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On the entropy of partitions in product MV algebras

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Abstract Generalizing the entropy of fuzzy partitions, the entropy of a partition in recently introduced product MV algebra has been studied. The least common refinement of two partitions is defined and the algebraic properties of the entropies and conditional entropies are examined.

Key words Product MV algebra, entropy of a partition

1 Introduction

If (Ω, S, P) is a probability space, then a finite measurable partition ξ is a finite family of non-empty disjoint measurable sets covering Ω:

$$\xi = \{A_1, \ldots, A_n\}, \quad \bigcup_{i=1}^{n} A_i = \Omega, \ \ A_i \cap A_j = \emptyset \quad (i, j = 1, \ldots, n; \ i \neq j) \ .$$

The entropy $H(\xi)$ of ξ is defined by the formula

$$H(\xi) = -\sum_{i=1}^{n} \varphi(P(A_i)) \ ,$$

where

$$\varphi(x) = \begin{cases} x\log(x), & \text{if } x > 0, \\ 0, & \text{if } x = 0 \ . \end{cases}$$

If $\xi = \{A_1, \ldots, A_n\}$ and $\eta = \{B_1, \ldots, B_k\}$ are two partitions, then the common refinement is defined by the formula

$$\xi \vee \eta = \{A_i \cap B_j : i = 1, \ldots, n; \ j = 1, \ldots, k\} \ .$$

In [8] the fuzzy partition has been used instead of a partition, i.e. a family

$$\xi = \{f_1, \ldots, f_n\} \quad \text{of functions} \quad f_i : \Omega \to \langle 0, 1\rangle$$

such that

$$\sum_{i=1}^{n} f_i = 1_\Omega \ ,$$

(see also [2, 7, 9]. The entropy of the partition ξ is the number

$$H(\xi) = -\sum_{i=1}^{n} \varphi(m(f_i)) \ ,$$

where

$$m(f) = \int_\Omega f \, \mathrm{d}P \ .$$

If $\xi = \{f_1, \ldots, f_n\}$ and $\eta = \{g_1, \ldots, g_k\}$ are two fuzzy partitions, then the common refinement is defined by the formula

$$\xi \vee \eta = \{f_i \cdot g_j : i = 1, \ldots, n; j = 1, \ldots, k\} \ .$$

In this paper we shall consider the entropy of a partition in an arbitrary product MV algebra. The product is necessary for the definition of the common refinement of two partitions.

A prototype of an MV algebra is a set F of fuzzy sets

$$f_i : \Omega \to \langle 0, 1\rangle$$

closed under the two binary operations $\oplus$ and $\odot$ and a binary operation $*$ defined by

$$f \oplus g = \min(f + g, 1_\Omega),$$
$$f \odot g = \max(f + g - 1_\Omega, 0_\Omega),$$
$$f^* = 1_\Omega - f$$

A motivating example for $\oplus$ is the composition $f \oplus g$ of two grey pictures

$$f : \Omega \to \langle 0, 1\rangle \quad \text{and} \quad g : \Omega \to \langle 0, 1\rangle \ .$$

Evidently

$$1_\Omega = f \oplus f^* \in F \quad \text{and} \quad 0_\Omega = 1_\Omega^* \in F \ .$$

Generally, an MV algebra is an algebraic system

$$(M, \oplus, \odot, *, 1, 0) \ ,$$

where M is a set, $\oplus$ and $\odot$ are binary operations, * is a unary operation, 1 and 0 are fixed elements, satisfying some properties:

(i) $a \oplus b = b \oplus a$,
(ii) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
(iii) $a \oplus 0 = a$,
(iv) $a \oplus 1 = 1$,
(v) $(a^*)^* = a$,
(vi) $0^* = 1$,
(vii) $a \oplus a^* = 1$,
(viii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$,
(ix) $a \odot b = (a^* \oplus b^*)^*$.

Of course, by the Mundici theorem [11] any MV algebra can be represented by a commutative l-group. Recall that a commutative l-group is an algebraic system $(G, +, \leq)$,

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where $(G,+)$ is a commutative group, $(G,\leq)$ is a partially ordered set being a lattice and $a \leq b$ implies $a+c \leq b+c$. Let $(G,+,\leq)$ be a commutative l-group, 0 be a neutral element of $(G,+)$ and $u \in G$, $u > 0$. On the interval $\langle 0,u\rangle$ we define the following operations:

$$a \oplus b = (a+b) \wedge u,$$

$$a \odot b = (a+b-u) \vee 0,$$

$$a^* = u - a \ .$$

Then

$$MG = (\langle 0,u\rangle, \oplus, \odot, {}^*, u, 0)$$

becomes an MV algebra. The Mundici theorem states that to any MV algebra there exists a commutative l-group G with a strong unit u (i.e., to any $a \in G$ there exists $n \in \mathbb{N}$ such that $a \leq nu$) such that

$$M \simeq MG \ .$$

Definition 1 A product MV algebra is an algebraic system $(M,\oplus,\odot,\cdot,{}^*,u,0)$, where $(M,\oplus,\odot,{}^*,u,0)$ is an MV algebra and $\cdot$ is a binary operation satisfying the following conditions:

(i) $u \cdot u = u$,
(ii) operation $\cdot$ is commutative and associative,
(iii) if $a+b \leq u$, then $c\cdot(a\oplus b) = c\cdot a \oplus c \cdot b$ for any $c \in M$,
(iv) if $a_n \searrow 0$, $b_n \searrow 0$, then $a_n \cdot b_n \searrow 0$.

We will use only (i), (ii), (iii). (See [1, 13])

2
Entropy of partitions

We shall consider a product MV algebra $(M,\oplus,\odot,\cdot,{}^*,u,0)$. Moreover, we shall consider a finitely additive state

$$m : M \to \langle 0,1\rangle$$

satisfying the following conditions:

1. $m(u) = 1$,
2. $m(a\cdot u) = m(a)$ for any $a \in M$,
3. if $a = \sum_{i=1}^n a_i$ then $m(a) = \sum_{i=1}^n m(a_i)$.

Definition 2 A partition in a product MV algebra is a set $A = \{a_1,\ldots,a_n\}$, where $\sum_{i=1}^n a_i = u$. If $A = \{a_1,\ldots,a_n\}$ and $B = \{b_1,\ldots,b_k\}$ are partitions, then the common refinement is defined by the formula:

$$A \vee B = \{a_i \cdot b_j : i = 1,\ldots,n;\ j = 1,\ldots,k\} \ .$$

Proposition 1 If $A = \{a_1,\ldots,a_n\}$ and $B = \{b_1,\ldots,b_k\}$ are partitions, then $A \vee B$ is a partition too.

Proof. Evidently

$$\sum_{i=1}^n \sum_{j=1}^k a_i \cdot b_j = \sum_{i=1}^n a_i \cdot \sum_{j=1}^k b_j = \sum_{i=1}^n a_i \cdot u = u \ .$$

Definition 3 If $A = \{a_1,\ldots,a_n\}$ is a partition, then its entropy $H(A)$ is defined by

$$H(A) = -\sum_{i=1}^n \varphi(m(a_i)) \ .$$

Proposition 2 If $A = \{a_1,\ldots,a_n\}$ and $B = \{b_1,\ldots,b_k\}$ are fuzzy partitions, then

$$H(A \vee B) \leq H(A) + H(B) \ .$$

Proof. Put

$$\alpha_i = m(b_i), \quad x_i = \frac{m(a_j \cdot b_i)}{m(b_i)}, \ (m(b_i) > 0,\ j \text{ is fixed}) \ .$$

Since

$$\sum_{i=1}^k \alpha_i = \sum_{i=1}^k m(b_i) = m\left(\sum_{i=1}^k b_i\right) = m(u) = 1$$

and φ is convex we have

$$\begin{aligned}
\varphi(m(a_j)) &= \varphi\left(m\left(a_j\left(\sum_{i=1}^k b_i\right)\right)\right) \\
&= \varphi\left(m\left(\sum_{i=1}^k a_j \cdot b_i\right)\right) \\
&= \varphi\left(\sum_{i=1}^k m(a_j\cdot b_i)\right) \\
&= \varphi\left(\sum_{i=1}^k m(b_i)\cdot \frac{m(a_j\cdot b_i)}{m(b_i)}\right) = \varphi\left(\sum_{i=1}^k \alpha_i \cdot x_i\right) \\
&\leq \sum_{i=1}^k \alpha_i \cdot \varphi(x_i) = \sum_{i=1}^k m(b_i)\cdot\varphi(x_i) \ .
\end{aligned}$$

Therefore

$$\begin{aligned}
H(A) &= -\sum_j \varphi(m(a_j)) \\
&\geq -\sum_i \sum_j m(b_i)\cdot\varphi(x_i) \\
&= -\sum_i \sum_j \varphi(m(a_j\cdot b_i)) - \Big(-\sum_i \varphi(m(b_i))\Big) \\
&= H(A\vee B) - H(B) \ .
\end{aligned}$$

3
Conditional entropy

Definition 4 If $A = \{a_1,\ldots,a_n\}$ and $B = \{b_1,\ldots,b_k\}$ are two partitions, then we define

$$B \geq A \text{ (B is a refinement of A)} \ ,$$

if there exists a partition $\{I(1),\ldots,I(n)\}$ of the set $\{1,\ldots,k\}$ such that

$$m(a_i) = m\left(\sum_{j\in I(i)} b_j\right) \quad \text{for every } i = 1,\ldots,n \ .$$

Definition 5 If $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_k\}$ are two partitions, then we define the conditional entropy by the formula

$$H(A|B) = -\sum_{i=1}^{n}\sum_{j=1}^{k} m(b_j)\cdot\varphi\left(\frac{m(a_i\cdot b_j)}{m(b_j)}\right) \ .$$

Proposition 3 Let A, B, C be partitions, C is a refinement of B. Then it holds:

$$H(A|C) \le H(A|B) \ .$$

Proof. Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_r\}$ and $C = \{c_1, \ldots, c_p\}$,

$$b_j = \sum_{k\in I(j)} c_k \quad \text{where } \{I(1), \ldots, I(k)\} \text{ is the partition}$$

and

$$\alpha_k = \frac{m(c_k)}{m(b_j)}, \quad j \text{ is fixed} \ .$$

Then

$$\sum_{k\in I(j)} \alpha_k = \frac{1}{m(b_j)}\cdot m\left(\sum_{k\in I(j)} c_k\right) = 1 \ ,$$

hence by the convexity of φ

$$\sum_k \alpha_k\cdot\varphi(x_k) \ge \varphi\left(\sum_k \alpha_k\cdot x_k\right) \quad \text{for every } x_k \ .$$

Therefore

$$\begin{aligned} H(A|C) &= -\sum_i\sum_j m(b_j)\sum_{k\in I(j)}\frac{m(c_k)}{m(b_j)}\varphi\left(\frac{m(a_i\cdot c_k)}{m(c_k)}\right) \\ &\le -\sum_i\sum_j m(b_j)\cdot\varphi\left(\sum_{k\in I(j)}\frac{m(a_i\cdot c_k)}{m(b_j)}\right) \\ &= -\sum_i\sum_j m(b_j) \\ &\quad \times\varphi\left(\frac{1}{m(b_j)}\cdot m\left(a_i\cdot\sum_{k\in I(j)} c_k\right)\right) \\ &= H(A|B) \ . \end{aligned}$$

Proposition 4 For all fuzzy partitions A, B, C it holds:

$$H(B\vee C|A) = H(C|A\vee B) + H(B|A) \ .$$

Proof. From the definitions we obtain:

$$\begin{aligned} H(B\vee C|A) &= -\sum_i\sum_j\sum_k m(a_i) \\ &\quad\cdot\varphi\left(\frac{m(b_j\cdot c_k\cdot a_i)\cdot m(b_j\cdot a_i)}{m(a_i)\cdot m(b_j\cdot a_i)}\right) \\ &= -\sum_i\sum_j\sum_k m(a_i)\cdot\frac{m(b_j\cdot c_k\cdot a_i)}{m(a_i)} \\ &\quad\times\log\frac{m(b_j\cdot c_k\cdot a_i)}{m(a_i)}\cdot\frac{m(b_j\cdot a_i)}{m(b_j\cdot a_i)} \\ &= -\sum_i\sum_j\sum_k m(b_j\cdot c_k\cdot a_i) \\ &\quad\times\left(\log\frac{m(b_j\cdot c_k\cdot a_i)}{m(b_j\cdot a_i)} + \log\frac{m(a_i\cdot b_j)}{m(a_i)}\right) \\ &= H(C|B\vee A)\oplus\sum_i\sum_j m\left(\sum_k b_j\cdot c_k\cdot a_i\right) \\ &\quad\times\log\frac{m(a_i\cdot b_j)}{m(a_i)} \\ &= H(C|B\vee A)\oplus\sum_i\sum_j m(a_i\cdot b_j)\cdot\log\frac{m(a_i\cdot b_j)}{m(a_i)} \\ &= H(C|B\vee A) + H(B|A) \ , \end{aligned}$$

because $\sum_k c_k = u$. Thus

$$H(B\vee C|A) = H(C|A\vee B) + H(B|A) \ .$$

Corollary 5 For arbitrary fuzzy partitions B, C it holds:

$$H(B\vee C) = H(B) + H(C|B) \ .$$

Proof. It is sufficient to put $A = \{u\}$ in Proposition 4 and by the relation

$$H(B\vee C|A) = H(C|A\vee B) + H(B|A) \ ,$$

we get

$$H(B\vee C) = H(C|B) + H(B) \ .$$

Proposition 6 For arbitrary fuzzy partitions B, C it holds:

$$H(B\vee C) \ge H(B) \ .$$

Proof. By Corollary 5

$$H(B\vee C) = H(B) + H(C|B) \ge H(B) \ .$$

Proposition 7 For arbitrary fuzzy partitions A, B, C it holds:

$$H(B\vee C|A) \le H(B|A) + H(C|A) \ .$$

Proof. Since $A\vee B$ is a refinement of the partition A, according to the Proposition 3 it holds:

$$H(C|A\vee B) \le H(C|A) \ .$$

By the Proposition 4:

$$\begin{aligned} H(B\vee C|A) &= H(C|A\vee B) + H(B|A) \\ &\le H(C|A) + H(B|A) \ . \end{aligned}$$

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