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The Dirichlet-to-Neumann Operator on Exterior Domains

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Abstract We define two versions of the Dirichlet-to-Neumann operator on exterior domains and study convergence properties when the domain is truncated.

Keywords Dirichlet-to-Neumann operator · Resolvent convergence · Harmonic function

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1 Introduction

On a bounded open connected set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial\Omega$, the Dirichlet-to-Neumann operator A_Ω is fairly well studied, see for example [1, 4, 7, 8, 10, 11, 13, 16, 17]. It is a self-adjoint operator on $L_2(\Gamma)$, which is defined as follows. A function $\varphi \in L_2(\Gamma)$ is in the domain $D(A_\Omega)$ of A_Ω if and only if the solution u of the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 && \text{weakly on } \Omega \\ u &= \varphi && \text{on } \Gamma \end{aligned}$$

has a normal derivative $\partial_\nu u$ which is in $L_2(\Gamma)$, and then one sets $A_\Omega\varphi = \partial_\nu u$. The aim of this paper is to study the Dirichlet-to-Neumann operator on an exterior domain. It is again a self-adjoint operator on $L_2(\Gamma)$, where the boundary Γ of the exterior domain is a compact set which we endow with the surface measure. A new feature for exterior domains is the behaviour of the harmonic function u at infinity. We consider a Dirichlet and a Neumann boundary condition, which lead to different Dirichlet-to-Neumann operators. One of our

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main results (Theorem 5.9) shows that the difference of these two operators is a bounded rank-one operator. It is natural to compare our Dirichlet-to-Neumann operators on the exterior domain Ω with Dirichlet-to-Neumann operators on the truncated domain $\Omega_R = \Omega \cap B_R$, where B_R is a large ball of radius R. We will prove convergence as $R \to \infty$ for both boundary conditions (Theorems 5.5 and 5.6). Even though in Potential Theory exterior domains are a classical subject (see e.g. [15]), so far no systematic study of the Dirichlet-to-Neumann operator on such domains seems to be known. For a modified operator, however, numerical calculations on exterior domains have been obtained by Brown–Marletta [12] and Marletta [22].

We now explain our results in more detail. Throughout the paper we consider a connected exterior domain $\Omega \subset \mathbb{R}^d$, with $d \geq 3$, of the form $\Omega = \mathbb{R}^d \setminus \overline{\Omega_0}$, where Ω_0 is a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary Γ. Thus the boundary Γ of Ω_0 coincides with the boundary $\partial\Omega$ of Ω. Of great importance for our study are the Sobolev spaces

$$W(\Omega) = \left\{ u \in H^1_{\mathrm{loc}}(\Omega) : \int_\Omega |\nabla u|^2 < \infty \right\}$$

and

$$W^D(\Omega) = W(\Omega) \cap L_p(\Omega),$$

where $\frac{1}{p} = \frac{1}{2} - \frac{1}{d}$. It turns out that $W^D(\Omega)$ is the space of all $u \in W(\Omega)$ with 'average' zero. We consider two versions of the Dirichlet-to-Neumann on the exterior domain Ω. The version A^D with *Dirichlet boundary conditions at infinity* is as follows. Let $\varphi \in L_2(\Gamma)$. Then $\varphi \in D(A^D)$ if there exists a $u \in W^D(\Omega)$ such that $\Delta u = 0$ weakly on Ω, with trace $\mathrm{Tr}\, u = \varphi$ and u has a normal derivative $\partial_\nu u \in L_2(\Gamma)$. Then we set $A^D \varphi = \partial_\nu u$. It turns out that A^D is a positive self-adjoint operator. The alluded approximation with truncated domains is as follows. Let $\psi \in L_2(\Gamma)$ and $\lambda > 0$. Let $R > 0$ be such that $\overline{\Omega_0} \subset B_R$. Then there exists a unique $u_R \in W^D(\Omega)$ such that

$$\begin{aligned} \Delta u_R &= 0 && \text{weakly on } \Omega \cap B_R, \\ u_R(x) &= 0 && \text{for all } x \in \mathbb{R}^d \text{ with } |x| \geq R, \\ \lambda\, \mathrm{Tr}\, u_R + \partial_\nu u_R &= \psi && \text{on } \Gamma. \end{aligned} \tag{1}$$

Moreover, there exists a unique $u \in W^D(\Omega)$ such that

$$\begin{aligned} \Delta u &= 0 && \text{weakly on } \Omega \cap B_R, \\ \lambda\, \mathrm{Tr}\, u_R + \partial_\nu u_R &= \psi && \text{on } \Gamma. \end{aligned}$$

We shall prove in Theorem 4.3 that

$$\lim_{R \to \infty} u_R = u \tag{2}$$

in $W^D(\Omega)$, that is $\lim_{R\to\infty} \int_\Omega |\nabla(u_R - u)|^2 = 0$. Hence $\left(\lambda I + A^D\right)^{-1} \psi = \lim_{R\to\infty} \mathrm{Tr}\, u_R$ in $L_2(\Gamma)$. Moreover, if there exists a $q \in (d, \infty)$ such that $\psi \in L_q(\Gamma)$, then we show in Theorem 4.12 that $u_R, u \in C(\overline{\Omega})$ and Eq. 2 is valid uniformly on Ω.

We also consider the Dirichlet-to-Neumann operator A on Ω with *Neumann boundary conditions at infinity*, replacing $W^D(\Omega)$ by $W(\Omega)$. Then in Theorem 5.6 we prove resolvent convergence of a truncated version of the Dirichlet-to-Neumann operator on $\Omega \cap B_R$ with Neumann boundary conditions on $\{x \in \mathbb{R}^d : |x| = R\}$ instead of Eq. 1.

Let S^D and S be the semigroups on $L_2(\Gamma)$ generated by A^D and A. We prove in Section 5 that both S^D and S are real positive submarkovian and irreducible.

The outline of this paper is as follows. In Section 2 we consider the spaces $W(\Omega)$ and $W^D(\Omega)$. These were introduced by Lu-Ou [21]. In Section 3 we consider the Neumann problem on $\Omega \cap B_R$ with Dirichlet and Neumann boundary conditions at ∂B_R and in Section 4 we prove various convergence properties. In Section 5 we investigate the two versions of the Dirichlet-to-Neumann operator on Ω. Finally in Section 6 we compare A and A^D with the Dirichlet-to-Neumann operator A_{Ω_0} on Ω_0 in case Ω_0 is a ball.

2 Spaces

For an exterior domain the H^1 Sobolev space is too small for our purpose. Therefore we first start with a suitable Sobolev type space on $\mathbb{R}^d$. Throughout this paper we fix $d \geq 3$. Define the space

$$W(\mathbb{R}^d) = \left\{ u \in H^1_{\text{loc}}\left(\mathbb{R}^d\right) : \int_{\mathbb{R}^d} |\nabla u|^2 < \infty \right\}.$$

Moreover, throughout this paper let $p = 2^*$, i.e. $p \in (2, \infty)$ is such that $\frac{1}{p} = \frac{1}{2} - \frac{1}{d}$. Then $W^{1,2}\left(\mathbb{R}^d\right) \subset L_p\left(\mathbb{R}^d\right)$ by the Sobolev embedding theorem (see, for example [14] Théorème IX.9). Hence there exists a $c_s > 0$ such that

$$\|u\|_p \leq c_s \|u\|_{W^{1,2}(\mathbb{R}^d)}$$

for all $u \in W^{1,2}\left(\mathbb{R}^d\right)$. For all $R > 0$ define $B_R = \left\{x \in \mathbb{R}^d : |x| < R\right\}$. The following theorem of Lu and Ou is used frequently in this paper.

Theorem 2.1 *Let $u \in W\left(\mathbb{R}^d\right)$. Then the limit*

$$\langle u \rangle = \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} u \tag{3}$$

exists. Moreover, $u - \langle u \rangle \in L_p\left(\mathbb{R}^d\right)$ and

$$\|u - \langle u \rangle\|^2_{L_p(\mathbb{R}^d)} \leq c_s^2 \int_{\mathbb{R}^d} |\nabla u|^2.$$

Proof See [21] Theorem 1.1, or [23] Theorem 1.78. □

Throughout the remainder of the paper we define $\langle u \rangle$ by Eq. 3 for all $u \in W\left(\mathbb{R}^d\right)$. We define the norm on $W\left(\mathbb{R}^d\right)$ by

$$\|u\|_{W(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\nabla u|^2 + |\langle u \rangle|^2 \right)^{1/2}.$$

Then $W\left(\mathbb{R}^d\right)$ is a Hilbert space by [21] Theorem 2.1.

We wish to establish an equivalent norm on $W\left(\mathbb{R}^d\right)$. For the proof we need the next lemma, which we will use several times in the sequel.

Lemma 2.2 *Let $U \subset \mathbb{R}^d$ be a bounded connected open set with Lipschitz boundary.*

(a) *Let $A \subset U$ be a measurable set with $0 < |A|$. Then the norms $\|\cdot\|_{H^1(U)}$ and $u \mapsto \left(\int_U |\nabla u|^2 + \int_A |u|^2 \right)^{1/2}$ are equivalent on $H^1(U)$.*

(b) *Let $\Lambda \subset \overline{U}$ be a measurable set with $0 < \sigma(\Lambda) < \infty$, where σ is the $(d-1)$-dimensional Hausdorff measure. Suppose that the map $u \mapsto u|_\Lambda$ from $H^1(U) \cap C(\overline{U})$*

into $L_2(\Lambda)$ *extends continuously to a compact map* $\mathrm{tr} : H^1(U) \to L_2(\Lambda)$. *Then the norms* $\|\cdot\|_{H^1(U)}$ *and* $u \mapsto \left(\int_U |\nabla u|^2 + \int_\Lambda |\mathrm{tr}\ u|^2 \right)^{1/2}$ *are equivalent on* $H^1(U)$.

Proof We first show (b). Since the trace $\mathrm{tr}\colon H^1(U) \to L_2(\Lambda)$ is continuous, it is obvious that there exists a $c > 0$ such that

$$\int_U |\nabla u|^2 + \int_\Lambda |\mathrm{tr}\ u|^2 \leq c\, \|u\|^2_{H^1(U)}$$

for all $u \in H^1(U)$.

We next show that there exists a $c > 0$ such that

$$\int_U |u|^2 \leq c\Big(\int_U |\nabla u|^2 + \int_\Lambda |\mathrm{tr}\ u|^2 \Big)$$

for all $u \in H^1(U)$. If not, then for all $n \in \mathbb{N}$ there exists a $u_n \in H^1(U)$ such that

$$\int_U |\nabla u_n|^2 + \int_\Lambda |\mathrm{tr}\ u_n|^2 < \frac{1}{n} \int_U |u_n|^2.$$

Without loss of generality we may assume that $\int_U |u_n|^2 = 1$ for all $n \in \mathbb{N}$. Then $\|u_n\|^2_{H^1(U)} \leq 2$ for all $n \in \mathbb{N}$. Passing to a subsequence if necessary, there exists a $u \in H^1(U)$ such that $\lim u_n = u$ weakly in $H^1(U)$. Then $\lim u_n = u$ strongly in $L_2(U)$, so $\|u\|_{L_2(U)} = 1$. Moreover,

$$\int_U |\nabla u|^2 \leq \liminf_{n\to\infty} \int_U |\nabla u_n|^2 \leq \liminf_{n\to\infty} \frac{1}{n} = 0.$$

Therefore u is constant. Since the trace $\mathrm{tr} : H^1(U) \to L_2(\Lambda)$ is compact by assumption, one deduces that $\int_\Lambda |\mathrm{tr}\ u|^2 = \lim_{n\to\infty} \int_\Lambda |\mathrm{tr}\ u_n|^2 = 0$. So $u = 0$. This is a contradiction.

We now show (a). Obviously $\int_U |\nabla u|^2 + \int_A |u|^2 \leq \|u\|^2_{H^1(U)}$ for all $u \in H^1(U)$. Since the map $u \mapsto u|_A$ from $H^1(U)$ into $L_2(A)$ is compact, it follows as above that there exists a $c > 0$ such that $\int_A |u|^2 \leq c\, \|u\|^2_{H^1(U)}$ for all $u \in H^1(U)$. □

Occasionally we need an equivalent norm on $W\left(\mathbb{R}^d\right)$.

Lemma 2.3 *Let* $A \subset \mathbb{R}^d$ *be bounded and measurable with* $0 < |A|$. *Then the norm*

$$u \mapsto \Big(\int_{\mathbb{R}^d} |\nabla u|^2 + \int_A |u|^2 \Big)^{1/2}$$

is equivalent with $\|\cdot\|_{W(\mathbb{R}^d)}$.

Proof Write $|||u||| = \left(\int_{\mathbb{R}^d} |\nabla u|^2 + \int_A |u|^2 \right)^{1/2}$ for all $u \in W\left(\mathbb{R}^d\right)$. We first show that $\left(W(\mathbb{R}^d), |||\cdot|||\right)$ is complete.

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(W(\mathbb{R}^d), |||\cdot|||\right)$. Let $R > 0$ with $A \subset B_R$. By Lemma 2.2(a) the sequence $(u_n|_{B_R})_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space $H^1(B_R)$, so it converges. By a diagonal argument, there exists a $u\colon \mathbb{R}^d \to \mathbb{R}$ such that $\lim_{n\to\infty} u_n|_{B_R} = u|_{B_R}$ in $H^1(B_R)$ for all $R > 0$ with $A \subset B_R$. Let $n \in \mathbb{N}$. Then

$$\int_{B_R} |\nabla(u - u_n)|^2 = \lim_{m\to\infty} \int_{B_R} |\nabla(u_m - u_n)|^2 \leq \lim_{m\to\infty} \|u_m - u_n\|^2_{W(\mathbb{R}^d)}$$

for all $R > 0$ with $A \subset B_R$. So

$$\int_{\mathbb{R}^d} |\nabla(u - u_n)|^2 \le \lim_{m\to\infty} \|u_m - u_n\|^2_{W(\mathbb{R}^d)}.$$

Therefore $\nabla(u_n - u) \in L_2\left(\mathbb{R}^d\right)^d$ if n is large enough. Hence $u \in W\left(\mathbb{R}^d\right)$. Since also $\lim_{n\to\infty} \int_A |u_n - u|^2 = 0$ one deduces that $\lim u_n = u$ in $\left(W(\mathbb{R}^d), |||\cdot|||\right)$.

Secondly, let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W\left(\mathbb{R}^d\right)$. We shall show that it is also a Cauchy sequence in $\left(W(\mathbb{R}^d), |||\cdot|||\right)$. Define $v_n = u_n - \langle u_n\rangle \in W\left(\mathbb{R}^d\right)$ for all $n \in \mathbb{N}$. Then it follows from Theorem 2.1 that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_p\left(\mathbb{R}^d\right)$. Hence $(v_n|_A)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_p(A)$. Since $p > 2$ and $|A| < \infty$ one deduces that $(v_n|_A)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_2(A)$. Clearly $(\langle u_n\rangle \mathbb{1}_A)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_2(A)$. Therefore $(u_n|_A)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_2(A)$. Hence $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(W(\mathbb{R}^d), |||\cdot|||\right)$.

Thirdly, there exists a $c > 0$ such that $|||u||| \le c\,\|u\|_{W(\mathbb{R}^d)}$ for all $u \in W\left(\mathbb{R}^d\right)$. If not, then for all $n \in \mathbb{N}$ there exists a $u_n \in W\left(\mathbb{R}^d\right)$ such that $\|u_n\|_{W(\mathbb{R}^d)} \le \frac{1}{n}$ and $|||u_n||| \ge n$, which contradicts the second step.

Finally, by the closed graph theorem the two norms are equivalent. □

Let $W^D\left(\mathbb{R}^d\right)$ be the closure of the space $\mathcal{D}\left(\mathbb{R}^d\right)$ in $W\left(\mathbb{R}^d\right)$. We provide $W^D\left(\mathbb{R}^d\right)$ with the norm

$$\|u\|_{W^D(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\nabla u|^2\right)^{1/2}.$$

Then $W^D\left(\mathbb{R}^d\right)$ is a Hilbert space by [21] page 8.

Proposition 2.4 *The space $W^D\left(\mathbb{R}^d\right)$ has codimension 1 in $W\left(\mathbb{R}^d\right)$. Moreover, $W^D\left(\mathbb{R}^d\right) = \left\{u \in W\left(\mathbb{R}^d\right) : \langle u\rangle = 0\right\} = W\left(\mathbb{R}^d\right) \cap L_p\left(\mathbb{R}^d\right)$.*

Proof See [21], Proposition 2.2 and Theorem 2.3. □

So $W^D\left(\mathbb{R}^d\right)$ has the induced norm of $W\left(\mathbb{R}^d\right)$. Moreover, $H^1\left(\mathbb{R}^d\right) \subset W^D\left(\mathbb{R}^d\right)$.

Throughout the rest of this paper we fix a bounded open set $\Omega_0 \subset \mathbb{R}^d$ with Lipschitz boundary and set

$$\Omega = \mathbb{R}^d \setminus \overline{\Omega_0}.$$

We assume throughout that Ω is connected. Moreover, set $\Gamma = \partial\Omega = \partial\Omega_0$. We provide Γ with the $(d-1)$-dimensional Hausdorff measure. For all $R > 0$ define $\Omega_R = \Omega \cap B_R$. Define

$$W(\Omega) = \left\{u \in H^1_{\mathrm{loc}}(\Omega) : \int_\Omega |\nabla u|^2 < \infty\right\}.$$

Fix $R_0 > 3$ such that $\overline{\Omega_0} \subset B_{R_0-3}$. Note that $u|_{\Omega_R} \in H^1(\Omega_R)$ for all $R \ge R_0$ by Lemma 1.1.11 in [24], because Ω_R has a Lipschitz boundary. So

$$W(\Omega) = \{u\colon \Omega \to \mathbb{C} : u \text{ is measurable, } u|_{\Omega_R} \in H^1(\Omega_R) \text{ for all } R \ge R_0 \text{ and } \int_\Omega |\nabla u|^2 < \infty\}.$$

We shall use this equality frequently in the sequel.

Since Ω and hence Ω_{R_0} has a Lipschitz boundary, there exists a continuous extension operator $E_0 \colon H^1(\Omega_{R_0}) \to H^1(B_{R_0})$. Define $E \colon W(\Omega) \to W\left(\mathbb{R}^d\right)$ by

$$(Eu)(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^d \setminus B_{R_0}, \\ (E_0(u|_{\Omega_{R_0}}))(x) & \text{if } x \in B_{R_0}. \end{cases}$$

For all $u \in W(\Omega)$ define $\langle u \rangle = \langle Eu \rangle$. Recall that Ω_0 is bounded. Hence

$$\langle u \rangle = \lim_{R\to\infty} \frac{1}{|\Omega_R|} \int_{\Omega_R} u$$

for all $u \in W(\Omega)$. Consequently, by [21] Theorem 5.2 there exists a $c > 0$ such that

$$\|u - \langle u \rangle\|^2_{L_p(\Omega)} \le c^2 \int_\Omega |\nabla u|^2 \tag{4}$$

for all $u \in W(\Omega)$. We provide $W(\Omega)$ with the norm

$$\|u\|_{W(\Omega)} = \Big(\int_\Omega |\nabla u|^2 + |\langle u \rangle|^2 \Big)^{1/2}.$$

In the next proposition we define several equivalent norms on $W(\Omega)$.

We need various traces. Let $R \ge R_0$. Then Ω_R has a Lipschitz boundary. We denote by $\mathrm{Tr}_{\Omega_R} \colon H^1(\Omega_R) \to L_2(\partial\Omega_R)$ the trace on $H^1(\Omega_R)$. Define $\mathrm{Tr}_R \colon H^1(\Omega_R) \to L_2(\Gamma)$ by $\mathrm{Tr}_R u = (\mathrm{Tr}_{\Omega_R} u)|_\Gamma$. Moreover, define the map $\mathrm{Tr} \colon W(\Omega) \to L_2(\Gamma)$ by $\mathrm{Tr}\, u = \mathrm{Tr}_R(u|_{\Omega_R})$. Note that $\mathrm{Tr}\, u$, defined above, is independent of R.

Proposition 2.5 (a) *The space $W(\Omega)$ is a Hilbert space.*
(b) *The map $Tr \colon W(\Omega) \to L_2(\Gamma)$ is compact.*
(c) *The extension operator $E \colon W(\Omega) \to W(\mathbb{R}^d)$ is continuous.*
(d) *The norm*

$$u \mapsto \Big(\int_\Omega |\nabla u|^2 + \int_\Gamma |Tr\, u|^2 \Big)^{1/2} \tag{5}$$

is equivalent with $\|\cdot\|_{W(\Omega)}$.
(e) *Let $A \subset \Omega$ be a bounded measurable set with $0 < |A|$. Then the norm*

$$u \mapsto \Big(\int_\Omega |\nabla u|^2 + \int_A |u|^2 \Big)^{1/2}$$

is equivalent with $\|\cdot\|_{W(\Omega)}$.

Proof Define $|||\cdot|||_A \colon W(\Omega) \to [0, \infty)$ by

$$|||u|||_A = \Big(\int_\Omega |\nabla u|^2 + \int_A |u|^2 \Big)^{1/2}.$$

Then $(W(\Omega), |||\cdot|||_A)$ is complete by the same arguments as in the first step of Lemma 2.3, if one replaces B_R by Ω_R.

We next show that $|||\cdot|||_A$ and the norm (5) on $W(\Omega)$ are equivalent. Let $R \ge R_0$ be such that $A \subset B_R$. Then the map $u \mapsto u|_{\Omega_R}$ is continuous from $(W(\Omega), |||\cdot|||_A)$ into $H^1(\Omega_R)$ by Lemma 2.2(a). Clearly the map $u \mapsto u|_A$ is continuous from $H^1(\Omega_R)$ into $L_2(A)$ and the map $u \mapsto \mathrm{Tr}_R u$ is continuous from $H^1(\Omega_R)$ in $L_2(\Gamma)$. Hence in view of Lemma 2.2(b), $|||\cdot|||_A$ and the norm (5) on $W(\Omega)$ are equivalent. Note that this implies that the norm $|||\cdot|||_A$ does not depend, up to equivalence, on the set A.

Next we show that the extension operator E is continuous from $(W(\Omega), |||\cdot|||_{\Omega_{R_0}})$ into $W(\mathbb{R}^d)$. Here $|||\cdot|||_{\Omega_{R_0}} = |||\cdot|||_A$ with the choice $A = \Omega_{R_0}$. Now let $u \in W(\Omega)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla Eu|^2 + \int_{\Omega_{R_0}} |Eu|^2 &= \int_{\mathbb{R}^d\setminus B_{R_0}} |\nabla Eu|^2 + \int_{B_{R_0}} |\nabla Eu|^2 + \int_{\Omega_{R_0}} |Eu|^2 \\
&\leq \int_{\mathbb{R}^d\setminus B_{R_0}} |\nabla Eu|^2 + \|(Eu)|_{B_{R_0}}\|^2_{H^1(B_{R_0})} \\
&\leq \int_{\mathbb{R}^d\setminus B_{R_0}} |\nabla u|^2 + \|E_0\|\, \|u|_{\Omega_{R_0}}\|^2_{H^1(\Omega_{R_0})} \\
&\leq (1 + \|E_0\|)|||u|||^2_{\Omega_{R_0}}.
\end{aligned}$$

Hence by Lemma 2.3 the extension operator E is continuous from $(W(\Omega), |||\cdot|||_{\Omega_{R_0}})$ into $W\left(\mathbb{R}^d\right)$.

If $u \in W(\Omega)$ then

$$\|u\|^2_{W(\Omega)} = \int_\Omega |\nabla Eu|^2 + |\langle Eu\rangle|^2 \leq \|Eu\|^2_{W(\mathbb{R}^d)} \leq \|E\|^2_{(W(\Omega),|||\cdot|||_{\Omega_{R_0}})\to W(\mathbb{R}^d)}\, |||u|||^2_{\Omega_{R_0}}.$$

Hence the inclusion from $\left(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\right)$ into $W(\Omega)$ is continuous. In particular, the map $u \mapsto \langle u\rangle$ is continuous from $(W(\Omega), |||\cdot|||_{\Omega_{R_0}})$ into $\mathbb{C}$.

Now we are able to prove Statement (a). Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W(\Omega)$. Then it follows as in the second step in the proof of Lemma 2.3, but now using Eq. 4 instead of Theorem 2.1, that that $(u_n|_{\Omega_{R_0}})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_2\left(\Omega_{R_0}\right)$. Hence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\left(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\right)$ and there exists a $u \in W(\Omega)$ such that $\lim u_n = u$ in $\left(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\right)$. Then $\lim\langle u_n - u\rangle = 0$ by the above and $\lim u_n = u$ in $W(\Omega)$.

By the closed graph theorem the norms $\|\cdot\|_{W(\Omega)}$ and $|||\cdot|||_{\Omega_{R_0}}$ are equivalent. Then Statements (c), (d) and (e) follow by the previously proved equivalences of norms.

Finally we prove Statement (b). The restriction map $u \mapsto u|_{\Omega_{R_0}}$ from $\left(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\right)$ into $H^1(\Omega_{R_0})$ is continuous and the map $\mathrm{Tr}_{R_0}\colon H^1(\Omega_{R_0}) \to L_2(\Gamma)$ is compact. Hence the composition Tr is compact. The proof is complete. □

Let $W^D(\Omega)$ be the closure of the space $\left\{u|_\Omega : u \in \mathcal{D}(\mathbb{R}^d)\right\}$ in $W(\Omega)$. We provide $W^D(\Omega)$ with the inner product

$$(u, v)_{W^D(\Omega)} = \int_\Omega \nabla u \cdot \nabla v.$$

Then $W^D(\Omega)$ is a Hilbert space with the inner product $(\cdot,\cdot)_{W^D(\Omega)}$ by [21] page 8.

Proposition 2.6 *The space $W^D(\Omega)$ has codimension 1 in $W(\Omega)$. Moreover, $W^D(\Omega) = W(\Omega) \cap L_p(\Omega) = \{u \in W(\Omega) : \langle u\rangle = 0\}$.*

Proof The codimension follows easily once we have the equality of the three spaces.

Clearly $W^D(\Omega) \subset \{u \in W(\Omega) : \langle u\rangle = 0\}$. It follows from Eq. 4 that $\{u \in W(\Omega) : \langle u\rangle = 0\} \subset W(\Omega) \cap L_p(\Omega)$. Finally, suppose that $u \in W(\Omega) \cap L_p(\Omega)$. Then $Eu \in W(\mathbb{R}^d) \cap L_p\left(\mathbb{R}^d\right) = W^D\left(\mathbb{R}^d\right)$ by Proposition 2.4. Hence there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R}^d)$ such that $\lim u_n = Eu$ in $W\left(\mathbb{R}^d\right)$. Then $\lim u_n|_\Omega = (Eu)|_\Omega = u$ in $W(\Omega)$. □

Remark 2.7 If the exterior domain in Proposition 2.6 is replaced by a cylinder, then the has codimension of $W^D(\Omega)$ in $W(\Omega)$ is more than one. Indeed, if the axis of the cylinder is parallel to e_1, then there exists a C^∞-function u on the cylinder for which $u(x) = 1$ whenever $x_1 > 1$ and $u(x) = 2$ whenever $x_1 < -1$. Then $u \in W(\Omega)$, but $u \notin W^D(\Omega) \oplus \operatorname{span}\{\mathbb{1}_\Omega\}$.

Note that the Sobolev embedding gives $H^1(\Omega) \subset W(\Omega) \cap L_p(\Omega) = W^D(\Omega)$.

In the remaining part of this section we consider the ordering on the real space

$$W(\Omega, \mathbb{R}) = \{u \in W(\Omega) : u \text{ is real valued}\}.$$

Define similarly $W^D(\Omega, \mathbb{R})$. Recall that for every open $U \subset \mathbb{R}^d$ and $u \in W^{1,1}_{\mathrm{loc}}(U, \mathbb{R})$ one has $u^+ \in W^{1,1}_{\mathrm{loc}}(U, \mathbb{R})$ and $\partial_k u^+ = \mathbb{1}_{[u>0]} \partial_k u$ for all $k \in \{1, \ldots d\}$ by [18] Lemma 7.1. We shall next show that a similar result is valid for the spaces $W(\Omega, \mathbb{R})$ and $W^D(\Omega, \mathbb{R})$.

Proposition 2.8 (a) *If $u \in W(\Omega, \mathbb{R})$ then $u^+, u^-, |u| \in W(\Omega)$ and $\| \, |u| \, \|_{W(\Omega)} = \|u\|_{W(\Omega)}$.*
(b) *If $u \in W^D(\Omega, \mathbb{R})$ then $u^+, u^-, |u| \in W^D(\Omega)$.*
(c) *The maps $u \mapsto u^+$, $u \mapsto u^-$ and $u \mapsto |u|$ are continuous from $W(\Omega, \mathbb{R})$ into $W(\Omega)$.*
(d) *If $u \in W(\Omega, \mathbb{R})$ then $\mathrm{Tr}\,(u^+) = (\mathrm{Tr}\,u)^+$, $\mathrm{Tr}\,(u^-) = (\mathrm{Tr}\,u)^-$ and $\mathrm{Tr}\,|u| = |\mathrm{Tr}\,u|$.*

Proof We only prove part (c). Let $u, u_1, u_2, \ldots \in W(\Omega, \mathbb{R})$ and suppose that $\lim u_n = u$ in $W(\Omega)$. Since $\| \, |u_n| \, \|_{W(\Omega)} = \|u_n\|_{W(\Omega)}$ for all $n \in \mathbb{N}$ it follows that (passing to a subsequence if necessary) the sequence $(|u_n|)_{n\in\mathbb{N}}$ converges weakly in $W(\Omega)$. But $(|u_n|)_{n\in\mathbb{N}}$ converges to $|u|$ in $L_{2,\mathrm{loc}}(\Omega)$. Hence $(|u_n|)_{n\in\mathbb{N}}$ converges weakly to $|u|$ in $W(\Omega)$. In addition,

$$\lim_{n\to\infty} \| \, |u_n| \, \|_{W(\Omega)} = \lim_{n\to\infty} \|u_n\|_{W(\Omega)} = \|u\|_{W(\Omega)} = \| \, |u| \, \|_{W(\Omega)}.$$

Hence $\lim_{n\to\infty} |u_n| = |u|$ in $W(\Omega)$. □

3 The Robin Problem on an Exterior Domain

In this section we consider the Dirichlet problem on an exterior domain Ω with Robin boundary conditions on the boundary Γ. This means we look for a harmonic function $u \in W(\Omega)$ satisfying the boundary condition on Γ, see Eq. 8. The parameter λ occurring in the boundary condition is motivated by our final aim, namely to study the resolvent of the Dirichlet-to-Neumann operator on Ω. At infinite we may consider Neumann or Dirichlet boundary conditions. If $u \in L_{1,\mathrm{loc}}(\Omega)$ then we denote throughout this paper by $\Delta u \in \mathcal{D}(\Omega)'$ the distributional Laplacian applied to u. We next define the normal derivative of a function in $W(\Omega)$, if it exists.

Definition 3.1 Let $u \in W(\Omega)$ and $\psi \in L_2(\Gamma)$. We say that u has **normal derivative ψ on Γ** if there exists an $R \geq R_0$ such that $\Delta u|_{\Omega_R} \in L_2(\Omega_R)$ and

$$\int_{\Omega_R} \nabla u \cdot \overline{\nabla v} + \int_{\Omega_R} (\Delta u) \, \overline{v} = \int_\Gamma \psi \, \overline{\mathrm{Tr}\, v} \tag{6}$$

for all $v \in C_c^\infty(B_R)$. Obviously ψ has at most one normal derivative and we write $\partial_\nu u = \psi$.

It is easy to verify that the above definition has the following extension of the type 'one implies all'. If $u \in W(\Omega)$ and $S \geq R_0$ are such that $\Delta u|_{\Omega_S} \in L_2(\Omega_S)$, then u has a normal derivative if and only if Eq. 6 is valid for all $R \in [R_0, S]$. Moreover, if Eq. 6 is valid for all $v \in C_c^\infty(B_R)$, then by density, Eq. 6 is valid for all $v \in H^1(\Omega_R)$ with $\mathbb{1}_{\partial B_R} \operatorname{Tr}_{\Omega_R} v = 0$.

If $u \in W(\Omega)$ with $\Delta u \in L_2(\Omega)$ has normal derivative $\psi \in L_2(\Gamma)$, then

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (\Delta u)\, \overline{v} = \int_\Gamma \psi\, \overline{\operatorname{Tr} v} \tag{7}$$

for all $v \in \left\{ w|_\Omega : w \in C_c^\infty(\mathbb{R}^d) \right\}$ and then by density for all $v \in W^D(\Omega)$.

Given $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$ we wish to solve

$$\begin{aligned} \Delta u = 0 \quad & \text{on } \Omega \\ \lambda \operatorname{Tr} u + \partial_\nu u = \psi \quad & \text{on } \Gamma \end{aligned} \tag{8}$$

with two different boundary conditions at infinity, namely Dirichlet and Neumann boundary conditions.

Definition 3.2 Let $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$. A **solution of Eq. 8 with Dirichlet boundary conditions at infinity** is a function $u \in W^D(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \lambda \int_\Gamma \operatorname{Tr} u\, \overline{\operatorname{Tr} v} = \int_\Gamma \psi\, \overline{\operatorname{Tr} v} \tag{9}$$

for all $v \in W^D(\Omega)$.

A **solution of Eq. 8 with Neumann boundary conditions at infinity** is a function $u \in W(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \lambda \int_\Gamma \operatorname{Tr} u\, \overline{\operatorname{Tr} v} = \int_\Gamma \psi\, \overline{\operatorname{Tr} v} \tag{10}$$

for all $v \in W(\Omega)$.

Thus Dirichlet boundary conditions at infinity corresponds to the test space $W^D(\Omega)$ and Neumann boundary conditions at infinity corresponds to the test space $W(\Omega)$. Note that if $u \in W^D(\Omega)$ then u is a solution of Eq. 8 with Dirichlet boundary conditions at infinity if and only if Eq. 8 is satisfied. A similar statement for Neumann boundary conditions at infinity is false.

Since the constant function $\mathbb{1}_\Omega$ is in $W(\Omega)$, a solution of Eq. 8 with Neumann boundary conditions can only exist for $\lambda = 0$ if $\int_\Gamma \psi = 0$.

Because Tr is continuous from $W(\Omega)$ into $L_2(\Gamma)$ one deduces existence and uniqueness of solutions from the Lax–Milgram theorem. We formulate this as a proposition.

Proposition 3.3 (a) *Let $\lambda \in [0, \infty)$. Then for all $\psi \in L_2(\Gamma)$ there exists a unique $u \in W^D(\Omega)$ such that u is a solution of* Eq. 8 *with Dirichlet boundary conditions. Define $B_\lambda^D \psi := u$. Then B_λ^D is a continuous operator from $L_2(\Gamma)$ into $W^D(\Omega)$.*

(b) *Let $\lambda \in (0, \infty)$. Then for all $\psi \in L_2(\Gamma)$ there exists a unique $u \in W(\Omega)$ such that u is a solution of* Eq. 8 *with Neumann boundary conditions. Define $B_\lambda \psi := u$. Then B_λ is a continuous operator from $L_2(\Gamma)$ into $W(\Omega)$.*

There is a 'simple' relation between B_λ^D and $B_\lambda \psi$.

Proposition 3.4 *Let $\lambda \in (0, \infty)$ and $\psi \in L_2(\Gamma)$. Then*

$$B_\lambda^D \psi = B_\lambda \psi + \langle B_\lambda \psi \rangle \Big(\lambda\, B_\lambda^D \mathbb{1}_\Gamma - \mathbb{1}_\Omega \Big).$$

Proof Write $u = B_\lambda \psi$. Then $u - \langle u \rangle \mathbb{1}_\Omega \in W^D(\Omega)$. Moreover, if $v \in W^D(\Omega)$ then

$$\begin{aligned}\int_\Omega \nabla(u - \langle u \rangle \mathbb{1}_\Omega) \cdot \overline{\nabla v} + \lambda \int_\Gamma \mathrm{Tr}\,(u - \langle u \rangle \mathbb{1}_\Omega)\, \overline{\mathrm{Tr}\, v} \\ = \int_\Omega \nabla u \cdot \overline{\nabla v} + \lambda \int_\Gamma \mathrm{Tr}\, u\, \overline{\mathrm{Tr}\, v} - \int_\Gamma \langle u \rangle\, \lambda\, \mathbb{1}_\Gamma \cdot \overline{\mathrm{Tr}\, v} \\ = \int_\Gamma \Big(\psi - \langle u \rangle\, \lambda\, \mathbb{1}_\Gamma\Big) \overline{\mathrm{Tr}\, v}.\end{aligned}$$

So

$$B_\lambda^D(\psi - \langle u \rangle\, \lambda\, \mathbb{1}_\Gamma) = u - \langle u \rangle \mathbb{1}_\Omega$$

and the proposition follows. □

We next consider positivity and monotonicity properties of B_λ^D and B_λ.

Proposition 3.5 *Let $\psi \in L_2(\Gamma)$ with $\psi \geq 0$. Then $B_\lambda^D \psi \geq 0$ for all $\lambda \in [0, \infty)$ and $B_\lambda \psi \geq 0$ for all $\lambda \in (0, \infty)$.*

Proof Let $\psi \in L_2(\Gamma)$ and suppose that $\psi \leq 0$. Let $\lambda \in (0, \infty)$ and consider $u = B_\lambda \psi$. Then $u \in W(\Omega, \mathbb{R})$ and $v := u^+ \in W(\Omega)$ by Proposition 2.8(a). Hence Eq. 10 gives

$$\int_\Omega |\nabla u^+|^2 + \lambda \int_\Gamma |\mathrm{Tr}\, u^+|^2 = \int_\Gamma \psi\, \mathrm{Tr}\, u^+ \leq 0.$$

So $\|u^+\|_{W(\Omega)} = 0$ by Proposition 2.5(d) and $u^+ = 0$. Similarly, if $\lambda \in [0, \infty)$ and $u = B_\lambda^D \psi$, then $\int_\Omega |\nabla u^+|^2 = 0$ and since $u^+ \in W^D(\Omega)$ on deduces again that $u^+ = 0$. □

Proposition 3.6 *Let $\lambda \in (0, \infty)$ and $\psi \in L_2(\Gamma)$ with $\psi \geq 0$. Then*

$$B_\lambda^D \psi \leq B_\lambda \psi.$$

Proof Set $u = B_\lambda^D \psi$ and $w = B_\lambda \psi$. Then $u \geq 0$ and $w \geq 0$ by Proposition 3.5. We wish to show that $u \leq w$, or equivalently that $(u - w)^+ = 0$. It follows from Eqs. 9 and 10 that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \lambda \int_\Gamma \mathrm{Tr}\, u\, \overline{\mathrm{Tr}\, v} = \int_\Gamma \psi\, \overline{\mathrm{Tr}\, v}$$

for all $v \in W^D(\Omega)$ and

$$\int_\Omega \nabla w \cdot \overline{\nabla v} + \lambda \int_\Gamma \mathrm{Tr}\, w\, \overline{\mathrm{Tr}\, v} = \int_\Gamma \psi\, \overline{\mathrm{Tr}\, v}$$

for all $v \in W(\Omega)$. So

$$\int_\Omega \nabla(u - w) \cdot \overline{\nabla v} + \lambda \int_\Gamma \mathrm{Tr}\,(u - w)\, \overline{\mathrm{Tr}\, v} = 0 \tag{11}$$

for all $v \in W^D(\Omega)$. Since $(u - w)^+ \leq u \in L_p(\Omega)$ it follows from Proposition 2.6 that $(u - w)^+ \in W^D(\Omega)$. Choose $v = (u - w)^+$ in Eq. 11. Then

$$\int_\Omega |\nabla(u - w)^+|^2 + \lambda \int_\Gamma |\mathrm{Tr}\,(u - w)^+|^2 = 0.$$

Hence $(u - w)^+ = 0$ as required. □

Proposition 3.7 (a) *Let $0 \leq \lambda_1 < \lambda_2$. Then $B_{\lambda_2}^D \psi \leq B_{\lambda_1}^D \psi$ for all $\psi \in L_2(\Gamma)$ with $\psi \geq 0$.*

(b) *Let* $0 < \lambda_1 < \lambda_2$. *Then* $B_{\lambda_2}\psi \leq B_{\lambda_1}\psi$ *for all* $\psi \in L_2(\Gamma)$ *with* $\psi \geq 0$.

Proof We only prove Statement (b). Set $u_1 = B_{\lambda_1}\psi$ and $u_2 = B_{\lambda_2}\psi$. Then

$$\int_\Omega \nabla u_1 \cdot \overline{\nabla v} + \lambda_1 \int_\Gamma \mathrm{Tr}\, u_1 \, \overline{\mathrm{Tr}\, v} = \int_\Gamma \psi \, \mathrm{Tr}\, v \quad \text{and}$$
$$\int_\Omega \nabla u_2 \cdot \overline{\nabla v} + \lambda_2 \int_\Gamma \mathrm{Tr}\, u_2 \, \overline{\mathrm{Tr}\, v} = \int_\Gamma \psi \, \overline{\mathrm{Tr}\, v}$$

for all $v \in W(\Omega)$. Let $v \in W(\Omega)$ with $v \geq 0$. Then

$$\begin{aligned}
&\int_\Omega \nabla(u_2 - u_1) \cdot \nabla v + \lambda_1 \int_\Gamma \mathrm{Tr}\,(u_2 - u_1) \cdot \mathrm{Tr}\, v \\
&\quad \leq \int_\Omega \nabla(u_2 - u_1) \cdot \nabla v + \lambda_1 \int_\Gamma \mathrm{Tr}\,(u_2 - u_1) \cdot \mathrm{Tr}\, v + (\lambda_2 - \lambda_1) \int_\Gamma \mathrm{Tr}\, u_2 \cdot \mathrm{Tr}\, v \\
&\quad = \int_\Omega \nabla(u_2 - u_1) \cdot \nabla v + \int_\Gamma \mathrm{Tr}\,(\lambda_2 u_2 - \lambda_1 u_1) \cdot \mathrm{Tr}\, v = 0.
\end{aligned}$$

In particular, if one chooses $v = (u_2 - u_1)^+$ then

$$\int_\Omega |\nabla(u_2 - u_1)^+|^2 + \lambda_1 \int_\Gamma |\mathrm{Tr}\,(u_2 - u_1)^+|^2 \leq 0$$

and $(u_2 - u_1)^+ = 0$. □

The operators B_λ and B_λ^D are submarkovian in the following sense.

Corollary 3.8 (a) *If* $\lambda \in [0, \infty)$ *then* $\lambda\, B_\lambda^D \mathbb{1}_\Gamma \leq \mathbb{1}_\Omega$.
(b) *If* $\lambda \in (0, \infty)$ *then* $\lambda\, B_\lambda \mathbb{1}_\Gamma = \mathbb{1}_\Omega$.
(c) *If* $\lambda \in (0, \infty)$, $R \in [R_0, \infty)$ *and* $q \in [2, \infty]$ *then* $\psi \mapsto \mathbb{1}_{\Omega_R} B_\lambda^D \psi$ *is continuous from* $L_q(\Gamma)$ *into* $L_q(\Omega)$.
(d) *If* $\lambda \in (0, \infty)$, $R \in [R_0, \infty)$ *and* $q \in [2, \infty]$ *then* $\psi \mapsto \mathbb{1}_{\Omega_R} B_\lambda \psi$ *is continuous from* $L_q(\Gamma)$ *into* $L_q(\Omega)$.

Proof For all $\lambda > 0$ the function $\frac{1}{\lambda}\mathbb{1}_\Omega$ is a solution of Eq. 8 with Neumann boundary conditions at infinity if $\psi = \mathbb{1}_\Gamma$. So $B_\lambda \mathbb{1}_\Gamma = \frac{1}{\lambda}\mathbb{1}_\Omega$. This proves Statement (b). Then Statement (a) follows from Proposition 3.7. Since $|B_\lambda^D \psi| \leq \|\psi\|_\infty\, B_\lambda^D \mathbb{1}_\Gamma$ by Proposition 3.5 and $0 \leq B_\lambda^D \mathbb{1}_\Gamma \leq \frac{1}{\lambda}\mathbb{1}_\Omega$, it follows that B_λ^D maps $L_\infty(\Gamma)$ continuously into $L_\infty(\Omega)$. Then Statement (c) follows by interpolation. The proof of Statement (d) is similar. □

Proposition 3.9 *The operator* B_λ^D *is compact for all* $\lambda \in [0, \infty)$ *and the operator* B_λ *is compact for all* $\lambda \in (0, \infty)$.

Proof Let $\lambda \in (0, \infty)$, let $\psi, \psi_1, \psi_2, \ldots \in L_2(\Gamma)$ and suppose that $\lim \psi_n = \psi$ weakly in $L_2(\Gamma)$. Write $u_n = B_\lambda \psi_n$ and $u = B_\lambda \psi$ for all $n \in \mathbb{N}$. Since B_λ is continuous, it follows that $\lim u_n = u$ weakly in $W(\Omega)$. Moreover, $\lim \mathrm{Tr}\, u_n = \mathrm{Tr}\, u$ in $L_2(\Gamma)$ because Tr is compact. Let $n \in \mathbb{N}$ and choose $v = u_n$ in Eq. 10. Then

$$\int_\Omega |\nabla u_n|^2 = \int_\Gamma \psi \, \overline{\mathrm{Tr}\, u_n} - \lambda \int_\Omega |\mathrm{Tr}\, u_n|^2.$$

So

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 = \int_\Gamma \psi\, \overline{\operatorname{Tr} u} - \lambda \int_\Omega |\operatorname{Tr} u|^2 = \int_\Omega |\nabla u|^2,$$

where we use Eq. 10 with $v = u$ in the last step. Therefore $\lim \|u_n\|_{W(\Omega)} = \|u\|_{W(\Omega)}$ by Proposition 2.5(d) and consequently $\lim u_n = u$ in $W(\Omega)$. □

4 Convergence Properties of the Robin Operators

Fix $\psi \in L_2(\Gamma)$. Let $R \geq R_0$. Consider the problem in $H^1(\Omega_R)$ given by

$$\begin{aligned} &\Delta u_R = 0 && \text{on } \Omega_R, \\ &\lambda \operatorname{Tr}_R u_R + \partial_\nu u_R = \psi && \text{on } \Gamma, \\ &\operatorname{Tr}_{\Omega_R} u_R = 0 && \text{on } \partial B_R. \end{aligned}$$

This means that we impose Robin boundary conditions at Γ as before and Dirichlet boundary conditions at ∂B_R. We shall show that $\lim_{R\to\infty} u_R = B_\lambda^D \psi$ in an appropriate sense. This justifies our terminology 'Dirichlet boundary conditions at infinity'. We start with introducing the precise spaces.

Let $R \geq R_0$. Then Ω_R has a Lipschitz boundary. Define the closed subspace

$$W_R^D(\Omega) = \{u \in W^D(\Omega) : u|_{\Omega\setminus\Omega_R} = 0\}$$

of $W^D(\Omega)$ with induced norm.

Lemma 4.1 *Let $R \geq R_0$. Then $W_R^D(\Omega)$ is a Hilbert space. Moreover, the map $u \mapsto u|_{\Omega_R}$ is a continuous bijection from $W_R^D(\Omega)$ onto $\{v \in H^1(\Omega_R) : \mathbb{1}_{\partial B_R} Tr_{\Omega_R} v = 0\}$.*

Proof One has the equalities $H_0^1(B_R) = \{u|_{B_R} : u \in H^1(\mathbb{R}^d) \text{ and } u|_{\mathbb{R}^d\setminus B_R} = 0\} = \{u \in H^1(B_R) : \operatorname{Tr}_{B_R} u = 0\}$. From this one deduces the claim easily.

Let $\lambda \geq 0$, $\psi \in L_2(\Gamma)$ and $R \geq R_0$. It follows from the Lax–Milgram theorem and Lemma 4.1 that there exists a unique $u \in W_R^D(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \lambda \int_\Gamma \operatorname{Tr} u\, \overline{\operatorname{Tr} v} = \int_\Gamma \psi\, \overline{\operatorname{Tr} v}$$

for all $v \in W_R^D(\Omega)$. We write $B_\lambda^D(R)\psi := u$. Then $B_\lambda^D(R)$ is a continuous operator from $L_2(\Gamma)$ into $W_R^D(\Omega)$. Since u is harmonic on Ω_R the function u is C^∞ on Ω_R and in particular continuous on Ω_R. We show next that $\lim_{|x|\uparrow R} u(x) = 0$. Since $u|_{\mathbb{R}^d\setminus\Omega_R} = 0$ a.e., it follows that u may be chosen continuous on Ω.

Proposition 4.2 *Let $\psi \in L_2(\Gamma)$, $\lambda \in [0, \infty)$ and $R \in [R_0, \infty)$. Then $B_\lambda^D(R)\psi \in C(\Omega)$.*

Proof Without loss of generality we may assume that ψ is real valued. Write $u = B_\lambda^D(R)\psi$. We may assume that $u(x) = 0$ for all $x \in \partial B_R$. Clearly $u|_{\Omega\setminus\Omega_R}$ and $u|_{\Omega_R}$ are continuous. Consider the annulus

$$A = \{x \in \mathbb{R}^d : R_0 - 1 < |x| < R\}.$$

Then it remains to show that $u|_{\overline{A}}$ is continuous.

Recall that $\overline{\Omega_0} \subset B_{R_0-3}$. There exists a $\chi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that $\chi(x) = 0$ for all $x \in \mathbb{R}^d$ with $|x| < R_0 - 2$ or $|x| > R_0 - \frac{1}{2}$, and $\chi(x) = 1$ for all x in a neighbourhood of

∂B_{R_0-1}. Define $G\colon \mathbb{R}^d \to \mathbb{R}$ by

$$G(x) = \begin{cases} (\chi\, u)(x) & \text{if } x \in \Omega, \\ 0 & \text{if } \ x \notin \Omega. \end{cases}$$

Then $G \in C_c^\infty\left(\mathbb{R}^d\right)$ and $\mathrm{Tr}_A(G|_A) = \mathrm{Tr}_A(u|_A)$, where $\mathrm{Tr}_A\colon\, H^1(A) \to L_2(\partial A)$ is the trace map. There exists a unique $v \in H_0^1(A)$ such that $-\Delta v = \Delta G$ on A. Extend v by zero to a function on $\mathbb{R}^d$, which we still denote by v. Set $w = v + G$. It follows from [3] Theorem 1.1 that $w|_{\overline{A}}$ is the Perron solution of the Dirichlet problem

$$\Delta h = 0 \qquad \text{on } A,$$
$$h|_{\partial A} = G|_{\partial A},$$
$$h \in C(\overline{A}) \cap \mathcal{H}(A),$$

where $\mathcal{H}(A)$ is the space of all harmonic functions on A. Since A is Dirichlet regular, it follows that $w|_{\overline{A}}$ is continuous on $\overline{A}$. We next show that $w|_A = u|_A$. Clearly $w \in H^1(A)$. Moreover, $\mathrm{Tr}_A(w|_A) = \mathrm{Tr}_A(G|_A) = \mathrm{Tr}_A(u|_A)$. So $\mathrm{Tr}_A((w-u)|_A) = 0$. Therefore $(w - u)|_A \in H_0^1(A)$. In addition, $\Delta(w-u) = 0$ on A. So $(w-u)|_A = 0$ in $H_0^1(A)$ and $w|_A = u|_A$ pointwise. Since $w|_{\partial B_R} = 0 = u|_{\partial B_R}$ it follows that $u|_{\overline{A}} = w|_{\overline{A}}$ is continuous. □

We shall prove in Theorem 4.12 that $B_\lambda^D(R)\psi$ extends continuously to a function on $\overline{\Omega}$ if $\psi \in L_q(\Gamma)$ for some $q \in (d-1, \infty]$. The first main result of this section is the following convergence result.

Theorem 4.3 *Let* $\lambda \in [0, \infty)$. *Then* $\lim_{R\to\infty} B_\lambda^D(R) = B_\lambda^D$ *in* $\mathcal{L}\left(L_2(\Gamma), W^D(\Omega)\right)$.

Proof Let $R_1, R_2, \ldots \in [R_0, \infty)$ and suppose that $\lim R_n = \infty$. We shall show that $\lim_{n\to\infty} B_\lambda^D(R_n) = B_\lambda^D$ in $\mathcal{L}(L_2(\Gamma), W^D(\Omega))$. Let $\psi, \psi_1, \psi_2, \ldots \in L_2(\Gamma)$ and suppose that $\lim \psi_n = \psi$ weakly in $L_2(\Gamma)$. Since B_λ^D is compact by Proposition 3.9 it suffices to show that $\lim B_\lambda^D(R_n)\psi_n = B_\lambda^D\psi$ in $W^D(\Omega)$.

There exists a $c > 0$ such that

$$\|\mathrm{Tr}\, v\|_{L_2(\Gamma)} \le c\, \|v\|_{W^D(\Omega)}$$

for all $v \in W^D(\Omega)$. Write $u_n = B_\lambda^D(R_n)\psi_n$ and $u = B_\lambda^D\psi$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Then

$$\int_\Omega \nabla u_n \cdot \overline{\nabla v} + \lambda \int_\Gamma \mathrm{Tr}\, u_n\, \overline{\mathrm{Tr}\, v} = \int_\Gamma \psi_n\, \overline{\mathrm{Tr}\, v} \tag{12}$$

for all $v \in W_{R_n}^D(\Omega)$. Hence

$$\begin{aligned} \|u_n\|_{W^D(\Omega)}^2 + \lambda \int_\Gamma |\mathrm{Tr}\, u_n|^2 &= \int_\Omega |\nabla u_n|^2 + \lambda \int_\Gamma |\mathrm{Tr}\, u_n|^2 \\ &= \int_\Gamma \psi_n\, \overline{\mathrm{Tr}\, u_n} \le c\, \|\psi_n\|_{L_2(\Gamma)}\, \|u_n\|_{W^D(\Omega)}. \end{aligned} \tag{13}$$

Therefore the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^D(\Omega)$. Passing to a subsequence if necessary, we can assume that there exists a $u_0 \in W^D(\Omega)$ such that $\lim u_n = u_0$ weakly in $W^D(\Omega)$. Then $\lim \mathrm{Tr}\, u_n = \mathrm{Tr}\, u_0$ in $L_2(\Gamma)$ by the compactness of Tr. Let $R \ge R_0$ and $v \in W_R^D(\Omega)$. Then $v \in W_{R_n}^D(\Omega)$ for all large $n \in \mathbb{N}$. Take the limit $n \to \infty$ in Eq. 12. One deduces that

$$\int_\Omega \nabla u_0 \cdot \overline{\nabla v} + \lambda \int_\Gamma \mathrm{Tr}\, u_0\, \overline{\mathrm{Tr}\, v} = \int_\Gamma \psi\, \overline{\mathrm{Tr}\, v}. \tag{14}$$

Since $\bigcup_{R \geq R_0} W_R^D(\Omega)$ is dense in $W^D(\Omega)$ it follows that Eq. 14 is valid for all $v \in W^D(\Omega)$ and $u_0 = B_\lambda^D \psi = u$. Moreover, Eq. 13 gives

$$\begin{aligned} \lim_{n\to\infty} \|u_n\|_{W^D(\Omega)}^2 &= \lim_{n\to\infty} \left(\int_\Gamma \psi_n \, \overline{\operatorname{Tr} u_n} - \lambda \int_\Gamma |\operatorname{Tr} u_n|^2 \right) \\ &= \int_\Gamma \psi \, \overline{\operatorname{Tr} u} - \lambda \int_\Gamma |\operatorname{Tr} u|^2 = \|u\|_{W^D(\Omega)}^2. \end{aligned}$$

Hence $\lim u_n = u$ in $W^D(\Omega)$. □

Next we consider domination.

Proposition 4.4 (a) *Let $\lambda \in [0, \infty)$, $\psi \in L_2(\Gamma)$ and $R \geq R_0$ with $\psi \geq 0$. Then $B_\lambda^D(R)\psi \geq 0$.*
(b) *Let $\lambda \in [0, \infty)$, $\psi \in L_2(\Gamma)$ and $R \geq R_0$ with $\psi \geq 0$. Then $B_\lambda^D(R)\psi \leq B_\lambda^D \psi$.*
(c) *Let $\lambda \in [0, \infty)$, $\psi \in L_2(\Gamma)$ and $R_2 \geq R_1 \geq R_0$ with $\psi \geq 0$. Then $B_\lambda^D(R_1)\psi \leq B_\lambda^D(R_2)\psi$.*

Proof The proof is similar to the proof of Propositions 3.5 and 3.6. □

Corollary 4.5 *Let $\lambda \in (0, \infty)$, $R \geq R_0$ and $q \in [2, \infty]$. Then $B_\lambda^D(R)$ maps $L_q(\Gamma)$ continuously into $L_q(\Omega)$.*

Proof The proof is similar to the proof of Corollary 3.8(c). Note that the cut-off function is not needed here. □

Together with Theorem 4.3 one obtains locally uniform convergence.

Corollary 4.6 *Let $K \subset \Omega$ compact and $\lambda \in (0, \infty)$. Then*

$$\lim_{R\to\infty} \sup_{\|\psi\|_{L_2(\Gamma)} \leq 1} \| \left(B_\lambda^D(R)\psi \right)|_K - \left(B_\lambda^D \psi \right)|_K \|_{C(K)} = 0.$$

In particular, $\lim_{R\to\infty} B_\lambda^D(R)\psi = B_\lambda^D \psi$ *locally uniformly on Ω for all $\psi \in L_2(\Gamma)$.*

Proof Let $q \in \left(\frac{d}{2}, \infty\right)$. There exists a bounded open $U_1, U_2 \subset \Omega$ such that $K \subset U_1 \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset \Omega$. By the Sobolev embedding theorem and elliptic regularity there exists a $c > 0$ such that

$$c^{-1} \|u\|_{C(K)} \leq \|u\|_{W^{1,q}(U_1)} \leq c\Big(\|\Delta u\|_{L_q(U_2)} + \|u\|_{L_2(U_2)} \Big)$$

for all $u \in W^{2,q}(U_2)$. Hence

$$c^{-1} \|B_\lambda^D(R)\psi - B_\lambda^D \psi\|_{C(K)} \leq c \, \|B_\lambda^D(R)\psi - B_\lambda^D \psi\|_{L_2(U_2)}$$

for all $R \geq R_0$ and $\psi \in L_2(\Gamma)$. Then the statement follows from Theorem 4.3. □

Next we show that $B_\lambda^D \psi$ vanishes at infinity with a certain rate for all $\psi \in L_\infty(\Gamma)$.

Theorem 4.7 *Let $\psi \in L_\infty(\Gamma)$ and $\lambda \in (0, \infty)$. Then*

$$|(B_\lambda^D \psi)(x)| \leq \frac{R_0^{d-2} \|\psi\|_\infty}{\lambda} \cdot \frac{1}{|x|^{d-2}} \tag{15}$$

for all $x \in \Omega \setminus \Omega_{R_0}$.

Proof of Theorem 4.7 Since $|B_\lambda^D \psi| \leq \|\psi\|_\infty \, B_\lambda^D \mathbb{1}_\Gamma$ by Proposition 3.5 we may assume that $\psi = \mathbb{1}_\Gamma$. Set $u = B_\lambda^D \mathbb{1}_\Gamma$. For all $R \geq R_0$ set $u_R = B_\lambda^D(R)\mathbb{1}_\Gamma$. Then $0 \leq u_R \leq u$ by Proposition 4.4(b). Moreover, $u \leq \frac{1}{\lambda} \mathbb{1}_\Omega$ by Corollary 3.8(a). In particular, $u_R|_{\partial B_{R_0}} \leq \frac{1}{\lambda} \mathbb{1}_{\partial B_{R_0}}$ and by definition $u_R|_{\partial B_R} = 0$. For all $R > R_0$ let

$$w_R \colon \Big\{x \in \mathbb{R}^d : R_0 \leq |x| \leq R\Big\} \to \mathbb{R}$$

be the continuous function whose restriction to the annulus $A_{R_0,R} := \big\{x \in \mathbb{R}^d : R_0 < |x| < R\big\}$ is harmonic and such that $w_R|_{\partial B_{R_0}} = \frac{1}{\lambda} \mathbb{1}_{\partial B_{R_0}}$ and $w_R|_{\partial B_R} = 0$. Then $u_R|_{\partial A_{R_0,R}} \leq w_R|_{\partial A_{R_0,R}}$. By the maximum principle one deduces that $u_R|_{A_{R_0,R}} \leq w_R|_{A_{R_0,R}}$. But $w_R(x) = \frac{1}{\lambda} R_0^{d-2} \left(|x|^{-(d-2)} - R^{-(d-2)}\right)$ for all $x \in \overline{A_{R_0,R}}$. So

$$u_R(x) \leq \frac{1}{\lambda} R_0^{d-2} \left(|x|^{-(d-2)} - R^{-(d-2)}\right) \leq \frac{1}{\lambda} R_0^{d-2} \, |x|^{-(d-2)}$$

first for all $x \in A_{R_0,R}$ and then obviously for all $x \in \Omega \setminus \Omega_{R_0}$. Since $\lim_{R\to\infty} u_R(x) = u(x)$ for all $x \in \Omega \setminus \Omega_{R_0}$ by Corollary 4.6 the estimate (15) follows. □

Corollary 4.8 *Let* $\psi \in L_\infty(\Gamma)$ *and* $\lambda \in (0, \infty)$. *Then*

$$|(B_\lambda \psi)(x) - \langle B_\lambda \psi \rangle| \leq R_0^{d-2} \Big(\frac{\|\psi\|_\infty}{\lambda} + |\langle B_\lambda \psi \rangle|\Big) \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus \Omega_{R_0}$.

Proof Since $(B_\lambda \psi)(x) - \langle B_\lambda \psi \rangle = (B_\lambda^D \psi)(x) - \langle B_\lambda \psi \rangle \lambda \, B_\lambda^D \mathbb{1}_\Gamma$ by Proposition 3.4 the estimate is a consequence of Theorem 4.7. □

Corollary 4.9 *Let* $u \in W(\Omega)$ *and suppose that* $\Delta u = 0$. *Then*

$$\lim_{R\to\infty} \sup_{|x|\geq R} |u(x) - \langle u \rangle| = 0.$$

Proof Define $\psi \colon \partial B_{R_0} \to \mathbb{R}$ by $\psi(x) = u(x) + \frac{1}{R_0} x \cdot (\nabla u)(x)$. Then $\psi \in L_\infty(\partial B_{R_0})$. Choose $\Omega' = \mathbb{R}^d \setminus \overline{B_{R_0}}$ and $\lambda = 1$. If B_λ is the operator defined with Ω', and the trace and normal derivative are with respect to the boundary of Ω', then $\mathrm{Tr}\, u = u|_{\partial\Omega'}$ and $(\partial_\nu u)(x) = \frac{1}{R_0} x \cdot (\nabla u)(x)$ for all $x \in \partial\Omega'$. So $B_\lambda \psi = u|_{\Omega'}$. Now apply Corollary 4.8. □

Corollary 4.10 *Let* $u \in W^D(\Omega)$ *and suppose that* $\Delta u = 0$ *and* $Tr u = 0$. *Then* $u = 0$.

Proof Without loss of generality we may assume that u is real valued. Let $\varepsilon > 0$. Because $\langle u \rangle = 0$ it follows from Corollary 4.9 that $(u - \varepsilon)^+$ vanished outside a bounded set. By assumption $\mathrm{Tr}\, u = 0$. Hence $(u - \varepsilon)^+ \in H_0^1(\Omega)$. Since $\Delta u = 0$ on Ω one deduces that

$$0 = \int_\Omega \nabla u \cdot \nabla \left((u - \varepsilon)^+\right) = \int_\Omega |\nabla \left((u - \varepsilon)^+\right)|^2.$$

So $(u - \varepsilon)^+$ is constant. Also $\mathrm{Tr}\, \left((u - \varepsilon)^+\right) = 0$. So $(u - \varepsilon)^+ = 0$ and consequently $u \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary we deduce that $u \leq 0$. Similarly, $-u \leq 0$ and thus $u = 0$. □

In the remaining part of this section we wish to show that the solution $u = B^D_\lambda \psi$ extends continuously to $\overline{\Omega}$. In the proof we use the following result of Nittka [26] Proposition 3.6. It is the Nash–De Giorgi result on the regularity of weak solutions extended to the boundary by a reflection argument.

Proposition 4.11 *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $q \in (d-1, \infty]$. Then there exist $\gamma \in (0,1)$ and $c > 0$ such that for every $\psi \in L_q(\partial U)$ and every solution $u \in H^1(U)$ of*

$$\Delta u = 0 \quad \text{on } U$$

$$\partial_\nu u = \psi \quad \text{on } \partial U$$

the function u is Hölder continuous of order γ. Moreover,

$$|u(x)| \le c \, \big(\|u\|_{L_2(U)} + \|\psi\|_{L_q(\partial U)}\big)$$

and

$$|u(x) - u(y)| \le c \, \big(\|u\|_{L_2(U)} + \|\psi\|_{L_q(\partial U)}\big) \, |x-y|^\gamma$$

for all $x, y \in U$.

Theorem 4.12 *Let $q \in (d-1, \infty]$, $\psi \in L_q(\Gamma)$ and $\lambda \in (0,\infty)$. Then $B^D_\lambda \psi$ and $B^D_\lambda(R)\psi$ extend continuously to $\overline{\Omega}$ for all $R \ge R_0$. Moreover,*

$$\lim_{R\to\infty} B^D_\lambda(R) = B^D_\lambda$$

in $\mathcal{L}(L_\infty(\Gamma), L_\infty(\Omega))$.

We need a lemma.

Lemma 4.13 *Let $S \in [R_0 - 2, \infty)$ and $q \in (d-1, \infty]$. Then there exists a $c > 0$ such that for every $\psi \in L_q(\Gamma)$ and $u \in H^1(\Omega_{S+2})$ with $\Delta u = 0$ on Ω_{S+2} and $\partial_\nu u = \psi$ on Γ it follows that $u|_{\Omega_S}$ extends to a continuous function on $\overline{\Omega_S}$ and*

$$|u(x)| \le c \, (\|\psi\|_{L_q(\Gamma)} + \|u\|_{H^1(\Omega_{S+1})})$$

for all $x \in \Omega_S$.

Proof There exists a $c_1 > 0$ such that

$$\Big(\int_{\partial B_1} |v|^2 \Big)^{1/2} \le c_1 \, \|v\|_{H^1(B_1)}$$

for all $v \in H^1(B_1) \cap C\big(\overline{B_1}\big)$.

Let $c > 0$ be as in Proposition 4.11 with $U = \Omega_{S+2}$. Let $\psi \in L_q(\Gamma)$, $u \in H^1(\Omega_{S+2})$ and suppose that $\Delta u = 0$ on Ω_{S+2} and $\partial_\nu u = \psi$ on Γ. Then u is harmonic on Ω_{S+2}. Let $x_0 \in \partial B_S$. Then

$$u(x) = \frac{1}{d\,\omega_d} \int_{\partial B_1(x_0)} \frac{1 - |x - x_0|^2}{|x-y|^d} \, u(y) \, d\sigma(y)$$

for all $x \in B_1(x_0)$. Hence

$$(\partial_k u)(x) = -\frac{1}{d\,\omega_d} \int_{\partial B_1(x_0)} \left(\frac{2(x-x_0)_k}{|x-y|^d} + \frac{d\, x_k \, (1 - |x-x_0|^2)}{|x-y|^{d+2}} \right) u(y) \, d\sigma(y)$$

for all $x \in B_1(x_0)$ and $k \in \{, \dots, d\}$. Therefore

$$(\partial_k u)(x_0) = -\frac{1}{\omega_d} \int_{\partial B_1(x_0)} (x_0)_k \, u(y) \, d\sigma(y)$$

and

$$\begin{aligned} \|\partial_\nu u\|_{L_\infty(\partial B_S)} &\leq d^2 \, S\Big(\frac{1}{d\,\omega_d} \int_{\partial B_1(x_0)} |u(y)|^2\Big)^{1/2} \\ &\leq c_1 \, d^2 \, S \, \|u\|_{H^1(B_1(x_0))} \leq c_1 \, d^2 \, S \, \|u\|_{H^1(\Omega_{S+1})}. \end{aligned} \tag{16}$$

Define $\tau \in L_\infty(\partial\Omega_S)$ by

$$\tau(x) = \begin{cases} \psi(x) & \text{if } x \in \Gamma, \\ (\partial_\nu u)(x) & \text{if } x \in \partial B_S. \end{cases}$$

Then $u|_{\Omega_S} \in H^1(\Omega_S)$ is a solution of

$$\begin{aligned} \Delta v &= 0 \ \text{ on } \Omega_S, \\ \partial_\nu v &= \tau \ \text{ on } \partial\Omega_S. \end{aligned}$$

Consequently, the function $u|_{\Omega_S}$ is Hölder continuous on Ω_S by Proposition 4.11. In particular, $u|_{\Omega_S}$ extends continuously to $\overline{\Omega_S}$. Moreover,

$$\begin{aligned} |u(x)| &\leq c \, \big(\|u\|_{L_2(\Omega_S)} + \|\tau\|_{L_q(\partial\Omega_S)}\big) \\ &\leq c \, \Big(\|u\|_{L_2(\Omega_S)} + \|\psi\|_{L_q(\Gamma)} + c_1 \, d^2 \, S \, |\sigma(\partial B_S)|^{1/q} \, \|u\|_{H^1(\Omega_{S+1})}\Big) \end{aligned}$$

for all $x \in \Omega_S$, where we used Eq. 16 in the last step. □

Proof of Theorem 4.12 Without loss of generality we may assume that $q < \infty$. Let $S \in [R_0 - 2, \infty)$. Let $c > 0$ be as in Lemma 4.13. Let $\psi \in L_q(\Gamma)$. Set $u = B_\lambda^D \psi$ and $u_R = B_\lambda^D(R)\psi$ for all $R \in [S + 2, \infty)$. Then it follows from Lemma 4.13 that $u|_{\Omega_S}$ and $u_R|_{\Omega_S}$ extend continuously to $\overline{\Omega_S}$. Note that $u_R \in C(\Omega)$ by Proposition 4.2. Hence u and u_R extend continuously to $\overline{\Omega}$. Moreover,

$$\|u - u_R\|_{L_\infty(\Omega_S)} \leq c \, \Big(\|\lambda \mathrm{Tr}\,(u - u_R)\|_{L_q(\Gamma)} + \|u - u_R\|_{H^1(\Omega_{S+1})}\Big) .$$

But

$$\|\mathrm{Tr}\,(u - u_R)\|_{L_q(\Gamma)} \leq \|\mathrm{Tr}\,(u - u_R)\|_{L_2(\Gamma)}^{2/q} \, \|\mathrm{Tr}\,(u - u_R)\|_{L_\infty(\Gamma)}^{(q-2)/q}$$

and $A\,B \leq \frac{1}{\alpha}\,\delta^{-\alpha}\,A^\alpha + \frac{1}{\beta}\,\delta^\beta\,B^\beta$ for all $A, B \geq 0$, $\delta > 0$ and $\alpha, \beta \in [1, \infty]$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Choose $\alpha = q/2$ and $\beta = q/(q-2)$. Then

$$\|\mathrm{Tr}\,(u - u_R)\|_{L_q(\Gamma)} \leq \frac{2}{q}\,\delta^{-q/2}\,\|\mathrm{Tr}\,(u - u_R)\|_{L_2(\Gamma)} + \frac{q-2}{q}\,\delta^{q/(q-2)}\,\|\mathrm{Tr}\,(u - u_R)\|_{L_\infty(\Gamma)}$$

for all $\delta > 0$. Moreover, $\|\mathrm{Tr}\,(u - u_R)\|_{L_\infty(\Gamma)} \leq \|u - u_R\|_{L_\infty(\Omega_S)}$. Choose $\delta > 0$ such that $c\,\lambda\,\frac{q-2}{q}\,\delta^{q/(q-2)} = \frac{1}{2}$. Note that δ is independent of ψ and R. Then

$$\begin{aligned} \|u - u_R\|_{L_\infty(\Omega_S)} &\leq 2c \, \Big(\frac{2\lambda}{q}\,\delta^{-q/2}\,\|\mathrm{Tr}\,(u - u_R)\|_{L_2(\Gamma)} + \|u - u_R\|_{H^1(\Omega_{S+1})}\Big) \\ &\leq 2c \, \Big(1 + \frac{2\lambda}{q}\,\delta^{-q/2}\,\|\mathrm{Tr}_{S+1}\|_{\mathcal{L}(H^1(\Omega_{S+1}), L_2(\Gamma))}\Big) \, \|u - u_R\|_{H^1(\Omega_{S+1})}. \end{aligned}$$

The restriction $v \mapsto v|_{\Omega_{S+1}}$ is continuous from $W^D(\Omega)$ into $H^1(\Omega_{S+1})$. Let

$$M = \sup\Big\{ \|v|_{\Omega_{S+1}}\|_{H^1(\Omega_{S+1})} : v \in W^D(\Omega),\ \|v\|_{W^D(\Omega)} \le 1 \Big\}.$$

Then

$$\begin{aligned} \|u - u_R\|_{L_\infty(\Omega_S)} &\le c_1 \, \|B_\lambda^D - B_\lambda^D(R)\|_{\mathcal{L}(L_2(\Gamma), W^D(\Omega))} \, \|\psi\|_{L_2(\Gamma)} \\ &\le c_1 \, |\sigma(\Gamma)|^{1/2} \, \|B_\lambda^D - B_\lambda^D(R)\|_{\mathcal{L}(L_2(\Gamma), W^D(\Omega))} \, \|\psi\|_{L_\infty(\Gamma)}, \end{aligned} \tag{17}$$

where $c_1 = 2c\, M\, (1 + \frac{2\lambda}{q}\, \delta^{-q/2}\, \|\mathrm{Tr}_{S+1}\|_{\mathcal{L}(H^1(\Omega_{S+1}), L_2(\Gamma))})$. Note that

$$\lim_{R \to \infty} \|B_\lambda^D - B_\lambda^D(R)\|_{\mathcal{L}(L_2(\Gamma), W^D(\Omega))} = 0$$

by Theorem 4.3.

If $R \ge S + 2$, then

$$|\Big(B_\lambda^D(R)\psi\Big)(x)| \le \Big(B_\lambda^D|\psi|\Big)(x) \le \frac{R_0^{d-2}\, \|\psi\|_\infty}{\lambda} \cdot \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus \Omega_{R_0}$ by Proposition 4.4(b) and Theorem 4.7. So

$$\|u - u_R\|_{L_\infty(\Omega \setminus \Omega_S)} \le 2 \frac{R_0^{d-2}\, \|\psi\|_\infty}{\lambda\, S^{d-2}}$$

for all $R \ge S + 2$. Together with the estimate in Eq. 17 and Theorem 4.3 it follows that $\lim_{R \to \infty} \|B_\lambda^D - B_\lambda^D(R)\|_{\mathcal{L}(L_\infty(\Gamma), L_\infty(\Omega))} = 0$. □

Corollary 4.14 *Let $q \in (d-1, \infty]$, $\psi \in L_q(\Gamma)$ and $\lambda \in (0, \infty)$. Then $B_\lambda \psi$ extends continuously to $\overline{\Omega}$.*

Proof This follows from Theorem 4.12 and Proposition 3.4. □

5 The Dirichlet-to-Neumann Operator on $L_2(\Gamma)$

Finally we introduce two versions of the Dirichlet-to-Neumann operator on $L_2(\Gamma)$ with respect to the exterior domain Ω with boundary Γ.

Define the form $\mathfrak{a}\colon W(\Omega) \times W(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v}.$$

Then $\mathfrak{a}$ is a continuous symmetric sesquilinear form and $\mathfrak{a}$ is Tr-elliptic. We use the terminology and basic generation results of [6] (see also [5]). Let A be the self-adjoint operator in $L_2(\Gamma)$ associated with $(\mathfrak{a}, \mathrm{Tr})$. We call A the **Dirichlet-to-Neumann operator with Neumann boundary conditions at infinity**. The Dirichlet version is as follows. Define the form $\mathfrak{a}^D\colon W^D(\Omega) \times W^D(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}^D = \mathfrak{a}|_{W^D(\Omega) \times W^D(\Omega)}.$$

Then also $\mathfrak{a}^D$ is $\mathrm{Tr}\,|_{W^D(\Omega)}$-elliptic. Let A^D be the self-adjoint operator in $L_2(\Gamma)$ associated with $(\mathfrak{a}^D, \mathrm{Tr}\,|_{W^D(\Omega)})$. We call A^D the **Dirichlet-to-Neumann operator with Dirichlet boundary conditions at infinity**.

Proposition 5.1 *Let $\varphi, \psi \in L_2(\Gamma)$.*

(a) *The following are equivalent.*

(i) $\varphi \in D\left(A^D\right)$ *and* $\psi = A^D\varphi$.

(ii) *There exists a function* $u \in W^D(\Omega)$ *such that* $Tr\, u = \varphi$, $\Delta u = 0$ *and* $\partial_\nu u = \psi$. *and* $\partial_\nu u = \psi$.

(b) *The following are equivalent.*

(i) $\varphi \in D(A)$ *and* $\psi = A\varphi$.

(ii) *There exists a function* $u \in W(\Omega)$ *such that* $Tr\, u = \varphi$, $\Delta u = 0$, $\partial_\nu u = \psi$ *and* $\int_\Gamma \psi = 0$.

Proof The equivalence of (i) and (ii) in (a) and (b) follows as in Subsection 4.4 in [6] using Eq. 7. □

Remark 5.2 It is not possible to replace $W(\Omega)$ or $W^D(\Omega)$ by $H^1(\Omega)$. Precisely, let $\mathfrak{b} = \mathfrak{a}|_{H^1(\Omega)\times H^1(\Omega)}$. Then the form $\mathfrak{b}$ is not Tr-elliptic, since there are no $\mu, \omega > 0$ such that $\mu\,\|u\|^2_{H^1(\Omega)} \le \mathfrak{b}(u)+\omega\,\|\mathrm{Tr}\,u\|^2_{L_2(\Gamma)}$ for all $u \in H^1(\Omega)$, because the inequality $\mu\,\|u\|^2_{L_2(\Omega)} \le \mathfrak{b}(u)+\omega\,\|\mathrm{Tr}\,u\|^2_{L_2(\Gamma)}$ is not valid for all $u \in H^1(\Omega)$. In order to give a counter example, we may assume that $0 \notin \overline{\Omega}$. Fix $\tau \in C_c^\infty\left(\mathbb{R}^d\right)$ such that $\tau|_{B_1} = \mathbb{1}$ and for large $R \in [1,\infty)$ consider $u \in H^1(\Omega)$ given by $u(x) = |x|^{-(d-1)/2}\,\tau\left(\frac{1}{R}\,x\right)$.

If $\lambda \in (0,\infty)$ then $(\lambda I + A)^{-1} = \mathrm{Tr}\circ B_\lambda$ is a compact operator by Proposition 3.9. Similarly, $(\lambda I + A^D)^{-1} = \mathrm{Tr}\circ B^D_\lambda$ is a compact operator for all $\lambda \in [0,\infty)$. So both A and A^D have compact resolvents. Clearly $\ker A = \mathrm{span}(\mathbb{1}_\Gamma)$ and $\ker A^D = \{0\}$. So $A \ne A^D$. Let S and S^D be the semigroups generated by $-A$ and $-A^D$.

Proposition 5.3 *The semigroups* S *and* S^D *are real, positive and submarkovian. In particular, both* S *and* S^D *extend consistently to* C_0*-semigroups on* $L_q(\Gamma)$ *for all* $q \in [1,\infty)$.

Proof This follows from [6] Proposition 2.9 (see also Corollary 3.13 in [6]). □

The following domination result holds.

Proposition 5.4 *If* $t > 0$ *and* $\varphi \in L_2(\Gamma)$ *with* $\varphi \ge 0$, *then* $S^D_t\varphi \le S_t\varphi$.

Proof It follows from Proposition 3.6 that $(\lambda I + A^D)^{-1}\varphi = \mathrm{Tr}\,B^D_\lambda\varphi \le \mathrm{Tr}\,B_\lambda\varphi = (\lambda I + A)^{-1}\varphi$ for all $\lambda > 0$. Hence $S^D_t\varphi = \lim_{n\to\infty}\left(\frac{t}{n}\,I + A^D\right)^{-n}\varphi \le S_t\varphi$. □

Next let $R \in [R_0,\infty)$. Define the form $\mathfrak{a}^D_R \colon W^D_R(\Omega)\times W^D_R(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}^D_R = \mathfrak{a}|_{W^D_R(\Omega)\times W^D_R(\Omega)}.$$

Then again $\mathfrak{a}^D_R$ is $\mathrm{Tr}\,|_{W^D_R(\Omega)}$-elliptic. Let A^D_R be the self-adjoint operator in $L_2(\Gamma)$ associated with $\left(\mathfrak{a}^D_R, \mathrm{Tr}\,|_{W^D_R(\Omega)}\right)$. We call A^D_R the **Dirichlet-to-Neumann operator with Dirichlet boundary conditions at** ∂B_R. Clearly A^D_R is a positive invertible operator. Moreover, the semigroup $S^{D,R}$ generated by $-A_R^D$ is real positive and submarkovian by [6] Proposition 2.9. Hence $S^{D,R}$ extends consistently to a C_0-semigroup on $L_q(\Gamma)$ for all $q \in [1, \infty)$. With abuse of notation we denote by $-A_R^D$ the generator in $L_q(\Gamma)$. In addition we denote by $-A^D$ the generator of the semigroup S^D in $L_q(\Gamma)$.

The reason that we call A^D and A the Dirichlet-to-Neumann operator with Dirichlet or Neumann conditions at infinity is motivated by the next two convergence theorems.

Theorem 5.5 *Let* $\lambda \in [0, \infty)$ *and* $q \in [1, \infty)$. *Then* $\lim_{R\to\infty} \big(\lambda I + A_R^D\big)^{-1} = \big(\lambda I + A^D\big)^{-1}$ *in* $\mathcal{L}\big(L_q(\Gamma)\big)$.

Proof Since $\big(\lambda I + A_R^D\big)^{-1} = \mathrm{Tr} \circ B_\lambda^D(R)$, the theorem follows from Theorems 4.3 and 4.12, together with duality and interpolation. □

We also wish to consider the Dirichlet-to-Neumann operator with Neumann boundary conditions at ∂B_R. Let $R \in [R_0, \infty)$. Define the form $\mathfrak{a}_R \colon H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ by

$$\mathfrak{a}_R(u, v) = \int_{\Omega_R} \nabla u \cdot \overline{\nabla v}.$$

Then $\mathfrak{a}_R$ is a Tr_R-elliptic symmetric continuous sesquilinear form by Lemma 2.2. Let A_R be the positive self-adjoint operator in $L_2(\Gamma)$ associated with $(\mathfrak{a}_R, \mathrm{Tr}_R)$. We call A_R the **Dirichlet-to-Neumann operator with Neumann boundary conditions at** ∂B_R. The resolvents of the operators A_R also converge uniformly.

Theorem 5.6 *Let* $\lambda \in (0, \infty)$. *Then* $\lim_{R\to\infty}(\lambda I + A_R)^{-1} = (\lambda I + A)^{-1}$ *in* $\mathcal{L}(L_2(\Gamma))$.

Proof Let $R_1, R_2, \ldots \in [R_0, \infty)$ and $\psi, \psi_1, \psi_2, \ldots \in L_2(\Gamma)$ with $\lim R_n = \infty$ and $\lim \psi_n = \psi$ weakly in $L_2(\Gamma)$. We shall show that $\lim(\lambda I + A_{R_n})^{-1}\psi_n = (\lambda I + A)^{-1}\psi$ in $L_2(\Gamma)$. Then the theorem follows since $(\lambda I + A)^{-1}$ is compact.

For all $n \in \mathbb{N}$ set $\varphi_n = (\lambda I + A_{R_n})^{-1}\psi_n$. Let $n \in \mathbb{N}$. There exists a $u_n \in H^1(\Omega_{R_n})$ such that $\mathrm{Tr}_{R_n} u_n = \varphi_n$ and

$$\int_{\Omega_{R_n}} \nabla u_n \cdot \overline{\nabla v} + \lambda \int_\Gamma \varphi_n \, \overline{\mathrm{Tr}_{R_n} v} = \int_\Gamma \psi_n \, \overline{\mathrm{Tr}_{R_n} v} \tag{18}$$

for all $v \in H^1(\Omega_{R_n})$. Choose $v = u_n$. Then

$$\int_{\Omega_{R_n}} |\nabla u_n|^2 + \lambda \int_\Gamma |\varphi_n|^2 = \int_\Gamma \psi_n \, \overline{\varphi_n} \le \|\psi_n\|_{L_2(\Gamma)} \, \|\varphi_n\|_{L_2(\Gamma)}.$$

Therefore

$$M = \sup_{n\in\mathbb{N}} \int_{\Omega_{R_n}} |\nabla u_n|^2 + \|\varphi_n\|_{L_2(\Gamma)}^2 < \infty. \tag{19}$$

Let $R \in [R_0, \infty) \cap \mathbb{N}$. Then $v \mapsto \Big(\int_{\Omega_R} |\nabla v|^2 + \int_\Gamma |\mathrm{Tr}_R v|^2\Big)^{1/2}$ is a norm on $H^1(\Omega_R)$ which is equivalent with $\|\cdot\|_{H^1(\Omega_R)}$ by Lemma 2.2. Hence $(u_n|_{\Omega_R})_{n\in\mathbb{N},\, R_n \ge R}$ is bounded in $H^1(\Omega_R)$. Using a diagonal argument and passing to a subsequence if necessary, we may assume that for all $R \in [R_0, \infty)$ the sequence $\big(u_n|_{\Omega_R}\big)_{n\in\mathbb{N}}$ is weakly convergent in $H^1(\Omega_R)$. Therefore there exists a $u \in H^1_{\mathrm{loc}}(\Omega)$ such that $u|_{\Omega_R} \in H^1(\Omega_R)$ and $\lim u_n|_{\Omega_R} = u|_{\Omega_R}$

weakly in $H^1(\Omega_R)$ for all $R \in [R_0, \infty)$. It follows from Eq. 19 that

$$\int_{\Omega_R} |\nabla u|^2 \le \liminf_{n\to\infty} \int_{\Omega_R} |\nabla u_n|^2 \le M$$

for all $R \in [R_0, \infty)$. Hence $\int_\Omega |\nabla u|^2 \le M < \infty$ and $u \in W(\Omega)$.

By Proposition 2.5(b), that is the compactness of Tr, it follows that $\lim \varphi_n = \lim \operatorname{Tr} u_n = \operatorname{Tr} u$ in $L_2(\Gamma)$. Let $v \in C_c^\infty(\mathbb{R}^d)$. Then Eq. 18 gives

$$\int_\Omega \nabla u_n \cdot \overline{\nabla(v|_\Omega)} + \lambda \int_\Gamma \varphi_n \, \overline{\operatorname{Tr}(v|_\Omega)} = \int_{\Omega_{R_n}} \nabla u_n \cdot \overline{\nabla(v|_\Omega)} + \lambda \int_\Gamma \varphi_n \, \overline{\operatorname{Tr}_{R_n}(v|_\Omega)} = \int_\Gamma \psi_n \, \overline{\operatorname{Tr} v}$$

for all $n \in \mathbb{N}$ with $\operatorname{supp} v \subset B_{R_n}$. Take the limit $n \to \infty$. Then

$$\int_\Omega \nabla u \cdot \overline{\nabla(v|_\Omega)} + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr}(v|_\Omega)} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}. \tag{20}$$

By density Eq. 20 is valid for all $v \in W^D(\Omega)$. Finally, choose $v = \mathbb{1}_{\Omega_{R_n}}$ in Eq. 18. Then $\lambda \int_\Gamma \varphi_n = \int \psi_n$. Take the limit $n \to \infty$. Then $\lambda \int_\Gamma \operatorname{Tr} u = \int \psi$. So Eq. 20 is also valid if $v = \mathbb{1}_\Omega$. By linearity and Proposition 2.6 one deduces that (20) is valid for all $v \in W(\Omega)$. Hence $B_\lambda \psi = u$ and $(\lambda I + A)^{-1} \psi = \operatorname{Tr} u$. The proof is complete. □

Clearly $\mathbb{1}_\Gamma \in D(A)$ and $A \mathbb{1}_\Gamma = 0$. We next show that also $\mathbb{1}_\Gamma \in D\left(A^D\right)$.

Proposition 5.7 $\mathbb{1}_\Gamma \in D\left(A^D\right)$.

Proof Let $\tau \in C_c^\infty\left(\mathbb{R}^d, \mathbb{R}\right)$ be such that $\tau|_{\Omega_{R_0}} = \mathbb{1}$. For all $n \in \mathbb{N}$ with $n \ge R_0$ there exists a $u_n \in H_0^1(\Omega_n)$ such that $-\Delta u_n = \Delta \tau$ on Ω_n. Then

$$\begin{aligned}\int_{\Omega_n} |\nabla u_n|^2 = -\int_{\Omega_n} (\Delta u_n)\, u_n &= \int_{\Omega_n} (\Delta \tau)\, u_n \\ &= -\int_{\Omega_n} (\nabla \tau) \cdot \nabla u_n \le \frac{1}{2} \int_{\Omega_n} |\nabla u_n|^2 + \frac{1}{2} \int_{\Omega_n} |\nabla \tau|^2.\end{aligned}$$

So

$$\int_{\Omega_n} |\nabla u_n|^2 \le \int_{\Omega_n} |\nabla \tau|^2$$

for all $n \in \mathbb{N}$ with $n \ge R_0$. Extend u_n by zero to a function on Ω, which we still denote by u_n. Then $u_n \in W^D(\Omega)$ and $\|u_n\|^2_{W^D(\Omega)} \le \int_{\Omega_n} |\nabla \tau|^2$. So $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^D(\Omega)$. Passing to a subsequence if necessary, there exists a $u \in W^D(\Omega)$ such that $\lim u_n = u$ weakly in $W^D(\Omega)$. Then $\operatorname{Tr} u = \lim \operatorname{Tr} u_n = 0$ in $L_2(\Gamma)$. Let $v \in C_c^\infty(\Omega)$. Then

$$\int_\Omega \nabla u \cdot \overline{\nabla v} = \lim \int_\Omega \nabla u_n \cdot \overline{\nabla v} = \lim \int_{\Omega_n} \nabla u_n \cdot \overline{\nabla v} = \lim \int_{\Omega_n} (-\Delta u_n) \overline{v} = \lim \int_{\Omega_n} (\Delta \tau) \overline{v}$$

and $-\Delta u = \Delta \tau$ on Ω. Let $\chi \in C_c^\infty\left(B_{R_0+1}\right)$ be such that $\chi|_{B_{R_0}} = \mathbb{1}$. Then $\chi\, u \in H_0^1\left(\Omega_{R_0+1}\right)$ and $\Delta(\chi\, u) = (\Delta \chi)\, u + \chi\, (\Delta u) + 2 \nabla \chi \cdot \nabla u \in L_2(\Omega_{R_0+1})$. By [20] Theorem B.2 it follows that $\chi\, u \in H^{3/2}\left(\Omega_{R_0+1}\right)$. Then by Lemma 2.4 in [16] one deduces that $\partial_\nu(\chi\, u) \in L_2\left(\partial \Omega_{R_0+1}\right)$. Hence $\partial_\nu u \in L_2(\Gamma)$.

Finally, set $w = u + \tau$. Then $w \in W^D(\Omega)$, $\operatorname{Tr} w = \mathbb{1}_\Gamma$ and $\Delta w = 0$ on Ω. Moreover, $\partial_\nu w = \partial_\nu u + \partial_\nu \tau \in L_2(\Gamma)$. So $\mathbb{1}_\Gamma \in D\left(A^D\right)$. □

Corollary 5.8 $A^D \mathbb{1}_\Gamma \ne 0$ *and* $\int_\Gamma A^D \mathbb{1}_\Gamma \ne 0$.

Proof If $\int_\Gamma A^D \mathbb{1}_\Gamma = 0$ then $(A^D \mathbb{1}_\Gamma, \mathbb{1}_\Gamma)_{L_2(\Gamma)} = 0$, so $\min \sigma\left(A^D\right) = 0$ and A^D is not invertible. This is a contradiction. □

Note that $A^D \mathbb{1}_\Gamma = \lim_{t\downarrow 0} t^{-1}\left(I - S_t^D\right)\mathbb{1}_\Gamma \geq 0$. We next show that A^D is a bounded rank-one perturbation of A.

Theorem 5.9 *$D\left(A^D\right) = D(A)$ and*

$$A^D \varphi = A\varphi + \frac{1}{\beta}\left(\varphi, A^D \mathbb{1}_\Gamma\right)_{L_2(\Gamma)} A^D \mathbb{1}_\Gamma$$

for all $\varphi \in D(A)$, where $\beta = \int_\Gamma A^D \mathbb{1}_\Gamma$.

Proof Let $\varphi \in D(A)$. Write $\psi = A\varphi$. By definition there exists a unique $u \in W(\Omega)$ such that $\mathrm{Tr}\, u = \varphi$ and $\mathfrak{a}(u, v) = (A\varphi, \mathrm{Tr}\, v)_{L_2(\Gamma)}$ for all $v \in W(\Omega)$. Then $\int_\Gamma A\varphi = 0$ since $\mathbb{1}_\Omega \in W(\Omega)$. Clearly $u - \langle u\rangle \mathbb{1}_\Omega \in W^D(\Omega)$. Moreover, if $v \in W^D(\Omega)$ then

$$\mathfrak{a}^D(u - \langle u\rangle \mathbb{1}_\Omega, v) = \mathfrak{a}(u - \langle u\rangle \mathbb{1}_\Omega, v) = \mathfrak{a}(u, v) = \int_\Gamma \psi\, \overline{\mathrm{Tr}\, v}.$$

So

$$\varphi - \langle u\rangle \mathbb{1}_\Gamma = \mathrm{Tr}\,(u - \langle u\rangle \mathbb{1}_\Omega) \in D\left(A^D\right)$$

and $A^D(\varphi - \langle u\rangle \mathbb{1}_\Gamma) = \psi$. Since $\mathbb{1}_\Gamma \in D\left(A^D\right)$ by Proposition 5.7 it follows that $\varphi \in D\left(A^D\right)$ and

$$A^D\varphi - \langle u\rangle A^D \mathbb{1}_\Gamma = \psi = A\varphi. \tag{21}$$

Therefore

$$0 = \int_\Gamma A\varphi = \int_\Gamma A^D\varphi - \langle u\rangle \int_\Gamma A^D \mathbb{1}_\Gamma = (A^D\varphi, \mathbb{1}_\Gamma)_{L_2(\Gamma)} - \beta\,\langle u\rangle = (\varphi, A^D \mathbb{1}_\Gamma)_{L_2(\Gamma)} - \beta\,\langle u\rangle$$

and so $\langle u\rangle = \beta^{-1}(\varphi, A^D \mathbb{1}_\Gamma)_{L_2(\Gamma)}$. Hence Eq. 21 gives

$$A^D \varphi = A\varphi + \frac{1}{\beta}\left(\varphi, A^D \mathbb{1}_\Gamma\right)_{L_2(\Gamma)} A^D \mathbb{1}_\Gamma$$

for all $\varphi \in D(A)$.

Define $B\colon L_2(\Gamma) \to L_2(\Gamma)$ by $B\varphi = \frac{1}{\beta}\left(\varphi, A^D \mathbb{1}_\Gamma\right)_{L_2(\Gamma)} A^D \mathbb{1}_\Gamma$. Then B is bounded and $A \subset A^D - B$. Since both $-A$ and $-\left(A^D - B\right)$ generate a C_0-semigroup it follows that $A = A^D - B$. In particular $D(A) = D\left(A^D\right)$ and the proof of the theorem is complete. □

Corollary 5.10 *The operator $A - A^D$ is bounded.*

Finally we consider irreducibility. Since ker A is one-dimensional, Proposition 2.2 in [4] implies the next proposition.

Proposition 5.11 *The semigroup S is irreducible.*

The irreducibility of S^D is much deeper and requires a couple of preliminary steps involving the Dirichlet-to-Neumann operator A_R^D with Dirichlet boundary conditions at ∂B_R, where $R \in [R_0, \infty)$. The next proof is inspired by the proof of Theorem 4.2 in [8], where the domain was bounded.

Proposition 5.12 *Let $R \in [R_0, \infty)$, $t > 0$ and $\varphi \in L_2(\Gamma)$ with $\varphi \geq 0$. Then $0 \leq e^{-tA_R^D}\varphi \leq S_t^D\varphi$.*

Proof Let $\lambda > 0$. Then $\left(\lambda I + A_R^D\right)^{-1}\varphi = \operatorname{Tr} B_\lambda^D(R)\varphi$. Hence Proposition 4.4(a) and (b) imply that $0 \leq \left(\lambda I + A_R^D\right)^{-1}\varphi \leq \left(\lambda I + A^D\right)^{-1}\varphi$. Then the proposition follows by the Euler formula. □

We shall show that the semigroup $\left(e^{-tA_R^D}\right)_{t>0}$ is irreducible. As a tool we will consider the Laplacian Δ_β^R on $L_2(\Omega_R)$ with Dirichlet boundary conditions on ∂B_R and Robin boundary conditions on Γ. The following result is a special case of [2] Theorem 1.3. We give a direct proof for convenience.

Proposition 5.13 *Let $U \subset \mathbb{R}^d$ be open and let A_1 and A_2 be two lower-bounded self-adjoint operators with compact resolvent. Suppose that $0 \leq e^{-tA_1}u \leq e^{-tA_2}u$ for all $t > 0$ and $u \in L_2(U)$ with $u \geq 0$. Moreover, assume that the semigroup $(e^{-tA_1})_{t>0}$ is irreducible. If the lowest eigenvalues of A_1 and A_2 are equal, then $A_1 = A_2$.*

Proof We may assume that the lowest eigenvalues of A_1 and A_2 are zero. Since A_1 and A_2 have compact resolvents and the semigroups $\left(e^{-tA_1}\right)_{t>0}$ and $\left(e^{-tA_2}\right)_{t>0}$ are positive, it follows from the Krein–Rutman theorem that there are $u_1, u_2 \in L_2(U)$ with $u_1, u_2 \geq 0$ and $u_1 \neq 0 \neq u_2$ such that $e^{-tA_1}u_1 = u_1$ and $e^{-tA_2}u_2 = u_2$ for all $t > 0$. Because the semigroup $\left(e^{-tA_1}\right)_{t>0}$ and hence also $(e^{-tA_2})_{t>0}$ is irreducible, it follows from [25] Chapter C-III, Theorem 3.2.(b) that $u_1(x) > 0$ and $u_2(x) > 0$ for almost every $x \in U$. Let $t > 0$. Then the domination assumption gives $e^{-tA_2}u_1 - e^{-tA_1}u_1 \geq 0$. Moreover,

$$\left(e^{-tA_2}u_1 - e^{-tA_1}u_1, u_2\right) = \left(u_1, e^{-tA_2}u_2\right) - \left(e^{-tA_1}u_1, u_2\right) = (u_1, u_2) - (u_1, u_2) = 0.$$

So $e^{-tA_2}u_1 - e^{-tA_1}u_1 = 0$. Now let $f \in L_2(U)$ with $f \geq 0$. Then $e^{-tA_2}f - e^{-tA_1}f \geq 0$ and

$$\left(e^{-tA_2}f - e^{-tA_1}f, u_1\right) = \left(f, e^{-tA_2}u_1 - e^{-tA_1}u_1\right) = 0.$$

So $e^{-tA_2}f - e^{-tA_1}f = 0$ and $e^{-tA_1}f = e^{-tA_2}f$. Hence by linearity $e^{-tA_1} = e^{-tA_2}$. This is for all $t > 0$, so $A_1 = A_2$. □

We also need the next known lemma, whose short proof we include for self-containment.

Lemma 5.14 *Let $U \subset \mathbb{R}^d$ be open and let T be a positive irreducible semigroup in $L_2(U)$ with generator $-B$. Assume that B is self-adjoint and has compact resolvent. Let λ be an eigenvalue of B with a positive eigenfunction. Then λ is the smallest eigenvalue of B.*

Proof Let λ_1 be the smallest eigenvalue of $-B$. We may assume that $\lambda_1 = 0$. Then $\lambda \geq 0$. Let $u \in D(B)$ with $Bu = \lambda u$, $u \geq 0$ and $u \neq 0$. Since T is positive, holomorphic and irreducible it follows from the Krein–Rutman theorem and [25] Chapter C-III, Theorem 3.2.(b) that there exists a $u_1 \in D(B)$ such that $Bu_1 = 0$ and $u_1(x) > 0$ for almost every $x \in U$. Then $\lambda\,(u, u_1) = (Bu, u_1) = (u, Bu_1) = 0$. So $\lambda = 0$ or $(u, u_1) = 0$. But if $(u, u_1) = 0$, then $u = 0$ almost everywhere, which is a contradiction. So $\lambda = 0$. □

For the remainder of the proof of irreducibility of S^D we need the Sobolev space used in Lemma 4.1. For all $R \geq R_0$ define

$$V_R = \left\{ v \in H^1(\Omega_R) : \mathbb{1}_{\partial B_R} \operatorname{Tr}_{\Omega_R} v = 0 \right\}.$$

Let $\beta \in L_\infty(\Gamma, \mathbb{R})$. Define the form $\mathfrak{a}_{\beta,R} \colon V_R \times V_R \to \mathbb{R}$ by

$$\mathfrak{a}_{\beta,R}(u, v) = \int_{\Omega_R} \nabla u \cdot \overline{\nabla v} - \int_\Gamma \beta \operatorname{Tr}_R u \, \overline{\operatorname{Tr}_R v}.$$

Then $\mathfrak{a}_{\beta,R}$ is a densely defined closed symmetric form in $L_2(\Omega_R)$. Let $-\Delta_{\beta,R}$ be the self-adjoint operator associated with $\mathfrak{a}_{\beta,R}$. Then $\Delta_{\beta,R}$ has compact resolvent and $e^{t\Delta_{\beta,R}} u \geq 0$ for all $t > 0$ and $u \in L_2(\Omega_R)$ with $u \geq 0$. It follows from Ouhabaz' criterion [28] Corollary 2.11 (see also [9] Theorems 10.1.5 and 11.2.1) that the semigroup $(e^{t\Delta_{\beta,R}})_{t>0}$ is irreducible. By the domination theorem [27] Theorem 3.7 one deduces that

$$e^{t\Delta_{\beta_1,R}} u \leq e^{t\Delta_{\beta_2,R}} u$$

for all $\beta_1, \beta_2 \in L_\infty(\Gamma, \mathbb{R})$ and $u \in L_2(\Omega_R)$ with $\beta_1 \leq \beta_2$ and $u \geq 0$.

Recall that Γ has the $(d-1)$-dimensional Hausdorff measure.

Proposition 5.15 *Let $R \in [R_0, \infty)$ and $\beta \in L_\infty(\Gamma, \mathbb{R})$. Let λ_1 be the smallest eigenvalue of $-\Delta_{\beta,R}$. Let $u \in D(-\Delta_{\beta,R})$ be an eigenfunction with eigenvalue λ_1 and $u \geq 0$ (which exists by the Krein–Rutman theorem). Then $(Tr_R u)(z) > 0$ for almost all $z \in \Gamma$*

Proof Since $\left(e^{t\Delta_{\beta,R}}\right)_{t>0}$ is irreducible, one deduces from Theorem 3.2.(b) in Chapter C-III in [25] that $u(x) > 0$ for almost every $x \in \Omega_R$. Hence $\operatorname{Tr}_R u \geq 0$. Let $\Gamma_2 = \{z \in \Gamma : (\operatorname{Tr}_R u)(z) = 0\}$. Then $\beta_1 \in L_\infty(\Gamma)$ and

$$\mathfrak{a}_{\beta_1,R}(u, v) = \mathfrak{a}_{\beta,R}(u, v) = (\lambda_1 \, u, v)_{L_2(\Omega_R)}$$

for all $v \in V_R$. So $u \in D(\Delta_{\beta_1,R})$ and $-\Delta_{\beta_1,R} u = \lambda_1 \, u$. Since $u \geq 0$ and $u \neq 0$, it follows from Lemma 5.14 that λ_1 is also the smallest eigenvalue of $-\Delta_{\beta_1,R}$. Moreover, $0 \leq e^{t\Delta_{\beta,R}} v \leq e^{t\Delta_{\beta_1,R}} v$ for all $v \in L_2(\Omega_R)$ with $v \geq 0$. Hence $\Delta_{\beta,R} = \Delta_{\beta_1,R}$ by Proposition 5.13. Since $\operatorname{Tr}_R(V_R)$ is dense in $L_2(\Gamma)$ it follows that $\beta = \beta_1$ almost everywhere. This implies that Γ_2 is a null set. □

Proposition 5.16 *Let $R \in [R_0, \infty)$. Then the semigroup $\left(e^{-tA_R^D}\right)_{t>0}$ is irreducible.*

Proof Let $\Gamma_1 \subset \Gamma$ be measurable and assume that $e^{-tA_R^D} L_2(\Gamma_1) \subset L_2(\Gamma_1)$ for all $t > 0$. Here $L_2(\Gamma_1) = \{\varphi \in L_2(\Gamma) : \varphi(z) = 0$ for almost every $z \in \Gamma \setminus \Gamma_1\}$. Let μ be the smallest eigenvalue of A_R^D. By the Krein–Rutman theorem there exists a $\varphi \in D\left(A_R^D\right)$ such that $\varphi \geq 0$, $\varphi \neq 0$ and $A_R^D \varphi = \mu \, \varphi$. Hence by Lemma 4.1 there exists a $u \in V_R$ such that $\operatorname{Tr}_R(u) = \varphi$ and

$$\int_{\Omega_R} \nabla u \cdot \overline{\nabla v} = \int_\Gamma \mu \operatorname{Tr}_R u \, \overline{\operatorname{Tr}_R v} \tag{22}$$

for all $v \in V_R$.

We first show that $u \geq 0$. Note that $(\operatorname{Tr}_{\Omega_R}(u^-))|_\Gamma = \operatorname{Tr}_R(u^-) = \varphi^- = 0$ and $(\operatorname{Tr}_{\Omega_R}(u))|_{\partial B_R} = 0$. So $\operatorname{Tr}_{\Omega_R}(u^-) = 0$ and $u^- \in H_0^1(\Omega_R) \subset V_R$. Choose $v = u^-$ in Eq. 22. Then $\int_{\Omega_R} |\nabla(u^-)|^2 = 0$ and u^- is constant. Since $\operatorname{Tr}_R(u^-) = 0$ it follows that $u^- = 0$ and therefore $u \geq 0$.

Choose $\beta = \mu$ constant. Then Eq. 22 is equivalent with $\mathfrak{a}_{\beta,R}(u, v) = 0$ for all $v \in V_R$. Then $-\Delta_{\beta,R} u = 0$. Hence the smallest eigenvalue of $-\Delta_{\beta,R}$ is zero by Lemma 5.14. Therefore Proposition 5.15 implies that $\varphi(z) = (\mathrm{Tr}_R u)(z) > 0$ for almost every $z \in \Gamma$. Thus $\Gamma \setminus \Gamma_1$ is a null set. $\square$

As a corollary of Proposition 5.12 we finally obtain the irreducibility of S^D.

Theorem 5.17 *The semigroup S^D is irreducible.*

This theorem is remarkable, since Γ does not need to be connected. We only require that Ω is connected.

6 Example

In this section we present an example such that the Dirichlet-to-Neumann operator D_0 on Ω_0, the Dirichlet-to-Neumann operator A with Neumann boundary conditions at infinity and and the Dirichlet-to-Neumann operator A^D with Dirichlet boundary conditions at infinity are all different.

Choose $\Omega_0 = B(0, 1) \subset \mathbb{R}^d$, the unit ball. If $u : \Omega_0 \to \mathbb{R}$ is a function then the Kelvin transform $Ku : \Omega \to \mathbb{R}$ is defined by

$$(Ku)(x) = u\left(\frac{x}{|x|^2}\right)\frac{1}{|x|^{d-2}}.$$

Then Ku is a harmonic function on Ω whenever u is a harmonic function on $\Omega_0 \setminus \{0\}$, see [19] Theorem 2.8.1. We recall that for a function on Ω we use the outward normal derivative with respect to Ω, and for a function on Ω_0 we use the outward normal derivative with respect to Ω_0

Lemma 6.1 *Let $u \in H^1(\Omega_0)$ and suppose that u has a normal derivative in $L_2(\Gamma)$. Moreover, suppose that there exist $c > 0$ and $r < 1$ such that $|u(x)| \leq c\,|x|$ for all $x \in B(0, r)$. Then $Ku \in W^D(\Omega)$ and $\partial_\nu Ku = \partial_\nu u + (d-2)Tr\, u$.*

Proof Clearly $Ku \in H^1_{\mathrm{loc}}(\Omega)$. Let $k \in \{1, \ldots, d\}$. Then

$$\Big(\partial_k(Ku)\Big)(x) = -(d-2)\,u\left(\frac{x}{|x|^2}\right)\frac{x_k}{|x|^d} + \sum_{l=1}^{d}(\partial_l u)\Big(\frac{x}{|x|^2}\Big)\frac{1}{|x|^{d-2}}\left(\frac{\delta_{kl}}{|x|^2} - \frac{2x_k\,x_l}{|x|^4}\right)$$

for a.e. $x \in \Omega$. Hence

$$\left|\Big(\partial_k(Ku)\Big)(x)\right| = (d-2)\left|u\left(\frac{x}{|x|^2}\right)\right|\frac{|x_k|}{|x|^d} + \left|(\partial_k u)\left(\frac{x}{|x|^2}\right)\right|\frac{1}{|x|^d} + 2\left|(\nabla u)\left(\frac{x}{|x|^2}\right)\right|\frac{|x_k|}{|x|^{d+1}}$$

and

$$\int_\Omega |\nabla(Ku)|^2 \leq 3(d-2)^2\int_\Omega\left|u\left(\frac{x}{|x|^2}\right)\right|^2\frac{|x|^2}{|x|^{2d}}\,dx + 15\int_\Omega\left|(\nabla u)\left(\frac{x}{|x|^2}\right)\right|^2\frac{1}{|x|^{2d}}\,dx.$$

We estimates the terms. First

$$\begin{aligned}\int_\Omega \Big| u\Big(\frac{x}{|x|^2}\Big)\Big|^2 \frac{|x|^2}{|x|^{2d}} &= \int_1^\infty \int_{S_{d-1}} |u\Big(\tfrac{1}{r}\,\omega\Big)|^2\, r^{-(2d-2)}\, r^{d-1}\, d\omega\, dr \\ &= \int_1^\infty \int_{S_{d-1}} |u\Big(\tfrac{1}{r}\,\omega\Big)|^2\, r^{-(d-1)}\, d\omega\, dr \\ &= \int_0^1 \int_{S_{d-1}} |u(s\,\omega)|^2\, s^{d-1}\, d\omega\, s^{-2}\, ds \\ &= \int_{\Omega_0} \frac{|u(x)|^2}{|x|^2}\, dx < \infty\end{aligned}$$

since $|u(x)| \leq c\,|x|$ for all $x \in B(0, r)$. Similarly,

$$\int_\Omega \Big|(\nabla u)\Big(\frac{x}{|x|^2}\Big)\Big|^2 \frac{1}{|x|^{2d}} = \int_{\Omega_0} |\nabla u|^2 < \infty.$$

Hence $\int_\Omega |\nabla(Ku)|^2 < \infty$ and $Ku \in W(\Omega)$. Then also $Ku \in W^D(\Omega)$.

Next, if $x \in \Gamma$, then the outer normal of Ω is equal to $-x$. Therefore

$$\begin{aligned}(\partial_\nu Ku)(x) &= \sum_{k=1}^d -x_k\Big(\partial_k(Ku)\Big)(x) \\ &= (d-2)\,(\mathrm{Tr}\,u)(x) - \sum_{k=1}^d x_k(\partial_k u)(x) + 2\sum_{k=1}^d x_k(\partial_k u)(x) \\ &= (\partial_\nu u)(x) + (d-2)\,(\mathrm{Tr}\,u)(x)\end{aligned}$$

as required. □

We next determine the relation between D_0, A and A^D. Note that for D_0 we use the outward normal derivative with respect to Ω_0.

Theorem 6.2 $A^D = D_0 + (d-2)\,I$.

Proof First, define $w\colon \Omega \to \mathbb{R}$ by $w(x) = \frac{1}{|x|^{d-2}}$. Then $w \in W^D(\Omega)$, $\Delta w = 0$ weakly, $\mathrm{Tr}\,w = \mathbb{1}_\Gamma$ and $\partial_\nu w = (d-2)\,\mathbb{1}_\Gamma$. So

$$A^D \mathbb{1}_\Gamma = (d-2)\,\mathbb{1}_\Gamma. \tag{23}$$

Let $\varphi \in D(D_0)$ and write $\psi = D_0\varphi$. Then there exists a $u \in H^1(\Omega_0)$ such that $\mathrm{Tr}\,u = \varphi$, $\Delta u = 0$ weakly on Ω_0 and $\partial_\nu u = \psi$, where ∂_ν is the outward normal with respect to Ω_0. Set $u_0 = u - u(0)\,\mathbb{1}_{\Omega_0}$. Then $\mathrm{Tr}\,u_0 = \varphi - u(0)\,\mathbb{1}_\Gamma$. Moreover, $\partial_\nu u_0 = \partial_\nu u = \psi$. It follows from the Taylor theorem and Lemma 6.1 that $Ku_0 \in W^D(\Omega)$ and $\Delta K u_0 = 0$ on Ω. Moreover, $\partial_\nu K u_0 = \partial_\nu u_0 + (d-2)\,\mathrm{Tr}\,u_0 \in L_2(\Gamma)$. So $\varphi - u(0)\,\mathbb{1}_\Gamma = \mathrm{Tr}\,Ku_0 \in D\big(A^D\big)$ and $A^D(\varphi - u(0)\,\mathbb{1}_\Gamma) = \partial_\nu K u_0$. Since $\mathbb{1}_\Gamma \in D(A^D)$ by Proposition 5.7 it follows that $\varphi \in D(A^D)$. Then Eq. 23 gives

$$\begin{aligned}A^D\varphi - (d-2)\,u(0)\,\mathbb{1}_\Gamma &= A^D(\varphi - u(0)\,\mathbb{1}_\Gamma) = \partial_\nu K u_0 = \psi + (d-2)\,\mathrm{Tr}\,u_0 \\ &= \psi + (d-2)\,(\varphi - u(0)\,\mathbb{1}_\Gamma).\end{aligned}$$

So

$$A^D\varphi = \psi + (d-2)\,\varphi = D_0\varphi + (d-2)\,\varphi$$

and the theorem follows. □

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