Journal of Sound and Vibration (1982) **82**(3), 401–410

PARAMETRIC EXCITATION STABILITY VIA HAMILTON'S ACTION PRINCIPLE

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(*Received* 19 *February* 1981, *and in revised form* 7 *October* 1981)

A Hamilton's principle based direct variational method for the asymptotic determination of the well-known stability/instability boundaries of Mathieu's equation is presented. The presence of time-dependent parameters in the system of the Lagrangian necessitates a generalization of the conventional Hamilton's principle: this consists in treating these variable system parameters as additional generalized co-ordinates, and subjecting them to similar variations. The interpretation of the resulting energetic expressions leads to the formulation of a "new" time-integral-of-energy stability criterion, and a "parametric invariance" theorem. Its relation/equivalence with existing non-conservative system energy tests is pointed out. The historical roots of the method are finally traced.

1. INTRODUCTION

It is well known that the linear parametric excitation theories lead to the study of the Hill–Mathieu equation, a second order homogeneous differential equation with a time-periodic coefficient (see, e.g., references [1, section 25], [2, chapter VI] and [3, sections 7.6 and 8.5]). As the theory of this equation shows, the boundedness or not of its solutions, for arbitrary (small) initial conditions, depends only on the relative values of the two time-dependent coefficient parameters; the first of these is the constant part, whereas the second is the time-dependent (cosinusoidal) term amplitude; in the sequel, they will be referred to as the "static" and the "kinetic" parameters respectively. On the two-parameter plane, the famous Strutt diagram, the regions of bounded or stable solutions are separated from those of unbounded or unstable ones by *transition* curves. The (linear) stability study then reduces to that of calculating the equations of these curves.

The standard method for the approximate determination of these stability/instability boundaries, for small kinetic parameter values, is to substitute into the equation of motion asymptotic expressions for the displacement and the static parameter in terms of powers of the kinetic parameter, in the well known Poincaré–Lindstedt sense; the periodicity requirements for the various orders remove the secular terms and supply the coefficients of these asymptotic transition curve expressions.

In what follows here these coefficients instead will be determined by using not the equation of motion, but Hamilton's Principle of Analytical–Variational mechanics for general non-contemporaneous (small) co-ordinate variations between the (originally) *constant coefficient conservative* motion of the system, and a specially chosen, neighbouring *time-dependent coefficient non-conservative* one. This will necessitate *treating the system's parameters, such as elasticity and/or inertia, on an equal footing with the system's generalized co-ordinates*: i.e., subjecting them to similar variations! This procedure is not meant to compete with other available ones: i.e., there is no particular *computational* advantage of this method as compared with, say, that of canonical perturbations, or

0022–460X/82/110401+10 $03.00/0 © 1982 Academic Press Inc. (London) Limited

multiple scales. The author has been aiming instead at conceptual clarification and, hopefully, greater physical understanding. It is also meant as a contribution to the small, but growing, family of variational principles for non-conservative systems.

The necessary tools are developed in sufficient generality in section 2. In section 3 this variational formalism is applied to the special case at hand, i.e., the Hill–Mathieu equation; the discussion is preceded by a summary of background information concerning this equation. Finally, in section 4 the results are discussed and interpreted, and compared with those of other time-integral treatments of non-conservative systems, and their origins are traced. An Appendix describes the procedure for the calculation of higher order coefficients.

2. HAMILTON'S ACTION PRINCIPLE

Consider a mechanical system described by the Lagrangian L,

$$L(q, \dot{q}; c) = T(q, \dot{q}) - V(q; c). \tag{1}$$

Here T, V, $q = q(t)$ and $\dot{q} = \dot{q}(t)$ denote, respectively, the kinetic and potential energies, and the generalized co-ordinate and velocity, and $c = c(t)$ stands for any *parameter* of the system—its stiffness, for example. The explicit dependence of c on the time t obviously makes the system non-conservative; this point will be fully discussed in section 3.2, following equations (16). Instead of the real time one may think of t as any other convenient evolution parameter; also $(\ldots)^{\cdot} \equiv \mathrm{d}(\ldots)/\mathrm{d}t$.

Furthermore, in the light of the formalism that follows, no extra difficulty would result if T too depended on some additional system parameter, say, an explicitly time-dependent inertia.

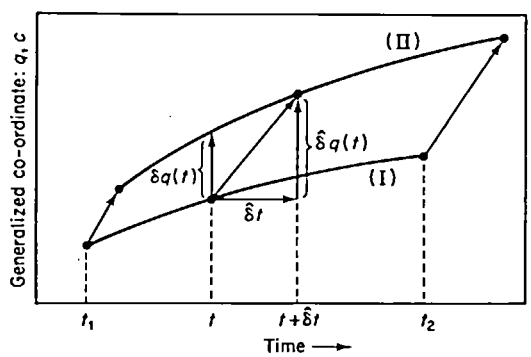

Figure 1. On the concept of *non-contemporaneous* variation $\hat{\delta}q(t) \equiv q_{II}(t+\hat{\delta}t) - q_I(t) = \delta q(t) + \dot{q}_I(t)\hat{\delta}t$ of a generalized co-ordinate $q(t)$ or $c(t)$.

Consider now the following two neighbouring paths of the system (see Figure 1): (i) Path I: the system moves continuously from the configuration $q_I(t_1)$ *and* $c_I(t_1)$ at time t_1, to the configuration $q_I(t_2)$ *and* $c_I(t_2)$ at time t_2; (ii) Path II: the system moves continuously from the configuration $q_{II}(t_1+\hat{\delta}t_1)$ and $c_{II}(t_1+\hat{\delta}t_1)$ at time $t_1+\hat{\delta}t_1$, to the configuration $q_{II}(t_2+\hat{\delta}t_2)$ and $c_{II}(t_2+\hat{\delta}t_2)$ at time $t_2+\hat{\delta}t_2$. A few clarifications are in order here. (i) The (first) *non-contemporaneous* and *contemporaneous* path variation operators $\hat{\delta}(\ldots)$ and $\delta(\ldots)$, respectively, are related at any time by the well known equations [3, section 9.1; 4, section 1.3]

$$\hat{\delta}(\ldots) = \delta(\ldots) + [\mathrm{d}(\ldots)/\mathrm{d}t]\hat{\delta}t, \qquad \hat{\delta}t \neq 0. \tag{2}$$

(ii) *The system's parameter $c(t)$ is treated just like another generalized co-ordinate.*

For both paths one evaluates the action functional $A = \int L\,dt$ between the corresponding time end points. The *total non-contemporaneous variation* of the action, from Path I, is defined as

$$\hat{\Delta}A = \int_{t_1+\hat{\delta}t_1}^{t_2+\hat{\delta}t_2} L(\mathrm{II})\,dt - \int_{t_1}^{t_2} L(\mathrm{I})\,dt. \tag{3}$$

Now, the part of $\hat{\Delta}A$ that is *linear* in $\hat{\delta}q(t) \equiv q_{\mathrm{II}}(t+\hat{\delta}t) - q_{\mathrm{I}}(t)$, $\hat{\delta}c(t) = c_{\mathrm{II}}(t+\hat{\delta}t) - c_{\mathrm{I}}(t)$, and $\hat{\delta}t$, is *the (first) non-contemporaneous variation* of $A(\mathrm{I})$, and according to standard rules of the calculus of variations is given by [5, section 13; 6, chapter IV, section 8; 4, section 1.3; 7, chapter II, sections 24, 25; 8, section 12.9]

$$\hat{\delta}A = \hat{\delta}\int_{t_1}^{t_2} L(\mathrm{I})\,dt = \int_{t_1}^{t_2} \delta L\,dt + [L(\mathrm{I})\hat{\delta}t]_{t_1}^{t_2}, \tag{4}$$

$$\delta L = \{(\partial L/\partial q)_{\mathrm{I}}\delta q + (\partial L/\partial \dot{q})_{\mathrm{I}}\delta\dot{q}\} + (\partial L/\partial c)_{\mathrm{I}}\delta c. \tag{5}$$

Partial integration in equation (4), and use of the commutation rule $\delta\dot{q} = (\delta q)^{\cdot}$ gives

$$\hat{\delta}A = \int_{t_1}^{t_2} \left\{\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right\}_{\mathrm{I}} \delta q\,dt + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial c}\right)_{\mathrm{I}} \delta c\,dt + \left[\left\{L - \left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q}\right\}_{\mathrm{I}} \hat{\delta}t + \left(\frac{\partial L}{\partial \dot{q}}\right)_{\mathrm{I}} \hat{\delta}q\right]_{t_1}^{t_2}. \tag{6}$$

Equation (6) constitutes the basic tool for the variational treatment of systems with time-dependent parameters (properties). Indeed from it one can obtain several special results by imposing particular conditions on L, the variations, and the end point values. Specifically, assuming that Path I is an actual, i.e., dynamical, trajectory, i.e.,

$$\left\{\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right\}_{\mathrm{I}} = 0, \tag{7}$$

and recalling that the first term of the integrated-out part in equation (6) (with a minus sign) equals the system's total energy E along Path I [9, section 12], whereas the second is the generalized momentum $p(t)$, yields

$$\hat{\delta}A = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial c}\right)_{\mathrm{I}} \delta c\,dt + [p_{\mathrm{I}}\hat{\delta}q - E(\mathrm{I})\hat{\delta}t]_{t_1}^{t_2}. \tag{8}$$

Assuming further that *both motions* I *and* II *are periodic with respective periods* τ *and* $\tau+\hat{\delta}\tau$, setting $t_1 = 0$, $t_2 = \tau$, $\hat{\delta}t_1 = 0$ and $\hat{\delta}t_2 = \hat{\delta}\tau$ and since in this case $p_{\mathrm{I}}(\tau) = p_{\mathrm{I}}(0)$, $q_{\mathrm{I}}(\tau) = q_{\mathrm{I}}(0)$, and $q_{\mathrm{II}}(\tau+\hat{\delta}\tau) = q_{\mathrm{II}}(0)$ (i.e., $\hat{\delta}q(\tau) = \hat{\delta}q(0)$), one can finally reduce equation (8)

$$\hat{\delta}A = \hat{\delta}\int_0^{\tau} L(\mathrm{I})\,dt = \int_0^{\tau} \left(\frac{\partial L}{\partial c}\right)_{\mathrm{I}} \delta c\,dt - E_{\mathrm{I}}(\tau)\hat{\delta}\tau. \tag{9}$$

3. APPLICATION TO THE HILL–MATHIEU EQUATION

3.1. BACKGROUND INFORMATION

Consider a mechanical system, the evolution of a typical generalized co-ordinate $q(t)$ of which is governed by the Hill–Mathieu equation

$$\ddot{q} + [s + k\cos t]q = 0; \tag{10}$$

the two constants s and k will be called the *static* and *kinetic* parameters respectively.

Along the transition curves oscillatory purely periodic functions $q(t)$, of period 2π or 4π, can be found, and for these combinations of s and k that describe them the system is in a state of "neutral stability". For the technically important case

$$|k| \ll s, \tag{11}$$

one can, therefore, (in the Poincaré–Lindstedt sense) assume along these curves the following asymptotic representations:

$$q(t;k) = q_0(t) + kq_1(t) + k^2 q_2(t) + \cdots, \qquad s = s(k) = s_0 + ks_1 + k^2 s_2 + \cdots. \tag{12, 13}$$

As the theory of equation (10) shows [2, 3], on the s–k plane (i) the stable regions are connected together at the points

$$s(0) = s_0 = (n/2)^2, \qquad n = 0, 1, 2, \ldots, \tag{14}$$

which correspond to the (linearly independent) periodic solutions $\{\sin(nt/2), \cos(nt/2)\}$ of equation (10), and (ii) from each of the points given by equations (14) *two* branches (symmetric with respect to the s axis) emerge, except for $n = 0$ where there is only one, on each pair of these branches $q(t;k)$ of equation (12) has period 2π (n even) or 4π (n odd), and (apart from inessential constant factors) for $|k| \to 0$

$$q(t;k) \to q_0(t) = \begin{Bmatrix} \sin(nt/2) \\ \cos(nt/2) \end{Bmatrix}. \tag{15}$$

The (linearized) stability problem then reduces to that of determining $s_1, s_2, \ldots$; the functions $q_1(t)$, $q_2(t)$, ... are needed only to the extent that they are necessary for the determination of $s_1, s_2, \ldots$. Here, however, the calculation will be restricted to that of the first (dominant) non-zero coefficient after s_0 in equation (13), and to the first pair of transition branches: i.e., for $n = 1$. It will soon become clear that no loss of generality will result from these restrictions: the methodology of the next subsection shall prove applicable to any pair of branches, and any coefficient of equation (13) (see the Appendix).

3.2. THE VARIATIONAL CHARACTERIZATION OF THE TRANSITION CURVES

The equation of motion (10) obviously derives from the Lagrangian

$$L[q, \dot{q}; c(t)] = T(\dot{q}) - V[q; c(t)] = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}c(t)q^2, \tag{16}$$

where $c(t) = s + kf(t)$, and $f(t + 2\pi) = f(t)$; $f(t)$ will be later identified with cost. The explicit dependence of this Lagrangian on time obviously makes the Mathieu's equation system non-conservative. A direct derivation of this fact can also be easily obtained as follows: multiplying the corresponding equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{16a}$$

with $\dot{q}$ and rearranging terms one readily obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} - L\right\} = -\frac{\partial L}{\partial t}, \tag{16b}$$

or, since the left-side bracketed expression represents the total energy $E = T + V$ of the system [9, section 12; 10, chapter V, section 3]

$$\frac{\mathrm{d}}{\mathrm{d}t}E = -\frac{\partial}{\partial t}(T - V) = -\left(\frac{\partial}{\partial c}(T - V)\right)\dot{c} = \left(\frac{\partial V}{\partial c}\right)\dot{c}\left(\text{here } \frac{\partial T}{\partial t} = 0\right) = \tfrac{1}{2}k\dot{f}(t)q^2 \quad (\neq 0). \tag{16c}$$

Now the connection with section 2 is made through the following identifications.

(i) The *conservative* periodic motion, corresponding to $k=0$, shall be taken as Path I. In this case

$$q_{\mathrm{I}}(t)\to q_0(t)=\begin{Bmatrix}\sin(t/2)\\ \cos(t/2)\end{Bmatrix},\quad c_{\mathrm{I}}(t)\to s_0|_{n=1}=1/4, \tag{17a, b}$$

$$\tau\to 4\pi,\ E(\mathrm{I})\to E_0=T_0+V_0=\tfrac{1}{2}\dot{q}_0^2+\tfrac{1}{2}s_0q_0^2=\text{constant}. \tag{17c, d}$$

(ii) The *slightly non-conservative* periodic motion corresponding to small $|k|$ values, shall be taken as Path II. Then, to the first order in k,

$$q_{\mathrm{II}}(t)\to q(t;k)=q_0(t)+kq_1(t),\ c_{\mathrm{II}}(t)\to s_0+ks_1+kf(t),\ \tau\to 4\pi, \tag{18a–c}$$

or

$$\delta q_0=kq_1(t),\qquad \delta c=k[s_1+f(t)], \tag{19a, b}$$

$$\hat{\delta}c=\delta c+\dot{c}_1(t)\hat{\delta}t=\delta c\ (\text{due to expression (17b)}),\qquad \hat{\delta}\tau=0; \tag{19c, d}$$

equation (19d) simply expresses the fact that here both motions I and II have the same period τ.

Combining expressions (16)–(19) one finally specializes equation (9) to

$$\hat{\delta}A=\int_0^{\tau=4\pi}(-\partial V/\partial c)_{\mathrm{I}}\delta c\,\mathrm{d}t=\int_0^{\tau=4\pi}(-\tfrac{1}{2}q_0^2)\{k[s_1+f(t)]\}\,\mathrm{d}t. \tag{20}$$

Now, since along the transition lines q_0 and δq_0 are $\tau=4\pi$ –periodic, and as equations (1), (5), (6) and (7) show

$$\int_0^{\tau=4\pi}[\delta T_0-(\partial V/\partial q)_{\mathrm{I}}\delta q_0]\,\mathrm{d}t=0, \tag{20a}$$

Hamilton's principle (or the integral of d'Alembert's equation) yields

$$\delta A=\int_0^{\tau=4\pi}(\delta T_0-\delta V_0)\,\mathrm{d}t=\{(\partial T/\partial\dot{q})_{\mathrm{I}}\delta q_0\}_0^{\tau=4\pi}=0,\ \text{or}$$

$$\int_0^{\tau=4\pi}-(\partial V/\partial c)_{\mathrm{I}}\delta c\,\mathrm{d}t=0. \tag{20b}$$

Equation (20a) shows that the (average) kinetic energy change is produced *solely* by the (average) work $(\partial V/\partial q)_{\mathrm{I}}\delta q_0$, and therefore *the (average) contribution of the additional work term* $(\partial V/\partial c)_{\mathrm{I}}\delta c$ (of δV_0) *to the kinetic energy change must reduce to zero,* i.e., equation (20b); more on the physical interpretation of these terms will appear later.

Collecting the above results, and since as equations (4)–(6) with expression (19d) show $\hat{\delta}A=\delta A$, one obtains the following fundamental variational equation:

$$\hat{\delta}A=0\ (\text{along transition lines}). \tag{21}$$

Combination of equations (20) and (21) (and since, as expression (19b) shows, the variation $\delta c(t)$ is not arbitrary!) gives

$$k\int_0^{4\pi}(-\tfrac{1}{2}q_0^2)[s_1+f(t)]\,\mathrm{d}t=0,$$

$$\text{or}\quad s_1=-\int_0^{4\pi}q_0^2f(t)\,\mathrm{d}t\Big/\int_0^{4\pi}q_0^2\,\mathrm{d}t. \tag{22}$$

This equation clearly shows the *averaging and balancing* effect of the small stiffness change ks_1.

Utilization of equation (17) in equation (22), with $f(t) = \cos t$, immediately gives the well known initial slopes s_1 of the two $s(k)$ curves emerging from $s(0) = s_0 = 1/4$:

$$s(k) = \left\{ \begin{matrix} \frac{1}{4} + \frac{1}{2}k + O_2(k),\ q_0 \sim \sin(t/2) = \textit{odd branch} \\ \frac{1}{4} - \frac{1}{2}k + O_2(k),\ q_0 \sim \cos(t/2) = \textit{even branch} \end{matrix} \right\}. \tag{23}$$

Conversely, if one moves away from the conservative Path I ($k = 0$) *along the* (*non-conservative*) *transition curves* with slopes $s_1 = \pm\frac{1}{2}$, then $\hat{\delta}A = 0$. The extension of this methodology to the calculation of the higher order coefficients $s_2, s_3, \ldots$ of equation (13) is briefly described in the Appendix.

4. DISCUSSION AND CONCLUSIONS

An engineering interpretation of the action's stationary, equations (20)–(22), is the following. Path I is an actual mechanical motion with generalized co-ordinate/parameter $c_1(t)$ ($= s_0 = 1/4$). A corresponding generalized force $-C(\mathrm{I})$, where

$$C(\mathrm{I}) = (\partial L/\partial c)_{\mathrm{I}} = -(\partial V/\partial c)_{\mathrm{I}} = -\tfrac{1}{2}q_0^2, \tag{24}$$

must then be acting *on* the system from its environment; conversely, *the* system reacts back *to* its environment, with respect to c_1 changes, with a force equal to $C(\mathrm{I})$. In this light, the work product

$$(\partial L/\partial c)_{\mathrm{I}}\delta c = -(\partial V/\partial c)_{\mathrm{I}}\delta c = C(\mathrm{I})\delta c = (-\tfrac{1}{2}q_0^2)\{k[s_1 + f(t)]\}, \tag{25}$$

represents *the elementary work* $(\delta W)_{\mathrm{nc}}$ *done by all the system's non-conservative agents, during the small change* $\delta c(t)$ *of its parameter*, at time t. Equations (20b) and (21) can then be restated as

$$\oint (\delta W)_{\mathrm{nc}}\,\mathrm{d}t = 0 \quad \left(\oint \equiv \int_0^{\tau} \left\{ \begin{matrix} \tau = 4\pi,\ n \text{ odd} \\ \tau = 2\pi,\ n \text{ even} \end{matrix} \right\}\right), \tag{26}$$

$$\text{or} \quad \underbrace{\oint (C)(ks_1)\,\mathrm{d}t}_{\substack{\text{average work (action)} \\ \text{of } \textit{internal} \text{ forces}}} = \underbrace{\oint (-C)(kf(t))\,\mathrm{d}t.}_{\substack{\text{average work (action)} \\ \text{of } \textit{external} \text{ forces}}} \tag{27}$$

This can be interpreted as follows. Consider a conservative system in steady state oscillation. Suppose now that small periodic non-conservative agents are acting on the system; these may have the form of time-dependent loads or variable characteristics such as stiffness or inertia. Then *the new* (*slightly*) *non-conservative system remains in that steady state if and only if the average non-conservative work in and out of the system, over its oscillation period, is zero.* If this "zero net average work" condition is not satisfied, and depending on the relative values of s and k, one has either (flutter-like) instability or stability: i.e., this condition supplies the "threshold" line equations!

Several energy-based techniques have been applied to other areas of (mainly autonomous) non-linear oscillations, for example in the *averaging/energy balance* method of the approximate limit cycle amplitude determination of the van der Pol oscillator [11, chapter 6; 12, chapters 2–5]. The foregoing interpretation, however, helps one understand why in the Mathieu equation case (as well as for more general non-autonomous systems) one is compelled to use a *variational* rather than the conventional *energetic* approach:

expanding the energy equation (16c) into k powers yields

$$\dot{E}(\mathrm{II}) = [E(\mathrm{I}) + \Delta E(\mathrm{I})]^{\cdot} = [E_0 + kE_1 + O_2]^{\cdot}$$
$$= (\delta E_0)^{\cdot} + O_2 \text{ (due to expression (17d))} = k[\tfrac{1}{2}\dot{f}(t)q_0^2] + O_2. \tag{28}$$

The periodicity of the system along the transition lines, however, requires that

$$\oint \dot{E}(\mathrm{II})\, \mathrm{d}t = 0, \tag{29}$$

and this, thanks to equation (28), leads to the following first-order "conservation" condition:

$$\oint (\delta E_0)\ \mathrm{d}t = \oint (\delta T_0 + \delta V_0)^{\cdot}\, \mathrm{d}t$$
$$= \frac{k}{2} \oint (\dot{f} q_0^2)\, \mathrm{d}t = 0; \tag{30}$$

equations (29) and (30) express energy conservation in an average (normed), rather than the conventional (pointwise), sense.

Although the above condition is satisfied for f and q_0, it does not supply, however, any information toward the objective of this investigation: i.e., the determination of s_1 $(s_2, \ldots)$.

In concluding this interpretation it is instructive to derive the action-based counterpart/complement of equation (30): as simple transformations show,

$$\oint (\delta E_0)\, \mathrm{d}t = \oint (\delta T_0 + \delta V_0)\, \mathrm{d}t = \oint (\delta T_0 - \delta V_0)\, \mathrm{d}t + \oint (2\delta V_0)\, \mathrm{d}t$$
$$= \delta A + \oint 2[(\partial V/\partial q)_{\mathrm{I}} \delta q_0 + (\partial V/\partial c)_{\mathrm{I}} \delta c]\, \mathrm{d}t = \oint 2[(\partial V/\partial q)_{\mathrm{I}} \delta q_0]\, \mathrm{d}t, \tag{30a}$$

and invoking equations (20a) and (20b) finally yields

$$\oint (\delta E_0)\, \mathrm{d}t = 2\oint (\partial V/\partial q)_{\mathrm{I}} \delta q_0\, \mathrm{d}t = 2\oint (\delta T_0)\, \mathrm{d}t = 2\oint (s_0 q_0)\delta q_0\, \mathrm{d}t = 2\oint (\dot{q}_0)\delta \dot{q}_0\, \mathrm{d}t. \tag{30b}$$

Thus, along the transition lines the Mathieu equation satisfies an averaged energy equation similar to that of the ordinary (i.e., constant coefficient) harmonic oscillator. Routh, in (the second part of) his famous treatise on dynamics, and as an application of variational mechanics to the study of *cyclical* motions, proposed the following problem: "If the period of complete recurrence of a dynamical system is not altered by the addition of energy, prove that this additional energy is equally distributed into potential and kinetic energies" [13, chapter X, Articles 461 and 462]. Equations (30a) and (30b) constitute a proof of this for the Mathieu system.

A physicist might interpret the matter as follows: one has successively

$$\hat{\delta} A = \hat{\delta}[\overline{L(\mathrm{I})}(\tau)] = [\hat{\delta}\, \overline{L(\mathrm{I})}](\tau) = [\overline{C(\mathrm{I})\delta c}](\tau) = 0, \tag{31}$$

$$\text{or} \quad \hat{\delta}[\overline{L(\mathrm{I})}] = \overline{C(\mathrm{I})\delta c} = 0, \tag{32}$$

where

$$(\overline{\ldots}) \equiv (\tau)^{-1} \oint (\ldots)\, \mathrm{d}t \tag{33}$$

is the average value of (...) over a period $\tau(=4\pi$, or $2\pi)$; i.e., the first part of equation (32) states that *along the transition lines, the average (time dependent parameter!) Lagrangian* $\overline{L[q, \dot{q}; c(t)]}$ *is a "parametric" invariant.* And this is the closest to a "conservation" equation one can get for this non-conservative system! Similar invariants have been discovered in other areas of analytical mechanics: e.g., *adiabatic* (for very slow variation of $c(t)$), and *integral* (for slightly differing initial conditions of the equations of motion) [14, section 49; 3, section 9.4].

The idea behind this "kinetic" stability criterion (the italic parts in both interpretations above) is not new. Several investigators, including this author, have independently arrived at it via either energy or variational (d'Alembert's equation) methods; see reference [15] and references cited therein. What is important here, however, is the development of a unifying variational principle formalism that is capable of dealing with *time-dependent parameter systems*; also the energetic interpretation of the stiffness change in equation (22) throws more light into the energy flux mechanism of the Hill–Mathieu equation. Hamilton's Principle was presented in its most general (i.e., non-contemporaneous/variable end point variations) form (6), not only to show all the steps and assumptions involved, *but also to keep the door open for the treatment of wider classes of functions* $f(t)$ *in equation* (16) *and/or path variations,* as a genuine method should: for more general $c(t)$ changes $\hat{\delta}\tau$ of equations (9), and (19d) is not zero, and therefore $\hat{\delta}A \neq 0$ even when the perturbed non-conservative system oscillates periodically [14, section 49].

Finally, the crucial idea behind this particular Hamiltonian formulation, i.e., that of *treating the system's parameters as additional generalized co-ordinates,* dates back to the (second half of the) nineteenth century attempts to describe the various thermo- and electro-mechanical (i.e., non-conservative!) phenomena with the methods and concepts of analytical mechanics. At this point, this author would like to acknowledge his indebtedness to the lively presentation of Polak [16] on the evolution of variational principles in mechanics and physics, and to the relevant epoch-making contributions of Boltzmann, Clausius, Thomson *et al.*; see references given in reference [16, chapter V], and also reference [17, chapters III and IV].

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APPENDIX

The higher-order stability coefficients $s_2, s_3, \ldots$ of equation (13) can be determined by successively setting the second, third, ... variations of the Action equal to zero, i.e., $\delta^2 A = 0$, $\delta^3 A = 0, \ldots$; as usual, the nth variation $\delta^n A$ is defined as the homogeneous functional of degree n in k resulting from the following Taylor-like expansion of equation (3), and equation (16) (with utilization of equations (12) and (13)):

$$\Delta A = \delta A + \frac{1}{2!}\delta^2 A + \frac{1}{3!}\delta^3 A + \cdots. \tag{A1}$$

For example, the second-order condition $\delta^2 A = 0$, with the help of equations (16)–(19), yeilds successively

$$\begin{aligned}\delta^2 A \equiv \delta(\delta A) &= \int_0^\tau \delta\left[\left(\frac{\partial L}{\partial c}\right)_I \delta c\right] \mathrm{d}t \\ &= \int_0^\tau \left\{\left[\left(\frac{\partial^2 L}{\partial q \partial c}\right)_I \delta q + \left(\frac{\partial^2 L}{\partial \dot{q} \partial c}\right)_I \delta \dot{q} + \left(\frac{\partial^2 L}{\partial c^2}\right)_I \delta c\right] \delta c + \left(\frac{\partial L}{\partial c}\right)_I \delta^2 c\right\} \mathrm{d}t \\ &= \int_0^\tau \{[(-q_0)(kq_1) + 0 + 0](k\{s_1 + f(t)\}) + (-\tfrac{1}{2}q_0^2)(2k^2 s_2)\}\, \mathrm{d}t = 0,\end{aligned} \tag{A2}$$

from which

$$s_2 = -\int_0^\tau \{s_1 + f(t)\} q_0 q_1 \, \mathrm{d}t \Big/ \int_0^\tau q_0^2 \, \mathrm{d}t. \tag{A3}$$

In the spirit of equation (26), this translates to

$$\delta^2 A = \oint (\delta^2 W)_{\mathrm{nc}} \, \mathrm{d}t = 0, \tag{A4}$$

and corresponds to equation (A7) of reference [15], with a minus sign. In each step of this process, one invokes the 4π-periodicity of $q_0, q_1, q_2, \ldots$, for n odd, plus appropriate initial normalizing conditions, and the equations of motion derived in all previous steps; for n even, one uses 2π-periodicity. Thus, *along the transition lines* (and so long as equation (10) is physically acceptable), $\Delta A = 0$. Particulars of the calculation are given in the Appendix of reference [15].

In closing, it should be remarked that the methodology described in this paper, along with its energetic interpretation, can be extended without much difficulty to the calculation of the stability boundaries of the general Hill's equation, i.e.,

$$\ddot{q} + [s + kf_1(t) + k^2 f_2(t) + \ldots]q = 0, \tag{A5}$$

where $f_i(t)$ $(i = 1, 2, \ldots)$ are periodic with the *same* period; here, too, one seeks approximate solutions in the form given by equations (12) and (13).

Correction. In reference [15], on the first line of page 506, the second term of the expression for $q_1(t)$ should be $(1/4) \sin (3t/2)$, instead of $(1/4) \cos (3t/2)$: i.e., the cosine should be replaced by the sine.