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Spectral Analysis and Spectral Synthesis on Polynomial Hypergroups

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Abstract. This paper presents some recent results concerning spectral analysis and spectral synthesis on polynomial hypergroups. The main results show that both discrete spectral analysis and discrete spectral synthesis hold for any polynomial hypergroup.

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1. Introduction

Spectral synthesis deals with the description of translation invariant function spaces over topological groups. Classical spectral synthesis problems have been considered in $L^1(G)$, where G is a locally compact abelian group. These problems are closely related to Wiener–Tauberian theory. In this work, however, our basic function space is the set of all continuous complex valued functions instead of L^1, hence maybe we should use the terms "continuous spectral analysis" and "continuous spectral synthesis". In the group case this means that a locally compact abelian group is given and we consider the set of all continuous complex valued functions on it, equipped with the pointwise linear operations and with the topology of uniform convergence on compact sets. In order to set up the problem of spectral analysis and spectral synthesis in this context we have to define exponential functions and exponential monomials on commutative topological groups. Continuous homomorphisms of such groups into the additive topological group of complex numbers, and into the multiplicative topological group of nonzero complex numbers are called *additive functions*, and *exponential functions*, or *exponentials*, respectively. A *polynomial* on such a group is a polynomial of additive functions. An *exponential monomial* is a product of a polynomial and an exponential function. An *exponential polynomial* is a sum of exponential monomials. Now the problem of spectral analysis and spectral synthesis can be formulated: is it true, that any nonzero, closed, translation invariant linear subspace of

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the space mentioned above (in other words a *variety*) contains an exponential function (*spectral analysis*), and is it true, that in any variety the linear hull of all exponential monomials is dense (*spectral synthesis*)? It is not hard to show that if a variety contains an exponential monomial, then it contains the corresponding exponential, too. Hence for nonzero varieties spectral synthesis implies spectral analysis. The spectral analysis question can be reformulated in the following way: is it true, that any variety contains a minimal (that means, one-dimensional) variety? The corresponding reformulation of the spectral analysis problem asks if any variety is the sum of finite dimensional varieties. Concerning continuous spectral analysis and spectral synthesis problems on locally compact abelian groups the reader is referred to Schwartz [12], Lefranc [8], Székelyhidi [15], [16]. For more about L^1-spectral synthesis on hypergroups we refer to the paper [17]. In [3] a Wiener–Tauberian theorem is presented for commutative locally compact hypergroups, whose dual is a hypergroup under pointwise operations. For further references on L^1-spectral synthesis in hypergroups the reader is referred to [2], [6], [10].

In this paper we formulate the basic problems of spectral analysis and spectral synthesis on abelian hypergroups and solve these problems on some types of hypergroups.

2. Basic Concepts and Facts

The concept of DJS-hypergroup (according to the initials of Dunkl, Jewett and Spector) depends on a set of axioms which can be formulated in several different ways. The way of formulating these axioms we follow here is due to Lasser (see e.g. [11]). One begins with a locally compact Haussdorff space K, the space $\mathscr{M}(K)$ of all finite complex regular measures on K, the space $\mathscr{M}_c(K)$ of all compactly supported measures in $\mathscr{M}(K)$, the space $\mathscr{M}^1(K)$ of all probability measures in $\mathscr{M}(K)$, and the space $\mathscr{M}_c^1(K)$ of all compactly supported probability measures in $\mathscr{M}(K)$. The point mass concentrated at x is denoted by δ_x. Suppose that we have the following:

(a) (H^*) There is a continuous mapping $(x,y)\mapsto\delta_x*\delta_y$ from $K\times K$ into $\mathscr{M}_c^1(K)$, the latter being endowed with the weak*-topology with respect to the space of compactly supported complex valued continuous functions on K. This mapping is called *convolution*.

(b) $(H^\vee)$ There is an involutive homeomorphism $x\mapsto x^\vee$ from K to K. This mapping is called *involution*.

(c) (He) There is a fixed element e in K. This element is called *identity*.

Identifying x by δ_x the mapping in (H^*) has a unique extension to a continuous bilinear mapping from $\mathscr{M}(K)\times\mathscr{M}(K)$ to $\mathscr{M}(K)$. The involution on K extends to an involution on $\mathscr{M}(K)$. Then a *DJS-hypergroup*, or simply *hypergroup* is a quadruple $(K, *, \vee, e)$ satisfying the following axioms: for any x, y, z in K we have

(H1) $\delta_x*(\delta_y*\delta_z)=(\delta_x*\delta_y)*\delta_z$;
(H2) $(\delta_x*\delta_y)^\vee=\delta_{y^\vee}*\delta_{x^\vee}$;
(H3) $\delta_x*\delta_e=\delta_e*\delta_x=\delta_x$;

(H4) e is in the support of $\delta_x * \delta_{y^\vee}$ if and only if $x = y$;

(H5) the mapping $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space of nonvoid compact subsets of K is continuous, the latter being endowed with the Michael-topology (see [1]).

If the topology of K is discrete, then we call the hypergroup *discrete*. If $\delta_x * \delta_y = \delta_y * \delta_x$ holds for all x, y in K, then we call the hypergroup *commutative*. If $x^\vee = x$ holds for all x in K then we call the hypergroup *Hermitian*. By (H2) any Hermitian hypergroup is commutative. For instance, if $K = G$ is a locally compact Haussdorff-group, $\delta_x * \delta_y = \delta_{xy}$ for all x, y in K, $x^\vee$ is the inverse of x, and e is the identity of G, then we obviously have a hypergroup $(K, *, \vee, e)$, which is commutative if and only if the group G is commutative. However, not every hypergroup originates in this way.

In any hypergroup K we identify x by δ_x and we define the *right translation operator* T_y by the element y in K according to the formula:

$$T_y f(x) = \int_K f \, d(\delta_x * \delta_y),$$

for any f integrable with respect to $\delta_x * \delta_y$. In particular, T_y is defined for any continuous complex valued function on K. Similarly, we can define *left translation operators*. Sometimes one uses the suggestive notation

$$f(x * y) = \int_K f \, d(\delta_x * \delta_y),$$

for any x, y in K. Obviously, in case of commutative hypergroups the simple term *translation operator* is used. The function $T_y f$ is *the translate of f by y.*

The presence of translation operators on commutative hypergroups leads to the concept of variety. Let K be a locally compact Haussdorff-space and let $\mathscr{C}(K)$ denote the locally convex topological vector space of all continuous, complex valued functions on K, equipped with the pointwise linear operations and with the topology of uniform convergence on compact sets. The dual of this space can be identified with $\mathscr{M}_c(K)$, the latter being endowed with the weak*-topology with respect to the space of complex valued continuous functions on K (see e.g. [5], Vol. I.). If, in addition, K is equipped with a commutative hypergroup structure, then a subset H of $\mathscr{C}(K)$ is called *translation invariant*, if for any f in H the function $T_y f$ belongs to H for all y in K. A nonzero, closed, translation invariant subspace of $\mathscr{C}(K)$ is called a *variety*. For any f in $\mathscr{C}(K)$ *the variety generated by f* is the closed subspace generated by all translates of f, which is denoted by $\tau(f)$.

The dual of $\mathscr{C}(K)$ is a locally convex topological vector space, which bears a natural algebra structure, corresponding to the convolution of measures (see e.g. [5], Vol. I.). It is easy to see ([1]), that for any continuous function f in $\mathscr{C}(K)$ the function $(x, y) \mapsto f(x * y)$ is continuous. For any measures μ, ν in $\mathscr{M}_c(K)$ and for any f in $\mathscr{C}(K)$ we let

$$(\mu * \nu)(f) = \int_K \int_K f(x * y) \, d\mu(x) \, d\nu(y).$$

Then $\mu * \nu$ is an element of $\mathscr{M}_c(K)$, which is called the *convolution* of μ and ν. The space $\mathscr{M}_c(K)$ equipped with the pointwise linear operations and with the convolution is a commutative algebra with unit.

For any closed linear subspace V in $\mathscr{C}(K)$ its *annihilator* $V^\perp$ in $\mathscr{M}_c(K)$ is the set of all measures from $\mathscr{M}_c(K)$ which vanish on V. Clearly, it is a closed linear subspace of $\mathscr{M}_c(K)$. The dual correspondence is also true: the annihilator $I^\perp$ of any closed linear subspace I of $\mathscr{M}_c(K)$, that is, the set of all elements of $\mathscr{C}(K)$, which belong to the kernel of all linear functionals in I, is a closed linear subspace of $\mathscr{C}(K)$. By the Hahn–Banach-theorem we have the obvious relations $V = V^{\perp\perp}$ and $I = I^{\perp\perp}$ for any closed linear subspace V of $\mathscr{C}(K)$ and for any closed linear subspace I of $\mathscr{M}_c(K)$. In the case of varieties the annihilators can be characterized.

Theorem 1. *Let K be a commutative hypergroup, V a variety in $\mathscr{C}(K)$ and I a proper closed ideal in $\mathscr{M}(K)$. Then $V^\perp$ is a proper closed ideal in $\mathscr{M}(K)$ and $I^\perp$ is a variety in $\mathscr{C}(K)$ (see* [15]).

Proof. If f is an element of V, μ is an element of $V^\perp$ and ν is arbitrary in $\mathscr{M}_c(K)$, then by definition

$$(\mu * \nu)(f) = \int_K \int_K f(x * y)\, d\mu(x)\, d\nu(y) = \int_K \left[\int_K T_y f(x)\, d\mu(x) \right] d\nu(y) = 0,$$

hence $\mu * \nu$ belongs to $V^\perp$. As V is nonzero, hence $V^\perp$ is a proper ideal. Conversely, if $I = I^{\perp\perp}$ is a proper closed ideal in $\mathscr{M}_c(K)$ then for any μ in I, f in $I^\perp$ and ν in $\mathscr{M}_c(K)$ we have

$$0 = \int_K f\, d(\mu * \nu) = \int_K \left[\int_K f(x * y)\, d\mu(x) \right] d\nu(y),$$

that is, the function $y \mapsto \mu(T_y f)$ annihilates $\mathscr{M}_c(K)$. This means that this function vanishes, and by definition, $T_y f$ belongs to $I^\perp$ for all y in K. □

Let V be a variety in $\mathscr{C}(K)$. We say that *spectral analysis holds for V*, if V contains a minimal (that is, one dimensional) variety. If spectral analysis holds for any variety in $\mathscr{C}(K)$, then we say that *spectral analysis holds for the hypergroup K*.

Let K be a commutative hypergroup. The function $\varphi : K \to \mathbb{C}$ is called an *exponential*, if it is nonidentically zero, and satisfies

$$\varphi(x * y) = \varphi(x)\varphi(y)$$

for all x, y in K. Obviously, for any exponential φ the variety $\tau(\varphi)$ is of one dimension. Conversely, any one dimensional variety is generated by an exponential. Hence spectral analysis holds for a given variety V if and only if it contains an exponential. According to Theorem 1, spectral analysis holds for a given variety if and only if its annihilator ideal is contained in a maximal ideal.

Let V be a variety in $\mathscr{C}(K)$. We say that *spectral synthesis holds for V*, if V is the sum of finite dimensional varieties. This means, that there is a set $(V_\gamma)_{\gamma \in \Gamma}$ of finite dimensional subvarieties of V such that any element f of V can be represented in the form

$$f = f_{\gamma_1} + f_{\gamma_2} + \cdots + f_{\gamma_n}$$

with some positive integer n, with some elements $\gamma_1, \gamma_2, \ldots, \gamma_n$ in Γ and with some functions f_{γ_i} in V_{γ_i}. If K is a locally compact abelian group, convolution is defined by $\delta_x * \delta_y = \delta_{xy}$, involution is defined by $x^\vee = -x$ and e is the zero element of the group, then finite dimensional varieties are spanned by exponential monomials (see e.g. [14]), hence spectral synthesis holds for a variety in the hypergroup-sense if and only if the linear subspace of the variety spanned by its exponential monomials is dense in the variety.

If spectral synthesis holds for any variety in $\mathscr{C}(K)$, then we say that *spectral synthesis holds for the hypergroup K*.

We shall see, that – similarly to the case of groups – the Fourier–Laplace-transform on $\mathscr{M}_c(K)$ can be used successfully in the study of spectral synthesis. If K is a commutative hypergroup and μ is an element of $\mathscr{M}_c(K)$ then for any exponential φ on K we define

$$\hat{\mu}(\varphi) = \int_K \varphi(x^\vee)\, d\mu(x).$$

Then $\hat{\mu}$ is a complex valued function defined on the set of all exponentials on K. The mapping $\mu \mapsto \hat{\mu}$ is obviously linear. It also has the important property

$$(\mu * \nu)\hat{} = \hat{\mu}\hat{\nu}$$

for all μ, ν in $\mathscr{M}_c(K)$, which is called the *convolution formula*.

3. Spectral Analysis on Polynomial Hypergroups

An important special class of Hermitian hypergroups is closely related to orthogonal polynomials.

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be real sequences with the following properties: $c_n > 0$, $b_n \geqslant 0$, $a_{n+1} \geqslant 0$ for all n in $\mathbb{N}$, moreover $a_0 = b_0 = 0$, and $a_n + b_n + c_n = 1$ for all n in $\mathbb{N}$. We define the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ by $P_0(x) = 1$, $P_1(x) = x$, and by the recursive formula

$$xP_n(x) = a_nP_{n-1}(x) + b_nP_n(x) + c_nP_{n+1}(x)$$

for all $n \geqslant 1$ and x in $\mathbb{R}$. The following theorem holds (see [1]).

Theorem 2. *If the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ satisfies the above conditions, then there exist constants $c(n, m, k)$ for all n, m, k in $\mathbb{N}$ such that*

$$P_nP_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)P_k$$

holds for all n, m in $\mathbb{N}$.

Proof. By the theorem of Favard (see [4], [13]) the conditions on the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ imply that there exists a probability measure μ on $[-1, 1]$ such that $(P_n)_{n\in\mathbb{N}}$ forms an orthogonal system on $[-1, 1]$ with respect to μ. As P_n has degree n, we have

$$P_nP_m = \sum_{k=0}^{n+m} c(n,m,k)P_k$$

for all n, m in $\mathbb{N}$, where

$$c(n,m,k) = \frac{\int_{-1}^{1} P_k P_n P_m \, d\mu}{\int_{-1}^{1} P_k^2 \, d\mu}$$

holds for all n, m, k in $\mathbb{N}$. The orthogonality of $(P_n)_{n \in \mathbb{N}}$ with respect to μ implies $c(n, m, k) = 0$ for $k > n + m$ or $n > m + k$ or $m > n + k$. Hence our statement is proved. □

The formula in the theorem is called *linearization formula*, and the coefficients $c(n, m, k)$ are called *linearization coefficients*. The recursive formula for the sequence $(P_n)_{n \in \mathbb{N}}$ implies $P_n(1) = 1$ for all n in $\mathbb{N}$, hence we have

$$\sum_{k=|n-m|}^{n+m} c(n, m, k) = 1$$

for all n in $\mathbb{N}$. If the linearization is *nonnegative*, that is, the linearization coefficients are nonnegative: $c(n, m, k) \geqslant 0$ for all n, m, k in $\mathbb{N}$, then we can define a hypergroup structure on $\mathbb{N}$ by the following rule:

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c(n, m, k)\delta_k$$

for all n, m in $\mathbb{N}$, with involution as the identity mapping and with e as 0. The resulting discrete Hermitian (hence commutative) hypergroup is called *the polynomial hypergroup associated with the sequence* $(P_n)_{n \in \mathbb{N}}$. We can denote it by $(\mathbb{N}, (P_n)_{n \in \mathbb{N}})$.

We mention here an easy consequence of the linearization formula in polynomial hypergroups. Namely, let $\varphi(n) = P_n^{(k)}(\lambda)$ for all n in $\mathbb{N}$ with some nonnegative integer k and complex number λ. Then we have

$$\varphi(n * 1) = \lambda P_n^{(k)}(\lambda) + k P_n^{(k-1)}(\lambda) = \lambda \varphi(n) + k P_n^{(k-1)}(\lambda)$$

for all n in $\mathbb{N}$. (Here $P_n^{(-1)}$ is meant to be 0.)

In this section we show that spectral analysis holds for any polynomial hypergroup. First we need the general form of exponential functions on polynomial hypergroups, which is well-known (see [1], [7]).

Theorem 3. *Let K be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$. The function $\varphi : \mathbb{N} \to \mathbb{C}$ is an exponential on K if and only if there exists a complex number λ such that*

$$\varphi(n) = P_n(\lambda)$$

holds for all n in $\mathbb{N}$.

Proof. First of all we remark that if a sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ satisfies a recursion of the form

$$P_n(x) P_m(x) = \sum_{k=0}^{n+m} c(n, m, k) P_k(x)$$

with some real or complex coefficients $c(n, m, k)$ for all real x, then the recursion holds for all complex x. Let λ be a complex number and let $\varphi(n) = P_n(\lambda)$ for any n in $\mathbb{N}$. Then by the definition of convolution we have for any m, n in $\mathbb{N}$

$$\begin{aligned}\varphi(\delta_n * \delta_m) &= \sum_{k=|n-m|}^{n+m} c(n,m,k)\varphi(k) \\ &= \sum_{k=|n-m|}^{n+m} c(n,m,k)P_k(\lambda) = P_n(\lambda)P_m(\lambda) = \varphi(n)\varphi(m),\end{aligned}$$

hence φ is exponential.

Conversely, let φ be an exponential on K and we define $\lambda = \varphi(1)$. By the exponential property we have for all positive integer n that

$$\begin{aligned}\lambda\varphi(n) = \varphi(1)\varphi(n) = \varphi(\delta_1 * \delta_n) &= \sum_{k=n-1}^{n+1} c(n,1,k)\varphi(k) \\ &= c(n,1,n-1)\varphi(n-1) + c(n,1,n)\varphi(n) + c(n,1,n+1)\varphi(n+1).\end{aligned}$$

As the same recursion holds for $n \mapsto P_n(\lambda)$, further $\varphi(0) = 1 = P_0(\lambda)$ and $\varphi(1) = \lambda = P_1(\lambda)$, hence $\varphi(n) = P_n(\lambda)$ for all n in $\mathbb{N}$ and the theorem is proved. □

Now we can prove that spectral analysis holds for any polynomial hypergroup.

Theorem 4. *Spectral analysis holds for any polynomial hypergroup.*

Proof. Let K be the hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ and let V be an arbitrary variety in $\mathscr{C}(K)$. We remark, that $\mathscr{C}(K)$ is the set of all complex valued functions on $\mathbb{N}$, equipped with the pointwise linear operations and with the topology of pointwise convergence. Accordingly, $\mathscr{M}_c(K)$ is the set of all finitely supported complex measures on $\mathbb{N}$. By Theorem 1 the annihilator $V^\perp$ of V is a proper closed ideal in $\mathscr{M}_c(K)$. By the convolution formula the Fourier–Laplace-transforms of the elements of $V^\perp$ form a proper ideal in the ring of the Fourier–Laplace-transforms of all elements of $\mathscr{M}_c(K)$. By Theorem 3 the set of all exponentials of K can be identified by $\mathbb{C}$. For any μ in $\mathscr{M}_c(K)$ and for any λ in $\mathbb{C}$ we have

$$\hat{\mu}(\lambda) = \int_{\mathbb{N}} P_n(\lambda)\, d\mu(n).$$

As μ is finitely supported, hence $\hat{\mu}$ is a complex polynomial on $\mathbb{C}$. We can see easily, that any complex polynomial on $\mathbb{C}$ can be written in the form $\hat{\mu}$ with some μ in $\mathscr{M}_c(K)$. Indeed, if p is a complex polynomial on $\mathbb{C}$ of degree n, then it can be written in the form $p = \sum_{k=0}^{n} c_k P_k$ with some complex constants c_k ($k = 0, 1, \ldots, n$). Then we have

$$p = \Big(\sum_{k=0}^{n} c_k\delta_k\Big)^{\wedge}.$$

This means that the Fourier–Laplace-transforms of the elements of $V^{\perp}$ form a proper ideal in the ring of all complex polynomials on $\mathbb{C}$. It is known, that in this case there exists a complex λ_0 such that $\hat{\mu}(\lambda_0) = 0$ for all μ in $V^{\perp}$, which means by definition, that the exponential $n \mapsto P_n(\lambda_0)$ belongs to V and the theorem is proved. □

4. Exponential Polynomials on Polynomial Hypergroups

In order to study spectral synthesis on hypergroups it is necessary to introduce a reasonable concept for exponential monomials. As the product of exponentials is not necessarily an exponential, it turns out that the above mentioned concept is not the appropriate one. In the group case an exponential monomial can be characterized by the property, that the variety it generates contains exactly one exponential and is of finite dimension. One dimensional varieties are clearly the ones generated by a single exponential. An additive function can be characterized by the properties that it vanishes at zero, the variety it generates is two dimensional and the only exponential contained in it is the function identically 1. In the case of polynomial hypergroups it seems to be reasonable to introduce the following concept. Let K be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. We call the function $\varphi : \mathbb{N} \to \mathbb{C}$ an *exponential monomial*, if it has the form

$$\varphi(n) = \sum_{j=0}^{k} c_j P_n^{(j)}(\lambda)$$

for all n in $\mathbb{N}$, where k is a nonnegative integer and λ, c_j $(j = 0, 1, \ldots, k)$ are complex numbers. The forthcoming theorems will justify this definition. Then a sum of exponential monomials is called an *exponential polynomial*. It is easy to see (see e.g. [9]), that on a polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ the additive functions have the general form $n \mapsto cP_n'(1)$, where c is any complex constant. Now we list some basic properties of exponential monomials and polynomials on polynomial hypergroups. In what follows K is a fixed polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$.

Theorem 5. *Let k be a nonnegative integer and λ a complex number. Then the functions $n \mapsto P_n^{(j)}(\lambda)$ $(j = 0, 1, \ldots, k)$ are linearly independent.*

Proof. Suppose that

$$c_0 P_n(\lambda) + c_1 P_n'(\lambda) + \cdots + c_k P_n^{(k)}(\lambda) = 0$$

holds for all n in $\mathbb{N}$ with some complex numbers c_j $(j = 0, 1, \ldots, k)$. Substituting $n = 0$ we have $c_0 = 0$. Supposing that we have proved $c_0 = c_1 = \cdots = c_m = 0$ for some $0 \leqslant m < k$, then substituting $n = m + 1$ we have $c_{m+1} = 0$. Hence by induction we have the statement. □

Theorem 6. *Let V be a variety over K, k a positive integer, m_i a nonnegative integer and λ_i, $c_{i,j}$ complex numbers $(i=1, 2, \ldots, k, j=0, 1, \ldots, m_i)$. Suppose that $\lambda_s \neq \lambda_t$ for $s \neq t$. Let*

$$\varphi(n) = \sum_{i=1}^{k} \sum_{j=0}^{m_i} c_{i,j} P_n^{(j)}(\lambda_i)$$

for all n in $\mathbb{N}$. If φ belongs to V, then the function $n \mapsto c_{i,m_i} P_n^{(j)}(\lambda_i)$ belongs to V for $i=1, 2, \ldots, k$ and $j=0, 1, \ldots, m_i$.

Proof. Obviously we may suppose that $c_{i,m_i} \neq 0$ for $i=1, 2, \ldots, k$. We prove the statement by induction on $m_1 + \cdots + m_k$. If $m_1 + \cdots + m_k = 0$, then

$$\varphi(n) = c_{1,0} P_n(\lambda_1) + \cdots + c_{k,0} P_n(\lambda_k)$$

holds for all n in $\mathbb{N}$. We have to show that the function $n \mapsto P_n(\lambda_i)$ belongs to V for $i=1, 2, \ldots, k$. We prove this statement by induction on k. As it is obvious for $k=1$ suppose that it has been proved for some $k \geqslant 1$ and let

$$\varphi(n) = c_{1,0} P_n(\lambda_1) + \cdots + c_{k,0} P_n(\lambda_k) + c_{k+1,0} P_n(\lambda_{k+1})$$

for all n in $\mathbb{N}$.

Let for all n in $\mathbb{N}$

$$\psi(n) = \varphi(n*1) - \lambda_{k+1}\varphi(n).$$

Then ψ belongs to V. On the other hand

$$\psi(n) = \sum_{i=1}^{k} c_{i,0}(\lambda_i - \lambda_{k+1}) P_n(\lambda_i)$$

holds for all n in $\mathbb{N}$, and hence our statement follows.

Now suppose that the theorem is proved for some $m_1 + \cdots + m_k \geqslant 1$ and let

$$\varphi(n) = \sum_{i=1}^{k-1} \sum_{j=0}^{m_i} c_{i,j} P_n^{(j)}(\lambda_i) + \sum_{j=0}^{m_k} c_{k,j} P_n^{(j)}(\lambda_k) + c_{k,m_k+1} P_n^{(m_k+1)}(\lambda_k)$$

for all n in $\mathbb{N}$. (For $k=1$ the first – empty – sum is zero.) Again we set for all n in $\mathbb{N}$

$$\psi(n) = \varphi(n*1) - \lambda_k \varphi(n).$$

Then ψ belongs to V. On the other hand

$$\begin{aligned}\psi(n) &= \sum_{i=1}^{k-1} \sum_{j=0}^{m_i} c_{i,j} \big[\lambda_i P_n^{(j)}(\lambda_i) + j P_n^{(j-1)}(\lambda_i) - \lambda_k P_n^{(j)}(\lambda_i)\big] \\ &\quad + \sum_{j=0}^{m_k} j c_{k,j} P_n^{(j-1)}(\lambda_k) + c_{k,m_k+1}(m_k+1) P_n^{(m_k)}(\lambda_k) \\ &= \sum_{i=1}^{k} \sum_{j=0}^{m_k} b_{i,j} P_n^{(j)}(\lambda_i)\end{aligned}$$

holds for all n in $\mathbb{N}$, where

$$b_{i,m_i} = c_{i,m_i}(\lambda_i - \lambda_k)$$

for $i = 1, 2, \ldots, k-1$, and

$$b_{k,m_k} = c_{k,m_k}(m_k + 1),$$

hence our statement follows. □

A special case of this theorem is the following.

Theorem 7. *Let V be a variety over K, k a positive integer and λ a complex number. If the function $n \mapsto P_n^{(k)}(\lambda)$ belongs to V then so do the functions $n \mapsto P_n^{(j)}(\lambda)$ for $j = 0, 1, \ldots, k-1$.*

Another important special case of Theorem 6 reads as follows.

Theorem 8. *Let K be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$. If k is a positive integer, $m_1, m_2, \ldots, m_k$ are nonnegative integers and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different complex numbers, then the functions $n \mapsto P_n^{(j)}(\lambda_i)$ are linearly independent for $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, m_i$.*

Proof. Take $V = \{0\}$ in Theorem 6. □

5. Spectral Synthesis on Polynomial Hypergroups

In this section we show that spectral synthesis holds for any polynomial hypergroup in the sense that the linear hull of all exponential monomials is dense in any variety. Actually, we shall prove that any variety on a polynomial hypergroup is finite dimensional, and it is generated by functions of the form $n \mapsto P_n^{(j)}(\lambda)$ with some finite set of j's and some finite set of λ's. We use the notation of the previous section.

Theorem 9. *Spectral synthesis holds for any polynomial hypergroup.*

Proof. Let $(P_n)_{n \in \mathbb{N}}$ be the sequence of polynomials with which the polynomial hypergroup K is associated. First we show that the variety generated by the function $n \mapsto P_n^{(k)}(\lambda)$ is finite dimensional for any nonnegative integer k and for any complex number λ. Let $\psi(n) = P_n^{(k)}(\lambda)$ for any n, k in $\mathbb{N}$ and λ in $\mathbb{C}$. Then by the linearization formula we have

$$\psi(n * m) = \sum_{j=0}^{k} \binom{k}{j} P_n^{(j)}(\lambda) P_m^{(k-j)}(\lambda)$$

for all m, n in $\mathbb{N}$, which yields the statement.

Now we know that for any variety V in $\mathscr{C}(K)$ the Fourier–Laplace-transforms of the elements of $V^{\perp}$ form a proper ideal in the ring of all complex polynomials on $\mathbb{C}$. We denote this ideal by J. It is known that in this case there exist complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ and nonnegative integers $m_1, m_2, \ldots, m_k$ such that a polynomial p belongs to J if and only if $p^{(j)}(\lambda_i) = 0$ holds for $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, m_i$. This means, that the measure μ in $\mathscr{M}_c(K)$ annihilates V if and only if $\hat{\mu}^{(j)}(\lambda_i) = 0$ holds for $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, m_i$, that is, if and only if the functions $n \mapsto P_n^{(j)}(\lambda_i)$ are annihilated by μ for $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, m_i$. It follows that V is the closure of the linear hull of these functions. As these functions generate finite dimensional varieties, our statement is proved. $\square$

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