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BOUNDARY CONTROL OF MKDV-BURGERS EQUATION *

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Abstract: The boundary control of MKdV-Burgers equation was considered by feedback control on the domain [0,1]. The existence of the solution of MKdV-Burgers equation with the feedback control law was proved. On the base, priori estimates for the solution was given. At last, the existence of the weak solution of MKdV-Burgers equation was proved and the global-exponential and asymptotic stability of the solution of MKdV-Burgers equation was given.

Key words: boundary control; feedback control law; MKdV-Burgers equation

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Introduction

Because of extensively applicable background of boundary control which is described by nonlinear PDE, the boundary control of nonlinear PDE attracts more attention of Physics, Mathematics and control engineers. With the periodic boundary conditions, some results were obtained by Russell and Zhang[1]. With the domain of the equation being whole real line, some results were obtained by Naumkin[2]. Weijiu-Liu and Krstic[3] attained some results of boundary control and numerical analysis of KdV equation by feedback control. On the above results, we will consider boundary control of MKdV-Burgers equation by feedback control.

In this work, we will consider boundary control of the following MKdV-Burgers equation:

$$u_t - \varepsilon u_{xx} + u_{xxx} + 6u^2 u_x = 0,\ 0 < x < 1,\ t > 0, \tag{1}$$

$$u(0,t) = u_x(1,t) = 0, \quad u(x,0) = u^0(x),\ 0 < x < 1,\ t > 0, \tag{2}$$

$$u_{xx}(1,t) = k_1 u(1,t)^3 + k_2 u(1,t),\ t > 0. \tag{3}$$

Definition 1 *If a function $u(x,t) \in C([0,T], H_0^1(0,1))$ satisfies the following equation*

$$\begin{aligned}&\langle u_t(t), \varphi\rangle - \varepsilon\langle u_x, \varphi_x\rangle - \langle u_x, \varphi_{xx}\rangle - \langle uu_x, \varphi\rangle \\ &= \langle [k_1 u(1,t)^3 + k_2 u(1,t)], \varphi(1,t)\rangle - \langle u^0, \varphi(0)\rangle,\end{aligned} \tag{4}$$

$u(x,t)$ is called a weak solution of Eqs.(1)–(3).

The rest part of this work is organized as follows. Section 1 is given the main theorems and main denotes. Section 2 is devoted to existence and stability of solution of system described by Eqs.(1)–(3).

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1 Main Signals and Main Theorems

Suppose $V(t)=\int_0^1 u_x(x,t)^2dx$ is high order energy, $H^s(0,1)$ is usual Sobolev space. For arbitrary $s\in R$, set $H_0^1=\{\varphi\in H^1(0,1):\varphi(0)=0\}$, $H_0^2(0,1)=\{\varphi\in H^2(0,1):\varphi(0)=\varphi_x(1)=0\}$, $H_0^3(0,1)=\{\varphi\in H^3(0,1):\varphi(0)=\varphi_x(1)=\varphi_{xx}(1)=0\}$. Let X is a Banach space, $\|\cdot\|$ and $(\cdot,\cdot)$ respectively denote norm and scalar product on $L^2(0,1)$. We define operator L, $Lu=-u_{xxx}+\varepsilon u_{xx}$, whose domain is $D(L)=H_0^3(0,1)$. So we may easily prove adjoint operator L^* of L given by $L^*u=u_{xxx}+\varepsilon u_{xx}$ with the domain $D(L^*)=\{u\in H_0^3(0,1)\}$. It is easa to be known that operators L and L^* are close, dissipative, dense linear operator on the domain $L^2(0,1)$ and L is infinitesimal generator of C_0 contraction semigroup on the domain $L^2(0,1)$. Suppose

$$W=\{\omega\in C^2([0,T];H_0^1(0,1)):\omega(x,0)=u^0(x),\omega_t(x,0)=\varepsilon u^0_{xx}(x)-u^0_{xxx}(x)-6u^{0^2}(x)u^0_x(x)\},$$

$$\|\omega\|_W=[\max_{0\le t\le T}(\|\omega_x(t)\|^2+\|\omega_{xt}(t)\|^2+\|\omega_{xtt}(t)\|^2)]^{\frac{1}{2}}.$$

W is a Banach space obviously.

Theorem 1 (1) *For initial value $u^0(x)\in H_0^1(0,1)$, there exists a unique weak solution u of Eqs.(1)–(3), which u satisfies the following L^2 global-exponential stability estimate:*

$$\|u(t)\|^2\le\left\|u^0\right\|^2 e^{-2\varepsilon t},\ \forall t\ge 0, \tag{5}$$

and the H^1 global-asymptotic and semiglobal-exponential stability estimate is got as follows:

$$\begin{aligned}\max_{0\le x\le 1}u(x,t)^2&\le\|u_x(t)\|^2\\&\le M_1\left\|u_x^0\right\|^2\exp(M_2\left\|u^0\right\|^2)e^{-\varepsilon t},\ \forall t\ge 0,\end{aligned} \tag{6}$$

where M_1 and M_2 are positive real constants.

(2) *For the initial value $u^0(x)\in H_0^3(0,1)$ which satisfies compatibility condition $u^0_{xx}(1)=k_1u^0(1)^3+k_2u^0(1)$, there exists a global classical solution u satisfying:*

$$u\in C([0,T];H_0^3(0,1))\cap C^1([0,T];H_0^1(0,1))$$

and

$$\|u\|^2_{H^3}\le M_3\left\|u^0\right\|^2_{H^3}\exp[M_4F(\left\|u^0\right\|^2_{H^3})]e^{-\varepsilon t}. \tag{7}$$

Theorem 2 *Set $\omega=\frac{\varepsilon+8}{8},k=\max\{2,(\frac{12}{\varepsilon+8})^{\frac{\omega}{\varepsilon}}\},F(r)=r^2+r^{4+\frac{4\omega}{\varepsilon}}$.*

(1) *For initial value $u^0(x)\in H_0^1(0,1)$, there exists a solution u of Eqs.(1)–(3), which u satisfies the following global-asymptotic and semiglobal-exponential stability estimate:*

$$\|u(t)\|^2\le K(\left\|u^0\right\|^2+\left\|u^0\right\|^{2+\frac{2\omega}{\varepsilon}})e^{-2\omega t},\ \forall t\ge 0, \tag{8}$$

and

$$\begin{aligned}\max_{0\le x\le 1}u(x,t)^2&\le\|u_x(t)\|^2\\&\le cF(\left\|u^0\right\|_{H^1})\exp[cF(\left\|u^0\right\|)]e^{-\omega t},\ \forall t\ge 0.\end{aligned} \tag{9}$$

(2) *For the initial value $u^0(x)\in H_0^2(0,1)\cap H^3(0,1)$ which satisfies compatibility condition $u^0_{xx}(1)=k_1u^0(1)^3+k_2u^0(1)$, there exists a global classical solution u satisfying the following global-asymptotic and semiglobal-exponential stability estimate:*

$$\|u\|^2_{H^3}\le c\sum_i^3F^i(\left\|u^0\right\|_{H^3})\exp[cF(\left\|u^0\right\|)]e^{-\omega t},\ \forall t\ge 0. \tag{10}$$

2 Proof of Theorems

In order to prove Theorem 1 and Theorem 2, we first verify the following Lemma 1.

Lemma 1 *Assume* $u^0(x) \in H_0^2(0,1) \cap H^7(0,1)$ *and* $u_{xx}^0(1) = k_1 u^0(1)^3 + k_2 u^0(1)$, *then there exists a unique local classical solution* u *of Eqs.*(1)–(3) *when* $T > 0$.

Proof First, we verify that for an arbitrary fixed $\omega \in W$, the following equations have a solution $u \in W$:

$$u_t - \varepsilon u_{xx} + u_{xxx} + 6\omega^2 \omega_x = 0, \quad 0 < x < 1, \quad t > 0, \tag{11}$$

$$u(0,t) = u_x(1,t) = 0, \quad u(x,0) = u^0, \quad u_{xx}(1,t) = k_1 u(1,t)^3 + k_2 u(1,t), \quad t > 0. \tag{12}$$

Define nonlinear transform A such that$A\omega = u$. If we can show that A has a unique fixed point u^* by Banach fixed theorem, then we have that u^* is the unique solution of Eqs.(1)–(3). Set

$$\psi = \frac{1}{2} x(x-1)^2 [k_1 \omega^3(1,t) + k_2 \omega(1,t)], \quad \upsilon = u - \psi.$$

From the definition of operator L, Eqs.(11) and (12) are transformed to an abstract Cauchy problem as follows:

$$\upsilon_t = L\upsilon + f, \quad \upsilon(0) = \upsilon^0.$$

Assume ω, f are differential on both x and t and the linear operator$L$ on $L^2(0,1)$ is infinitesimal of C_0 semigroup of contraction. According to semigroup theory, Eqs.(11) and (12) have a unique solution $\upsilon \in C^1([0,T]; L^2(0,1)) \cap C([0,T]; H_0^3(0,1))$. Then it can be drawn that Eqs.(11) and (12) exist a unique solution $u = \upsilon + \psi \in C^1([0,T]; L^2(0,1)) \cap C([0,T])$.

Further suppose that if initial value u^0 is sufficiently regular,u is also sufficiently regular. Assume that $\omega, u^0$are sufficiently smooth and $u_{1,} u_2$ are respectively solutions of Eqs.(11) and (12) according to $\omega_{1,} \omega_2$ and initial value u_1^0, u_2^0. Set

$$z = u_1 - u_2, \quad z^0 = u_1^0 - u_2^0, \quad \eta = \omega_1 - \omega_2.$$

Then

$$z_t - \varepsilon z_{xx} + z_{xxx} + 6\omega_1^2 \eta_x + 6\eta(\omega_1 + \omega_2)\omega_{2x} = 0, \quad 0 < x < 1, \quad 0 < t < T. \tag{13}$$

Assume $C(s_1, s_2)$is a general continuous linear function. For the general function $\phi = \phi(x,t)$, define

$$\begin{aligned} \|\phi\|_\infty &= \max_{0 \le x \le 1, 0 \le t \le T} |\phi(x,t)|, \\ \|\phi\|_{0,\infty} &= \max_{0 \le t \le T} \|\phi(t)\|, \\ \|\phi\|_{1,\infty} &= \|\phi_t\|_\infty + \max_{0 \le t \le T} \|\phi_x(t)\|. \end{aligned}$$

For Eq.(13) integrating by part, it can be gotten:

$$\begin{aligned} \frac{d}{dt} \int_0^1 z(t)^2 dx \quad &= 2 \int_0^1 z[\varepsilon z_{xx} - z_{xxx} - 6\omega_1^2 \eta_x - 6\eta(\omega_1 + \omega_2)\omega_{2x}] dx \\ &\le (\|\eta\|_\infty^2 + \|\eta_x\|_{0,\infty}^2) C(\|\omega_1\|_W, \|\omega_2\|_W). \end{aligned}$$

It means that

$$\|z(t)\|^2 \le T \|\eta\|_W C(\|\omega_1\|_W, \|\omega_2\|_W) + \left\|z^0\right\|^2.$$

Analogue to the above, it can be drawn:

$$\|z_x\|^2 \le T\,\|\eta\|_W \cdot C(\|\omega_1\|_W\,, \|\omega_2\|_W) + \left\|z_x^0\right\|^2. \tag{14}$$

So they can be also concluded the same results of $\|z_{xt}\|^2$, $\|z_t\|^2$, $\|z_{tt}\|^2$ and $\|z_{xtt}\|^2$.

For $\omega \in W, \varphi^0 \in H_0^2(0,1) \cap H^7(0,1)$ and utilizing dense theorem, the solution of Eqs.(11) and (12) satisfies $u \in C^2([0,T], H_0^1(0,1))$. Moreover, since

$$u(x,0) = u^0(x),\ \ u_t(x,0) = \varepsilon u_{xx}^0(x) - u_{xxx}^0(x) - 6(u^0(x))^2 u_x^0(x),$$

it can be drawn $u \in W$ and A: $W \to W$, which W satisfies Eq.(14). Supposing $\omega_2 = 0, u_2^0 = 0$, we have

$$\begin{aligned}\|A\omega\|_W^2 \le{} & \|u(0)\|^2 + \left\|u_x^0(0)\right\|^2 + \|u_t(0)\|^2 + \|u_{xt}(0)\|^2 \\ & + \|u_{tt}(0)\|^2 + \|u_{xtt}(0)\|^2 + T\,\|\omega\|_w^2\, C(\|\omega\|_W) \\ \le{} & R^2 + T\,\|\omega\|_W^2\, C(\|\omega\|_W)\end{aligned}$$

where $R = R(\|u^0\|_{H^7})$ is a positive real constant. Moreover , letting $B(0,2R) = \{\omega \in W : \|\omega\|_W \le 2R\ \}$ and T is sufficiently small. Then $\|A\omega\|_W^2 \le R^2 + TR^2C(R) \le 4R^2$ holds . Then we have $A : B(0,2R) \to B(0,2R)$. Further, assuming T being sufficiently small to make $TC(R) < 1$, it can be drawn that A is a contraction injection. So utilizing Banach fixed point theorem, A has a unique fixed point $u^* \in W$. Then for T being sufficiently small, there exists a unique solution $u^*$of Eqs.(1)–(3). From the above, the proof of Lemma 1 is finished.

The proof of Theorem 1 and Theorem 2

In order to prove the existence and uniqueness of the solution, we first give priori estimates for the solution of Eq.(1).

(1) Stability estimate

From Lemma 1, the systems (1)–(3) have a local solution u. Inequality (5) can easily be attained by energy inequality. In fact, we have

$$\begin{aligned}\frac{d}{dt}\int_0^1 u^2 dx ={}& 2\int_0^1 u u_t dx \\ \le{}& -2\varepsilon\,\|u_x\|^2 \\ \le{}& -2\varepsilon \int_0^1 u^2 dx.\end{aligned}$$

So $\|u\|^2 \le \left\|u^0\right\| \mathrm{e}^{-2\varepsilon t}$ holds. Therefore Eq.(5) holds.

In the following we will prove Eq.(8) by Lyapunov function $\int_0^1 (x+1)u(t)^2 dt$. From Eq.(1), we have

$$\begin{aligned}\frac{d}{dt}\int_0^1 (x+1)u^2 dx &\le \varepsilon \int_0^1 (u^2 + u_x^2)dx - 2\varepsilon \int_0^1 u_x^2 dx + 3\,\|u_x\|^2 \int_0^1 u^2 dx - 4\int_0^1 u_x^2 dx \\ &\le -(\frac{\varepsilon}{2} + 4)\int_0^1 u_x^2 dx + 3\,\|u_x\|^2 \cdot \left\|u^0\right\| \mathrm{e}^{-2\varepsilon t} \\ &= -(\frac{\varepsilon}{2} + 4 - 3\left\|u^0\right\| \mathrm{e}^{-2\varepsilon t})\int_0^1 u_x^2 dx \\ &= -\frac{\varepsilon + 8 - 6\left\|u^0\right\|^2 \mathrm{e}^{-2\varepsilon t}}{2}\int_0^1 u_x^2 dx.\end{aligned} \tag{15}$$

Let

$$T_0 = \begin{cases} 0, & \left\|u^0\right\|^2 \le \frac{1}{12}(\varepsilon+8), \\ \frac{1}{2\varepsilon}\ln(\frac{12\left\|u^0\right\|^2}{\varepsilon+8}), & \left\|u^0\right\|^2 > \frac{1}{12}(\varepsilon+8). \end{cases}$$

It can be attained

$$\begin{aligned} \int_0^1 u^2dx &\le \int_0^1 (x+1)u^2dx \\ &\le 2\left\|\mu^0\right\| \mathrm{e}^{-2\varepsilon T_0}e^{-2\omega t}, \quad \forall t \ge T_0, \\ \int_0^1 u^2dx &\le \left\|u^0\right\|^2 \mathrm{e}^{-2\varepsilon t} \\ &\le (\frac{12}{\varepsilon+8})^{\frac{\omega}{\varepsilon}} \left\|u^0\right\|^{2+\frac{2\omega}{\varepsilon}} \mathrm{e}^{-2\omega t}, \quad 0 \le t \le T_0. \end{aligned} \tag{16}$$

Let $K = \max\{2, (\frac{12}{\varepsilon+8})^{\frac{\omega}{\varepsilon}}\}$. From Eq.(16), Eq.(8) can be attained. To prove Eq.(9), let us to estimate the following function:

$$B(t) = k_2u(1,t)^2 + (k_1+6)u(1,t)^4 + (\varepsilon+4)\int_0^1 u_x^2dx + u_x^2(0,t).$$

From Eq.(15), we attain

$$\begin{aligned} \frac{d}{dt}\int_0^1 (x+1)u^2dx + B(t) &\le 3\int_0^1 u^4dx + \varepsilon\int_0^1 u^2dx \\ &\le 3(\int_0^1 u^2dx)^2 + \varepsilon\int_0^1 u^2dx. \end{aligned} \tag{17}$$

To multiply Eq.(17) by $\mathrm{e}^{\omega t}$, we have

$$\frac{d}{dt}(\mathrm{e}^{\omega t}\int_0^1 (x+1)u^2dx) + \mathrm{e}^{\omega t}B(t) \le C(\left\|u^0\right\|^2 + \left\|u^0\right\|^{4+\frac{4\omega}{\varepsilon}})\mathrm{e}^{-\omega t}, \tag{18}$$

where $C = C(\varepsilon, \omega)$ is general continuous linear function. Integrating Eq.(18) from 0 to ∞, so

$$\int_0^\infty \mathrm{e}^{\omega s}B(s)ds \le cF(\left\|u^0\right\|). \tag{19}$$

From the definition of $V(t)$, $u(x,t)^2 = (\int_0^x u_xdx)^2 \le x\int_0^1 u_x^2dx \le V(t)$ and

$$\begin{aligned} 12\int_0^1 u^2u_xu_{xx}dx &\le \frac{18}{\varepsilon}\int_0^1 u^4u_x^2dx + 2\varepsilon\int_0^1 u_{xx}^2dx \\ &\le \frac{18}{\varepsilon}V(t)^3 + 2\varepsilon\int_0^1 u_{xx}^2dx, \end{aligned}$$

we attain

$$\dot{V}(t) \le 2k_1^2V(t)u(1,t)^4 + \frac{18}{\varepsilon}V(t)^3 + 2k_2^2V(t). \tag{20}$$

Multiplying Eq.(20) by $\mathrm{e}^{\omega t}$, we have

$$\frac{d}{dt}(\mathrm{e}^{\omega t}V(t)) \le \omega\mathrm{e}^{\omega t}V(t) + [2k_1^2u(1,t)^4 + \frac{18}{\varepsilon}V(t)^2 + 2k_2^2]V(t)\mathrm{e}^{\omega t}. \tag{21}$$

To integrate Eq.(21) from 0 to t and to utilize Eq.(19), it can be drawn

$$e^{\omega t}V(t) \le cF(\left\|u^0\right\|_{H^1})\exp[cF(\left\|u^0\right\|)].$$

From the embedding theorem, it can be drawn

$$\|u_x(t)\|^2 \le cF(\left\|u^0\right\|_{H^1})\exp[cF(\left\|u^0\right\|)]e^{-\omega t}, \quad \forall t \ge 0.$$

So Eq.(9) holds. Analogue to the above, we have

$$\begin{aligned}\|u_t\|^2 &\le 2\,\|u_t(0)\|^2 \exp(\int_0^t 162\,\|u_x(s)\|^4\,ds)e^{-\omega t}\\ &\le cF(\left\|u^0\right\|_{H^3})\exp(cF^2(\left\|u^0\right\|))e^{-\omega t}.\end{aligned} \tag{22}$$

So we have

$$\begin{aligned}\|u_{xx}\|^2 &\le c(\|u_t\|^2 + \|u_x\|^2 + \|u_x\|^4 + \|u_x\|^6 + \|u_x\|^8)\\ &\le c\sum_{i=1}^{4} F^i(\left\|u^0\right\|_{H^3})\exp[cF^2(\left\|u^0\right\|)]e^{-\omega t}.\end{aligned} \tag{23}$$

Analogue to Eq.(23) and utilizing Eq.(1), we can get that

$$\begin{aligned}\|u_{xxx}\|^2 &\le c(\|u_{xx}\|^2 + \|u_t\|^2 + \|u_x\|^6)\\ &\le c\sum_{i=1}^{4} F^i(\left\|u^0\right\|_{H^3})\exp[cF^2(\left\|u^0\right\|)]e^{-\omega t}.\end{aligned}$$

Hence，we have

$$\begin{aligned}\left\|u^0\right\|_{H^3}^2 &= \|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2\\ &\le 3\,\|u_{xx}\|^2 + \|u_{xxx}\|^2\\ &\le c\sum_{i=1}^{4} F^i(\left\|u^0\right\|_{H^3})\exp[cF^2(\left\|u^0\right\|)]e^{-\omega t}.\end{aligned}$$

It means Eq.(10) holds.

Analogue to the same procedure, we can obtain Eqs.(6) and (7).

(2) Continuous dependence of solutions with respect to initial value

From Eqs.(5), (6), (8), (9), we show that estimates (5)–(10) of the solution hold. In the following part we must establish the continuous dependence of solutions with respect to initial value.

To replace ω_1, ω_2 and η in Eq.(11) by u_1, u_2 and z, we have that

$$z_t - \varepsilon z_{xx} + z_{xxx} + 6u_1^2 z_x + 6z(u_1+u_2)u_{2x} = 0, \quad 0 < x < 1, \quad 0 < t < T.$$

Then we have that

$$\begin{aligned}\frac{d}{dt}\int_0^1 z(t)^2dx =& 2\int_0^1 z\cdot z_t dx\\ =& 2\int_0^1 z[\varepsilon z_{xx} - z_{xxx} - 6u_1^2 z_x - 6z(u_1+u_2)u_{2x}]dx\\ \le& -2z(1,t)z_{xx}(1,t) - z_x^2(0,t) - 2\varepsilon\,\|z_x\|^2 + 2\,\|z\|\,(6\left\|u_1^2\right\|\|z_x\|\\ &+ 6\,\|z\|\,\|u_1+u_2\|\,\|u_{2x}\|).\end{aligned}$$

From Eq.(19), we obtain that

$$\begin{aligned}\|u_1(t)-u_2(t)\|^2 &\leq 6\left\|u_1^0-u_2^0\right\|^2\exp(c\int_0^t\|u_{1x}(s)\|^2+\|u_{2x}(s)\|^2)ds)\\&\leq 6C(\left\|u_1^0\right\|,\left\|u_2^0\right\|)\left\|u_1^0-u_2^0\right\|^2,\quad t\geq 0.\end{aligned}\tag{24}$$

Also we have

$$\int_0^\infty(z_x(0,t)^2+\|z_x(t)\|^2)dt\leq C(\left\|u_1^0\right\|,\left\|u_2^0\right\|)\left\|u_1^0-u_2^0\right\|^2.\tag{25}$$

Moreover, we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 z_x^2(t)dx=&2\int_0^1 z_{xx}(-\varepsilon z_{xx}+z_{xxx}+6u_1^2z_x+6z(u_1+u_2)u_{2x})dx\\\leq& z_{xx}^2(1,t)-z_{xx}^2(0,t)-2\varepsilon\|z_{xx}\|^2\\&+2\|z_{xx}\|(\left\|u_1^2\right\|\|z_x\|+\|u_{2x}\|\|z\|(\|u_1+u_2\|)).\end{aligned}\tag{26}$$

Utilizing Eqs.(19) and (25), we have that

$$\|u_{1x}(t)-u_{2x}(t)\|^2\leq C(\left\|u_1^0\right\|,\left\|u_2^0\right\|)\left\|u_1^0-u_2^0\right\|^2.\tag{27}$$

Further, we obtain that

$$\begin{aligned}\frac{d}{dt}\int_0^1 z_t(t)^2dx=&2\int_0^1 z_t\cdot z_{tt}dx\\\leq&-2z_t(1,t)z_{xxt}(1,t)-z_{xt}^2(0,t)-2\varepsilon\|z_{xt}\|^2\\&+2\|z_t\|(6\left\|u_{1t}^2\right\|\|z_{xt}\|+6\|z_t\|\|u_{1t}+u_{2t}\|\|u_{2xt}\|)\\\leq& c\|z\|^2\sum_{i=1}^2(\|u_{it}\|^2+\|u_{ixt}\|^2)+c\|z_x\|^2\sum_{i=1}^2\|u_{it}\|^2\\&+c\|z_t\|^2\sum_{i=1}^2(\|u_{ix}\|^2+\|u_{ix}\|^2).\end{aligned}\tag{28}$$

Further we show

$$\int_0^\infty(\sum_{i=1}^2\|u_{ixt}\|^2)dt\leq C(\left\|u_1^0\right\|_{H^3},\left\|u_2^0\right\|_{H^3}).$$

Therefore from Eqs.(10), (22), (24), (25), (28), we obtain that

$$\|u_{1x}(t)-u_{2x}(t)\|^2\leq C(\left\|u_1^0\right\|_{H^3},\left\|u_2^0\right\|_{H^3})\left\|u_1^0-u_2^0\right\|_{H^3}^2,\quad t\geq 0.\tag{29}$$

So we have that

$$\|u_1(t)-u_2(t)\|_{H^3}^2\leq C(\left\|u_1^0\right\|_{H^3},\left\|u_2^0\right\|_{H^3})\left\|u_1^0-u_2^0\right\|_{H^3}^2,\quad t\geq 0.\tag{30}$$

(3) The proof of existence and uniqueness of local weak solution and classical solution

For $u^0(x)\in H_0^1(0,1)$ utilizing Eqs.(27), (29), (30) and density argument, it can be proved that there exists a local weak and local classical solution of system described by Eqs.(1)–(3). Moreover for $u^0(x)\in H_0^2(0,1)\cap H^3(0,1)$, it can be proved that there exists a local classical solution of system described by Eqs.(1)–(3). The uniqueness of solution is directly drawn by Eq.(27).

(4) Existence and uniqueness of global weak solution and classical solution

From Eqs.(5), (8), (9), we obtain that the solution doesn't blow up in finite time and the local solution can be continued to infinity. So we have the existence and uniqueness of global weak and classical solution. From now on, we have finished all proofs.

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