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ON THE CONSTRUCTION OF CUBATURE FORMULAS INVARIANT UNDER DIHEDRAL GROUPS

É. A. Shamsiev

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We study cubature formulas invariant under the dihedral group of order $16p$.

In 1961, Sobolev proposed a method for the construction of cubature formulas for a two-dimensional sphere invariant under finite rotation groups [1]. Later, this method was developed by Salikhov [2, 3], Lebedev [4, 5], and Mysovskikh [6–8]. The main results are given in [9]. It is worth noting that the formulas constructed in the indicated works are invariant either under rotation groups or under groups of transformations of regular polyhedra. In [10–12], for the construction of cubature formulas, a dihedral group of order $8(2p+1)$ was used. The present paper is devoted to the investigation of cubature formulas invariant under the dihedral group of order $16p$.

It is known [13] that the group of transformations G_{4p} of a regular $4p$-gon into itself is generated by reflections, and the ring of its invariant forms is generated by basis invariant forms of degrees 2 and $4p$. By virtue of the orthogonality of the group G_{4p}, it is obvious that the invariant form of degree 2 is $r^2 = x^2 + y^2$. Let $\Pi_{4p}(x, y)$ denote a basis invariant form of degree $4p$ and let $a^{(k)}$ and $b^{(k)}$ denote, respectively, the vertices and the projections of centers of edges of a regular $4p$-gon onto the unit circle S_1:

$$a^{(k)} = \left(\cos\frac{k\pi}{2p}, \sin\frac{k\pi}{2p}\right), \qquad b^{(k)} = \left(\cos\frac{2k-1}{4p}\pi, \sin\frac{2k-1}{4p}\pi\right),$$

$$k = 1, 2, 3, \ldots, 4p.$$

We need the value of the integral

$$\int_{S_1} \Pi_{4p}(x, y)\, dS = \int_0^{2\pi} \Pi_{4p}(\cos\varphi, \sin\varphi)\, d\varphi.$$

To determine this value, we use the fact that the polynomial $(x+iy)^{4p}$ (i is the imaginary unit) is invariant under the rotation group $\tilde{G}_{4p}$ of a regular $4p$-gon (a subgroup of index 2 of the group G_{4p}), and the integral of it over S_1 is equal to zero by virtue of orthogonality. It is easy to verify that

$$(x+iy)^{4p} = r^{4p} - 8p^2\Pi_{4p}(x, y) + i4p\Delta_{4p}(x, y),$$

Tashkent Technical University, Tashkent, Uzbekistan.

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where $\Delta_{4p}(x, y)$ is the Jacobian of the basis invariant forms r^2 and $\Pi_{4p}(x, y)$. Thus,

$$\int_{S_1}(x+iy)^{4p}\,dS = \int_{S_1}(r^{4p} - 8p^2\Pi_{4p}(x, y))\,dS + i\int_{S_1}4p\Delta_{4p}(x, y)\,dS = 0,$$

which implies that

$$\int_{S_1} r^{4p}\,dS = \int_0^{2\pi} d\varphi = 2\pi = 8p^2\int_{S_1}\Pi_{4p}(x, y)\,dS = 8p^2\int_0^{2\pi}\Pi_{4p}(\cos\varphi, \sin\varphi)\,d\varphi,$$

i.e.,

$$\int_0^{2\pi}\Pi_{4p}(\cos\varphi, \sin\varphi)\,d\varphi = \frac{\pi}{4p^2}.$$

Taking into account that an invariant polynomial takes equal values at points of the same orbit, and $a^{(4p)} = (1, 0)$, we obtain

$$\Pi_{4p}(a^{(k)}) = \Delta_{4p}(a^{(k)}) = \Delta_{4p}(b^{(k)}) = 0$$

because the product of the left-hand sides of the equations of axes of symmetry coincides, up to a constant, with $\Delta_{4p}(x, y)$, and the points $a^{(k)}$ and $b^{(k)}$ are located on the axes of symmetry.

To determine $\Pi_{4p}(b^{(k)})$, we perform the following procedure: Together with the polynomial $(x+iy)^{4p}$, the polynomial

$$(x-iy)^{4p} = r^{4p} - 8p^2\Pi_{4p}(x, y) - i4p\Delta_{4p}(x, y)$$

is invariant under the group $\tilde{G}_{4p}$. Multiplying the polynomials $(x+iy)^{4p}$ and $(x-iy)^{4p}$ together, we obtain

$$(x+iy)^{4p}(x-iy)^{4p} = [r^{4p} - 8p^2\Pi_{4p}(x, y)]^2 - i^2 16p^2\Delta_{4p}^2(x, y),$$

or

$$(x^2+y^2)^{4p} = r^{8p} = [r^{4p} - 8p^2\Pi_{4p}(x, y)]^2 + 16p^2\Delta_{4p}^2(x, y).$$

Substituting the value of $b^{(k)}$ for (x, y), we get $1 = [1 - 8p^2\Pi_{4p}(b^{(k)})]^2$. This yields

$$1 - 8p^2\Pi_{4p}(b^{(k)}) = -1, \qquad \Pi_{4p}(b^{(k)}) = \frac{1}{4p^2}.$$

Note that $1 - 8p^2\Pi_{4p}(b^{(k)})$ cannot be equal to 1 because, in this case, $8p^2\Pi_{4p}(b^{(k)}) = 0$, which contradicts the quadrature formula

$$\int_{S_1} f(x, y)\, dS \cong \frac{\pi}{4p} \sum_{k=1}^{4p} [f(a^{(k)}) + f(b^{(k)})],$$

which is invariant under the group G_{8p} and has the $(8p-1)$th accuracy degree. The contradiction lies in the fact that

$$\int_{S_1} \Pi_{4p}(x, y)\, dS \neq 0,$$

whereas the quadrature sum is equal to 0.

In the three-dimensional real space R^3, we consider the group G_{4p} complemented with symmetry transformation with respect to the plane Oxy. We obtain the dihedral group DG_{4p} of order $16p$ generated by reflections [14, pp. 17–20]. It is obvious that the ring of invariant forms of the group DG_{4p} is generated by the basis invariant forms r^2, $\Pi_{4p}(x, y)$, and z^2. On the surface of the sphere

$$S_2 = \{ (x, y, z) \in R^3 \,|\, x^2 + y^2 + z^2 = 1 \},$$

one of the basis invariant forms of the second degree is linearly expressed in terms of the second form, e.g., $r^2 = 1 - z^2$. Therefore, in this case, the basis is formed by the polynomials $\Pi_{4p}(x, y)$ and z^2. Passing to the spherical coordinate system, we easily obtain

$$\int_{S_2} z^{2q}\, dS = 2\pi \int_0^{2\pi} \cos^{2q}\theta \sin\theta\, d\theta = \frac{4\pi}{2q+1},$$

$$\int_{S_2} \Pi_{4p}(x, y) z^{2q}\, dS = \int_0^{2\pi} \Pi_{4p}(\cos\varphi, \sin\varphi)\, d\varphi \int_0^{\pi} \sin^{4p}\theta \cos^{2q}\theta \sin\theta\, d\theta$$

$$= \frac{1}{4p^2} \frac{2(4p)!!(2q-1)!!}{(4p+2q+1)!} \pi, \quad q = 0, 1, 2, \ldots,$$

where $(2m)!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2m$ and $(2m+1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m+1)$.

1. We write all linearly independent invariant polynomials of the group DG_{4p} up to degree $(4p + 4s - 3)$ on S_2, $s \le p$:

$$1, z^2, z^4, z^6, \ldots, z^{4p+4s-4}, \Pi_{4p}(x, y), \Pi_{4p}(x, y) \cdot z^2, \Pi_{4p}(x, y) \cdot z^4, \ldots, \Pi_{4p}(x, y) \cdot z^{4s-4}. \quad (1)$$

We construct a cubature formula of accuracy degree $4p + 4s - 3$ in the form

$$\int_{S_2} f(x, y, z)dS \cong A_0 \sum_{k=1}^{4p} f\left(\cos\frac{(2k-1)\pi}{4p}, \sin\frac{(2k-1)\pi}{4p}, 0\right)$$

$$+ \sum_1^2 \left[B_0 f(0,0,\pm 1) + \sum_{i=1}^{s-1} A_i \sum_{k=1}^{4p} f\left(\sqrt{1-m_i^2}\cos\frac{(2k-1)\pi}{4p}, \sqrt{1-m_i^2}\sin\frac{(2k-1)\pi}{4p}, \pm m_i\right)\right.$$

$$+ \sum_{j=1}^{p+s-1} B_j \sum_{k=1}^{4p} f\left(\sqrt{1-n_j^2}\cos\frac{k\pi}{2p}, \sqrt{1-n_j^2}\sin\frac{k\pi}{2p}, \pm n_j\right), \tag{2}$$

where the outer sum in the second term corresponds to the change in the sign of the last coordinate of nodes, and A_0, B_0, A_i, m_i, B_j, and n_j are the parameters to be determined $(0 < m_i$ and $n_j < 1)$.

The requirement that the cubature formula (2) be exact for polynomials (1) leads to the following system of nonlinear algebraic equations:

$$2B_0 + 8p\sum_{j=1}^{p+s-1} B_j + 4pA_0 + 8p\sum_{i=1}^{s-1} A_i = 4\pi,$$

$$2B_0 + 8p\left[\sum_{j=1}^{p+s-1} n_j^{2q} B_j + \sum_{i=1}^{s-1} m_i^{2q} A_i\right] = \frac{4\pi}{2q+1}, \quad q = 1, 2, \dots, 2(p+s-1),$$

$$4pA_0 + 8p\sum_{i=1}^{s-1}(1-m_i^2)^{2p} A_i = \frac{2(4p)!!}{(4p+1)!!}\pi,$$

$$8p\sum_{i=1}^{s-1}(1-m_i^2)^{2p} m_i^{2q} A_i = \frac{2(4p)!!(2q-1)\pi}{(4p+2q+1)!!}, \quad q = 1, 2, \dots, 2(s-1).$$

Performing the substitutions

$$T_0 = \frac{4p}{\pi}A_0, \quad T_i = \frac{8p}{\pi}(1-m_i^2)^{2p} A_i, \quad t_i = m_i^2, \quad i = 1, 2, 3, \dots, s-1,$$

we obtain the following relations from the last $2s-1$ equations of the system:

$$T_0 + \sum_{i=1}^{s-1} T_i = 2\frac{(4p)!!}{(4p+1)!!},$$

$$\sum_{i=1}^{s-1} t_i^q T_i = \frac{2(4p)!!(2q-1)!!}{(4p+2q+1)!}, \quad q = 1, 2, \dots, 2s-2.$$

This implies that T_0, T_i, and t_i are parameters of the Gauss–Markov quadrature formula for the segment $[0, 1]$ with weight $\dfrac{(1-t)^{2p}}{\sqrt{t}}$ and one fixed node $t_0 = 0$:

$$\int_0^1 \frac{(1-t)^{2p}}{\sqrt{t}} \varphi(t)\,dt \cong \sum_{i=1}^{s-1} T_i \varphi(t_i) + T_0 \varphi(0).$$

Thus,

$$A_0 = \frac{\pi}{4p} T_0, \quad A_i = \frac{\pi}{8p(1-t)^{2p}} T_0, \quad m_i = \sqrt{t_i}, \quad i = 1, 2, 3, \dots, s-1.$$

We determine the last parameters of the cubature formula (2) from the system

$$2B_0 + 8p \sum_{j=1}^{p+s-1} B_j = 4\pi - \pi \left[\sum_{i=1}^{s-1} \frac{1}{(1-t_i)^{2p}} T_i + T_0 \right],$$

$$2B_0 + 8p \sum_{j=1}^{p+s-1} n_j^{2q} B_j = \frac{4\pi}{2q+1} - \pi \sum_{i=1}^{s-1} \frac{t_i^q}{(1-t_i)^{2p}} T_i, \quad q = 1, 2, \dots, 2(p+s-1).$$

After the introduction of new notation, this system is solved in the standard way [9, pp. 105–107]). The number $N = 4p(2p + 4s - 3) + 2$ of nodes of the cubature formula (2) is greater than the lower bound of the number of nodes by $4p^2 + 8ps - 4s^2 - 10p + 2s + 2$ [9, p. 203]. We give the values of the parameters of the cubature formula for two special cases:

(a) $p = 1$, $s = 1$:

$$A_0 = \frac{4\pi}{15}, \quad B_0 = \frac{4\pi}{15}, \quad B_1 = \frac{3\pi}{10}, \quad n_1 = \sqrt{\frac{1}{3}};$$

(b) $p = 2$, $s = 1$:

$$A_0 = \frac{2\pi}{3^2 \cdot 5 \cdot 7}, \quad B_0 = \frac{1807954451\pi}{3164988099},$$

$$B_1 = \frac{66624616453 - 36295045\sqrt{610177}}{759597143760}\pi, \quad n_1 = \sqrt{\frac{16789 - \sqrt{158127145}}{50058}},$$

$$B_2 = \frac{66624616453 + 36295045\sqrt{610177}}{759597143760}\pi, \quad n_2 = \sqrt{\frac{16789 + \sqrt{158127145}}{50058}}.$$

By analogy, we obtain the following cubature formula of accuracy degree $4p + 4s - 3$, which is also invariant under the group DG_{4p} and contains $4p - 2$ nodes more than formula (2):

$$\int_{S_2} f(x,y,z)\,dS \cong B_0 \sum_{k=1}^{4p} f\left(\cos\frac{k\pi}{2p}, \sin\frac{k\pi}{2p}, 0\right) + A_0 \sum_{k=1}^{4p} f\left(\cos\frac{(2k-1)\pi}{4p}, \sin\frac{(2k-1)\pi}{4p}, 0\right)$$

$$+ \sum_{1}^{2}\left[\sum_{j=1}^{p+s-1} D_j \sum_{k=1}^{4p} f\left(\sqrt{1-l_j^2}\cos\frac{k\pi}{2p}, \sqrt{1-l_j^2}\sin\frac{k\pi}{2p}, \pm l_j\right)\right.$$

$$\left.+ \sum_{i=1}^{s-1} A_i \sum_{k=1}^{4p} f\left(\sqrt{1-m_i^2}\cos\frac{(2k-1)\pi}{4p}, \sqrt{1-m_j^2}\sin\frac{(2k-1)\pi}{4p}, \pm m_i\right)\right]. \tag{3}$$

Here, A_0, A_i, and m_i are the same as in the cubature formula (2). The parameters D_j and l_j are determined from the system

$$4pB_0 + 8p\sum_{j=1}^{p+s-1} D_j = 4\pi - \pi\left[\sum_{i=1}^{s-1}\frac{1}{(1-t_i)^{2p}}T_i + T_0\right],$$

$$8p\sum_{j=1}^{p+s-1} l_j^{2q} D_j = \frac{4\pi}{2q+1} - \pi\sum_{i=1}^{s-1}\frac{t_i^q}{(1-t_i)^{2p}}T_i, \quad q = 1, 2, \ldots, 2(p+s-1).$$

The cubature formula (3) for $s = 1$ and an arbitrary p has the parameters

$$A_0 = \frac{(4p)!!}{2p(4p+1)!!}\pi,$$

$$B_0 = \frac{\pi}{p}\left[1 - \frac{(4p)!!}{2(4p+1)!!} - \frac{1}{2}\sum_{j=1}^{p}\frac{1}{\tau_j}K_j\right],$$

$$l_j = \sqrt{\tau_j}, \qquad D_j = \frac{\pi}{4p\tau_j}K_j,$$

where τ_j and K_j are the parameters of a Gauss-type quadrature formula for the segment $[0, 1]$ with weight $\sqrt{t}$, namely,

$$\int_0^1 \sqrt{t}\varphi(t)\,dt \cong \sum_{j=1}^{p} K_j\varphi(\tau_j). \tag{4}$$

2. We now pass to the construction of a cubature formula of accuracy degree $4p + 4s - 1$. We seek it in the form

$$\int_{S_2} f(x, y, z)\,dS \;\cong\; A_0 \sum_{k=1}^{4p} f\left(\cos\frac{k\pi}{2p},\ \sin\frac{k\pi}{2p},\ 0\right)$$

$$+ \sum_{1}^{2}\left[B_0 f(0, 0, \pm 1) + \sum_{j=1}^{p+s-1} B_j \sum_{k=1}^{4p} f\left(\sqrt{1-n_j^2}\cos\frac{k\pi}{2p}, \sqrt{1-n_j^2}\sin\frac{k\pi}{2p}, \pm n_j\right)\right.$$

$$\left. + \sum_{i=1}^{s} A_i \sum_{k=1}^{4p} f\left(\sqrt{1-m_i^2}\cos\frac{(2k-1)\pi}{4p}, \sqrt{1-m_i^2}\sin\frac{(2k-1)\pi}{4p}, \pm m_i\right)\right]. \tag{5}$$

Requiring that the cubature formula (5) exactly integrate polynomials (1) and $\Pi_{4p}(x, y)z^{4s-2}$, $z^{4p+4s-2}$, we obtain the system of equations

$$2B_0 + 4pA_0 + 8p\left[\sum_{j=1}^{p+s-1} B_j + \sum_{i=1}^{s} A_i\right] = 4\pi,$$

$$2B_0 + 8p\left[\sum_{j=1}^{p+s-1} n_j^{2q} B_j + \sum_{i=1}^{s} m_i^{2q} C_i\right] = \frac{4\pi}{2q+1}, \quad q = 1, 2, \dots, 2(p+s)-1,$$

$$8p\sum_{i=1}^{s-1}(1-m_i^2)^{2p} m_i^{2q} A_i = \frac{2(4p)!!(2q-1)!!}{(4p+2q+1)!!}\pi, \quad q = 0, 1, 2, \dots, 2s-1.$$

Performing the substitutions

$$T_i = \frac{8p}{\pi}(1-m_i^2)A_j \qquad \text{and} \qquad t_i = n_i^2, \quad i = 1, 2, \dots, s,$$

we establish that T_i and t_i are the parameters of the Gauss quadrature formula for the segment $[0, 1]$ with weight $\dfrac{(1-t)^{2p}}{\sqrt{t}}$:

$$\int_0^1 \frac{(1-t)^{2p}}{\sqrt{t}}\varphi(t)\,dt \;\cong\; \sum_{i=1}^{s} T_i \varphi(t_i).$$

Hence,

$$m_i = \sqrt{t_i}, \qquad A_i = \frac{\pi}{8p(1-t_i)^{2p}}T_i.$$

The last parameters of the cubature formula are determined from the system

$$2B_0 + 4pA_0 + 8p\sum_{j=1}^{p+s-1} B_j = 4\pi - \pi\sum_{i=1}^{s}\frac{1}{(1-t_i)^{2p}}T_i,$$

$$2B_0 + 8p\sum_{j=1}^{p+s-1} n_j^{2q}B_j = \frac{4\pi}{2q+1} - \pi\sum_{i=1}^{s}\frac{t_i^q}{(1-t_i)^{2p}}T_i, \quad q = 1, 2, \dots, 2(p+s)-1.$$

The number $N = 4p(2p+4s-1)+2$ of nodes of the cubature formula (5) is greater than the lower bound of the number of nodes by $4p^2 - 4s^2 + 8ps - 6p - 2s + 2$. For $s = 0$, we have the known formula obtained by the method of repeated application of quadrature formulas [9, pp. 119–123]. We give the values of the parameters of the cubature formula (5) for two special cases:

(a) $p = s = 1$:

$$A_0 = \frac{\pi}{5}, \quad B_0 = \frac{4\pi}{27}, \quad A_1 = \frac{49\pi}{270}, \quad B_1 = \frac{49\pi}{270}, \quad m_1 = \sqrt{\frac{1}{7}}, \quad n_1 = \sqrt{\frac{4}{7}};$$

the number $N = 22$ of nodes is greater than the corresponding lower bound by 2;

(b) $p = 2$, $s = 1$:

$$A_0 = -\frac{5581\pi}{3^3\cdot5^5\cdot7}, \quad B_0 = \frac{11096\pi}{3^2\cdot5^3\cdot7}, \quad A_1 = \frac{14641\pi}{3^2\cdot5^5\cdot7},$$

$$B_1 = \frac{1403780618 + 9541127\sqrt{606}}{2^4\cdot3^3\cdot5^5\cdot7\cdot23\cdot101}\pi, \quad m_1 = \sqrt{\frac{1}{11}}, \quad n_1 = \sqrt{\frac{141 - 2\sqrt{606}}{253}},$$

$$B_2 = \frac{1403780618 - 9541127\sqrt{606}}{2^4\cdot3^3\cdot5^5\cdot7\cdot23\cdot101}\pi, \quad n_2 = \sqrt{\frac{141 + 2\sqrt{606}}{253}}.$$

Another cubature formula of accuracy degree $4p + 4s - 1$, which is also invariant under the group DG_{4p}, has the form

$$\int_{S_2} f(x, y, z)\,dS \cong \sum_1^2\Bigg[\sum_{j=1}^{p+s} D_j \sum_{k=1}^{4p} f\left(\sqrt{1-l_j^2}\cos\frac{k\pi}{2p}, \sqrt{1-l_j^2}\sin\frac{k\pi}{2p}, \pm l_j\right)$$

$$+ \sum_{i=1}^{s} A_i \sum_{k=1}^{4p} f\left(\sqrt{1-m_i^2}\cos\frac{(2k-1)\pi}{4p}, \sqrt{1-m_i^2}\sin\frac{(2k-1)\pi}{4p}, \pm m_i\right)\Bigg], \tag{6}$$

where A_i and m_i take the same values as in the cubature formula (5). The values of the parameters D_j and l_j are determined from the system

$$8p \sum_{j=1}^{p+s-1} D_j = 4\pi - \pi \sum_{i=1}^{s} \frac{1}{(1-t_i)^{2p}} T_i ,$$

$$8p \sum_{j=1}^{p+s} l_j^{2q} D_j = \frac{4\pi}{2q+1} - \pi \sum_{i=1}^{s} \frac{t_i^q}{(1-t_i)^{2p}} T_i , \quad q = 1, 2, \dots, 2(p+s)-1.$$

The number of nodes of the cubature formula (6) is greater than the number of nodes of the cubature formula (5) by $4p-2$.

Note that, for $4p-2$, the cubature formula (6) is also known [9, pp. 119–123].

We give the values of the parameters in the cubature formula (6) for $s = 0$:

$$A_1 = \frac{49\pi}{270}, \quad D_1 = \frac{3053 - 29\sqrt{71}}{1917}\pi,$$

$$D_2 = \frac{3053 + 29\sqrt{71}}{1917}\pi, \quad m_1 = \sqrt{\frac{1}{7}},$$

$$l_1 = \sqrt{\frac{32 - 3\sqrt{71}}{77}}, \quad l_2 = \sqrt{\frac{32 + 3\sqrt{71}}{77}}.$$

The following statement is true:

Theorem 1. *There does not exist a cubature formula of the form (5) or (6) invariant under the group DG_{4p} that has the algebraic accuracy degree greater than $8p-1$.*

Proof. The planes of symmetry of the group DG_{4p} are defined by the equations [15]

$$x \sin\frac{k\pi}{4p} - y\cos\frac{k\pi}{4p} = 0, \quad k = 0, 1, 2, \dots, 4p-1,$$

and $z = 0$. Multiplying the left-hand sides of the first $4p$ equations together, we obtain the polynomial

$$P(x, y) = \prod_{k=0}^{4p-1} \left(x\sin\frac{k\pi}{4p} - y\cos\frac{k\pi}{4p} \right)$$

of degree $4p$. The polynomial $P^2(x, y)$ is nonnegative everywhere on S_2, and the integral of this polynomial over the sphere is positive. However, the cubature formula gives the value zero independently of the number of sets of nodes because the nodes lie on planes of symmetry.

The theorem is proved.

In conclusion, we present the following cubature formula of accuracy degree $4p + 1$, which is derived from formulas (2) and (3) for $s = 1$:

$$\int_{S_2} f(x, y, z)\,dS \cong B_0 \sum_{k=1}^{4p} f\left(\cos\frac{k\pi}{2p},\ \sin\frac{k\pi}{2p},\ 0\right) + A_0 \sum_{k=1}^{4p} f\left(\cos\frac{(2k-1)\pi}{4p},\ \sin\frac{(2k-1)\pi}{4p},\ 0\right)$$

$$+ \sum_{1}^{2} \sum_{j=1}^{p} A_j f\left(\sqrt{1-m_j^2}\cos\frac{(2k-1)\pi}{4p},\ \sqrt{1-m_j^2}\sin\frac{(2k-1)\pi}{4p},\ \pm m_j\right),$$

where

$$A_0 = \frac{\pi}{2p}\left[\frac{(4p)!!}{(4p+1)!!} - \sum_{j=1}^{p}\frac{1-\tau_j}{\tau_j}K_j\right],$$

$$B_0 = \frac{\pi}{2p}\left[2 - \frac{(4p)!!}{(4p+1)!!} + \sum_{j=1}^{p}\frac{(1-\tau_j)^{2p}}{\tau_j}K_j - \sum_{j=1}^{p}\frac{1}{\tau_j}K_j\right],$$

and the parameters K_j and τ_j are determined by the quadrature formula (4).

We should also mention the work [16], where an algorithm was proposed for the construction of weight cubature formulas for a two-dimensional sphere that are invariant under the group DG_m (the group $\overline{D}_m$ in [16]). In these formulas, the algebraic accuracy degree n is independent of the number m because not only points lying on the planes of symmetry of the group DG_m but also points in general position may be nodes. As an invariant form of degree m, the polynomial $\mathrm{Re}\,(x+iy)^m$ is taken, which, for even values of m, contains a power of the polynomial $x^2 + y^2$ ($(x^2+y^2)^{m/2}$ is taken with plus sign if $m = 4p$ or with minus sign if $m = 4p + 2$; the remaining expression is the basis invariant form $\Pi_m(x, y)$ with a certain coefficient). Moreover, the process of determination of the values of the polynomial $\mathrm{Re}\,(x+iy)^m$ at the nodes of the cubature formula becomes more complicated, whereas the determination of the integral containing this polynomial is simplified (it is equal to zero due to the orthogonality of $\mathrm{Re}\,(x+iy)^m$). For this reason, the algorithm is presented in the moment form. The system for the determination of the parameters of the cubature formula is reduced to several subsystems, which are solved turn by turn. For the solution of these subsystems, a method more stable than the known methods is proposed. The quality of the relations obtained is estimated by the efficiency factor

$$\eta = \frac{(n+1)^2}{3N},$$

where $(n+1)^2$ is the number of spherical polynomials exactly integrable by the cubature formula and N is the number of nodes. The authors note that their algorithm enables one to obtain cubature formulas with efficiency factors

$$\eta = \frac{2}{3} \quad \text{and} \quad \eta = \frac{8}{9} \quad \text{as} \quad n \to \infty.$$

In [16], surfaces for which this method can be used are also indicated. In conclusion, it is noted that Kazakov realized this algorithm in the QR software package for IBM-compatible personal computers.

Our method proposed in the present paper enables one to construct cubature formulas invariant under the group DG_{2m} with algebraic accuracy of at most $4m - 1$. Due to a successful selection of a set of nodes and the use of a simplified form of the polynomial $\Pi_{2m}(x, y)$, the system for the determination of the parameters of a cubature formula is decomposed into two standardly solvable systems represented in explicit form. The cases of even and odd m are different because, in one case, $\Pi_{2m}(x, y)$ is equal to zero at the nodes whose projections to the plane Oxy lie on the radius vectors of the vertices of a $2m$-triangle, and, in the other case, they lie on the radius vectors of the centers of edges. The cubature formulas contain the parameter s, $0 \le s \le p$. Depending on the value of this parameter, the efficiency factor of the asymptotics of the cubature formula can take values from $\eta = \frac{2}{3}$ $(s = 0)$ to $\eta = \frac{8}{9}$ $(s = p)$.

In conclusion, we also note the work [1], where dihedral groups were used for the construction of cubature formulas for a torus, and the work [18], where our algorithm was realized for a disk.

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