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Portfolio selection under institutional procedures for short selling: Normative and market-equilibrium considerations

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Abstract

In view of the acceptance of short selling of stocks as an investment tool in the portfolio context by a growing number of institutional investors in recent years, the present study considers both normative and market-equilibrium aspects of portfolio selection with short selling. Under the full-information covariance structure of security returns, the study accurately captures institutional procedures for short selling without sacrificing analytical tractability. While short selling enhances the portfolio's risk–return trade-off from a normative perspective, the equilibrium analysis reveals that there is a continuum of market-clearing prices within two boundaries for each security. Economic implications of the equilibrium pricing relationship are also explored in the study.

JEL classification: G11; G12

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1. Introduction

Short selling of stocks, being traditionally perceived to be highly speculative, has been subject to various regulatory restrictions, as well as institutional biases against its application as an investment tool. Given such a perception, much of the

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media attention to short selling has been on its speculative aspect. Not surprisingly, in bearish markets there have been many success stories in the financial press about this form of equity trading. In bullish markets, in contrast, media reports of disastrous performance of short-sale activities have also been plentiful. [1] Short selling in a portfolio context, however, may not be speculative, considering that it can reduce or even neutralize investors' exposure to market and industry risks.

In recent years, a growing number of institutional investors have shown their acceptance of short selling as a viable investment tool. As Hansell (1992) observes, "(a)lthough no one has bet the bank on market-neutral strategies, a number of leading pension funds and endowments have started putting from 2 to 5 percent of their assets in them" (p. 58). [2] The Internal Revenue Service (IRS) of the U.S., in its ruling 95-8 released in January 1995, has provided further tax incentives for tax-exempt institutional investors to engage in short-sale activities. Anand (1995) indicates that, under the IRS ruling, since profits for tax-exempt investors made from short selling publicly traded stocks through brokerage firms are no longer treated as taxable income from unrelated businesses, there should be a substantial broadening of interests in long–short strategies among these investors.

Accompanying the gradual acceptance of short selling by investors, there have been renewed academic interests in normative portfolio modeling, now with emphasis on accurately capturing short-sale transactions. Specifically, Alexander (1993, 1995) and Kwan (1995) extend the Elton et al. (1976) analysis for optimal portfolio selection. Prior to the Alexander (1993) study, short-sale transactions were typically assumed in two alternative ways in portfolio modeling. One assumption is that the short seller not only provides no collateral for the borrowed security but also can use the short-sale proceeds to purchase other securities. The alternative assumption, following Lintner (1965), is that the short-sale proceeds are held as collateral and the short seller, who provides a 100% margin deposit, earns a risk-free return on both the proceeds and the deposit. [3] In practice the short-sale proceeds are indeed held as collateral for the borrowed stock as Lintner portrays. The margin deposit, which is currently at least 50% of the share value, can be provided by cash or securities that the short seller owns. While all short

[1] See, for example, Shao and Weiss (1991), Marcial (1993), Hamilton (1993), Weiss (1994), and Scholl (1995) for some anecdotic reports. Recent empirical evidence of the profitability of short selling has been mixed. While Choie and Hwang (1994) report that short sellers are indeed rewarded with profits for their contributions in enhancing market efficiency, Woolridge and Dickinson (1994) strongly reject the notion that short sellers earn abnormal profits at the expense of less informed traders.

[2] See also White (1990) for another media report on institutional investors' gradual acceptance of short selling of stocks as a viable investment tool.

[3] See Elton and Gruber (1995, Ch. 6) for portfolio models based on these assumptions of short selling.

sellers can earn a risk-free return on the margin deposit that is provided by cash or Treasury bills, only those with sizeable accounts can get any rebate of interest that the brokerage firm earns from the short-sale proceeds. [4]

The effect of institutional procedures for short selling on portfolio selection, therefore, is like having transaction costs on negative holdings of securities. Because of such costs, a security that fails to contribute adequately to an investor's portfolio is not necessarily desirable for short selling, from the perspective of the same investor. Yet, in portfolio models using the above two alternative assumptions, such costs are completely ignored and consequently, with the short-sale benefit overstated, any security disqualified for purchase is automatically held short; that is, no security ever considered can be excluded from the portfolio.

By adhering to institutional procedures for short selling and by using the Sharpe (1963) single index model to simplify the covariance structure of security returns, Alexander (1993, 1995) and Kwan (1995) are able to filter out undesirable securities from any given set of securities in normative portfolio construction. With the short seller always providing a cash deposit, as a pre-determined fraction of share values to be held short, Alexander (1993) transforms the portfolio selection problem into an analytically equivalent no-short-sale case for which the Elton et al. (1976) solution method is directly applicable. Kwan presents an alternative solution method which allows the use of purchased securities to satisfy part or all of the margin requirement for short selling, with any required cash deposit determined iteratively. The problem of estimation risk, while briefly considered in Kwan, is addressed in detail in Alexander (1995).

Of particular interest to the present study is Alexander's (1993) insight that, with the short sale of a security viewed as the purchase of an artificially constructed security, a portfolio problem allowing short sales (based on n securities) can be transformed into an analytically equivalent no-short-sale case (based on the same n securities and n artificial securities). Although the single index model has been used for analytical convenience, Alexander's (1993) insight does not require any simplification of the covariance structure of security returns. Drawing on Alexander's (1993) insight, the present study considers both normative and market-equilibrium aspects of portfolio selection with short selling under the full-information covariance structure. In order to capture more accurately institutional features that are relevant to portfolio investment, the normative analysis also accommodates margin purchases and the potential use of purchased securities to fulfill the margin requirement for short selling.

The abundance of asset pricing models in the literature notwithstanding, any attempt to account for institutional features of short selling in a model has been indirect. Specifically, in a single-period model examining the impact of transaction

[4] See Sharpe and Alexander (1990, Ch. 2), Brent et al. (1990), Weiss (1991), and Jacobs and Levy (1993) for further details of institutional features of short selling.

costs on security pricing, Mayshar (1981) uses different transaction costs for long and short positions, thus indirectly capturing the impact of institutional procedures for short selling on security pricing. The Mayshar (1981) model, however, assumes that each investor has a prior decision as to which of the available securities are to be held long, and which to be held short. Although information costs and fixed transaction costs for trading different securities can explain why an investor would consider only a subset of available securities, the assumption of a prior selection among the securities considered regardless of price, unfortunately, has distorted the influence of price on the demand for each security.

The equilibrium analysis in the present study, not requiring a prior selection of securities, draws on the Sharpe (1991) Nobel lecture on asset pricing. Sharpe (1991) considers equilibrium pricing of securities in a single-period market as in the CAPM but without short selling. With the short sale of each security treated as the purchase of a corresponding artificial security, the present analysis has many common features with the no-short-sale case. The availability of three choices (*long*, *short*, or *out*) for individual investors respect to each security considered, instead of two (*in* or *out*) as in the Sharpe (1991) case, actually provides a more convenient analytical setting for establishing an equilibrium pricing relationship.

In what follows, Section 2 presents the normative analysis, and Section 3 derives the market-equilibrium pricing relationship and explores its economic implications. Finally, Section 4 concludes the present study.

2. A normative analysis

Consider a set of n securities where each security i, for $i = 1,2,\dots,n$, has a beginning-of-period price P_{i0} and random end-of-period share value (including both price and dividend) P_{i1}. The random return on security i without considering any transaction costs is $R_i = (P_{i1} - P_{i0})/P_{i0}$. Let $\bar{R}_i$ be the expected value of R_i, and σ_{ij} the covariance of returns between securities i and j, for $i, j = 1,2,\dots,n$. Let N_i (≥ 0) and N_{n+i} (≥ 0) be the numbers of shares of security i that are held long and short, respectively, and assume that trading of fractional shares is permissible. Although the present formulation allows both N_i and N_{n+i} to be positive in the same portfolio (that is, security i to be held both long and short), it will soon be shown that, as in Alexander (1993), this scenario is ruled out under optimality conditions.

As counter-intuitive as it may seem, it will also soon be clear that it is analytically convenient to incorporate in the model some transaction costs beyond what are implicit under institutional procedures for short selling. While in practice transaction costs for trading each security have a fixed component and a variable component which depends on the share price and the number of shares traded, some simplifying assumptions about the fee schedules have to be made for maintaining analytical tractability. Specifically, for all securities regardless of their price ranges, let the constant ϕ be the beginning-of-period proportional transaction costs as a fraction of the share values traded. Assume further that there are neither fixed transaction costs nor end-of-period proportional transaction costs. [5]

2.1. The case of cash purchases of securities in long positions

Suppose that the margin requirement for either long or short positions is a constant fraction m of the corresponding share values. Assuming for now that cash purchases apply to all securities in long positions, the required payment amounts to $\Sigma_{i=1}^{n} N_i P_{i0}$ plus transaction costs of $\phi \Sigma_{i=1}^{n} N_i P_{i0}$. The total short-sale proceeds are $\Sigma_{i=1}^{n} N_{n+i} P_{i0}$, and the corresponding transaction costs are $\phi \Sigma_{i=1} N_{n+i} P_{i0}$. As long as the condition of $(1-m)\Sigma_{i=1}^{n} N_i P_{i0} \geq m \Sigma_{i=1}^{n} N_{n+i} P_{i0}$ is satisfied, the margin requirement for short selling can be fulfilled by using purchased securities alone. Otherwise, a cash deposit equal to a fraction λ of the total short-sale proceeds must be provided so that $(1-m)\Sigma_{i=1}^{n} N_i P_{i0} + \lambda \Sigma_{i=1}^{n} N_{n+i} P_{i0} \geq m \Sigma_{i=1}^{n} N_{n+i} P_{i0}$.

As indicated earlier, the short seller with a sizeable account at the brokerage firm may get a rebate of interest that the latter earns on the short-sale proceeds. Letting r_ℓ be the interest rate for risk-free lending and $0 \leq \nu < 1$ be the fraction of interest rebated, the total end-of-period rebate will be $\nu r_\ell \Sigma_{i=1}^{n} N_{n+i} P_{i0}$. The short seller also earns the interest rate r_ℓ on any cash deposit $(= \lambda \Sigma_{i=1}^{n} N_{n+i} P_{i0})$. While in the short position, the short seller of each share of security i must pay the brokerage firm an amount equal to any cash dividend subsequently declared by the issuer of the security.

With investment funds allocated to the n securities in long and short positions being

$$W_0 = \Sigma_{i=1}^{n} \left[(1+\phi) N_i P_{i0} + (\lambda + \phi) N_{n+i} P_{i0} \right], \tag{1}$$

the corresponding end-of-period random value of the portfolio will be

$$W_1 = \Sigma_{i=1}^{n} N_i P_{i1} + N_{n+i} \left[-P_{i1} + (1 + \nu r_\ell) P_{i0} + (1 + r_\ell) \lambda P_{i0} \right]. \tag{2}$$

Although λ is one of the $2n+1$ decision variables in the above formulation, the remaining being N_i and N_{n+i}, for $i = 1,2,\ldots,n$, it is analytically more convenient to treat λ as a parameter. As will be shown in Section 2.4, optimal values of λ for individual efficient portfolios can be determined via a simple

[5] Regarding the end-of-period transaction costs, a convenient assumption for maintaining analytical tractability is to have proportional transaction costs on both price and dividend components of the share values. Alternatively, the costs are applied to prices only; dividends are assumed to be risk-free. In either case, except for some algebraic changes throughout the analysis, there is no change in the nature of the portfolio problem, nor in the solution method. Therefore, without any loss of generality, the algebraic details of these cases are omitted.

iterative procedure. For cases where the optimal λ is zero, margin purchases may be possible or even desirable. The analytical detail will be provided in Section 2.5.

2.2. Formulation as an analytically equivalent case without short selling

For an alternative set of decision variables, let $x_i = (1+\phi)N_i P_{i0}/W_0$ (≥ 0), for $i = 1,2,\ldots,n$, be the fraction of investment funds for the beginning-of-period cash purchase of security i, and $x_{n+i} = (\lambda+\phi)N_{n+i}P_{i0}/W_0$ (≥ 0) the beginning-of-period investment for short selling security i, also as a fraction of investment funds. With $\sum_{i=1}^{n}(x_i + x_{n+i}) = 1$ and with $\overline{W}_1$ being the expected value of W_1, the portfolio's expected return and variance of returns are, respectively,

$$\overline{R}_p = \frac{\overline{W}_1 - W_0}{W^0} = \sum_{i=1}^{n}\left\{x_i\left(\frac{\overline{R}_i - \phi}{1+\phi}\right) + x_{n+i}\left[\frac{-\overline{R}_i - \phi + (\lambda+\nu)r_\ell}{\lambda+\phi}\right]\right\} \tag{3}$$

and

$$\begin{aligned}\sigma_p^2 &= \mathrm{Var}\left(\frac{W_1 - W_0}{W_0}\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{x_i x_j}{(1+\phi)^2} - \frac{x_i x_{n+j} + x_{n+i}x_j}{(1+\phi)(\lambda+\phi)} + \frac{x_{n+i}x_{n+j}}{(\lambda+\phi)^2}\right]\sigma_{ij}.\end{aligned} \tag{4}$$

In order to express $\overline{R}_p$ and σ_p^2 in more familiar forms, let

$$\hat{R}_i = \frac{\overline{R}_i - \phi}{1+\phi}, \tag{5}$$

$$\hat{R}_{n+i} = \frac{-\overline{R}_i - \phi + (\lambda+\nu)r_\ell}{\lambda+\phi}, \tag{6}$$

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{(1+\phi)^2}, \tag{7}$$

$$\hat{\sigma}_{i,n+j} = \frac{-\sigma_{ij}}{(1+\phi)(\lambda+\phi)} = \hat{\sigma}_{n+i,j}, \tag{8}$$

and

$$\hat{\sigma}_{n+i,n+j} = \frac{\sigma_{ij}}{(\lambda+\phi)^2}, \quad \text{for} \quad i,j = 1,2,\ldots,n. \tag{9}$$

With these substitutions, Eqs. (3) and (4) become, respectively,

$$\bar{R}_p = \sum_{i=1}^{2n} x_i \hat{R}_i \tag{10}$$

and

$$\sigma_p^2 = \sum_{i=1}^{2n} \sum_{j=1}^{2n} x_i x_j \hat{\sigma}_{ij}. \tag{11}$$

In Eqs. (10) and (11), since $x_i \geq 0$, for $i = 1,2,\ldots,2n$, satisfying $\sum_{i=1}^{2n} x_i = 1$, optimal portfolio selection with short selling based on securities $1,2,\ldots,n$ is analytically equivalent to that without short selling based on the same n securities and n artificially constructed securities, labeled as securities $n+1, n+2, \ldots, 2n$. Drawing on Alexander's (1993) insight, the short sale of a regular security i, for $i = 1,2,\ldots,n$, can be viewed as the purchase of its artificial counterpart, security $n+i$. Each security $n+i$ has been constructed in such a way that, while it is riskier than security i, it is perfectly negatively correlated in returns with security i. The higher risk, however, is due to the leverage effect analogous to that of margin purchases of securities, rather than the effect of short selling per se. [6]

In the above formulation, it is obvious that $\lambda + \phi$ must be strictly positive for all input parameters and decision variables to be properly defined. Since λ can be zero or positive, depending on whether the margin requirement for short selling is satisfied by purchased securities alone, a positive ϕ is required for the analysis. Although a positive ϕ can easily be justified for practical reasons, an alternative formulation which can accommodate the case of $\phi = 0$ is actually more convenient in formally justifying an iterative procedure to determine the optimal λ, as proposed in Section 2.4. While the analytical detail of the alternative formulation is omitted from the main text, it is provided in some footnotes. [7]

Returning to the original formulation, if each summation in Eqs. (10) and (11) is over securities 1 to n instead of over all $2n$ securities, along with $\sum_{i=1}^{n} x_i = 1$, the portfolio problem reduces to a no-short-sale case based on the same set of n regular securities. Thus, the impact of allowing short sales on portfolio selection is like augmenting the opportunity set from n to $2n$ securities in a no-short-sale case.

[6] As can be inferred from Section 2.5, if the margin provided for the purchase of a regular security i is a fraction τ of its share value, the standard deviation of returns of the investment is $\sigma_i/(\tau + \phi)$ which increases with decreasing τ. This inverse relationship is the same as that between $\hat{\sigma}_{n+i}$ and λ according to Eq. (9).

[7] As an alternative formulation, define $\hat{x}_i = N_i P_{i0} / W_0 \geq 0$ and $\hat{x}_{n+i} = N_{n+i} P_{i0} / W_0 \geq 0$, for $i = 1,2,\ldots,n$, as the values of individual securities selected for the portfolio with each dollar's worth of investment funds. It follows that $\bar{R}_p = \sum_{i=1}^{n} \{\hat{x}_i(\bar{R}_i - \phi) + \hat{x}_{n+i}[-\bar{R}_i - \phi + (\lambda + \nu) r_\ell]\}$ and $\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (\hat{x}_i \hat{x}_j - \hat{x}_i \hat{x}_{n+j} - \hat{x}_{n+i} \hat{x}_j + \hat{x}_{n+i} \hat{x}_{n+j}) \sigma_{ij}$, with $\sum_{i=1}^{n} [(1+\phi)\hat{x}_i + (\lambda + \phi)\hat{x}_{n+i}] = 1$.

This alone should have the potential to improve the portfolio's risk–return trade-off. More importantly, since all artificial securities are perfectly negatively correlated in returns with their corresponding regular securities, a positive (negative) correlation between two regular securities always produces a negative (positive) correlation if either security is replaced by its artificial counterpart. In a typical portfolio selection problem where most regular securities are positively correlated in returns, the presence of many negative correlations in the augmented opportunity set should lead to good improvements in the portfolio's risk–return trade-off. Of course, for the portfolio solution to exist, the covariance matrix of all regular and artificial securities to be included in the portfolio must be non-singular. This requires that any security i and its artificial counterpart, security $n+i$, not both be included in the same portfolio. In order to show that this requirement is always satisfied, the portfolio problem is further formulated as follows.

2.3. Tangency portfolios and optimality conditions

Given that the portfolio problem has been formulated as an analytically equivalent no-short-sale case on the basis of $2n$ securities, each efficient portfolio p can be constructed by maximizing $\theta = (\bar{R}_p - r)/\sigma_p$, subject to the allocation constraints of $\sum_{i=1}^{2n} x_i = 1$ and $x_i \geq 0$, for $i = 1,2,\ldots,2n$. Here, r is the intercept, on the $\bar{R}$-axis of the $(\bar{R}, \sigma)$-plane, of the line that is tangent to the efficient frontier at portfolio p. Following Elton et al. (1978), the entire efficient frontier, in principle, can be traced out by varying the parameter r. As long as the investor can allocate his/her investment funds among the $2n$ risky securities and a risk-free security which yields a return of r_f, the efficient frontier has a non-linear segment (corresponding to $r > r_f$ and zero risk-free investments) and a linear extension (corresponding to $r = r_f$ and non-zero risk-free investments).

By defining z_i as $x_i(\bar{R}_p - r)/\sigma_p^2$, the optimality conditions for the tangency portfolio for a given r are [8]

$$\sum_{j=1}^{2n} \hat{\sigma}_{ij} z_j = \hat{R}_i - r + \delta_i, \tag{12}$$

$$z_i \geq 0, \quad \delta_i \geq 0, \text{ and } z_i \delta_i = 0, \quad \text{for} \quad i = 1,2,\ldots,2n, \tag{13}$$

[8] Incorporating the constraint $\sum_{i=1}^{2n} x_i = 1$ into the objective function by writing r as $r\sum_{i=1}^{2n} x_i$ yields $\theta = \sum_{i=1}^{2n} x_i(\hat{R}_i - r)/(\sum_{i=1}^{2n}\sum_{j=1}^{2n} x_i x_j \hat{\sigma}_{ij})^{1/2}$. As shown by Elton et al. (1976), θ being pseudo-concave, the Kuhn–Tucker conditions are necessary and sufficient for the optimality of the solution. With $\partial\theta/\partial z_i + \delta_i = 0$, for $i = 1,2,\ldots,2n$, Eq. (12) follows. See also Elton and Gruber (1995, Ch. 6) for analytical details.

each δ_i being a slack variable. Eq. (12) can be written for the n regular securities and the n artificial securities separately as

$$\frac{\sum_{j=1}^{n} \sigma_{ij} z_j}{(1+\phi)^2} - \frac{\sum_{j=1}^{n} \sigma_{ij} z_{n+j}}{(1+\phi)(\lambda+\phi)} = \frac{\bar{R}_i - \phi}{1+\phi} - r + \delta_i \tag{14}$$

and

$$-\frac{\sum_{j=1}^{n} \sigma_{ij} z_j}{(1+\phi)(\lambda+\phi)} + \frac{\sum_{j=1}^{n} \sigma_{ij} z_{n+j}}{(\lambda+\phi)^2}$$
$$= \frac{-\bar{R}_i - \phi + (\lambda+\nu) r_l}{\lambda+\phi} - r + \delta_{n+i} \quad \text{for } i = 1,2,\ldots,n. \tag{15}$$

Combining Eqs. (14) and (15) leads to

$$(1+\phi)\delta_i + (\lambda+\phi)\delta_{n+i}$$
$$= 2\phi(1+r) + (1+\lambda)\left[r - \left(\frac{\nu+\lambda}{1+\lambda}\right) r_\ell\right], \text{ for } i = 1,2,\ldots,n. \tag{16}$$

With $r \geq r_\ell$, $\lambda \geq 0$, $1 > \nu \geq 0$ and $\phi > 0$, the sum of $(1+\phi)\delta_i$ and $(\lambda+\phi)\delta_{n+i}$ must be positive, implying that δ_i and δ_{n+i} cannot both be zero, for $i = 1,2,\ldots,n$. Given complementarity conditions (13), z_i and z_{n+i} (equivalently, x_i and x_{n+i}, or N_i and N_{n+i}) cannot both be positive. [9] Thus, as long as the covariance matrix based on the n regular securities is non-singular, so is the covariance matrix based on all selected securities for the tangency portfolio, among the $2n$ regular and artificial securities considered. The perfect negative correlation between each regular security and its artificial counterpart is not a concern because, as shown above, they cannot both be selected for the same portfolio under optimality conditions.

2.4. *An iterative procedure for solving the portfolio problem*

Having formulated the portfolio optimization problem with short selling as an analytically equivalent no-short-sale case by augmenting the opportunity set with artificial securities, the next step is to construct portfolios corresponding to

[9] Under the alternative formulation as described in footnote 7, where the problem is to maximize $\theta = (\bar{R}_p - r)/\sigma_p$, subject to $\sum_{i=1}^{n}[(1+\phi)\hat{x}_i + (\lambda+\phi)\hat{x}_{n+i}] = 1$ and $\hat{x}_i, \hat{x}_{n+i} \geq 0$, for $i = 1,2,\ldots,n$. With $\hat{z}_i = \hat{x}_i(\bar{R}_p - r)/\sigma_p^2 \geq 0$, $\hat{\delta}_i \geq 0$, and $\hat{z}_i \hat{\delta}_i = 0$, for $i = 1,2,\ldots,2n$, each $\hat{\delta}_i$ being a slack variable, the optimality conditions lead to $\hat{\delta}_i + \hat{\delta}_{n+i} = 2\phi(1+r) + (1+\lambda)[r - r_\ell(\nu-\lambda)/(1+\lambda)] > 0$, for $i = 1,2,\ldots,n$, providing the same conclusion as Eq. (16) does, even with both ϕ and λ being zero.

different values of r ($\geq r_\ell$) by using optimality conditions (12) and (13). Among the currently available solution methods, the approach by Kwan and Yuan (1993) that applies the Markowitz (1956, 1959, 1987) critical line algorithm to tangency portfolios is particularly suitable for the present formulation. Suppose that the tangency portfolio to be constructed is for $r = \hat{r}$. The Kwan and Yuan iterative procedure starts with the portfolio consisting only of the security with the highest expected return $\hat{R}_i$ among $i = 1,2,\ldots,2n$. The security's expected return provides the starting value of parameter r. Accompanying gradual decreases of r, all securities to be included in each corresponding tangency portfolio for $r \geq \hat{r}$ can be identified in the following manner. Of particular interest here is the portfolio corresponding to $r = \hat{r}$.

For the initial portfolio, as well as all other tangency portfolios considered subsequently, δ_i for each included security i is set to zero; so is z_i for each excluded security i. Eq. (12), as a system of $2n$ linear equations, is used to solve the remaining z_i's and δ_i's in terms of r. By allowing r to decrease, the onset of any violation of complementarity conditions (13) by a security, which produces a critical value of r, leads to a change in the set of included securities, as caused by that security. Upon revising the portfolio, r is allowed to decrease again until the onset of another violation of the optimality conditions, which also provides a new critical value of r. All efficient portfolios between two adjacent critical values of r consist of the same set of securities. The procedure continues until a critical value of r is less than or equal to $\hat{r}$. [10]

Now, with all securities in the portfolio corresponding to $r = \hat{r}$ identified, the allocation of investment funds can be determined from $x_i = z_i / \sum_{j=1}^{2n} z_j$, for $i = 1,2,\ldots,2n$. [11] As Kwan and Yuan illustrate, the analysis can be performed via simple matrix manipulations and without any traditional programming requirements. Practitioners in portfolio management who are familiar with spreadsheet operations on microcomputers should find the analysis very accessible.

To use the Kwan and Yuan iterative procedure for portfolio construction, however, the value of λ for each given $\hat{r}$ must be pre-determined. The next task,

[10] The same procedure also applies to the alternative formulation as described in footnotes 7 and 9, except for how the tangency portfolio is initialized. The selection of a regular security i to the initial portfolio will result in $\hat{x}_i = 1/(1+\phi)$ and $\bar{R}_p = (\bar{R}_i - \phi)/(1+\phi)$, and the selection of an artificial security $n+i$ instead will result in $\hat{x}_{n+i} = 1/(\lambda+\phi)$ and $\bar{R}_p = [-\bar{R}_i - \phi + (\lambda+\nu)r_\ell]/(\lambda+\phi)$. Thus, the security to be selected must be the one corresponding to the maximum among the $2n$ cases of these $\bar{R}_p$'s. For the case where both λ and ϕ are zero, define $R^*_{n+i} = -\bar{R}_i + \nu r_\ell$, for $i = 1,2,\ldots,n$. If none of the R^*_{n+i}'s are positive, a regular security providing the highest $\bar{R}_p$ is selected; otherwise, an artificial security corresponding to the maximum of the R^*_{n+i}'s is selected. If there is no regular security in the portfolio, the zero value of λ (or ϕ) must be approximated by a very small positive fraction to avoid singularities in matrix manipulations. As soon as a regular security enters the portfolio, such an approximation is no longer necessary.

[11] Under the alternative formulation as described in footnotes 7 and 9, each $\hat{x}_i$ is given by $\hat{z}_i / \sum_{j=1}^{n}[(1+\phi)\hat{z}_j + (\lambda+\phi)\hat{z}_{n+j}]$ instead.

therefore, is to find the value of λ that maximizes $\theta = (\bar{R}_p - \hat{r})/\sigma_p$ under allocation constraints. Intuitively, for any given λ, the portfolio to be obtained from the constrained maximization of θ is analytically equivalent to the tangency portfolio with risk-free lending and borrowing at the same interest rate $\hat{r}$. While the margin deposit that the short seller provides earns a risk-free return r_{ℓ}, an alternative use of the same fund earns the return $\hat{r}$ which is also risk-free. Clearly, if $\hat{r} = r_{\ell}$, the value of λ does not have any impact on how the investment funds are allocated among the risky securities considered. If $\hat{r} > r_{\ell}$, however, an increase in λ increases the opportunity cost to the short seller. Then, within the present analytical framework, it is rational for the investor to provide the lowest acceptable margin deposit. [12]

In order to find out whether a cash deposit is necessary, the portfolio for $r = \hat{r}$ to be constructed via the Kwan and Yuan iterative procedure can start with the case of $\lambda = 0$. As long as $(1 - m)\Sigma_{i=1}^{n} N_i P_{i0} \geq m\Sigma_{i=1}^{n} N_{n+i} P_{i0}$ or, equivalently, $(1 - m)(\Sigma_{i=1}^{n} x_i)/(1 + \phi) \geq (m\Sigma_{i=1}^{n} x_{n+i})/\phi$, no cash deposit is necessary. Otherwise, a cash deposit must be provided to satisfy the margin requirement. A simple way to determine the lowest acceptable λ is to repeat the same analysis for successively higher values of λ ($\leq m$) until the condition of $(1 - m)(\Sigma_{i=1}^{n} \hat{x}_i) \geq (m - \lambda)(\Sigma_{i=1}^{n} \hat{x}_{n+i})$ is satisfied. Here, $\hat{x}_i = x_i/(1 + \phi)$ and $\hat{x}_{n+i} = x_{n+i}/(\lambda + \phi)$, for $i = 1,2,\ldots,n$, are the values of individual securities selected for each dollar's worth of investment funds. An illustrative example of the above analysis is provided in the appendix.

2.5. *The case of margin purchases of securities in long positions*

If the optimal λ is zero in the above analysis, the investor not only can provide no cash deposit for the shorted securities, but also can use margin purchases, instead of cash purchases, for possible enhancement of his/her portfolio's risk–return trade-off. [13] To accommodate margin purchases in the analysis, however, some algebraic changes are required. Specifically, let τ represent the margin provided for the purchases of regular securities as a fraction of their beginning-of-

[12] In principle, for a given value of r ($= \hat{r}$), if λ is allowed to vary gradually from 0 to m, a series of its critical values can be produced. All tangency portfolios based on different values of λ between two adjacent critical values must contain the same subset of the $2n$ securities considered. Each critical λ indicates the onset of a change in the subset of securities. Consider two tangency portfolios with their λ's infinitesimally greater than and less than a critical λ. The corresponding θ's must converge to the same value at this critical λ because investment funds are not allocated among securities not common to both portfolios but are allocated in the same way among the common securities for both portfolios. Thus, to justify formally the statements made in the main text, what needs to be established is how θ varies with λ between any two adjacent critical values of λ. It can be shown that, under the alternative formulation as described in footnotes 7 and 9, $\partial\theta/\partial\lambda = (r_{\ell} - \hat{r})(\Sigma_{i=1}^{n} \hat{x}_{n+i})/\sigma_p$ which is zero for $\hat{r} = r_{\ell}$ and negative for $\hat{r} > r_{\ell}$.

[13] See Fama (1976, Ch. 8) for a discussion of risk-free borrowing under a margin requirement.

period values, and r_b ($> r_\ell$) the interest rate on the loan, $(1-\tau)\Sigma_{i=1}^{n} N_i P_{i0}$, as provided by the brokerage firm. With $\lambda = 0$ and $m \leq \tau \leq 1$, Eqs. (1) and (2) can be rewritten, respectively, as

$$W_0 = \sum_{i=1}^{n} \left[(\tau + \phi) N_i P_{i0} + \phi N_{n+i} P_{i0} \right] \tag{17}$$

and

$$W_1 = \sum_{i=1}^{n} \left\{ N_i \left[P_{i1} - (1 + r_b)(1 - \tau) P_{i0} \right] + N_{n+i} \left[-P_{i1} + (1 + \nu r_\ell) P_{i0} \right] \right\}. \tag{18}$$

Letting $x_i = (\tau + \phi) N_i P_{i0}/W_0$, $x_{n+i} = \phi N_{n+i} P_{i0}/W_0$, $\hat{R}_i = [\bar{R}_i - \phi - (1 - \tau) r_b]/(\tau + \phi)$, $\hat{R}_{n+i} = (-\bar{R}_i - \phi + \nu r_\ell)/\phi$, $\hat{\sigma}_{ij} = \sigma_{ij}/(\tau + \phi)^2$, $\hat{\sigma}_{i,n+j} = -\sigma_{ij}/[(\tau + \phi)\phi] = \hat{\sigma}_{n+i,j}$, and $\hat{\sigma}_{n+i,n+j} = \sigma_{ij}/\phi^2$, for $i, j = 1,2,\ldots,n$, instead, Eqs. (10) and (11), where $\Sigma_{i=1}^{2n} x_i = 1$, along with optimality conditions (12) and (13), remain unchanged. Given the above algebraic changes, Eq. (16) becomes

$$\begin{aligned} &(\tau + \phi)\delta_i + \phi \delta_{n+i} \\ &= 2\phi(1 + r) + (r_b - \nu r_\ell)\left[1 - \tau\left(\frac{r_b - r}{r_b - \nu r_\ell}\right)\right], \text{ for } i = 1,2,\ldots,n. \end{aligned} \tag{19}$$

With $\tau \leq 1$, $r_b > r_\ell$, $r \geq r_\ell$, and $1 > \nu \geq 0$, it follows that $r_b - \nu r_\ell > r_b - r_\ell \geq r_b - r$ and that, regardless of the sign of $r_b - r$, the sum of $(\tau + \phi)\delta_i$ and $\phi\delta_{n+i}$ must be positive. Thus, δ_i and δ_{n+i} cannot both be zero, implying that securities i and $n+i$ cannot both be selected for the same portfolio under optimality conditions (12) and (13).[14]

Suppose that the portfolio to be constructed is for $r = \hat{r}$. With τ treated as a parameter, the portfolio corresponding to $\theta = (\bar{R}_p - \hat{r})/\sigma_p$ is analytically equivalent to the tangency portfolio with risk-free lending and borrowing at the same interest rate $\hat{r}$. Intuitively, if $\hat{r} < r_b$ (if $\hat{r} > r_b$), it would be more (less) costly for the investor to have margin purchases by carrying a loan, as provided by the brokerage firm, at the interest rate r_b, than to have cash purchases, made possible by borrowing the same amount of funds elsewhere at the interest rate $\hat{r}$. Therefore,

[14] As an alternative formulation with $\hat{x}_i$ and $\hat{x}_{n+i}$, for $i = 1,2,\ldots,n$, as defined in footnote 7, the expected return of portfolio p becomes $\bar{R}_p = \Sigma_{i=1}^{n}\{\hat{x}_i[\bar{R}_i - \phi - (1-\tau)r_b] + \hat{x}_{n+i}[-\bar{R}_i - \phi + \nu r_\ell]\}$. Its variance of returns σ_p^2 remains the same as what is shown in footnote 7. Under the allocation constraint of $\Sigma_{i=1}^{n}[(\tau + \phi)\hat{x}_i + \phi\hat{x}_{n+i}] = 1$, and with $\hat{z}_i$ and $\hat{\delta}_i$, for $i = 1,2,\ldots,2n$, as defined in footnote 9, the optimality conditions lead to $\hat{\delta}_i + \hat{\delta}_{n+i}$ being the same as the right-hand side of Eq. (19).

if $\hat{r} < r_b$, the investor would prefer cash purchases; if $\hat{r} > r_b$, he/she would prefer margin purchases with the lowest acceptable τ. In the case where $\hat{r} = r_b$, however, he/she would be indifferent between cash purchases and margin purchases. [15]

As long as the optimal λ is zero for the tangency portfolio at $\hat{r} = r_b$ under the formulation of cash purchases of regular securities, the efficient frontier on the $(\bar{R},\sigma)$-plane corresponding to the same $\hat{r}$ with margin purchases has a linear segment that covers values of $\tau \leq 1$ satisfying the condition of $(\tau - m)\Sigma_{i=1}^{n}\hat{x}_i \geq m\Sigma_{i=1}^{n}\hat{x}_{n+i}$. Here, $\hat{x}_i = x_i/(\tau + \phi)$ and $\hat{x}_{n+i} = x_{n+i}/\phi$, for $i = 1,2,\ldots,n$, are the values of individual securities selected. [16] The efficient frontier corresponding to values of $\hat{r} > r_b$, however, is non-linear. With the algebraic changes as indicated in the text following Eqs. (17) and (18), the same iterative procedure in Section 2.4 can be used to determine the lowest acceptable τ instead for any given $\hat{r} > r_b$. Specifically, starting with $\tau = 1$, the procedure is repeated with successively lower values of $\tau(\geq m)$ just before the condition of $(\tau - m)\Sigma_{i=1}^{n}\hat{x}_i \geq m\Sigma_{i=1}^{n}\hat{x}_{n+i}$ is violated.

3. A market-equilibrium analysis

To extend the normative analysis above to a market-equilibrium setting, consider a market with K investors and n risky securities where trading of fractional shares is permissible. There are Q_i shares of each security i, for $i = 1,2,\ldots,n$. Each investor k with initial wealth W_0^k, for $k = 1,2,\ldots,K$, is assumed to have constant absolute risk aversion A^k or, equivalently, constant absolute risk tolerance $1/A^k$. Let P_{i0} be the beginning-of-period price of security i, and N_i^k (N_{n+i}^k) the number of shares of security i, for $i = 1,2,\ldots,n$, that is held long (short) by investor k. It is further assumed that, as in Mayshar (1981), while investors can differ in their expectations about the absolute levels of end-of-period values of individual securities, they have the same opinion regarding the relative dispersions and co-movements of these values. That is, while $\bar{P}_{i1}^k$ represents each investor k's expectation of the end-of-period share value P_{i1} of security i, the covariances $\Gamma_{ij} = \mathrm{Cov}(P_{i1}, P_{j1})$, for $i,j = 1,2,\ldots,n$, apply to all K investors.

Risk-free lending and borrowing opportunities are still available as in the

[15] Based on the same idea as described in footnote 12, a formal proof regarding the desirability of margin purchases can be established under the alternative formulation as described in footnote 14. In principle, for a given value of $r(=\hat{r})$, if τ is allowed to vary gradually over the range of m to 1, a series of its critical values can be obtained. Analogous to the case in footnote 12, what needs to be established is how θ varies with τ between any two adjacent critical values of τ. Since $\partial\theta/\partial\tau = (r_b - \hat{r})(\Sigma_{i=1}^{n}\hat{x}_i)/\sigma_p$, the desirability of margin purchases depends on whether $\hat{r}$ is less than, equal to, or greater than r_b, as explained in the main text.

[16] Under the alternative formulation as described in footnote 14, the values of individual securities selected are given by $\hat{x}_i = \hat{z}_i/\Sigma_{j=1}^{n}[(\tau+\phi)\hat{z}_j + \phi\hat{z}_{n+j}]$, for $i = 1,2,\ldots,2n$.

normative analysis. A single interest rate r $(= r_\ell = r_b)$, however, is assumed for the sake of analytical tractability. Let the constant ϕ^k be investor k's beginning-of-period proportional transaction costs as a fraction of share values traded. Again, it is assumed that there are no fixed transaction costs for trading different securities and no end-of-period proportional transaction costs. All beginning-of-period trades are assumed to take place as in a Walrasian auction market. Although the movements towards market-clearing prices for all n securities require successive trades by each investor k, the transaction costs incurred are assumed to be based only on his/her eventual holding (that is, $\Sigma_{i=1}^n N_i P_{i0}$ and $\Sigma_{i=1}^n N_{n+i} P_{i0}$) at market-clearing prices.

Assume that the investor's initial wealth W_0^k is always fully allocated among the n risky securities and the risk-free security. Assume also that cash purchases apply to all securities in long positions and that, in the case of any short selling, a margin deposit equal to a fraction λ^k of the total short-sale proceeds is provided. A short seller may get a rebate of interest that the brokerage firm earns on the short-sale proceeds. For each investor k, let $0 \le \nu^k < 1$ be the fraction of interest rebated. Then, the expected value and the variance of the end-of-period wealth are, respectively,

$$\begin{aligned}\overline{W}_1^k &= \sum_{i=1}^{n}\left\{N_i^k\overline{P}_{i1}^k + N_{n+i}^k\left[-\overline{P}_{i1}^k + (1+\nu^k r)P_{i0} + (1+r)\lambda^k P_{i0}\right]\right\}\\ &\quad+\left\{W_0^k - \sum_{i=1}^{n}\left[(1+\phi^k)N_i^k + (\lambda^k+\phi^k)N_{n+i}\right]P_{i0}\right\}(1+r)\\ &= W_0^k(1+r) + \sum_{i=1}^{n}\left\{N_i^k\left[\overline{P}_{i1}^k - (1+\phi^k)(1+r)P_{i0}\right]\right.\\ &\quad\left.-N_{n+i}^k\left[\overline{P}_{i1}^k - (1+\nu^k r)P_{i0} + \phi^k(1+r)P_{i0}\right]\right\} \end{aligned} \tag{20}$$

and

$$\operatorname{Var}(W_1^k) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(N_i^k - N_{n+i}^k\right)\left(N_j^k - N_{n+j}^k\right)\Gamma_{ij}. \tag{21}$$

Since the same interest rate r applies to risk-free lending and borrowing and the margin deposit, they complement or counteract the effect of each other, allowing the investor to achieve any desired level of risk-free investment. Thus, the absence of λ^k from the above expressions is a natural outcome. [17]

[17] The problem can be formulated alternatively to accommodate margin purchases, with τ^k representing the margin provided by investor k for each dollar of purchase among the n securities. As expected, given the same interest rate r for risk-free lending and borrowing, Eqs. (20) and (21) remain unchanged under the alternative formulation; these equations are unaffected by τ^k.

The investor's optimal portfolio can be achieved by maximizing his/her certainty equivalent of the end-of-period wealth, [18]

$$CEQ^k = \overline{W}_1^k - \frac{A^k}{2}\mathrm{Var}(W_1^k), \tag{22}$$

subject to N_i^k, $N_{n+i}^k \geq 0$, for $i = 1,2,\dots,n$. This being a familiar quadratic programming problem, the optimality conditions can be obtained from $\partial(CEQ^k)/\partial N_i^k + \delta_i^k = 0$ and $\partial(CEQ^k)/\partial N_{n+i}^k + \delta_{n+i}^k = 0$, where δ_i^k and δ_{n+i}^k are slack variables. That is,

$$\overline{P}_{i1}^k - (1+\phi^k)(1+r)P_{i0} - A^k\sum_{j=1}^n \Gamma_{ij}\left(N_j^k - N_{n+j}^k\right) + \delta_i^k = 0, \tag{23}$$

$$\overline{P}_{i1}^k + \left[\phi^k(1+r) - (1+\nu^k r)\right]P_{i0} - A^k\sum_{j=1}^n \Gamma_{ij}\left(N_j^k - N_{n+j}^k\right) - \delta_{n+i}^k = 0, \tag{24}$$

$$N_i^k \geq 0,\ \ \delta_i^k \geq 0,\ \ N_i^k\delta_i^k = 0,\ \ N_{n+i}^k \geq 0,\ \ \delta_{n+i}^k \geq 0, \text{ and } N_{n+i}^k\delta_{n+i}^k = 0,$$
$$\text{for } i = 1,2,\dots,n. \tag{25}$$

Combining Eqs. (23) and (24) leads to

$$\delta_i^k + \delta_{n+i}^k = \left[2\phi^k(1+r) + (1-\nu^k)r\right]P_{i0} > 0. \tag{26}$$

As in the normative analysis, the implication is that, for each security i, at least one of δ_i^k and δ_{n+i}^k must be positive and, consequently, N_i^k and N_{n+i}^k cannot both be positive.

3.1. A market-clearing pricing relationship

To clear the market, the supply of each security i, Q_i, must be matched exactly by the aggregate demand for it by all K investors, $\Sigma_{k=1}^K(N_i^k - N_{n+i}^k)$. Aggregating Eqs. (23) and (24) over all K investors yields, respectively,

$$\hat{P}_{i1} - \left(1+\hat{\phi}\right)(1+r)P_{i0} - \hat{A}\sum_{j=1}^n \Gamma_{ij}Q_j + \hat{\delta}_i = 0 \tag{27}$$

and

$$\hat{P}_{i1} - \left[(1+\hat{\nu}r) - \hat{\phi}(1+r)\right]P_{i0} - \hat{A}\sum_{j=1}^n \Gamma_{ij}Q_j - \hat{\delta}_{n+i} = 0,$$
$$\text{for } i = 1,2,\dots,n. \tag{28}$$

[18] The equilibrium analysis here follows the Eun and Janakiramanan (1986) formulation for the derivation of pricing relationships. The formulation differs from that by Sharpe (1991) in that prices, rather than returns, are used here in establishing market equilibrium.

With $1/\hat{A} = \Sigma_{k=1}^{K} 1/A^k$ being the aggregate risk tolerance for the market, $\hat{P}_{i1} = \hat{A}\Sigma_{k=1}^{K} \bar{P}_{i1}^k/A^k$, $\hat{\nu} = \hat{A}\Sigma_{k=1}^{K} \nu^k/A^k$, $\hat{\phi} = \hat{A}\Sigma_{k=1}^{K} \phi^k/A^k$, $\hat{\delta}_i = \hat{A}\Sigma_{k=1}^{K} \delta_i^k/A^k$, and $\hat{\delta}_{n+i} = \hat{A}\Sigma_{k=1}^{K} \delta_{n+i}^k/A^k$ are various averages, as weighted by individual investors' risk tolerance measures.

Since market clearing of security i, for $i = 1,2,\ldots,n$, cannot be achieved if the security is simultaneously held short by all K investors, the case of $\hat{\delta}_{n+i} = 0$ must be ruled out. Now, with $\hat{\delta}_i \geq 0$ and $\hat{\delta}_{n+i} > 0$, Eqs. (27) and (28) can be combined as

$$P_{i0}^{\min} \leq P_{i0} < P_{i0}^{\max}, \quad \text{for} \quad i = 1,2,\ldots,n, \tag{29}$$

where

$$P_{i0}^{\min} = \frac{1}{(1+\hat{\phi})(1+r)}\left(\hat{P}_{i1} - \hat{A}\sum_{j=1}^{n} \Gamma_{ij} Q_j\right), \tag{30}$$

and

$$P_{i0}^{\max} = \frac{1}{(1+\hat{\nu} r) - \hat{\phi}(1+r)}\left(\hat{P}_{i1} - \hat{A}\sum_{j=1}^{n} \Gamma_{ij} Q_j\right). \tag{31}$$

As can be inferred from the above pricing relationship, the equilibrium prices are unique if the conditions of $A^1 = A^2 = \ldots = A^K$ and $\bar{P}_{i1}^1 = \bar{P}_{i1}^2 = \ldots = \bar{P}_{i1}^K$, for $i = 1,2,\ldots,n$, are satisfied. The reason is as follows. Under these conditions, all K investors must have the same allocation of wealth among the n securities, according to optimality conditions (23)–(25). If security i is held long (short) by one investor, it must be held long (short) by all K investors; if it fails to be selected by one investor, it must not be selected by any other investor. The only scenario leading to market clearing is that the security is held long by all K investors. With $\delta_i^k = 0$, for $k = 1,2,\ldots,K$, and consequently $\hat{\delta}_i = 0$, Eq. (27) leads to $P_{i0} = P_{i0}^{\min}$, for $i = 1,2,\ldots,n$; that is, the market-clearing prices for the n securities are unique. [19]

Under the homogeneity conditions regarding the K investors, an increase (a decrease) in $\hat{\phi}$ causes the market to clear at a lower (higher) price P_{i0} for each security i. In the special case where $\phi^k = 0$, for $k = 1,2,\ldots,K$, or, equivalently, $\hat{\phi} = 0$, the market-clearing prices are also provided by the security market line of the CAPM: [20]

$$P_{i0} = \frac{1}{1+r}\left(\hat{P}_{i1} - \hat{A}\sum_{j=1}^{n} \Gamma_{ij} Q_j\right), \quad \text{for} \quad i = 1,2,\ldots,n. \tag{32}$$

[19] With $\delta_i^k = 0$ and, from Eq. (26), $\delta_{n+i}^k = [2\phi^k(1+r) + (1-\nu^k)r]P_{i0}$, for $i = 1,2,\ldots,n$ and $k = 1,2,\ldots,K$, Eq. (28) also leads to $P_{i0} = P_{i0}^{\min}$.

[20] To write Eq. (32) in a more traditional form of the security market line, let $\mu_i = \hat{P}_{i1}/P_{i0} - 1$, $\sigma_{ij} = \Gamma_{ij}/(P_{i0}P_{j0})$, $W_m = \Sigma_{j=1}^{n} Q_j P_{j0}$, $X_i = Q_i P_{i0}/W_m$, and $\sigma_{im} = \Sigma_{j=1}\sigma_{ij}X_j$. Then, with Eq. (32) written as $\mu_i = r + \hat{A}W_m\sigma_{im}$, let $\mu_m = \Sigma_{i=1}^{n} X_i\mu_i$ and $\sigma_m^2 = \Sigma_{i=1}^{n} X_i\sigma_{im}$. As $\mu_m = r + \hat{A}W_m\sigma_m^2$ or, equivalently, $\hat{A}W_m = (\mu_m - r)/\sigma_m^2$, it follows that $\mu_i = r + (\mu_m - r)\beta_i$, where $\beta_i = \sigma_{im}/\sigma_m^2$.

In the CAPM, the portfolio allocation based on the n securities is the same among all investors, and thus the only way for a security to clear the market is that it is held long by all investors. The market price of each security must be low enough to allow this to happen. Since $P_{i0}^{\min}$ is the price at which security i is held long by all investors, it is a natural outcome that, in the special case of $\hat{\phi}=0$, $P_{i0}^{\min}$ matches the corresponding price in the CAPM.

If any of the conditions of $A^1 = A^2 = \ldots = A^K$ and $\overline{P}_{i1}^1 = \overline{P}_{i1}^2 = \ldots = \overline{P}_{i1}^K$, for $i=1,2,\ldots,n$, is violated, however, the market-clearing prices are no longer unique. Then, there is a continuum of prices within the two boundaries, $P_{i0}^{\min}$ and $P_{i0}^{\max}$, for each security i that all allow the aggregate demand by the K investors to match the supply, Q_i.[21] Indeed, exactly what the market-clearing price of each security will be is unpredictable prior to the beginning-of-period trading in the market.

Regarding the effects of $\hat{\phi}$ and $\hat{\nu}$ on the pricing relationship, notice that $\partial(P_{i0}^{\min})/\partial\hat{\phi}<0$, $\partial(P_{i0}^{\max})/\partial\hat{\phi}>0$, and $\partial(P_{i0}^{\max})/\partial\hat{\nu}<0$, for $i=1,2,\ldots,n$. An increase of market impediments in the form of higher proportional transaction costs for security trading or lower interest rebates on the short-sale proceeds to short sellers, therefore, will widen the gap between the two price boundaries:[22] Intuitively, an investor k who faces higher proportional costs of transacting, as caused by a higher ϕ^k (a higher ϕ^k and/or a lower ν^k), tends to require a lower (higher) P_{i0} to justify the purchase (short sale) of security i. Then, as $\hat{\phi}$ increases and/or $\hat{\nu}$ decreases, any achievable market clearing of the security, at a given P_{i0}, tends to be from the participation of fewer investors. This feature is consistent with the results of the analyses by Brennan (1975) and Mayshar (1979) that an investor facing higher fixed costs of transacting has fewer securities in his/her optimal portfolio.

Within the same analytical framework but without short selling, it can be shown in an analogous manner that market-clearing prices for each security i are

[21] Notice that the presence of different interest rates for risk-free lending and borrowing in a market is the manifestation of a form of transaction costs. In an equilibrium analysis with divergent lending and borrowing rates but without any restriction on short selling, Brennan (1971) finds that there are infinitely many efficient portfolios, all as combinations of lending and borrowing portfolios, and that the expected return (or, in the present setting, the beginning-of-period price) of a security in equilibrium is affected by who the lenders and the borrowers are, among all investors. Since an investor's portfolio decision, including his/her decision on lending or borrowing, in turn, is affected by the beginning-of-period prices, what can be inferred from the analysis of Brennan (1971), therefore, is that the equilibrium prices are also not unique.

[22] Decreasing market impediments to the extreme with $\hat{\phi}$ being zero and $\hat{\nu}$ approaching one, the gap between the two pricing boundaries would almost disappear. The pricing relationship would approach that of the CAPM. This is expected because $\hat{\nu}$ is exactly one under Lintner's treatment of short-sale transactions in the CAPM.

also given by Eq. (27) but with $Q_j = \Sigma_{k=1}^{K} N_j^k$ in the equation. [23] Since $N_i^k \geq 0$, $\delta_i^k \geq 0$, and $N_i^k \delta_i^k = 0$, for $i = 1,2,\ldots,n$ and $k = 1,2,\ldots,K$, and, consequently, $\hat{\delta}_i = \hat{A}\Sigma_{k=1}^{K} \delta_i^k / A^k \geq 0$, the same condition that $P_{i0} \geq P_{i0}^{\min} = (\hat{P}_{i1} \hat{A}\Sigma_{j=1}^{n} \Gamma_{ij} Q_j)/[(1+\hat{\phi})(1+r)]$ still holds. For the special case of $\hat{\phi} = 0$, the pricing relationship is equivalent to that in the Sharpe (1991) model without short selling where single-period returns, rather than prices, are used. In either formulation, the security market line of the CAPM still provides the lower boundary of market-clearing prices for each security.

Again, if the conditions of $A^1 = A^2 = \ldots = A^K$ and $\bar{P}_{i1}^1 = \bar{P}_{i1}^2 = \ldots = \bar{P}_{i1}^K$, for $i = 1,2,\ldots,n$, are satisfied, the market-clearing prices are unique; that is, $P_{i0} = P_{i0}^{\min}$. Like the short-sale case, violations of any of these conditions also lead to a continuum of market-clearing prices for each security. The difference from the short-sale case, however, is in the range of achievable market-clearing prices for each security. While the price range for the short-sale case is well specified, exactly how high each P_{i0} in the no-short-sale case can reach is unclear. Intuitively, with short selling disallowed, security i can still clear the market at prices well above $P_{i0}^{\min}$, provided that the supply, Q_i, is matched by the aggregate demand of one or more investors who are highly optimistic about the end-of-period price of the security. The allowance of short selling prevents P_{i0} from reaching excessively high levels because some investors who are pessimistic about the end-of-period price of the security will sell it short as soon as P_{i0} is high enough to justify the transactions.

3.2. Economic implications

Any limitations of the single-period framework notwithstanding, the lack of uniqueness of market-clearing prices, in general, should be a robust outcome, considering that a different price of a security could result in a different combination of investors who choose to invest in the security. The lack of uniqueness of market-clearing prices has undesirable consequences. In a single-period market, a firm's cost of equity capital can be inferred directly from investors' expected return for holding its stock. The indeterminateness of prices would make it difficult for investors to know what returns to expect for given risk exposures of their investments in the capital market. Then, the firm's cost of equity capital is also unpredictable prior to the beginning-of-period trading. Not being able to pinpoint the cost of capital, the firm must have difficulties in investment decisions. Overestimation (underestimation) of the cost of capital causes the rejection

[23] For each investor k, since $\bar{W}_1^k = W_0(1+r) + \Sigma_{i=1}^{n} N_i^k [\bar{P}_{i1}^k - (1+\phi^k)(1+r)P_{i0}]$ and $\mathrm{Var}(W_1^k) = \Sigma_{i=1}^{n}\Sigma_{j=1}^{n} N_i^k N_j^k \Gamma_{ij}$, the optimality conditions, as achieved by maximizing $CEQ^k = \bar{W}_1^k - (A^k/2)Var(W_1^k)$ subject to $N_i^k \geq 0$, for $i = 1,2,\ldots,n$, are $\bar{P}_{i1}^k - (1+\phi^k)(1+r)P_{i0} - A^k \Sigma_{j=1}^{n} \Gamma_{ij} N_j^k + \delta_i^k = 0$, $N_i^k \geq 0$, $\delta_i^k \geq 0$, and $N_i^k \delta_i^k = 0$. Upon aggregation over all K investors to clear the market, Eq. (27) follows.

(acceptance) of worthwhile (inferior) projects and, consequently, inefficient allocation of the firm's resources is inevitable.

In view of the above economic implications, a relevant question is whether and to what extent the allowance of short selling under institutional procedures can enhance the efficiencies in the allocation of resources in the economy. For a market where short selling is prohibited, while the lower price boundary for each security is known, it is unclear as to how high individual prices can be before the market fails to clear. The difficulties in the prediction of market-clearing prices, prior to the beginning-of-period trading, and hence in the determination of each firm's cost of equity capital are less severe if short selling is allowed instead. The upper price boundary for each security in a market allowing short sales serves to prevent prices from reaching excessively high levels, thus narrowing the range of achievable market-clearing prices.

Within the analytical frameworks of the Sharpe (1991) paper and the present study, market impediments against security trading are in the form of total disallowance of short selling, proportional transaction costs, or zero/partial rebates of interest on short-sale proceeds to short sellers. The less market impediments, the closer are the two price boundaries for each security. Consequently, market-clearing prices and each firm's cost of capital are more predictable, and the economic allocation of resources in the market is more efficient. [24]

4. Conclusion

In view of the acceptance of short selling as a viable investment tool by a small but growing number of institutional investors in recent years, the present study considers portfolio selection with short selling under a full-information covariance structure of security returns. Both normative and market-equilibrium perspectives are considered. Since the analysis can accurately capture institutional procedures for short selling, it is an improvement over previous portfolio models with short selling that retains the full-information covariance structure.

From a normative perspective, short selling has good potential to improve the portfolio's risk–return trade-off. For a given set of securities for portfolio consideration, the present analysis is able to filter out securities which are desirable

[24] In a study examining effects of short-sale constraints on the speed of adjustment of security prices to private information, Diamond and Verrecchia (1987) show analytically that short-sale prohibition reduces informational efficiency. An empirical implication of Diamond and Verrecchia's analysis is that reducing the cost of short selling increases the speed of adjustment to private information, and hence informational efficiency. The present equilibrium analysis, in turn, argues that a reduction of market impediments, by increasing the short-sale benefit, enhance the efficiency in the economic allocation of resources. Thus, the present study complements very well Diamond and Verrecchia's analysis with respect to the effects of short-sale constraints on the market.

neither for purchasing nor for short selling. The analysis is also able to accommodate cash purchases, margin purchases, and the use of purchased securities to satisfy part or all of the margin requirement for short selling. The analysis can be implemented using spreadsheets on microcomputers without relying on sophisticated algebraic manipulations and traditional forms of computer programming. The easy accessibility of the analysis and its accuracy in capturing institutional procedures for short selling should further enhance the practical appeal of portfolio modeling to practitioners in portfolio management.

The equilibrium analysis in the present study provides a pricing relationship under institutional procedures for short selling. Within a single-period framework, the market-clearing prices are in general not unique but are confined within two boundaries. The gap between the price boundaries widens with increasing proportional transaction costs for security trading, as well as with decreasing rebates of interest, to short-sellers, that brokerage firms earn from short-sale proceeds.

Market impediments, in the form of total disallowance of short selling, transaction costs, or zero/partial rebates of interest on short-sale proceeds to short sellers, all contribute to the indeterminateness of security prices and each firm's cost of equity capital, and hence inefficiencies in the allocation of resources. Many institutional regulations that restrict or discourage investors from engaging in short-sale activities are intended for safeguarding the stability of the equity market, under the assumption that short selling would exacerbate the situation when security prices are falling. Market impediments against short selling, however, would dampen investors' response to unfavorable information and interfere with useful market signals that could be revealed in investors' response. The analytical result in the present study, therefore, raises the question of whether market impediments against short selling are really desirable.

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Appendix A

A numerical illustration of the normative analysis for the case of $n = 2$ is presented below. Panel A of Table 1 shows the input data. The tangency portfolio considered is for $r = \hat{r} = 0.06$ only. To construct the entire efficient frontier, a wide range of values of $\hat{r} \geq r_{\ell}$ would be required. With λ initially set at zero, the

Table 1
An illustrative example

A. *Input data* $n = 2$, $\hat{r} = 0.06$, $r_f = 0.05$, $\nu = 0.7$, $\phi = 0.01$, $m = 0.5$

i	$\bar{R}_i$	σ_i	$\sigma_{ij}/(\sigma_i\sigma_j)$	
			$j=1$	$j=2$
1	0.14	0.16	1.0	
2	−0.02	0.08	0.3	1.0

B. *The case of* $\lambda = 0$

i	$\hat{R}_i$	σ_i	$\hat{\sigma}_{ij}/(\hat{\sigma}_i\hat{\sigma}_j)$			
			$j=1$	$j=2$	$j=3$	$j=4$
1	0.1287	0.1584	1.0			
2	−0.0297	0.0792	0.3	1.0		
3	−11.5000	16.0000	−1.0	−0.3	1.0	
4	4.5000	8.0000	−0.3	−1.0	0.3	1.0

The first iteration

$z_1 = 0$, $\delta_1 = -0.15545 + 1.00594\ r \geq 0$ if $r \geq 0.15453\ (= r_c)$
$z_2 = 0$, $\delta_2 = -0.01485 + 1.00990\ r \geq 0$ if $r \geq 0.01471$
$z_3 = 0$, $\delta_3 = 14.20000 + 0.40000\ r \geq 0$ if $r \geq -35.50000$
$\delta_4 = 0$, $z_4 = 0.07031 - 0.01563\ r \geq 0$ if $r \leq 4.50000$

The second iteration

$\delta_1 = 0$, $z_1 = 6.80675 - 44.04877\ r \geq 0$ if $r \leq 0.15453$
$z_2 = 0$, $\delta_2 = -0.01485 + 1.00990\ r \geq 0$ if $r \geq 0.01471$
$z_3 = 0$, $\delta_3 = -1.50000 + 102.00000\ r \geq 0$ if $r \geq 0.01471$
$\delta_4 = 0$, $z_4 = 0.11075 - 0.27730\ r \geq 0$ if $r \leq 0.39938$

C. *Tangency portfolios*

λ	$x_1(\hat{x}_1)$	$x_2(\hat{x}_2)$	$x_3(\hat{x}_3)$	$x_4(\hat{x}_4)$	$\bar{R}_p$	σ_p	θ	y
0.0000	0.9779	0.0000	0.0000	0.0221	0.2253	0.1970	0.8390	−0.6210
	(0.9682)	(0.0000)	(0.0000)	(2.2102)				
0.1000	0.8028	0.0000	0.0000	0.1972	0.1930	0.1606	0.8278	−0.3196
	(0.7949)	(0.0000)	(0.0000)	(1.7926)				
0.2000	0.6835	0.0000	0.0000	0.3165	0.1709	0.1358	0.8167	−0.1138
	(0.6767)	(0.0000)	(0.0000)	(1.5073)				
0.3000	0.5970	0.0000	0.0000	0.4030	0.1548	0.1177	0.8056	0.0356
	(0.5911)	(0.0000)	(0.0000)	(1.2999)				
0.2735	0.6175	0.0000	0.0000	0.3825	0.1586	0.1220	0.8086	0.001
	(0.6114)	(0.0000)	(0.0000)	(1.3491)				

present portfolio problem is formulated as a no-short-sale case based on four securities, two regular and two artificial, whose data, according to Eqs. (5)–(9), are shown in Panel B of the table.

Since $\hat{R}_4 = 4.5000$ is the highest among $\hat{R}_i$, for $i = 1,2,3,4$, security 4 is selected for the initial portfolio. Setting $z_1 = z_2 = z_3 = 0$ and $\delta_4 = 0$ for the first iteration, the remaining variables are solved in terms of r using Eq. (12), as a system of linear equations. The results, as shown also in Panel B, indicate that complementarity conditions (13) are satisfied as long as $4.5000 \geq r \geq 0.15453$. For $r < 0.15453$ (labeled as r_c, a critical value of r), δ_1 becomes negative. Given that $r_c > \hat{r}$, the procedure continues by adding security 1 to the portfolio, thereby enabling r to decrease further.

With $\delta_1 = \delta_4 = 0$ and $z_2 = z_3 = 0$ for the second iteration, Eq. (12) is used again to solve for the remaining variables in terms of r. As the results (also in Panel B) show, the optimality conditions are satisfied as long as $0.15453 \geq r \geq 0.01471$. Since this range encompasses $r = \hat{r}$, no further iteration is necessary. Next, the values of z_1 and z_4 for $r = \hat{r}$ are obtained using the corresponding expressions from the second iteration. With $x_i = z_i / \Sigma_{j=1}^{4} z_j$, for $i = 1,2,3,4$, values of $\hat{x}_i = x_i/(1+\phi)$, for $i = 1,2$, and $\hat{x}_i = x_i/(\lambda + \phi)$, for $i = 3,4$, are then deduced. Results of x_i and $\hat{x}_i$, for $i = 1,2,3,4$, along with $\overline{R}_p$, σ_p, and $\theta = (\overline{R}_p - \hat{r})/\sigma_p$, for the case of $\lambda = 0$, are shown at the top of Panel C of the table. Also shown is the excess margin, defined as $y = (1-m)\Sigma_{i=1}^{n} \hat{x}_i - (m-\lambda)\Sigma_{i=1}^{n} \hat{x}_{n+i}$, where $n = 2$ in this example.

Since $y < 0$ for $\lambda = 0$, the same analysis with positive values of λ is repeated. For each new λ, the input data of $\hat{R}_3$, $\hat{R}_4$, $\hat{\sigma}_3$, and $\hat{\sigma}_4$ are reestablished using Eqs. (6) and (9). The portfolio results for $\lambda = 0.1$, 0.2, and 0.3 are also shown in Panel C. As λ increases, y increases from negative to positive values. Further iterations for $\lambda > 0.2$ with smaller increments of λ reach $y \approx 0$ at $\lambda = 0.2735$. The final portfolio result, based on this λ and $\hat{R}_3 = -0.3574$, $\hat{R}_4 = 0.2070$, $\hat{\sigma}_3 = 0.5644$, and $\hat{\sigma}_4 = 0.2822$, is shown at the bottom of Panel C. [25]

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[25] Since $\lambda > 0$ in the present example, cash purchases of regular securities are required (that is, $\tau = 1$). If, instead, $y > 0$ for $\lambda = 0$, margin purchases would become feasible. The search procedure for the lowest acceptable margin τ, for $\hat{r} > r_b$, as described in Section 2.5, is analogous, and thus no numerical illustration is provided.

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