Ann Oper Res (2014) 217:399–423
DOI 10.1007/s10479-014-1590-x

Analysis of inventory policies for perishable items with fixed leadtimes and lifetimes

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Published online: 6 April 2014

Abstract This paper deals with a continuous review inventory system with perishable items and Poisson demand. Lifetimes and leadtimes are assumed to be fixed. First, a quite general base-stock model is developed where a number of combinations between backorder- and lost sales policies are evaluated and optimized. The solution technique for all these combinations is exact. Secondly, we consider the case with non-negligible ordering costs and assume that the inventory policy employed is the commonly used (R, Q) policy. We develop a new heuristic approach for evaluating and analyzing the proposed (R, Q) model and compare our results with those obtained by related papers. This heuristic approach uses the base-stock model developed as a building block. The results reveal that our approach works reasonably well in all cases considered.

Keywords Inventory · Perishable items · Batch ordering policy · Partial backorders

1 Introduction and literature review

One important problem which several companies face is how to control their inventories of products with limited shelf-life. For example, consider products in the agro-food industry. Some typical products in this class of perishables are bread, fresh fruit, ready-to-cook vegetables, dairy products, etc. In the automotive industry rubber materials and steel coils (steel coils may rust) are typical examples of products which become unusable after too long time in stock. Other types of perishables are medicines, blood in blood banks, batteries and fashion clothes.

As pointed out in Minner and Transchel (2010), in a 2008 survey, half of grocery retailers indicated that more than 5 % of the inventory is near expiration at any given time, and approximately one-third of grocery retailers indicated that more than 10 % of the inventory is near expiration at any given time (see Grocery Manufacturer Association 2008). To operate

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a supply chain efficiently, it is evident that expiration dates should be taken into account when deciding inventory target levels. However, in practice, many companies today are controlling their inventories with simple ad-hoc rules or with "standard procedures", i.e., by assuming infinite lifetime of products, which may lead to quite large cost increases. This may be the case since when dealing with perishable products large inventories may lead to very high costs due to outdating and inventory holding costs, while too small inventories may cause poor customer service (or equivalently, high penalty costs). Moreover, since the expiration of products is not taken into account in infinite lifetime models, calculated inventory levels may be quite inaccurate. Hence, there is a pressing need for more sophisticated ways to manage perishable goods in supply chains.

In this paper, we extend the literature concerning inventory systems with perishable items in several directions. First, in Sect. 3, we consider perishable base-stock continuous review inventory models with combined lost sales and backordering. An interesting and important application of such models is in the pharmacy industry. For example, in 2009 the Swedish pharmacy market was privatized, which gave rise to fierce competition between the new players in the pharmacy business. Due to this new competition customers are not, to the same extent as before, willing to wait for their medicines if there is no stock on hand. However, to the best of our knowledge, customers typically tend to accept a relatively short waiting time since it may be inconvenient and also time consuming to re-direct the demand and order the product from another pharmacy company. Hence, there is a practical need for being able to control perishable inventory models with combined lost sales and backordering in an efficient way.

Concerning the use of a base-stock policy, we believe that the base-stock case deserves some attention since there do, indeed, exist low demanded expensive products with a limited shelf life. From discussions with companies, we know that some types of pharmaceutical products fall into this category. Other products with limited shelf life where the base-stock policy may be suitable is, for example, low demanded steel coils for the automotive industry. Below, when we discuss relevant related literature, we will elaborate in more detail about the novelty concerning this line of research.

Our second aim is to use the base-stock policy as a building block in order to develop a new heuristic method for batch-ordering policies. More specifically, in this particular case the inventory system consists of a single location with continuous review, fixed leadtimes and non-negligible ordering costs. In our model we consider mainly backorder costs per unit and unit time, but also the case of backorder costs per unit. Since ordering costs are non-negligible, the system is controlled by an (R, Q) policy. A paper by Chiu (1995), which we will discuss later, presents a technique for approximate evaluation of the same system as we consider, but only for backorder costs per unit. Hence, for this special case, we will also compare our results to those obtained by Chiu (1995).

Inventory control of perishable items is in general a hard problem, especially in case of continuous review. First of all, the structure of the optimal policy for an inventory system with perishable items under continuous review and constant leadtimes is still unknown. As mentioned in previous work, e.g. Schmidt and Nahmias (1985), an optimal policy would be very complicated and therefore not directly applicable for practical purposes. This justifies the assumption of relatively simple alternative policies, such as (R, Q) or (s, S) policies.

The literature concerning inventory problems with perishable items under continuous review is relatively limited compared to the literature on corresponding periodic review models. In a zero leadtime inventory model with continuous review and perishable items Weiss (1980) derives optimal (s, S) policies for fixed lifetime of items under the assumption of Poisson demand. Kalpakam and Sapna (1995) consider an (s, S) policy and relax the zero leadtime assumption by assuming that leadtimes are exponentially distributed. In their model unsatisfied customers are lost, and only one outstanding order is allowed. Other inventory models with perishable items under the assumption of exponentially distributed leadtimes are Liu and Yang (1999), and Chakravarthy and Daniel (2004).

Schmidt and Nahmias (1985) deal with the more realistic case of deterministic leadtimes and lifetimes and assume that the inventory system is operating under an $(S-1, S)$ continuous review lost sales policy. The modeling technique in Schmidt and Nahmias (1985) is exact, but the evaluation of the expected total cost rate requires numerical integration. Perry and Posner (1998) consider a similar system as Schmidt and Nahmias (1985), but allow general mixtures of lost sales and backorders. However, as also mentioned in Karaesmen et al. (2011), the results obtained in Perry and Posner (1998) are not, in general, computationally tractable since the solution procedure hinges upon solving a complex functional equation. Furthermore, they do not develop any explicit cost formulations. Nahmias (2011) also emphasizes that Perry and Posner (1998) do not provide any calculations to compare their results to those obtained in Schmidt and Nahmias (1985). A recent paper by Olsson and Tydesjö (2010) analyzes a similar model as Schmidt and Nahmias (1985), but allow full backordering instead of lost sales. In this paper we develop a perishable base-stock model that generalizes Schmidt and Nahmias (1985), and Olsson and Tydesjö (2010) by allowing combinations of lost sales and backordering. In contrast to Perry and Posner (1998), we develop a computationally tractable model and we will also compare our results to those obtained by Schmidt and Nahmias (1985). Moreover, our solution is exact and does not require numerical integration (as in Schmidt and Nahmias 1985). Other papers that deal with partial backorder inventory models with perishable items are, e.g., Abad (1996) and Chang and Dye (1999).

There are very few studies available in the literature which incorporate ordering costs when assuming continuous review, fixed leadtimes and fixed lifetimes. An early work by Nahmias and Wang (1979) consider a lot size model with fixed ordering costs, fixed leadtime where items deteriorate exponentially. Chiu (1995) develops a heuristic solution for a continuous review (R, Q) model with fixed leadtimes and lifetimes. Moreover, he assumes full backordering and that backorder costs are charged per unit (i.e., *not* per unit time). Berk and Gürler (2008) provide an exact (and elegant) solution to a similar model as Chiu (1995), but assume that all unsatisfied demands are lost. In their model, only one outstanding order is allowed in the system. Other interesting continuous review (R, Q) inventory models for non-perishables are, e.g., Federgruen and Zheng (1992), Johansen and Thorstenson (1996), and Katehakis and Smit (2012).

For more complete literature reviews concerning perishable items in inventory systems see, e.g., Nahmias (1982) for early papers and Karaesmen et al. (2011) and Nahmias (2011) for more recent ones.

This paper is organized as follows. In Sect. 2 we formulate our models and introduce some useful notation. In Sect. 3, we develop an exact solution technique for a base-stock inventory model with perishable items and partial backordering. In Sect. 4, a heuristic batch ordering model is developed for the complete backordering case. Our numerical results are shown and discussed in Sect. 5, and some concluding remarks are presented in Sect. 6.

2 Model formulation

We consider a continuous review inventory systems with fixed leadtimes and fixed lifetimes. Aging of items is assumed to begin when the order is placed (an equivalent formulation would be to assume that the lifetime begins when the item arrives in stock). Outdated items are immediately discarded from the system. In our model, items leave inventory according to a FIFO (first in first out) policy, i.e., the oldest items are used first.

First, in Sect. 3, we consider the case where ordering costs are negligible, which means that a base-stock inventory policy is reasonable. Hence, a new item is ordered every time when a customer arrives or when an item is outdated. The only exception from this rule is when a customer is lost. That is, a new item is *not* ordered if a customer is lost. In this base-stock model we will develop an exact solution procedure for a number of partial backorder policies. To this end, we assume that customer demands follow a doubly stochastic Poisson process (see, e.g., Kingman 1964) with time dependent intensity $\eta(t)$, where t is the age of the oldest item in the system (not yet assigned for any customer demand). Hence, depending on the explicit formulation of $\eta(t)$, various combinations between backorder and lost sales models can be analyzed.

In the second case, in Sect. 4, we assume that ordering costs are included in the model, which means that replenishments are in batches. More specifically, the system operates under the commonly used (R, Q) policy. In this specific case it means that whenever the inventory position (stock on hand + outstanding orders − backorders) drops to or below the reorder point R, a batch of fixed size Q is ordered and arrives in inventory after a fixed leadtime. Hence, if a batch (or a part of a batch) in inventory is outdated and the inventory position drops to or below R, a new batch is ordered. It should be noted that the (R, Q) policy is not the optimal policy in this particular setting. The optimal policy would certainly be very complex and difficult to derive since we have to take current residual lifetimes of items on order and in stock into account. Also the current inventory level and inventory position will of course affect the decision policy. Moreover, we have to relax the assumption that a new order can only be initiated when a demand occurs or if a part of a batch is outdated. Clearly, it may be optimal to order a new batch *before* the oldest batch in the system has perished. As mentioned, a very complex optimal policy would hardly be of interest for practical purposes. Therefore, we will focus on the commonly used (R, Q) policy.

Let us conclude by introducing some useful notation:

L Leadtime for an item to arrive in inventory,

T Fixed lifetime $(T > L)$,

S Base-stock level,

R Reorder point,

Q Order quantity,

IL Inventory level,

IP Inventory position,

OF Order frequency,

B Number of backorders generated per unit time,

h Holding cost per unit and unit time,

b_1 Backorder cost per unit and unit time,

b_2 Backorder cost per unit,

θ Lost sales cost per unit,

p Perishing cost per unit,

A Ordering cost per batch,

$\propto$ Proportional to.

3 Perishable base-stock inventory models with combined lost sales and backordering

Here we consider a base-stock policy with a base-stock level equal to S. Note that, in a pure lost sales inventory system there are always exactly S items in the system (here the system constitutes of the items in stock and the items which are outstanding). However, if backorders are allowed, there is no upper bound on the number of items outstanding. For example, if there are $S+n$ items in the system, the oldest n items are already assigned for waiting customers. Hence, regarding future customers, we are only interested in the timing of the S youngest items in the system. For this reason, let $T_1, T_2, \ldots, T_S$ represent the ages of the items in the system which are not assigned for waiting customers, i.e., the S youngest items in the system. We define T_1 as the age of the oldest item and T_S as the age of the youngest item. Hence, we have the order $0 \le T_S < T_{S-1} < \cdots < T_1 < T$, where T is the fixed lifetime. The system is viewed in stationarity.

As mentioned in the previous section, demands follow a doubly stochastic Poisson process with time dependent intensity $\eta(t)$. Note that, in cases where the customer demand rate varies due to the age of the inventory, it is clear that the rate is only a function of the age of the oldest item in the system since the issuing policy is FIFO. In this specific case we assume that the demand rate switches between two pre-determined levels, λ and μ respectively, depending on the age of the oldest item in the system. That is, for $T_1 = t_1$, we have

$$\eta(t_1) = \begin{cases} \lambda, & \text{for } 0 \le t_1 < \tau \\ \mu, & \text{for } \tau \le t_1 < T, \end{cases} \tag{1}$$

Here, τ is the break point where the demand rate switches to another constant value. It is clear that a relatively large number of combinations between backorder and lost sales models can be achieved by explicitly changing the values of λ, μ and τ in (1). Let us give a few possible combinations.

Example 1 Let us consider the following five backordering and/or lost sales cases:

(1) Assume $\lambda \equiv 0$, $\mu > 0$ and $\tau \equiv L$. Then we have the complete lost sales model considered in Schmidt and Nahmias (1985).
(2) Assume $\lambda \equiv \mu > 0$. Then we have the complete backordering model considered in Olsson and Tydesjö (2010). Note that τ can be chosen arbitrarily in this case.
(3) Assume $\lambda \equiv 0$, $\mu > 0$ and $0 < \tau < L$. Then we have a combined lost sales and backordering model. In other words, a customer is backordered if the waiting time is less than $L - \tau$ units of time, and lost otherwise.
(4) Assume $\lambda \equiv \alpha\mu$ and $\tau \equiv L$, where α is the probability that an arriving customer is backordered. This is a finite lifetime version of the (infinite lifetime) model considered in Moinzadeh (1989).
(5) Consider the scenario where a customer is not willing to purchase a product if its residual lifetime is too short. This means that an arriving customer should be satisfied by a sufficiently younger item, instead of the oldest one, if the age of the oldest item has exceeded some threshold value τ, i.e., $T_1 > \tau$. However, this also implies that the oldest item will only continue aging and finally perish at age $T_1 = T$, which is clearly sub-optimal. Hence, in this particular scenario, it is better to discard an item from the system when its age has reached τ (instead of T), in order to avoid having non-desirable items in the system which only incur inventory holding costs. Then, by discarding an item from the system at age τ (instead of T), we again have the model considered in case 2 above, but with a lifetime equal to τ (instead of T). □

We continue by deriving the limiting joint density of $T_1, T_2, \ldots, T_S$, $f_{T_1,T_2,\ldots,T_S}(t_1, t_2, \ldots, t_S)$, which will be used later when deriving average stock on hand, average number of backorders, etc. For this task we will use a similar approach as in Schmidt and Nahmias (1985). Other similar methods can be found in, e.g., Moinzadeh (1989), Moinzadeh and Schmidt (1991) and Olsson (2011).

Proposition 1

$$f_{T_1,T_2,\ldots,T_S}(t_1, t_2, \ldots, t_S) \propto \exp\left\{-\int_0^{t_1} \eta(t)dt\right\} = \begin{cases} e^{-\lambda t_1} & \text{for } 0 \le t_1 < \tau \\ e^{-(\lambda\tau+\mu(t_1-\tau))} & \text{for } \tau \le t_1 \le T. \end{cases} \tag{2}$$

The proof of Proposition 1, together with all other proofs, can be found in Appendix 1. Obviously, Proposition 1 states that

$$f_{T_1,T_2,\ldots,T_S}(t_1, t_2, \ldots, t_S) = \begin{cases} Ce^{-\lambda t_1} & \text{for } 0 \le t_1 < \tau \\ Ce^{-(\lambda\tau+\mu(t_1-\tau))} & \text{for } \tau \le t_1 \le T, \end{cases} \tag{3}$$

where C is the normalizing constant. In the following analysis we will use the marginal density of T_1. Using (3), this density is determined as

$$\begin{aligned} f_{T_1}(t_1) &= \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{S-1}} C\exp\left\{-\int_0^{t_1}\eta(t)dt\right\} dt_S \ldots dt_2 \\ &= C\exp\left\{-\int_0^{t_1}\eta(t)dt\right\}\frac{t_1^{S-1}}{(S-1)!}, \end{aligned}$$

which simplifies to

$$f_{T_1}(t_1) = \begin{cases} Ce^{-\lambda t_1} t_1^{S-1}/(S-1)! & \text{for } 0 \le t_1 < \tau \\ Ce^{-(\lambda\tau+\mu(t_1-\tau))} t_1^{S-1}/(S-1)! & \text{for } \tau \le t_1 \le T. \end{cases} \tag{4}$$

Finally, the constant C can be obtained by integrating the marginal density of T_1 to unity over its domain. We get

$$\begin{aligned} \frac{1}{C} &= \int_0^{\tau} e^{-\lambda t_1}\frac{t_1^{S-1}}{(S-1)!}dt_1 + \int_{\tau}^{T} e^{-(\lambda\tau+\mu(t_1-\tau))}\frac{t_1^{S-1}}{(S-1)!}dt_1. \\ &= \frac{1}{\lambda^S}\left(1-\sum_{n=0}^{S-1}\frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau}\right) + \frac{e^{\tau(\mu-\lambda)}}{\mu^S}\left(\sum_{n=0}^{S-1}\frac{(\mu\tau)^n}{n!}e^{-\mu\tau} - \sum_{n=0}^{S-1}\frac{(\mu T)^n}{n!}e^{-\mu T}\right). \end{aligned} \tag{5}$$

By using (3)–(5) we will, in the next sub-section, derive expressions for average inventory holding costs, average backorder costs and lost sales costs. However, in order to calculate perishing costs we first need to derive the average outdating rate, π (the average rate at which items perish). Similar to Schmidt and Nahmias (1985) and Olsson and Tydesjö (2010) we obtain

$$\pi = \lim_{\delta\to 0}\frac{P(T_1 \le T) - P(T_1 \le T-\delta)}{\delta} = f_{T_1}(T) = Ce^{-(\lambda\tau+\mu(T-\tau))}\frac{T^{S-1}}{(S-1)!}. \tag{6}$$

3.1 Cost structures and performance measures

When defining a cost structure it is clear that we first need to formulate the customer behavior in our model more explicitly. In this section we will consider the cost structure of all four cases considered in Example 1. Define $E(IL)^+$ as the average stock on hand, $E(IL)^-_{CB}$ as the average number of backorders when applying complete backordering, and $E(IL)^-_{PB}$ as the average number of backorders if customers are willing to wait maximum $L-\tau$ units of time (this corresponds to Case 3 in Example 1, where customers are partially backordered).

Assuming for the moment that the expressions for $E(IL)^+$, $E(IL)^-_{CB}$ and $E(IL)^-_{PB}$ are given, we proceed by presenting the expected cost per unit of time, EC, for all cases mentioned in Example 1.

Cost structures:

(1) Assume $\lambda \equiv 0$, $\mu > 0$ and $\tau \equiv L$. Hence, in this case customers are lost and no backorders are allowed. This gives

$$EC = h \cdot E(IL)^+ + \theta \cdot \mu \cdot P(T_1 < L) + p \cdot \pi. \tag{7}$$

(2) Assume $\lambda \equiv \mu > 0$. Hence, in this case all unsatisfied demands are backordered, i.e., no lost sales exist. This gives

$$EC = h \cdot E(IL)^+ + b_1 \cdot E(IL)^-_{CB} + p \cdot \pi, \tag{8}$$

(3) Assume $\lambda \equiv 0$, $\mu > 0$ and $0 < \tau < L$. Hence, in this case a customer is backordered if the waiting time is less than $L-\tau$ units of time, otherwise the customer is lost. This gives

$$EC = h \cdot E(IL)^+ + b_1 \cdot E(IL)^-_{PB} + \theta \cdot \mu \cdot P(T_1 < \tau) + p \cdot \pi, \tag{9}$$

(4) Assume $\lambda \equiv \alpha\mu$ and $\tau \equiv L$, where α is the probability that an arriving customer is backordered. Hence, in this case we have a combined backordering and lost sales model, similar to case 3 above. This gives

$$EC = h \cdot E(IL)^+ + b_1 \cdot E(IL)^-_{CB} + (1-\alpha) \cdot \mu \cdot \theta \cdot P(T_1 < L) + p \cdot \pi. \tag{10}$$

(5) As argued in Example 1, this case is modified such that it becomes equivalent to case 2 above, but with a lifetime equal to τ (instead of T).

Here, $P(T_1 < t)$ is readily obtained by using (4). In the following proposition we will derive expressions for $E(IL)^+$, $E(IL)^-_{CB}$ and $E(IL)^-_{PB}$, which were assumed to be given when deriving the cost structures. Recall that $E(IL)^-_{CB}$ is used in the cost function for Cases 2 and 4 in Example 1, which means that $\tau = L$ when deriving the expression for $E(IL)^-_{CB}$. Similarly, $E(IL)^-_{PB}$ only appears in the cost function for Case 3 in Example 1. Hence, $\lambda = 0$ when deriving the expression for $E(IL)^-_{PB}$. We get

Proposition 2

$$E(IL)^+ = C \cdot e^{\tau(\mu-\lambda)} \left[\frac{S}{\mu^S} \left(\sum_{n=0}^{S} \frac{(\mu L)^n}{n!} e^{-\mu L} - \sum_{n=0}^{S} \frac{(\mu T)^n}{n!} e^{-\mu T} \right) \right.$$
$$\left. - \frac{L}{\mu^{S-1}} \left(\sum_{n=0}^{S-1} \frac{(\mu L)^n}{n!} e^{-\mu L} - \sum_{n=0}^{S-1} \frac{(\mu T)^n}{n!} e^{-\mu T} \right) \right] + (T-L)\pi, \quad (11)$$

$$E(IL)^-_{CB} = \frac{C \cdot L}{\lambda^{S-1}} \left(1 - \sum_{n=0}^{S-1} \frac{(\lambda L)^n}{n!} e^{-\lambda L} \right) - \frac{C \cdot S}{\lambda^S} \left(1 - \sum_{n=0}^{S} \frac{(\lambda L)^n}{n!} e^{-\lambda L} \right), \quad (12)$$

$$E(IL)^-_{PB} = C \cdot e^{-\mu\tau} \left[\frac{L}{\mu^{S-1}} \left(\sum_{n=0}^{S-1} \frac{(\mu\tau)^n}{n!} e^{-\mu\tau} - \sum_{n=0}^{S-1} \frac{(\mu L)^n}{n!} e^{-\mu L} \right) \right.$$
$$\left. - \frac{S}{\mu^S} \left(\sum_{n=0}^{S} \frac{(\mu\tau)^n}{n!} e^{-\mu\tau} - \sum_{n=0}^{S} \frac{(\mu L)^n}{n!} e^{-\mu L} \right) \right]. \quad (13)$$

A related and important performance measure is the probability function for the inventory level, $P(IL = k)$. For example, $P(IL = k)$ may be of great interest when calculating the fill rate, β, where $\beta = P(IL > 0) = \sum_{k=1}^{S} P(IL = k)$. For this reason we will concentrate on positive inventory levels.

Now, by using the limiting joint density, $f_{T_1,T_2,\dots,T_S}(t_1, t_2, \dots, t_S)$, we obtain the following closed expression for the probability function for positive inventory levels:

Proposition 3 *The probability that the on hand inventory level is equal to $k > 0$ is given by*

$$P(IL = k) = \frac{C \cdot L^{S-k} \cdot e^{(\mu-\lambda)\tau - \mu L}}{\mu^k (S-k)!} \cdot \left(e^{-\mu(\tau-L)} \sum_{n=0}^{k-1} \frac{(\mu(\tau-L))^n}{n!} \right.$$
$$\left. - e^{-\mu(T-L)} \sum_{n=0}^{k-1} \frac{(\mu(T-L))^n}{n!} \right)$$
$$+ \frac{C \cdot L^{S-k} \cdot e^{-\lambda L}}{\lambda^k (S-k)!} \left(1 - e^{-\lambda(\tau-L)} \sum_{n=0}^{k-1} \frac{(\lambda(\tau-L))^n}{n!} \right). \quad (14)$$

Regarding the optimization procedure for finding the optimal value of S, the same procedure as developed in Olsson and Tydesjö (2010) can be used.

4 A heuristic (R, Q) model for the complete backorder case

We will now consider a more general ordering policy than the base-stock policy, namely the well known (R, Q) policy. In this particular case we will, for simplicity reasons, concentrate on the complete backorder case. Our main contribution is to derive a new heuristic for the case with backorder costs per unit and unit time. However, as mentioned, we will also consider the case of backorder costs per unit and compare our results against those obtained in Chiu (1995).

4.1 The base-stock policy as a building block

The main idea in our solution procedure is to use the steady state distribution from the base-stock case. For this purpose we will use results from the previous section, but also results from Olsson and Tydesjö (2010). For the complete backorder case (Case 2 in Example 1) the probability function for positive inventory levels in (14) degenerates to (recall that $\lambda \equiv \mu$)

$$P(IL = j) = \frac{CL^{S-j}e^{-\lambda L}}{\lambda^j (S-j)!}\left(1 - e^{-\lambda(T-L)}\sum_{n=0}^{j-1}\frac{(\lambda(T-L))^n}{n!}\right), \tag{15}$$

where

$$\frac{1}{C} = \frac{1}{\lambda^S}\left(1 - \sum_{n=0}^{S-1} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\right),$$

according to (5).

Considering the probability distribution for non-positive inventory levels, we will use the approximation developed in Olsson and Tydesjö (2010), that is

$$P(IL = j) = \frac{S!}{(S-j)!}(\lambda L)^{-j} P(IL = 0), \quad j = -1, -2, \ldots \tag{16}$$

and

$$P(IL = 0) = \frac{(\lambda L)^S}{S!} \cdot \frac{1 - \sum_{n=1}^{S} P(IL = n)}{e^{\lambda L} - \sum_{n=0}^{S-1}\frac{(\lambda L)^n}{n!}}. \tag{17}$$

Note that the inventory position, IP, is always equal to S in the base-stock model.

4.2 The (R, Q) policy

We will use the base-stock policy as a building block when constructing the heuristic (R, Q) policy and assume for a start that the distribution of the inventory position is known (this distribution is derived in Sect. 4.2.1). By conditioning on IP, say $IP = k$, the steady state probabilities of inventory levels can be calculated by using the law of total probability,

$$P(IL = j) = \sum_{k=\max(R+1, j)}^{R+Q} P(IL = j|IP = k)P(IP = k), \quad j \le R + Q, \tag{18}$$

where $P(IL = j|IP = k)$ is the probability function for the inventory level in the base-stock case given the base-stock level k, i.e.,

$$P(IL = j|IP = k) = \begin{cases} \frac{CL^{k-j}e^{-\lambda L}}{\lambda^j (k-j)!}\left(1 - e^{-\lambda(T-L)}\sum_{n=0}^{j-1}\frac{(\lambda(T-L))^n}{n!}\right), & j > 0, \\ \frac{k!}{(k-j)!}(\lambda L)^{-j} P(IL = 0|IP = k), & j = -1, -2, \ldots \\ \frac{(\lambda L)^k}{k!} \cdot \frac{1-\sum_{n=1}^{k} P(IL=n|IP=k)}{e^{\lambda L} - \sum_{n=0}^{k-1}\frac{(\lambda L)^n}{n!}}, & j = 0. \end{cases} \tag{19}$$

Similarly, the average outdating rate when using an (R, Q) policy, π_{RQ}, is obtained as

$$\pi_{RQ}(R, Q) = \sum_{k=R+1}^{R+Q} \pi(k)P(IP = k), \tag{20}$$

where $\pi(k)$, the average outdating rate given the base-stock level k, is obtained in complete analogy with (6),

$$\begin{aligned}\pi(k) &= \lim_{\delta\to 0}\frac{P(T_1 \le T|IP=k) - P(T_1 \le T-\delta|IP=k)}{\delta} = f_{T_1|IP=k}(T)\\ &= Ce^{-\lambda T}\frac{T^{k-1}}{(k-1)!}.\end{aligned} \tag{21}$$

Here, $f_{T_1|IP=k}(t)$ is the marginal density of T_1 for the base-stock model, see (4), given the base-stock level $IP \equiv S = k$.

It is important to realize that (18) and (20) are approximations. In the base-stock case, each individual item has its own residual lifetime. However, when dealing with (R, Q) policies in this particular model, all items in a batch have the same residual lifetime. This means that our approach by averaging over base-stock policies will not lead to exact results for the batch-ordering model. Note also, in the base-stock case the inventory level decreases one by one either by demand or outdating events, which means that an order is initiated when the inventory position hits the reorder point exactly. However, in our batch-ordering model, an order may be initiated when the inventory position has declined strictly below the reorder point due to depletion of a batch (or a part of a batch). Since (18) is based on base-stock policies we do not take this possible "undershoot" of R into consideration in our model. One clear consequence of disregarding the undershoot of R is that we will, in our model, underestimate the average number of backorders and overestimate the average stock on hand. However, it should be noted that the model is exact when $T \to \infty$. Hence, we should of course expect better results in cases with long lifetimes rather than short ones.

4.2.1 Distribution of the inventory position

Recall that IP is uniformly distributed on the integers $R+1, R+2, \ldots, R+Q$ in the traditional case with infinite lifetime of items, see e.g. Axsäter (2006). However, it is interesting to note that this is not the case in our model where items have finite lifetime. Consider, for example, the case when the shelflife of items is very short and the leadtime is relatively long. Then, for example, the probability $P(IP = Q)$ should be rather high compared to other inventory position probabilities, because the proportion of time when there is no stock on hand and one order outstanding will be large. Hence, the shorter the lifetime (or equivalently, the shelflife) is, the worse will the approximation of uniformly distributed inventory position be. We continue by deriving an approximate distribution of IP.

We will use the connection between IP and IL in order to derive the distribution of IP. First, note that $P(IL = j)$ in (18) is a function of $P(IP = k)$, for all possible values of k. Hence, by expressing $P(IP = k)$, $k = R+1, \ldots, R+Q$, in terms of inventory level probabilities we obtain a linear system of equations.

Note that, in the backorder case there is no limit on the number of outstanding orders. However, in order to obtain a linear system of finite dimension we will only allow a finite number of outstanding orders, N. This means that we have an upper limit on the number of backorders. For most practical settings this is not a very restrictive assumption. In our problem, N is chosen sufficiently large such that our model can be regarded as a full backordering model.

The unconditional inventory level distribution is obtained by using (19) in (18). What remains is to derive the distribution for the inventory position. This can be done by solving the following linear system of equations of size $Q+1$.

$$P(IP=k)=\sum_{i=0}^{N} P(IL=k-iQ), \quad k=R+1,\dots,R+Q \tag{22}$$

$$\sum_{k=R+1}^{R+Q} P(IP=k)=1$$

In (22) we use that the number of outstanding units must be a multiple of Q. If there are no outstanding orders then obviously $IP=IL$, and if there is one outstanding order (i.e., Q units outstanding) then $IP=IL+Q$, etc.

This general procedure for deriving the distribution of the inventory position can also be used in other inventory problems where the inventory position is no longer uniformly distributed. For example, when introducing so called lateral transshipments in an inventory system, the inventory position is in general no longer uniformly distributed.

4.2.2 Backorder costs per unit and unit time

Given the steady state inventory level probabilities, $P(IL=k)$, and the average outdating rate, π_{RQ}, we can formulate the total expected cost per unit time, C, as

$$C(R,Q)=h\sum_{k=1}^{R+Q} kP(IL=k)+b_1\sum_{k=1}^{\infty} kP(IL=-k)+\frac{A(\lambda+\pi_{RQ})}{Q}+p\pi_{RQ}, \tag{23}$$

when considering backorder costs per unit and unit time, b_1.

4.2.3 Backorder costs per unit

When considering backorder costs per unit, b_2, the total expected cost per unit time can be formulated as

$$C(R,Q)=h\sum_{k=1}^{R+Q} kP(IL=k)+b_2E(B)+\frac{A(\lambda+\pi_{RQ})}{Q}+p\pi_{RQ}, \tag{24}$$

where $E(B)=\lambda P\{IL\le 0\}$ is the expected number of backorders generated per unit time.

4.2.4 Optimization

Regarding the optimization procedure for finding optimal values of R and Q we will use a procedure based on the assumption that the inventory position is uniformly distributed. When the inventory position is uniformly distributed, i.e., $P(IP=k)=1/Q$, we have:

Lemma 1 *$\pi_{RQ}(R,Q)$ in (20) has the following properties:*

(i) $\pi_{RQ}(R,Q)$ is increasing in R and Q separately,
(ii) $\pi_{RQ}(R,Q)\to\infty$ as $R\to\infty$ or $Q\to\infty$.

The proof is given in Appendix 1. The properties from Lemma 1 are used to determine optimal values of R and Q in the following simple bounding procedure described below. Note also that the reorder point R is bounded below by $-Q$.

1. Set $Q := 1$, $R := -Q$, and $C_{\min} = \infty$ and go to step 2.
2. Calculate $C(R, Q)$. If $C(R, Q) < C_{\min}$ then set $C_{\min} := C(R, Q)$. Set $R := R + 1$ and go to step 3.
3. Calculate $\Pi(R, Q) = p \cdot \pi_{RQ}(R, Q)$. If $\Pi(R, Q) \leq C_{\min}$ return to step 2, otherwise go to step 4.
4. Set $Q := Q + 1$ and $R := -Q$. If $\Pi(R, Q) > C_{\min}$ stop the procedure, otherwise return to step 2.

The intuition behind this procedure is that we search over a grid of values of R and Q and stop the search whenever $\Pi(R, Q)$ exceeds the minimum cost so far. Obviously, there is no meaning of continue the search if $\Pi(R, Q)$ exceeds the minimum cost so far, due to the fact that $\Pi(R, Q)$ is increasing in both R and Q. By using this simple procedure we will always find the optimal values of R and Q, without using any convexity properties, which are hard to prove. Although this procedure is based on the assumption that the inventory position is uniformly distributed, it found the optimal values for all considered numerical examples in Sect. 5. Hence, the procedure appears to be insensitive regarding this assumption.

5 Numerical results

We start by evaluating a number of numerical examples for the base-stock model with partial backorders considered in Sect. 3 (Case 3 in Example 1). In Table 1 we consider a very similar set of parameters as in Schmidt and Nahmias (1985). We have $\mu = 50$ items per year, $h = \$20$ per unit and year, $p = \$10$ per unit, $L = 0.1$ years and $b_1 = \$100$ per unit and year. In Table 1 we vary the lost sales cost θ (we consider $\theta \in \{150, 200, 300, 600\}$), the lifetime

Table 1 Optimization results for Case 3 in Example 1

T	θ			
	150	200	300	600
Optimization results for $\tau = 0.8L$:				
$L + 0.01$	14; 1117.6	15; 1202.1	16; 1318.5	18; 1523.4
$L + 0.05$	11; 481.6	11; 516.8	12; 561.8	13; 644.83
$L + 0.1$	10; 269.6	11; 288.3	11; 311.4	12; 354.6
$L + 1.0$	11; 135.2	11; 140.0	11; 149.6	12; 159.5
Optimization results for $\tau = 0.5L$:				
$L + 0.01$	8; 515.1	9; 542.8	9; 583.7	10; 655.0
$L + 0.05$	8; 261.2	8; 277.6	9; 299.3	9; 337.6
$L + 0.1$	8; 153.6	8; 163.5	9; 176.2	9; 195.6
$L + 1.0$	9; 92.7	9; 94.8	9; 99.1	10; 109.1
Optimization results for $\tau = L$:				
$L + 0.01$	24; 2819.6	28; 3101.1	32; 3499.6	40; 4185.4
$L + 0.05$	14; 743.6	14; 798.8	16; 874.4	17; 1005.2
$L + 0.1$	12; 377.7	12; 406.2	13; 438.5	14; 500.5
$L + 1.0$	12; 166.2	13; 173.4	13; 180.0	14; 194.2

The first entry represents the optimal value of S, and the second entry is the corresponding cost. In all problems we have $\mu = 50$, $h = 20$, $p = 10$ and $b_1 = 100$

Table 2 Results from our approximation with backorder costs per unit and unit time

Parameters				Approximation (Olsson)		Optimum by simulation		Infinite lifetime	
λ	A	b_1	p	R^*, Q^*	$C_{\text{sim}}(R^*, Q^*)$	R_s^*, Q_s^*	$C_{\text{sim}}(R_s^*, Q_s^*)$	R^*, Q^*	$C_{\text{sim}}(R^*, Q^*)$
4	3	8	15	3, 5	8.95 (0.02)	3, 4	8.85 (0.03)	4, 6	11.13 (0.06)
4	3	8	30	3, 4	9.79 (0.03)	3, 4	9.79 (0.03)	4, 6	15.55 (0.08)
4	3	16	15	4, 5	11.18 (0.04)	4, 4	10.62 (0.03)	5, 6	14.41 (0.04)
4	3	16	30	4, 4	12.54 (0.03)	4, 3	12.27 (0.05)	5, 6	20.98 (0.07)
4	6	8	15	3, 6	11.82 (0.03)	3, 5	11.42 (0.03)	3, 9	16.71 (0.02)
4	6	8	30	2, 6	13.10 (0.05)	2, 5	12.62 (0.05)	3, 9	24.21 (0.09)
4	6	16	15	4, 5	13.63 (0.04)	4, 5	13.63 (0.04)	4, 9	21.30 (0.07)
4	6	16	30	3, 5	15.66 (0.06)	3, 5	15.66 (0.06)	4, 9	31.19 (0.09)
8	3	8	15	8, 8	10.23 (0.04)	8, 7	10.06 (0.03)	8, 9	10.72 (0.03)
8	3	8	30	7, 8	10.58 (0.04)	7, 8	10.58 (0.04)	8, 9	12.16 (0.05)
8	3	16	15	9, 7	11.61 (0.03)	9, 7	11.61 (0.03)	9, 9	12.64 (0.06)
8	3	16	30	9, 7	12.65 (0.06)	9, 6	12.36 (0.05)	9, 9	14.77 (0.09)
8	6	8	15	7, 10	13.14 (0.04)	7, 9	13.00 (0.04)	7, 12	14.59 (0.06)
8	6	8	30	7, 9	13.72 (0.06)	7, 8	13.62 (0.06)	7, 12	17.25 (0.08)
8	6	16	15	8, 10	15.31 (0.04)	8, 9	14.93 (0.04)	9, 11	17.19 (0.03)
8	6	16	30	8, 9	16.21 (0.06)	8, 8	15.85 (0.04)	9, 11	21.53 (0.08)

In all problems we have $h = 1$, $L = 1$ and $T = 3$. Standard deviation of simulation results in parenthesis

T (we consider $T \in \{L + 0.01, L + 0.05, L + 0.1, L + 1.0\}$) and the acceptable customer waiting time $L - \tau$ (we consider $L - \tau \in \{0, 0.02, 0.05\}$).

The results from Table 1 show that, even if the acceptable waiting time is as short as 20 % of the leadtime, the optimal base-stock level can differ quite substantially when comparing a partial backordering policy with a pure lost sales policy. Clearly, the difference tends to get smaller as $\tau \to L$ (recall that we have a pure lost sales model when $\tau = L$). Hence, we can conclude that an optimal policy obtained from a pure lost sales model is, in general, not appropriate to use as an approximation for a corresponding model with partial backorders.

Let us continue by evaluating the performance of the heuristics developed in Sect. 4. In Tables 2, 3, and 4 we evaluate our approximation for the case of backordering costs per unit and unit time, and in Tables 5, 6, and 7 we consider a cost structure with backordering costs per unit (as in Chiu 1995). In all test problems we compare our results with the results obtained by simulation. The simulation model has been implemented in Matlab and all cases have been simulated for 10 replications. The simulation time for each replication was chosen such that the standard deviation of the mean was sufficiently low. The optimal values of R and Q obtained from our approximation and from simulation are denoted by (R^*, Q^*) and (R_s^*, Q_s^*), respectively. The cost obtained from simulation when using R and Q is denoted by $C_{\text{sim}}(R, Q)$. The standard deviation of the simulated costs are given in parenthesis. The most important feature of the performance of an approximation is that the difference in costs for the approximate policy and for the optimal policy is low. Hence, we introduce the relative cost increase

$$CI = \frac{C_{\text{sim}}(R^*, Q^*) - C_{\text{sim}}(R_s^*, Q_s^*)}{C_{\text{sim}}(R_s^*, Q_s^*)},$$

as an indicator of the quality of our approximation.

Table 3 A comparison between the results from our approximation and simulation for the same test problems as in Table 2

Parameters				R_s^*, Q_s^*	Approximation (Olsson)				Simulation			
λ	A	b_1	p		$E(IL)^+$	$E(IL)^-$	π_{RQ}	$E(OF)$	$E(IL)^+$	$E(IL)^-$	π_{RQ}	$E(OF)$
4	3	8	15	3, 4	1.82	0.37	0.047	1.01	1.82 (0.005)	0.38 (0.003)	0.065 (0.002)	1.02 (0.002)
4	3	8	30	3, 4	1.82	0.37	0.047	1.01	1.82 (0.005)	0.38 (0.003)	0.065 (0.002)	1.02 (0.002)
4	3	16	15	4, 4	2.60	0.18	0.093	1.02	2.60 (0.005)	0.19 (0.002)	0.13 (0.001)	1.03 (0.002)
4	3	16	30	4, 3	2.17	0.23	0.060	1.35	2.16 (0.004)	0.24 (0.002)	0.073 (0.001)	1.36 (0.002)
4	6	8	15	3, 5	2.23	0.30	0.076	0.82	2.21 (0.006)	0.32 (0.002)	0.11 (0.002)	0.82 (0.001)
4	6	8	30	2, 5	1.53	0.57	0.038	0.81	1.51 (0.004)	0.59 (0.004)	0.05 (0.001)	0.81 (0.002)
4	6	16	15	4, 5	3.01	0.15	0.14	0.83	2.98 (0.006)	0.17 (0.002)	0.20 (0.002)	0.84 (0.001)
4	6	16	30	3, 5	2.23	0.30	0.076	0.82	2.21 (0.006)	0.32 (0.002)	0.11 (0.002)	0.82 (0.001)
8	3	8	15	8, 7	4.21	0.23	0.023	1.15	4.19 (0.01)	0.24 (0.003)	0.038 (0.002)	1.15 (0.002)
8	3	8	30	7, 8	3.82	0.34	0.020	1.00	3.80 (0.006)	0.34 (0.003)	0.035 (0.001)	1.00 (0.001)
8	3	16	15	9, 7	5.09	0.13	0.040	1.15	5.07 (0.01)	0.14 (0.003)	0.065 (0.003)	1.15 (0.002)
8	3	16	30	9, 6	4.63	0.15	0.027	1.34	4.62 (0.008)	0.16 (0.002)	0.045 (0.002)	1.34 (0.002)
8	6	8	15	7, 9	4.27	0.30	0.031	0.89	4.26 (0.01)	0.31 (0.003)	0.055 (0.001)	0.90 (0.002)
8	6	8	30	7, 8	3.82	0.34	0.020	1.00	3.80 (0.006)	0.34 (0.003)	0.035 (0.001)	1.00 (0.001)
8	6	16	15	8, 9	5.13	0.18	0.052	0.89	5.11 (0.008)	0.18 (0.002)	0.094 (0.002)	0.90 (0.001)
8	6	16	30	8, 8	4.67	0.20	0.036	1.00	4.67 (0.01)	0.21 (0.003)	0.062 (0.003)	1.00 (0.002)

Table 4 Additional results from our approximation with backorder costs per unit and unit time

Parameters				Approximation (Olsson)	Optimum by simulation	Infinite lifetime
λ	A	b_1	p	R^*, Q^*	R_s^*, Q_s^*	R^*, Q^*
25	20	8	15	21, 35	21, 35	21, 36
25	20	8	30	21, 34	21, 34	21, 36
25	20	16	15	24, 34	24, 34	24, 35
25	20	16	30	24, 33	24, 33	24, 35
25	40	8	15	19, 44	19, 44	19, 50
25	40	8	30	19, 42	19, 42	19, 50
25	40	16	15	23, 41	23, 41	23, 48
25	40	16	30	23, 39	23, 39	23, 48
50	20	8	15	45, 51	45, 51	45, 51
50	20	8	30	45, 51	45, 51	45, 51
50	20	16	15	49, 50	49, 50	49, 50
50	20	16	30	49, 50	49, 50	49, 50
50	40	8	15	42, 70	42, 70	42, 70
50	40	8	30	42, 70	42, 70	42, 70
50	40	16	15	47, 69	47, 69	47, 69
50	40	16	30	47, 68	47, 68	47, 69

In all problems we have $h = 1$, $L = 1$ and $T = 3$

In Tables 2 and 3 we consider all combinations of $\lambda \in \{4, 8\}$, $A \in \{3, 6\}$, $b_1 \in \{8, 16\}$ and $p \in \{15, 30\}$. In all numerical problems in Tables 2 and 3 we have $h = 1$, $L = 1$ and $T = 3$. This means that the shelflife in these examples is equal to $T - L = 2$ units of time (i.e., slightly shorter than the shelflife considered in the numerical examples in Chiu 1995). Considering the results in Table 2, we see that our method performs rather well for all cases considered. The average value of CI over all problems in Table 2 is 1.7 %. It is interesting to note that, by using our heuristic method, we find the true optimal value of R for all cases considered. Regarding the optimal value of Q in Table 2, our solution method finds the true optimal value of Q or slightly overestimates it. In order to understand why Q^* is slightly overestimated in some cases, we present the average stock on hand, average number of backorders, average rate of outdating and average order frequency in Table 3 for the same numerical examples as in Table 2. From Table 3 it is clear that, in our model, the average number of backorders are underestimated while the average stock on hand is overestimated. However, the values obtained from our approximation are very close to the simulated values. The reason for underestimating the average number of backorders and overestimating the average stock on hand is related to the undershoot of R. Since our model is developed by averaging over base stock policies, we do not take the undershoot of R into account. Therefore, we can expect that the average number of backorders are underestimated and the average stock on hand is overestimated, which is confirmed in Table 3. We also see in Table 3 that the estimation of the average outdating rate, π_{RQ}, is not as good as the other approximations, and will therefore have a greater influence on the optimal values of R and Q obtained from our approximation. The fact that π_{RQ} is underestimated can explain why Q^* is overestimated. Clearly, since π_{RQ} is increasing in Q it is clear that we can allow for higher values of Q if π_{RQ} is underestimated. The average value of CI over all problems in Table 2 when using traditional inventory control without taking the finite lifetime into consideration

Table 5 Results from our and Chiu's approximation

Parameters				Approximation (Olsson)		Approximation (Chiu)		Optimum by simulation		Infinite lifetime	
λ	A	b_2	p	R^*, Q^*	$C_{sim}(R^*,Q^*)$	R^*, Q^*	$C_{sim}(R_s^*,Q_s^*)$	R_s^*, Q_s^*	$C_{sim}(R^*,Q^*)$	R^*, Q^*	$C_{sim}(R^*,Q^*)$
4	3	8	15	9,5	16.95 (0.05)	7,5	17.96 (0.03)	9,4	16.31 (0.05)	10,7	21.30 (0.06)
4	3	8	30	8,5	20.43 (0.05)	6,6	21.08 (0.04)	8,5	20.43 (0.05)	10,7	31.58 (0.07)
4	3	16	15	10,5	20.98 (0.05)	9,4	22.61 (0.06)	12,3	20.72 (0.06)	12,6	24.76 (0.07)
4	3	16	30	10,4	26.23 (0.06)	8,4	27.76 (0.07)	10,3	25.33 (0.06)	12,6	38.42 (0.08)
4	6	8	15	8,7	20.38 (0.04)	6,7	20.73 (0.04)	8,5	19.46 (0.05)	10,9	27.25 (0.08)
4	6	8	30	7,6	23.18 (0.06)	5,7	24.14 (0.06)	8,4	22.50 (0.04)	10,9	41.72 (0.11)
4	6	16	15	10,6	25.61 (0.06)	8,6	27.20 (0.06)	10,5	23.64 (0.04)	11,9	33.04 (0.07)
4	6	16	30	9,6	32.41 (0.06)	8,5	30.86 (0.07)	9,4	29.42 (0.06)	11,9	49.75 (0.11)
8	3	8	15	19,8	19.55 (0.05)	18,7	19.59 (0.08)	19,8	19.55 (0.05)	20,10	22.40 (0.11)
8	3	8	30	19,6	21.27 (0.07)	17,7	22.44 (0.08)	19,6	21.27 (0.07)	20,10	30.95 (0.12)
8	3	16	15	21,7	22.41 (0.06)	20,6	23.31 (0.09)	21,7	22.41 (0.06)	22,9	26.45 (0.09)
8	3	16	30	20,7	27.32 (0.07)	19,6	27.49 (0.10)	20,5	26.28 (0.10)	22,9	37.05 (0.11)
8	6	8	15	18,10	23.24 (0.06)	17,8	23.58 (0.08)	18,9	22.64 (0.07)	19,13	29.14 (0.12)
8	6	8	30	18,8	25.85 (0.06)	17,7	26.17 (0.04)	18,8	25.85 (0.06)	19,13	40.37 (0.11)
8	6	16	15	20,9	27.48 (0.06)	19,7	27.77 (0.09)	21,7	26.30 (0.07)	21,13	36.15 (0.12)
8	6	16	30	20,8	32.42 (0.09)	19,6	31.52 (0.09)	20,6	30.75 (0.09)	21,13	51.88 (0.13)

In all problems we have $h = 1$, $L = 2$ and $T = 4$. Standard deviation of simulation results in parenthesis

Table 6 A comparison between the results from our approximation and simulation for the same test problems as in Table 5

Parameters				R_s^*, Q_s^*	Approximation (Olsson)				Simulation			
λ	A	b_2	p		$E(IL^+)$	$E(B)$	π_{RQ}	$E(OF)$	$E(IL)^+$	$E(B)$	π_{RQ}	$E(OF)$
4	3	8	15	9,4	3.33	0.70	0.20	1.05	3.24 (0.001)	0.77 (0.008)	0.25 (0.003)	1.06 (0.001)
4	3	8	30	8,5	3.00	0.90	0.17	0.83	2.90 (0.006)	1.00 (0.008)	0.24 (0.002)	0.85 (0.001)
4	3	16	15	12,3	5.14	0.21	0.45	1.49	5.10 (0.01)	0.21 (0.007)	0.51 (0.003)	1.50 (0.002)
4	3	16	30	10,3	3.69	0.54	0.23	1.41	3.61 (0.01)	0.58 (0.01)	0.27 (0.002)	1.42 (0.002)
4	6	8	15	8,5	3.00	0.90	0.17	0.83	2.90 (0.006)	1.00 (0.008)	0.24 (0.002)	0.85 (0.001)
4	6	8	30	8,4	2.64	1.04	0.13	1.03	2.61 (0.01)	1.07 (0.01)	0.17 (0.04)	1.04 (0.002)
4	6	16	15	10,5	4.42	0.39	0.35	0.87	4.34 (0.008)	0.44 (0.003)	0.46 (0.004)	0.89 (0.001)
4	6	16	30	9,4	3.33	0.70	0.20	1.05	3.24 (0.001)	0.77 (0.008)	0.25 (0.003)	1.06 (0.001)
8	3	8	15	19,8	7.26	0.56	0.18	1.02	7.03 (0.002)	0.64 (0.01)	0.27 (0.004)	1.03 (0.001)
8	3	8	30	19,6	6.42	0.72	0.12	1.35	6.32 (0.03)	0.77 (0.01)	0.16 (0.004)	1.36 (0.002)
8	3	16	15	21,7	8.52	0.28	0.27	1.18	8.36 (0.02)	0.30 (0.007)	0.38 (0.006)	1.20 (0.001)
8	3	16	30	20,5	6.85	0.56	0.13	1.63	6.79 (0.02)	0.59 (0.01)	0.18 (0.004)	1.63 (0.002)
8	6	8	15	18,9	6.85	0.73	0.16	0.91	6.71 (0.02)	0.78 (0.01)	0.28 (0.004)	0.92 (0.001)
8	6	8	30	18,8	6.43	0.81	0.13	1.02	6.26 (0.02)	0.88 (0.01)	0.21 (0.004)	1.03 (0.001)
8	6	16	15	21,7	8.52	0.28	0.27	1.18	8.36 (0.02)	0.30 (0.007)	0.38 (0.006)	1.20 (0.001)
8	6	16	30	20,6	7.27	0.49	0.16	1.36	7.12 (0.02)	0.52 (0.01)	0.23 (0.006)	1.37 (0.002)

Table 7 A comparison between the results from Chiu's approximation and simulation for the same test problems as in Table 5

Parameters				R_s^*, Q_s^*	Approximation (Chiu)				Simulation			
λ	A	b_2	p		$E(IL^+)$	$E(B)$	π_{RQ}	$E(OF)$	$E(IL)^+$	$E(B)$	π_{RQ}	$E(OF)$
4	3	8	15	9,4	3.00	0.80	0.50	1.12	3.24 (0.001)	0.77 (0.008)	0.25 (0.003)	1.06 (0.001)
4	3	8	30	8,5	2.50	0.98	0.41	0.88	2.90 (0.006)	1.00 (0.008)	0.24 (0.002)	0.85 (0.001)
4	3	16	15	12,3	5.50	0.24	1.54	1.85	5.10 (0.01)	0.21 (0.007)	0.51 (0.003)	1.50 (0.002)
4	3	16	30	10,3	3.50	0.65	0.61	1.54	3.61 (0.01)	0.58 (0.01)	0.27 (0.002)	1.42 (0.002)
4	6	8	15	8,5	2.50	0.98	0.41	0.88	2.90 (0.006)	1.00 (0.008)	0.24 (0.002)	0.85 (0.001)
4	6	8	30	8,4	2.00	1.20	0.29	1.07	2.61 (0.01)	1.07 (0.01)	0.17 (0.04)	1.04 (0.002)
4	6	16	15	10,5	4.50	0.43	1.05	1.01	4.34 (0.008)	0.44 (0.003)	0.46 (0.004)	0.89 (0.001)
4	6	16	30	9,4	3.00	0.80	0.50	1.12	3.24 (0.001)	0.77 (0.008)	0.25 (0.003)	1.06 (0.001)
8	3	8	15	19,8	7.00	0.59	0.57	1.07	7.03 (0.002)	0.64 (0.01)	0.27 (0.004)	1.03 (0.001)
8	3	8	30	19,6	6.00	0.77	0.33	1.39	6.32 (0.03)	0.77 (0.01)	0.16 (0.004)	1.36 (0.002)
8	3	16	15	21,7	8.50	0.30	0.92	1.27	8.36 (0.02)	0.30 (0.007)	0.38 (0.006)	1.20 (0.001)
8	3	16	30	20,5	6.50	0.62	0.39	1.68	6.79 (0.02)	0.59 (0.01)	0.18 (0.004)	1.63 (0.002)
8	6	8	15	18,9	6.50	0.76	0.51	0.95	6.71 (0.02)	0.78 (0.01)	0.28 (0.004)	0.92 (0.001)
8	6	8	30	18,8	6.00	0.85	0.39	1.05	6.26 (0.02)	0.88 (0.01)	0.21 (0.004)	1.03 (0.001)
8	6	16	15	21,7	8.50	0.30	0.92	1.27	8.36 (0.02)	0.30 (0.007)	0.38 (0.006)	1.20 (0.001)
8	6	16	30	20,6	7.00	0.52	0.50	1.42	7.12 (0.02)	0.52 (0.01)	0.23 (0.006)	1.37 (0.002)

is 39 %. We can conclude that, from a managerial perspective, it is clear that controlling the system by disregarding the finite lifetime of items (i.e., by using standard methods with infinite lifetime) may lead to large cost increases. As an additional test, it would be interesting to investigate how our heuristic performes when R^* and Q^* are relatively large. Note that, in order to obtain larger values of R^* we must consider larger values of λ. Similarly, larger values of Q^* typically means that the ordering cost must be increased. Hence, to test the performance of our heuristic for larger values of R^* and Q^*, we consider, in Table 4, all combinations of $\lambda \in \{25, 50\}$, $A \in \{20, 40\}$, $b_1 \in \{8, 16\}$ and $p \in \{15, 30\}$. In Table 4, we still have $h = 1$, $L = 1$ and $T = 3$. From Table 4, we can conclude that our heuristic still performs very well. Note also that when λ is large, quite few items will perish. This means that the solution from the classical model with infinite lifetimes often resembles with the solution based on our heuristic, as can be seen in Table 4.

Next, in Tables 5, 6, and 7, we consider instead backorder costs per unit and evaluate the performance of our heuristic and compare our results with those obtained by Chiu (1995). In Table 5 the optimal values of R and Q are tabulated together with the corresponding simulated costs. The same parameter setting as in Tables 2 and 3 are used in Tables 5, 6, and 7, but with $h = 1$, $L = 2$ and $T = 4$. Hence, we consider a slightly longer leadtime, L, in Tables 5, 6, and 7 compared with Tables 2 and 3. This means that we introduce more stochastic variations into the system. As in the previous case (i.e., backordering costs per unit and unit time) we see again, in Table 5, that Q^* is in some cases slightly overestimated. In this case the optimal value of R is underestimated in a few cases. The performance of the heuristic developed in Chiu (1995) shows the same pattern in most cases. Considering our heuristic, the average value of CI over all problems in Table 5 is 3.2 % while Chiu's heuristic gives a corresponding value of 5.9 %. Hence, these results indicate that our approach gives slightly better results than Chiu's method.

Precisely as in the latter problems in Table 3, we observe in Table 6 that the average number of backorders are underestimated and the average stock on hand is overestimated. The approximate values of the order frequency are very accurate, while the approximation of the expected outdating rate is slightly worse. In Table 7 the corresponding results from Chiu's heuristic are shown.

6 Summary and conclusions

In this paper we extend the literature concerning continuous review inventory systems with perishable items in several directions. First, we generalize Schmidt and Nahmias (1985) and Olsson and Tydesjö (2010) by formulating an exact closed form solution for a continuous review base-stock inventory model with perishable items and partial backordering. The partial backorder model with infinite lifetimes considered in Moinzadeh (1989) is also a special case of our model. Moreover, in opposite to Perry and Posner (1998), our solution is tractable and may be used in practice. From the numerical results we can conclude that a partial backordering model, in this context, can generate policies that differ significantly from those obtained assuming that all customers are lost if there is no stock on hand when a customer arrives at the inventory.

Secondly, we provide a framework for evaluating and analyzing an inventory system with perishable items in a setting with non-negligible ordering costs. Under the assumption of a continuous review (R, Q) policy and Poisson demands we develop a heuristic method for finding near optimal values of R and Q for different types of backordering cost structures. The performance of our heuristic is evaluated in a simulation study, from which we can conclude that our heuristic method is reasonably accurate. In the special case of backorder costs per unit, we can also conclude that our approximate method seems to significantly outperform the heuristic suggested by Chiu (1995) (who only deals with case of backorder costs per unit). An important conclusion from a practical point of view is that, by using our method, the cost reductions is in general very large compared to standard inventory control methods based on the assumption of non-deteriorating goods. The results indicate that the ideas and techniques used in this paper may be used in other related problems. More specifically, it may be worth investigating the use of base-stock policies as a building block for evaluating more complicated ordering policies in related inventory problems.

An interesting topic for future research may be to investigate the impact of assuming an (R, Q) policy instead of the true optimal policy. Another line of research could be to relax the assumption of pure Poisson demand, and allow for more complex demand structures, e.g., compound Poisson.

Appendix 1

Proof of Proposition 1

Let us compare the state of the process at time points t and $t+\delta$, where δ is a small positive number. In the light of (1), we consider two different time intervals. Consider first the case when $0 \le T_1 < \tau$ and $T_S \neq 0$. Now, if there is no demand during $[t, t+\delta)$, all units in the system will age δ time units. As a result, we have

$$\begin{aligned} f_{T_1,T_2,\ldots,T_S}(t+\delta, t_1, t_2, \ldots, t_S) = f_{T_1,T_2,\ldots,T_S}&(t, t_1-\delta, t_2-\delta, \ldots, t_S-\delta) \\ &\cdot (1-\lambda\delta + o(\delta)) + o(\delta). \end{aligned} \tag{25}$$

Note that (25) can be reformulated as a sum of partial derivatives of $t, t_1, \ldots, t_S$ by using the standard trick of telescoping sums, and by taking the limit as $\delta \to 0$. That is, by adding and subtracting terms we get

$$\begin{aligned} &\frac{f_{T_1,T_2,\ldots,T_S}(t+\delta, t_1, t_2, \ldots, t_S) - f_{T_1,T_2,\ldots,T_S}(t, t_1, t_2, \ldots, t_S)}{\delta} \\ &\quad + \frac{1}{\delta}\sum_{k=1}^{S}\{f_{T_1,T_2,\ldots,T_S}(t, t_1-\delta, \ldots, t_{k-1}-\delta, t_k, \ldots, t_S) \\ &\quad - f_{T_1,T_2,\ldots,T_S}(t, t_1-\delta, \ldots, t_k-\delta, t_{k+1}, \ldots, t_S)\} \\ &= -\lambda \cdot f_{T_1,T_2,\ldots,T_S}(t, t_1-\delta, \ldots, t_S-\delta). \end{aligned} \tag{26}$$

As mentioned, by taking the limit as $\delta \to 0$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} f_{T_1,T_2,\ldots,T_S}(t, t_1, \ldots, t_S) + \sum_{k=1}^{S} \frac{\partial}{\partial t_k} f_{T_1,T_2,\ldots,T_S}(t, t_1, \ldots, t_S) \\ &\quad = -\lambda \cdot f_{T_1,T_2,\ldots,T_S}(t, t_1, \ldots, t_S) \end{aligned} \tag{27}$$

In order to obtain the limiting distribution of $f_{T_1,T_2,\ldots,T_{S_i}}(t, t_1, \ldots, t_S)$, take the limit as $t \to \infty$. This implies that

$$\sum_{k=1}^{S} \frac{\partial}{\partial t_k} f_{T_1,T_2,\ldots,T_S}(t_1, \ldots, t_S) = -\lambda \cdot f_{T_1,T_2,\ldots,T_S}(t_1, \ldots, t_S). \tag{28}$$

Similarly as (28) we obtain

$$\sum_{k=1}^{S} \frac{\partial}{\partial t_k} f_{T_1,T_2,\ldots,T_S}(t_1,\ldots,t_S) = -\mu \cdot f_{T_1,T_2,\ldots,T_S}(t_1,\ldots,t_S) \tag{29}$$

for the case when $\tau \le t_1 < T$ and $t_S \ne 0$.

In order to derive the solutions of (28) and (29), note that (28) and (29) are first order linear partial differential equations. In general, it is relatively easy to solve such equations by using, e.g., the characteristic method, see e.g., Strauss (1992). By using this method, the solutions of (28) and (29) can in a compact way be stated as

$$f_{T_1,T_2,\ldots,T_S}(t_1,t_2,\ldots,t_S) = \varphi(t_1-t_2, t_2-t_3,\ldots,t_{S-1}-t_S)\exp\left\{-\int_0^{t_1}\eta(t)dt\right\}, \tag{30}$$

where $\varphi(t_1-t_2, t_2-t_3,\ldots,t_{S-1}-t_S)$ is a differentiable function determined by the boundary conditions describing the process when a demand occurs or when an item perishes. It turns out that $\varphi(t_1-t_2, t_2-t_3,\ldots,t_{S-1}-t_S)$, in this specific case, is a constant. To prove this fact, it is possible to use exactly the same proof as in Schmidt and Nahmias (1985) with small modifications. Hence, we refer to Schmidt and Nahmias (1985) for these details.

Using the fact that $\varphi(t_1-t_2, t_2-t_3,\ldots,t_{S-1}-t_S)$ is a just a constant, C, we can express (30) more explicitly as

$$f_{T_1,T_2,\ldots,T_S}(t_1,t_2,\ldots,t_S) = \begin{cases} Ce^{-\lambda t_1} & \text{for } 0 \le t_1 < \tau \\ Ce^{-(\lambda\tau+\mu(t_1-\tau))} & \text{for } \tau \le t_1 \le T. \end{cases}$$

□

Proof of Proposition 2

In order to calculate the average stock on hand we will use the well known Little's law. Note that, if an item has been consumed by a customer demand, the average time the item has spent in stock is $\int_L^T (t_1-L) f_{T_1}(t_1)dt_1$. However, if an item has perished, the item has spent exactly $T-L$ units of time in stock. Hence, the average stock on hand is obtained as

$$E(IL)^+ = \mu\int_L^T (t_1-L) f_{T_1}(t_1)dt_1 + (T-L)\pi. \tag{31}$$

The integral in (31) can be expanded as

$$\begin{aligned}\int_L^T (t_1-L) f_{T_1}(t_1)dt_1 &= \int_L^T (t_1-L)Ce^{-(\lambda\tau+\mu(t_1-\tau))}\frac{t_1^{S-1}}{(S-1)!}dt_1 \\ &= e^{(\mu-\lambda)\tau}\left(\frac{C\cdot S}{\mu^{S+1}}\int_L^T \frac{e^{-\mu t_1}t_1^S\mu^{S+1}}{S!}dt_1\right. \\ &\quad \left. -\frac{C\cdot L}{\mu^S}\int_L^T \frac{e^{-\mu t_1}t_1^{S-1}\mu^S}{(S-1)!}dt_1\right). \end{aligned} \tag{32}$$

By recognizing $P(X \leq x) = \int_0^x e^{-\mu t_1} \mu^k {t_1}^{k-1}/(k-1)! dt_1 = 1 - \sum_{n=0}^{k-1} e^{-\mu x} (\mu x)^n / n!$ as the distribution function of a r.v. $X \in \text{Erlang}(\mu, k)$, the integrals in (32) can be simplified as

$$\int_L^T \frac{e^{-\mu t_1} t_1^S \mu^{S+1}}{S!} dt_1 = \sum_{n=0}^{S} e^{-\mu L} \frac{(\mu L)^n}{n!} - \sum_{n=0}^{S} e^{-\mu T} \frac{(\mu T)^n}{n!}$$

$$\int_L^T \frac{e^{-\mu t_1} t_1^{S-1} \mu^S}{(S-1)!} dt_1 = \sum_{n=0}^{S-1} e^{-\mu L} \frac{(\mu L)^n}{n!} - \sum_{n=0}^{S-1} e^{-\mu T} \frac{(\mu T)^n}{n!}.$$

To conclude, the average stock on hand is obtained as

$$\begin{aligned} E(IL)^+ &= \mu \int_L^T (t_1 - L) f_{T_1}(t_1) dt_1 + (T-L)\pi \\ &= e^{\tau(\mu-\lambda)} \left[\frac{C \cdot S}{\mu^S} \left(\sum_{n=0}^{S} \frac{(\mu L)^n}{n!} e^{-\mu L} - \sum_{n=0}^{S} \frac{(\mu T)^n}{n!} e^{-\mu T} \right) \right. \\ &\quad \left. - \frac{C \cdot L}{\mu^{S-1}} \left(\sum_{n=0}^{S-1} \frac{(\mu L)^n}{n!} e^{-\mu L} - \sum_{n=0}^{S-1} \frac{(\mu T)^n}{n!} e^{-\mu T} \right) \right] + (T-L)\pi. \end{aligned} \tag{33}$$

Let us proceed with the derivation of the average number of backorders when applying complete and partial backordering, respectively. Recall that for the complete backorder (CB) cost structures in (8) and (10), we have $\eta(t_1) = \lambda$ for $0 \leq t_1 < L$. Moreover, for the partial backorder (PB) cost structure in (9), we have $\eta(t_1) = 0$ for $0 \leq t_1 < \tau$ (i.e., $\lambda = 0$), and $\eta(t_1) = \mu$ for $\tau \leq t_1 < L$. Hence, by using (4) and applying Little's law we obtain

$$E(IL)^-_{CB} = \lambda \int_0^L (L - t_1) f_{T_1}(t_1) dt_1 = \lambda \int_0^L (L - t_1) C e^{-\lambda t_1} \frac{t_1^{S-1}}{(S-1)!} dt_1,$$

$$E(IL)^-_{PB} = \mu \int_\tau^L (L - t_1) f_{T_1}(t_1) dt_1 = \mu \int_\tau^L (L - t_1) C e^{-\mu(t_1 - \tau)} \frac{t_1^{S-1}}{(S-1)!} dt_1$$

In view of the derivation of $E(IL)^+$ in (33), the rest of the proof is straightforward calculus and therefore omitted. □

Proof of Proposition 3

It is clear that k units in stock is equivalent to $0 \leq T_S < T_{S-1} < \cdots < T_{k+1} < L \leq T_k < \cdots < T_1 < T$. Hence, we have

$$\begin{aligned} P(IL = k) &= \int_L^\tau \left(\int_L^{t_1} \int_L^{t_2} \cdots \int_L^{t_{k-1}} \int_0^L \int_0^{t_{k+1}} \cdots \int_0^{t_{S-1}} C \cdot e^{-\lambda t_1} dt_S dt_{S-1} \ldots dt_2 \right) dt_1 \\ &\quad + \int_\tau^T \left(\int_L^{t_1} \int_L^{t_2} \cdots \int_L^{t_{k-1}} \int_0^L \int_0^{t_{k+1}} \cdots \int_0^{t_{S-1}} C \cdot e^{-(\lambda\tau + \mu(t_1 - \tau))} dt_S dt_{S-1} \ldots dt_2 \right) dt_1 \end{aligned}$$

$$
\begin{aligned}
&= \frac{C \cdot L^{S-k}}{(S-k)!(k-1)!} \cdot \int_{L}^{\tau} e^{-\lambda t_1}(t_1 - L)^{k-1} dt_1 \\
&\quad + \frac{C \cdot L^{S-k} \cdot e^{(\mu-\lambda)\tau}}{(S-k)!(k-1)!} \cdot \int_{\tau}^{T} e^{-\mu t_1}(t_1 - L)^{k-1} dt_1
\end{aligned} \tag{34}
$$

The rest of the proof is finished by simple calculus. The integral $\int_L^\tau e^{-\lambda t_1}(t_1 - L)^{k-1} dt_1$ in (34) can be simplified as

$$
\begin{aligned}
\int_{L}^{\tau} e^{-\lambda t_1}(t_1 - L)^{k-1} dt_1 &= e^{-\lambda L} \int_{0}^{\tau - L} e^{-\lambda t_1} {t_1}^{k-1} dt_1 \\
&= \frac{e^{-\lambda L}(k-1)!}{\lambda^k} \int_{0}^{\tau - L} \frac{e^{-\lambda t_1} \lambda^k {t_i}^{k-1}}{(k-1)!} dt_1 \\
&= \frac{e^{-\lambda L}(k-1)!}{\lambda^k} \left(1 - e^{-\lambda(\tau - L)} \sum_{n=0}^{k-1} \frac{(\lambda(\tau - L))^n}{n!} \right)
\end{aligned} \tag{35}
$$

by recognizing $P(X \leq \tau - L) = \int_0^{\tau - L} e^{-\lambda t_1} \lambda^k {t_1}^{k-1}/(k-1)! dt_1$ as the distribution function of a r.v. $X \in \text{Erlang}(\lambda, k)$. By similar calculations, the integral $\int_\tau^T e^{-\mu t_1}(t_1 - L)^{k-1} dt_1$ in (34) can be stated as

$$
\begin{aligned}
\int_{\tau}^{T} e^{-\mu t_1}(t_1 - L)^{k-1} dt_1 = \frac{e^{-\mu L}(k-1)!}{\mu^k} \cdot \\
\left(e^{-\mu(\tau - L)} \sum_{n=0}^{k-1} \frac{(\mu(\tau - L))^n}{n!} - e^{-\mu(T - L)} \sum_{n=0}^{k-1} \frac{(\mu(T - L))^n}{n!} \right).
\end{aligned} \tag{36}
$$

Inserting (35) and (36) in (34) yields the desired result. □

Proof of Lemma 1

Proof of (i): By assuming that the inventory position is uniformly distributed over the integers $R+1, \ldots, R+Q$ we have

$$
\pi_{RQ}(R, Q) = \frac{1}{Q} \sum_{k=R+1}^{R+Q} \pi(k).
$$

From Olsson and Tydesjö (2010) we know that $\pi(k)$ is increasing in k. Hence, keeping R fixed we get

$$
\begin{aligned}
\pi_{RQ}(R, Q+1) - \pi_{RQ}(R, Q) &= \frac{1}{Q+1} \sum_{k=R+1}^{R+Q+1} \pi(k) - \frac{1}{Q} \sum_{k=R+1}^{R+Q} \pi(k) \\
&= \frac{Q \sum_{k=R+1}^{R+Q+1} \pi(k) - (Q+1) \sum_{k=R+1}^{R+Q} \pi(k)}{Q(Q+1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{Q\pi(R+Q+1) - \sum_{k=R+1}^{R+Q} \pi(k)}{Q(Q+1)} \\
&> \frac{Q[\pi(R+Q+1) - \pi(R+Q)]}{Q(Q+1)} > 0.
\end{aligned}
$$

Here we have used that $\pi(k)$ is increasing in k. Hence, $\pi_{RQ}(R, Q+1) > \pi_{RQ}(R, Q)$ for all positive integer values of Q. To show that π_{RQ} is increasing in R for a fixed Q is straightforward and can be done in a similar way. □

Proof of (ii): From (4) and (21), $\pi(k)$ can be stated as

$$\pi(k) = \frac{e^{-\lambda T} T^{k-1}}{\int_0^T t^{k-1} e^{-\lambda t} dt}.$$

Since $\lambda > 0$ we note that $e^{-\lambda t} < 1$ for $t > 0$. This gives an upper bound on the integral,

$$\int_0^T t^{k-1} e^{-\lambda t} dt \leq \int_0^T t^{k-1} dt = \frac{T^k}{k}.$$

This means that

$$\pi(k) \geq e^{-\lambda T} \frac{k}{T}$$

which results in

$$
\begin{aligned}
\pi_{RQ}(R, Q) &= \frac{1}{Q} \sum_{k=R+1}^{R+Q} \pi(k) \geq \frac{e^{-\lambda T}}{T} \cdot \frac{1}{Q} \sum_{k=R+1}^{R+Q} k = \frac{e^{-\lambda T}}{T} \cdot \frac{1}{Q} (2R+Q+1) Q \\
&= \frac{e^{-\lambda T}}{T} (2R+Q+1) \to \infty \text{ when } R \to \infty \text{ or } Q \to \infty.
\end{aligned}
$$

□

References

Abad, P. L. (1996). Optimal pricing and lot-sizing under conditions of perishability and partial backordering. *Management Science*, *42*, 1093–1104.

Axsäter, S. (2006). *Inventory control*. New York: Springer.

Berk, E., & Gürler, Ü. (2008). Analysis of the (Q, r) inventory model for perishables with positive lead times and lost sales. *Operations Research*, *56*, 1238–1246.

Chakravarthy, S. R., & Daniel, J. K. (2004). A Markovian inventory system with random shelf time and back orders. *Computers & Industrial Engineering*, *47*, 315–337.

Chang, H. J., & Dye, C. Y. (1999). An EOQ model for deteriorating items with time varying demand and partial backlogging. *The Journal of the Operational Research Society*, *50*, 1176–1182.

Chiu, H. N. (1995). An approximation to the continuous review inventory model with perishable items and leadtimes. *European Journal of Operational Research*, *87*, 93–108.

Federgruen, A., & Zheng, Y. S. (1992). An efficient algorithm for finding an optimal (r, Q) policy in continuous review stochastic inventory systems. *Operations Research*, *40*, 808–813.

Grocery Manufacturer Association. (2008). 2008 Joint industry unsaleables report: The real causes and actionable solutions. http://www.gmabrands.com/publications/UnsaleablesFINAL091108

Johansen, S. G., & Thorstenson, A. (1996). Optimal (r, Q) inventory policies with poisson demands and lost sales; discounted and undiscounted cases. *International Journal of Production Economics*, *46–47*, 359–371.

Kalpakam, S., & Sapna, K. P. (1995). $(S-1, S)$ Perishable systems with stochastic leadtimes. *Mathematical and Computer Modeling*, *21*, 95–104.

Karaesmen, I., Scheller-Wolf, A., & Deniz, B. (2011). Managing perishable and aging inventories: review and future research directions, planning production and inventories in the extended enterprise. In K. Kempf, P. Keskinocak, & P. Uzsoy (Eds.), *International series in operations research and management science*. New York: Springer.

Katehakis, M. N., & Smit, L. C. (2012). On computing optimal (Q, r) replenishment policies under quantity discounts. *Annals of Operations Research*, *200*, 279–298.

Kingman, J. F. C. (1964). On doubly stochastic poisson processes. *Mathematical Proceedings of the Cambridge Philosophical Society*, *60*, 923–930.

Liu, L., & Yang, T. (1999). An (s, S) random lifetime inventory model with a positive leadtime. *European Journal of Operational Research*, *113*, 52–63.

Minner, S., & Transchel, S. (2010). Periodic review inventory-control for perishable products under service-level constraints. *OR Spectrum*, *32*, 979–996.

Moinzadeh, K. (1989). Operating characteristics of the $(S-1, S)$ inventory system with partial backorders and constant resupply times. *Management Science*, *35*, 472–477.

Moinzadeh, K., & Schmidt, C. P. (1991). An $(S-1, S)$ inventory system with emergency orders. *Operations Research*, *39*, 308–321.

Nahmias, S., & Wang, S. S. (1979). A heuristic lot size reorder point model for decaying inventories. *Management Science*, *25*, 90–97.

Nahmias, S. (1982). Perishable inventory theory: A review. *Operations Research*, *30*, 680–708.

Nahmias, S. (2011). *Perishable inventory systems*. New York: Springer.

Olsson, F., & Tydesjö, P. (2010). Inventory problems with perishable items: Fixed lifetimes and backlogging. *European Journal of Operational Research*, *202*, 131–137.

Olsson, F. (2011). *Emergency lateral transshipments in inventory systems with positive transshipment leadtimes*. Working paper, Lund University, Lund.

Perry, D., & Posner, M. J. M. (1998). An $(S-1, S)$ inventory system with fixed shelf life and constant leadtimes. *Operations Research*, *46*, S65–S71.

Schmidt, C. P., & Nahmias, S. (1985). $(S-1, S)$ policies for perishable inventory. *Management Science*, *31*, 719–728.

Strauss, W. A. (1992). *Partial Differential Equations: An Introduction*. New York: Wiley.

Weiss, H. J. (1980). Optimal Ordering Policies for Continuous Review Perishable Inventory Models. *Operations Research*, *28*, 365–374.