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Monotonic models and cycles

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Abstract A partitional model of knowledge is monotonic if there exists a linear order on the state space such that, for every player, each element of her partition contains only a sequence of consecutive states. In monotonic models, the absence of alternating cycles is equivalent to the property that, for every pair of players, the join of their partitions contains only singletons. Under these equivalent conditions any set of posteriors for the players is consistent (i.e., there is a common prior). When checking for consistency in a monotonic model, it is not necessary to evaluate all cycle equations; if the cycle equations corresponding to cycles of length two hold, then there is a common prior.

Keywords Acyclicity · Cycle · Knowledge · Monotonicity · Partition · Posterior · Prior

JEL Classification C02 · D80 · D82 · D83.

1 Introduction

This paper studies discrete information structures, which are usually present in games of incomplete information. It investigates the implications of imposing a new condition on players' knowledge partitions, named monotonicity. This paper focuses on the problem of common prior existence (i.e., consistency of posteriors) and the connec-

[1] See Rodrigues-Neto (2009, 2012), for more details on cycles and common prior existence.

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tions between monotonicity and acyclicity.[1] The results suggest that these connections are non-trivial.

A model refers to a non-empty, finite state space, a non-empty, finite set of players, and their respective partitions of the state space.[2] A model is monotonic if its state space admits a linear order and all players' partitions respect this order in the sense that, for each player, each element of her partition consists of consecutive states. These concepts, and their associated results, do not depend on any probabilistic information. However, the analysis is, at least in part, motivated by Harsanyi consistency problem (i.e., common prior existence), a question related to players' beliefs and, consequently, to probabilities.

How is monotonicity linked to Harsanyi consistency problem? The literature explains that the only potential difficulty in obtaining consistency in finite space models arises from the existence of cycles in the meet-join diagram.[3] Each cycle is associated to a cycle equation. Posteriors are consistent if and only if all cycle equations are satisfied. Hence, in acyclic models, consistency holds regardless of the specific values of the posteriors.

In monotonic models, acyclicity is equivalent to the property that all joins of two players at a time contain only singletons. The join of two players is composed of singletons if and only if the knowledge, at any state, obtained by performing direct communication among these two players completely reveals the true state. In monotonic models where all joins of two players have only singletons, every collection of players' posteriors is consistent, i.e., there is a common prior.

Monotonicity helps us to check for consistency by decreasing the computational complexity of the checking procedure. Unlike the case of some models that are not monotonic, in every monotonic model, any arbitrary cycle equation can be deduced from cycle equations corresponding to cycles of length two. In other words, if the model is monotonic and all cycle equations corresponding to cycles of length two hold, all other cycle equations also hold. In this case, posteriors must be consistent; i.e., there is a common prior. This result simplifies the procedure for consistency checking proposed by Rodrigues-Neto (2009) if the model is monotonic. Instead of monotonicity, Hellwig (2013) proposes a different hypothesis on partitions under which, to check for consistency, one needs to inspect only the cycle equations corresponding to cycles of length four or less.[4] Rodrigues-Neto (2012) defines cycle spaces for generic knowledge models. If a monotonic model has at least one cycle, we provide a procedure for finding a basis for its cycle space that contains only cycles of length two. Hence, every cycle can be decomposed as a linear combination of cycles in this basis. Edges

[2] For details of how partitions can represent knowledge of players, see Aumann (1976, 1999).

[3] For more on the common prior assumption and the consistency problem, see Harsanyi (1967–1968), Feinberg (2000), and the references therein. Rodrigues-Neto (2009) introduces meet-join diagrams, cycles and cycle equations.

[4] Harsanyi type spaces are models such that each player has one of many possible types, the state space is the Cartesian product of all sets of possible types, and partitions are such that each player knows only her own type. The condition imposed by Hellwig (2013) makes his class of models more general than Harsanyi spaces because he requires only that for every pair of players i, j and every element π^i of i's partition and π^j of j's partition, then $\pi^i \cap \pi^j \neq \varnothing$. Intuitively, Harsanyi spaces could be imagined as hyperrectangles (generalization of rectangles for higher dimensions), while a monotonic model could be imagined as a line.

linking consecutive states form the cycles of this basis. This construction sheds more light on the mathematical structure of cycle spaces in monotonic models.

In a seminal contribution, Harsanyi (1967–1968) develops the basic ideas that allow us to study games of incomplete information. Aumann (1976) introduces the partitional approach to studying knowledge and common knowledge. Morris (1994) has the first converse to Aumann's agreement theorem. Samet (1998a,b) bring other necessary and sufficient conditions for consistency, while Feinberg (2000) presents the syntactic characterization of common priors.

The next section describes knowledge, partitions, orders, and monotonic models. Section 3 establishes the links between monotonic models, joins and acyclicity. Section 4 reveals that in monotonic models one can simplify the test for consistency by checking only cycle equations corresponding to cycles of length two. Appendix presents all proofs.

2 Partitions, orders, and monotonic models

The state space is a finite set with $n \geq 2$ elements, denoted Ω. There is a non-empty, finite set J of players. Each player j's knowledge is characterized by her partition $\Pi^j = \left\{\pi_1^j, \pi_2^j, \ldots, \pi_{L^j}^j\right\}$ of Ω. For any collection of partitions $\left\{\Pi^j \mid j \in J\right\}$, their join, denoted $\underset{j\in J}{\vee}\Pi^j$, is their coarsest common refinement, and their meet is their finest common coarsening. A model $M = \left(\Omega, J, \left\{\Pi^j \mid j \in J\right\}\right)$ is a triple with a state space Ω, a set of players J, and a collection of partitions $\left\{\Pi^j \mid j \in J\right\}$.

A partial order $\succsim$ on an arbitrary non-empty set A is a binary relation on A that is reflexive, antisymmetric and transitive. A linear order on the set A is any partial order on A such that we can always compare any two points of A. A strict linear order $\succ$ on A refers to the non-reflexive part of a linear order $\succsim$; that is, for any $a, b \in A$: $a \succ b$ if and only if $a \succsim b$ and it is not the case that $b \succsim a$.

Let $(A, \succ)$ be a strictly linearly ordered set, and $a, b \in A$. Elements a and b are consecutive according to $\succ$ if: (i) $a \succ b$ and there is no other element $x \in A$ such that $a \succ x \succ b$; or (ii) $b \succ a$ and there is no other element $x \in A$ such that $b \succ x \succ a$. If the context is clear, we just say that a and b are consecutive. Being consecutive is a symmetric relation.

Fix a strict linear order $\succ$ on Ω. Let $\pi^j(\omega)$ denote the element of j's partition containing state ω. Partition Π^j is compatible with $(\Omega, \succ)$ if for any pair of states $\hat{\omega}$ and $\bar{\omega}$ such that $\hat{\omega} \in \pi^j(\bar{\omega})$, there is a finite sequence of states $\omega_0, \omega_1, \omega_2, \ldots, \omega_k \in \pi^j(\bar{\omega})$ satisfying: (i) $\omega_0 = \hat{\omega}$; (ii) $\omega_k = \bar{\omega}$; (iii) ω_s and ω_{s+1} are consecutive, for every $s \in \{0, 1, \ldots, k-1\}$.

Model $M = \left(\Omega, J, \left\{\Pi^j \mid j \in J\right\}\right)$ is monotonic if there exists a strict linear order $\succ$ on Ω such that the partition of each player $j \in J$ is compatible with $(\Omega, \succ)$. In general, a partition Π of Ω is a monotonic partition of $(\Omega, \succ)$ if Π is compatible with $(\Omega, \succ)$. Let $\Im^M(\Omega, \succ)$ denote the set of all monotonic partitions of $(\Omega, \succ)$. There may exist several strict linear orders making a model monotonic. If it is important to choose a particular strict linear order $\succ$, then we write $M = \left(\Omega, \succ, J, \left\{\Pi^j \mid j \in J\right\}\right)$.

Any model of a single player is monotonic. Any model of two players, where one of the players has full information (i.e., her partition has singletons only), is an example of a monotonic model because the partition of the fully informed player does not impose any restriction on the relative order of states. In general, any model of exactly $z+1$ players, where z players have full information, is monotonic. Example 1 ahead presents a model that is not monotonic.

3 Monotonicity, joins, and alternating cycles

Given a model $M = \left(\Omega, J, \left\{\Pi^j \mid j \in J\right\}\right)$, define a multigraph G_0, named the meet-join diagram, as follows: the set of nodes is Ω. An oriented j-edge $\langle j; \omega_1, \omega_2\rangle$ is a triple composed of a player $j \in J$ and an ordered pair of distinct states $\omega_1 \neq \omega_2$ belonging to the same element of j's partition. For any $i, j \in J$, the edge $\langle j; \omega_3, \omega_4\rangle$ is said to be consecutive to the edge $\langle i; \omega_1, \omega_2\rangle$ if and only if $\omega_3 = \omega_2$. The opposite of the j-edge $\langle j; \omega_1, \omega_2\rangle$, denoted $-\langle j; \omega_1, \omega_2\rangle$, is the j-edge that switches the orders of the states; that is, $-\langle j; \omega_1, \omega_2\rangle = \langle j; \omega_2, \omega_1\rangle$. A path p is a sequence of consecutive edges. Let $-p$ denote the opposite path of p; that is, the path obtained from p by considering the opposites of all of its edges.

Path $c = \{\langle j_1; \omega_1, \omega_2\rangle, \langle j_2; \omega_2, \omega_3\rangle, \ldots, \langle j_k; \omega_k, \omega_{k+1}\rangle\}$, $k \geq 2$, is said to be a cycle if and only if: (i) path c is closed; that is, $\omega_{k+1} = \omega_1$; and (ii) path c contains two edges $\left\langle j; \omega_p, \omega_q\right\rangle$ and $\left\langle j'; \omega'_p, \omega'_q\right\rangle$ such that $j \neq j'$. Cycle c is said to be an alternating cycle if $j_1 \neq j_k$, and $j_s \neq j_{s+1}$, $s \in \{1, 2, \ldots, k-1\}$. An acyclic model is one having no cycle. It turns out that if a model has no alternating cycle, it must be acyclic.[5] Cycles are important because they provide necessary and sufficient conditions for consistency (i.e., existence of a common prior). Rodrigues-Neto (2009) associates a cycle equation to each cycle in G_0. Posteriors are consistent if and only if all cycle equations hold.[6]

The property that joins of two players contain only singletons means that if two players perform direct communication, then, at any state, they find out the true state. In other words, a pair of players always finds the true state by sharing information if

[5] If a model has a cycle c, then there is an alternating cycle. If c is alternating we are done. Otherwise, by the definition of cycle, part (ii), the cycle c contains consecutive edges $\left\langle j; \omega'_1, \omega'_2\right\rangle, \left\langle j; \omega'_2, \omega'_3\right\rangle, \ldots, \left\langle j; \omega'_{s-1}, \omega'_s\right\rangle, \left\langle j'; \omega'_s, \omega'_{s+1}\right\rangle$ such that $s \geq 3$ and $j \neq j'$. Because states ω'_i and ω'_{i+1}, for $i = 1, \ldots, s$–1, belong to the same element of Π^j, then ω'_1 and ω'_s belong to the same element of Π^j. Define the cycle c' from c by replacing the collection of edges $\left\langle j; \omega'_1, \omega'_2\right\rangle, \left\langle j; \omega'_2, \omega'_3\right\rangle, \ldots, \left\langle j; \omega'_{s-1}, \omega'_s\right\rangle$ by $\left\langle j; \omega'_1, \omega'_s\right\rangle$. If the result is alternate, we are done. Otherwise, repeat the process. This process decreases the number of edges in cycle. Because c has only finitely many edges, eventually, the resulting cycle will be alternate.

[6] To obtain consistency it suffices to check cycle equations associated with alternating cycles. This is true because if all cycle equations in this subset hold, all other cycle equations must also hold. Indeed, every equation associated to a cycle that is not alternating has one or more common factors on both sides. After cancelling out all of these factors, the resulting equation becomes equal to a cycle equation of an alternating cycle. For instance, if a cycle c has the j-edges $\langle j; \omega_0, \omega_1\rangle$ and $\langle j; \omega_1, \omega_2\rangle$, the factor $\theta^j_{\omega_1}$ appears in both sides of its cycle equation. By cancelling this common factor, the simplified equation corresponds to the cycle obtained from c by replacing j-edges $\langle j; \omega_0, \omega_1\rangle$ and $\langle j; \omega_1, \omega_2\rangle$ and their opposites with, respectively, $\langle j; \omega_0, \omega_2\rangle$ and its opposite.

and only if their join has only singletons. The information structure of Rubinstein's e-mail game (see Rubinstein 1989) is a monotonic model whose join has singletons only.

Proposition 1 *Suppose that* $\left(\Omega, J, \left\{\Pi^j \mid j \in J\right\}\right)$ *is a monotonic model with two or more players. The model is acyclic (equivalently, the meet-join diagram has no alternating cycle) if and only if all joins of two players at a time contain singletons only; that is, for every* $j_1, j_2 \in J$*, if* $j_1 \neq j_2$*, then* $\Pi^{j_1} \vee \Pi^{j_2} = \{\{\omega\} \mid \omega \in \Omega\}$*. Therefore, in monotonic models where all joins of two players have only singletons, there is a common prior; that is, posteriors are consistent.*

Remark 1 The Proof of Proposition 1 does not use the fact that Ω is finite in a crucial way. It requires only the finiteness of the number of edges in alternating cycles.

Proposition 1 is valid regardless of the number of players. When a monotonic model has exactly two players, this result proves that alternating cycles may only exist inside elements of the join. In a monotonic model of two players where the join has only singletons, there is no alternating cycle. If there is an alternating cycle in a model of two players, then this cycle is contained in an element of the join, or else the model is not monotonic (as in Example 1).

Example 1 Let $\Omega = \{a, b, c, d\}$, $J = \{1, 2\}$, $\Pi^1 = \{\{a, b\}, \{c, d\}\}$ and $\Pi^2 = \{\{a, c\}, \{b, d\}\}$. This model has an alternating cycle, namely $\{\langle 1; a, b\rangle, \langle 2; b, d\rangle, \langle 1; d, c\rangle, \langle 2; c, a\rangle\}$. However, the join $\Pi^1 \vee \Pi^2$ has only singletons. By Proposition 1, this model is not monotonic.

In a monotonic model of two players, the join has only singletons if and only if there is no alternating cycle. Thus, a natural intuition would be that monotonic models and acyclic models are closely related. This intuition is "tricky", as the next two examples suggest.

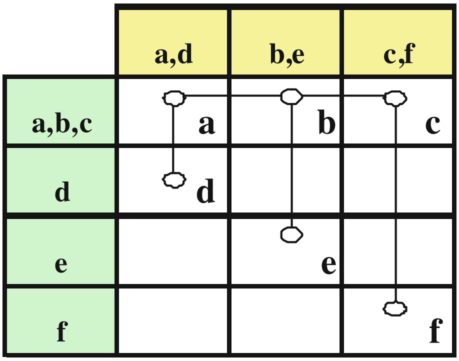
Example 2 Let $\Omega = \{a, b, c, d\}$, $\Pi^1 = \{\{a\}, \{b, c, d\}\}$, $\Pi^2 = \{\{a, b\}, \{c, d\}\}$, and $\Pi^3 = \{\{a, b, c\}, \{d\}\}$. This model is monotonic, the join $\Pi^1 \vee \Pi^2 \vee \Pi^3$ contains only singletons, but there are at least three alternating cycles: $\{\langle 1; c, d\rangle, \langle 2; d, c\rangle\}$, $\{\langle 1; b, c\rangle, \langle 3; c, b\rangle\}$, and $\{\langle 2; a, b\rangle, \langle 3; b, a\rangle\}$. Thus, in Proposition 1 we really need to look at the joins of two players at a time.

Example 3 Consider a model with two players and partitions $\Pi^1 = \{\{a, b, c\}, \{d\}, \{e\}, \{f\}\}$ and $\Pi^2 = \{\{a, d\}, \{b, e\}, \{c, f\}\}$. Figure 1 reveals that the join contains only singletons. If this model were monotonic, states a, b and c would have to be consecutive because they are together in partition Π^1, but this is impossible given their location in partition Π^2. Hence, the model is not monotonic, but there is no alternating cycle; i.e., the model is acyclic.

4 Cycles of length two

A posterior for player $j \in J$ is a vector $\theta^j = (\theta^j_1, \ldots, \theta^j_n)$, where, for each $\omega \in \Omega$, $\theta^j_\omega > 0$ denotes player j's posterior probability that the true state is ω when she is

Fig. 1 No cycles in a non-monotonic model



informed that the true state belongs to the unique element $\pi^j(\omega)$ of her partition Π^j that contains ω. Hence, $\sum_{\omega' \in \pi^j(\omega)} \theta^j_{\omega'} = 1$, for every $\omega \in \Omega$. A profile of posteriors is a collection $\theta = (\theta^1, \ldots, \theta^{|J|})$ of posteriors, one for each player. Consider a probability measure $\mu^j = \left(\mu^j_1, \ldots, \mu^j_n\right)$ over the space Ω, and let $\mu^j(B) = \sum_{\omega \in B} \mu^j_\omega$, for any event $B \subset \Omega$. Measure μ^j is a (full support) prior for player j if $\mu^j_\omega > 0$ and $\theta^j_\omega = \mu^j_\omega / \mu^j\left(\pi^j(\omega)\right)$, for every $\omega \in \Omega$.

Each cycle has a cycle equation associated to it. Let p be an arbitrary cycle:

$$p = \{\langle j_1; \omega_1, \omega_2\rangle, \langle j_2; \omega_2, \omega_3\rangle, \langle j_3; \omega_3, \omega_4\rangle, \ldots, \langle j_{k-1}; \omega_{k-1}, \omega_k\rangle, \langle j_k; \omega_k, \omega_1\rangle\}.$$

The cycle equation associated with (or corresponding to) cycle p is:

$$\theta^{j_1}_{\omega_2} \cdot \theta^{j_2}_{\omega_3} \cdots \theta^{j_{k-2}}_{\omega_{k-1}} \cdot \theta^{j_{k-1}}_{\omega_k} \cdot \theta^{j_k}_{\omega_1} = \theta^{j_1}_{\omega_1} \cdot \theta^{j_2}_{\omega_2} \cdot \theta^{j_3}_{\omega_3} \cdots \theta^{j_{k-1}}_{\omega_{k-1}} \cdot \theta^{j_k}_{\omega_k}.$$

Given a set of posteriors, one for each player, the consistency problem asks if there is a common prior; i.e., if any single prior can generate all posteriors. Posteriors are consistent if and only if all cycle equations are satisfied. Depending on players' partitions, there may be redundant cycle equations. Hence, it may not be necessary to check all cycle equations.[7]

4.1 Checking for consistency in monotonic models

In monotonic models, to find out if posteriors are consistent, it suffices to check the cycle equations associated with cycles of length two. When these cycle equations are satisfied, then, necessarily, all other cycle equations must also be satisfied.

Proposition 2 *Suppose that Ω is finite and $M = \left(\Omega, J, \left\{\Pi^j \mid j \in J\right\}\right)$ is a monotonic model with two or more players. Fix players' posteriors θ^j, $j \in J$. If all cycle equations corresponding to cycles of length two hold, then all other cycle equations must hold as well. In this case, there is a common prior.*

[7] See Hellwig (2013) or Rodrigues-Neto (2012).

Example 4 Let $\Omega = \{1, 2, 3, 4, 5\}$, $J = \{A, B, C\}$, $\Pi^A = \{\{1, 2, 3, 4, 5\}\}$, $\Pi^B = \{\{1, 2\}, \{3, 4\}, \{5\}\}$, and $\Pi^C = \{\{1, 2, 3\}, \{4, 5\}\}$. This is a monotonic model under the usual order of the real numbers. Consider the cycle $p = \{\langle A; 2, 5\rangle, \langle C; 5, 4\rangle, \langle B; 4, 3\rangle, \langle C; 3, 2\rangle\}$. Its cycle equation is:

$$\theta_5^A \cdot \theta_4^C \cdot \theta_3^B \cdot \theta_2^C = \theta_3^C \cdot \theta_4^B \cdot \theta_5^C \cdot \theta_2^A.$$

Consider the sets of edges $U_0 = \{\langle A; 2, 5\rangle\}$ and $D_0 = \{\langle C; 5, 4\rangle, \langle B; 4, 3\rangle, \langle C; 3, 2\rangle\}$ decomposing path p. Set U_0 contains all edges that "move up", i.e., edges $\langle j; \omega, \omega^*\rangle$ such that state ω^* is larger than ω according to the order on Ω. Set D_0 has all edges that "move down". Next, decompose each edge in U_0 so that we end up with edges connecting consecutive states only. For instance, instead of $\langle A; 2, 5\rangle$, use the edges $\langle A; 2, 3\rangle, \langle A; 3, 4\rangle, \langle A; 4, 5\rangle$. Similarly, decompose the edges of D_0, so that, from sets U_0 and D_0 we obtain sets $U = \{\langle A; 2, 3\rangle, \langle A; 3, 4\rangle, \langle A; 4, 5\rangle\}$ and $D = \{\langle C; 5, 4\rangle, \langle B; 4, 3\rangle, \langle C; 3, 2\rangle\}$, where each edge has consecutive states. Because p is a closed path, there is a bijective function $F : U \to D$, defined by $F(\langle A; 2, 3\rangle) = \langle C; 3, 2\rangle$, $F(\langle A; 3, 4\rangle) = \langle B; 4, 3\rangle$, and $F(\langle A; 4, 5\rangle) = \langle C; 5, 4\rangle$. Consider the following three different cycles of length two and their corresponding cycle equations:

$$\begin{aligned} p_1 &= \{\langle A; 2, 3\rangle, \langle C; 3, 2\rangle\}, \quad \theta_3^A \cdot \theta_2^C = \theta_2^A \cdot \theta_3^C, \\ p_2 &= \{\langle A; 3, 4\rangle, \langle B; 4, 3\rangle\}, \quad \theta_4^A \cdot \theta_3^B = \theta_3^A \cdot \theta_4^B, \\ p_3 &= \{\langle A; 4, 5\rangle, \langle C; 5, 4\rangle\}, \quad \theta_5^A \cdot \theta_4^C = \theta_4^A \cdot \theta_5^C. \end{aligned}$$

Multiplying the three cycle equations corresponding to cycles p_1, p_2, and p_3, after cancelling out the common factors θ_3^A and θ_4^A, we obtain the cycle equation corresponding to cycle p.

Does the converse of Proposition 2 hold? Suppose that all cycles can be decomposed in cycles of length two; would this imply that the model is monotonic? The answer is no. To see why, consider the model in Example 3, in which every cycle can be decomposed in cycles of length two (because the model is acyclic). However, the model is not monotonic.

4.2 Cycle space of a monotonic model

Fix a monotonic model $M = \left(\Omega, \succ, J, \left\{\Pi^j \mid j \in J\right\}\right)$. Without loss of generality, assume that $\Omega = \{1, 2, \ldots, n\}$, and $n \succ n-1 \succ \cdots \succ 2 \succ 1$.

Define a multigraph $\vec{G}$, named the ordered version of M, as follows: (i) the nodes of $\vec{G}$ are the states in Ω; and (ii) for every $j \in J$, a j-edge of $\vec{G}$ is a triple $\left\langle j; \omega_p, \omega_q\right\rangle$ composed of player j and two consecutive states, namely ω_p and $\omega_q = \omega_p + 1$, belonging to the same element of partition Π^j; that is, $\omega_q \in \pi^j(\omega_p)$. State ω_p is the beginning point of the j -edge and ω_q is the end point of the j-edge.[8] The set $\vec{E}$ of edges of $\vec{G}$ is the set of all j-edges of $\vec{G}$, for all $j \in J$. Clearly, $\vec{E}$ is a subset of the set of edges of the meet-join diagram. A path $p = \{\langle j_1; \omega_1, \omega_2\rangle, \langle j_2; \omega_2, \omega_3\rangle, \ldots, \langle j_k; \omega_k, \omega_{k+1}\rangle\}$ is increasing if $\omega_{k+1} \succ \omega_k \succ \cdots \succ \omega_2 \succ \omega_1$. Path p is decreasing if path $-p$ is increasing.

Define the edge space of $\vec{G}$, denoted $V(\vec{G})$, to be the real vector space spanned by $\vec{E}$. An edge $x \in \vec{E}$ can be interpreted as a vector of $V(\vec{G})$. For this, think of x as the vector $\sum_{y\in\vec{E}} \alpha_y y$, with coefficients $\alpha_y = 1$, if $y = x$; and $\alpha_y = 0$, if $y \neq x$. Multiplication of the j-edge $x = \langle j; \omega_p, \omega_q\rangle$ by the scalar -1 is associated with $-x = \langle j; \omega_q, \omega_p\rangle$.[9] Set $\vec{E}$ is linearly independent because we can never express any edge of $\vec{G}$ as a linear combination of other edges of $\vec{E}$. By definition, $\vec{E}$ spans $V(\vec{G})$. Hence, $\vec{E}$ is a basis for $V(\vec{G})$, and $\dim V(\vec{G}) = \left|\vec{E}\right|$. Every vector $v \in V(\vec{G})$ is a linear combination of edges with real coefficients, i.e., there are unique scalars $\beta_x \in \mathbb{R}$, with $x \in \vec{E}$, such that $v = \sum_{x\in\vec{E}} \beta_x x$. For every $v_1 = \sum_{x\in\vec{E}} \alpha_x x$ and $v_2 = \sum_{x\in\vec{E}} \beta_x x$, with $\alpha_x, \beta_x \in \mathbb{R}$, define the sum in $V(\vec{G})$ as $v_1 + v_2 = \sum_{x\in\vec{E}}(\alpha_x + \beta_x)x$, and the product of $v_1 \in V(\vec{G})$ by $\lambda \in \mathbb{R}$ as $\lambda v_1 = \sum_{x\in\vec{E}}(\lambda\alpha_x)x$. The cycle space of $\vec{G}$, denoted $W(\vec{G})$, is the sub-vector space of $V(\vec{G})$ spanned by all cycles of $\vec{G}$.

4.3 A special collection of cycles of length two

For each index $i \in \{1, 2, \ldots, n-1\}$, and each pair of distinct players $j \neq j' \in J$, define the cycle of length two $c_\omega(j, j') = \{\langle j; \omega, \omega+1\rangle, \langle j'; \omega+1, \omega\rangle\}$, if the state $\omega + 1 \in \pi^j(\omega) \cap \pi^{j'}(\omega)$. This means that states ω and $\omega + 1$ belong to the same element of partition Π^j and also belong to the same element of partition $\Pi^{j'}$. When $\omega+1 \notin \pi^j(\omega)\cap\pi^{j'}(\omega)$, the cycles $c_\omega(j, j')$ and $c_\omega(j', j)$ are not defined. If $\omega+1 \in \pi^j(\omega) \cap \pi^{j'}(\omega)$, then $c_\omega(j, j') = -c_\omega(j', j)$, and the cycles $c_\omega(j, j')$ and $c_\omega(j', j)$ have the same cycle equation. By construction, one of the edges of $c_\omega(j, j')$, and the opposite of the other edge, belong to $\vec{E}$.

Let $\Im^*$ denote the collection of all length two cycles $c_\omega(j, j')$, for $\omega \in \Omega$ and $j, j' \in J$. The cycles in collection $\Im^*$ span the entire cycle space $W(\vec{G})$. However, the collection $\Im^*$ may be linearly dependent, as the following example illustrates.

Example 5 Let $\Omega = \{1, 2\}, 2 \succ 1, J = \{A, B, D\}$, and $\Pi^A = \Pi^B = \Pi^D = \{\Omega\}$. Then, $c_1(A, B) = \{\langle A; 1, 2\rangle, \langle B; 2, 1\rangle\}$, $c_1(A, D) = \{\langle A; 1, 2\rangle, \langle D; 2, 1\rangle\}$, and $c_1(B, D) = \{\langle B; 1, 2\rangle, \langle D; 2, 1\rangle\}$. Because $\langle B; 1, 2\rangle = -\langle B; 2, 1\rangle$, then $c_1(A, B) + c_1(B, D) = c_1(A, D)$. Thus, the collection $\{c_1(A, B), c_1(B, D), c_1(A, D)\} \subset \Im^*$ is linearly dependent.

8 The j-edges of $\vec{G}$ inside each element π_r^j of partition Π^j connect, directly or indirectly, all nodes in π_r^j. For every pair of distinct nodes of each π_r^j, there is a unique increasing path of j-edges of $\vec{G}$ connecting them.

9 Formally, $-x$ is not a j-edge of graph $\vec{G}$, but it is a j-edge of the meet-join diagram, and it belongs to the edge space, $V(\vec{G})$. See more details about edge spaces and cycle spaces in Rodrigues-Neto (2012).

Depending on the model, it is possible that the collection $\Im^*$ is empty. This is the case if and only if $\omega + 1 \notin \pi^j(\omega) \cap \pi^{j'}(\omega)$, for any $\omega \in \{1, 2, \ldots, n-1\}$ and any pair of distinct players $j \neq j'$ in J. This means that $\Im^* = \varnothing$ if and only if every pair of partitions Π^j and $\Pi^{j'}$ has a join made only of singletons. In this case, by Proposition 1, the model has no alternating cycle, and then, it must be acyclic; so, there is a common prior, regardless of the posteriors.

Proposition 3 *In a monotonic model, every cycle can be decomposed as a linear combination of cycles in the collection $\Im^*$. If $\Im^*$ is non-empty, then $\Im^*$ contains a non-empty subset of linearly independent cycles that span the cycle space; i.e., a basis for the cycle space.*

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Appendix: Proofs

Proof of Proposition 1

In a monotonic model, suppose that, for every pair of different partitions $\Pi^j \neq \Pi^{j'}$, their join contains only singletons; $\Pi^j \vee \Pi^{j'} = \{\{\omega\} \mid \omega \in \Omega\}$. Then, there is no pair of different states belonging to the same element of Π^j and also to the same element of $\Pi^{j'}$. In particular, there are no two *consecutive* states belonging to the same element of Π^j and, at the same time, belonging to the same element of $\Pi^{j'}$. This means that there is no cycle of length two; i.e., $\Im^*$ is empty. Hence, this monotonic model is acyclic because, by Proposition 3, the existence of any cycle would imply the existence of a cycle of length two.

Conversely, suppose that there is no alternating cycle. Let $j_1 \neq j_2 \in J$ be two arbitrary players. Suppose by contradiction that there are two distinct states ω_1 and ω_2 in the same element of the join $\Pi^{j_1} \vee \Pi^{j_2}$. Because the join is a refinement of Π^{j_1}, then, ω_1 and ω_2 belong to the same element of Π^{j_1}. Hence, the meet-join diagram has the j_1-edge $\langle j_1; \omega_1, \omega_2 \rangle$. Similarly, ω_1 and ω_2 are in the same element of Π^{j_2}, and the meet-join diagram has the j_2-edge $\langle j_2; \omega_2, \omega_1 \rangle$. Consider the path $c = \{\langle j_1; \omega_1, \omega_2 \rangle, \langle j_2; \omega_2, \omega_1 \rangle\}$. Clearly, c is an alternating cycle, which contradicts the hypothesis.

If in a monotonic model all joins of two players have only singletons, there is no cycle. By Rodrigues-Neto (2009), all posteriors of are consistent; that is, there is a common prior.

Proof of Proposition 2

If there is no cycle, the result is trivial. Consider any arbitrary cycle p of arbitrary length $k \geq 2$. Without loss of generality (see footnote 5), p is an alternating cycle. Fix any arbitrary strict linear order $\succ$ on Ω that makes the model monotonic. Let U_0 be the set of edges $\left\langle j; \omega_\alpha, \omega_\beta \right\rangle$ of cycle p such that $\omega_\beta \succ \omega_\alpha$. Let D_0 be the set of edges $\langle j; \omega_\alpha, \omega_\beta \rangle$ of cycle p such that $\omega_\alpha \succ \omega_\beta$. Because cycle p is a closed path, $U_0 \neq \varnothing$ and $D_0 \neq \varnothing$. Let U be the multiset of edges obtained from the edges of U_0 in the following way: for each edge $\langle j; \omega_\alpha, \omega_\beta \rangle \in U_0$, include in U the edges $\langle j; \omega_1, \omega_2 \rangle, \langle j; \omega_2, \omega_3 \rangle, \ldots, \langle j; \omega_{k-1}, \omega_k \rangle$, such that $\omega_1 = \omega_\alpha$, $\omega_k = \omega_\beta$, and for every $i \in \{1, \ldots, k-1\}$, ω_{i+1} and ω_i are consecutive according to $\succ$, with $\omega_{i+1} \succ \omega_i$. The monotonicity of the model implies that $\omega_1, \omega_2, \ldots, \omega_k$ belong to the same element of player j's partition. Hence, every $\langle j; \omega_i, \omega_{i+1} \rangle$ belongs to the meet-join diagram. Construct a multiset D from the set D_0 in an analogous way. It is crucial to define U and D as multisets (instead of sets), so they can include multiple copies of the same edge, if necessary. Because p is a closed path and because we defined U and D as multisets, they must always have the same number of elements, $|U| = |D|$. Because $U_0 \neq \varnothing$ and $D_0 \neq \varnothing$, then $U \neq \varnothing$ and $D \neq \varnothing$. Because p is a closed path, there is a bijection F from U to D such that each edge $u = \langle j_U; \omega_i, \omega_{i+1} \rangle$ of U is associated to edge $F(u) = \langle j_D; \omega_{i+1}, \omega_i \rangle$ of D, $j_U \neq j_D$, (edge u "moves up" from ω_i to ω_{i+1}; edge $F(u)$ "goes down" from ω_{i+1} to ω_i),[10] or else, in this process, there is a link between two consecutive states ω_i to ω_{i+1} such that $j_U = j_D$. In the first case, every $\{u, F(u)\}$, for $u \in U$, is a cycle of length two, and the collection of cycle equations corresponding to the collection of cycles $\{\{u, F(u)\} \mid u \in U\}$ generates the cycle equation corresponding to cycle p. Hence, if all cycle equations corresponding to the collection of (length two) cycles $\{\{u, F(u)\} \mid u \in U\}$ are satisfied, the cycle equation corresponding to cycle p must also be satisfied. In the second case, there are edges $u = \langle j; \omega_i, \omega_{i+1} \rangle \in U$ and $\langle j; \omega_{i+1}, \omega_i \rangle \in D$, and then, $\{u, F(u)\}$ is not a cycle, but only a closed path. Let Z be the sub-collection of $\{\{u, F(u)\} \mid u \in U\}$ such that each $\{u, F(u)\}$ in Z is a cycle. This sub-collection has only cycles of length two (by construction) and is non-empty because p is a cycle. Moreover, if all cycle equations corresponding to cycles in Z are satisfied, the cycle equation corresponding to cycle p must also be satisfied.[11] In both cases, all cycle equations hold. Because Ω is finite, then, by Rodrigues-Neto (2009), there is a common prior.

Proof of Proposition 3

This proof follows similar steps to those in the previous argument. Given any arbitrary cycle c, the decomposition of c proposed in the Proof of Proposition 2 leads to cycles of length two belonging to the collection $\Im^*$. If $\Im^*$ is non-empty, it must contain a basis for the cycle space because the previous part of this proposition states that $\Im^*$ spans the cycle space. Every maximal linearly independent subset of $\Im^*$ is a basis for the cycle space. There is at least one maximal linearly independent subset of $\Im^*$ because the collection $\Im^*$ is non-empty and finite.

[10] Function F might not be unique, but the argument stands.

[11] If we were to insist in writing a cycle equation for the closed path $\{\langle j; \omega_i, \omega_{i+1} \rangle, \langle j; \omega_{i+1}, \omega_i \rangle\}$, this equation would be trivial: $\theta^j_{\omega_{i+1}} \cdot \theta^j_{\omega_i} = \theta^j_{\omega_i} \cdot \theta^j_{\omega_{i+1}}$.

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