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Scattering of elastic waves by a crack in a isotropic plate

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Abstract

A T-matrix and boundary element hybrid method has been used to study the scattering of elastic waves by a crack in a plate. A fictitious surface is added to the crack to separate the infinite plate into two semi-infinite plates. Expressions for the components of Green's tensors are written in the form of expansions in the normal modes of the plate. The reflection and transmission coefficients of the normal modes are calculated numerically as functions of frequency. © 1997 Elsevier Science B.V.

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1. Introduction

The scattering of normal modes by a crack in a plate is an important practical problem in ultrasonic nondestructive testing and evaluation. One method of solving this scattering problem when the crack surface is parallel to the surface of the plate is discussed by Kasathin [1]. Auld [2,10] applies lumped equivalent network elements to represent scattering of the dominant symmetrical Lamb wave by a symmetrical pair thin slots or crack in a plate. Rokhlin [3,11] uses the Wiener–Hopf method to analyze the resonance phenomena of Lamé waves scattering by a finite crack whose surface is parallel to the surface of the solid layer. Using the T-matrix method, Karlsson [4] has calculated the sphere void scatterering problem in a plate. He uses the general Green's function in a unbounded solid. The propagation and excitation of normal modes have been investigated exhaustively by ray or mode method [5]. Kharitonov [6] expands the Green's tensors in the forms of normal modes and calculated the excitation by a system of volume or surface forces.

The solution of the crack scattering problem in unbounded medium or half space medium has been investigated by in Refs. [7,8]. The difficulty in this problem is to deal with the higher-order singularity that occurs in the crack-opening displacement (COD) approach. Bostrom and Olsson [9] studied the nonplanar crack by the T-matrix method where a fictitious surface is added to the crack so as to obtain a closed surface and the edge conditions of the crack tip are explicitly included in their formulations. In present paper, we use a method similar to that of Bostrom and and Olsson [9]. By introducing a fictitious surface, the infinite plate is separated into two semi-infinite plates. The T-matrix method is used to calculate the reflection and transmission coefficients of the normal modes.

2. Theory

Consider a scattering geometry as shown in Fig. 1, in an elastic plate with density ρ and Lamé parameters λ, μ, and with an embedded crack. The left edge of the crack is (x_0, z_0) and the right is (x_1, z_1). The surface of the crack is divided into C_+, C_- and two fictitious surfaces C_{w1} and C_{w2} are added to separated the infinite plate. Only time harmonic conditions with an angular frequency ω are considered and the time factor is suppressed throughout. An application of the divergence theorem to Betti–Rayleigh reciprocal relationships and the elastodynamic Green's functions in a plate gives the

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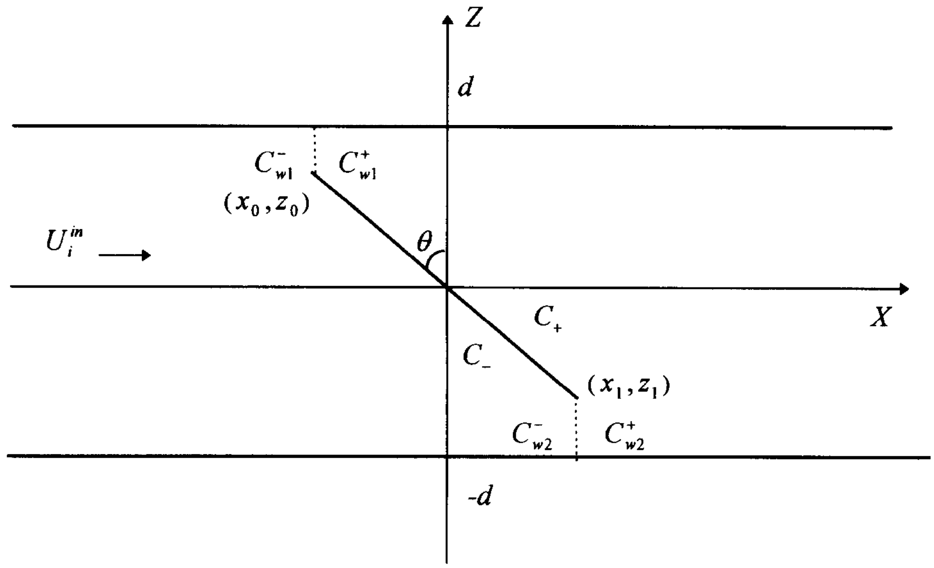


Fig. 1. The geometry of a crack in a plate.

following integral equations:

$$U_i^{\text{in}}(r)+\int_{C_-+C_w^-}(P_{ik}(r,r')U_k(r')-U_{ik}(r,r')P_k(r'))\,\mathrm{d}s'$$

$$=\begin{cases}U_i(r), & r\in\text{left side of } C_-+C_w^-,\\ 0, & r\in\text{right side of } C_-+C_w^-,\end{cases}\tag{1a}$$

and

$$\int_{C_++C_w^+}(P_{ik}(r,r')U_k(r')-U_{ik}(r,r')P_k(r'))\,\mathrm{d}s',$$

$$=\begin{cases}U_i(r), & r\in\text{right side of } C_++C_w^+,\\ 0, & r\in\text{left side of } C_++C_w^+,\end{cases}\tag{1b}$$

where $U_{ik}(r, r')$ are the elastodynamic displacement Green's functions. The corresponding stress Green's functions $P_{ik}(r, r')$ are related to $U_{ik}(r, r')$ by Hooke's law. The explicit functions of $U_{ik}(r, r')$ and $P_{ik}(r, r')$ are given in Appendix A.

The normal modes of the plate define a set of basis functions. The incident field U_i^{in}, the reflected field U_i^{re} and the transmitted field U_i^{t} can be expanded as follows:

$$\boldsymbol{U}^{\text{in}}(r)=\sum_{\text{s,a}} b_{\text{s,a}}\boldsymbol{U}_{\text{s,a}}^+(r),$$

$$\boldsymbol{U}^{\text{re}}(r)=\sum_{\text{s,a}} f_{\text{s,a}}\boldsymbol{U}_{\text{s,a}}^-(r)\quad\text{if } x<x_0,$$

$$\boldsymbol{U}^{\text{t}}(r)=\sum_{\text{s,a}} t_{\text{s,a}}\boldsymbol{U}_{\text{s,a}}^+(r)\quad\text{if } x>x_1.\tag{2}$$

$\boldsymbol{U}_\text{s}^+$, $\boldsymbol{U}_\text{a}^+$, $\boldsymbol{U}_\text{s}^-$ and $\boldsymbol{U}_\text{a}^-$ are displacement vectors of normal modes for the plate. The subscripts 's' and 'a' represent the symmetric and antisymmetric Lamb waves and the superscripts '+' and '−' represent the positive and negative propagation directions of the x-axis. $\sum_{\text{s,a}}$ denotes the sum of all the symmetry and antisymmetry modes. The explicit formulations for the normal modes are given in Appendix A. In the region $x<x_0$, the reflected field contains only the negative propagation modes and in the region $x>x_1$, the transmitted wave contains only the negative propagation modes.

Expanding the Green's functions $U_{ik}(r, r')$, $P_{ik}(r, r')$ in the form of normal modes, we obtain:

$$U_{ik}(r,r')=\frac{j}{\mu}\sum_{\text{s,a}}\frac{k_t}{k_{\text{s,a}}}\,U_{\text{is,a}}^{(\mp)}(r)U_{\text{ks,a}}^{(\pm)}(r'),\tag{3a}$$

$$P_{ik}(r,r')=\frac{j}{\mu}\sum_{\text{s,a}}\frac{k_t}{k_{\text{s,a}}}\,U_{\text{is,a}}^{(\mp)^+}(r)D_{\text{ks,a}}^{(\pm)}(r'),\tag{3b}$$

where $K_t^2=\rho\omega^2/\mu$, $k_l^2=(\rho\omega^2)/(\lambda+2\mu)$ are the wave numbers of normal modes (see Appendix A). In Eqs. (3a) and (3b), the upper sign of the superscript must be chosen for $x'>x$ and the lower sign for $x'<x$.

Substituting Eqs. (3a) and (3b) and Eq. (2) into Eqs. (1a) and (1b) and using the orthogonality relationship for the normal modes we can obtain:

$$f_{\text{s,a}}=\frac{jk_t}{\mu_{\text{s,a}}}\int_{C_-+C_w^-}(D_{\text{is,a}}^+(r')U_i(r')-U_{\text{is,a}}^+(r')P_i(r'))\,\mathrm{d}s,\tag{4a}$$

$$b_{\text{s,a}}=\frac{jk_t}{\mu_{\text{s,a}}}\int_{C_-+C_w^-}(D_{\text{is,a}}^-(r')U_i(r')-U_{\text{is,a}}^-(r')P_i(r'))\,\mathrm{d}s,\tag{4b}$$

$$0=\frac{jk_t}{\mu_{s,a}}\int_{C_++C_w^+}(D_{is,a}^{+}(r')U_i(r')-U_{is,a}^{+}(r')P_i(r'))\,ds, \tag{4c}$$

$$t_{s,a}=\frac{jk_t}{\mu_{s,a}}\int_{C_++C_w^+}(D_{is,a}^{-}(r')U_i(r')-U_{is,a}^{-}(r')P_i(r'))\,ds. \tag{4d}$$

C_w^+ represents the sum of C_{w1}^+ and C_{w2}^+ and C_w^- represents the sum of C_{w1}^- and C_{w2}^-. Following Bostrom and Olsson [9], we introduce an auxiliary field $\mathbf{V}(r)$ in order to avoid the singularity of the tractions at the crack tips

$$V(r)=\begin{cases}\mathbf{Z}^{-1}(U_+(r)-U_-(r)) & \text{on } C,\\ \mathbf{Z}^{-1}\boldsymbol{P}_+ & \text{on } C_w.\end{cases} \tag{5}$$

Here $\boldsymbol{U}_+(r)$ and $\boldsymbol{U}_-(r)$ are the displacement fields on C_+ and C_- respectively and $\boldsymbol{P}_+$ denotes the traction at the fictitious surface C_w. The function $\mathbf{Z}^{-1}(r)$ is chosen so that $V(r)$ is continuous. In Eqs. (4a), (4b), (4c) and (4d) with the introduction of the field $V(r)$ and the boundary condition, traction free on C_+ and C_- and displacement and traction continuity across C_w, we have two unknown surface fields $V(r)$ and $U_-(r)$. The two unknown fields can be expanded in a function basis, $\boldsymbol{A}_n(r)$, such as normal mode basis with coefficients α_n and β_n, respectively:

$$\boldsymbol{U}_-(r)=\sum_n \alpha_n \boldsymbol{A}_n(r),$$
$$V(r)=\sum_n \beta_n \boldsymbol{A}_n(r),\quad r\in C \text{ or } C_w. \tag{6}$$

Inserting Eqs. (6) and (5) into Eqs. (4a), (4b), (4c) and (4d), we obtain:

$$f_{s,a}=Q^{0+}_{s,a,n}\alpha_n-Q^{1+}_{s,a,n}\beta_n, \tag{7a}$$

$$b_{s,a}=Q^{0-}_{s,a,n}\alpha_n-Q^{1-}_{s,a,n}\beta_n, \tag{7b}$$

$$0=-Q^{0+}_{s,a,n}\alpha_n+(Q^{1+}_{s,a,n}-Q^{2+}_{s,a,n})\beta_n, \tag{7c}$$

$$t_{s,a}=-Q^{0-}_{s,a,n}\alpha_n+(Q^{1-}_{s,a,n}-Q^{2-}_{s,a,n})\beta_n, \tag{7d}$$

where

$$Q^{0+}_{s,a,n}=\frac{jk_t}{\mu k_{s,a}}\int_{C_-+C_w}D^{+}_{is,a}(r')A_{in}(r')\,ds,$$

$$Q^{1+}_{s,a,n}=\frac{jk_t}{\mu k_{s,a}}\int_{C_w}U^{+}_{is,a}(r')Z_{ij}(r')A_{jn}(r')\,ds,$$

$$Q^{0-}_{s,a,n}=\frac{jk_t}{\mu k_{s,a}}\int_{C_-+C_w}D^{-}_{is,a}(r')A_{in}(r')\,ds,$$

$$Q^{1-}_{s,a,n}=\frac{jk_t}{\mu k_{s,a}}\int_{C_w}U^{-}_{is,a}(r')Z_{ij}(r')A_{jn}(r')\,ds,$$

$$Q^{2+}_{s,a,n}=\frac{jk_t}{\mu k_{s,a}}\int_{C_+}D^{+}_{is,a}(r')Z_{ij}(r')A_{jn}(r')\,ds,$$

$$Q^{2-}_{s,a,n}=\frac{jk_t}{\mu k_{s,a}}\int_{C_+}D^{-}_{is,a}(r')Z_{ij}(r')A_{jn}(r')\,ds.$$

The solution to Eqs. (7a), (7b), (7c) and (7d) is:

$$\boldsymbol{f}=\mathbf{R}\boldsymbol{b},\quad \boldsymbol{t}=\mathbf{T}\boldsymbol{b},$$

where the transition matrix is

$$\mathbf{Q}=\mathbf{Q}^{0-}(\mathbf{Q}^{0+})^{-1}(\mathbf{Q}^{2+}+\mathbf{Q}^{1+})-\mathbf{Q}^{1-},$$

$$\mathbf{R}=\mathbf{Q}^{2+}\mathbf{Q}^{-1},$$

$$\mathbf{T}=(\mathbf{Q}^{2-}\mathbf{Q}^{2-}-\mathbf{I}).$$

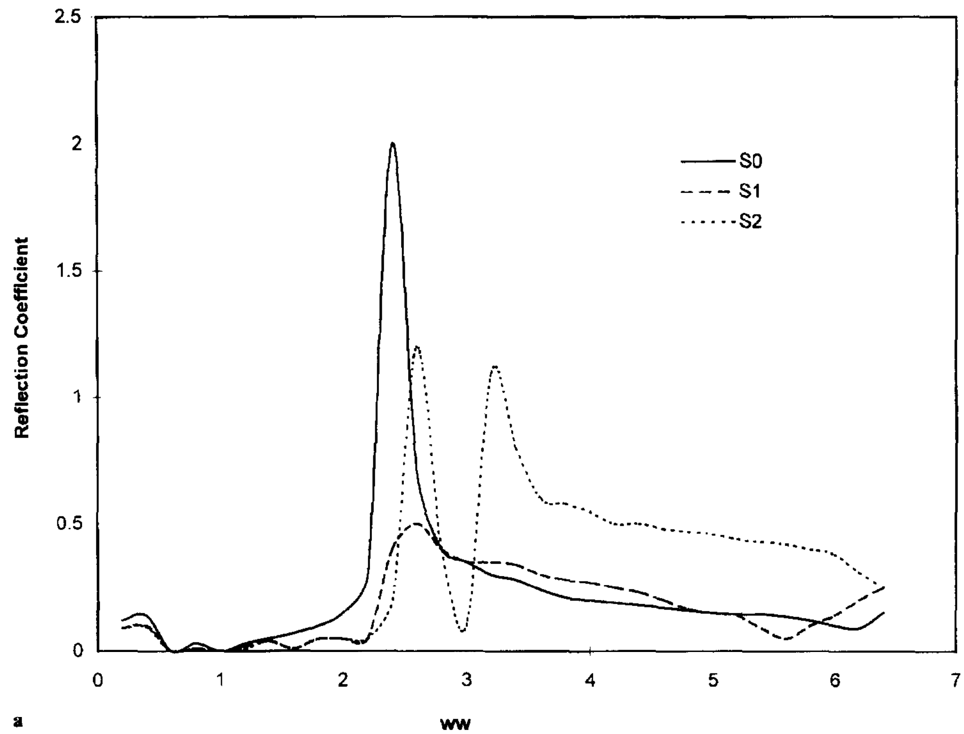
The $\mathbf{T}$ and $\mathbf{R}$ matrices contain all the relevant properties of the scattering problem, i.e. given any incident field $\boldsymbol{b}$ (normal mode or their sum), it is possible to compute the reflection $\boldsymbol{f}$ and transmission $\boldsymbol{t}$ in the form of normal modes.

3. Numerical result

The reflection $\mathbf{R}$ and transmission $\mathbf{T}$ coefficients are calculated numerically in a glass plate. Its density, longitudinal wave speed and transverse wave speed are 2.2 g/cm^3, and 6.21 and 3.47 mm/µs, respectively. Firstly, all real and some complex wave numbers of the normal modes are calculated using equation of $\Delta_{s,a}$ at a fixed frequency. The boundary C_+, C_-, C_w are subdivided into finite elements. The interpolation polynomials are chosen as the basis function for $V(r)$ and $U_-(r)$, then α_n and β_n correspond to the value $V(r)$ and $U_-(r)$ at point n. The number of elements are determined by trial and error. The numerical procedure is the same as that applied in boundary element method (BEM). The normal modes are so chosen as to include all the propation modes (real wave number) and some non-propagation modes (complex wave number).

In Fig. 2, the crack is centrally placed in the plate with a length of d and angle (θ) of 0° relative to the $+z$-axis. The incident mode is a Lamb mode. The coefficients are plotted as a function of the dimensionless paramenter $ww=\omega d/c_t$ and there is a sharp resonance at $ww=\pi c_l/(2c_t)$. The reflection and transmission coefficients of S2 also have obvious resonance at π. In Fig. 3, the length of the crack is 1.5d. It has more obvious resonance points than Fig. 2. In Fig. 4, the crack is centrally placed in the plate with a length, d, and angle of 4°. the incident mode is an S0 mode

These figures show that resonance happens at $ww=n\pi/2$ and $n\pi c_l/(2c_t)$. Because of at these values, the wave number, i.e. k_s, k_a, equals zero and some components of the Green's function become infinite. When the crack size and angle increase then resonance becomes more obvious. However, numerical instabilities may happen at large angle because of the nonpropagating normal mode.



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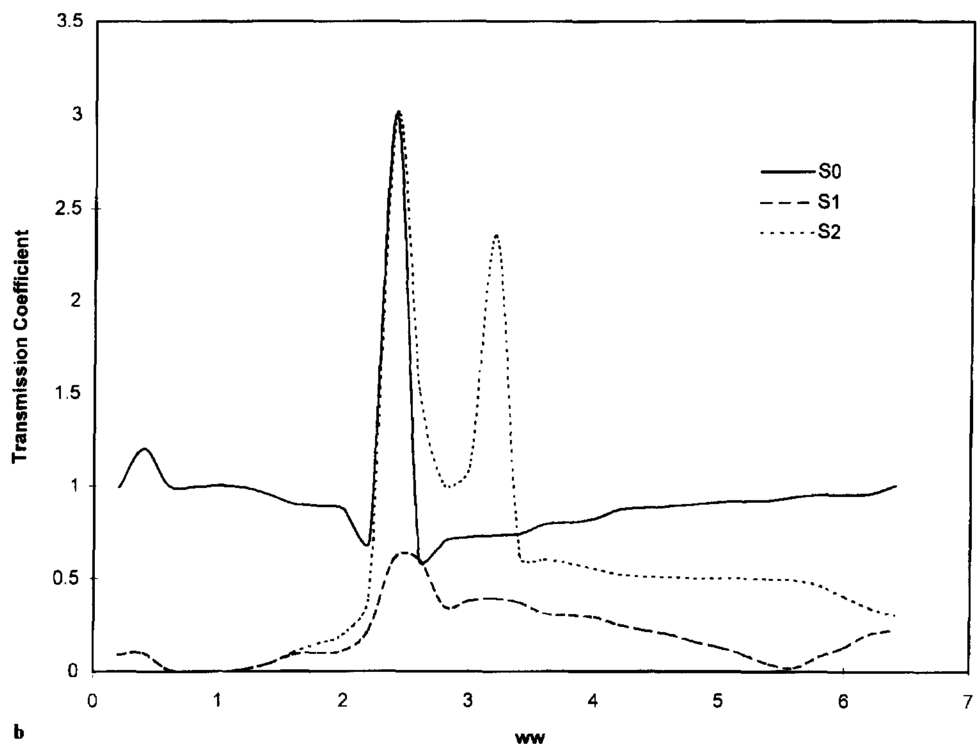

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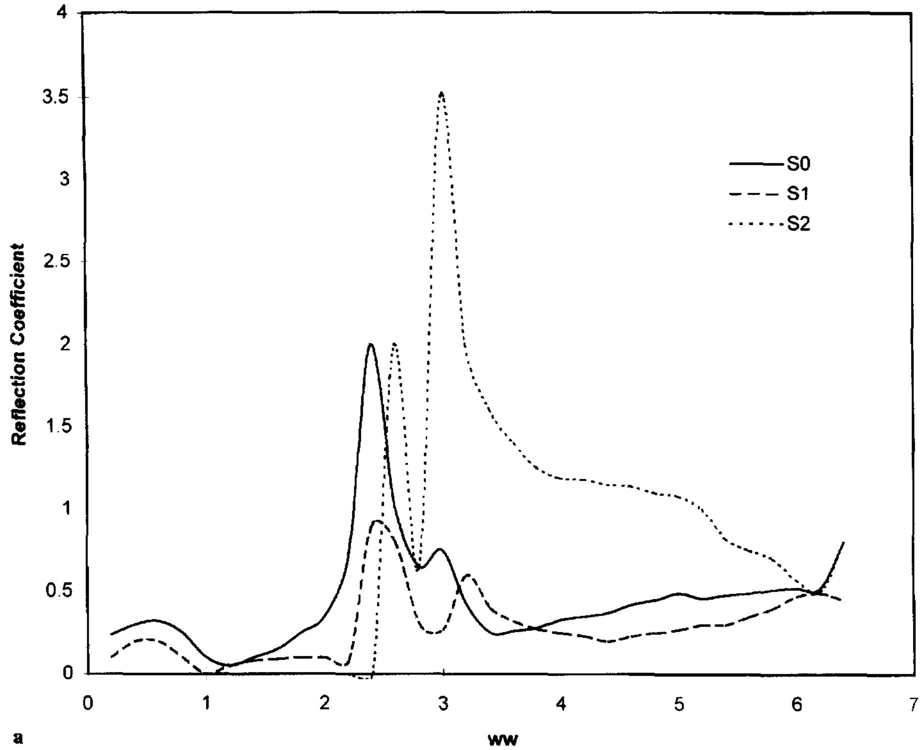
Fig. 2. (a) The refection coefficient of normal modes for the incident of the S0 mode. (b) The transmission coefficient of normal modes for the incident of the S0 mode.

4. Conclusion

The solution of scattering of elastic waves by a planar crack in a plate has been presented by a T-matrix and boundary element hybrid approach. The results show the resonance effects in the presence of the crack. The voluminous and rough crack will be considered in the future. By using the Green's functions in the plate, the reflection



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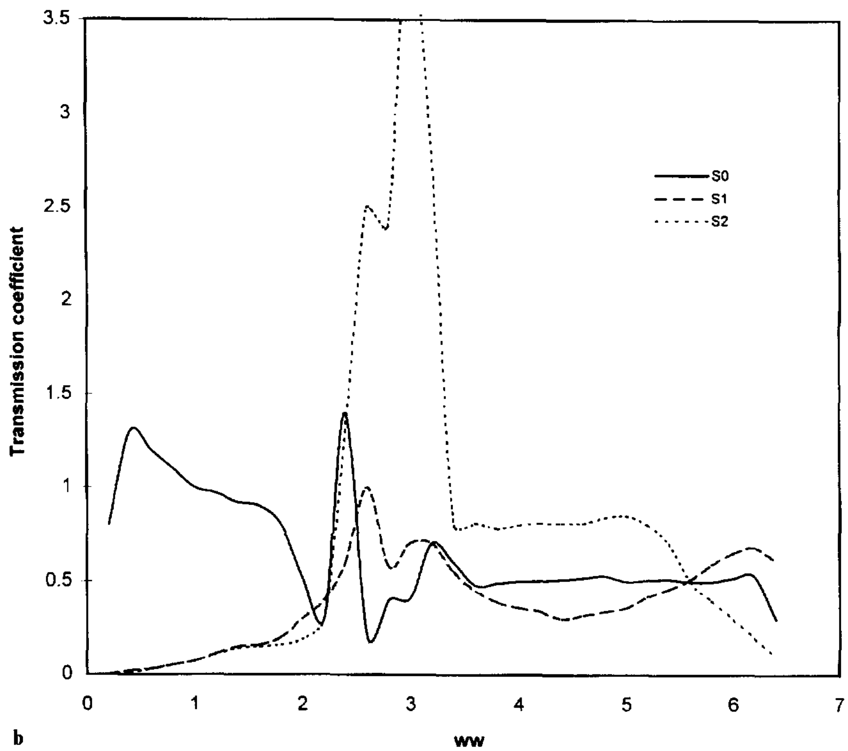

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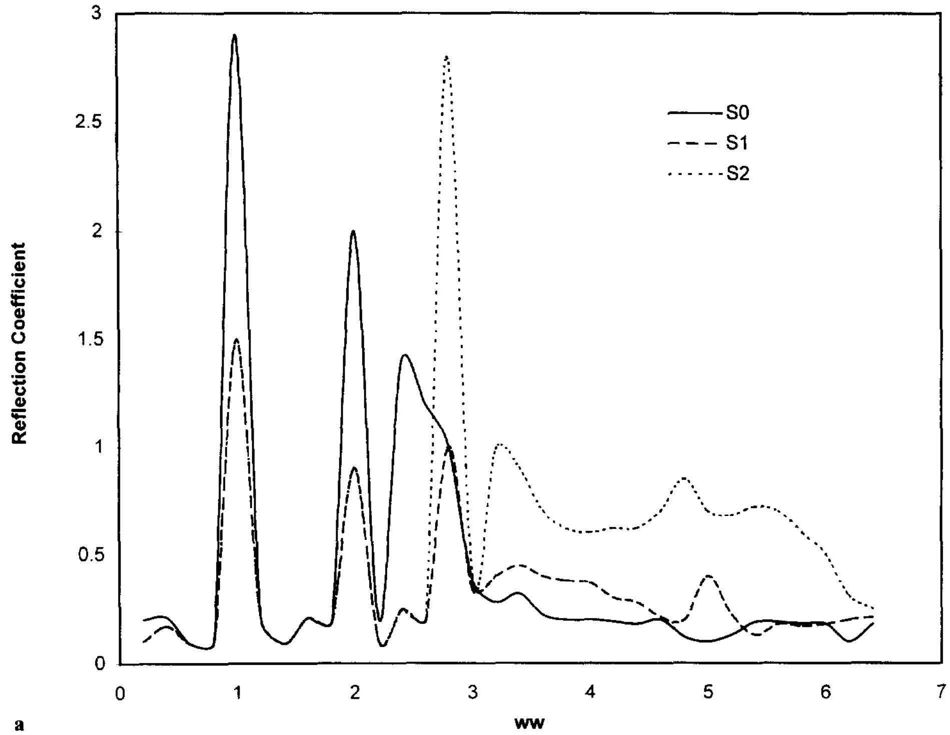
Fig. 3. (a) The reflection coefficient of the normal modes for the incident of the S0 mode. (b) The transmission coefficient of the normal modes for the incident of the S0 mode.

and transmission coefficients of normal modes can be more easily calculated than using the Green's functions in the unbounded medium. These curves of the coefficients against frequency may be very useful in practical nondestructive testing. However, the inversion of the characters of the crack (orientation, size) needs further investigation.



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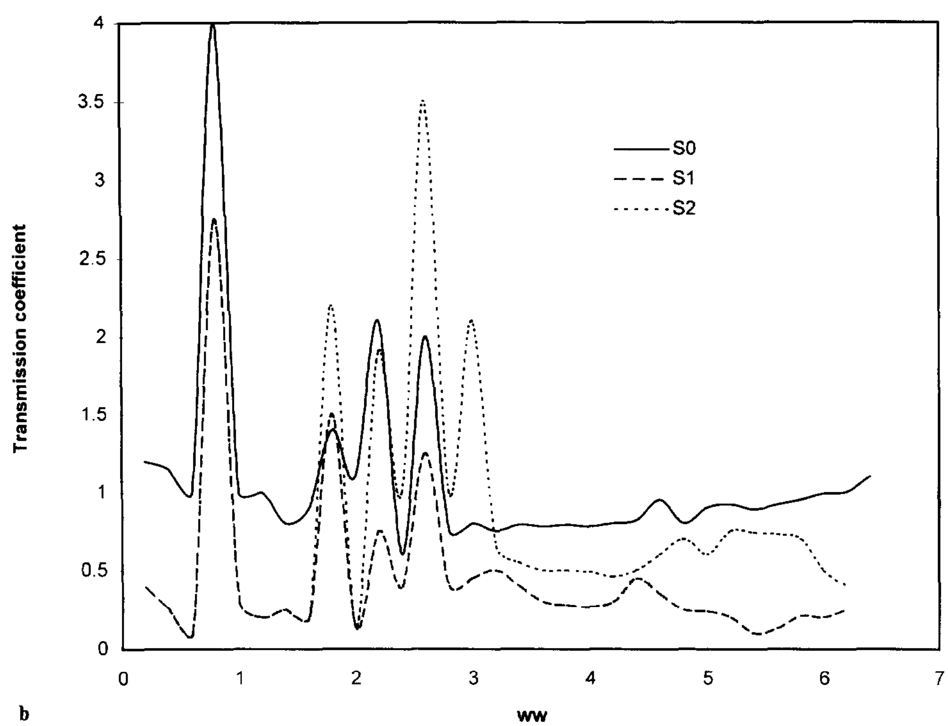

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Fig. 4. (a) The reflection coefficient of normal modes for the incident of the S0 mode. (b) The transmission coefficient of normal modes for the incident of the S0 mode.

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Appendix A

The components of the Green's tensor in a plate can be written as [3]:

$$U_{ij}(r,r')=\frac{j}{\mu}\sum_{s,a}\frac{k_t}{k_{s,a}}U^{(\mp)}_{is,a}(r)U^{(\pm)}_{js,a}(r'),$$

$$P_{ij}(r,r')=\frac{j}{\mu}\sum_{s,a}\frac{k_t}{k_{s,a}}U^{(\mp)}_{is,a}(r)D^{(\pm)}_{js,a}(r').$$

The components of the eigenfunctions have the form:

$$U^{(\pm)}_{1s}(r)=(\pm)jN_s\left[\frac{k_s^2}{k_t^2}\frac{\operatorname{ch}q_s z}{\operatorname{ch}q_s d}-\frac{2k_s^2-k_t^2}{2k_t^2}\frac{\operatorname{ch}r_s z}{\operatorname{ch}r_s d}\right]e^{(\pm)jk_s x},$$

$$U^{(\pm)}_{2s}(r)=N_s\frac{k_s}{k_t}\left[\frac{q_s}{k_t}\frac{\operatorname{sh}q_s z}{\operatorname{ch}q_s d}-\frac{2k_s^2-k_t^2}{2k_t r_s}\frac{\operatorname{sh}r_s z}{\operatorname{ch}r_s d}\right]e^{(\pm)jk_s x},$$

$$U^{(\pm)}_{1a}(r)=N_a\frac{k_a}{k_t}\left[\frac{2k_a^2-k_t^2}{k_t q_a}\frac{\operatorname{sh}q_a z}{\operatorname{ch}q_a d}-\frac{r_a}{k_t}\frac{\operatorname{sh}r_a z}{\operatorname{ch}r_a d}\right]e^{(\pm)jk_a x},$$

$$U^{(\pm)}_{2a}(r)=(\pm)jN_a\left[\frac{k_a^2}{k_t}\frac{\operatorname{ch}r_a z}{\operatorname{ch}r_a d}-\frac{2k_a^2-k_t^2}{2k_t^2}\frac{\operatorname{ch}q_a z}{\operatorname{ch}q_a d}\right]e^{(\pm)jk_a x},$$

where

$$N_{s,a}=\frac{\sqrt{k_t\operatorname{ch}q_{s,a}d\operatorname{ch}r_{s,a}d}}{\Delta'_{s,a}},\quad \Delta'_{s,a}=\left[\frac{\partial\Delta_{s,a}}{\partial(k^2)}\right]_{k=k_{s,a}},$$

$$q_{s,a}=\sqrt{k_{s,a}^2-k_l^2},\quad r_{s,a}=\sqrt{k_{s,a}^2-k_t^2},$$

$$\Delta_s=4qk^2\operatorname{sh}qd\operatorname{ch}rd-\frac{(2k^2-k_t^2)^2}{q}\operatorname{sh}rd\operatorname{ch}qd,$$

$$\Delta_a=4rk^2\operatorname{sh}rd\operatorname{ch}qd-\frac{(2k^2-k_t^2)^2}{q}\operatorname{sh}qd\operatorname{ch}rd,$$

$$q=\sqrt{k^2-k_l^2},\quad r=\sqrt{k^2-k_t^2},$$

where k_s and k_a are the roots of equations $\Delta_s(k)$, $\Delta_a(k)$ and $\operatorname{Im}(k_{s,a})\geq 0$.

$$D^{(\pm)}_{is,a}(r')=\left(\lambda\delta_{ij}\frac{\partial U^{(\pm)}_{is,a}(r')}{\partial x'_l}+\mu\left(\frac{\partial U^{(\pm)}_{is,a}(r')}{\partial x'_j}\frac{\partial U^{(\pm)}_{is,a}(r')}{\partial x'_i}\right)\right)n_j(r').$$

The weighing dyadic Z_{ij} an be chosen to be:

$$\mathbf{Z}=\begin{bmatrix}\dfrac{1}{\sqrt{k_t|r-r_c|}} & 0\\ 0 & \dfrac{1}{\sqrt{k_t|r-r_c|}}\end{bmatrix}$$

if $r\in C_+$, C_-, and

$$\mathbf{Z}=\begin{bmatrix}\dfrac{\mu k_t}{4(1-\nu)\sqrt{k_t|r-r_c|}} & 0\\ 0 & \dfrac{\mu k_t}{4\sqrt{k_t|r-r_c|}}\end{bmatrix}$$

if $r\in C_w$. r_c is the point (x_0, z_0) or (x_1, z_1) which is closest to r.

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