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ORIGINAL ARTICLE

On the limit set of discrete subgroups of PU(2, 1)

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Abstract Let G be a discrete subgroup of PU(2, 1); G acts on $P^2_{\mathbb{C}}$ preserving the unit ball $\mathbf{H}^2_{\mathbb{C}}$, equipped with the Bergman metric. Let $L(G) \subset S^3 = \partial\mathbf{H}^2_{\mathbb{C}}$ be the limit set of G in the sense of Chen–Greenberg, and let $\Lambda(G) \subset P^2_{\mathbb{C}}$ be the limit set of the G-action on $P^2_{\mathbb{C}}$ in the sense of Kulkarni. We prove that $L(G) = \Lambda(G) \cap S^3$ and $\Lambda(G)$ is the union of all complex projective lines in $P^2_{\mathbb{C}}$ which are tangent to S^3 at a point in $L(G)$.

Keywords Limit set · Discrete subgroup · Complex hyperbolic plane · Complex hyperbolic geometry · Complex projective plane

Mathematics Subject Classification (2000) 51M10

0 Introduction

Let G be a discrete subgroup of $\mathrm{PU}(2, 1) \cong \mathrm{U}(2, 1)/\mathrm{U}(1)$. This group acts on the complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ by holomorphic isometries, where we are thinking of $\mathbf{H}^2_{\mathbb{C}}$ as being the ball

$$\{[z_1\colon z_2\colon z_3] \in P^2_{\mathbb{C}} \,\big|\, |z_1|^2 + |z_2|^2 < |z_3|^2\},$$

equipped with the Bergman metric. The limit set $L(G)$ of G in the sense of Chen and Greenberg [2] is the set of accumulation points of the G-orbits in $H^2_{\mathbb{C}}$; just as for Kleinian groups in real hyperbolic geometry, $L(G)$ is contained in the "sphere at infinity" $S^3 = \partial\mathbf{H}^2_{\mathbb{C}}$. On the other hand, Kulkarni [8] introduced a notion of "limit set" for discrete group actions in general, which has the important property of granting that the action on its complement is discontinuous. We denote by $\Lambda(G) \subset P^2_{\mathbb{C}}$ the limit set of G in the sense of Kulkarni, now regarding G as a group of automorphisms of $P^2_{\mathbb{C}}$. In this article we compare these two limit sets. We prove:

Theorem *The limit set $L(G)$ is the intersection*

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$$L(G) = \Lambda(G) \cap \partial \mathbf{H}^2_{\mathbb{C}},$$

and $\Lambda(G)$ *is the union of all complex projective lines* l_z *tangent to* $\partial \mathbf{H}^2_{\mathbb{C}}$ *at points in* $L(G)$*:*

$$\Lambda(G) = \bigcup_{z \in L(G)} l_z.$$

Furthermore, if G is non-elementary then the action of G is minimal on $L(G) \subset \partial \mathbf{H}^2_{\mathbb{C}}$*, i.e. all orbits are dense, while the orbit of each line* l_z *is dense in* $\Lambda(G)$ (*though the G-action on* $\Lambda(G)$ *is not minimal*).

We refer to Section 1 below for the definition of Kulkarni's limit set $\Lambda(G)$. This theorem is proved in Section 4, where we actually strengthen the first statement (see Theorem 4.5). In Sections 2 and 3 we give some elementary properties of the Chen and Greenberg [2] limit set $L(G)$.

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1 The limit set according to Kulkarni

This section is mostly a summary of the material in Kulkarni's article [8] that we need in the sequel. Kulkarni considers a very general setting of discrete group actions on spaces. Here we restrict the discussion to the case relevant for us, where the group G is a subgroup of PU(2, 1) acting on the complex projective plane $P^2_{\mathbb{C}}$. Kulkarni's limit set $\Lambda(G)$ provides a canonical choice of a closed G-invariant set in $P^2_{\mathbb{C}}$ such that the G-action on its complement $P^2_{\mathbb{C}} - \Lambda(G)$ is properly discontinuous. Let us motivate this with an example.

Let $g \in \mathrm{PU}(2, 1)$ be a loxodromic element (see [5]), let G be the cyclic group generated by g and let z_r, z_s, z_a its three fixed points in $P^2_{\mathbb{C}}$. Since these points are fixed by g, the isotropy group of G at each of them is all of G; obviously we can think of $L_0(G) = \{z_r, z_s, z_a\}$ as being the closure of the set of points in $P^2_{\mathbb{C}}$ with infinite isotropy group.

Now let $\overleftrightarrow{z_r z_s}$ be the projective line determined by these two points, and similarly for the others. These three points can be distinguished by the dynamics of g:

- For every $z \in P^2_{\mathbb{C}} - \overleftrightarrow{z_r z_s}$ we have $\lim_{n\to\infty} g^n(z) = z_a$; i.e., z_a is an *attractor* of g.
- For every $z \in P^2_{\mathbb{C}} - \overleftrightarrow{z_s z_a}$ we have $\lim_{n\to-\infty} g^n(z) = z_r$; i.e., z_r is *repellent*.
- The fixed point z_s is neither an attractor nor repellent but a *saddle point*.

We recall that if $\{A_\beta\}$ is a family of subsets of $P^2_{\mathbb{C}}$, indexed by β in some infinite set B then a point $p \in P^2_{\mathbb{C}}$ is an *accumulation* (or *cluster*) *point* of this family if every neighborhood of p meets an infinite number of sets $\{A_\beta\}$ in the family. Now, following [8], let $L_1(G)$ be the closure of the set of accumulation points of the orbits of all points in $P^2_{\mathbb{C}} - L_0(G)$. It follows from the above observations on the dynamics of g that in this example we have $L_1(G) = L_0(G)$. Hence this set is a natural candidate to be called the "limit set" of G. Notice however that the G-action on $P^2_{\mathbb{C}} - \{z_r, z_s, z_a\}$ is not properly discontinuous. In fact, if we let $W \subset P^2_{\mathbb{C}} - \{z_r, z_s, z_a\}$ be any 3-sphere in $P^2_{\mathbb{C}}$ which bounds a ball around the saddle point z_s, then $g^n(W) \cap W \neq \varnothing$ for every $n \in \mathbb{Z}$. So we must remove some other points in $P^2_{\mathbb{C}}$ besides those in $L_1(G) = L_0(G)$ in order to get a properly discontinuous action on its complement.

Let $L_2(G) \subset P^2_{\mathbb{C}}$ be the closure of the set of accumulation points of all G-orbits $\{g(K)\}_{g\in G}$ of compact subsets $K \subset P^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$. It is an exercise to see that in our example

$L_2(G)$ is the union of the lines $\overleftrightarrow{z_r z_s}$ and $\overleftrightarrow{z_s z_a}$, and that the action of G on $P^2_{\mathbb{C}} - (L_0(G) \cup L_1(G) \cup L_2(G))$ is properly discontinuous.

This motivates Kulkarni's definition of the limit set. Given a discrete subgroup of $\mathrm{PSL}(3, \mathbb{C})$, let $L_0(G)$ be the closure of the set of points in $P^2_{\mathbb{C}}$ with infinite isotropy group. Now let $L_1(G)$ be the closure of the set of accumulation points of all G-orbits in $P^2_{\mathbb{C}} - L_0(G)$. That is, for each point $z \in P^2_{\mathbb{C}} - L_0(G)$ we consider the set of cluster points of the orbit $\{g(z)\}_{g \in G}$, and $L_1(G)$ is the closure of the union of all of these sets. And finally let $L_2(G)$ be the closure of the set of cluster points of all the orbits $\{g(K)\}_{g \in G}$, where K runs over compact subsets of $P^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$. That is, for each compact set $K \subset P^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$ we consider the set of accumulation points of the orbit $\{g(K)\}_{g \in G}$, and $L_2(G)$ is the closure of the union of all of these sets.

Definition 1.1 ([8]) The limit set of G is the union

$$\Lambda(G) = L_0(G) \cup L_1(G) \cup L_2(G).$$

The set $\Omega(G) = P^2_{\mathbb{C}} - \Lambda(G)$ is the domain of discontinuity of G.

We remark that in classical hyperbolic geometry one considers the limit set to be only the closure of the accumulation points of the orbits, and the action on the complement turns out to be discontinuous. This is a consequence of deep properties of conformal geometry which are implicit within hyperbolic geometry.

Recall that a Kleinian group in hyperbolic geometry can be defined as a discrete subgroup of isometries of the real hyperbolic n-space whose action on the sphere at infinity has non-empty region of discontinuity (see [1]). Similarly one has:

Definition 1.2 ([8, 12, 13]) The group $G \subset \mathrm{PSL}(3, \mathbb{C})$ is complex Kleinian if $\Omega(G) \neq \emptyset$.

One has (2.4 below) that every discrete subgroup of $\mathrm{PSL}(3, \mathbb{C})$ which is contained in $\mathrm{PU}(2, 1)$ is automatically complex Kleinian, since it preserves the unit ball in $P^2_{\mathbb{C}}$ and this ball is contained in $\Omega(G)$.

Proposition 1.3 ([8]) *If G is equipped with the compact open topology. Then $L_0(G)$, $L_1(G)$, $L_2(G)$ and $\Omega(G)$ are G-invariant and G acts properly discontinuously on $\Omega(G)$. If G is complex Kleinian then G is discrete and countable.*

We remark that the set $\Omega(G)$ is not always the maximal open subset of $P^2_{\mathbb{C}}$ where the action is properly discontinuous. For instance in the previous example the action of G is properly discontinuous on $P^2_{\mathbb{C}} - (\overleftrightarrow{z_r z_s} \cup \{z_a\})$ and also on $P^2_{\mathbb{C}} - (\overleftrightarrow{z_s z_a} \cup \{z_r\})$. In Section 4, Corollary 4.13, we prove that this does cannot happen for "non-elementary" groups.

The sets $L_0(G)$, $L_1(G)$ and $L_2(G)$, whose union is $\Lambda(G)$, can be quite different amongst them, as shown below.

Example We use the terminology of W. Goldman in [5].

(i) Let G be a cyclic subgroup of $\mathrm{PU}(2, 1)$ generated by a parabolic element having a line of fixed points. That is, we consider an element $g \in \mathrm{PU}(2, 1)$ having a lifting to $\mathrm{SU}(2, 1)$ whose canonical Jordan form is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $L_0(G)$ is the line consisting of the fixed points of g. This line is tangent at one point p to $\partial \mathbf{H}^2_{\mathbb{C}}$, the boundary of the unit ball, and one has $\{p\} = L_1(G) = L_2(G)$. Hence in this case one has $L_0(G) \supsetneq L_1(G) = L_2(G)$.

(ii) Now let G be generated by an ellipto-parabolic element $g \in \mathrm{PU}(2,1)$ having a lifting to $\mathrm{SU}(2,1)$ with canonical Jordan form:

$$\begin{pmatrix} e^{2\pi i\theta} & 1 & 0 \\ 0 & e^{2\pi i\theta} & 0 \\ 0 & 0 & e^{-4\pi i\theta} \end{pmatrix}, \quad \theta \notin \mathbb{Q},$$

then $L_0(G)$ consists of the two fixed points p_1, p_2 of g in $P^2_{\mathbb{C}}$; one of these points is contained in $\partial\mathbf{H}^2_{\mathbb{C}}$ while the other is in $P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$. The set $L_1(G)$ is now the line $\overleftrightarrow{p_1 p_2}$. This line is tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at p_1 and $L_2(G) = \{p_1\}$. Thus in this example we have $L_0(G) \subsetneq L_1(G)$, $L_2(G) \subsetneq L_1(G)$ and $L_2(G) \subsetneq L_0(G)$.

Finally we remark that if G_1 and G_2 are discrete subgroups of $\mathrm{PU}(2,1)$ and G_1 is contained in G_2 then one has $\Lambda(G_1) \subset \Lambda(G_2)$, but to prove this claim we need some machinery developed below, so we will return to this point later, in Section 4.

2 The limit set according to Chen and Greenberg

In the previous section we considered discrete subgroups $G \subset \mathrm{PU}(2,1)$ and their action on the whole of $P^2_{\mathbb{C}}$, i.e., thinking of them as subgroups of $\mathrm{PSL}(3,\mathbb{C})$. These groups have the important property of preserving the set of points of $P^2_{\mathbb{C}}$ whose homogeneous coordinates satisfy $|z_1|^2 + |z_2|^2 < |z_3|^2$. This is a ball that can be equipped with the Bergman metric (see [5]) and serves as a model for the 2-dimensional complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$, with $\mathrm{PU}(2,1)$ being the corresponding group of automorphisms. It is thus natural to look at the action of $G \subset \mathrm{PU}(2,1)$ restricted to $\mathbf{H}^2_{\mathbb{C}}$ and try to use this to get information about its action on the whole $P^2_{\mathbb{C}}$. This is what we do in this and the following sections.

For discrete groups $G \subset \mathrm{PU}(2,1)$ there is in [2] the following definition of limit set inspired by the corresponding definition in real hyperbolic geometry:

Definition 2.1 The limit set $L(G)$ (of Chen and Greenberg) is the set of points in $\partial\mathbf{H}^2_{\mathbb{C}}$ that are accumulation points of the G-orbit of some point in $\mathbf{H}^2_{\mathbb{C}}$.

The following lemma is a slight generalization of Lemma 4.3.1 in [2]. It says that the set $L(G)$ does not depend on the choice of the point $x \in \mathbf{H}^2_{\mathbb{C}}$. This lemma is also used in our proof of 2.4 below.

Lemma 2.2 *Let p be a point in $\mathbf{H}^2_{\mathbb{C}}$ and $\{g_n\}_{n\in\mathbb{N}}$ a sequence of elements of* $\mathrm{PU}(2,1)$ *such that $g_n(p) \to q \in \partial\mathbf{H}^2_{\mathbb{C}}$ as $n \to \infty$. Then for all $p' \in \mathbf{H}^2_{\mathbb{C}}$ we have $\lim_{n\to\infty} g_n(p') = q$. Furthermore, if $K \subset \mathbf{H}^2_{\mathbb{C}}$ is a compact set, then the sequence of functions $g_n|_K$ converges uniformly to the constant function with value equal to q.*

Proof Let $B_h(x,\rho)$ denote the ball with center $x \in \mathbf{H}^2_{\mathbb{C}}$ and radius ρ, with respect to the Bergman metric in $\mathbf{H}^2_{\mathbb{C}}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathbf{H}^2_{\mathbb{C}}$ such that $x_n \to q \in \partial\mathbf{H}^2_{\mathbb{C}}$, and $\rho > 0$ is fixed, then the Euclidean diameter of $B_h(x_n,\rho)$ goes to 0 as $n \to \infty$. The proof follows from the reasoning above and Lemma 4.3.1 of [2]. □

The proof of the following lemma is contained in [2]. This says that the limit set of Chen–Greenberg is a minimal set in some sense (compare with 4.7).

Lemma 2.3 *Let X be a closed subset of $\partial\mathbf{H}^2_{\mathbb{C}}$ which contains more than one point and is invariant under G. Then $L(G) \subset X$.*

Notice that 2.3 implies that whenever $L(G)$ has more than two points, there is at most one point in $\partial\mathbf{H}^2_{\mathbb{C}}$ which is fixed by the whole group G.

By a well-known theorem of Siegel [14], a subgroup $G \subset \mathrm{PU}(2,1)$ is discrete if and only if G acts properly discontinuously on $\mathbf{H}^2_{\mathbb{C}}$. This implies that $L_0(G) \cap \mathbf{H}^2_{\mathbb{C}} = \varnothing = L_1(G) \cap \mathbf{H}^2_{\mathbb{C}}$, whenever G is discrete. In other words, the sets $L_0(G)$, $L_1(G)$ are contained in $P^2_{\mathbb{C}} - \mathbf{H}^2_{\mathbb{C}}$. We remark again that the domain of discontinuity as defined by Kulkarni is not necessarily the maximal domain of discontinuity. However we will prove that $L_2(G) \cap \mathbf{H}^2_{\mathbb{C}} = \varnothing$ whenever G is discrete, and so we have:

Proposition 2.4 *If $G \subset \mathrm{PU}(2,1)$ is a discrete subgroup then G is a complex Kleinian group. In fact, $\mathbf{H}^2_{\mathbb{C}} \subset \Omega(G)$.*

Proof Let x be a point in $\mathbf{H}^2_{\mathbb{C}}$ and suppose that there exists a compact set $K \subset P^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$, such that x is a cluster point of the family of compact sets $\{g(K)\}_{g\in G}$. Then, there exists a sequence g_n of distinct elements of G and a sequence k_n of elements in K such that $k_n \to k \in K$ and $g_n(k_n) \to x \in \mathbf{H}^2_{\mathbb{C}}$. Furthermore, we can suppose that $(k_n) \subset \mathbf{H}^2_{\mathbb{C}}$ and then $k \in \overline{\mathbf{H}^2_{\mathbb{C}}}$. If k is a point in $\mathbf{H}^2_{\mathbb{C}}$, then there exists a subsequence of $(g_n(k))$ which converges to $q \in \overline{\mathbf{H}^2_{\mathbb{C}}}$; this implies that $q \in \partial\mathbf{H}^2_{\mathbb{C}}$, because $L_1(G) \cap \mathbf{H}^2_{\mathbb{C}} = \varnothing$.

Now, using the Bergman metric, the length of the geodesic segment connecting k_n to k is equal to the length of the geodesic segment connecting $g_n(k_n)$ to $g_n(k)$. As $n \to \infty$ this length tends to the length of the geodesic segment connecting x to q, which is equal to infinity. This contradicts $k_n \to k$, so $k \in \partial\mathbf{H}^2_{\mathbb{C}}$.

We can suppose, taking a subsequence if needed, that g_n^{-1} converges uniformly in compact subsets of $\mathbf{H}^2_{\mathbb{C}}$ to a constant function with value $p \in \partial\mathbf{H}^2_{\mathbb{C}}$ (Lemma 2.2). In particular, taking the compact subset of $\mathbf{H}^2_{\mathbb{C}}$ given by $\{g_n(k_n)\} \cup \{x\}$, we have that $k_n = g_n^{-1}(g_n(k_n)) \to p$, which means that $k = p \in L_1(G)$, contradiction. Hence $L_2(G) \cap \mathbf{H}^2_{\mathbb{C}} = \varnothing$. □

3 Some useful properties of $L(G)$

In this section we give some properties of the Chen–Greenberg limit set $L(G)$ which are analogous to well-known properties of the limit set of Kleinian groups in real hyperbolic geometry. These are used later to prove Theorem 4.5.

Let $z_1, z_2 \in \mathbf{H}^2_{\mathbb{C}}$ be two distinct points. The *bisector* $\mathfrak{E}\{z_1, z_2\}$ is defined as the set $\{z \in \mathbf{H}^2_{\mathbb{C}} | \rho(z_1, z) = \rho(z_2, z)\}$, where ρ is the Bergman metric. Let $\Sigma \subset \mathbf{H}^2_{\mathbb{C}}$ be the complex geodesic spanned by z_1 and z_2; we call Σ the *complex spine of* $\mathfrak{E}\{z_1, z_2\}$. The *spine* of $\mathfrak{E}\{z_1, z_2\}$ equals $\sigma = \mathfrak{E}\{z_1, z_2\} \cap \Sigma = \{z \in \Sigma | \rho(z_1, z) = \rho(z_2, z)\}$; that is, the orthogonal bisector of the geodesic segment joining z_1 and z_2 in Σ. (see [5])

The slice decomposition of the bisector $\mathfrak{E}\{z_1, z_2\}$ says that

$$\mathfrak{E}\{z_1, z_2\} = \Pi_{\Sigma}^{-1}(\sigma),$$

where $\Pi_\Sigma : \mathbf{H}^2_{\mathbb{C}} \to \Sigma$ is the orthogonal projection onto Σ (see [4, 10]). This implies that the half-space $\{z \in \mathbf{H}^2_{\mathbb{C}} | \rho(z, z_1) \geq \rho(z, z_2)\}$ is equal to the set $\Pi_\Sigma^{-1}\{z \in \Sigma \mid \rho(z, z_1) \geq \rho(z, z_2)\}$; we use this equality to prove:

Lemma 3.1 *Let (x_n) be a sequence of elements of $\mathbf{H}^2_{\mathbb{C}}$ such that $x_n \to q \in \partial\mathbf{H}^2_{\mathbb{C}}$. In order to simplify notation, we identify $\mathbf{H}^2_{\mathbb{C}}$ with the unit ball of $\mathbb{C}^2$, and $\mathbf{0}$ with the origin of $\mathbb{C}^2$.*

(i) *If S_n denotes the closed half-space $\{z \in \mathbf{H}^2_{\mathbb{C}} | \rho(z, \mathbf{0}) \geq \rho(z, x_n)\}$, and $\partial S_n \subset \partial\mathbf{H}^2_{\mathbb{C}}$ denotes its ideal boundary, then the Euclidean diameter of $S_n \cup \partial S_n$ goes to 0 as $n \to \infty$.*

(ii) *If (z_n) is a sequence such that $z_n \in S_n \cup \partial S_n$ for all $n \in \mathbb{N}$, then $z_n \to q$.*

Proof We prove (i) first. We can assume that $x_n = (r_n, 0), 0 < r_n < 1$ for all n, because the elements of $U(2)$ are isometries for the Bergman metric. It follows that the complex spine, Σ_n, of $\mathfrak{E}\{\mathbf{0}, x_n\}$ is equal to the disk $\mathbf{H}^1_{\mathbb{C}} \times \{0\}$ for each n, and the orthogonal projection, $\Pi_{\Sigma_n}\colon \mathbf{H}^2_{\mathbb{C}} \to \Sigma_n$ is given by $\Pi_{\Sigma_n}(z_1, z_2) = (z_1, 0)$. By the comments above concerning the slice decomposition, we have that $S_n = \Pi^{-1}_{\Sigma_n}(\{z \in \mathbf{H}^1_{\mathbb{C}} \times \{0\} \mid \rho(z, \mathbf{0}) \geq \rho(z, x_n)\})$. We notice that the set $\{z \in \mathbf{H}^1_{\mathbb{C}} \times \{0\} \mid \rho(z, \mathbf{0}) \geq \rho(z, x_n)\}$ is contained in the set $\{(z_1, 0) \in \mathbf{H}^1_{\mathbb{C}} \times \{0\} \mid Re(z_1) \geq m_n\}$ for each n, where m_n is the intersection point of the real axis in $\mathbf{H}^1_{\mathbb{C}} \times \{0\}$ and the spine, σ_n, of $\mathfrak{E}\{\mathbf{0}, x_n\}$. Then $S_n \subset \Pi^{-1}_{\Sigma}(\{(z_1, 0) \in \mathbf{H}^1_{\mathbb{C}} \times \{0\} \mid Re(z_1) \geq m_n\}) = \{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} \mid Re(z_1) \geq m_n\}$, therefore

$$S_n \cup \partial S_n \subset \{(z_1, z_2) \in \overline{\mathbf{H}^2_{\mathbb{C}}} \mid Re(z_1) \geq m_n\},$$

and the Euclidean diameter of the set $\{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} \mid Re(z_1) \geq m_n\}$ goes to zero when $n \to \infty$, because $m_n \to 1$ when $n \to \infty$. This proves (i). The proof of (ii) follows easily from (i). □

Proposition 3.2 *If (g_n) is a sequence of distinct elements of a discrete subgroup G of* PU(2, 1)*, then there exists a subsequence, still denoted (g_n), and elements $x, y \in L(G)$, such that $g_n(z) \to x$ uniformly on compact subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{y\}$.*

Proof Discarding a finite number of terms if necessary, we assume that $g_n(\mathbf{0}) \neq \mathbf{0}$ for all n. There exists a subsequence, still denoted g_n, such that $g_n(\mathbf{0}) \to x \in L(G)$ and $g_n^{-1}(\mathbf{0}) \to y \in L(G)$. Let

$$S_{g_n} = \{z \in \mathbf{H}^2_{\mathbb{C}} \mid \rho(z, \mathbf{0}) \geq \rho(z, g_n^{-1}(\mathbf{0}))\}$$

and

$$S_{g_n^{-1}} = \{z \in \mathbf{H}^2_{\mathbb{C}} \mid \rho(z, \mathbf{0}) \geq \rho(z, g_n(\mathbf{0}))\}.$$

The result now follows from statement (ii) in 3.1, together with the fact that the Euclidean diameters of the sets $S_{g_n} \cup \partial S_{g_n}$, $S_{g_n^{-1}} \cup \partial S_{g_n^{-1}}$ tend to zero as $n \to \infty$, and the maps g_n carry $\overline{\mathbf{H}^2_{\mathbb{C}}} - (S_{g_n} \cup \partial S_{g_n})$ into $S_{g_n^{-1}} \cup \partial S_{g_n^{-1}}$. □

Our proof of 3.2 is inspired by the proof in [9] of the analogous result in real hyperbolic geometry. The sets $S_{g_n} \cup \partial S_{g_n}$ play a role similar to that of the isometric circle. In fact, in [5] the boundary of ∂S_{g_n} in $\partial \mathbf{H}^2_{\mathbb{C}}$ is called the isometric sphere of g_n with respect to $\mathbf{0} \in \mathbf{H}^2_{\mathbb{C}}$. Some mild changes in these arguments yield the following result, which is essentially contained in [7]. See also [3].

Corollary 3.3 *Let G be as above. If (g_n) is a sequence of distinct elements of G, then there exists a subsequence (g_n) and points $x, y \in L(G)$ such that $g_n(z) \to x$ uniformly on compact subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{y\}$, and $g_n^{-1}(z) \to y$ uniformly on compact subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{x\}$.*

When $L(G)$ has at most two points, the group G is called *elementary*, otherwise G is called *non-elementary*. It is proved in [7] that $L(G)$ is a perfect set whenever G is non-elementary, an alternate proof can be given using Proposition 3.2 and imitating the proof for classical Kleinian groups. See, for example, Maskit's book [9].

Lemma 3.4 *If G is non-elementary, then there exist $x, y \in L(G)$, $x \neq y$, and a sequence, (g_n), of distinct elements of G such that $g_n(z) \to x$ uniformly on compact subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{y\}$, and $g_n^{-1}(z) \to y$ uniformly on compact subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{x\}$.*

Proof By Lemma 2.3, the group G fixes at most one point. Let x be a point of $L(G)$ not fixed by every element of G. By 3.3, there is a sequence g_n of distinct elements of G and a point $x' \in L(G)$, such that $g_n(z) \to x$ uniformly on compact subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{x'\}$ and $g_n^{-1}(z) \to x'$ uniformly on compact subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{x\}$. If $x \neq x'$ we are done. If $x = x'$, then there exists $g \in G$ such that $y = g(x') \neq x$. The elements $x, y \in L(G)$ and the sequence $(g_n g^{-1})$ satisfy the conditions of the lemma. □

Proposition 3.5 ([7]) *Let G be a discrete subgroup of* PU(2, 1) *such that $L(G)$ contains at least two distinct points, then there exists a loxodromic element in G.*

Proof First assume $L(G)$ has precisely two points, then we may consider G as a classical Fuchsian group, because the complex geodesic determined by the points in $L(G)$ is G-invariant. Moreover, $L(G)$ agrees with the classical limit set, so G has a loxodromic element.

If G is non-elementary then we apply Lemma 3.4 and use its notation. Choose two disjoint open 3-balls $D_x, D_y \subset \partial\mathbf{H}^2_{\mathbb{C}}$, such that $x \in D_x$, $y \in D_y$. There exists n such that $g_n(\partial\mathbf{H}^2_{\mathbb{C}} - D_y) \subset D_x$. The Brouwer fixed point theorem implies that g_n has a fixed point in D_x and a similar reasoning implies that g_n^{-1} has a fixed point in D_y. Now it is easy to see that g_n is a loxodromic element. □

Proposition 3.6 *Let G be a discrete subgroup of* PU(2, 1) *such that every element has finite order, then G is finite.*

Proof Assume G is infinite, then $|L(G)| \geq 1$. By Proposition 3.5, $L(G)$ contains only one point. Suppose $L(G) = \{x\}$, then the line l_x tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at x is invariant under G. Furthermore G acts as a finite classical Kleinian group on l_x, because otherwise G would contain an element of infinite order. Let $g \in G - \{1\}$ be an element acting as the identity on l_x, then it has finite order, and fixes $x \in \partial\mathbf{H}^2_{\mathbb{C}}$. Thus g is a complex reflection whose fixed complex geodesic contains x in its boundary. But there is no element in PU(2, 1) with these properties, and therefore G must be finite. □

4 Comparing the sets $\Lambda(G)$ and $L(G)$

In this section we state and prove the main result of this paper, Theorem 4.5. We need first several lemmas. The first of them is useful to relate the limit sets $\Lambda(G)$ and $L(G)$.

Lemma 4.1 *Let $(w_n) \subset P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$ be a sequence such that $w_n \to w$. Denote by Σ_n the complex geodesic which is polar to w_n. Assume (v_n) is a sequence such that $v_n \in \Sigma_n$ for every $n \in \mathbb{N}$, and $v_n \to v$.*

(a) *If $w \in \partial\mathbf{H}^2_{\mathbb{C}}$ then $w = v$.*
(b) *If $w \in P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$ then $v \in \Sigma \cup \partial\Sigma$, where Σ denotes the complex geodesic which is polar to w. In particular, if $v \in \partial\mathbf{H}^2_{\mathbb{C}}$ then $w \in l_v$, where l_v is the only complex line which is tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at v.*

Proof Let $\langle\cdot,\cdot\rangle$ denote the hermitian product in $\mathbb{C}^3$ given by

$$\langle(x_1,x_2,x_3),(y_1,y_2,y_3)\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 - x_3\bar{y}_3.$$

Let $\tilde{w}_n$ (respectively $\tilde{v}_n$) be a lift to $S^5 \subset \mathbb{C}^3$ of w_n (respectively v_n). Take subsequences such that $\tilde{v}_n \to \tilde{v}$ and $\tilde{w}_n \to \tilde{w}$. The elements $\tilde{v}$ and $\tilde{w}$ are lifts of v and w, respectively, to S^5; and the equations $\langle\tilde{v}_n,\tilde{w}_n\rangle = 0,\ n \in \mathbb{N}$, imply that $\langle\tilde{v},\tilde{w}\rangle = 0$.

In order to prove (a) we assume $w \in \partial\mathbf{H}^2_{\mathbb{C}}$, then the complex line in $P^2_{\mathbb{C}}$, induced by the two dimensional complex space $\{\tilde{w}\}^{\perp}$, is the only complex line which is tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at w. Given that $\tilde{v} \in \{\tilde{w}\}^{\perp}$ and $v \in \overline{\mathbf{H}^2_{\mathbb{C}}}$, we have that $v = w$.

For (b) we assume $w \in P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$, then the equation $\langle\tilde{v},\tilde{w}\rangle = 0$ implies that $v \in \Sigma \cup \partial\Sigma$, where Σ denotes the complex geodesic which is polar to w. In particular, when $v \in \partial\Sigma$, the equation $\langle\tilde{v},\tilde{w}\rangle = 0$ implies $w \in l_v$. □

Lemma 4.2 *If G is a discrete subgroup of* PU(2, 1) *then* $L(G) = L_0(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$.

Proof If $L_0(G) \cap \partial\mathbf{H}^2_{\mathbb{C}} = \varnothing$ then Proposition 3.6 implies that G is finite, so $L(G) = \varnothing$.

If $L_0(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$ consists of a single point, then this point is fixed by the whole group G, and Proposition 3.2 implies that it belongs to $L(G)$. If $L(G)$ has more than one point then there exists a loxodromic element in G, whose fixed points are in $L_0(G)$, a contradiction of our assumption. Therefore $L(G) = L_0(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$.

Finally, we assume $|L_0(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}| \geq 2$. Since $L_0(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$ is a G-invariant closed set, Lemma 2.3 implies that $L(G) \subset L_0(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$. The converse inclusion is easily obtained from Proposition 3.2. □

Lemma 4.3 *If G is a discrete subgroup of* PU(2, 1) *then* $L(G) = L_1(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$.

Proof If G is finite then both sets in the equality are empty. Assume G is infinite and notice that the definition of $L(G)$ and the fact that $L_0(G) \cap \mathbf{H}^2_{\mathbb{C}} = \varnothing$ show that $L(G) \subset L_1(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$.

Let us show the converse, i.e. that $L_1(G) \cap \partial\mathbf{H}^2_{\mathbb{C}} \subset L(G)$. Assume that $z \in \partial\mathbf{H}^2_{\mathbb{C}}$, suppose that there exist a sequence g_n of distinct elements of G and one point $\zeta \in P^2_{\mathbb{C}} - L_0(G)$, such that $g_n(\zeta) \to z$. We consider the following three cases:

(1) We suppose $\zeta \in \mathbf{H}^2_{\mathbb{C}} - L_0(G)$, then by definition $z \in L(G)$.
(2) If $\zeta \in \partial\mathbf{H}^2_{\mathbb{C}} - L_0(G)$, then Lemma 4.2 implies that $\zeta \in \partial\mathbf{H}^2_{\mathbb{C}} - L(G)$. By Proposition 3.2, we have that $z \in L(G)$.
(3) Finally we assume $\zeta \in P^2_{\mathbb{C}} - (\overline{\mathbf{H}^2_{\mathbb{C}}} \cup L_0(G))$. Let Σ denote the complex geodesic which is polar to ζ, then the complex geodesic $g_n(\Sigma)$ is polar to $g_n(\zeta)$. We take a point $x \in \Sigma$, and a subsequence of g_n, still denoted g_n, such that $g_n(x) \to q \in L(G)$. Lemma 4.1 a) implies that $z = q \in L(G)$. □

Lemma 4.4 *If G is a discrete subgroup of* PU(2, 1) *then* $L(G) = L_2(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$.

Proof We assume G is infinite, because when G is finite the equality is trivial. The definition of $L(G)$ and the fact that $(L_0(G) \cup L_1(G)) \cap \mathbf{H}^2_{\mathbb{C}} = \varnothing$ show that $L(G) \subset L_2(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$.

Let us show the converse, i.e. that $L_2(G) \cap \partial\mathbf{H}^2_{\mathbb{C}} \subset L(G)$. Let $K \subset P^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$ be a compact set. If $z \in \partial\mathbf{H}^2_{\mathbb{C}}$ is a cluster point of the orbit of K, then there exists a sequence $(k_n) \subset K$, such that $k_n \to k \in K$, and a sequence of distinct elements $(g_n) \subset G$ such that $g_n(k_n) \to z$. We consider 4 cases:

(1) If $k \in \mathbf{H}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$ then we can assume that $k_n \in \mathbf{H}^2_{\mathbb{C}}$ for all n. By Proposition 3.2, there exists a subsequence, g_n, and elements $x, y \in L(G)$ such that $g_n(\cdot) \to x$ uniformly on compact subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}} - \{y\}$. Then $z = x \in L(G)$.

(2) We assume $k \in \partial\mathbf{H}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$ and there exists a subsequence of k_n, still denoted k_n, such that $k_n \in \overline{\mathbf{H}^2_{\mathbb{C}}}$ for each n. By Lemmas 4.2 and 4.3 we have that $k \in \partial\mathbf{H}^2_{\mathbb{C}} - L(G)$, and Proposition 3.2 implies that $z \in L(G)$.

(3) We assume $k \in \partial\mathbf{H}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G)) = \partial\mathbf{H}^2_{\mathbb{C}} - L(G)$ and there exists a subsequence of k_n, still denoted k_n, such that $k_n \in P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$ for all n. We denote by Σ_n the complex geodesic which is polar to k_n, and let x_n be an element of Σ_n. We can assume, taking a subsequence, if needed, that $x_n \to x \in \overline{\mathbf{H}^2_{\mathbb{C}}}$. Lemma 4.1(a) implies that $x = k$. By Proposition 3.2, we can suppose, taking subsequences if needed, that $g_n(x_n) \to q \in L(G)$.
The complex geodesic $g_n(\Sigma_n)$ is polar to $g_n(k_n)$, and we know that $g_n(k_n) \to z$, thus Lemma 4.1(a) implies that $z = q \in L(G)$.

(4) Finally, if $k \in P^2_{\mathbb{C}} - (\overline{\mathbf{H}^2_{\mathbb{C}}} \cup L_0(G) \cup L_1(G))$ then we can assume, discarding a finite number of terms, that $k_n \in P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$ for all n. Let Σ_n (respectively Σ) be the complex geodesic which is polar to k_n (respectively k), and x_n a point in Σ_n. We assume, taking a subsequence, if needed, that $x_n \to x \in \Sigma \cup \partial\Sigma$. Also, we can choose the sequence in such way that $x \in \Sigma \subset \mathbf{H}^2_{\mathbb{C}}$. Once more, we apply Proposition 3.2 to find subsequences of g_n and x_n such that $g_n(x_n) \to q \in L(G)$. Finally, the complex geodesic $g_n(\Sigma_n)$ is polar to $g_n(k_n)$, and we know that $g_n(k_n) \to z$, then Lemma 4.1(a) implies that $z = q \in L(G)$. □

It follows immediately from Lemmas 4.2, 4.3 and 4.4 that $L(G) = \Lambda(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}$. We have thus proved the first statement of our main theorem:

Theorem 4.5 *Let G be a discrete subgroup of* PU(2, 1) *then*

(a) *The limit set $L(G)$ satisfies the following*

$$L(G) = L_0(G) \cap \partial\mathbf{H}^2_{\mathbb{C}} = L_1(G) \cap \partial\mathbf{H}^2_{\mathbb{C}} = L_2(G) \cap \partial\mathbf{H}^2_{\mathbb{C}} = \Lambda(G) \cap \partial\mathbf{H}^2_{\mathbb{C}}.$$

(b) *The limit set $\Lambda(G)$ is the union of all complex projective lines l_z tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at points in $L(G)$:*

$$\Lambda(G) = \bigcup_{z \in L(G)} l_z.$$

Furthermore, if G is non-elementary then the orbit of each line l_z, $z \in L(G)$, is dense in $\Lambda(G)$.

We need a few results and lemmas in order to prove statement (b). The first of them is proved in [6] and we state it without proof.

Proposition 4.6 ([6]) *Let $g \in$ PU(2, 1) be a loxodromic element. Let f be an element in* PU(2, 1) *which has one and only one fixed point in common with g, and assume that such a point lies in $\partial\mathbf{H}^2_{\mathbb{C}}$. Then the group generated by f and g is not discrete.*

Corollary 4.7 *If G is a non-elementary group then the orbit of any point in $L(G)$ is dense in $L(G)$.*

Proof By Lemma 2.3, the group G fixes at most one point. Assume $x \in L(G)$ is a point fixed by every element of G. There exists a loxodromic element $g \in G$ (by Proposition 3.5), this element fixes x and another point $y \in \partial \mathbf{H}^2_{\mathbb{C}} - \{x\}$, but there exists $f \in G$ such that $x \neq f(y) \neq y$, then g and fgf^{-1} are loxodromic elements with precisely one fixed point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$. Hence, Proposition 4.6 implies that G is not discrete, which is a contradiction. Therefore, there is no point in $L(G)$ fixed by G.

Now let z be a point in $L(G)$, then the closure of the orbit of z is contained in $L(G)$ since this set is closed and G-invariant. Conversely, Lemma 2.3 implies $L(G)$ is contained in the closure of the orbit of z. □

The following lemma is reminiscent of the λ-lemma in [11]; we use it to prove that the limit set $\Lambda(G)$ contains a line whenever G contains a loxodromic element. As before, we denote by $\overleftrightarrow{xy}$ the complex projective line determined by the points $x, y \in P^2_{\mathbb{C}}$.

Lemma 4.8 *Let $g \in \mathrm{PU}(2,1)$ be a loxodromic element with fixed points $z_r, z_s, z_a \in P^2_{\mathbb{C}}$, where z_r is a repelling fixed point for g, z_s is a saddle, and z_a is an attractor. Let $S \subset P^2_{\mathbb{C}}$ be any three sphere that does not contain neither z_r nor z_s and meets transversally the complex projective line $\overleftrightarrow{z_r z_s}$ in a circle. Then the set of cluster points of the family of compact sets $\{g^n(S)\}_{n\in\mathbb{N}}$ is equal to the whole complex projective line $\overleftrightarrow{z_s z_a}$.*

Proof We can assume, that up to conjugation in $\mathrm{SL}(3,\mathbb{C})$, g is induced by a matrix in $\mathrm{SL}(3,\mathbb{C})$ of the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $0 < |\lambda_1| < |\lambda_2| = 1 < |\lambda_3|$, so we have $z_r = [1:0:0]$, $z_s = [0:1:0]$, and $z_a = [0:0:1]$. The set of cluster points of the family $\{g^n(S)\}_{n\in\mathbb{N}}$ is contained in $\overleftrightarrow{z_s z_a}$ because the sequence of functions $\{d(g^n(\cdot), \overleftrightarrow{z_s z_a})\}_{n\in\mathbb{N}}$ converges to zero uniformly on S, where $d(\cdot,\cdot)$ denotes the Fubini–Study metric in $P^2_{\mathbb{C}}$.

Now we want to prove that every point in $\overleftrightarrow{z_s z_a}$ is a cluster point of the family $\{g^n(S)\}_{n\in\mathbb{N}}$. For this, we take $z \in \overleftrightarrow{z_s z_a} - \{z_a\}$ and $\epsilon > 0$. Since the sequence of functions $\{d(g^n(\cdot), \overleftrightarrow{z_s z_a})\}_{n\in\mathbb{N}}$ converges to zero uniformly on S, there exists $N_1 > 0$ such that $d(g^n(s), \overleftrightarrow{z_s z_a}) < \epsilon$ for every $n > N_1$ and $s \in S$.

Let $\Pi : P^2_{\mathbb{C}} - \{z_r\} \to \overleftrightarrow{z_s z_a}$ be the projection given by $[x_1 : x_2 : x_3] \mapsto [0 : x_2 : x_3]$. Given that $\Pi(S)$ is a neighborhood of z_s in $\overleftrightarrow{z_s z_a}$, we consider a disk D in $\overleftrightarrow{z_s z_a}$ with center z_s such that $D \subset \Pi(S)$. The transformation g acts on $\overleftrightarrow{z_s z_a}$ as a classic loxodromic transformation with z_s as a repelling point, therefore there exists $N_2 \in \mathbb{N}$ such that $z \in g^{N_2}(D)$.

Now, let $n > \max(N_1, N_2)$ then $g^{N_2}(D) \subset g^n(D) \subset g^n(\Pi(S))$, thus $z = g^n(\Pi(s))$ for some $s \in S$, and $d(g^n(s), z) = d(g^n(s), g^n(\Pi(s))) = d(g^n(s), \Pi(g^n(s))) = d(g^n(s), \overleftrightarrow{z_s z_a}) < \epsilon$. Then z is a cluster point of the family $\{g^n(S)\}_{n\in\mathbb{N}}$, and we have proved that the set $\overleftrightarrow{z_s z_a} - \{z_a\}$ is contained in the (closed) set of cluster points of $\{g^n(S)\}_{n\in\mathbb{N}}$, therefore the whole complex projective line $\overleftrightarrow{z_s z_a}$ is contained in this set of cluster points. □

Lemma 4.9 *If G is a discrete subgroup of* $\mathrm{PU}(2,1)$ *and $g \in G$ is a loxodromic element with fixed points $z_r, z_s, z_a \in P^2_{\mathbb{C}}$, where z_r is a repelling fixed point, z_s is a saddle fixed point, and z_a is an attracting fixed point, then*

$$\overleftrightarrow{z_r z_s} \subset \Lambda(G) \quad or \quad \overleftrightarrow{z_s z_a} \subset \Lambda(G).$$

Proof Clearly $\{z_r, z_s, z_a\} \subset L_0(G)$. If $\overleftrightarrow{z_r z_s} \subset (L_0(G) \cup L_1(G))$ then we are done. If $\overleftrightarrow{z_r z_s} \not\subseteq (L_0(G) \cup L_1(G))$ then $\overleftrightarrow{z_r z_s} - (L_0(G) \cup L_1(G)) \neq \varnothing$ is an open set in $\overleftrightarrow{z_r z_s}$. Thus, there exists a 3-sphere satisfying the hypothesis of Lemma 4.8, therefore $\overleftrightarrow{z_s z_a} \subset \Lambda(G)$. □

We remark, using the notation of Lemma 4.9, that $\overleftrightarrow{z_r z_s}$ (respectively $\overleftrightarrow{z_s z_a}$) is the only complex projective line tangent to $\partial \mathbf{H}^2_{\mathbb{C}}$ at the point z_r (respectively z_a). Therefore, when $G \subset \mathrm{PU}(2, 1)$ is a discrete group containing a loxodromic element, the limit set $\Lambda(G)$ contains at least one complex projective line tangent to $\partial \mathbf{H}^2_{\mathbb{C}}$ at a point in $L(G)$. In the sequel we denote by l_z the only complex projective line tangent to $\partial \mathbf{H}^2_{\mathbb{C}}$ at the point $z \in \partial \mathbf{H}^2_{\mathbb{C}}$.

Lemma 4.10 *If G is a discrete subgroup of* $\mathrm{PU}(2, 1)$ *then* $L_0(G) \subset \bigcup_{z \in L(G)} l_z$.

Proof Let ζ be a point in $P^2_{\mathbb{C}}$ whose isotropy subgroup has infinite order, and consider the two possible cases: either $\zeta \in \overline{\mathbf{H}^2_{\mathbb{C}}}$ or not. In the first case Proposition 2.4 and Theorem 4.5(a) imply that $\zeta \in L_0(G) \cap \partial \mathbf{H}^2_{\mathbb{C}} = L(G) \subset \bigcup_{z \in L(G)} l_z$.

If $\zeta \in P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$, let Σ denote the complex geodesic polar to ζ. Observe that $\partial\Sigma$ is invariant under the isotropy subgroup of ζ, so Lemma 2.3 implies that the (nonempty) Chen–Greenberg's limit set of this isotropy subgroup is contained in $\partial\Sigma$. Hence there exists a point $z \in L(G)$ such that $z \in \partial\Sigma$, then $\zeta \in l_z$ for some $z \in L(G)$.

Lemma 4.11 *If G is a discrete subgroup of* $\mathrm{PU}(2, 1)$ *then* $L_1(G) \subset \bigcup_{z \in L(G)} l_z$.

Proof Proposition 2.4 implies $L_0(G) \subset P^2_{\mathbb{C}} - \mathbf{H}^2_{\mathbb{C}}$. If $L_0(G) = P^2_{\mathbb{C}} - \mathbf{H}^2_{\mathbb{C}}$ then $L_1(G) = L(G) \subset \bigcup_{z \in L(G)} l_z$, and we have finished. Therefore, we assume that $L_0(G) \subsetneq P^2_{\mathbb{C}} - \mathbf{H}^2_{\mathbb{C}}$. We take $x \in P^2_{\mathbb{C}} - L_0(G)$ and there are two possible cases, according to whether $x \in \overline{\mathbf{H}^2_{\mathbb{C}}} - L_0(G) = \overline{\mathbf{H}^2_{\mathbb{C}}} - L(G)$ or not. In the first case Proposition 3.2 implies that the cluster points of the orbit of x are contained in $L(G) \subset \bigcup_{z \in L(G)} l_z$ proving the lemma in this case.

If $x \in P^2_{\mathbb{C}} - (\overline{\mathbf{H}^2_{\mathbb{C}}} \cup L_0(G))$, and g_n is a sequence of different elements of G such that $g_n(x) \to \xi \in L_1(G)$, then one has two possibilities: either $\xi \in \mathbf{H}^2_{\mathbb{C}}$ or not. In the first case Proposition 2.4 implies $\xi \in \partial \mathbf{H}^2_{\mathbb{C}}$, and Theorem 4.5(a) implies that $\xi \in L(G) \subset \bigcup_{z \in L(G)} l_z$ as claimed.

If $\xi \in P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$. Let Σ be the complex geodesic which is polar to x, then $g_n(\Sigma)$ is the complex geodesic polar to $g_n(x)$. Let Σ' be the complex geodesic polar to ξ. Using Proposition 3.2, take a subsequence of g_n, still denoted g_n, such that $g_n(z) \to p \in L(G)$ for all $z \in \mathbf{H}^2_{\mathbb{C}}$. In particular, if $\sigma \in \Sigma$ then $g_n(\sigma) \to p$, and $g_n(\sigma) \in g_n(\Sigma)$. Thus Lemma 4.1 (b) implies $p \in \partial\Sigma'$ and therefore $\xi \in l_p$. □

Lemma 4.12 *If G is a discrete subgroup of* $\mathrm{PU}(2, 1)$ *then* $L_2(G) \subset \bigcup_{z \in L(G)} l_z$.

Proof By Proposition 2.4, we have $L_0(G) \cup L_1(G) \subset P^2_{\mathbb{C}} - \mathbf{H}^2_{\mathbb{C}}$. If $L_0(G) \cup L_1(G)$ is equal to $P^2_{\mathbb{C}} - \mathbf{H}^2_{\mathbb{C}}$, then $L_2(G) = L(G) \subset \bigcup_{z \in L(G)} l_z$ and we have finished.

Assume $L_0(G) \cup L_1(G) \subsetneq P^2_{\mathbb{C}} - \mathbf{H}^2_{\mathbb{C}}$. Let z be a cluster point of the orbit of the compact set $K \subset P^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$, then there exists a sequence $(k_n) \subset K$, such that $k_n \to k \in K$, and a sequence of distinct elements $(g_n) \subset G$ such that $g_n(k_n) \to z$.

If $z \in \overline{\mathbf{H}^2_{\mathbb{C}}}$ then Proposition 2.4 and Theorem 4.5(a) imply that $z \in L_2(G) \cap \partial \mathbf{H}^2_{\mathbb{C}} = L(G) \subset \bigcup_{z \in L(G)} l_z$. Otherwise, if $z \notin \overline{\mathbf{H}^2_{\mathbb{C}}}$, then $k \notin \mathbf{H}^2_{\mathbb{C}}$, and we consider two cases:

(1) If $k \in \partial \mathbf{H}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$, then $k_n \in P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$ for almost every $n \in \mathbb{N}$, for otherwise there would exist a subsequence, still denoted k_n, such that $k_n \in \overline{\mathbf{H}^2_{\mathbb{C}}}$ for all n, and given that $g_n(k_n) \to z$ we have $z \in \overline{\mathbf{H}^2_{\mathbb{C}}}$, which is a contradiction.
We denote by Σ_n the complex geodesic which is polar to k_n. Let x_n be a point in Σ_n; we can assume, taking a subsequence if needed, that $x_n \to x \in \overline{\mathbf{H}^2_{\mathbb{C}}}$. Lemma 4.1(a) implies that $x = k$. By Proposition 3.2, we can suppose, taking subsequences if needed, that $g_n(x_n) \to q \in L(G)$. The complex geodesic $g_n(\Sigma_n)$ is polar to $g_n(k_n)$, and we know that $g_n(k_n) \to z$, then Lemma 4.1(b) implies that $z \in l_q$.

(2) If $k \in P^2_{\mathbb{C}} - (\overline{\mathbf{H}^2_{\mathbb{C}}} \cup L_0(G) \cup L_1(G))$, then we can assume, discarding a finite number of terms, that $k_n \in P^2_{\mathbb{C}} - \overline{\mathbf{H}^2_{\mathbb{C}}}$ for all n. Let Σ_n (respectively Σ) denote the complex geodesic which is polar to k_n (respectively k). Let x_n be an element of Σ_n. We assume, taking a subsequence if necessary, that $x_n \to x \in \Sigma \cup \partial\Sigma$. Also, we can choose the sequence in such way that $x \in \Sigma \subset \mathbf{H}^2_{\mathbb{C}}$. Once more, we apply Proposition 3.2 to find subsequences of g_n and x_n such that $g_n(x_n) \to q \in L(G)$. Finally, the complex geodesic $g_n(\Sigma_n)$ is polar to $g_n(k_n)$, and we know that $g_n(k_n) \to z$, then Lemma 4.1(b) implies that $z \in l_q$. $\square$

We are now ready to finish the proof of Theorem 4.5.

Proof of Theorem 4.5(b) First notice that Lemmas 4.10, 4.11 and 4.12 imply that $\Lambda(G) \subset \bigcup_{z \in L(G)} l_z$, so it suffices to prove the converse inclusion. If $L(G) = \varnothing$ then G is finite and there is nothing to do. Thus, we split the proof in three cases according to whether the cardinality of $L(G)$ is 1, 2 or more than 2. We begin with the last case which is the most interesting.

If $|L(G)| > 2$ then G is non-elementary. Let $g_0 \in G$ be a loxodromic element, and $p \in L(G)$ a fixed point of g_0 such that the line l_p, tangent to $\partial \mathbf{H}^2_{\mathbb{C}}$ at p, is contained in $\Lambda(G)$ (Lemma 4.9). Observe that $g(l_p) = l_{g(p)}$ for each $g \in G \subset \mathrm{PU}(2,1)$. So $\cup_{g \in G} l_{g(p)}$ is contained in $\Lambda(G)$, because $\Lambda(G)$ is a G-invariant closed set. Thus,

$$\Lambda(G) \supset \overline{\bigcup_{g \in G} l_{g(p)}} = \overline{\bigcup_{z \in \{g(p)\}_{g \in G}} l_z} = \bigcup_{z \in \overline{\{g(p)\}_{g \in G}}} l_z = \bigcup_{z \in L(G)} l_z.$$

Now notice that, by Corollary 4.7, if $l_z \subset \Lambda(G)$, then its orbit is dense on $\Lambda(G)$. This proves 4.5 (b) when G is non-elementary.

Assume now that $|L(G)| = 2$ and denote by x and y the two points in $L(G)$. Then G contains a loxodromic element and it is not difficult to see that $L_0(G) = \{x, y, z\} = L_1(G)$, where $z = l_x \cap l_y$, and $L_2(G) = l_x \cup l_y$.

Finally consider the case $|L(G)| = 1$ and denote by x the only point in $L(G)$. Proposition 3.6 implies that exists an element $g_0 \in G$ of infinite order which is parabolic and fixes x, and one has $\Lambda(G) = \Lambda(\langle g_0 \rangle) = l_x$. $\square$

Corollary 4.13 *Let G, G_1, G_2 be discrete subgroups of* $\mathrm{PU}(2,1)$. *Then:*

(a) *The limit set $\Lambda(G)$ is path connected.*
(b) *If $L(G)$ is all of $\partial \mathbf{H}^2_{\mathbb{C}}$, then $\Lambda(G) = P^2_{\mathbb{C}} - \mathbf{H}^2_{\mathbb{C}}$.*
(c) *If $G_1 \subset G_2$ then $\Lambda(G_1) \subset \Lambda(G_2)$.*
(d) *If G is non-elementary and W is a G-invariant open set such that G acts properly discontinuously on W, then $W \subseteq \Omega(G)$. In other words, $\Omega(G)$ is the maximal open set on which G acts properly discontinuously.*

We remark that the first statement in 4.13 holds for all subgroups of PU$(2,1)$, not only for discrete subgroups, since when G is not discrete one necessarily has $\Lambda(G) = P^2_{\mathbb{C}}$. This remark applies also to statement (c), for the same reason.

Proof

(a) Let x_1, x_2 be points in $\Lambda(G)$. By Theorem 4.5(b) there exist $z_1, z_2 \in L(G)$ such that $x_1 \in l_{z_1}$ and $x_2 \in l_{z_2}$. If $z_1 = z_2$ then we can join x_1 to x_2 by a path in $l_{z_1} \subset \Lambda(G)$. Finally, if $z_1 \neq z_2$ then we can join x_1 to x_2 by a path in $l_{z_1} \cup l_{z_2} \subset \Lambda(G)$, because any two complex lines in $P^2_{\mathbb{C}}$ have non-empty intersection.

(b) This claim follows immediately from 4.5(b). See also [15].

(c) This claim follows from Theorem 4.5(b) and the fact that for the Chen–Greenberg limit sets one obviously has $L(G_1) \subset L(G_2)$.

(d) If $W \not\subseteq \Omega(G)$ then $W \cap l_z \neq \varnothing$ for every $z \in L(G)$ (for otherwise there exists $z_0 \in L(G)$ such that $l_{z_0} \subset (P^2_{\mathbb{C}} - W)$. Theorem 4.5(b)) and the fact that $P^2_{\mathbb{C}} - W$ is a G-invariant closed set imply that $\Lambda(G) = \overline{\bigcup_{g\in G} g(l_{z_0})} \subset P^2_{\mathbb{C}} - W$. Hence, $W \subset \Omega(G)$, which is a contradiction to our assumption). In particular, if $g_0 \in G$ is a loxodromic element having fixed points $z_1, z_2 \in L(G)$ then $W \cap l_{z_1} \neq \varnothing \neq W \cap l_{z_2}$. But Lemma 4.8 says that G does not act properly discontinuously on W, so we get a contradiction and therefore $W \subseteq \Omega(G)$. □

References

1. Beardon, A.: The Geometry of Discrete Groups. Graduate Texts in Mathematics, vol. 91. Springer-Verlag, New York (1983)
2. Chen, S.S., Greenberg, L.: Hyperbolic Spaces. Contributions to Analysis, pp. 49–87. Academic Press, New York (1974)
3. Eberlein, P.: Geodesic flows on negatively curved manifolds I. Ann. Math. Num. **95**, 492–510 (1972)
4. Giraud, G.: Sur certaines fonctions automorphes de deux variables. Ann. Ecole Norm. (3) **38**, 43–164 (1921)
5. Goldman, W.: Complex Hyperbolic Geometry. Oxford Science Publications (1999)
6. Kamiya, S.: Notes on non-discrete subgroups of $\hat{U}(1,n;F)$. Hiroshima Math. J. Num. **13**, 501–506 (1982)
7. Kamiya, S.: Notes on elements of $U(1,n;\mathbb{C})$. Hiroshima Math. J. Num. **21**, 23–45 (1991)
8. Kulkarni, R.S.: Groups with domains of discontinuity. Math. Ann. Num. **237**, 253–272 (1978)
9. Maskit, B.: Kleinian Groups. A Series of Comprehensive Studies in Mathematics, vol. 287. Springer-Verlag, New York (1988)
10. Mostow, G.D.: On a remarkable class of polyhedra in complex hyperbolic space. Pacific J. Math. Num. **86**, 171–276 (1980)
11. Palis, J., de Melo, W.: Geometric Theory of Dynamical Systems: An Introduction. Springer-Verlag, New York (1982)
12. Seade, J., Verjovsky, A.: Actions of discrete groups on complex projective spaces. Contemp. Math. Num. **269**, 155–178 (2001)
13. Seade, J., Verjovsky, A.: Higher dimensional complex Kleinian groups. Math. Ann. Num. **322**, 279–300 (2002)
14. Siegel, C.L.: Discontinuous groups. Ann. Math. Num. **44**, 674–689 (1943)
15. Sienra, G.: Complex Kleinian groups and limit sets in P^2. Complex Variables **49**(10), 689–701 (2004)