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Global Pointwise Error Estimates for Uniformly Convergent Finite Element Methods for the Elliptic Boundary Layer Problem

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Abstract—This paper continues our discussion for the anisotropic model problem $-(\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + a(x,y)u = f(x,y)$ in [1]. There we constructed a bilinear finite element method on a Shishkin type mesh. The method was shown to be convergent, independent of the small parameter ε, in the order of $N^{-2}\ln^2 N$ in the L^2-norm, where N^2 is the total number of mesh points. In this paper, the method is shown to be convergent, independent of ε, in the order of $N^{-2}\ln^3 N$ in the L^∞-norm in the whole computational domain, which explains the uniform convergence phenomena we found in the numerical results in [1]. Another numerical experiment is presented here, which confirms our theoretical analysis. Published by Elsevier Science Ltd.

Keywords—Finite element methods, Singularly perturbed problems, Elliptic partial differential equations, Pointwise error estimates.

1. INTRODUCTION

In this paper, we will consider a uniformly convergent finite element method (FEM) for the problem:

$$L_\varepsilon u \equiv -\left(\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + a(x,y)u = f(x,y), \qquad \text{on } \Omega \equiv (0,1)\times(0,1), \tag{1}$$

$$u = g(x,y), \qquad \text{on } \partial\Omega, \tag{2}$$

where $\varepsilon \in (0,1]$ is a small positive parameter. The functions a, f, and g are assumed to be sufficiently smooth on Ω, with

$$a(x,y) \geq \alpha^2 > 0, \qquad \text{on } \Omega.$$

This problem is a typical model of singularly perturbed (SP) problems [2,3]. Usually these problems will have sharp boundary layers or interior layers. For example, problem (1),(2) has elliptic boundary layers at sides $x = 0$ and $x = 1$. Because of these boundary layers or interior layers, classical methods do not work well for these problems. As we know, the classical FEM generally gives the following global error estimates:

$$|||u - u_h|||_\Omega \leq Ch^m ||u||_{H^n(\Omega)},$$

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where $H^n(\Omega)$ is the usual Sobolev space [3] with norm $||\cdot||_{H^n(\Omega)}$ and $|||\cdot|||_\Omega$ denotes some norm on Ω. But for the SP problems, we have $||u||_{H^n(\Omega)} \le C\varepsilon^{-k}$ (see, e.g., [11]), where k is a positive integer. Hence, to ensure the global convergence, the mesh size h must be less than or equal to ε^p, where p is a positive number, which is impossible in practice, since ε can be as small as 10^{-10}. Therefore, many special methods have been investigated, such as *Adaptive FEM* [6], *hp-FEM* [15], and *Streamline Diffusion FEM* [8], to name but a few. For more details, see [2, Chapter 5,6,12,13, and 21, Chapter 12]. Among them, the most robust method is the globally uniform convergent (GUC) method. By GUC, we mean that the error between the analytic solution u and the computed FEM solution u_h satisfies:

$$|||u-u_h|||_\Omega \le Ch^m$$

for some positive constant C that is independent of ε and h.

Recently, we obtained GUC error estimates in L^2-norm by FEM for some SP problems [1,14,15]. In [1], we constructed a bilinear FEM on a Shishkin type mesh [12,13] for problem (1),(2) and proved that our method is GUC in L^2-norm. Even though the numerical experiments there showed that our method also has global uniform convergence in L^∞ -norm, the theoretical proof was left open [1, p. 21]. In the present paper, we will prove that the global uniform convergence in L^∞-norm actually holds true for our method for problem (1),(2). To the best of our knowledge, many local error estimates were obtained, e.g., [8,16,17]. But such global pointwise error estimates for FEM were only obtained by Guo and Stynes [18] for a model time-dependent convection-diffusion problem in one space dimension and Stynes and O'Riordan [19] for a linear convection-dominated, convection-diffusion problem in two dimensions.

The organization of this paper is as follows. In Section 2, we present the FEM method we used in [1]. Then, in Section 3, we introduce a discrete Green's function and prove that our method is also GUC in the order of $N^{-2}\ln^3 N$ in L^∞-norm, where the total number of mesh points is $O(N^2)$. Finally, numerical results are presented in Section 4, which confirms our theoretical analysis.

Throughout the paper, we will use C, sometimes subscripted, to denote a generic positive constant that is independent of ε and of the mesh size.

2. FINITE ELEMENT METHOD FOR (1)

Without loss of generality, we consider homogeneous boundary conditions, i.e., $g = 0$. The weak formulation of (1) is: find $u \in H_0^1(\Omega)$ such that

$$B(u,v) \equiv \left(\varepsilon^2 u_x, v_x\right) + (u_y, v_y) + (au, v) = (f, v), \qquad \forall\, v \in H_0^1(\Omega)\,, \tag{3}$$

where $(\cdot,\cdot)$ denotes the usual $L^2(\Omega)$ inner product and $H_0^1(\Omega)$ is the usual Sobolev space.

Denote the energy norm

$$|||v||| \equiv \left\{\varepsilon^2\|v_x\|^2 + \|v_y\|^2 + \|v\|^2\right\}^{1/2}, \qquad \forall\, v \in H_0^1(\Omega),$$

where $\|\cdot\|$ denotes the usual L^2 norm. It is easy to see that

$$B(v,v) = \varepsilon^2\|v_x\|^2 + \|v_y\|^2 + (av,v) \ge \min\left(1,\alpha^2\right)|||v|||^2 \tag{4}$$

for any $v \in H_0^1(\Omega)$.

Let N and M be two positive integers. To construct a Shishkin type mesh, we assume N is divisible by 4.

In the y-direction, we discretize $[0,1]$ as $0 = y_0 < y_1 < \cdots < y_M = 1$, where all the mesh sizes $y_j - y_{j-1}$ are in the order of M^{-1}.

In the x-direction, we divide the interval $[0,1]$ into the subintervals

$$[0,\sigma], \qquad [\sigma, 1-\sigma], \qquad [1-\sigma, 1].$$

Then uniform meshes are used on each subinterval, with $N/4$ points on each of $[0,\sigma]$ and $[1-\sigma,1]$, and $N/2$ points on $[\sigma, 1-\sigma]$. Here σ is defined by

$$\sigma = \min\left\{\frac{1}{4}, 2\alpha^{-1}\varepsilon \ln N\right\}.$$

More explicitly, we have

$$0 = x_0 < x_1 < \cdots < x_{i_0} < \cdots < x_{N-i_0} < \cdots < x_N = 1,$$

with $i_0 = N/4$, $x_{i_0} = \sigma$, $x_{N-i_0} = 1-\sigma$, and

$$\begin{aligned} h_i &= 4\sigma N^{-1}, && \text{for } i = 1,\cdots,i_0, N-i_0+1,\cdots,N,\\ h_i &= 2(1-2\sigma)N^{-1}, && \text{for } i = i_0+1,\cdots,N-i_0, \end{aligned}$$

where $h_i = x_i - x_{i-1}$.

Let $S_h(\Omega)$ be the standard bilinear finite element space [20]. Our finite element method is: find $u^h \in S_h$ such that

$$B(u^h, v) \equiv \left(\varepsilon^2 u_x^h, v_x\right) + \left(u_y^h, v_y^h\right) + \left(au^h, v\right) = (f, v), \qquad \forall v \in S_h. \tag{5}$$

Let us express the standard bilinear interpolate of u as

$$\Pi u \equiv \sum_{i=0}^{N}\sum_{j=0}^{M} u_{ij} l_i(x) l_j(y),$$

and denote $\Pi_x u$ and $\Pi_y u$ as the linear interpolate in the x-direction and y-direction, respectively. Here, $l_i(x)$ is the well-known "hat" function [20].

Let $\Omega_c = (\sigma, 1-\sigma)\times(0,1)$ and $\Omega_f = \Omega \setminus \overline{\Omega}_c$, i.e., Ω_c is the part on which the mesh is coarse, while the mesh is fine on Ω_f.

To simplify the notation, we assume $M = N$ in the rest of this paper. It is easy to see that all the results are still true, provided that the ratios N/M and M/N are bounded by some constants.

Following the same proofs as Lemmas 4.1 and 4.2 in [1], we have the following interpolation estimates.

LEMMA 2.1. *For the solution u of (1) and (2), we have*

$$\|u - \Pi u\|_{L^\infty(\Omega_f)} + \|\Pi_x(u_y) - u_y\|_{L^\infty(\Omega_f)} + \|\Pi_y(\varepsilon u_x) - \varepsilon u_x\|_{L^\infty(\Omega_f)} \le CN^{-2}\ln^2 N, \tag{6}$$

$$\|u - \Pi u\|_{L^\infty(\Omega_c)} + \|\Pi_x(u_y) - u_y\|_{L^\infty(\Omega_c)} + \|\Pi_y(\varepsilon u_x) - \varepsilon u_x\|_{L^\infty(\Omega_c)} \le CN^{-2}. \tag{7}$$

REMARK 2.1. To prove Lemma 2.1, we assume that u is sufficiently smooth, i.e., some compatibility conditions are implied [13,21].

3. MAIN RESULTS

Let $G \in S_h$ be the discrete Green's function associated with node (x_i, y_j); i.e., G satisfies

$$B(v, G) = v(x_i, y_j), \qquad \forall\, v \in S_h. \tag{8}$$

By (4), we know that G is uniquely defined. Also, G has the following bounds.

LEMMA 3.1. *For $(x_i, y_j) \in \Omega_c$, we have*

$$||G|| + ||G_y|| + \varepsilon ||G_x|| \le C.$$

PROOF. By (4) and (8),

$$C_1 |||G|||^2 \le B(G, G) = G(x_i, y_j) = -\int_{y_j}^{1} G_y(x_i, y)\, dy. \tag{9}$$

Since $G(x, y)$ is bilinear, we have the following expansion:

$$G_y(x_i, y) = G_y(x, y) + (x_i - x) G_{xy}(x, y). \tag{10}$$

Integrating both sides from σ to $1-\sigma$ and from y_j to 1 with respect to x and y, respectively, we obtain

$$\begin{aligned}(1 - 2\sigma) \int_{y_j}^{1} G_y(x_i, y)\, dy &= \int_{\sigma}^{1-\sigma} \int_{y_j}^{1} G_y(x_i, y)\, dy\, dx \\ &= \int_{\sigma}^{1-\sigma} \int_{y_j}^{1} G_y(x, y)\, dy\, dx + \int_{\sigma}^{1-\sigma} \int_{y_j}^{1} (x_i - x) G_{xy}(x, y)\, dy\, dx.\end{aligned}$$

Note that

$$\begin{aligned}\int_{\sigma}^{1-\sigma} \int_{y_j}^{1} (x_i - x) G_{xy}(x, y)\, dy\, dx &= \sum_{i=i_0+1}^{N-i_0} \int_{x_{i-1}}^{x_i} \int_{y_j}^{1} (x_i - x) G_{xy}(x, y)\, dy\, dx \\ &\le \sum_{i=i_0+1}^{N-i_0} (x_i - x_{i-1}) \int_{x_{i-1}}^{x_i} \int_{y_j}^{1} |G_{xy}(x, y)|\, dy\, dx \\ &\le C \sum_{i=i_0+1}^{N-i_0} \int_{x_{i-1}}^{x_i} \int_{y_j}^{1} |G_y(x, y)|\, dy\, dx \\ &= \int_{\sigma}^{1-\sigma} \int_{y_j}^{1} |G_y(x, y)|\, dy\, dx,\end{aligned}$$

where, in the last inequality, we used the inverse estimate $|G|_{W^{2,1}(\Omega)} \le Ch^{-1} |G|_{W^{1,1}(\Omega)}$ [3, Theorem 3.2.6], where $W^{k,j}(\Omega)$ denotes the usual Sobolev space [4].

Therefore, by the Cauchy-Schwarz inequality, we have

$$C_1 |||G|||^2 \le \frac{C}{1 - 2\sigma} \int_{\sigma}^{1-\sigma} \int_{y_j}^{1} |G_y(x, y)|\, dy\, dx \tag{11}$$

$$\le \frac{C}{1 - 2\sigma} (1 - 2\sigma)^{1/2} (1 - y_j)^{1/2} ||G_y|| \le ||G_y||, \tag{12}$$

where we used the fact that $1/2 \le 1 - 2\sigma < 1$ and $1 - y_j \le 1$, from which our proof finishes. ∎

LEMMA 3.2. *For $(x_i, y_j) \in \Omega_f$, we have*

$$||G|| + ||G_y|| + \varepsilon ||G_x|| \le C \varepsilon^{-1/2} \ln^{1/2} N.$$

PROOF. When $x_i \ge 1 - \sigma$, by (4) and (8),

$$C_1 |||G|||^2 \le B(G, G) = G(x_i, y_j) = -\int_{x_i}^{1} G_x(x, y_j)\, dy.$$

Integrating the expansion $G_x(x, y_j) = G_x(x, y) + (y_j - y)G_{xy}(x, y)$ from x_i to 1 and from 0 to 1 with respect to x and y, respectively, we have

$$\int_{x_i}^1 G_x(x, y_j)\,dx = \int_0^1 \int_{x_i}^1 G_x(x, y)\,dy\,dx + \int_0^1 \int_{x_i}^1 (y_j - y)G_{xy}(x, y)\,dy\,dx.$$

By the same arguments as Lemma 3.1, we can obtain

$$\begin{aligned}\int_0^1 \int_{x_i}^1 (y_j - y)G_{xy}(x, y)\,dy\,dx &= \sum_{j=1}^N \int_{y_{j-1}}^{y_j} \int_{x_i}^1 (y_j - y)G_{xy}(x, y)\,dy\,dx \\ &\le \sum_{j=1}^N (y_j - y_{j-1}) \int_{y_{j-1}}^{y_j} \int_{x_i}^1 |G_{xy}(x, y)|\,dy\,dx \\ &\le C \int_0^1 \int_{x_i}^1 |G_x(x, y)|\,dy\,dx.\end{aligned}$$

Therefore,

$$|||G|||^2 \le C \int_0^1 \int_{x_i}^1 |G_x(x, y)|\,dy\,dx \tag{13}$$

$$\le C(1 - x_i)^{1/2}||G_x|| \le C(\varepsilon \ln N)^{1/2}||G_x||, \tag{14}$$

where we used the fact that $1 - x_i \le \sigma \le C\varepsilon \ln N$.

Similarly, if $x_i \le \sigma$, we have

$$C_1|||G|||^2 \le B(G, G) = G(x_i, y_j) = \int_0^{x_i} G_x(x, y_j)\,dy \tag{15}$$

$$\le C \int_0^1 \int_0^{x_i} |G_x(x, y)|\,dy\,dx \le Cx_i^{1/2}||G_x|| \le C(\varepsilon \ln N)^{1/2}||G_x||. \tag{16}$$

The proof is finished by combining inequalities (13)–(16). ∎

Using Lemmas 3.1 and 3.2, we can obtain the following pointwise error estimates.

THEOREM 3.1.

$$|(u - u_h)(x_i, y_j)| \le CN^{-2} \ln^3 N.$$

PROOF. Note that

$$(u - u_h)(x_i, y_j) = (\Pi u - u_h)(x_i, y_j) = B(\Pi u - u_h, G) \tag{17}$$

$$= B(\Pi u - u, G) \tag{18}$$

$$= \varepsilon^2((\Pi u - u)_x, G_x) + ((\Pi u - u)_y, G_y) + (a(\Pi u - u), G). \tag{19}$$

Furthermore, by Lemma 2.1 and the Cauchy-Schwarz inequality, we have

$$(a(\Pi u - u), G) \le C||\Pi u - u||_{L^\infty(\Omega_f)}||G||_{L^1(\Omega_f)} + C||\Pi u - u||_{L^\infty(\Omega_c)}||G||_{L^1(\Omega_c)} \tag{20}$$

$$\le CN^{-2} \ln^2 N\,(\text{meas } (\Omega_f))^{1/2}\,||G||_{L^2(\Omega_f)} + CN^{-2}||G||_{L^2(\Omega_c)} \tag{21}$$

$$\le CN^{-2} \ln^2 N(\varepsilon \ln N)^{1/2}\varepsilon^{-1/2} \ln^{1/2} N + CN^{-2} \tag{22}$$

$$\le CN^{-2} \ln^3 N. \tag{23}$$

By the very special properties of Π and Lemma 2.1, we have [1]:

$$\begin{aligned}((\Pi u - u)_y, G_y) &= ((\Pi_y \Pi_x u - u)_y, G_y) = ((\Pi_x u - u)_y, G_y) = (\Pi_x(u_y) - u_y, G_y) \\ &\le ||\Pi_x(u_y) - u_y||_{L^\infty(\Omega_f)}||G_y||_{L^1(\Omega_f)} + ||\Pi_x(u_y) - u_y||_{L^\infty(\Omega_c)}||G_y||_{L^1(\Omega_c)} \\ &\le CN^{-2} \ln^2 N(\text{meas } (\Omega_f))^{1/2}||G_y||_{L^2(\Omega_f)} + CN^{-2}||G_y||_{L^2(\Omega_c)} \\ &\le CN^{-2} \ln^3 N,\end{aligned}$$

Similarly,

$$\begin{aligned}\varepsilon^2((\Pi u - u)_x, G_x) &= \varepsilon^2(\Pi_y(u_x) - u_x, G_x) = (\Pi_y(\varepsilon u_x) - \varepsilon u_x, \varepsilon G_x)\\ &\le ||\Pi_y(\varepsilon u_x) - \varepsilon u_x||_{L^\infty(\Omega_f)}||\varepsilon G_x||_{L^1(\Omega_f)} + ||\Pi_y(\varepsilon u_x) - \varepsilon u_x||_{L^\infty(\Omega_c)}||\varepsilon G_x||_{L^1(\Omega_c} \\ &\le CN^{-2}\ln^3 N.\end{aligned}$$

Combining the above inequalities, we have

$$|(u - u_h)(x_i, y_j)| \le CN^{-2}\ln^3 N, \tag{24}$$

which completes our proof. ∎

Table 1. Errors in L^2 norm.

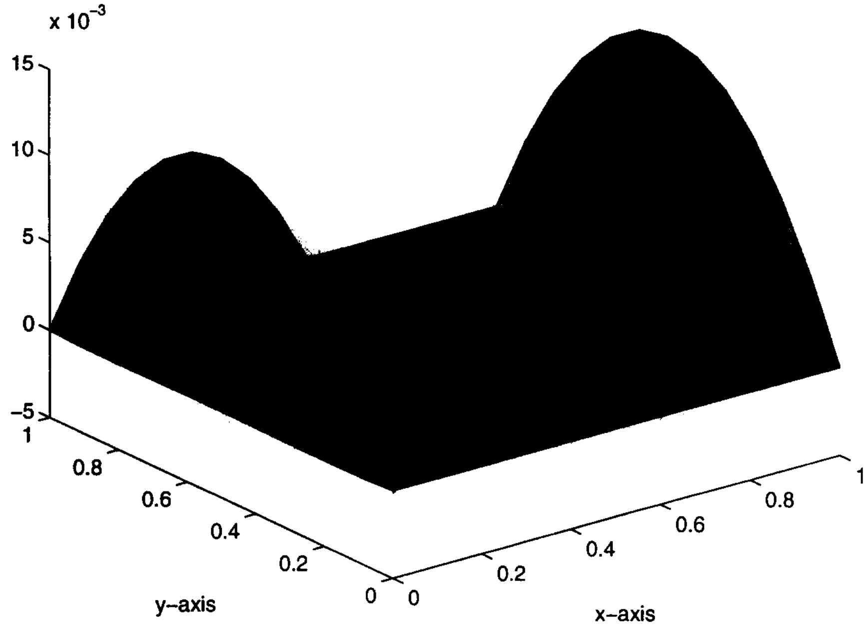
ε	N			
	12	24	48	72
$1.0D-02$	$1.96983D-03$	$7.43811D-04$	$2.55818D-04$	$1.33999D-04$
$1.0D-03$	$8.05838D-04$	$2.52062D-04$	$8.24954D-05$	$4.27799D-05$
$1.0D-04$	$5.71297D-04$	$1.20780D-04$	$3.06899D-05$	$1.47619D-05$
$1.0D-05$	$5.42294D-04$	$9.84569D-05$	$1.88567D-05$	$7.53294D-06$
$1.0D-06$	$5.39308D-04$	$9.59394D-05$	$1.72321D-05$	$6.37406D-06$
$1.0D-07$	$5.39008D-04$	$9.56840D-05$	$1.70629D-05$	$6.25393D-06$
$1.0D-08$	$5.38978D-04$	$9.56584D-05$	$1.70441D-05$	$6.23420D-06$

Table 2. Pointwise errors $u_h - u$ in L^∞ norm.

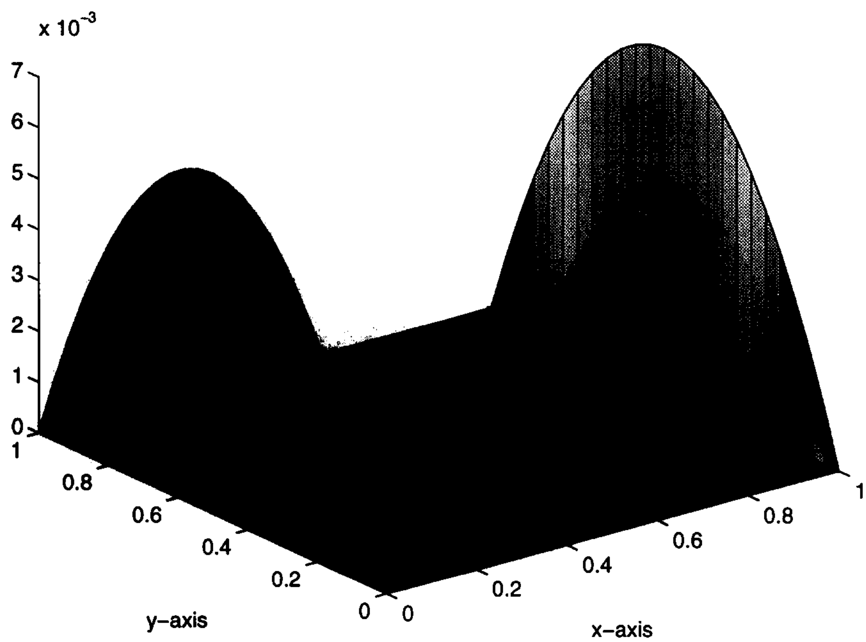
ε	N			
	12	24	48	72
$1.0D-02$	$1.433696500D-02$	$6.697284583D-03$	$2.525962802D-03$	$1.288998828D-03$
$1.0D-03$	$1.434287854D-02$	$6.697285272D-03$	$2.525963440D-03$	$1.288998829D-03$
$1.0D-04$	$1.434347900D-02$	$6.697285316D-03$	$2.525963473D-03$	$1.288998749D-03$
$1.0D-05$	$1.434353925D-02$	$6.697285396D-03$	$2.525963475D-03$	$1.288998750D-03$
$1.0D-06$	$1.434354527D-02$	$6.697285314D-03$	$2.525963475D-03$	$1.288998748D-03$
$1.0D-07$	$1.434354587D-02$	$6.697285350D-03$	$2.525796634D-03$	$1.290275350D-03$
$1.0D-08$	$1.434354679D-02$	$6.697286184D-03$	$2.525798007D-03$	$1.290276723D-03$
$N^{-2}\ln^3 N$	$1.06553D-01$	$5.57264D-02$	$2.51799D-02$	$1.50887D-02$

Table 3. Convergence rates R_ε^N in L^∞ norm.

ε	N		
	12	24	48
$1.0D-02$	1.0981	1.4067	1.6592
$1.0D-03$	1.0987	1.4067	1.6592
$1.0D-04$	1.0987	1.4067	1.6592
$1.0D-05$	1.0988	1.4067	1.6592
$1.0D-06$	1.0988	1.4067	1.6592
$1.0D-07$	1.0988	1.4068	1.6566
$1.0D-08$	1.0988	1.4068	1.6566



(a) $N = 12$.



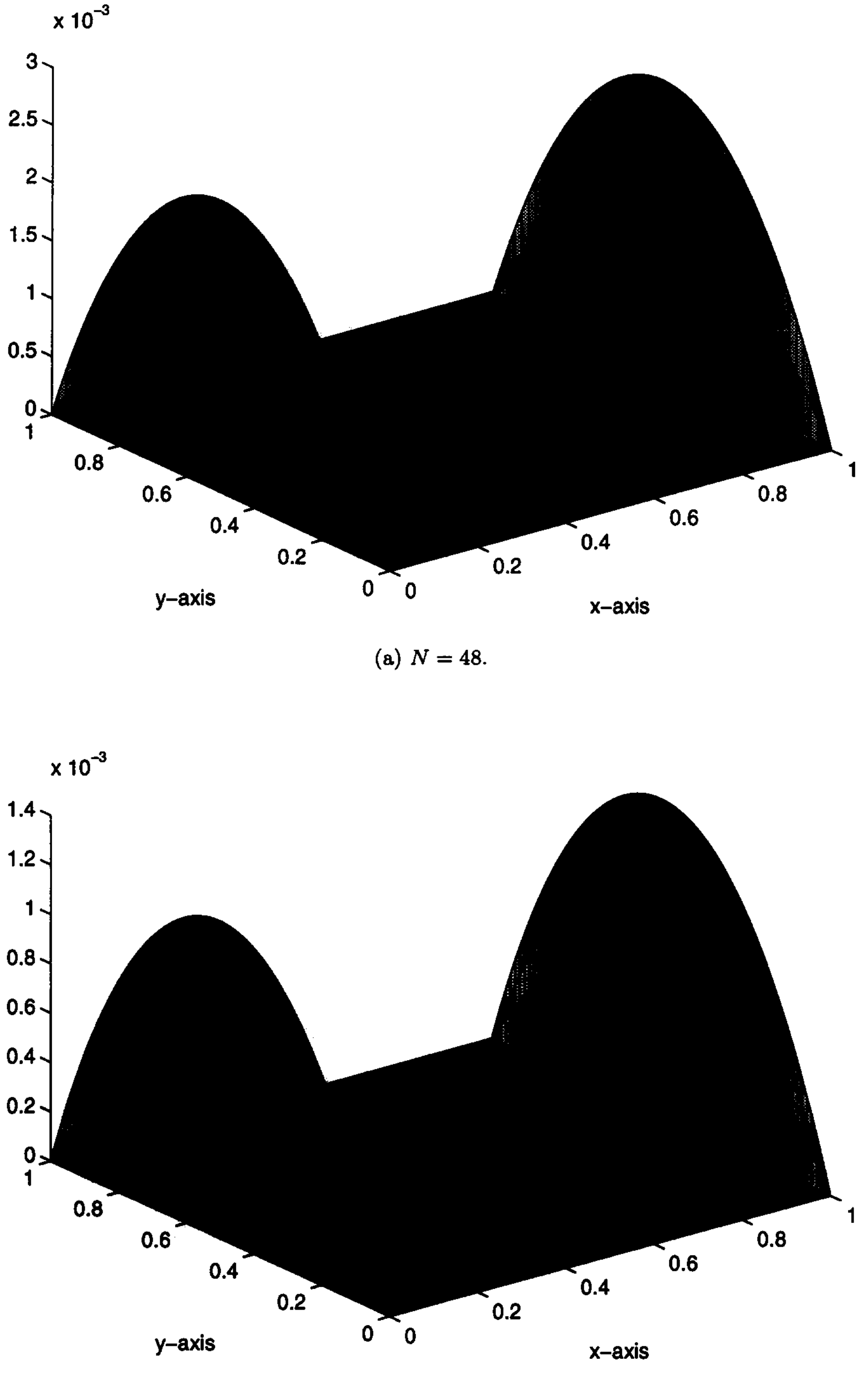
(b) $N = 24$.

Figure 1. Pointwise error $u_h - u$ for $\varepsilon = 10^{-5}$.

4. NUMERICAL EXPERIMENTS

In this section, we will present an example for our method applied to problem (1),(2), where $a = 1$ and f is properly chosen so that the solution is

$$u(x, y) = \left(1 - \frac{e^{-x/\varepsilon} + e^{-(1-x)/\varepsilon}}{1 + e^{-1/\varepsilon}}\right) y(1 - y).$$



(a) $N = 48$.

(b) $N = 72$.

Figure 2. Pointwise error $u_h - u$ for $\varepsilon = 10^{-5}$.

This u has the typical boundary layers at $x = 0$ and $x = 1$. We choose the bilinear interpolation Πf of f in our calculation and $M = N$. Our numerical results are shown in Tables 1–3.

The global uniform convergence (i.e., independent of ε) in both L^2-norm and L^∞-norm is shown very clearly in Tables 1 and 2. To investigate the convergence rate more accurately, let $e_\varepsilon^{N_i} = u_{N_i} - u$ be the pointwise errors between the analytic solution u and the computed FEM solution u_{N_i} on a mesh, which has a total number of mesh points $O(N_i^2)$, where $i = 0, 1$. We listed the computed convergence rate

$$R_\varepsilon^N = \frac{\left(\ln e_\varepsilon^{N_2} - \ln e_\varepsilon^{N_1}\right)}{\ln(N_1/N_2)}$$

in Table 3. It shows that the convergence rate is close to $O(N^{-2})$ when N becomes larger. It agrees with our theoretical analysis.

To show the error distribution more clearly, we plotted the pointwise error $u_h - u$ in Figures 1 and 2. Since the pointwise error is independent of ε, we presented the graphs only for $\varepsilon = 10^{-5}$ with different N values, where $N = 12, 24, 48,$ and 72. Figures 1 and 2 show that the maximum pointwise error comes from the boundary layers at sides $x = 0$ and $x = 1$, which is consistent with our analysis, as we already, noted in the proof of Theorem 3.1 that the error would be in the order of N^{-2} in the domain away from the boundary layers and in the order of $N^{-2}\ln^3 N$ inside the boundary layers.

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