on } \Gamma. \tag{2.4}$$

Define $V = \{v \in (L^2(\Omega))^2 : \nabla\cdot v \in L^2(\Omega)\}$, $W = L^2(\Omega)$. Problem (2.2)–(2.4) is recast in the mixed form as follows: find $(\sigma, u) \in V \times W$ such that

$$(\nabla\cdot\sigma, w) + (du, w) = (f, w) \quad \forall w \in W, \tag{2.5}$$

$$(a^{-1}\sigma, v) - (u, \nabla\cdot v) + (a^{-1}bu, v) = 0 \quad \forall v \in V. \tag{2.6}$$

Let $V_h \times W_h \subset V \times W$ denote the lowest-order Raviart–Thomas triangular element space [15] associated with a quasi-uniform partition $\mathscr{T}_h$ of Ω into open triangles E, $\Omega_h = \bigcup_{E\in\mathscr{T}_h} E$, $\partial\mathscr{T}_h$ denote the set of all interior edges of $\mathscr{T}_h$. The constraint $V_h \subset V$ says that the normal components of members of V_h are continuous across the interior boundaries in $\partial\mathscr{T}_h$. Define

$$\tilde{V}_h = \{v \in (L^2(\Omega))^2 : v|_E \in V_h|_E \ \forall E \in \mathscr{T}_h\},$$

$$L_h = \left\{\mu \in L^2\left(\bigcup_{e \in \partial\mathscr{T}_h} e\right) : \mu|_e \in V_h \cdot n|_e \ \forall e \in \partial\mathscr{T}_h\right\},$$

where n denotes the outer unit normal along e on E.

The mixed element method of (2.1) is to find $(\sigma_h, u_h) \in V_h \times W_h$ satisfying

$$(\nabla \cdot \sigma_h, w) + (du_h, w) = (f, w) \quad \forall w \in W_h, \tag{2.7}$$

$$(a^{-1}\sigma_h, v) - (u_h, \nabla \cdot v) + (a^{-1}bu_h, v) = 0 \quad \forall v \in V_h. \tag{2.8}$$

The hybrid form of mixed element method (2.7) and (2.8) is to find $(\sigma_h, u_h, \lambda_h) \in \tilde{V}_h \times W_h \times L_h$ such that

$$\sum_{E \in \mathscr{T}_h} (\nabla \cdot \sigma_h, w)_E + (du_h, w) = (f, w) \quad \forall w \in W_h, \tag{2.9}$$

$$(a^{-1}\sigma_h, v) - \sum_{E \in \mathscr{T}_h} \left((u_h, \nabla \cdot v)_E - (\lambda_h, v \cdot n_E)_{\partial E \setminus \Gamma}\right)$$

$$+ (a^{-1}bu_h, v) = 0 \quad \forall v \in \tilde{V}_h, \tag{2.10}$$

$$\sum_{E \in \mathscr{T}_h} (\sigma_h \cdot n_E, \mu)_{\partial E \setminus \Gamma} = 0 \quad \forall \mu \in L_h, \tag{2.11}$$

where and hereinafter n_E denotes the outer unit normal along ∂E on E. Note that σ_h and u_h are identical in the two formulations.

Let $P_k(E)$ or $P_k(e)$ denote the space of polynomials of total degree less than or equal to k defined on E or e, respectively. We will make use of the barycentric coordinates l_i, $i = 1, 2, 3$, defined on E to be the unique linear functions that take the value one at vertex i, and the value zero on the opposite edge. Let $B_h(E)$ be the span of P_3-bubble function $\beta_E^3 = l_1 l_2 l_3$, which vanishes on each edge.

The lowest-order Raviart–Thomas space on triangles [15] is defined by

$$V_h|_E = (P_0(E))^2 \oplus ((x_1, x_2)^{\mathrm{T}} P_0(E)),$$

$$W_h|_E = P_0(E),$$

where and hereinafter T denotes the transpose of a vector. From the definition of L_h, we know $L_h|_e = P_0(e)$.

Define nonconforming element space M_h on $\mathscr{T}_h$ (see [1,2]) as follows:

$$M_h|_E = P_1(E) \oplus B_h(E),$$

with unisolvent degrees of freedom which vanish on Γ, described by

(DF1) $|E|^{-1}(\xi, w)_E$ for all w in a basis of $W_h(E)$,
$|E|$ denotes the area of E,

(DF2) $|e|^{-1}(\xi, \mu)_e$ for all μ in a basis of $L_h(e)$,
$|e|$ denotes the length of e.

Let $P_{W_h} : L^2(\Omega) \to W_h$ denote usual local $L^2(\Omega)$-projection, $P_{L_h} : L^2(\bigcup_{e\in\partial\mathscr{T}_h} e) \to L_h$ denote usual local $L^2(\bigcup_{e\in\partial\mathscr{T}_h} e)$-projection, and $\tilde{P}_{V_h} : (L^2(\Omega))^2 \to \tilde{V}_h$ denote the weighted local $(L^2(\Omega))^2$-projection defined $\forall\varphi \in (L^2(\Omega))^2$ by

$$(a^{-1}(\varphi - \tilde{P}_{V_h}\varphi), v)_E = 0 \quad \forall v \in \tilde{V}_h.$$

Let $M = H_0^1(\Omega)$. Define

$$\|v\|_{1,h} = \left(\sum_{E\in\mathscr{T}_h} \|v\|^2_{H^1(E)}\right)^{1/2}, \qquad |v|_{1,h} = \left(\sum_{E\in\mathscr{T}_h} |v|^2_{H^1(E)}\right)^{1/2}$$

for any $v \in M \cup M_h$, and

$$a_p(v, w) = \sum_{E\in\mathscr{T}_h} (a\nabla v, \nabla w)_E \quad \forall v,\ w \in M \cup M_h,$$

$$a_L(v, w) = \sum_{E\in\mathscr{T}_h} (bv, \nabla w)_E + (dv, w) \quad \forall v,\ w \in M \cup M_h,$$

$$a(v, w) = a_p(v, w) + a_L(v, w) \quad \forall v,\ w \in M \cup M_h,$$

$$a_{h,p}(v, w) = \sum_{E\in\mathscr{T}_h} (\tilde{P}_{V_h}(a\nabla v), \nabla w)_E \quad \forall v,\ w \in M \cup M_h,$$

$$a_{h,L}(v, w) = \sum_{E\in\mathscr{T}_h} (\tilde{P}_{V_h}(bP_{W_h}v), \nabla w)_E + (dP_{W_h}v, P_{W_h}w) \quad \forall v,\ w \in M \cup M_h,$$

$$a_h(v, w) = a_{h,p}(v, w) + a_{h,L}(v, w) \quad \forall v,\ w \in M \cup M_h.$$

The weak form of problem (2.1) is: find $u \in H_0^1(\Omega)$ satisfying

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{2.12}$$

We assume that for any given $f \in L^2(\Omega)$, problem (2.12) has a unique solution $u \in H_0^1(\Omega)$ such that $\|u\|_1 \leqslant C\|f\|_0$.

The so-called projection nonconforming element method of problem (2.1) is to seek $\psi_h \in M_h$ satisfying

$$a_h(\psi_h, \xi) = (f, P_{W_h}\xi) \quad \forall \xi \in M_h. \tag{2.13}$$

3. Some lemmas

Lemma 3.1. *For any* $E \in \mathscr{T}_h$, $v \in (L^2(E))^2$ *and* $w \in (H^1(E))^2$, *we have*

$$\|\tilde{P}_{V_h} v\|_{(L^2(E))^2} \leqslant C_1 \|v\|_{(L^2(E))^2},$$
$$\|w - \tilde{P}_{V_h} w\|_{(L^2(E))^2} \leqslant C_2 h |w|_{(H^1(E))^2}.$$

Proof. From the definition of $\tilde{P}_{V_h}$ and Lemma 4.5 in [9], we know this lemma holds. □

Lemma 3.2. *For any* $E \in \mathscr{T}_h$, $v \in L^2(E)$ *and* $w \in H^1(E)$, *we have*

$$\|P_{W_h} v\|_{L^2(E)} \leqslant C_1 \|v\|_{L^2(E)},$$
$$\|w - P_{W_h} w\|_{L^2(E)} \leqslant C_2 h |w|_{H^1(E)}.$$

Proof. Using scaling argument, we can prove this lemma easily. □

Lemma 3.3. *If* $v \in H^1(E)$, $P_0^E \psi = (1/|E|) \int_E \psi \, \mathrm{d}x$, $R_0^E \psi = \psi - P_0^E \psi$, *then we have*

$$\int_e |R_0^E \psi|^2 \, \mathrm{d}s \leqslant Ch |\psi|^2_{H^1(E)} \quad \forall e \in \partial E.$$

Proof. See Lemma 4.6 in [9]. □

Lemma 3.4. $C_1 |v|^2_{1,h} \leqslant a_{h,p}(v,v) \leqslant C_2 |v|^2_{1,h} \quad \forall v \in M_h.$

Proof. From Lemma 2.1 in [2] and Corollary 5.2 in [9], we obtain this lemma immediately. □

Lemma 3.5. $a_h(v,v) \geqslant C_1 \|v\|^2_{1,h} - C_0 \|v\|^2_0 \quad \forall v \in M_h.$

Proof. See Theorem 5.1 in [9]. □

Now we consider the duality problem of problem (2.12): find $\varphi \in H_0^1(\Omega)$ such that

$$a(v, \varphi) = (g, v) \quad \forall v \in H_0^1(\Omega). \tag{3.1}$$

Assume for each $g \in L^2(\Omega)$, problem (3.1) has a unique solution $\varphi \in H_0^1(\Omega)$ satisfying $\|\varphi\|_1 \leqslant C\|g\|_0$. For problem (3.1), we have the following result.

Lemma 3.6. *For any given* $\epsilon > 0$, *we can find an* $h_0 > 0$, *such that if* $h \leqslant h_0$, *then for any* $g \in L^2(\Omega)$ *we have*

$$\inf_{v_h \in M_h} \|\varphi - v_h\|_{1,h} \leqslant \epsilon \|g\|_0, \tag{3.2}$$

$$\sup_{w_h \in M_h} \frac{|a(w_h, \varphi) - (g, w_h)|}{\|w_h\|_{1,h}} \leqslant C\epsilon \|g\|_0, \tag{3.3}$$

$$|a(\varphi, v) - a_h(\varphi, v)| \leqslant C\epsilon \|g\|_0 \|v\|_{1,h} \quad \forall v \in M_h \cup M, \tag{3.4}$$

$$|a(v, \varphi) - a_h(v, \varphi)| \leqslant C\epsilon \|v\|_{1,h} \|g\|_0 \quad \forall v \in M_h \cup M, \tag{3.5}$$

where $\varphi = T^* f \in H_0^1(\Omega)$ *is the solution of problem* (3.1)

Proof. From Lemma 2 in [17], we know inequality (3.2) holds. Arguing as Lemma 2.7 in [10], we can show inequality (3.3) holds. Finally, arguing as in Lemma 3.7 in [11], we can obtain inequalities (3.4) and (3.5). $\square$

Lemma 3.7. *For any* $v, w \in M_h$, *we have*

$$|a_{h,L}(v, w)| \leqslant C_1 \|v\|_0 \|w\|_{1,h}, \tag{3.6}$$

$$|a_{h,L}(v, w)| \leqslant C_2 \|v\|_{1,h} \|w\|_0. \tag{3.7}$$

Proof. From Lemmas 3.1 and 3.2, we know inequality (3.6) holds. For proving inequality (3.7), we first prove the following inequality:

$$|a_L(v, w)| \leqslant C \|v\|_{1,h} \|w\|_0 \quad \forall v, w \in M_h. \tag{3.8}$$

Using integration formula by parts, we get

$$a_L(v, w) = \sum_{E \in \mathscr{T}_h} \int_{\partial E} vwb \cdot n_E \, \mathrm{d}s + \sum_{E \in \mathscr{T}_h} (\nabla \cdot (bv), w)_E + (dv, w), \tag{3.9}$$

where n_E denote the unit outer normal. We only need to estimate the first term. To this end, let $b = (b^1, b^2)^{\mathrm{T}}$ and $n_E = (n_E^1, n_E^2)^{\mathrm{T}}$. Define

$$T_l(b, v, w) = \sum_{E \in \mathscr{T}_h} \int_{\partial E} b^l v w n_E^l \, \mathrm{d}s, \quad l = 1, 2,$$

$$P_0^e w|_e = \frac{1}{|e|} \int_e w \, \mathrm{d}s, \qquad R_0^e w|_e = w|_e - P_0^e w|_e.$$

Noting $b^l \in W_\infty^1(\Omega)$ and the definition of M_h, we have

$$T_l(b, P_0^e v, P_0^e w) = 0, \qquad T_l(b, v, P_0^e w) = T_l(b, R_0^e v, P_0^e w). \tag{3.10}$$

Since

$$\int_{\partial E} R_0^e v P_0^e w n_E^l \, \mathrm{d}s = \int_{\partial E} (v - P_0^e v) P_0^e w n_E^l \, \mathrm{d}s = 0,$$

by (3.10) and the definitions of P_0^E and R_0^E in Lemma 3.3, we obtain

$$
\begin{aligned}
T_l(b, v, P_0^e w) &= \sum_{E\in\mathscr{T}_h} \int_{\partial E} (b^l - P_0^E b^l) R_0^e v P_0^e w n_E^l \,\mathrm{d}s \\
&= \sum_{E\in\mathscr{T}_h} \int_{\partial E} R_0^E b^l R_0^e v P_0^e w n_E^l \,\mathrm{d}s. \qquad (3.11)
\end{aligned}
$$

Because $P_0^e : M_h \to P_0(e)$ is a $L^2(e)$-projection, we have

$$\|R_0^e v\|_{L^2(e)} \leqslant \|R_0^E v\|_{L^2(e)}. \qquad (3.12)$$

Using Schwarz inequality, inequality (3.12), Lemma 3.3, trace theorem and inverse inequality (see [6]), we get

$$
\begin{aligned}
\left| \int_e R_0^E b^l R_0^e v P_0^e w n_E^l \,\mathrm{d}s \right| &\leqslant \|P_0^e w\|_{L^\infty(e)} \|R_0^E b^l\|_{L^2(e)} \|R_0^E v\|_{L^2(e)} \\
&\leqslant Ch^{-1/2} \|w\|_{L^2(e)} h^{1/2} |b^l|_{H^1(E)} h^{1/2} |v|_{H^1(E)} \\
&\leqslant Ch^{1/2} \left(h^{-1} \|w\|^2_{L^2(E)} + h|w|^2_{H^1(E)}\right)^{1/2} |v|_{H^1(E)} \\
&\leqslant C\|w\|_{L^2(E)} |v|_{H^1(E)}.
\end{aligned}
$$

Summing over all $e \in \partial E$, $E \in \mathscr{T}_h$, using Schwarz inequality and (3.11), we obtain

$$|T_l(b, v, P_0^e w)| \leqslant C\|v\|_{1,h} \|w\|_0. \qquad (3.13)$$

Noting

$$\int_e R_0^e w n_E^l \,\mathrm{d}s = \int_e (w - P_0^e w) n_E^l \,\mathrm{d}s = 0,$$

we have

$$
\begin{aligned}
T_l(b, v, R_0^e w) &= \sum_{E\in\mathscr{T}_h} \int_{\partial E} (b^l v - P_0^E(b^l v)) R_0^e w n_E^l \,\mathrm{d}s \\
&= \sum_{E\in\mathscr{T}_h} \int_{\partial E} R_0^E(b^l v) R_0^e w n_E^l \,\mathrm{d}s.
\end{aligned}
$$

Using Schwarz inequality, inequality (3.12), Lemma 3.3 and inverse inequality, noting $b^l \in W^1_\infty(\Omega)$, we get

$$
\begin{aligned}
|T_l(b, v, R_0^e w)| &\leqslant \sum_{E\in\mathscr{T}_h} \|R_0^E(b^l v)\|_{L^2(\partial E)} \|R_0^e w\|_{L^2(\partial E)} \\
&\leqslant C \sum_{E\in\mathscr{T}_h} h|b^l v|_{H^1(E)} |w|_{H^1(E)} \\
&\leqslant C\|v\|_{1,h} \|w\|_0. \qquad (3.14)
\end{aligned}
$$

From inequalities (3.13) and (3.14), we obtain

$$|T_l(b,v,w)| \leqslant |T_l(b,v,R_0^e w)| + |T_l(b,v,P_0^e w)| \leqslant C\|v\|_{1,h}\|w\|_0.$$

Combining this inequality with (3.9), we have

$$|a_L(v,w)| \leqslant C\|v\|_{1,h}\|w\|_0. \tag{3.15}$$

By (3.15), Lemmas 3.1 and 3.2 and inverse inequality, we get

$$\begin{aligned}|a_{h,L}(v,w)| &\leqslant |a_{h,L}(v,w) - a_L(v,w)| + |a_L(v,w)| \\ &\leqslant \sum_{E\in\mathscr{T}_h} |(\tilde{P}_{V_h}(bP_{W_h}v),\nabla w)_E - (bv,\nabla w)_E| + C\|v\|_{1,h}\|w\|_0 \\ &\leqslant C_2\|v\|_{1,h}\|w\|_0. \qquad \square\end{aligned}$$

Remark. In order to prove inequality (3.8), authors in Refs. [8,10] used strong assumption for coefficient b, i.e., b is continuous differentiable and piecewise C^2 with the second-order derivative over piece being bounded. Here we only assume $b \in W_\infty^1(\Omega)$.

Lemma 3.8. *For sufficiently small h, the mixed element method* (2.9)–(2.11) *has a unique solution* $(\sigma_h, u_h, \lambda_h) \in \tilde{V}_h \times W_h \times L_h$, *and the mixed element method* (2.9)–(2.11) *is equivalent to the projection nonconforming element method* (2.13) *by the following relations*:

$$\sigma_h = -\tilde{P}_{V_h}(a\nabla\psi_h + bP_{W_h}\psi_h), \tag{3.16}$$

$$u_h = P_{W_h}\psi_h, \tag{3.17}$$

$$\lambda_h = P_{L_h}\psi_h, \tag{3.18}$$

where $\psi_h \in M_h$ *is the solution of problem* (2.13).

Proof. See Theorem 4.2 in [11]. $\square$

4. V-cycle multigrid method

Thanks to Lemma 3.8, in order to solve problem (2.9)–(2.11), we only need to solve problem (2.13). Once the solution of problem (2.13) is obtained, we can get the solution of problem (2.9)–(2.11) immediately by the relations (3.16)–(3.18). In this section, we propose a V-cycle multigrid method for solving problem (2.13) and prove the uniform convergence of the V-cycle multigrid method under minimal regularity assumption.

Given the coarsest triangular quasi-uniform partition $\mathscr{T}_1$ with maximum diameter h_1, $\mathscr{T}_k$ is constructed with maximum diameter h_k by connecting the midpoints of three edges of triangles in $\mathscr{T}_{k-1}$, $k = 2,3,\ldots,J$. Let $\mathscr{T}_h \equiv \mathscr{T}_J$, $M_J \equiv M_h$ (see Section 2) and $M_k \subset H_0^1(\Omega)$ be P_1 conforming element space defined on $\mathscr{T}_k$, $k = 1,2,\ldots,J-1$.

Define operators $A_k : M_k \to M_k$, $\hat{A}_k : M_k \to M_k$, $P_k : H_0^1(\Omega) \cup M_J \to M_k$, $\hat{P}_k : H_0^1(\Omega) \cup M_J \to M_k$, $Q_k : L^2(\Omega) \to M_k$, $k = 1, 2, \ldots, J$ by

$$(A_k v, w) = a_h(v, w) \quad \forall v,\ w \in M_k,$$

$$(\hat{A}_k v, w) = a_{h,p}(v, w) \quad \forall v,\ w \in M_k,$$

$$a_h(P_k v, w) = a_h(v, w) \quad \forall w \in M_k,$$

$$a_{h,p}(\hat{P}_k v, w) = a_{h,p}(v, w) \quad \forall w \in M_k,$$

$$(Q_k v, w) = (v, w) \quad \forall w \in M_k.$$

The projection nonconforming element method (2.13) can be written as: find $\psi_h \in M_h$ such that

$$A_J \psi_h = Q_J P_{W_h} f. \tag{4.1}$$

Let $R_k : M_k \to M_k$ be linear smoothing operators, $T_k = R_k A_k P_k$, $\hat{T}_k = R_k \hat{A}_k \hat{P}_k$, $\hat{K}_k = I - R_k \hat{A}_k$, $k = 2, \ldots, J$, $T_1 = P_1$ and $\hat{T}_1 = \hat{P}_1$. Denote the largest eigenvalue of $\hat{A}_k$ by λ_k. Define $\hat{K}_k^*$ to be the adjoint of $\hat{K}_k$ with respect to the inner product $a_{h,p}(\cdot, \cdot)$. We assume that

(C.1) $(v, v) \leqslant C\lambda_k(\bar{R}_k v, v)$ $\forall v \in M_k$, where $\bar{R}_k = (I - \hat{K}_k^* \hat{K}_k)\hat{A}_k^{-1}$.

(C.2) $a_{h,p}(\hat{T}_k v, \hat{T}_k v) \leqslant \theta a_{h,p}(\hat{T}_k v, v)$ $\forall v \in M_k$, where $\theta \in (0, 2)$.

(C.3) $(\hat{T}_k v, \hat{T}_k v) \leqslant C\lambda_k^{-1} a_{h,p}(\hat{T}_k v, v)$ $\forall v \in M_k$.

In Ref. [3], authors have given some methods to construct smoother R_k which satisfies the conditions (C.1)–(C.3).

Now we describe the simplest V-cycle multigrid algorithm for iteratively computing the solution ψ_h of problem (4.1). Given an initial approximate solution $\psi_0 \in M_J$, we define a sequence approximating ψ_h by

$$\psi_{i+1} = MG_J(\psi_i, Q_J P_{W_h} f),$$

where $MG_J(\cdot, \cdot)$ is a map of $M_J \times M_J$ into M_J and is defined as follows:

Definition MG. Set $MG_1(v, w) = A_1^{-1} w$. Let $k > 1$ and v, w be in M_k. Assume that $MG_{k-1}(\cdot, \cdot)$ has been defined, we define $MG_k(v, w)$ by

(1) $x_k = v + R_k(w - A_k v)$.

(2) $MG_k(v, w) = x_k + q$, where q is defined by

$$q = MG_{k-1}(0, Q_{k-1}(w - A_k x_k)).$$

From Definition MG we can deduce

$$\psi_h - \psi_{i+1} = (I - T_1)(I - T_2) \cdots (I - T_J)(\psi_h - \psi_i) \equiv E_J(\psi_h - \psi_i).$$

Thus we only need to study the norm of operator E_J.

If we replace A_k by $\hat{A}_k$ in Definition MG, we can obtain V-cycle multigrid method for the following symmetric positive definite problem:

$$\hat{A}_J \hat{\psi}_h = Q_J P_{W_h} f. \tag{4.2}$$

Meanwhile we have

$$\hat{\psi}_h - \hat{\psi}_{i+1} = (I - \hat{T}_1)(I - \hat{T}_2)\cdots(I - \hat{T}_J)(\hat{\psi}_h - \hat{\psi}_i) \equiv \hat{E}_J(\hat{\psi}_h - \hat{\psi}_i),$$

where $\hat{\psi}_i, \hat{\psi}_{i+1}$ are approximate solutions of $\hat{\psi}_h$ generated by multigrid method in ith and $(i+1)$th terms respectively.

Let N_J denote the set of vertices of all elements E in $\mathscr{T}_J$. For any $x \in N_J$, N_x denotes the set of midpoints of the edges with x as one of their endpoints. Let $\tilde{M}_J \subset H_0^1(\Omega)$ be P_1 conforming finite element space defined on $\mathscr{T}_J$. Define $\pi_J : M_J \to \tilde{M}_J$ by

$$(\pi_J v)(x) = \begin{cases} 0 & \text{if } x \in N_J \cap \Gamma, \\ \frac{1}{|N_x|} \sum\limits_{y \in N_x} v(y) & \text{if } x \in N_J \setminus \Gamma, \end{cases}$$

where $|N_x|$ denotes the number of points in N_x.

Lemma 4.1. $\|v - \pi_J v\|_0 + h_J |\pi_J v|_1 \leqslant C h_J |v|_{1,h} \quad \forall v \in M_J.$

Proof. See Lemmas 6.2 and 6.3 in [9]. $\square$

Lemma 4.2. *For $i \leqslant j$, any $v \in M_i$ and $w \in M_j$, we have*

$$a_p(v, w) \leqslant C h_j^{-1/2} h_i^{-1/2} \|v\|_1 \|w\|_0. \tag{4.3}$$

Proof. If $i = j$, we can obtain (4.3) easily by using inverse inequality. If $i < j < J$, (4.3) is well-known for P_1 conforming elements (see [18]). Now we only need to show (4.3) in the case $i < j = J$.

Given $K \in \mathscr{T}_i$, noting $v \in M_i$ to be linear on K, by Green's identity, Schwarz inequality, the assumption on coefficient a, Lemma 4.7 in [9], trace theorem and inverse inequality, we obtain

$$\begin{aligned}
\sum_{\substack{E \in \mathscr{T}_J \\ E \subset K}} \int_E a \nabla v \cdot \nabla w \, dx &= -\int_K ((\nabla \cdot a) \cdot \nabla v) w \, dx + \int_{\partial K} (a \nabla v) \cdot n_K w \, ds \\
&\quad + \sum_{\substack{E \in \mathscr{T}_J \\ E \subset K}} \int_{\partial E \setminus \partial K} (a \nabla v) \cdot n_E w \, ds \\
&\leqslant C\Big(\|v\|_{H^1(K)} \|w\|_{L^2(K)} + \|\nabla v\|_{L^2(\partial K)} \|w\|_{L^2(\partial K)} \Big) \\
&\quad + C h_J |v|_{H^1(K)} \left(\sum_{\substack{E \in \mathscr{T}_J \\ E \subset K}} |w|^2_{H^1(E)} \right)^{1/2} \\
&\leqslant C h_J^{-1/2} h_i^{-1/2} \|v\|_{H^1(K)} \|w\|_{L^2(K)}.
\end{aligned}$$

Summing over all $K \in \mathscr{T}_i$, using Schwarz inequality, we get (4.3). $\square$

Lemma 4.3. *For any* $\gamma \in (0, 1/2)$, *we have*

$$a_{h,p}(\hat{T}_k v, v) \leqslant C(h_k/h_i)^{\gamma} \|v\|_1^2 \quad \forall v \in M_i, \ i \leqslant k. \tag{4.4}$$

Proof. If $i = k$, we can obtain (4.4) from (C.2). In the following, we only consider the case $i < k$.

Since $\lambda_k = \mathrm{O}(h_k^{-2})$, by Lemma 4.2 and (C.2), (C.3) we get

$$\begin{aligned} a_p(\hat{T}_k v, v) &\leqslant C h_k^{-1/2} h_i^{-1/2} \|v\|_1 \|\hat{T}_k v\|_0 \\ &\leqslant C h_k^{1/2} h_i^{-1/2} \|v\|_1^2. \end{aligned}$$

For any $\gamma \in (0, 1/2)$ and any $v \in M_i, i < J$, we have $v \in H_0^{1+\gamma}(\Omega)$ (see [18]). Using this fact, Lemma 4.8 in [9] and fractional inverse inequality (see [18]), we obtain

$$\begin{aligned} a_{h,p}(\hat{T}_k v, v) &\leqslant |a_p(\hat{T}_k v, v)| + |a_{h,p}(\hat{T}_k v, v) - a_p(\hat{T}_k v, v)| \\ &\leqslant C(h_k^{1/2} h_i^{-1/2} \|v\|_1^2 + h_J^{\gamma} |\hat{T}_k v|_{1,h} |v|_{H^{1+\gamma}(\Omega)}) \\ &\leqslant C(h_k^{1/2} h_i^{1/2} \|v\|_1^2 + h_J^{\gamma} h_i^{-\gamma} \|v\|_1^2) \\ &\leqslant C(h_k/h_i)^{\gamma} \|v\|_1^2. \qquad \square \end{aligned}$$

For problem (4.2), we have the following result.

Theorem 4.4. *There exists a positive constant* $\hat{\delta} < 1$ *independent of* J *such that*

$$a_{h,p}(\hat{E}_J v, \hat{E}_J v) \leqslant \hat{\delta}^2 a_{h,p}(v, v) \quad \forall v \in M_J.$$

Proof. According to Theorem 3.2 in [4], we only need to verify the following two hypotheses:

(H1) $a_{h,p}(v, v) \leqslant C\Big(a_{h,p}(\hat{P}_1 v, v) + \sum_{k=2}^{J} \lambda_k^{-1} \|\hat{A}_k \hat{P}_k v\|_0^2\Big) \ \forall v \in M_J$.

(H2) $a_{h,p}(\tilde{T}_k v, v) \leqslant (C\epsilon^{k-l})^2 a_{h,p}(v, v) \ \forall v \in M_l, \ l \leqslant k$,

where $\epsilon \in (0, 1)$, $\tilde{T}_k = \lambda_k^{-1} \hat{A}_k \hat{P}_k$, $k = 2, \ldots, J$, $\tilde{T}_1 = \hat{P}_1$.

First we verify (H1). By Schwarz inequality and Lemma 3.4, for any $v \in M_J$, we have

$$
\begin{aligned}
a_{h,p}(v,v) &= a_{h,p}(\hat{P}_1 v, Q_1 \pi_J v) + \sum_{k=2}^{J} (\hat{A}_k \hat{P}_k v, (Q_k - Q_{k-1})\pi_J v) \\
&\quad + (\hat{A}_J \hat{P}_J v, v - \pi_J v) \\
&\leqslant C\left(a_{h,p}(\hat{P}_1 v, v) + \sum_{k=2}^{J} \lambda_k^{-1} \|\hat{A}_k \hat{P}_k v\|_0^2 \right)^{1/2} \Big(|Q_1 \pi_J v|_1^2 \\
&\qquad + \sum_{k=2}^{J} \lambda_k \|(Q_k - Q_{k-1})\pi_J v\|_0^2 + \lambda_J \|v - \pi_J v\|_0^2 \Big)^{1/2}. \qquad (4.5)
\end{aligned}
$$

Since $\pi_J v \in \tilde{M}_J$, it is well known that

$$
|Q_1 \pi_J v|_1^2 + \sum_{k=2}^{J} \lambda_k \|(Q_k - Q_{k-1})\pi_J v\|_0^2 \leqslant C |\pi_J v|_1^2. \qquad (4.6)
$$

Combining (4.5) with (4.6), using Lemmas 3.4 and 4.1, we obtain

$$
a_{h,p}(v,v) \leqslant C\left(a_{h,p}(\hat{P}_1 v, v) + \sum_{k=2}^{J} \lambda_k^{-1} \|\hat{A}_k \hat{P}_k v\|_0^2 \right)^{1/2} a_{h,p}^{1/2}(v,v).
$$

From this, we know (H1) holds.

Next we verify (H2). From (C.1) we can deduce

$$
\lambda_k^{-1}(v,v) \leqslant C(R_k v, v) \quad \forall v \in M_l, \quad l \leqslant k.
$$

By this and Lemma 4.3 we get

$$
\begin{aligned}
a_{h,p}(\tilde{T}_k v, v) &= \lambda_k^{-1}(\hat{A}_k \hat{P}_k v, \hat{A}_k \hat{P}_k v) \leqslant C a_{h,p}(\hat{T}_k v, v), \\
&\leqslant C(h_k/h_l)^{\gamma} a_{h,p}(v,v) \\
&\leqslant (C\epsilon^{k-l})^2 a_{h,p}(v,v) \quad \forall v \in M_l, \quad l \leqslant k,
\end{aligned}
$$

where $\gamma \in (0, 1/2)$, $\epsilon = 2^{-\gamma/2} \in (0,1)$. Thus we get (H2) and complete the proof. □

To deal with non-self-adjoint and indefinite problem (4.1) under minimal regularity assumption, we need the following result:

Lemma 4.5. *For sufficiently small $\epsilon > 0$, we can find an $h_0 > 0$, such that if $h_1 \leqslant h_0$, we have*

$$
\|P_1 v_h - v_h\|_0 \leqslant C\epsilon \|v_h\|_{1,h} \quad \forall v_h \in M_J.
$$

Proof. For any $g \in L^2(\Omega)$, using Eq. (3.1), we have

$$\begin{aligned}(P_1 v_h - v_h, g) &= a(P_1 v_h, \varphi) - (v_h, g)\\ &= (a(P_1 v_h - v_h, \varphi) - a_h(P_1 v_h - v_h, \varphi))\\ &\quad + a_h(P_1 v_h - v_h, \varphi) + (a(v_h, \varphi) - (v_h, g))\\ &\equiv I_1 + I_2 + I_2.\end{aligned}$$

Using Lemma 3.6, we obtain that for any given $\epsilon > 0$, there exists an $\tilde{h}_0 > 0$, such that for all $h_J \leqslant \tilde{h}_0$

$$|I_1| \leqslant C_1 \epsilon \|P_1 v_h - v_h\|_{1,h} \|\varphi\|_1 \leqslant C_1 \epsilon \|P_1 v_h - v_h\|_{1,h} \|g\|_0.$$

According to the definition of operator P_1 and Lemma 3.6, we know that for any given $\epsilon > 0$, we can find an $h_0' > 0$, such that if $h_1 \leqslant h_0'$, we have

$$\begin{aligned}|I_2| &\leqslant C_2 \|P_1 v_h - v_h\|_{1,h} \inf_{w_h \in M_1} \|\varphi - w_h\|_1\\ &\leqslant C_2 \epsilon \|P_1 v_h - v_h\|_{1,h} \|g\|_0.\end{aligned}$$

Using Lemma 3.6 again, we obtain that for any given $\epsilon > 0$, there exists an $h_0'' > 0$, such that for all $h_1 \leqslant h_0''$

$$\begin{aligned}|I_3| &= |a(v_h - P_1 v_h, \varphi) - (v_h - P_1 v_h, g)|\\ &\leqslant \sup_{w_h \in M_J} \frac{|a(w_h, \varphi) - (w_h, g)|}{\|w_h\|_{1,h}} \|v_h - P_1 v_h\|_{1,h}\\ &\leqslant C_3 \epsilon \|v_h - P_1 v_h\|_{1,h} \|g\|_0.\end{aligned}$$

From the above inequalities, we know that for any given $\epsilon > 0$, we can find an $h_0 > 0$, such that for all $h_1 \leqslant h_0$

$$\|P_1 v_h - v_h\|_0 \leqslant C\epsilon \|P_1 v_h - v_h\|_{1,h}. \tag{4.7}$$

From Lemma 3.5 and the definition of operator P_1 we get

$$\begin{aligned}C_0 \|P_1 v_h - v_h\|_{1,h}^2 - C_2 \|P_1 v_h - v_h\|_0^2 &\leqslant a_h(P_1 v_h - v_h, P_1 v_h - v_h)\\ &= a_h(P_1 v_h - v_h, -v_h)\\ &\leqslant C \|P_1 v_h - v_h\|_{1,h} \|v_h\|_{1,h}.\end{aligned}$$

By inequality (4.7), we know that for sufficiently small $\epsilon > 0$, we can find an $h_0 > 0$, such that for all $h_1 \leqslant h_0$

$$\begin{aligned}&\|P_1 v_h - v_h\|_{1,h} \leqslant C \|v_h\|_{1,h},\\ &\|P_1 v_h - v_h\|_0 \leqslant C\epsilon \|v_h\|_{1,h}. \qquad \square\end{aligned}$$

Now we prove the main result of this section.

Theorem 4.6. *For sufficiently small* $\epsilon > 0$, *there exists an* $h_0 > 0$, *such that for all* $h_1 \leqslant h_0$

$$a_{h,p}(E_J v, E_J v) \leqslant \delta^2 a_{h,p}(v, v) \quad \forall v \in M_J, \tag{4.8}$$

where $\delta = \hat{\delta} + C(h_1 + \epsilon)$, *positive constant* $\hat{\delta} < 1$ *is given by Theorem* 4.4.

Proof. Consider the following perturbation operator:

$$Z_k = T_k - \hat{T}_k.$$

For $k = 1$, any $v, w \in M_J$, we have

$$\begin{aligned} a_{h,p}(P_1 v, w) &= a_{h,p}(P_1 v, \hat{P}_1 w) \\ &= a_h(v, \hat{P}_1 w) - a_{h,L}(P_1 v, \hat{P}_1 w) \\ &= a_{h,p}(\hat{P}_1 v, w) + a_{h,L}(v - P_1 v, \hat{P}_1 w). \end{aligned}$$

From this we get

$$a_{h,p}(Z_1 v, w) = a_{h,L}((I - P_1)v, \hat{P}_1 w). \tag{4.9}$$

For $k > 1$, any $v, w \in M_J$, we have

$$\begin{aligned} a_{h,p}(T_k v, w) &= (T_k v, \hat{A}_k \hat{P}_k w) = (A_k P_k v, R_k^{\mathrm{t}} \hat{A}_k \hat{P}_k w) \\ &= a_h(P_k v, \hat{T}_k^* w) = a_h(v, \hat{T}_k^* w) \\ &= a_{h,p}(\hat{T}_k v, w) + a_{h,L}(v, \hat{T}_k^* w), \end{aligned} \tag{4.10}$$

where R_k^{t} denote the transpose of R_k in inner product $(\cdot,\cdot)$ and $\hat{T}_k^* = R_k^{\mathrm{t}} \hat{A}_k \hat{P}_k$.

From (4.10) we get

$$a_{h,p}(Z_k v, w) = a_{h,L}(v, \hat{T}_k^* w). \tag{4.11}$$

Let $\|v\|_* = a_{h,p}^{1/2}(v, v)$ for any $v \in M_J$. From Lemma 3.4 and Poincaré inequality for nonconforming element space (see Corollary 5.2 in [9]), we know $\|\cdot\|_*$ is a norm equivalent to $\|\cdot\|_{1,h}$. By the definititon of $\hat{P}_1$, we can show that $\|\hat{P}_1\|_* \leqslant 1$ and $\|I - \hat{P}_1\|_* \leqslant 1$. Using (4.9), Lemmas 3.7 and 4.5, we obtain

$$|a_{h,p}(Z_1 v, w)| \leqslant C\|v - P_1 v\|_0 \|\hat{P}_1 w\|_1 \leqslant C\epsilon \|v\|_* \|w\|_*.$$

From this we know

$$\begin{aligned} &\|Z_1\|_* \leqslant C\epsilon, \\ &\|I - P_1\|_* \leqslant \|I - \hat{P}_1\|_* + \|Z_1\|_* \leqslant 1 + C\epsilon. \end{aligned} \tag{4.12}$$

According to Remark 3.1 in [3], (C.2) and (C.3) also hold with $\hat{T}_k^*$ replacing $\hat{T}_k$. Thus for any $w \in M_k$ we have

$$
\begin{aligned}
(\hat{T}_k^* w, \hat{T}_k^* w) &\leqslant C h_k^2 a_{h,p}(\hat{T}_k^* w, w), \\
a_{h,p}(\hat{T}_k^* w, w) &\leqslant a_{h,p}^{1/2}(\hat{T}_k^* w, \hat{T}_k^* w) a_{h,p}(w, w) \\
&\leqslant C a_{h,p}^{1/2}(\hat{T}_k^* w, w) a_{h,p}(w, w), \\
a_{h,p}(\hat{T}_k^* w, w) &\leqslant C a_{h,p}(w, w).
\end{aligned}
$$

By (4.11), Lemma 3.7 and the above inequalities, we obtain

$$
\begin{aligned}
|a_{h,p}(Z_k v, w)| &\leqslant C \|v\|_* \|\hat{T}_k^* w\|_0 \\
&\leqslant C h_k \|v\|_* a_{h,p}^{1/2}(\hat{T}_k^* w, w) \\
&\leqslant C h_k \|v\|_* \|w\|_*,
\end{aligned}
$$

$$
\|Z_k\|_* \leqslant C h_k, \tag{4.13}
$$

$$
\|I - T_k\|_* \leqslant \|I - \hat{T}_k\|_* + \|Z_k\|_* \leqslant 1 + C h_k. \tag{4.14}
$$

Let $E_k = (I - T_1)(I - T_2) \cdots (I - T_k)$, $\hat{E}_k = (I - \hat{T}_1)(I - \hat{T}_2) \cdots (I - \hat{T}_k)$. By (4.12) and (4.14), we get

$$
\|E_k\|_* \leqslant (1 + C\epsilon) \prod_{i=2}^{k} (1 + C h_i) \leqslant C.
$$

From this inequality, (4.13), (4.14) and Theorem 4.4, we obtain for $k > 1$,

$$
\begin{aligned}
\|E_k - \hat{E}_k\|_* &= \|(E_{k-1} - \hat{E}_{k-1})(I - \hat{T}_k) - E_{k-1} Z_k\|_* \\
&\leqslant \|E_{k-1} - \hat{E}_{k-1}\|_* (1 + C h_k) + C h_k \\
&\leqslant \|E_{k-1} - \hat{E}_{k-1}\|_* + C h_k.
\end{aligned}
$$

Noting

$$
\|E_1 - \hat{E}_1\|_* = \|Z_1\|_* \leqslant C\epsilon,
$$

we obtain

$$
\|E_J - \hat{E}_J\|_* \leqslant C\left(\epsilon + \sum_{k=2}^{J} h_k\right) \leqslant C(\epsilon + h_1).
$$

Combining this with Theorem 4.4, we get

$$
\|E_J\|_* \leqslant \hat{\delta} + C(h_1 + \epsilon),
$$

where positive constant $\hat{\delta} < 1$ is given by Theorem 4.4. Let $\delta = \hat{\delta} + C(h_1 + \epsilon)$, we obtain (4.8). □

From Theorem 4.6, we know for sufficient small $h_1 > 0$, we have $0 < \delta < 1$, i.e., the V-cycle multigrid method is uniformly convergent.

5. Multilevel additive preconditioner

In this section, we use notations defined in Section 4. For smoothing operator R_k, replacing hypothesis (C.1) in Section 4, we assume that R_k is a symmetric operator with respect to $(\cdot,\cdot)$ and satisfies

$$(\mathrm{C.1}')\quad C_1\lambda_k^{-1}\|v\|_0^2 \leqslant (R_k v, v) \leqslant C_2\lambda_k^{-1}\|v\|_0^2 \quad \forall v \in M_k.$$

For problem (4.1), we construct the following preconditioner

$$B_J = A_1^{-1}Q_1 + \sum_{i=2}^{J} R_i Q_i.$$

Then the preconditioned problem is: find $\psi_h \in M_J$ such that

$$B_J A_J \psi_h = B_J Q_J P_{W_h} f. \tag{5.1}$$

One can easily verify that $B_J A_J = \sum_{i=1}^{J} T_i$. We will study the minimal eigenvalue of the symmetric part and the energy norm of the preconditioned operator $B_J A_J$, which are defined as follows, respectively,

$$\alpha_0 = \inf_{v\in M_J} \frac{a_{h,p}(B_J A_J v, v)}{a_{h,p}(v,v)}, \qquad \alpha_1 = \sup_{v\in M_J}\left\{\frac{a_{h,p}(B_J A_J v, B_J A_J v)}{a_{h,p}(v,v)}\right\}^{1/2}.$$

The asymptotic convergence rate with respect to the energy norm for GMRES method is $1-\alpha_0^2/\alpha_1^2$ (see [16]).

First we have the following result.

Lemma 5.1. $C_1 a_{h,p}(v,v) \leqslant a_{h,p}(\sum_{i=1}^{J} \hat{T}_i v, v) \leqslant C_2 a_{h,p}(v,v) \quad \forall v \in M_J.$

Proof. Since hypotheses (H1) and (H2) in Section 4 hold (see the proof of Theorem 4.4), arguing as Theorem 3.1 in [4], we can prove this lemma immediately. □

Next we prove the main result of this section.

Theorem 5.2. *There exists an* $h_0 > 0$, *such that if* $h_1 \leqslant h_0$, *we have*

$$a_{h,p}(B_J A_J v, v) \geqslant C_1 a_{h,p}(v,v) \quad \forall v \in M_J, \tag{5.2}$$

$$a_{h,p}(B_J A_J v, B_J A_J v) \leqslant C_2 a_{h,p}(v,v) \quad \forall v \in M_J. \tag{5.3}$$

Proof. From the proof of Theorem 4.6, we know

$$\|Z_1\|_* \leqslant C\epsilon, \quad \|Z_k\|_* \leqslant Ch_k, \quad k = 2,\ldots,J, \tag{5.4}$$

where $Z_i = T_i - \hat{T}_i,\ i = 1,2,\ldots,J.$

Since $\sum_{i=2}^{J} h_i \leqslant h_1$, by Lemma 5.1 and (5.4), for sufficiently small h_1, we have

$$\begin{aligned} a_{h,p}(B_J A_J v, v) &= a_{h,p}\left(\sum_{i=1}^{J} \hat{T}_i v, v\right) + a_{h,p}\left(\sum_{i=1}^{J} Z_i v, v\right) \\ &\geqslant C a_{h,p}(v,v) - \sum_{i=1}^{J} \|Z_i\|_* a_{h,p}(v,v) \\ &\geqslant C\left(1 - \epsilon - \sum_{i=2}^{J} h_i\right) a_{h,p}(v,v) \\ &\geqslant C_1 a_{h,p}(v,v), \end{aligned}$$

i.e., (5.2) holds. To prove (5.3), note that operator $\sum_{i=1}^{J} \hat{T}_i$ is symmetric positive definite with $a_{h,p}(\cdot,\cdot)$, by Lemma 5.1, we get

$$a_{h,p}\left(\sum_{i=1}^{J} \hat{T}_i v, \sum_{i=1}^{J} \hat{T}_i v\right) \leqslant C a_{h,p}(v,v) \quad \forall v \in M_J. \tag{5.5}$$

Using Schwarz inequality, (5.4) and (5.5), for sufficiently small h_1, we have

$$\begin{aligned} a_{h,p}(B_J A_J v, B_J A_J v) &= a_{h,p}\left(\sum_{i=1}^{J} \hat{T}_i v, \sum_{i=1}^{J} \hat{T}_i v\right) + a_{h,p}\left(\sum_{i=1}^{J} \hat{T}_i v, \sum_{i=1}^{J} Z_i v\right) \\ &\quad + a_{h,p}\left(\sum_{i=1}^{J} Z_i v, \sum_{i=1}^{J} \hat{T}_i v\right) + a_{h,p}\left(\sum_{i=1}^{J} Z_i v, \sum_{i=1}^{J} Z_i v\right) \\ &\leqslant C\left(1 + \epsilon + \sum_{i=2}^{J} h_i + \left(\sum_{i=2}^{J} h_i\right)^2\right) a_{h,p}(v,v) \\ &\leqslant C_2 a_{h,p}(v,v). \qquad \square \end{aligned}$$

From this theorem we know $\alpha_0 \geqslant C_1$, $\alpha_1 \leqslant C_2$ and the convergence rate for preconditioned GMRES method is optimal.

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