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A new inequality for bipartite distance-regular graphs

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Abstract

Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let θ denote an eigenvalue of Γ other than k and $-k$ and consider the associated cosine sequence, $\sigma_0, \sigma_1, \dots, \sigma_D$. We show

$$(\sigma_1 - \sigma_{i+1})(\sigma_1 - \sigma_{i-1}) \geqslant (\sigma_2 - \sigma_i)(\sigma_0 - \sigma_i)$$

for $1 \leqslant i \leqslant D-1$. We show the following are equivalent: (i) equality is attained above for $i = 3$, (ii) equality is attained above for $1 \leqslant i \leqslant D-1$, (iii) there exists a real scalar β such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1}$ is independent of i for $1 \leqslant i \leqslant D-1$. We say θ is three-term recurrent (or TTR) whenever (i)–(iii) are satisfied.

We discuss the connection between TTR eigenvalues and the Q-polynomial property. When an eigenvalue is TTR, we find formulae for the intersection numbers and eigenvalues of Γ in terms of at most two free parameters, classifying Γ if $\beta = \pm 2$. Among the eigenvalues of Γ in their natural order, we consider which can be TTR. We show Γ can have at most three distinct TTR eigenvalues. We show Γ has three distinct TTR eigenvalues if and only if Γ is 2-homogeneous in the sense of Curtin and Nomura. We show Γ has exactly two distinct TTR eigenvalues if and only if Γ is antipodal with diameter 5, but not 2-homogeneous.

Keywords: Cosine sequence; Q-polynomial; Association scheme; P-polynomial; Antipodal; Bipartite; Distance-regular graph

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1. Introduction

Let Γ denote a distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. (Definitions are contained in subsequent sections.) Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a cosine sequence of Γ.

In [12], Terwilliger found a series of inequalities involving $\sigma_0, \sigma_1, \ldots, \sigma_D$ and the intersection numbers of Γ. He showed equality holds in these inequalities with $\sigma_D \neq 1$ if and only if Γ is Q-polynomial with respect to $\sigma_0, \sigma_1, \ldots, \sigma_D$.

Terwilliger's inequalities are rather complicated, so we would like to find nicer ones. To simplify matters, we assume Γ is bipartite. For this case we find a series of attractive inequalities and examine the case of equality. We summarize our results as follows.

For the rest of this introduction, suppose Γ is bipartite. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of Γ. Let θ denote one of $\theta_1, \theta_2, \ldots, \theta_{D-1}$ and let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated cosine sequence. We show that for $1 \leqslant i \leqslant D-1$,

$$(\sigma_1 - \sigma_{i+1})(\sigma_1 - \sigma_{i-1}) \geqslant (\sigma_2 - \sigma_i)(\sigma_0 - \sigma_i). \tag{1}$$

We show equality holds in (1) for $i = 3$ if and only if equality holds in (1) for $1 \leqslant i \leqslant D-1$ if and only if there exist $\beta, \gamma \in \mathbb{C}$ such that

$$\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = \gamma \quad (1 \leqslant i \leqslant D-1). \tag{2}$$

We call θ *three-term recurrent* (TTR for short) whenever these equivalent conditions hold.

When θ is TTR, we use (2) to find formulae for the σ_i, the intersection numbers and the eigenvalues of Γ in terms of at most two free parameters.

To further study TTR eigenvalues, we make the following definitions. Given that the eigenvalue θ is TTR, we say it is *type AE* if $\sigma_D = 1$ and D is even, *type AO* if $\sigma_D = 1$ and D is odd, and *type Q* if $\sigma_D \neq 1$.

We show θ is type Q if and only if Γ is Q-polynomial with respect to θ. If θ is type AE or AO then Γ is antipodal and its quotient graph $\tilde{\Gamma}$ is Q-polynomial with respect to θ. Conversely, if Γ is antipodal with $D \neq 6$, and $\tilde{\Gamma}$ is Q-polynomial with respect to θ then θ is type AE or AO in Γ. This converse is not true if $D = 6$, since θ_4 in the 6-cube is a counterexample.

Examining the cases AE, AO and Q in turn, we obtain the following results.

Suppose θ is type AE. Then $\theta = \theta_2$. Moreover, $D = 4$ or Γ is the D-cube.

Now suppose θ is type AO. Recall D is odd and set $d := (D-1)/2$. First assume $\beta \notin \{-2, 2\}$. In this case, $\theta \in \{\theta_e, \theta_{D-1}\}$, where $e = d$ if d is even and $e = d+1$ if d is odd. We know of no examples of AO eigenvalues with $\beta \notin \{-2, 2\}$ and $D > 5$. If $\beta = 2$, then $\theta = \theta_2$. Moreover, Γ must be the double Hoffman–Singleton graph or the D-cube. If $\beta = -2$ then $\theta = \theta_{D-1}$. In this case, Γ must be the D-cube or the doubled Odd graph $2.O_k$.

Finally, assume θ is type Q and first suppose $\beta \notin \{-2, 2\}$. By a result of Caughman [3], $\theta \in \{\theta_1, \theta_{D-1}\}$. For $\theta = \theta_1$, the bipartite dual polar graphs and the Hemmeter graphs are examples but we have no classification. For $\theta = \theta_{D-1}$, the Hadamard graphs (excluding the 4-cube) are the only known examples, but again, we have no classification. Next suppose $\beta = 2$. Then $\theta = \theta_1$. In this case, Γ is the D-cube or the folded $2D$-cube. Finally, suppose $\beta = -2$. Then $\theta = \theta_{D-1}$. Moreover, D is even and Γ is the D-cube.

The graph Γ may have more than one TTR eigenvalue. We conclude our study by considering what combinations of TTR eigenvalues are possible. If Γ has at least one TTR eigenvalue, it is described in the table below. The details are as follows. We show Γ cannot have more than three TTR eigenvalues. Next we show Γ has a type AE eigenvalue and no other TTR eigenvalues if and only if $D = 4$ and Γ is an antipodal r-cover with $r > 2$. There are examples in which the sole TTR eigenvalue is type AO or Q, but we have no classification for these. Exactly two eigenvalues are TTR if and only if $D = 5$ and Γ is antipodal but not 2-homogeneous in the sense of Nomura [7]. In this case, both TTR eigenvalues are type AO. Exactly three eigenvalues of Γ are TTR if and only if Γ is 2-homogeneous. In this case, if D is even then Γ has one type AE and two type Q eigenvalues. If D is odd, two eigenvalues are type AO and one is type Q.

AE	AO	Q	Classification (Example)
1	0	0	$D = 4$, r-cover, $r > 2$
0	1	0	(Doubled Odd graphs only known)
0	0	1	(Dual polar graphs $D_D(q)$ and many others)
0	2	0	$D = 5$, 2-cover, not 2-homogeneous
1	0	2	D even, 2-homogeneous
0	2	1	D odd, 2-homogeneous

2. Distance-regular graphs and intersection numbers

Over the course of this paper, we use many results from the literature. We review the more basic of these in the next few sections; the remainder we simply cite as needed. For more complete background, we refer the reader to the books by Bannai and Ito [1] and Brouwer et al. [2].

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂, and diameter $D := \max\{\partial(x,y)\colon x, y \in X\}$. For all $x, y \in X$ and all integers i, j, we set $\Gamma_i(x) := \{z \in X\colon \partial(x,z) = i\}$ and $\Gamma_i^j(x,y) := \Gamma_i(x) \cap \Gamma_j(y)$.

We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \leqslant h, i, j \leqslant D)$ and all $x, y \in X$ with $\partial(x,y) = h$, the scalar $p_{ij}^h = |\Gamma_i^j(x,y)|$ is independent of x and y. The p_{ij}^h are called the *intersection numbers* of Γ. For notational convenience, set $c_i := p_{1i-1}^i$ $(1 \leqslant i \leqslant D)$, $a_i := p_{1i}^i$ $(0 \leqslant i \leqslant D)$, $b_i := p_{1i+1}^i$ $(0 \leqslant i \leqslant D-1)$, and $c_0 := b_D := 0$. We note $c_1 = 1$ and $b_i c_{i+1} \neq 0$ $(1 \leqslant i \leqslant D-1)$.

For the rest of this section, we let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D. We set $k := b_0$ and call this the *valency* of Γ. Observe

$$c_i + a_i + b_i = k \quad (0 \leqslant i \leqslant D). \tag{3}$$

By the *intersection array* of Γ we mean the sequence $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$.

We say Γ is *bipartite* whenever there exists a partition $X = X^+ \cup X^-$ such that each of X^+ and X^- does not contain an edge. We observe Γ is bipartite if and only if its intersection numbers satisfy

$$a_i = 0 \quad (0 \leqslant i \leqslant D). \tag{4}$$

We say Γ is *almost-bipartite* (or a *generalized Odd graph*) whenever $a_i = 0$ for $0 \leqslant i \leqslant D-1$ and $a_D > 0$.

This paper is about the case in which Γ is bipartite. We wish to point out, though, that many of our results have implications for the case in which Γ is almost-bipartite. These implications are obtained by applying our results to the bipartite double of Γ [2, Proposition 4.2.11].

3. The Bose–Mesner algebra—idempotents, eigenvalues and cosine sequences

Let $\mathbb{C}$ denote the complex numbers. Let X denote a nonempty finite set. By $Mat_X(\mathbb{C})$ we mean the $\mathbb{C}$-algebra of matrices whose entries are in $\mathbb{C}$ and whose rows and columns are indexed by X.

Let $V = \mathbb{C}^X$ denote the vector space of column vectors whose entries are in $\mathbb{C}$ and whose rows are indexed by X. We observe $Mat_X(\mathbb{C})$ acts on V by left multiplication. For each $x \in X$, let $\hat{x}$ denote the element of V that has a one in row x and zeros in all other rows. We endow V with the inner product $\langle u, v \rangle = u^t \bar{v}$ for all $u, v \in V$, where u^t means the transpose of u. We observe this inner product is positive definite. With respect to this inner product, $\{\hat{x}\colon x \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D. For each integer i $(0 \leqslant i \leqslant D)$, let A_i denote the matrix in $Mat_X(\mathbb{C})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \quad (x, y \in X).$$

We call A_i the *ith distance matrix*. We abbreviate $A := A_1$ and call this the *adjacency matrix*.

Let M denote the subalgebra of $Mat_X(\mathbb{C})$ generated by A. Observe $A_0, A_1, \ldots, A_D$ is a basis for M [2, p. 127]. We call M the *Bose–Mesner algebra* of Γ.

By Brouwer et al. [2, Theorem 2.6.1], M has a second basis $E_0, E_1, \ldots, E_D$ such that

$$E_i E_j = \delta_{ij} E_i \quad (0 \leqslant i, j \leqslant D). \tag{5}$$

Moreover, the entries in each of $E_0, E_1, \ldots, E_D$ are real. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ. For brevity, we will simply call them the *idempotents* of Γ.

Apparently, there exist real scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that

$$A = \sum_{i=0}^{D} \theta_i E_i. \tag{6}$$

Combining (5) and (6), we find that for $0 \leqslant i \leqslant D$, we have $AE_i = \theta_i E_i$. We call θ_i the *eigenvalue* of Γ associated with E_i. We note $\theta_0, \theta_1, \ldots, \theta_D$ are distinct since A generates M.

Let E denote an idempotent of Γ and let m denote its rank. Let θ denote the associated eigenvalue. Apparently, there exist real scalars $\sigma_0, \sigma_1, \ldots, \sigma_D$ such that

$$E = m|X|^{-1} \sum_{i=0}^{D} \sigma_i A_i.$$

We call the sequence $\sigma_0, \sigma_1, \ldots, \sigma_D$ the *cosine sequence* associated with θ, E. For notational convenience, we let σ_{-1} and σ_{D+1} denote indeterminates.

Lemma 3.1 (Terwilliger [12, Lemma 1.1]). *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D. Let θ denote an eigenvalue of Γ. Let E and $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated idempotent and cosine sequence.*

(i) *For all $x, y \in X$,*

$$\langle E\hat{x}, E\hat{y} \rangle = m|X|^{-1}\sigma_i, \tag{7}$$

where $i = \partial(x, y)$ and $m = \text{rank}\, E$.

(ii) *For all $x \in X$,*

$$\sum_{z \in \Gamma_1(x)} E\hat{z} = \theta E\hat{x}. \tag{8}$$

To state the following lemma we recall a definition. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a sequence of real scalars. By a *sign change* in $\sigma_0, \sigma_1, \ldots, \sigma_D$ we mean an ordered pair of integers (r, t) with $0 \leqslant r < t \leqslant D$ such that $\sigma_r \sigma_t < 0$ and $\sigma_s = 0$ for $r < s < t$.

Lemma 3.2 (Brouwer et al. [2, Corollary 4.1.2]). *Let Γ denote a distance-regular graph with diameter D. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ. For $0 \leqslant i \leqslant D$, the cosine sequence associated with θ_i has exactly i sign changes.*

For the rest of this section, we restrict our attention to bipartite distance-regular graphs.

Let Γ denote a bipartite distance-regular graph with diameter D. We will occasionally refer to the matrix

$$L := \begin{pmatrix} 0 & b_0 & & & & \\ c_1 & 0 & b_1 & & 0 & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & b_{D-1} \\ & & & & c_D & 0 \end{pmatrix}. \tag{9}$$

Lemma 3.3 (Brouwer et al. [2, pp. 128–132]). *Let Γ denote a bipartite distance-regular graph with diameter D and valency k. Let L be as in* (9) *and let $\theta, \sigma_0, \sigma_1, \dots, \sigma_D$ denote real scalars. The following are equivalent:*

(i) *$\sigma_0, \sigma_1, \dots, \sigma_D$ is a cosine sequence of Γ, and θ is the associated eigenvalue.*
(ii) *$(\sigma_0, \sigma_1, \dots, \sigma_D)^t$ is a right eigenvector of L with $\sigma_0 = 1$, and θ is the associated eigenvalue.*
(iii) *$\sigma_0 = 1$, and*

$$c_i\sigma_{i-1} + b_i\sigma_{i+1} = \theta\sigma_i \quad (0 \leqslant i \leqslant D). \tag{10}$$

(iv) *$\sigma_0 = 1$, $\theta = k\sigma_1$,*

$$\sigma_{D-1} = \sigma_1\sigma_D, \tag{11}$$

and

$$c_i(\sigma_{i-1} - \sigma_{i+1}) = k(\sigma_1\sigma_i - \sigma_{i+1}) \quad (1 \leqslant i \leqslant D-1). \tag{12}$$

(v) *$\sigma_0 = 1$, $\theta = k\sigma_1$, $\sigma_{D-1} = \sigma_1\sigma_D$, and*

$$b_i(\sigma_{i+1} - \sigma_{i-1}) = k(\sigma_1\sigma_i - \sigma_{i-1}) \quad (1 \leqslant i \leqslant D-1). \tag{13}$$

We list some well-known facts that follow immediately from Lemma 3.3.

Corollary 3.4. *Let Γ denote a bipartite distance-regular graph with diameter D and valency k. Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the eigenvalues of Γ.*

(i) *The matrix L from* (9) *is diagonalizable.*
(ii) *$\theta_0 = k$ and the associated cosine sequence is* $1, 1, \dots, 1$.
(iii) *$\theta_{D-i} = -\theta_i$ for $0 \leqslant i \leqslant D$.*
(iv) *Let θ denote an eigenvalue of Γ, with cosine sequence $\sigma_0, \sigma_1, \dots, \sigma_D$. The cosine sequence associated with $-\theta$ is $\sigma_0, -\sigma_1, \sigma_2, \dots, (-1)^D\sigma_D$.*

Definition 3.5. Let Γ denote a bipartite distance-regular graph with diameter D and valency k. Let θ denote an eigenvalue of Γ. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote the associated cosine sequence. We call θ (or $\sigma_0, \sigma_1, \dots, \sigma_D$) *trivial* whenever $\theta \in \{-k, k\}$.

We can use Lemma 3.3 to get formulae for the intersection numbers.

Lemma 3.6. *Let Γ denote a bipartite distance-regular graph with diameter D. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a cosine sequence of Γ. For $1 \leqslant i \leqslant D-1$, the following are equivalent:*

(i) $\sigma_{i-1} = \sigma_{i+1}$,
(ii) $\sigma_1\sigma_i = \sigma_{i-1}$,
(iii) $\sigma_1\sigma_i = \sigma_{i+1}$.

Proof. Follows immediately from (12) and (13). □

Lemma 3.7. *Let Γ denote a bipartite distance-regular graph with diameter $D\geqslant 3$ and valency $k\geqslant 3$. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a cosine sequence of Γ. The following are equivalent:*

(i) $\sigma_0, \sigma_1, \ldots, \sigma_D$ *is nontrivial.*
(ii) $\sigma_2 \neq 1$.
(iii) $\sigma_1^2 \neq \sigma_2$.
(iv) $\sigma_i \neq 1$ *for* $1 \leqslant i \leqslant D-1$.

Proof. (i)$\leftrightarrow$(ii): Immediate from Corollary 3.4.
(ii)$\leftrightarrow$(iii): Immediate from Lemma 3.6.
(i)$\rightarrow$(iv): Immediate from [2, Proposition 4.4.7].
(iv)$\rightarrow$(ii): Clear. □

Lemma 3.8 (Curtin [5, Lemma 8]). *Let Γ denote a bipartite distance-regular graph with diameter $D\geqslant 3$ and valency $k\geqslant 3$. Let θ denote a nontrivial eigenvalue of Γ with cosine sequence $\sigma_0, \sigma_1, \ldots, \sigma_D$. Then*

$$k = \frac{1-\sigma_2}{\sigma_1^2 - \sigma_2}. \tag{14}$$

(*Observe the denominator in* (14) *is nonzero by Lemma* 3.7.) *Let i denote an integer* $(1 \leqslant i \leqslant D-1)$, *and suppose* (i)–(iii) *in Lemma* 3.6 *do not hold. Then*

$$c_i = k\frac{\sigma_1\sigma_i - \sigma_{i+1}}{\sigma_{i-1} - \sigma_{i+1}}, \tag{15}$$

$$b_i = k\frac{\sigma_1\sigma_i - \sigma_{i-1}}{\sigma_{i+1} - \sigma_{i-1}}. \tag{16}$$

4. The antipodal property

Let Γ denote a distance-regular graph. We define Γ_i to be the graph with vertex set X where two vertices x, y are adjacent in Γ_i whenever $\partial(x,y) = i$ in Γ.

Definition 4.1. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \in \{2d, 2d+1\} \geqslant 2$. We say Γ is *antipodal* if Γ_D is a disjoint union of complete graphs.

In this case, there exists a distance-regular graph $\tilde{\Gamma} = (\tilde{X}, \tilde{R})$ with diameter d and a surjective map $\varepsilon : X \rightarrow \tilde{X}$, such that for all $x, y \in X$

$$\partial(x,y) \in \{i, D-i\} \quad \text{if and only if} \quad \partial(\varepsilon x, \varepsilon y) = i \quad (0 \leqslant i \leqslant d).$$

The graph $\tilde{\Gamma}$ is determined up to isomorphism by Γ and is known as the *quotient* graph of Γ. We say Γ is an *r-cover* of $\tilde{\Gamma}$, where r denotes the size of each complete graph in Γ_D. Note $|X| = r|\tilde{X}|$.

Lemma 4.2 (Brouwer et al. [2, p. 141]). *Let Γ denote a distance-regular graph with diameter $D \in \{2d, 2d+1\} \geqslant 2$ and valency $k \geqslant 3$ and intersection array $\{b_0, \dots, b_{D-1}; c_1, \dots, c_D\}$. Then Γ is antipodal if and only if $b_i = c_{D-i}$ for $0 \leqslant i \leqslant D$, $i \neq d$. In this case Γ is an antipodal r-cover of its quotient graph $\tilde{\Gamma}$, where*

$$r = 1 + b_d / c_{D-d}. \tag{17}$$

If $D \geqslant 3$ then $\tilde{\Gamma}$ is distance-regular with diameter d and intersection array

$$\{b_0, \dots, b_{d-1}; c_1, \dots, c_{d-1}, \alpha c_d\},$$

where

$$\alpha = r \quad \text{if } D = 2d, \quad \alpha = 1 \quad \text{if } D = 2d+1.$$

Suppose Γ is also bipartite. If D is even then $\tilde{\Gamma}$ is bipartite. If D is odd then $\tilde{\Gamma}$ is almost-bipartite and $r = 2$. In either case, if $D > 4$ then $\tilde{\Gamma}$ is not antipodal.

Lemma 4.3 (Brouwer et al. [2, pp. 142, 161]). *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a nontrivial cosine sequence of Γ and let θ and E denote the associated eigenvalue and idempotent. The following* (i)–(iii) *are equivalent*:

(i) $\sigma_D = 1$.
(ii) *Γ is antipodal and $\theta = \theta_i$, where i is even.*
(iii) *Γ is antipodal and θ is an eigenvalue of the quotient graph $\tilde{\Gamma}$.*

Suppose these statements hold. Then

$$\sigma_i = \sigma_{D-i} \quad (0 \leqslant i \leqslant D). \tag{18}$$

In $\tilde{\Gamma}$, the eigenvalue θ is associated with the cosine sequence $\sigma_0, \sigma_1, \dots, \sigma_d$, where $D \in \{2d, 2d+1\}$.

5. Krein parameters and the Q-polynomial property

Definition 5.1. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D and valency k. Let $E_0, E_1, \dots, E_D$ denote an ordering of the idempotents of Γ. Let $\circ$ denote entrywise multiplication and observe the Bose–Mesner algebra M is closed under $\circ$. For any i,j $(0 \leqslant i,j \leqslant D)$ we write

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h$$

and call the q_{ij}^h the *Krein parameters* of Γ.

We say $E_0, E_1, \ldots, E_D$ is a *Q-polynomial* ordering of the idempotents when E_0 is the idempotent associated with k and the following (i)–(ii) hold for $0 \leqslant h, i, j \leqslant D$:

(i) $q^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two.
(ii) $q^h_{ij} \neq 0$ if one of h, i, j is equal to the sum of the other two.

Let θ_i denote the eigenvalue associated with E_i for $0 \leqslant i \leqslant D$. We say $\theta_0, \theta_1, \ldots, \theta_D$ is a Q-polynomial ordering of the eigenvalues when $E_0, E_1, \ldots, E_D$ is a Q-polynomial ordering of the idempotents.

Let θ denote an eigenvalue of Γ and let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated cosine sequence. We say Γ is Q-polynomial with respect to θ (or $\sigma_0, \sigma_1, \ldots, \sigma_D$) when there exists a Q-polynomial ordering $\theta_0, \theta_1, \ldots, \theta_D$ of the eigenvalues such that $\theta = \theta_1$. We note if Γ is Q-polynomial with respect to a cosine sequence then those cosines are distinct [2, p. 135].

Finally, we say Γ itself is Q-polynomial when it is Q-polynomial with respect to at least one of its eigenvalues.

We present the distance-regular graphs with diameter 2 and the bipartite distance-regular graphs with diameter 3 as examples of graphs that have the Q-polynomial property.

Example 5.2 (Brouwer et al. [2, p. 59, Theorem 4.2.1]). Let Γ denote a distance-regular graph with diameter $D = 2$ and valency $k \geqslant 3$. Let $\theta_0 > \theta_1 > \theta_2$ denote the eigenvalues of Γ. Then Γ is Q-polynomial with respect to θ_1. If Γ is neither bipartite nor antipodal then Γ is also Q-polynomial with respect to θ_2.

Example 5.3 (Brouwer et al. [2, pp. 242, 432]). Let Γ denote a bipartite distance-regular graph with diameter $D = 3$ and valency k. Set $\mu := c_2$. The intersection array of Γ is

$$\{k, k-1, k-\mu; 1, \mu, k\}.$$

The eigenvalues of Γ are

$$\pm k, \pm\sqrt{k-\mu}.$$

Γ is Q-polynomial with respect to $\theta = \sqrt{k-\mu}$. The associated cosine sequence is

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = \left(1, \frac{\theta}{k}, \frac{\theta^2 - k}{k(k-1)}, \frac{\theta^2 - k}{\theta(k-1)}\right). \tag{19}$$

If $\mu \neq k-1$ (i.e. if Γ is not also antipodal) then Γ is also Q-polynomial with respect to $\theta = -\sqrt{k-\mu}$ with cosine sequence still given by (19).

Let Γ denote a distance-regular graph with diameter $D \geqslant 3$. Let θ denote an eigenvalue of Γ and suppose Γ is Q-polynomial with respect to θ. In [1, Theorem III.5.1], Bannai and Ito show this Q-polynomial structure must fall into one of seven cases, called types I, IA, II, IIA, IIB, IIC and III. In each case, they give formulae for the intersection numbers and eigenvalues of Γ in terms of a small set of parameters.

Lemma 5.4 (Terwilliger [12, Theorem 3.3]). *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geqslant 3$. Let θ denote an eigenvalue of Γ and let E and $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated idempotent and cosine sequence. The following are equivalent:*

(i) *Γ is Q-polynomial with respect to θ.*
(ii) *$\sigma_i \neq 1$ for $1 \leqslant i \leqslant D$ and for all $x, y \in X$ with $1 \leqslant \partial(x, y) \leqslant 3$,*

$$\sum_{z \in \Gamma_1^2(x,y)} E\hat{z} - \sum_{w \in \Gamma_2^1(x,y)} E\hat{w} \in Span(E\hat{x} - E\hat{y}). \tag{20}$$

Lemma 5.5 (Caughman [3, Theorem 1.1]). *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ. Let θ denote an eigenvalue of Γ and suppose Γ is Q-polynomial with respect to θ. Then $\theta \in \{\theta_1, \theta_{D-1}\}$.*

6. The 2-homogeneous property

Definition 6.1 (Curtin [5], Nomura [7]). Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$. We say Γ is *2-homogeneous* whenever for all integers i $(1 \leqslant i \leqslant D-1)$ and for all vertices $x, y, z \in X$ at distances $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$, the scalar $\gamma_i = |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|$ depends only on i.

Lemma 6.2 (Curtin [5, Theorems 34, 35, 42]). *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$. Then the following* (i)–(iii) *are equivalent*:

(i) *Γ is 2-homogeneous.*
(ii) *Γ is Q-polynomial and antipodal.*
(iii) *Either Γ is the D-cube or there exists $Q \in \mathbb{C}$ such that the intersection numbers of Γ are given by*

$$c_0 = 0, \tag{21}$$

$$c_i = \frac{(1 - Q^{2i})(Q^2 + Q^D)}{(1 - Q^2)(Q^{2i} + Q^D)} \quad (1 \leqslant i \leqslant D-1), \tag{22}$$

$$c_D = \frac{(1 - Q^D)(Q^2 + Q^D)}{Q^D(1 - Q^2)}, \tag{23}$$

$$b_0 = \frac{(1 - Q^D)(Q^2 + Q^D)}{Q^D(1 - Q^2)}, \tag{24}$$

$$b_i = \frac{(Q^{2i} - Q^{2D})(Q^2 + Q^D)}{Q^D(1 - Q^2)(Q^{2i} + Q^D)} \quad (1 \leqslant i \leqslant D - 1), \tag{25}$$

$$b_D = 0 \tag{26}$$

and such that the denominators in (22)–(25) *are nonzero*.

Suppose (i)–(iii) *hold. Then Γ is a* 2*-cover*.

Lemma 6.3 (Nomura [7, Theorem 1.2]). *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$. Then Γ is* 2*-homogeneous if and only if the intersection array of Γ is one of the following:*

(i) $\{k, k-1, 1; 1, k-1, k\}$ *for* $k \geqslant 4$.
(ii) $\{4\gamma, 4\gamma - 1, 2\gamma, 1; 1, 2\gamma, 4\gamma - 1, 4\gamma\}$ *for* $\gamma \geqslant 2$, *an integer*.
(iii) $\{k, k-1, k-\mu, \mu, 1; 1, \mu, k-\mu, k-1, k\}$, $k = \gamma(\gamma^2 + 3\gamma + 1), \mu = \gamma(\gamma + 1)$ *for* $\gamma \geqslant 2$, *an integer*.
(iv) $\{k, k-1, k-2, \ldots, 2, 1; 1, 2, \ldots, k-1, k\}$ *for* $k \geqslant 3$.

Arrays (i) *and* (iv) *are uniquely realized by the complement of the* $2 \times (k+1)$*-grid and the D-cube, respectively. The graphs with array* (ii) *are the Hadamard graphs of order* 4γ. *Array* (iii) *is uniquely realized by the double Higman–Sims graph when* $\gamma = 2$, *and no examples with* $\gamma \geqslant 3$ *are known*.

7. The inequality

In this section we produce an inequality involving the elements of a cosine sequence and investigate the case of equality. We begin with a long but straightforward lemma.

Lemma 7.1. *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geqslant 3$. Let θ denote an eigenvalue of Γ and let E and $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated idempotent and cosine sequence. Let m denote the rank of E. Pick an integer i $(1 \leqslant i \leqslant D-1)$ and let $x, y \in X$ be vertices such that $\partial(x, y) = i$. Set*

$$B := \sum_{z \in \Gamma_1^{i+1}(x,y)} E\hat{z} - \sum_{w \in \Gamma_{i+1}^1(x,y)} E\hat{w},$$

$$C := \sum_{z \in \Gamma_1^{i-1}(x,y)} E\hat{z} - \sum_{w \in \Gamma_{i-1}^1(x,y)} E\hat{w},$$

$$F := E\hat{x} - E\hat{y}.$$

Then the following hold:

(i)

$$\langle B, C \rangle = 2 b_i c_i m |X|^{-1} (\sigma_2 - \sigma_i),$$

$$\langle B, F \rangle = 2 b_i m |X|^{-1} (\sigma_1 - \sigma_{i+1}),$$

$$\langle C, F \rangle = 2c_i m|X|^{-1}(\sigma_1 - \sigma_{i-1}),$$
$$\langle F, F \rangle = 2m|X|^{-1}(\sigma_0 - \sigma_i).$$

(ii) $B + C = \theta F$.

(iii)
$$\begin{aligned}
&\langle C, C \rangle \langle F, F \rangle - \langle C, F \rangle \langle C, F \rangle \\
&\quad = 4b_i c_i m^2 |X|^{-2}((\sigma_1 - \sigma_{i+1})(\sigma_1 - \sigma_{i-1}) - (\sigma_2 - \sigma_i)(\sigma_0 - \sigma_i)) \\
&\quad = \langle B, B \rangle \langle F, F \rangle - \langle B, F \rangle \langle B, F \rangle.
\end{aligned}$$

Proof. (i): Immediate from (7) since Γ is bipartite.
(ii): Immediate from (8) since Γ is bipartite.
(iii): By (i) and (ii),

$$\begin{aligned}
&\langle C, C \rangle \langle F, F \rangle - \langle C, F \rangle \langle C, F \rangle \\
&\quad = \langle \theta F - B, C \rangle \langle F, F \rangle - \langle \theta F - B, F \rangle \langle C, F \rangle \\
&\quad = \theta \langle F, C \rangle \langle F, F \rangle - \langle B, C \rangle \langle F, F \rangle - \theta \langle F, F \rangle \langle C, F \rangle + \langle B, F \rangle \langle C, F \rangle \\
&\quad = \langle B, F \rangle \langle C, F \rangle - \langle B, C \rangle \langle F, F \rangle \\
&\quad = 4b_i c_i m^2 |X|^{-2}((\sigma_1 - \sigma_{i+1})(\sigma_1 - \sigma_{i-1}) - (\sigma_2 - \sigma_i)(\sigma_0 - \sigma_i)).
\end{aligned}$$

By (ii) and linearity, we find $\langle C, C \rangle \langle F, F \rangle - \langle C, F \rangle \langle C, F \rangle = \langle B, B \rangle \langle F, F \rangle - \langle B, F \rangle \langle B, F \rangle$. $\square$

We now present our main result.

Theorem 7.2. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 3$. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a cosine sequence of Γ. Then for any integer i $(1 \leqslant i \leqslant D-1)$,*

$$(\sigma_1 - \sigma_{i+1})(\sigma_1 - \sigma_{i-1}) \geqslant (\sigma_2 - \sigma_i)(\sigma_0 - \sigma_i). \tag{27}$$

Proof. Immediate from Lemma 7.1(iii) by Cauchy–Schwarz. $\square$

Equality always holds in (27) if $i \in \{1, 2\}$. So $i = 3$ is the first nontrivial case. By Corollary 3.4, if $\sigma_0, \sigma_1, \ldots, \sigma_D$ is trivial then equality holds in (27) for $1 \leqslant i \leqslant D-1$. In the next theorem we assume $\sigma_0, \sigma_1, \ldots, \sigma_D$ is nontrivial and consider when equality is attained in (27).

Theorem 7.3. *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geqslant 3$ and valency $k \geqslant 3$. Let E denote a nontrivial idempotent of Γ, with cosine* *sequence $\sigma_0, \sigma_1, \ldots, \sigma_D$. Then for any integer i $(1 \leqslant i \leqslant D-1)$, the following are equivalent:*

(i) *Equality holds in* (27).

(ii) *For all $x, y \in X$ such that $\partial(x,y) = i$,*
$$\sum_{z \in \Gamma_1^{i-1}(x,y)} E\hat{z} - \sum_{w \in \Gamma_{i-1}^1(x,y)} E\hat{w} = c_i \frac{\sigma_1 - \sigma_{i-1}}{\sigma_0 - \sigma_i}(E\hat{x} - E\hat{y}).$$

(iii) *There exist $x, y \in X$ such that $\partial(x,y) = i$, and such that*
$$E\hat{x} - E\hat{y}, \qquad \sum_{z \in \Gamma_1^{i-1}(x,y)} E\hat{z} - \sum_{w \in \Gamma_{i-1}^1(x,y)} E\hat{w}$$
are linearly dependent.

(iv) *For all $x, y \in X$ such that $\partial(x,y) = i$,*
$$\sum_{z \in \Gamma_1^{i+1}(x,y)} E\hat{z} - \sum_{w \in \Gamma_{i+1}^1(x,y)} E\hat{w} = b_i \frac{\sigma_1 - \sigma_{i+1}}{\sigma_0 - \sigma_i}(E\hat{x} - E\hat{y}).$$

(v) *There exist $x, y \in X$ such that $\partial(x,y) = i$, and such that*
$$E\hat{x} - E\hat{y}, \qquad \sum_{z \in \Gamma_1^{i+1}(x,y)} E\hat{z} - \sum_{w \in \Gamma_{i+1}^1(x,y)} E\hat{w}$$
are linearly dependent.

Proof. We use the notation of Lemma 7.1.

(i)$\rightarrow$(ii): By Lemmas 3.7 and 7.1(i), $F \neq 0$. By Lemma 7.1(iii) and Cauchy–Schwarz, there exists $t \in \mathbb{C}$ such that
$$C = tF. \tag{28}$$
Taking the inner product of each side of (28) with F, evaluating using Lemma 7.1(i) and rearranging in view of Lemma 3.7, we find
$$t = c_i \frac{\sigma_1 - \sigma_{i-1}}{\sigma_0 - \sigma_i}. \tag{29}$$
The result follows from (28) and (29).

(i)$\rightarrow$(iv): By Lemmas 3.7 and 7.1(i), $F \neq 0$. By Lemma 7.1(iii) and Cauchy–Schwarz, there exists $t \in \mathbb{C}$ such that
$$B = tF. \tag{30}$$
Taking the inner product of each side of (30) with F, evaluating using Lemma 7.1(i) and rearranging in view of Lemma 3.7, we find
$$t = b_i \frac{\sigma_1 - \sigma_{i+1}}{\sigma_0 - \sigma_i}. \tag{31}$$

The result follows from (30) and (31).

(ii) → (iii), (iv) → (v): Obvious.

(iii) → (i), (v) → (i): Immediate from Lemma 7.1(iii) and Cauchy–Schwarz. □

8. TTR eigenvalues/cosine sequences

Equality always holds in (27) if $D = 3$. In this section we suppose $D \geqslant 4$ and examine what happens when equality holds in (27) for $1 \leqslant i \leqslant D-1$. We begin with a technical lemma, which follows from the theory of recurrence relations.

Lemma 8.1. *Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a finite sequence of real scalars. Let $\beta, \gamma \in \mathbb{C}$ be given and suppose $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = \gamma$ for $1 \leqslant i \leqslant D-1$.*

(i) *Suppose $\beta \notin \{-2, 2\}$. Then there exist $a, b, c, q \in \mathbb{C}$ with $q \neq 0$ such that $\sigma_i = a + bq^i + cq^{-i}$ for $0 \leqslant i \leqslant D$.*
(ii) *Suppose $\beta = 2$. Then there exist $a, b, c \in \mathbb{C}$ such that $\sigma_i = a + bi + ci^2$ for $0 \leqslant i \leqslant D$.*
(iii) *Suppose $\beta = -2$. Then there exist $a, b, c \in \mathbb{C}$ such that $\sigma_i = a + b(-1)^i + ci(-1)^i$ for $0 \leqslant i \leqslant D$.*

Theorem 8.2. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a nontrivial cosine sequence of Γ. The following* (i)–(iv) *are equivalent*:

(i) *The equivalent conditions* (i)–(v) *of Theorem 7.3 hold for $1 \leqslant i \leqslant D-1$.*
(ii) *The equivalent conditions* (i)–(v) *of Theorem 7.3 hold for $i = 3$.*
(iii) *There exist $\beta, \gamma \in \mathbb{C}$ such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = \gamma$ for $1 \leqslant i \leqslant D-1$.*
(iv) *There exist $\beta, \gamma \in \mathbb{C}$ such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = \gamma$ for $1 \leqslant i \leqslant 3$.*

Moreover, when (i)–(iv) *hold, we have $\sigma_1 \neq \sigma_2$ and*

$$\beta = \frac{1 - \sigma_3}{\sigma_1 - \sigma_2} - 1, \quad \gamma = 1 - \beta\sigma_1 + \sigma_2. \tag{32}$$

Proof. (i) → (ii): Obvious.

(ii) → (iii): Recall $\sigma_i \neq 1$ for $1 \leqslant i \leqslant 3$ by Lemma 3.7. We claim $\sigma_1 \neq \sigma_2$. Suppose $\sigma_1 = \sigma_2$. By Theorem 7.3(i), equality holds in (27) for $i = 3$. So σ_3 also equals σ_2. Since $\sigma_1 = \sigma_3$ and by Lemma 3.6, we find $\sigma_1\sigma_2 = \sigma_3$. Equivalently, $\sigma_1\sigma_1 = \sigma_2$. By Lemma 3.7, this is a contradiction. We conclude $\sigma_1 \neq \sigma_2$ as desired. Now let β be as in (32).

To finish, we show $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1}$ is independent of i for $1 \leqslant i \leqslant D-1$. To do this, we pick i $(2 \leqslant i \leqslant D-1)$ and show

$$\sigma_{i-2} - \beta\sigma_{i-1} + \sigma_i = \sigma_{i-1} - \beta\sigma_i + \sigma_{i+1}. \tag{33}$$

Let X denote the vertex set of Γ. Let $x, y, w \in X$ denote any vertices with $\partial(x, y) = 3$, $\partial(x, w) = i + 1$, $\partial(y, w) = i - 2$. Let E denote the idempotent associated with

$\sigma_0, \sigma_1, \ldots, \sigma_D$ and set $m := \operatorname{rank} E$. Set

$$C := \sum_{z \in \Gamma_1^2(x,y)} E\hat{z} - \sum_{w \in \Gamma_2^1(x,y)} E\hat{w}, \quad F := E\hat{x} - E\hat{y}.$$

Using (7), we find

$$\langle \hat{w}, C \rangle = c_3 m |X|^{-1}(\sigma_i - \sigma_{i-1}), \quad \langle \hat{w}, F \rangle = m|X|^{-1}(\sigma_{i+1} - \sigma_{i-2}). \tag{34}$$

By Theorem 7.3(ii),

$$C = c_3 \frac{\sigma_1 - \sigma_2}{1 - \sigma_3} F. \tag{35}$$

Taking the inner product of each side of (35) with $\hat{w}$ using (34) and rearranging, we get (33), so we are done.

(iii)$\rightarrow$(iv): Obvious.

(iv)$\rightarrow$(ii), (iii)$\rightarrow$(i): Refer to Lemma 8.1 for the solution to the recurrence. Regardless of the value of β, equality holds in (27) for the appropriate value(s) of i. $\square$

Definition 8.3. Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let θ denote an eigenvalue of Γ and let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated cosine sequence. We say θ (or $\sigma_0, \sigma_1, \ldots, \sigma_D$) is *three-term recurrent* (or *TTR*) whenever θ is nontrivial and the equivalent conditions (i)–(iv) of Theorem 8.2 hold.

9. A cubic equation

In this section, we show a bipartite distance-regular graph can have at most three TTR eigenvalues. To do this, we examine the inequality (27) at $i = 3$.

Lemma 9.1. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency k. Let θ denote an eigenvalue of Γ and let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated cosine sequence. Setting $i = 3$ in* (27), *the left side minus the right side equals*

$$\frac{(\theta + k)(\theta - k)^2}{k^2 b_1^2 b_2^2 b_3} \tag{36}$$

times

$$(b_2 - b_3)\theta^3 + (b_2 - b_3 c_2)\theta^2 + (2b_3 b_2 - b_3 c_2 b_2 - b_2^2)\theta + b_2^2(b_3 - 1). \tag{37}$$

Proof. Repeatedly applying (10), we find

$$\sigma_0 = 1, \quad \sigma_1 = \frac{\theta}{k}, \quad \sigma_2 = \frac{\theta^2 - k}{kb_1}, \quad \sigma_3 = \frac{\theta(\theta^2 - k - c_2 b_1)}{kb_1 b_2},$$

$$\sigma_4 = \frac{\theta^4 - \theta^2(k + c_2 b_1 + c_3 b_2) + c_3 b_2 k}{kb_1 b_2 b_3}. \tag{38}$$

Let $i = 3$ in (27) and in the resulting equation let Δ denote the left side minus the right side. Eliminating $\sigma_0, \ldots, \sigma_4$ from Δ using (38) and simplifying the result using (3)–(4) and $k = c_2 + b_2$, we find Δ equals (36) times (37). □

Corollary 9.2. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$. Let θ denote a nontrivial eigenvalue of Γ. Then expression* (37) *is nonnegative.*

Proof. Let Δ denote the left side minus the right side of (27) with $i = 3$. By Theorem 7.2, Δ is nonnegative. But Lemma 9.1 indicates Δ equals (36) times (37). Since θ is nontrivial, Corollary 3.4 shows (36) is positive. So (37) is nonnegative. □

Lemma 9.3. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let θ denote a nontrivial eigenvalue of Γ. The following are equivalent:*

(i) *Expression* (37) *equals zero.*
(ii) *θ is TTR in the sense of Definition* 8.3.

Proof. (i) → (ii): By Lemma 9.1, the left side minus the right side of (27) equals zero when $i = 3$. So Theorem 7.3(i) holds for $i = 3$. Thus Theorem 8.2(ii) holds and θ is TTR.

(ii) → (i): By Definition 8.3, the equivalent conditions of Theorem 8.2 hold. In particular, Theorem 8.2(ii) holds, so the equivalent conditions of Theorem 7.3 hold with $i = 3$. Specifically, Theorem 7.3(i) holds with $i = 3$, so equality holds in (27) for $i = 3$. Let Δ denote the left side minus the right side of (27) with $i = 3$. On the one hand, $\Delta = 0$. On the other hand, Lemma 9.1 shows Δ equals (36) times (37). Since θ is nontrivial, (36) is nonzero. So (37) equals zero. □

Corollary 9.4. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. With reference to Definition* 8.3, *at most three eigenvalues of Γ are TTR.*

Proof. View (37) as a polynomial in θ. Suppose (37) is identically zero. Glancing at the constant term, we see $b_3 = 1$. Examining the θ^3 coefficient, we find $b_2 = 1$. Considering the θ^2 coefficient, we see $c_2 = 1$. Now $k = 2$, contrary to our assumption. So (37) is not identically zero.

By Lemma 9.3 and since (37) is cubic in θ, we now see Γ can have at most three TTR eigenvalues. □

10. The type of a TTR eigenvalue: AE, AO, Q

To aid our investigation of TTR eigenvalues, we divide them into three types.

Definition 10.1. Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let θ denote an eigenvalue of Γ and let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated cosine sequence. Suppose θ is TTR. We say θ (or $\sigma_0, \sigma_1, \ldots, \sigma_D$) is

(i) *type AE* if $\sigma_D = 1$ and D is even
(ii) *type AO* if $\sigma_D = 1$ and D is odd
(iii) *type Q* if $\sigma_D \neq 1$.

We note that if Γ has more than one TTR eigenvalue, they need not all be the same type.

Theorem 10.2. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let θ denote an eigenvalue of Γ. Consider the following statements:*

(i) *θ is type AE.*
(ii) *Γ is antipodal, the quotient graph $\tilde{\Gamma}$ is bipartite, θ is an eigenvalue of $\tilde{\Gamma}$, and $\tilde{\Gamma}$ is Q-polynomial with respect to θ.*

Then (i) $\rightarrow$ (ii). *Additionally, if $D \neq 6$, then* (ii) $\rightarrow$ (i).

Proof. (i) $\rightarrow$ (ii): Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ and E denote the cosine sequence and idempotent of Γ associated with θ. By Lemma 4.3, Γ is antipodal and (18) holds. Also, θ is an eigenvalue of $\tilde{\Gamma}$, with associated cosine sequence $\sigma_0, \sigma_1, \ldots, \sigma_d$, where $d = D/2$. By Lemma 4.2, the quotient $\tilde{\Gamma}$ is bipartite. We now show $\tilde{\Gamma}$ is Q-polynomial with respect to θ.

If $D < 6$ then $d = 2$, so we are done by Example 5.2. If $D = 6$ then $d = 3$. By Lemma 4.2, we see $\tilde{\Gamma}$ is not antipodal. Now we are done by Example 5.3.

So we assume $D > 6$ and apply Lemma 5.4. We see $\sigma_i \neq 1$ for $1 \leqslant i \leqslant d$ by Lemma 3.7. Let X and $\tilde{X}$ denote the vertex sets of Γ and $\tilde{\Gamma}$, respectively. Let $u, v \in \tilde{X}$ be any vertices of $\tilde{\Gamma}$ with $1 \leqslant \partial(u, v) \leqslant 3$ and set

$$\Delta := \sum_{z \in \tilde{\Gamma}^2_1(u,v)} \tilde{E}\hat{z} - \sum_{w \in \tilde{\Gamma}^1_2(u,v)} \tilde{E}\hat{w},$$

where $\tilde{E}$ is the idempotent of $\tilde{\Gamma}$ associated with θ. We must show $\Delta \in \mathrm{Span}(\tilde{E}\hat{u} - \tilde{E}\hat{v})$. Set $h := \partial(u, v)$. If $h = 1$ then $\Delta = (\theta + 1)(\tilde{E}\hat{u} - \tilde{E}\hat{v})$ by (8) since $a_1 = 0$. If $h = 2$ then $\Delta = 0$ because $a_2 = 0$. In either of these cases, $\Delta \in \mathrm{Span}(\tilde{E}\hat{u} - \tilde{E}\hat{v})$. So suppose $h = 3$.

Set $\tilde{V} := \mathbb{C}^{\tilde{X}}$. Recall ε from Definition 4.1 and extend ε to a linear transformation $e : V \rightarrow \tilde{V}$. Using (7) and (18), observe $\tilde{E}e = eE$. Choose vertices $x, y \in X$ such that $\varepsilon x = u$, $\varepsilon y = v$ and $\partial(x, y) = 3$. Consider the restrictions

$$\varepsilon|_{\Gamma^2_1(x,y)} : \Gamma^2_1(x, y) \rightarrow \tilde{\Gamma}^2_1(u, v) \quad \text{and} \quad \varepsilon|_{\Gamma^1_2(x,y)} : \Gamma^1_2(x, y) \rightarrow \tilde{\Gamma}^1_2(u, v).$$

Recall $\varepsilon z = \varepsilon w$ for vertices $z, w \in X$ only if $\partial(z, w) \in \{0, D\}$. Since $D > 2$, this means the above restrictions are injections. Since $D > 6$, we know c_3 is the same in Γ and $\tilde{\Gamma}$ by Lemma 4.2. Now we see the injections are between sets of the same size and are thus bijections.

Since θ is TTR in Γ, we can apply Theorem 7.3(ii) at $i=3$ to show there is a scalar t such that

$$\sum_{z\in\Gamma_1^2(x,y)} E\hat{z} - \sum_{w\in\Gamma_2^1(x,y)} E\hat{w} = t(E\hat{x}-E\hat{y}).$$

Applying e to both sides of this equation, we find $\Delta\in\mathrm{Span}(\tilde{E}\hat{u}-\tilde{E}\hat{v})$ and we are done.

(ii)$\rightarrow$(i): If $D<6$, this follows by Lemmas 4.2 and 4.3 and Example 17.4. For $D>6$, see [11, Theorem 2]. $\square$

Note 10.3. *In Theorem* 10.2, *if* $D=6$ *then* (ii)$\nrightarrow$(i) *since the eigenvalue* -2 *is* Q*-polynomial in the folded* 6*-cube but not type* AE *in the* 6*-cube itself.*

Theorem 10.4. *Let* Γ *denote a bipartite distance-regular graph with diameter* $D\geqslant 4$ *and valency* $k\geqslant 3$. *Let* θ *denote an eigenvalue of* Γ. *The following are equivalent*:

(i) θ *is type* AO.
(ii) Γ *is antipodal, the quotient graph* $\tilde{\Gamma}$ *is almost-bipartite*, θ *is an eigenvalue of* $\tilde{\Gamma}$, *and* $\tilde{\Gamma}$ *is* Q*-polynomial with respect to* θ.

Proof. (i)$\rightarrow$(ii): Let $\sigma_0,\sigma_1,\dots,\sigma_D$ and E denote the cosine sequence and idempotent of Γ associated with θ. By Lemma 4.3, Γ is antipodal and (18) holds. Also, θ is an eigenvalue of $\tilde{\Gamma}$, with associated cosine sequence $\sigma_0,\sigma_1,\dots,\sigma_d$, where $d=(D-1)/2$. By Lemma 4.2, the quotient $\tilde{\Gamma}$ is almost-bipartite and not antipodal. We now show $\tilde{\Gamma}$ is Q-polynomial with respect to θ.

If $D<6$ then $d=2$, so we are done by Example 5.2.

So we assume $D>6$ and apply Lemma 5.4. We see $\sigma_i\neq 1$ for $1\leqslant i\leqslant d$ by Lemma 3.7. Let X and $\tilde{X}$ denote the vertex sets of Γ and $\tilde{\Gamma}$, respectively. Let $u,v\in\tilde{X}$ be any vertices of $\tilde{\Gamma}$ with $1\leqslant\partial(u,v)\leqslant 3$ and set

$$\Delta := \sum_{z\in\tilde{\Gamma}_1^2(u,v)} \tilde{E}\hat{z} - \sum_{w\in\tilde{\Gamma}_2^1(u,v)} \tilde{E}\hat{w},$$

where $\tilde{E}$ is the idempotent of $\tilde{\Gamma}$ associated with θ. We must show $\Delta\in\mathrm{Span}(\tilde{E}\hat{u}-\tilde{E}\hat{v})$. Set $h:=\partial(u,v)$. If $h=1$ then $\Delta=(\theta+1)(\tilde{E}\hat{u}-\tilde{E}\hat{v})$ by (8) since $a_1=0$. If $h=2$ then $\Delta=0$ because $a_2=0$. In either of these cases, $\Delta\in\mathrm{Span}(\tilde{E}\hat{u}-\tilde{E}\hat{v})$. So suppose $h=3$.

Set $\tilde{V}:=\mathbb{C}^{\tilde{X}}$. Recall ε from Definition 4.1 and extend ε to a linear transformation $e: V\rightarrow\tilde{V}$. Using (7) and (18), observe $\tilde{E}e=eE$. Choose vertices $x,y\in X$ such that $\varepsilon x=u$, $\varepsilon y=v$ and $\partial(x,y)=3$. Consider the restrictions

$$\varepsilon|_{\Gamma_1^2(x,y)}:\Gamma_1^2(x,y)\rightarrow\tilde{\Gamma}_1^2(u,v) \quad\text{and}\quad \varepsilon|_{\Gamma_2^1(x,y)}:\Gamma_2^1(x,y)\rightarrow\tilde{\Gamma}_2^1(u,v).$$

Recall $\varepsilon z=\varepsilon w$ for vertices $z,w\in X$ only if $\partial(z,w)\in\{0,D\}$. Since $D>2$, this means the above restrictions are injections. Since $D>6$, we know c_3 is the same in Γ and $\tilde{\Gamma}$ by

Lemma 4.2. Now we see the injections are between sets of the same size and are thus bijections.

Since θ is TTR in Γ, we can apply Theorem 7.3(ii) at $i=3$ to show there is a scalar t such that

$$\sum_{z \in \Gamma_1^2(x,y)} E\hat{z} - \sum_{w \in \Gamma_2^1(x,y)} E\hat{w} = t(E\hat{x} - E\hat{y}).$$

Applying e to both sides of this equation, we find $\Delta \in \mathrm{Span}(\tilde{E}\hat{u} - \tilde{E}\hat{v})$ and we are done.

(ii)$\to$(i): If $D<6$, this follows by Lemmas 4.2 and 4.3 and Example 17.5. For $D>6$, see [11, Theorem 2]. □

Theorem 10.5. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let θ denote an eigenvalue of Γ. The following are equivalent*:

(i) *θ is type Q.*
(ii) *Γ is Q-polynomial with respect to θ.*

Proof. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ and E denote the cosine sequence and idempotent associated with θ.

(i)$\to$(ii): By Definition 10.1 and Lemma 3.7, $\sigma_i \neq 1$ for $1 \leqslant i \leqslant D$. We have both Theorem 7.3(ii) with $i=3$ and Theorem 7.3(iv) with $i=1$. Since Γ is bipartite, this combination implies (20). Now we are done by Lemma 5.4.

(ii)$\to$(i): By Lemma 5.4, $\sigma_i \neq 1$ for $1 \leqslant i \leqslant D$. Also, Theorem 7.3(iii) holds for $i=3$, so we are done. □

11. TTR cosine sequence parametrization

In due course, we will treat the type AE, AO and Q cosine sequences separately. In this section, we describe some behavior they have in common.

Lemma 11.1. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a cosine sequence of Γ. Suppose $\sigma_0, \sigma_1, \ldots, \sigma_D$ is TTR and let β be as in* (32).

(i) *Suppose $\beta \notin \{-2, 2\}$. Then there exist $q, s^*, f \in \mathbb{C}$ with $qf \neq 0$ such that*

$$\sigma_i = 1 + f\frac{(1-q^i)(1-s^*q^{i+1})}{q^i} \quad (0 \leqslant i \leqslant D). \tag{39}$$

(ii) *Suppose $\beta = 2$. Then exactly one of* (a) *and* (b) *holds.*
 (a) *There exist $s^*, f \in \mathbb{C}$ with $f \neq 0$ such that*

$$\sigma_i = 1 + fi(i+1+s^*) \quad (0 \leqslant i \leqslant D). \tag{40}$$

(b) *There exists* $s^* \in \mathbb{C}$ *with* $s^* \neq 0$ *such that*

$$\sigma_i = 1 + s^* i \quad (0 \leqslant i \leqslant D). \tag{41}$$

(iii) *Suppose* $\beta = -2$. *Then there exist* $s^*, f \in \mathbb{C}$ *with* $f \neq 0$ *such that*

$$\sigma_i = 1 + f(s^* - 1 + (1 - s^* + 2i)(-1)^i) \quad (0 \leqslant i \leqslant D). \tag{42}$$

Proof. (i): By Definition 8.3, the recurrence in Theorem 8.2(iii) holds. Referring to Lemma 8.1, there exist $a, b, c, q \in \mathbb{C}$ with $q \neq 0$ such that $\sigma_i = a + bq^i + cq^{-i}$ for $0 \leqslant i \leqslant D$. Since $\sigma_0 = 1$, we have $a = 1 - b - c$. Since $\sigma_2 \neq 1$ by Lemma 3.7, both b and c cannot be zero. Replacing q by $1/q$ if necessary, we may assume $c \neq 0$. Setting $f := c$ and $s^* := b/(cq)$, we obtain (39).

(ii): Arguing as in (i), there exist $a, b, c \in \mathbb{C}$ such that $\sigma_i = a + bi + ci^2$ for $0 \leqslant i \leqslant D$. The sequence a, b, c is unique since $D \geqslant 2$. Since $\sigma_0 = 1$, we have $a = 1$. If $c \neq 0$ we set $f := c$ and $s^* := -1 + b/c$ and obtain (40). If $c = 0$ we set $s^* := b$ and obtain (41).

(iii): Arguing as in (i), there exist $a, b, c \in \mathbb{C}$ such that $\sigma_i = a + b(-1)^i + ci(-1)^i$ for $0 \leqslant i \leqslant D$. Note $c \neq 0$ since $\sigma_2 \neq 1$ by Lemma 3.7. Since $\sigma_0 = 1$, we have $a = 1 - b$. Set $f := c/2$ and $s^* := (c - 2b)/c$ to obtain (42). □

To deal with the case $\beta \notin \{-2, 2\}$ we will need the following result.

Lemma 11.2. *Let* Γ *denote a bipartite distance-regular graph with diameter* $D \geqslant 4$ *and valency* $k \geqslant 3$. *Let* $\sigma_0, \sigma_1, \ldots, \sigma_D$ *denote a TTR cosine sequence of* Γ *with* $\beta \notin \{-2, 2\}$ *and let* q, s^*, f *be as in Lemma* 11.1(i). *For* $0 \leqslant i, j \leqslant D$,

$$\sigma_i = \sigma_j \quad \text{if and only if} \quad q^{i-j} = 1 \quad \text{or} \quad s^* q^{i+j+1} = 1. \tag{43}$$

Moreover,

$$q^i \neq 1 \quad (1 \leqslant i \leqslant D - 1), \tag{44}$$

$$s^* q^i \neq 1 \quad (2 \leqslant i \leqslant D). \tag{45}$$

Proof. By (39), $\sigma_i - \sigma_j = f(1 - q^{i-j})(1 - s^* q^{i+j+1})/q^i$. Now (43) follows since $f \neq 0$. By (43), Lemma 3.7 and since $\sigma_0, \sigma_1, \ldots, \sigma_D$ is nontrivial, we get (44)–(45). □

12. Type AE

We now move on to a type-specific treatment of TTR eigenvalues, beginning with type AE from Definition 10.1. Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$. Suppose θ is a type AE eigenvalue of Γ. We show θ is the third-largest eigenvalue of Γ. Moreover, if $D > 4$ then Γ is the D-cube.

Theorem 12.1. *Let Γ denote a bipartite distance-regular graph with diameter $D=4$ and valency $k\geqslant 3$. Let $\theta_0>\theta_1>\cdots>\theta_D$ denote the eigenvalues of Γ. The following are equivalent*:

(i) *θ is a type AE eigenvalue of Γ.*
(ii) *$\theta=\theta_2$ and Γ is antipodal.*

Proof. (i)$\rightarrow$(ii): Since θ is nontrivial, we are done by Lemma 4.3. (ii)$\rightarrow$(i): See Example 17.4. $\square$

Theorem 12.2. *Let Γ denote a bipartite distance-regular graph with diameter $D>4$ and valency $k\geqslant 3$. Let $\theta_0>\theta_1>\cdots>\theta_D$ denote the eigenvalues of Γ. The following are equivalent*:

(i) *θ is a type AE eigenvalue of Γ.*
(ii) *$\theta=\theta_2$ and Γ is the D-cube.*

Proof. (i)$\rightarrow$(ii): Let $\sigma_0,\sigma_1,\dots,\sigma_D$ denote the cosine sequence associated with θ. Recall θ is nontrivial and Γ is antipodal with D even. Set $d:=D/2$.

First suppose $\beta\notin\{-2,2\}$. Let q,s^*,f be as in (39). Observe $\sigma_1=\sigma_{D-1}$ by (18), so by Lemma 11.2,

$$s^*=q^{-D-1}. \tag{46}$$

Observe $\sigma_{d-1}=\sigma_{d+1}$ by (18), so by Lemma 3.6 we have $\sigma_{d-1}=\sigma_1\sigma_d$. Combining this, (39) and (46),

$$\begin{aligned}0&=\sigma_1\sigma_d-\sigma_{d-1}\\&=\frac{f(1-q^d)(1-q)(f(1-q^d)(q-q^D)-q^D(q+q^d))}{q^{2D+1}}.\end{aligned}\tag{47}$$

By Lemma 11.2, $f(1-q^d)(1-q)\neq 0$. Setting the rightmost factor in (47) equal to zero and solving, we get

$$f=\frac{q^D(q+q^d)}{(1-q^d)(q-q^D)}. \tag{48}$$

Eliminating s^*,f from (39) using (46) and (48),

$$\sigma_i=1+\frac{q^D(q+q^d)(1-q^i)(1-q^{i-D})}{(1-q^d)(q-q^D)q^i}\quad(0\leqslant i\leqslant D).$$

By Lemma 3.7, we have $q\neq 0$, $q^{d-1}\neq -1$ and $q^i\neq 1$ for $1\leqslant i\leqslant D-1$.

Let Q be a scalar such that $q=Q^2$. We want to show (21)–(26). Lines (21) and (26) are clear. We get (23) and (24) from (14). If $q^d=-1$ then $k=2$, a contradiction, so $q^d\neq -1$. Since $q^d\neq 1$, we get $q^D\neq 1$. If $1\leqslant i\leqslant D-1$ and $i\neq d$ then

$$\sigma_{i-1}-\sigma_{i+1}=\frac{(1-q^2)(q+q^d)(q^{2i}-q^D)}{q^{i+1}(1-q^d)(q-q^D)}\neq 0,$$

so the conditions of Lemma 3.6 do not hold and we get (22) and (25) from (15)–(16).

Combining (17) and (3), we find $c_d = k/r$ and $b_d = k(r-1)/r$. Eliminating k using (24), we find

$$c_d = \frac{(1-q^d)(q+q^d)}{rq^d(1-q)}, \tag{49}$$

$$b_d = \frac{(1-q^d)(q+q^d)(r-1)}{rq^d(1-q)}. \tag{50}$$

By Curtin [5, Theorem 12] and Brouwer et al. [2, Lemma 4.1.7],

$$(b_{i-1}-1)(c_{i+1}-1) \geqslant (c_2-1)\frac{b_i(c_{i+1}-1)+c_i(b_{i-1}-1)}{c_2} \quad (1 \leqslant i \leqslant D-1). \tag{51}$$

Setting $i = d+1$ (resp. $i = d-1$) in (51) and applying (22)–(25) and (49)–(50), we find

$$\frac{q^3(1-q^D)(1+q^{d-1})(1-q^{d+1})(r-2)}{q^D(1-q^2)(1-q^4)r} \tag{52}$$

is nonnegative (resp. nonpositive). Since (52) is neither positive nor negative, it equals zero and thus $r = 2$. Now we have (21)–(26), so Γ is 2-homogeneous by Lemma 6.2. Since $D > 4$ is even, Γ is the D-cube by Lemma 6.3. But by Example 17.1, this contradicts $\beta \notin \{-2, 2\}$. So $\beta \in \{-2, 2\}$.

We cannot have $\beta = -2$ since (42) is inconsistent with $\sigma_D = 1$. So $\beta = 2$. Since (41) is inconsistent with $\sigma_D = 1$, we find there exist s^*, f such that (40) holds. Arguing as in the case $\beta \notin \{-2, 2\}$, we have $\sigma_1 = \sigma_{D-1}$ and $\sigma_{d-1} = \sigma_1\sigma_d$. Combining these facts with (40), we find $s^* = -1 - D$ and $f = 2/(d - 2d^2)$. Now by Lemmas 3.8 and 4.2, $c_i = i$ for $i \neq d$ and $c_d = D/r$. Note $b_i = D - c_i$ for $0 \leqslant i \leqslant D$. Setting $i = d+1$ (resp. $i = d-1$) in (51), we find

$$\frac{d(d+1)(r-2)}{2r} \tag{53}$$

is nonnegative (resp. nonpositive). So (53) is zero, making $r = 2$ and thus $c_d = d$. By Example 17.1, Γ is the D-cube and $\theta = \theta_2$.

(ii) $\rightarrow$ (i): Immediate from Example 17.1. $\square$

We mention that Theorem 12.2 provides a simple proof of the following result of Caughman.

Corollary 12.3 (Caughman [4]). *Let Γ denote a Q-polynomial bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. If Γ has a distance-regular antipodal cover then Γ is the folded $2D$-cube.*

Proof. Suppose Γ is Q-polynomial with respect to the eigenvalue θ, and suppose Γ' is a distance-regular antipodal cover of Γ. By Lemma 4.2 and since Γ is bipartite, Γ' is bipartite with diameter $2D$. By Theorem 10.2, θ is type AE in Γ'. Now Γ' is the $2D$-cube by Theorem 12.2. $\square$

13. Type AO

In this section we study the type AO cosine sequences of Definition 10.1. We restrict our attention to the case $\beta \notin \{-2, 2\}$, postponing our treatment of the remaining cases until Section 15.

Lemma 13.1. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a type AO cosine sequence with $\beta \notin \{-2, 2\}$. Then there exists $(q, s) \in \mathbb{C}^2$ such that*

$$\sigma_i = 1 + \frac{q^D(1 - sq^2)}{(q^D - q)(1 + sq)} \cdot \frac{(1 - q^i)(1 - q^{i-D})}{q^i} \quad (0 \leqslant i \leqslant D) \tag{54}$$

and the denominator is nonzero. We call (q, s) a **parameter pair** *associated with $\sigma_0, \sigma_1, \ldots, \sigma_D$.*

Proof. Let q, s^*, f be as in (39). Since $\sigma_D = 1$, we have (18), so $\sigma_1 = \sigma_{D-1}$. By this and Lemma 11.2,

$$s^* = q^{-D-1}. \tag{55}$$

Eliminating s^* in (39) using (55), we find

$$\sigma_i = 1 + f\,\frac{(1 - q^i)(1 - q^{i-D})}{q^i} \quad (0 \leqslant i \leqslant D). \tag{56}$$

Replacing q and f by $1/q$ and f/q^D if necessary, we may assume $f \neq q^{D+1}/(q - q^D)$. So we set

$$s := \frac{q^{D-1} - fq^{D-1} + f}{q^{D+1} + f(q^D - q)}. \tag{57}$$

We wish to solve (57) for f. To do this, we make sure certain quantities are nonzero. Observe

$$(1 + sq)(q^{D+1} + f(q^D - q)) = q^D(1 + q)$$

is nonzero by Lemma 11.2, so $1 + sq \neq 0$. Also by Lemma 11.2, $q^D - q \neq 0$. Solving (57) for f, we find

$$f = \frac{q^D(1 - sq^2)}{(q^D - q)(1 + sq)}. \tag{58}$$

Eliminating f from (56) using (58), we get (54). $\square$

We want to consider the extent to which the parameter pair associated with a given cosine sequence is unique. To this end, we make the following definition.

Definition 13.2. We define a binary relation $\approx$ on $\mathbb{C}^2$ by

$$(a,b)\approx(c,d) \text{ if and only if } (a,b)=(c,d) \text{ or } (ac,bd)=(1,1).$$

We observe $\approx$ is an equivalence relation.

Lemma 13.3. *Let Γ denote a bipartite distance-regular graph with diameter $D\geqslant 4$ and valency $k\geqslant 3$. Let $\sigma_0,\sigma_1,\ldots,\sigma_D$ denote a type AO cosine sequence with $\beta\notin\{-2,2\}$. Then the associated parameter pair is unique up to the relation $\approx$ from Definition* 13.2.

Proof. By (54), $q(\sigma_1-q)(1+sq)=1-q^2$, which is nonzero by Lemma 11.2. We solve for s to get

$$s=\frac{1-\sigma_1 q}{q^2(\sigma_1-q)}. \tag{59}$$

Suppose q_1,s_1 and q_2,s_2 both satisfy (54). Then $q_1+q_1^{-1}=\beta=q_2+q_2^{-1}$ in view of (32). So either $q_2=q_1$ or $q_2=1/q_1$. We find using (59) that in the first case, $s_2=s_1$ and in the second, $s_2=1/s_1$. If (q,s) is a parameter pair with $s\neq 0$, observe $(1/q,1/s)$ is also a parameter pair. $\square$

Theorem 13.4. *Let Γ denote a distance-regular graph with diameter $D\geqslant 4$ and valency $k\geqslant 3$. Let $(q,s)\in\mathbb{C}^2$ be given. The following* (i)–(ii) *are equivalent:*

(i) *Γ is bipartite and (q,s) is a parameter pair associated with a type AO cosine sequence with $\beta\notin\{-2,2\}$.*

(ii) *There exists $h\in\mathbb{C}$ such that the intersection numbers of Γ are given by*

$$c_0=0, \tag{60}$$

$$c_i=h\frac{(q^i-1)(1+sq^{D+1-i})}{q^i(1-q^{D-2i})} \quad (1\leqslant i\leqslant D-1), \tag{61}$$

$$c_D=h(1+sq), \tag{62}$$

$$b_0=h(1+sq), \tag{63}$$

$$b_i=h\frac{(1-q^{D-i})(1+sq^{i+1})}{q^i(1-q^{D-2i})} \quad (1\leqslant i\leqslant D-1), \tag{64}$$

$$b_D=0 \tag{65}$$

and the denominators are nonzero.

Suppose (i)–(ii) *hold. Then*

$$h = \frac{q^D - q^2}{q(1-q)(1+sq^D)}, \tag{66}$$

$$q \neq 0, \quad q^i \neq 1 \quad (1 \leqslant i \leqslant D-1), \quad sq^i \neq 1 \quad (2 \leqslant i \leqslant D-1),$$
$$sq^i \neq -1 \quad (1 \leqslant i \leqslant D). \tag{67}$$

Note Γ is an antipodal 2*-cover. Let θ denote the eigenvalue associated with the cosine sequence of* (i) *and observe the quotient graph $\tilde{\Gamma}$ is Q-polynomial with respect to θ. The eigenvalues of $\tilde{\Gamma}$, in the associated Q-polynomial order, are*

$$\tilde{\theta}_i = h\frac{1+sq^{2i+1}}{q^i} \quad (0 \leqslant i \leqslant d), \tag{68}$$

where $d = (D-1)/2$. The eigenvalues of Γ are these and their opposites.

Proof. (i)$\rightarrow$(ii): Lines (60) and (65) are clear. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the cosine sequence associated with (q, s) and observe this sequence is given by (54). Since $\sigma_i \neq 1$ for $1 \leqslant i \leqslant D-1$ by Lemma 3.7, we find $q \neq 0$, $sq \neq -1$, $sq^2 \neq 1$ and $q^i \neq 1$ for $1 \leqslant i \leqslant D-1$. For $1 \leqslant i \leqslant D-1$,

$$\sigma_{i+1} - \sigma_{i-1} = \frac{(1-q^2)(1-sq^2)}{q(q^D-q)(1+sq)} \cdot \frac{q^D - q^{2i}}{q^i}.$$

This is nonzero by the above inequalities, so the conditions of Lemma 3.6 do not hold. Applying Lemma 3.8, we get (61)–(64) with h as in (66).

(ii)$\rightarrow$(i): Throughout, evaluate the intersection numbers using (60)–(65). For $0 \leqslant i \leqslant D$ we find $c_i + b_i = k$, so $a_i = 0$ by (3). Thus Γ is bipartite. We find $hq \neq 0$, $q^i \neq 1$ for $1 \leqslant i \leqslant D-1$ and $sq^i \neq -1$ for $1 \leqslant i \leqslant D$.

We claim $sq^2 \neq 1$. Let L be as in (9). Define

$$\tau_i := \frac{(1-q^i)(1-q^{i-D})}{q^i} \quad (0 \leqslant i \leqslant D)$$

and let $T = (\tau_0, \tau_1, \ldots, \tau_D)^t$. We find $(L - kI)T = (v_0, v_1, \ldots, v_D)^t$, where

$$v_i = \frac{h(1-q)}{q^{D+1}}((1-sq^2)(q^i + q^{D-i}) - (1+q)(1-sq^2 \cdot q^{D-1})) \quad (0 \leqslant i \leqslant D).$$

Suppose $sq^2 = 1$. Then we see $(L-kI)^2T = 0$ but $(L-kI)T \neq 0$, contradicting Corollary 3.4(i).

Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ be as in (54). We see $\sigma_0 = 1$ and (11)–(12) hold. By Lemma 3.3, $\sigma_0, \sigma_1, \ldots, \sigma_D$ is a cosine sequence for Γ and $\theta = k\sigma_1$ is the associated eigenvalue. Observe $\sigma_2 \neq 1$, so θ is nontrivial by Lemma 3.7. We find Theorem 8.2(iv) holds with $\beta = q + q^{-1}$. Note $\sigma_D = 1$. We have now shown θ is type AO. Observe $\beta \notin \{-2, 2\}$ since $q^2 \neq 1$. We see (q, s) is a parameter pair for $\sigma_0, \sigma_1, \ldots, \sigma_D$ by the statement at the end of Lemma 13.1.

Suppose (i)–(ii) hold. We see Γ is an antipodal 2-cover by Lemma 4.2 and $\tilde{\Gamma}$ is Q-polynomial with respect to θ by Theorem 10.4. Using (54) and (60)–(65) in [2, Corollary 8.3.3] (with $i = l$), we get (68). These and their opposites are the eigenvalues of Γ by Lemma 4.3 and Corollary 3.4. We get $sq^i \neq 1$ for $2 \leqslant i \leqslant D-1$ since the eigenvalues of Γ are distinct. $\square$

Lemma 13.5. *With the notation of Theorem* 13.4, *suppose* (i)–(ii) *hold. Let* $\theta_0 > \theta_1 > \cdots > \theta_D$ *denote the eigenvalues of* Γ. *Then* $\tilde{\theta}_1, \tilde{\theta}_d$ *is a permutation of* θ_e, θ_{D-1}, *where* e *is the even integer in* $\{d, d+1\}$.

Proof. If $D = 5$ then this is immediate from Lemma 4.3, so suppose $D > 5$. We set $\{\theta, \theta'\} = \{\tilde{\theta}_1, \tilde{\theta}_d\}$ in [6, (35)] and find equality holds. By Lemma 6.3 and Example 17.1, we see Γ is not 2-homogeneous. Now the result follows by MacLean [6, Theorem 4.3, Lemma 4.4] and Lemma 4.3. $\square$

Note 13.6. *With the notation of Theorem* 13.4, *suppose* (i)–(ii) *hold and suppose* $d \geqslant 3$. *In the notation of Bannai and Ito, the eigenvalue order in* (68) *is type I with* $(q, r_1, r_2, s, s^*) = (q, -s, -q^{-d-1}, s, q^{-D-1})$.

Note 13.7. *We know of no example of a bipartite distance-regular graph with diameter* $D > 5$ *that contains a type AO eigenvalue with* $\beta \notin \{-2, 2\}$.

14. Type Q

We now examine type Q from Definition 10.1. Let Γ denote a bipartite distance-regular graph. If Γ is antipodal and contains a type Q eigenvalue then Γ is 2-homogeneous by Theorem 10.5 and Lemma 6.2. Since 2-homogeneous graphs and their type Q eigenvalues are discussed in Sections 6 and 17, in this section we only consider type Q eigenvalues of nonantipodal graphs. As in Section 13, we restrict our attention here to the case $\beta \notin \{-2, 2\}$.

We acknowledge that the formulae we derive here are special cases of those in [1]. We include them for completeness and because of the simplicity of their form and proof.

Lemma 14.1. *Let* Γ *denote a bipartite nonantipodal distance-regular graph with diameter* $D \geqslant 4$ *and valency* $k \geqslant 3$. *Let* $\sigma_0, \sigma_1, \ldots, \sigma_D$ *denote a type Q cosine sequence with* $\beta \notin \{-2, 2\}$. *Then there exists* $(q, s^*) \in \mathbb{C}^2$ *such that*

$$\sigma_i = 1 + \frac{q + q^D}{(q^D - 1)(1 - s^* q^2)} \cdot \frac{(1 - q^i)(1 - s^* q^{i+1})}{q^i} \quad (0 \leqslant i \leqslant D) \tag{69}$$

and the denominator is nonzero. We call (q, s^*) *a* **parameter pair** *associated with* $\sigma_0, \sigma_1, \ldots, \sigma_D$.

Proof. Let q, s^*, f be as in (39). Using (11) and (39) and simplifying, we obtain

$$0 = \sigma_{D-1} - \sigma_1\sigma_D$$
$$= \frac{f(1-q)(1-s^*q^{D+1})(f(q^D-1)(1-s^*q^2)-(q+q^D))}{q^{D+1}}. \tag{70}$$

We know $f \neq 0$. By Lemma 11.2 and since $\sigma_D \neq 1$, we find $q^D - 1$, $1 - s^*q^2$, $1 - q$ and $1 - s^*q^{D+1}$ are nonzero. Setting the last factor of the numerator in (70) equal to zero and solving for f, we obtain

$$f = \frac{q + q^D}{(q^D - 1)(1 - s^*q^2)}. \tag{71}$$

Eliminating f from (39) using (71) we get (69), as desired. $\square$

Lemma 14.2. *Let Γ denote a bipartite nonantipodal distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote a type Q cosine sequence with $\beta \notin \{-2, 2\}$. Then the associated parameter pair is unique up to the relation $\approx$ from Definition* 13.2.

Proof. By (69),

$$\sigma_D + q^{D-1} = \frac{(q - q^D)(q + q^D)}{q^{D+1}(s^*q^2 - 1)}, \tag{72}$$

which is nonzero since otherwise $\sigma_{D-1} = 1$. So we can solve (72) for s^* to get

$$s^* = \frac{1 + \sigma_D q^{D-1}}{q^{D+1}(\sigma_D + q^{D-1})}. \tag{73}$$

Suppose q_1, s_1^* and q_2, s_2^* both satisfy (69). Then $q_1 + q_1^{-1} = \beta = q_2 + q_2^{-1}$ in view of (32). So either $q_2 = q_1$ or $q_2 = 1/q_1$. We find using (73) that in the first case, $s_2^* = s_1^*$ and in the second, $s_2^* = 1/s_1^*$. If (q, s^*) is a parameter pair with $s^* \neq 0$, observe $(1/q, 1/s^*)$ is also a parameter pair. $\square$

Theorem 14.3. *Let Γ denote a nonantipodal distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $(q, s^*) \in \mathbb{C}^2$ be given. The following* (i)–(ii) *are equivalent*:

(i) *Γ is bipartite and (q, s^*) is a parameter pair associated with a type Q cosine sequence with $\beta \notin \{-2, 2\}$.*
(ii) *There exists $h \in \mathbb{C}$ such that the intersection numbers of Γ are given by*

$$c_0 = 0, \tag{74}$$

$$c_i = h\frac{(1 - q^i)(1 - s^*q^{D+i+1})}{1 - s^*q^{2i+1}} \quad (1 \leqslant i \leqslant D - 1), \tag{75}$$

$$c_D = h(1 - q^D), \tag{76}$$

$$b_0 = h(1 - q^D), \tag{77}$$

$$b_i = h\frac{(q^i - q^D)(1 - s^*q^{i+1})}{1 - s^*q^{2i+1}} \quad (1 \leqslant i \leqslant D-1), \tag{78}$$

$$b_D = 0 \tag{79}$$

and the denominators are nonzero.

Suppose (i)–(ii) *hold. Then*

$$h = \frac{1 - s^*q^3}{(1-q)(1 - s^*q^{D+2})}, \tag{80}$$

$$q \neq 0, \quad q^{D-1} \neq -1, \quad q^i \neq 1 \ (1 \leqslant i \leqslant D), \quad s^*q^i \neq 1 \ (2 \leqslant i \leqslant 2D),$$

$$s^*q^{D+1} \neq -1.$$

The eigenvalues of Γ, in the Q-polynomial order associated with the cosine sequence in (i), *are*

$$\theta_i = h(q^i - q^{D-i}) \quad (0 \leqslant i \leqslant D). \tag{81}$$

Proof. (i)$\to$(ii): Lines (74) and (79) are clear. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the cosine sequence associated with (q, s^*) and observe this sequence is given by (69). Since $\sigma_i \neq 1$ for $1 \leqslant i \leqslant D$ by Lemma 3.7, we find $q \neq 0$, $q^{D-1} \neq -1$, $q^i \neq 1$ for $1 \leqslant i \leqslant D$ and $s^*q^i \neq 1$ for $2 \leqslant i \leqslant D+1$. For $1 \leqslant i \leqslant D-1$,

$$\sigma_{i-1} - \sigma_1\sigma_i = \frac{(1-q^2)(q+q^D)(q^i-q^D)(1-s^*q^{i+1})}{(1-q^D)^2(1-s^*q^2)q^{i+1}}.$$

This is nonzero by the above inequalities, so the conditions of Lemma 3.6 do not hold. Applying Lemma 3.8, we get (75)–(78) with h as in (80).

(ii)$\to$(i): Throughout, evaluate the intersection numbers using (74)–(79). For $0 \leqslant i \leqslant D$ we find $c_i + b_i = k$, so $a_i = 0$ by (3). Thus Γ is bipartite. We find $hq \neq 0$, $q^i \neq 1$ for $1 \leqslant i \leqslant D$, and $s^*q^i \neq 1$ for $2 \leqslant i \leqslant D$ and $D+2 \leqslant i \leqslant 2D$. Also, $s^*q^{D+1} \neq \pm 1$ since otherwise $c_i = b_{D-i}$ for $0 \leqslant i \leqslant D$, making Γ antipodal, a contradiction.

We claim $q^{D-1} \neq -1$. Let L be as in (9). Define

$$\tau_i := \frac{(1-q^i)(1-s^*q^{i+1})}{q^i} \quad (0 \leqslant i \leqslant D)$$

and let $T = (\tau_0, \tau_1, \ldots, \tau_D)^t$. We find $(L - kI)T = (v_0, v_1, \ldots, v_D)^t$, where

$$v_i = \frac{h(1-q)}{q}((1+q)(1+s^*q^{D+1}) - (1+q^{D-1})(q^{1-i} + s^*q^{i+2})) \quad (0 \leqslant i \leqslant D).$$

Suppose $q^{D-1} = -1$. Then we see $(L-kI)^2T = 0$ but $(L-kI)T \neq 0$, contradicting Corollary 3.4(i).

Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ be as in (69). We see $\sigma_0 = 1$ and (11)–(12) hold. By Lemma 3.3, $\sigma_0, \sigma_1, \ldots, \sigma_D$ is a cosine sequence for Γ and $\theta = k\sigma_1$ is the associated eigenvalue. Observe $\sigma_2 \neq 1$, so θ is nontrivial by Lemma 3.7. We find Theorem 8.2(iv) holds with

$\beta = q + q^{-1}$. Note $\sigma_D \neq 1$. We have now shown θ is type Q. Observe $\beta \notin \{-2, 2\}$ since $q^2 \neq 1$. We see (q, s^*) is a parameter pair for $\sigma_0, \sigma_1, \ldots, \sigma_D$ by the statement at the end of Lemma 14.1.

Suppose (i)–(ii) hold. We see Γ is Q-polynomial with respect to $\sigma_0, \sigma_1, \ldots, \sigma_D$ by Theorem 10.5. Using (69) and (74)–(79) in [2, Corollary 8.3.3] (with $i = l$), we get (81). □

Lemma 14.4. *With the notation of Theorem* 14.3, *suppose* (i)–(ii) *hold. Then* θ_1 *is either the second-largest or second-smallest eigenvalue of* Γ.

Proof. Immediate from Lemma 5.5 and Theorem 10.5. □

Note 14.5. *With the notation of Theorem* 14.3, *suppose* (i)–(ii) *hold. In the notation of Bannai and Ito, the eigenvalue order in* (81) *is type I with* $(q, r_1, r_2, s, s^*) = (q, \sqrt{s^*}, -\sqrt{s^*}, -q^{-D-1}, s^*)$.

15. Types AO and Q with $\beta \in \{-2, 2\}$

In Sections 13 and 14, we treated type AO and Q eigenvalues with $\beta \notin \{-2, 2\}$. In this section we consider type AO and Q eigenvalues with $\beta \in \{-2, 2\}$. We classify the graphs containing such eigenvalues and find the location of those eigenvalues in the graphs' spectra.

Lemma 15.1. *Let* Γ *denote a bipartite antipodal distance-regular graph with diameter* 5 *and valency* $k \geqslant 3$. *Let* $\theta_0 > \theta_1 > \cdots > \theta_5$ *denote the eigenvalues of* Γ. *Let* θ *denote an eigenvalue of* Γ *and suppose* θ *is type AO with* $\beta = 2$. *Then* $\theta = \theta_2$. *Moreover*, Γ *is either the double Hoffman–Singleton graph or the* 5-*cube*.

Proof. We examine Example 17.5. Since θ is type AO with $\beta = 2$, we find $\theta = \theta_2$ and $\theta_1 = \beta + 1 = 3$. Observe $\theta_2 = \theta_1 - c_2$ is a positive integer less than θ_1, so $\theta_2 \in \{2, 1\}$. With $\theta_2 = 2$ (resp. 1), we find Γ has the intersection numbers of the double Hoffman–Singleton graph (resp. 5-cube). Now Γ is the double Hoffman–Singleton graph (resp. 5-cube) by Brouwer et al. [2, p. 418]. □

Theorem 15.2. *Let* Γ *denote a bipartite distance-regular graph with diameter* $D > 5$ *and valency* $k \geqslant 3$. *Let* $\theta_0 > \theta_1 > \cdots > \theta_D$ *denote the eigenvalues of* Γ. *Let* θ *denote an eigenvalue of* Γ *and suppose* θ *is type AO with* $\beta = 2$. *Then* $\theta = \theta_2$. *Moreover*, Γ *is the* D-*cube*.

Proof. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the cosine sequence associated with θ. By Lemma 11.1, either (40) or (41) holds. We cannot have (41) since $\sigma_D = 1$ and θ is nontrivial. So we have (40). Since $\sigma_D = 1$, we find $s^* = -D - 1$. By Lemmas 3.2 and 4.3 and the fact that $\sigma_0, \sigma_1, \ldots, \sigma_D$ has at most two sign changes, $\theta = \theta_2$. Also, Γ is antipodal and $\sigma_0, \sigma_1, \ldots, \sigma_d$ (with $d = (D-1)/2$) is a cosine sequence of the quotient $\tilde{\Gamma}$.

Combining (40), Lemma 3.8 and [2, Corollary 8.3.3] (with $i = l$), we see $\tilde{\Gamma}$ is type IIB. Now $\tilde{\Gamma}$ is the folded D-cube by [10, Theorem 2.3] (with $t = D$ and $x = 1/fd$), so Γ is the D-cube by Example 17.1. □

Theorem 15.3. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ. Let θ denote an eigenvalue of Γ and let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the associated cosine sequence. Suppose θ is type Q with $\beta = 2$. Then $\theta = \theta_1$. Moreover, exactly one of the following holds:*

(i) *We have* (40) *and Γ is the folded $2D$-cube.*
(ii) *We have* (41) *and Γ is the D-cube.*

Proof. By Lemma 11.1, either (40) or (41) holds. In either case, $\sigma_0, \sigma_1, \ldots, \sigma_D$ has at most two sign changes. By Theorem 10.5, Γ is Q-polynomial with respect to θ, so $\theta = \theta_1$ by Lemmas 3.2 and 5.5.

Suppose (40) holds. Using this in (11), we find $s^* = -2(1 + fD)/fD$. Now combining (40), Lemma 3.8 and [2, Corollary 8.3.3] (with $i = l$), we see Γ is type IIB. By Terwilliger [10, Theorem 2.3] (with $t = (fD + 2)/fD$ and $x = 1/fD$) and Example 17.2, Γ is the folded $2D$-cube.

Suppose (41) holds. By (32), $\gamma = 0$, so Γ is 2-homogeneous by Curtin [5, Lemma 27]. Since $s^* \neq 0$, we have $\sigma_1 \neq \sigma_3$, so we can combine (41) and Lemma 3.8 to get $c_2 = 2$. By Lemma 6.3, Γ is the D-cube. □

Theorem 15.4. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ. Let θ denote an eigenvalue of Γ and suppose θ is type AO with $\beta = -2$. Then $\theta = \theta_{D-1}$. Moreover, Γ is either the doubled Odd graph $2.O_k$ or the D-cube.*

Proof. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ denote the cosine sequence associated with θ. By Lemma 11.1, (42) holds. Since $\sigma_D = 1$ and D is odd, $s^* = 1 + D$. It is clear the conditions of Lemma 3.6 do not hold for any i, so we can use Lemma 3.8 to get the intersection numbers of Γ. We find

$$c_i = \begin{cases} c_2 \frac{i}{2} & \text{if } i \text{ is even}, \\ c_2 \frac{i-1}{2} + 1 & \text{if } i \text{ is odd}, \end{cases} \qquad 0 \leqslant i \leqslant D.$$

Observe $c_3 = c_2 + 1$. Combining this and [8, Theorem 1] (with $x = y = 3$), we find $c_2 = 1$ or $c_2 = 2$.

If $c_2 = 1$ (resp. $c_2 = 2$) then Γ has the intersection numbers of the doubled Odd graph (resp. the D-cube). By Example 17.3 (resp. 17.1), Γ is the doubled Odd graph (resp. the D-cube) and $\theta = \theta_{D-1}$. □

Theorem 15.5. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ. Let θ denote an* *eigenvalue of Γ and suppose θ is type Q with $\beta = -2$. Then $\theta = \theta_{D-1}$. Moreover, D is even and Γ is the D-cube.*

Proof. By Theorem 10.5, Γ is Q-polynomial with respect to θ. We observe θ is type III. By Terwilliger [9, Theorem 2], Γ is the D-cube and D is even. Now $\theta = \theta_{D-1}$ by Example 17.1. □

16. Multiple TTR eigenvalues

In this section we investigate the possibilities for a bipartite distance-regular graph to have more than one TTR eigenvalue. Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. We show if Γ has more than one TTR eigenvalue, then Γ is an antipodal 2-cover. Moreover, if $D \geqslant 6$ then Γ is the D-cube.

Theorem 16.1. *Let Γ denote a bipartite distance-regular graph with diameter $D \in \{4, 5\}$ and valency $k \geqslant 3$. Suppose Γ has more than one TTR eigenvalue. Then Γ is an antipodal 2-cover.*

Proof. Observe at least one of the following two cases occurs. (i) Γ has an eigenvalue that is type AE or AO. (ii) Γ has two type Q eigenvalues.

Suppose (i). Then Γ is antipodal by Lemma 4.3. We are done by Lemma 4.2 or Example 17.4.

Next suppose (ii). By Lemma 5.5 and Theorem 10.5, the type Q eigenvalues must be θ_1 and θ_{D-1}, i.e. the second-largest and second-smallest eigenvalues of Γ. Let $\sigma_0, \sigma_1, \ldots, \sigma_D$ (resp. $\tau_0, \tau_1, \ldots, \tau_D$) denote the cosine sequence associated with θ_1 (resp. θ_{D-1}). Observe $\sigma_0, \sigma_1, \ldots, \sigma_D$ are distinct.

Since θ_1 (resp. θ_{D-1}) is TTR, there exist β, γ (resp. β', γ') such that the recurrence in Theorem 8.2(iii) holds for $\sigma_0, \sigma_1, \ldots, \sigma_D$ (resp. $\tau_0, \tau_1, \ldots, \tau_D$). By Corollary 3.4, we have $\theta_{D-1} = -\theta_1$ and thus $\tau_i = (-1)^i \sigma_i$ for $0 \leqslant i \leqslant D$. Combining this with the recurrences, we find $\beta' = -\beta$ and $\gamma = \gamma' = 0$. Together with [5, Lemma 27], this implies Γ is 2-homogeneous. Now we are done by Lemma 6.2. □

Theorem 16.2. *Let Γ denote a bipartite distance-regular graph with diameter $D \geqslant 6$ and valency $k \geqslant 3$. Suppose Γ has more than one TTR eigenvalue. Then Γ is the D-cube.*

Proof. If Γ has a type AE eigenvalue then we are done by Theorem 12.2.

If Γ has an eigenvalue that is type AO and one that is type Q, then Γ is 2-homogeneous by Theorems 10.4 and 10.5 and Lemma 6.2. If Γ has two type Q eigenvalues then Γ is 2-homogeneous by the argument in the proof of Theorem 16.1. In either case, since $D \geqslant 6$ we are done by Lemma 6.3.

If Γ has a type AO eigenvalue with $\beta = \pm 2$, we are done by Theorems 15.2 and 15.4 and Example 17.3.

So suppose Γ has two type AO eigenvalues with $\beta \notin \{-2, 2\}$. With reference to Theorem 13.4 and by Lemma 13.5, they must be θ_e and θ_{D-1}, which are $\tilde{\theta}_1$ and $\tilde{\theta}_d$. Set $\theta = \tilde{\theta}_d$ in (37) and simplify using (60)–(66) and (68). Applying Lemma 9.3 and recalling $d > 2$ and (67), we find $s^2 q^{D+2} = 1$. Repeating this calculation with $\theta = -\tilde{\theta}_d$, we find it is also TTR and thus type Q. Now we are done by our earlier argument. $\square$

The reader might wonder what combinations of TTR eigenvalues are possible. We tabulate these at the end of Section 17.

17. Examples of bipartite distance-regular graphs with TTR eigenvalues

Here we discuss in detail some examples of bipartite distance-regular graphs which have at least one TTR eigenvalue. In most of the examples, we present the intersection numbers and eigenvalues. We then indicate which eigenvalues are TTR. For those that are, we give the types and the associated cosine sequences.

The intersection numbers, eigenvalues, and any uniqueness statement come from the source cited. Given any eigenvalue, we check whether it is TTR using Lemma 9.3. If it is, we check its type in Definition 10.1 after calculating the associated cosine sequence using Lemma 3.3.

Example 17.1 (Brouwer et al. [2, Section 9.2]). Let Γ denote the D-cube with $D \geqslant 4$.

(i) Γ is a bipartite antipodal 2-cover and is uniquely determined by its intersection numbers

$$c_i = i, \quad b_i = D - i \quad (0 \leqslant i \leqslant D).$$

If $D > 4$ then Γ is the only distance-regular cover of the folded D-cube.

(ii) The eigenvalues of Γ are

$$\theta_i = D - 2i \quad (0 \leqslant i \leqslant D).$$

(a) θ_0 is trivial.

(b) θ_1 is type Q with $\beta = 2$, $\gamma = 0$. The associated cosine sequence is

$$\sigma_i = 1 - \frac{2}{D} i \quad (0 \leqslant i \leqslant D).$$

(c) θ_2 is TTR (type AE if D is even, AO if D is odd) with $\beta = 2$, $\gamma \neq 0$. The associated cosine sequence is

$$\sigma_i = 1 + \frac{4}{D(D-1)} i(i - D) \quad (0 \leqslant i \leqslant D).$$

(d) None of $\theta_3, \ldots, \theta_{D-2}$ are TTR.

(e) θ_{D-1} is TTR (type Q if D is even, AO if D is odd) with $\beta = -2$, $\gamma = 0$. The associated cosine sequence is

$$\sigma_i = (-1)^i \left(1 - \frac{2}{D} i\right) \quad (0 \leqslant i \leqslant D).$$

(f) θ_D is trivial.

We note Γ is 2-homogeneous by Lemma 6.3.

Example 17.2 (Brouwer et al. [2, p. 264]). Let Γ denote the folded (i.e. quotient graph of the) $2D$-cube with $D \geqslant 4$.

(i) Γ is bipartite but not antipodal. Γ is uniquely determined by its intersection numbers

$$c_i = i, \quad b_i = 2D - i \ (0 \leqslant i \leqslant D-1), \quad c_D = 2D, \quad b_D = 0.$$

(ii) The eigenvalues of Γ are

$$\theta_i = 2D - 4i \quad (0 \leqslant i \leqslant D).$$

(a) θ_0 is trivial.
(b) θ_1 is type Q with $\beta = 2$, $\gamma \neq 0$. The associated cosine sequence is

$$\sigma_i = 1 + \frac{2}{D(2D-1)} i(i-2D) \quad (0 \leqslant i \leqslant D).$$

(c) None of $\theta_2, \ldots, \theta_{D-1}$ are TTR.
(d) θ_D is trivial.

Example 17.3 (Brouwer et al. [2, Section 9.1D]). Let Γ denote the doubled Odd graph $2.O_{d+1}$ with $d \geqslant 2$.

(i) Γ is a bipartite antipodal 2-cover with diameter $D = 2d+1$. Γ is uniquely determined by its intersection numbers

$$c_i = \left\lfloor \frac{i+1}{2} \right\rfloor, \quad b_i = d + 1 - c_i \quad (0 \leqslant i \leqslant D),$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.
(ii) The eigenvalues of Γ are

$$\theta_i = d + 1 - i \quad (0 \leqslant i \leqslant d), \quad \theta_i = -\theta_{D-i} \quad (d+1 \leqslant i \leqslant D).$$

(a) θ_0 is trivial.
(b) θ_1 is not TTR.

(c) If $D>5$ then θ_2 is not TTR.
If $D=5$ then θ_2 is type AO with $\beta=1$, $\gamma\neq 0$. The associated cosine sequence is $1, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, 1$.

(d) None of $\theta_3, \dots, \theta_{D-2}$ are TTR.

(e) θ_{D-1} is type AO with $\beta=-2$, $\gamma\neq 0$. The associated cosine sequence is

$$\sigma_i = \frac{(-1)^i D(D-2i)-1}{4d(d+1)} \quad (0\leqslant i\leqslant D).$$

(f) θ_D is trivial.

Example 17.4 (Brouwer et al. [2, p. 425]). Let Γ denote a bipartite distance-regular graph with diameter 4 and valency $k\geqslant 3$. Suppose Γ is an antipodal r-cover. Let $\theta_0>\theta_1>\cdots>\theta_4$ denote the eigenvalues of Γ.

(i) The intersection array of Γ is

$$\left\{k, k-1, k-\frac{k}{r}, 1; 1, \frac{k}{r}, k-1, k\right\}.$$

(ii) The eigenvalues of Γ are described in the following table.

i	θ_i	*Cosines*	*Type*	β	γ
0	k	$\sigma_i=1$	*Trivial*		
1	$\sqrt{k}$	$1, \frac{1}{\sqrt{k}}, 0, -\frac{1}{\sqrt{k}(r-1)}, -\frac{1}{r-1}$	Q iff $r=2$	$\sqrt{k}$	0
2	0	$1, 0, -\frac{1}{k-1}, 0, 1$	*AE*	$k-2$	*Not* 0
3	$-\sqrt{k}$	$1, -\frac{1}{\sqrt{k}}, 0, \frac{1}{\sqrt{k}(r-1)}, -\frac{1}{r-1}$	Q iff $r=2$	$-\sqrt{k}$	0
4	$-k$	$\sigma_i=(-1)^i$	*Trivial*		

We note by Lemma 6.3 that $r=2$ if and only if Γ is 2-homogeneous.

The bipartite antipodal 2-covers with diameter 4 are known as Hadamard graphs. There are many known examples [2, Section 1.8].

Example 17.5 (MacLean [6, Lemmas 8.1, 8.2]). Let Γ denote a bipartite antipodal distance-regular graph with diameter 5 and valency $k\geqslant 3$. Let $\theta_0>\theta_1>\cdots>\theta_5$ denote the eigenvalues of Γ.

(i) The intersection array of Γ is

$$\{k, k-1, k-\mu, \mu, 1; 1, \mu, k-\mu, k-1, k\}$$

where $k=\theta_1\theta_2+\theta_1-\theta_2$ and $\mu=\theta_1-\theta_2$. Note θ_1 and θ_2 are positive.

(ii) The eigenvalues of Γ are described in the following table. We abbreviate $\Delta := \theta_2^2 + 2\theta_2 - \theta_1$.

i	θ_i	*Cosines*	*Type*	β	γ
0	k	$\sigma_i = 1$	*Trivial*		
1	θ_1	$1, \frac{\theta_1}{k}, \frac{\mu}{(\theta_2+1)k}, -\frac{\mu}{(\theta_2+1)k}, -\frac{\theta_1}{k}, -1$	*Q iff* $\Delta = 0$	$\theta_2 + 1$	0
2	θ_2	$1, \frac{\theta_2}{k}, -\frac{\mu}{(\theta_1-1)k}, -\frac{\mu}{(\theta_1-1)k}, \frac{\theta_2}{k}, 1$	*AO*	$\theta_1 - 1$	*Not* 0
3	$-\theta_2$	$1, -\frac{\theta_2}{k}, -\frac{\mu}{(\theta_1-1)k}, \frac{\mu}{(\theta_1-1)k}, \frac{\theta_2}{k}, -1$	*Not TTR*		
4	$-\theta_1$	$1, -\frac{\theta_1}{k}, \frac{\mu}{(\theta_2+1)k}, \frac{\mu}{(\theta_2+1)k}, -\frac{\theta_1}{k}, 1$	*AO*	$-\theta_2 - 1$	0 *iff* $\Delta = 0$
5	$-k$	$\sigma_i = (-1)^i$	*Trivial*		

We note by Lemma 6.3 (with $\gamma = \theta_2$) that $\Delta = 0$ if and only if Γ is 2-homogeneous.

There are a number of known graphs that fit Example 17.5. See [2, p. 418] for details.

We encountered 2-homogeneous graphs in Examples 17.1, 17.4 and 17.5. In view of Lemma 6.3, we combine these examples to get the following summary.

Example 17.6. Let Γ denote a bipartite 2-homogeneous graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ.

(i) θ_0 is trivial.
(ii) θ_1 is type Q.
(iii) θ_2 is type AO (resp. AE) if D is odd (resp. even).
(iv) None of $\theta_3, \ldots, \theta_{D-2}$ are TTR.
(v) θ_{D-1} is type AO (resp. *Q*) if D is odd (resp. even).
(vi) θ_D is trivial.

Suppose Γ is a bipartite distance-regular graph with diameter $D \geqslant 4$ and valency $k \geqslant 3$ but is not one of the above examples. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ and suppose θ is a TTR eigenvalue of Γ. By Theorems 16.1 and 16.2, Γ has no other TTR eigenvalues. By Theorems 12.1 and 12.2, θ is type AO or Q. In either case, we must have $\beta \notin \{-2, 2\}$ by the results of Section 15. By Lemma 13.5, if θ is type AO then it is either θ_{D-1} or θ_e, where e is the even integer closest to $D/2$. Also, Γ is not a known graph. (By a known graph we mean those graphs listed in [2].) If θ is type Q then it is either θ_1 or θ_{D-1} by Lemma 14.4. If θ_{D-1} is type Q then Γ is not a known graph. We find θ_1 is type Q if Γ is the dual polar graph $D_D(q)$, but this is not the only possibility [2].

We conclude with a summary tabulation of the possible combinations of TTR eigenvalues. We recall Γ cannot have more than three TTR eigenvalues by Corollary 9.4. Of course, Γ cannot have both a type AE and a type AO eigenvalue.

TTR eigenvalue combinations

AE	AO	Q	Classification (Example)	Justification (Comment)
1	0	0	$D=4$, r-cover, $r>2$	Theorems 12.1, 12.2, Examples 17.1, 17.4
0	1	0	(Doubled Odd graphs)	(No other known examples)
0	0	1	(Dual polar graphs $D_D(q)$)	(Many other examples)
2	0	0	Impossible	Theorems 12.1, 12.2
0	2	0	$D=5$, 2-cover, not 2-homogeneous	Section 16, Examples 17.1, 17.5
0	0	2	Impossible	Section 16, Examples 17.1, 17.4, 17.5
1	0	1	Impossible	Section 16, Examples 17.1, 17.4
0	1	1	Impossible	Section 16, Examples 17.1, 17.5
3	0	0	Impossible	Theorems 12.1, 12.2
0	3	0	Impossible	Section 16, Examples 17.1, 17.5
0	0	3	Impossible	Section 16, Examples 17.1, 17.4, 17.5
1	0	2	D even, 2-homogeneous	Section 16, Examples 17.1, 17.4
2	0	1	Impossible	Theorems 12.1, 12.2
0	1	2	Impossible	Section 16, Examples 17.1, 17.5
0	2	1	D odd, 2-homogeneous	Section 16, Examples 17.1, 17.5

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