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Foliations and polynomial diffeomorphisms of $\mathbb{R}^3$

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Abstract Let $Y = (f, g, h)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map and let $\mathrm{Spec}(Y)$ denote the set of eigenvalues of the derivative DY_p, when p varies in $\mathbb{R}^3$. We begin proving that if, for some $\epsilon > 0$, $\mathrm{Spec}(Y) \cap (-\epsilon, \epsilon) = \emptyset$, then the foliation $\mathcal{F}(k)$, with $k \in \{f, g, h\}$, made up by the level surfaces $\{k = \text{constant}\}$, consists just of planes. As a consequence, we prove a bijectivity result related to the three-dimensional case of Jelonek's Jacobian Conjecture for polynomial maps of $\mathbb{R}^n$.

Keywords Three dimensional vector field · Global injectivity · Foliation

Mathematics Subject Classification (2000) 37C85 · 57R30

1 Introduction

Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map and let $\mathrm{Spec}(Y)$ be the set of (complex) eigenvalues of the derivative DY_p when p varies in $\mathbb{R}^3$. If for all $p \in \mathbb{R}^3$, DY_p is non singular, (that is, $0 \notin \mathrm{Spec}(Y)$) then it follows from the inverse function theorem that:
for each $k \in \{f, g, h\}$, *the level surfaces* $\{k = \text{constant}\}$ *make up a codimension one* C^2*-foliation* $\mathcal{F}(k)$ *on* $\mathbb{R}^3$.

We say that the foliation $\mathcal{F}(k)$ is *by planes* if every leaf is C^2 diffeomorphic to $\mathbb{R}^2$. Our first result is the following

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Theorem 1.1 *If, for some* $\epsilon > 0$, $\text{Spec}(Y) \cap (-\epsilon, \epsilon) = \emptyset$, *then* $\mathcal{F}(k)$, $k \in \{f, g, h\}$, *is a foliation by planes. Consequently, there is a foliation* F_k *in* $\mathbb{R}^2$ *such that* $\mathcal{F}(k)$ *is conjugate to the product of* F_k *by* $\mathbb{R}$.

To state our next results, we need to introduce some concepts. Let $Y : M \to N$ be a continuous map of locally compact spaces. We say that the mapping Y is *not proper at a point* $y \in N$, if there is no neighborhood U of the point y such that the set $Y^{-1}(\overline{U}))$ is compact.

The set S_Y of points at which the map Y is not proper indicates how the map Y differs from a proper map. In particular Y is proper if and only if this set is empty. Moreover, if $Y(M)$ is open, then S_Y contains the border of the set $Y(M)$. The set S_Y is the minimal set S with the property that the mapping $Y : M \backslash Y^{-1}(S) \to N \backslash S$ is proper.

Jelonek proved in [22] that: if $Y : \mathbb{R}^n \to \mathbb{R}^n$ is a real polynomial mapping with nonzero Jacobian everywhere and $\text{codim}(S_Y) \geq 3$, then Y is a bijection (and consequently $S_Y = \emptyset$).

On the other hand, the example of Pinchuk (see [28]) shows that there are real polynomial mappings, which are not injective, with nonzero Jacobian determinant everywhere and with $\text{codim}(S_Y) = 1$. Hence the only interesting case is that of $\text{codim}(S_Y) = 2$ and we can state:

Jelonek's Real Jacobian Conjecture. *Let* $Y : \mathbb{R}^n \to \mathbb{R}^n$ *be a real polynomial mapping with nonzero Jacobian everywhere. If* $\text{codim}(S_Y) \geq 2$ *then* Y *is a bijection (and consequently* $S_Y = \emptyset$*).*

Jelonek [22] proved that his conjecture is true in dimension two. Consequently, the first interesting case is $n = 3$ and $\dim(S_Y) = 1$.

Jelonek's Real Jacobian Conjecture is closely connected with the following famous Keller Jacobian Conjecture:

Jacobian Conjecture. *Let* $Y : \mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial mapping with nonzero Jacobian determinant everywhere, then* Y *is an isomorphism.*

More precisely, Jelonek proved in [22] that his Real Jacobian Conjecture in dimension $2n$ implies the Jacobian Conjecture in (complex) dimension n. The corresponding arguments of Jelonek and some well known results [2,10,12,31] will be used to obtain in Sect. 3 the following version of the Reduction Theorem

Theorem 1.2 *Let* $X_i : \mathbb{R}^n \to \mathbb{R}$ *denote the canonical i-th coordinate function. If* F*, with* $\text{codim}(S_F) \geq 2$*, is injective for all* $n \geq 2$ *and all polynomial maps* $F : \mathbb{R}^n \to \mathbb{R}^n$ *of the form*

$$F = (-X_1 + H_1, -X_2 + H_2, \ldots, -X_n + H_n)$$

where each $H_i : \mathbb{R}^n \to \mathbb{R}$ *is either zero or homogeneous of degree 3, and the Jacobian matrix* JH *(with* $H = (H_1, H_2, \ldots, H_n)$*) is nilpotent, then the Jacobian Conjecture is true.*

Notice that in theorem above $\text{Spec}(F) = \{-1\}$.

Related to Theorem 1.2 and Jelonek's conjecture we prove the following.

Theorem 1.3 *Let* $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ *be a polynomial map such that* $\text{Spec}(Y) \cap [0, \varepsilon) = \emptyset$*, for some* $\varepsilon > 0$*. If* $\text{codim}(S_Y) \geq 2$ *then* Y *is a bijection.*

This result partially extends also the two dimensional results of [9,13] (see also [5–8,14,17–25,27]).

2 Half-Reeb components and the spectral condition

Let us recall the definition of a vanishing cycle stated in conformity with our needs. Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map such that for all $p \in \mathbb{R}^3$ the derivative DY_p is non-singular. Given $k \in \{f, g, h\}$, a *vanishing cycle* for the foliation $\mathcal{F}(k)$ is a C^2-embedding $F_0 : S^1 \to \mathbb{R}^3$ such that:

(a) $F_0(S^1)$ is contained in a leaf L_0 but it is not homotopic to a point in L_0;
(b) F_0 can be extended to a C^2-embedding $F : [-1, 2]\times S^1 \to \mathbb{R}^3$, $F(t, x) = F_t(x)$, such that for all $t \in (0, 1]$ there is a 2-disc D_t contained in a leaf L_t such that $\partial D_t = F_t(S^1)$;
(c) for all $x \in S^1$, the curve $t \in [-1, 2] \mapsto F(t, x)$ is transverse to the foliation $\mathcal{F}(k)$ and for all $t \in (0, 1]$ the disc D_t depends continuously on t.

We say that the leaf L_0 *supports* the vanishing cycle F_0 and that F is the map associated to F_0.

The *half-Reeb component for* $\mathcal{F}(k)$ (or simply the *hRc* for $\mathcal{F}(k)$) associated to the vanishing cycle F_0 is the region

$$\mathcal{A} = \left(\bigcup_{t\in(0,1]} D_t \right) \cup L \cup F_0(S^1)$$

where L is the connected component of $L_0 - F_0(S^1)$ contained in the closure of $\cup_{t\in(0,1]} D_t$. The transversal section $A = F([0, 1]\times S^1)$ to the foliation $\mathcal{F}(k)$ is called the *compact face* of $\mathcal{A}$ and the leaf $L \cup F_0(S^1)$ of $\mathcal{F}(k)|_{\mathcal{A}}$ is called the *non-compact face* of $\mathcal{A}$. The boundary ∂A of A will be the set $F(\{0, 1\} \times S^1)$.

Remark 2.1

(1) It will be seen in Proposition 2.2, that if $\mathcal{F}(k)$, $k \in \{f, g, h\}$, has a leaf which is not homeomorphic to the plane, then $\mathcal{F}(k)$ has a half-Reeb component (Fig. 1).
(2) The connection between half-Reeb components and the spectral condition on Y (that is, $\mathrm{Spec}(Y) \cap (-\varepsilon, \varepsilon) = \emptyset$) is given by Theorem 1.1.
(3) The half-Reeb component is not necessarily a closed set. See, for example, the *hRc* obtained by rotating the Fig. 2 around the x-axis.

The following proposition is obtained by using classical arguments of Foliation Theory (see [4,15]). For the sake of completeness we give the main lines of its proof. Let $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ denote the closed 2-disc.

Fig. 1 A half-Reeb component

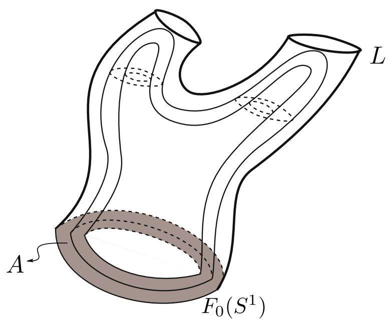
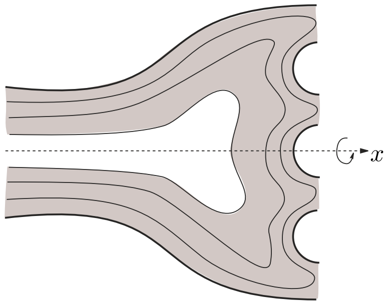


Fig. 2 A half-Reeb component in the plane that is symmetrical with respect to x-axis



Proposition 2.2 *If $\mathcal{F}(k)$, with $k \in \{f, g, h\}$, has a leaf L which is not homeomorphic to the plane, then $\mathcal{F}(k)$ has a vanishing cycle.*

Proof Let $\eta : S^1 \to L$ be a C^2-embedding which is not null homotopic in L. Since η is null homotopic in $\mathbb{R}^3$, we may extend it to a C^2-immersion $\eta : D^2 \to \mathbb{R}^3$, which is in general position with respect to $\mathcal{F}(k)$. In this way we are supposing that the contact set C_η, made up by the points of D^2 at which η meets tangentially $\mathcal{F}(k)$, is finite and is contained in $D^2 \setminus S^1$.

Via η, the foliation $\mathcal{F}(k)$ induces a foliation $\mathcal{G}$ (with singularities) on D^2. We claim that it is possible to construct a vector field G on D^2 such that the foliation $\mathcal{G}$ is induced by G. In fact, as η is in general position with respect to $\mathcal{F}(k)$, the foliation $\mathcal{G}$ has finitely many singularities each of which is locally topologically equivalent either to a center or to a saddle point of a vector field. This implies that $\mathcal{G}$ is locally orientable everywhere. As D^2 is simply connected, $\mathcal{G}$ is globally orientable. This proves the existence of the vector field G. Certainly, we may assume that η has been chosen so that no pair of singularities of G is taken by η into the same leaf of $\mathcal{F}(k)$; in other words, G has no saddle connections.

We claim that G has no limit cycles. In fact, otherwise, the Poincaré-Bendixon theorem would imply that there is an orbit of G which spirals towards a limit cycle C. Hence, the leaf of $\mathcal{F}(k)$ containing C would have a non trivial holonomy group. This contradiction proves our claim (Fig. 3).

Let $c_1, \ldots, c_\ell$ be the center singularities of G. Given $i \in \{1, \ldots, \ell\}$, there exists a G-invariant open 2-disc $D_i \subset D^2$ such that:

(a1) $c_i \in D_i$ and every orbit of G passing through a point in $D_i \setminus \{c_i\}$ is a closed orbit;
(a2) for every closed orbit $\gamma \subset D_i$ of G, $\eta(\gamma)$ is homotopic to a point in its corresponding leaf of $\mathcal{F}(k)$.
(a3) the 2-disc D_i is the biggest one satisfying properties (a1) and (a2) above.

Notice that the boundary γ_i of D_i has to be G-invariant. We claim that

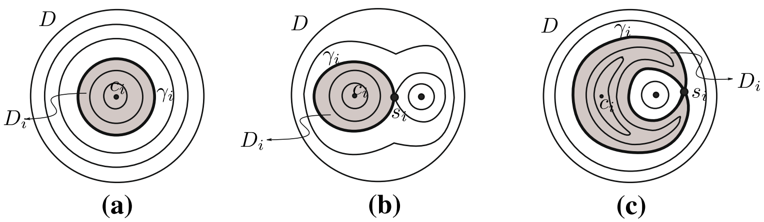


Fig. 3 Orbit structure of the vector field G

(b) if for some $i \in \{1, 2, \ldots, \ell\}$ the curve γ_i is a closed orbit of G, then $\eta(\gamma_i)$ is a vanishing cycle (see Fig. 3a), and the proposition is proved.

In fact, if γ_i is a closed orbit of G such that $\eta(\gamma_i)$ is homotopic to a point in its corresponding leaf, then, by a well known result of foliation theory, there exists a neighborhood $V_i \subset D^2$ of γ_i such that the image by η of every orbit of G contained in V_i is homotopic to a point in its corresponding leaf. This contradiction with the maximality of D_i proves (b).

Therefore, we may suppose from now on that:

(c) for every $i \in \{1, \ldots, \ell\}$ the curve γ_i is either the union of a saddle singularity s_i of G and one of its separatrices or the union of a saddle singularity s_i and its two separatrices, see (b) and (c) of Fig. 3.

By studying the phase portrait of G we may conclude that

(d) if (b) is not satisfied, there must exist $i \in \{1, 2, \ldots, \ell\}$ such that γ_i is the union of a saddle singularity s_i of G and one of its separatrices.

We claim that:

(e.1) If $\eta(\gamma_i)$ is homotopic to a point in its corresponding leaf, then η can be C^2-deformed to a C^2-immersion $\tilde{\eta} : D^2 \to \mathbb{R}^3$ which is in general position with respect to $\mathcal{F}(k)$ and such that $\#C_{\tilde{\eta}} < \#C_\eta$;

(e.2) If $\eta(\gamma_i)$ is a not homotopic to a point in its corresponding leaf, then η can be C^2-deformed to a C^2-immersion $\tilde{\eta} : D^2 \to \mathbb{R}^3$ for which (b) above is satisfied.

In fact, let us prove (e.1). By using Rosenberg's arguments (see [29, p. 137]), via a C^2-deformation of η, supported in a neighborhood of $\overline{D}_i$, we can eliminate the saddle singularity s_i and the center singularity c_i. The proof of (e.2) is similar and will be omitted.

Using (e.1) as many times as necessary, it follows from (d) that we will arrive at the situation considered in (e.2). This proves the proposition. □

Remark 2.3 Let $k \in \{f, g, h\}$. As k is a submersion, the foliation $\mathcal{F}(k)$ is without holonomy.

Lemma 2.4 *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 submersion and let $h : S^1 \times [-1, 1] \to \mathbb{R}^3$ be a C^2 embedding such that $\Sigma = h(S^1 \times [-1, 1])$ is transverse to $\mathcal{F}(f)$. Suppose that there are two discs D_c and D_d such that $D_c \cap \Sigma = \partial D_c$, $D_d \cap \Sigma = \partial D_d$ (are circles) and $f(D_c) = c < d = f(D_d)$, where c, d are constant. If K is the closure of the bounded connected component of $\mathbb{R}^3 \backslash (D_c \cup D_d \cup \Sigma)$, then*

(1) *for all $t \in [c, d]$, the level surface $\{f = t\}$ meets K in a disc D_t which together with D_c meet Σ from the same side (see Fig. 4(a) and satisfies $D_t \cap \Sigma = \partial D_t$;*
(2) *$\cup_{t \in [c,d]} D_t = K$ (in particular $f(K) = [c, d]$).*

Proof First we claim that

(a) $\mathcal{F}(f)$ has no compact leaves.

In fact, otherwise the complement of a compact leaf contains a bounded connected component M. Then, either $\inf\{f(p): p \in \overline{M}\}$ or $\sup\{f(p) : p \in \overline{M}\}$ is a critical value for f. This contradiction proves (a).

Now, we claim that

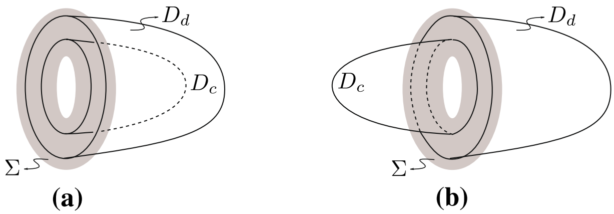


Fig. 4 Relative position of D_c and D_d with respect to Σ

(b) D_c and D_d meet Σ from the same side.

In fact, otherwise either D_c or D_d is contained in a compact leaf L of $\mathcal{F}(f)$ such that $L \subset K$. If the situation were as in Fig. 4(b), D_c would be contained in a compact leaf of $\mathcal{F}(f)$. This contradiction with (a) proves (b).

From (b) we have that the leaf $D_s = K \cap \{f = s\}$ of $\mathcal{F}(f)|_K$ is compact for all $s \in [c, d]$. Hence, as all leaves of $\mathcal{F}(f)$ are without holonomy, for all $s \in [c, d]$ the leaf D_s is a disc meeting Σ from the same side. Moreover, we have that $\cup_{s \in [c,d]} D_s = K$. □

Lemma 2.5 *Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 submersion such that $\mathcal{F}(f)$ has a hRc $\mathcal{A}$. There exists a neighborhood $\mathcal{U}$ of f in the Whitney C^2 topology such that every $\widetilde{f} \in \mathcal{U}$ has also a hRc $\widetilde{\mathcal{A}}$.*

Proof Let $\mathcal{U}$ be a neighborhood of f in the Whitney C^2-topology. By Proposition 2.2, it is sufficient to prove that if $\mathcal{U}$ is small enough and $\widetilde{f} \in \mathcal{U}$, then $\mathcal{F}(\widetilde{f})$ has a leaf which is not homeomorphic to the plane. Let A (resp. L) be the compact face (resp. non-compact face) of $\mathcal{A}$. Take a C^2-embedding $h : S^1 \times [-5, 5] \to \mathbb{R}^3$ such that $\Sigma_5 = h(S^1 \times [-5, 5])$ is transverse to $\mathcal{F}(f)$, $A = h(S^1 \times [-5, 0])$ and, for all $t \in [-5, 5]$, the circle $h(S^1 \times \{t\})$ is contained in a leaf of $\mathcal{F}(f)$.

For $t \in [0, 5]$, let $\Sigma_t = h(S^1 \times [-t, t])$. Let $\mathcal{G}(f)$ be the one dimensional foliation on Σ_5 induced by $\mathcal{F}(f)$. Without loss of generality we may assume that

(a) $f|_{h(S^1 \times [-5,0))} < 0$, $f|_L = 0$, $f|_{h(S^1 \times (0,5])} > 0$, and that, for some $p, q \in \Sigma_1$, $-8a = f(p) < 0 < f(q) = 8a$.

Recall that there is a compact disk $D_p(f)$ contained in a leaf of $\mathcal{F}(f)|_{\mathcal{A}}$ such that $C_p(f) = D_p(f) \cap \Sigma_1$ is the leaf of $\mathcal{G}(f)$ passing through p.

Under these circumstances, if $\mathcal{U}$ is small enough, we have that, for every $\widetilde{f} \in \mathcal{U}$,

(b.1) $\widetilde{f}(L) \subset (-a, a)$, $\widetilde{f}(p) < -7a$, $\widetilde{f}(q) > 7a$ and Σ_4 is transverse to $\mathcal{F}(\widetilde{f})$;

(b.2) p, q belong to circles $C_p(\widetilde{f}), C_q(\widetilde{f}) \subset \Sigma_2$ (respectively) which are leaves of $\mathcal{G}(\widetilde{f})$, where $\mathcal{G}(\widetilde{f})$ is the foliation on Σ_4 induced by $\mathcal{F}(\widetilde{f})$;

(b.3) there is a compact disk $D_p(\widetilde{f})$ contained in a leaf of $\mathcal{F}(\widetilde{f})$ such that $C_p(\widetilde{f}) = D_p(\widetilde{f}) \cap \Sigma_2$;

(b.4) $D_p(\widetilde{f})$ and $D_p(f)$ meet Σ_2 from the same side.

We claim that

(c) the leaf of $\mathcal{F}(\widetilde{f})$ passing through q is not homeomorphic to a plane.

In fact, we assume by contradiction that there is a compact disk $D_q(\widetilde{f})$ contained in a leaf of $\mathcal{F}(\widetilde{f})$ and such that $C_q(\widetilde{f}) = D_q(\widetilde{f}) \cap \Sigma_2$. Then by Lemma 2.4 and (b.4), $D_p(\widetilde{f})$, $D_q(\widetilde{f})$ and $D_p(f)$ are on the same side of Σ_2. Let K be the closure of the bounded connected component of $\mathbb{R}^3\backslash(D_p(\widetilde{f}) \cup D_q(\widetilde{f}) \cup \Sigma_2)$. By definition, L and $D_p(f)$ meet Σ_2 from the same side. Therefore L meets the interior of K. However, by (b.1), $L \cap \partial K = h(S^1 \times \{0\})$. This implies (by Lemma 2.4) that $L \subset K$ is bounded. This contradiction proves (c) and the lemma. □

The following remark will be used in the proof of the next lemma.

Remark 2.6 Let $Y = (f, g, h) \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map such that $0 \notin \mathrm{Spec}(Y)$ and let $\{i, j, k\}$ be an arbitrary permutation of $\{f, g, h\}$. If L is a leaf of $\mathcal{F}(i)$ and l is a leaf of $\mathcal{F}(j)|_L$ then $k|_l$ is regular; in this way $\mathcal{F}(j)|_L$ and $\mathcal{F}(k)|_L$ are transverse to each other.

Lemma 2.7 *Let $\mathcal{F}_i$, $i = 1, 2, 3$, be a C^2 foliation on $\mathbb{R}^3$ defined by the level surfaces of a C^2 function f_i such that the map $F = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ is not singular (that is, for all $p \in \mathbb{R}^3$, the derivative DF_p is a linear isomorphism). Let L be a leaf of $\mathcal{F}_1$. We have*:

(1) *If $i, j = 1, 2, 3$, and $i \neq j$, then $\mathcal{F}_i$ is transverse to $\mathcal{F}_j$,*
(2) *If $\mathcal{F}_2|_L$ is the foliation on L that is induced by $\mathcal{F}_2$, then every leaf of $\mathcal{F}_2|_L$ is homeomorphic to $\mathbb{R}$.*

Proof (1) is a direct consequence of the fact that F is not singular. Now we are going to prove (2). Suppose that there exists a leaf S of $\mathcal{F}_2|_L$, homeomorphic to S^1. The fact that F is not singular implies that $\mathcal{F}_2$ is without holonomy and $\mathcal{F}_3|_L$ is transverse to $\mathcal{F}_2|_L$. Hence, there exists a neighborhood C of S in L such that every leaf of $\mathcal{F}_2|_L$ passing through a point in C is homeomorphic to S^1 and is not homotopic to a point in L. Moreover, the leaves of $\mathcal{F}_3|_L$ restricted to C are curves starting at one connected component of ∂C, and ending at the other one.

Let D be a smoothly immersed open 2-disc containing S, which we may assume to be in general position with respect to $\mathcal{F}_3$. Let $\mathcal{G}_3$ be the foliation (with singularities) on D which is induced by $\mathcal{F}_3$. Then, $\mathcal{G}_3$ is transverse to S.

We claim that $\mathcal{G}_3$ has no limit cycles, otherwise, the Poincaré-Bendixon theorem implies that there is a leaf of $\mathcal{G}_3$ which spirals towards a limit cycle γ. Hence, the leaf of $\mathcal{F}_3$ containing γ would have a non trivial holonomy group. This contradiction proves our claim. It follows from the claim above that $\mathcal{G}_3$, has exactly one singularity. Since $\mathcal{G}_3$ is transverse to S, this singularity is an attractor. But D in general position with respect to $\mathcal{F}_3$ means that $\mathcal{G}_3$ has a finite number of singularities, each of which is either a center or a saddle point. This contradiction concludes the proof. □

For each $\theta \in \mathbb{R}$ let T_θ, $S_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations defined by the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},$$

respectively. Note that T_θ (resp. S_θ) restricted to the xy-plane (resp. xz-plane) is the rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Let $\Pi : \mathbb{R}^3 \to \mathbb{R}$ be given by $\Pi(x, y, z) = x$. The following proposition will be needed.

Proposition 2.8 *Let $Y = (f, g, h)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map such that $0 \notin \operatorname{Spec}(Y)$ and $\mathcal{A}$ be a hRc of $\mathcal{F}(f)$. If $\Pi(\mathcal{A})$ is bounded, then there is $\epsilon > 0$ and $K \in \{S, T\}$ such that for all $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$ the foliation $\mathcal{F}(f_\theta)$ has a hRc $\mathcal{A}_\theta$ such that $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length, where $(f_\theta, g_\theta, h_\theta) = K_\theta \circ Y \circ K_\theta^{-1}$.*

Proof If $\Pi(\mathcal{A})$ is bounded, then either $\{y : (x, y, z) \in \mathcal{A}\}$ or $\{z : (x, y, z) \in \mathcal{A}\}$ is an interval of infinite length. We are going to show that, if $\{y : (x, y, z) \in \mathcal{A}\}$ is an interval of infinite length, then, for $K = T$, $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length. The proof of the other case is analogous, in which case the proposition is satisfied for $K = S$. Then, assume that $\{y : (x, y, z) \in \mathcal{A}\}$ is an interval of infinite length.

(a) Let $\theta \in \mathbb{R}$ be such that, for all $m \in \mathbb{Z}$, $\theta \neq \frac{m\pi}{2}$. Then the foliations $\mathcal{F}_1 = \mathcal{F}(f_\theta)$, $\mathcal{F}_2 = T_\theta(\mathcal{F}(f)) = \mathcal{F}(f \circ T_{-\theta})$ and $\mathcal{F}_3 = T_\theta(\mathcal{F}(h)) = \mathcal{F}(h \circ T_{-\theta})$ satisfy the assumptions of Lemma 2.7.

In fact, since $Z = (f \cos\theta - g \sin\theta, f, h)$ is not singular, for all $\theta \in \mathbb{R}\backslash\{\frac{m}{2}\pi;\ m \in \mathbb{Z}\}$, then $F = Z \circ T_{-\theta} = (f_\theta, f \circ T_{-\theta}, h \circ T_{-\theta})$ is also not singular.

Let A (resp. L) be the compact face (resp. non-compact face) of $\mathcal{A}$. Take a C^2-embedding $\eta : S^1 \times [-2, 2] \to \mathbb{R}^3$ such that $\Sigma_2 = \eta(S^1 \times [-2, 2])$ is transverse to $\mathcal{F}(f)$, $A = \eta(S^1 \times [-2, 0])$ and, for all $t \in [-2, 2]$, the circle $\eta(S^1 \times \{t\})$ is contained in a leaf of $\mathcal{F}(f)$.

Let $\mathcal{G}(f)$ be the one dimensional foliation on Σ_2 induced by $\mathcal{F}(f)$. Recall that there is a compact disk $D(f)$ contained in a leaf of $\mathcal{F}(f)|_A$ and such that $C(f) = \eta(S^1 \times \{-1\}) = D(f) \cap \Sigma_2$ is a leaf of $\mathcal{G}(f)$.

Since, for θ small enough, Y_θ and T_θ are C^1 close on any given compact subset of $\mathbb{R}^3$ to Y and to the identity T_0, respectively, we can take Σ_2 so that, for some $\varepsilon > 0$ and, for all $\theta \in (-\varepsilon, \varepsilon)$,

(b1) $T_\theta(\Sigma_2)$ is transverse to both $T_\theta(\mathcal{F}(f))$ and $\mathcal{F}(f_\theta)$;

(b2) there is a disc $D(f_\theta)$ passing through $T_\theta(\eta((1, 0) \times \{-1\}))$ contained in a leaf of $\mathcal{F}(f_\theta)$, close to $T_\theta(D(f))$ such that $\partial(D(f_\theta)) = D(f_\theta) \cap T_\theta(\Sigma_2)$ (in particular $T_\theta(D(f))$ and $D(f_\theta)$ meet $T_\theta(\Sigma_2)$ from the same side).

Let $\mathcal{G}_\theta$ be the foliation in $T_\theta(\Sigma_2)$ which is induced by $\mathcal{F}(f_\theta)$. As $\mathcal{F}(f_\theta)$ is without holonomy, we can take $\varepsilon > 0$ so that

(c) for all $\theta \in (-\varepsilon, \varepsilon)$, there exist open cylinders A_θ^-, $A_\theta^+ \subset T_\theta(\Sigma_2)$ made up by closed trajectories of $\mathcal{G}_\theta$ such that $A_\theta^- \subset T_\theta(\eta(S^1 \times [-1, 0)))$, $A_\theta^+ \subset T_\theta(\eta(S^1 \times (0, 1]))$, and both A_θ^- and A_θ^+ are the biggest cylinders with these properties.

We claim that:

(d) every leaf of $\mathcal{G}_\theta$ contained in A_θ^+ is not homotopic to a point in its corresponding leaf of $\mathcal{F}(f_\theta)$.

In fact, assume by contradiction that there exist a leaf γ of $\mathcal{G}_\theta$ contained in A_θ^+ and bounding a closed 2-disc $D(\gamma)$ contained in a leaf of $\mathcal{F}(f_\theta)$. It follows from Lemma 2.5 that $D(f_\theta)$ and $D(\gamma)$ meet $T_\theta(\Sigma_2)$ from the same side. If L is the non-compact face of $\mathcal{A}$ then $D(f_\theta)$, $D(\gamma)$, $T_\theta(D(f))$ and $T_\theta(L)$ meet $T_\theta(\Sigma_2)$ from the same side (see (b2)); therefore, the unbounded set $T_\theta(L)$ meets the bounded connected component of $\mathbb{R}^3\backslash(D(\gamma) \cup D(f_\theta) \cup T_\theta(\Sigma_2))$ and so $T_\theta(L)$ has to meet $D(\gamma)$. In this way there exists a closed 2-disc $D_0(\gamma) \subset D(\gamma)$ such that $\partial D_0(\gamma) = D(\gamma) \cap T_\theta(L)$. Consequently, there is at least one point in $D_0(\gamma)$ where $T_\theta(\mathcal{F}(f))$ and $\mathcal{F}(f_\theta)$ are tangent, contradicting (a). This proves (d).

By using a similar argument we may also obtain that

(e) every leaf of $\mathcal{G}_\theta$ contained in A_θ^- is homotopic to a point in its corresponding leaf of $\mathcal{F}(f_\theta)$.

In what follows in this proof, every time that we refer to Lemma 2.7 we will be assuming that it is being applied to the three foliations $\mathcal{F}(f_\theta)$, $T_\theta(\mathcal{F}(f))$ and $T_\theta(\mathcal{F}(h))$ (see (a)).

From (c), (e) and Lemma 2.7 we obtain that there exists a leaf γ of $\mathcal{G}_\theta$ contained in $T_\theta(\Sigma_2)\setminus(A_\theta^- \cup A_\theta^+)$ which is a vanishing cycle of $\mathcal{F}(f_\theta)$.

From now on, fix $\theta \in (-\varepsilon, \varepsilon)$. Let $\mathcal{A}_\theta$ be the *hRc* of $\mathcal{F}(f_\theta)$ with non-compact face L_θ and compact face contained in $T_\theta(\Sigma_2)$, and such that $\gamma = L_\theta \cap T_\theta(\Sigma_2)$. Notice that $L = L_0$.

Finally, we claim that

(f) $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length.

In fact, we assume by contradiction that $\Pi(\mathcal{A}_\theta)$ is bounded. The fact of $\{y : (x, y, z) \in \mathcal{A}\}$ to be an interval of infinite length implies that $\Pi(T_\theta(\mathcal{A}))$ is also an interval of infinite length. Then, there exists a leaf D_d of $T_\theta(\mathcal{F}(f))|_{T_\theta(\mathcal{A})}$ which is a disc such that

$$\sup \Pi(D_d) > \sup \Pi(\mathcal{A}_\theta) \ \text{ and } \ (f \circ T_{-\theta})(D_d) = d.$$

On the other hand, there exist a leaf D_c of $T_\theta(\mathcal{F}(f))|_{T_\theta(\mathcal{A})}$ that is a disc whose frontier ∂D_c is contained in the interior of the compact face of $\mathcal{A}_\theta$ and, such that $(f \circ T_{-\theta})(D_c) = c$. In this situation, the Lemma 2.7 implies that D_c is contained in $\mathcal{A}_\theta \setminus L_\theta$.

Without loss of generality we can assume that $c < d$. Applying the Lemma 2.4 to $f \circ T_{-\theta}$, D_c and D_d we obtain $K = \cup_{t\in[c,d]} D_t$, a compact subset of $T_\theta(\mathcal{A})$, where for all $t \in [c, d]$ the leaf D_t is a disc whose frontier is contained in the compact face of $T_\theta(\mathcal{A})$ and $(f \circ T_{-\theta})(D_t) = t$.

Let I be the set of elements s of $[c, d]$ such that the disc D_s is contained in $\mathcal{A}_\theta \setminus L_\theta$. Clearly $c \in I$. The fact that D_t depends continuously on t together with the relation $\sup \Pi(D_d) > \sup \Pi(\mathcal{A}_\theta)$, implies that $I = [c, t_0)$ where $t_0 \in (c, d)$ is such that $D_{t_0} \subset \mathcal{A}_\theta$ and $D_{t_0} \cap L_\theta \neq \emptyset$. Hence, the foliations $T_\theta(\mathcal{F}(f))$ and $\mathcal{F}(f_\theta)$ are necessarily tangent in all the points of $D_{t_0} \cap L_\theta$. This contradiction with (a) proves (f) and concludes the proof of this proposition. □

Proof of Theorem 1.1 By Palmeira's theorem, see [26], it is sufficient to show that $\mathcal{F}(k)$, $k \in \{f, g, h\}$, is a foliation by planes. Suppose by contradiction that $\mathcal{F}(f)$ has a leaf which is not homeomorphic to $\mathbb{R}^2$. It follows, from Proposition 2.2, that $\mathcal{F}(f)$ has a half-Reeb component $\mathcal{A}$. Hereafter we will use the fact that the existence of a half-Reeb component (see Lemma 2.5) and the assumptions of Theorem 1.1 are open in the Whitney C^2 topology, in particular we shall assume, from now on, that Y is smooth. Let $\Pi : \mathbb{R}^3 \to \mathbb{R}$ be the orthogonal projection onto the first coordinate. By conjugating with a transformation T_θ or S_θ if necessary (see Proposition 2.8) we may assume that $\Pi(\mathcal{A})$ is an unbounded interval. To simplify matters, let us suppose that $[b, \infty) \subset \Pi(\mathcal{A})$ and that $\Pi(A) \cap [b, \infty) = \emptyset$, where A is the compact face of $\mathcal{A}$.

By Thom's Transversality Theorem for jets [16], we may assume that $\mathcal{F}(f)$ has generic contact with the foliation $\mathcal{F}(\Pi)$. In this way, as f is a submersion,

(a1) the contact manifold $T = \{(x, y, z) \in \mathbb{R}^3;\ f_y(x, y, z) = 0 = f_z(x, y, z)\}$ is a subset of $\{(x, y, z) \in \mathbb{R}^3 : f_x(x, y, z) \neq 0\}$ made up of regular curves;

(a2) there is a discrete subset Δ of T such that if $p \in T\setminus\Delta$, then Π, restricted to the leaf of $\mathcal{F}(f)$ passing through p, has a Morse-type singularity at p which is either a saddle point or an extremal (maximum or minimum) point (see Fig. 5).

Then, if $a > b$ is large enough,

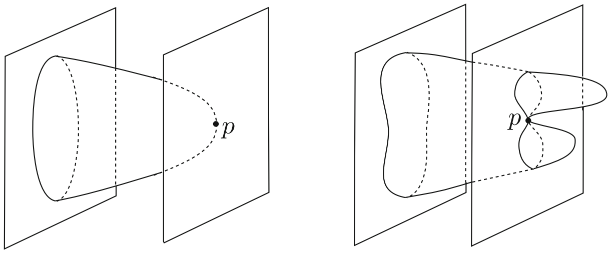


Fig. 5 Contact type of the leaf of $\mathcal{F}$ passing through p with the plane π = constant

(b) for any $x \geq a$, the plane $\Pi^{-1}(x)$ intersects exactly one leaf $L_x \subset \mathcal{A}$ of $\mathcal{F}(f)|_{\mathcal{A}}$ such that $\Pi(L_x) \cap (x, \infty) = \emptyset$. In other words, x is the supremum of the set $\Pi(L_x)$. Notice that L_x is a disc whose boundary is contained in the compact face of $\mathcal{A}$.
(c) if $x \geq a$ then $T_x = L_x \cap \Pi^{-1}(x)$ is contained in $T \cap \mathcal{A}$.
(d) if $p \in T_x$ then $p \in T \backslash \Delta$ is a maximum point for the restriction $\Pi|_{L_x}$.

Notice that T_x is a finite set disjoint from Δ for every $x \geq a$. Hence, the map $x \in [a, \infty) \longmapsto \#T_x$ is upper semi continuous, where $\#T_x$ denotes the cardinal number of T_x. To motivate what is claimed in (e) below, we observe that if, for some $x_0 \in [b, \infty)$ and for some $p \in T_{x_0}$, we had that $\#(T_{x_0}) > 1$ and $0 < f_x(p) < \min\{f_x(q) : q \in T_{x_0} \backslash \{p\}\}$, then we would obtain that, for some $\epsilon > 0$ and for every $x \in (x_0 - \epsilon, x_0) \cup (x_0, x_0 + \epsilon)$, $\#T_x = 1$; in this way, there would exist a smooth curve $\eta : (x_0 - \varepsilon, x_0 + \varepsilon) \mapsto T$ such that $\eta(x_0) = p \in T_{x_0}$ and for all $x \neq x_0$ the set $T_x = \{\eta(x)\}$.

Therefore, by $(b) - (d)$ and by using Thom's Transversality Theorem for jets, we may assume the following stronger statement:

(e) there is an increasing sequence $F = \{a_i\}_{i \geq 1}$ in $[a, +\infty)$, at most countable, such that if $x \in [a, +\infty) \backslash F$, then T_x is a one-point set.

If $x \in [a, +\infty) \backslash F$ and $T_x = \{(x, \eta_1(x), \eta_2(x))\}$, define $\eta : [a, +\infty) \backslash F \to T$ by $\eta(x) = (x, \eta_1(x), \eta_2(x))$. Observe that η is a smooth embedding and, since $f|_{\mathcal{A}}$ is continuous and bounded,

(f) $f \circ \eta$ extends continuously to a strictly monotone bounded map defined in $[a, +\infty)$ such that, for all $x \in [a, +\infty) \backslash F$, $f_x(\eta(x))$ has constant sign.

Therefore, there exists a real constant K such that

$$K = \int_{a_1}^{+\infty} \frac{d}{dx}(f \circ \eta)(x)dx$$
$$= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} \frac{d}{dx}(f \circ \eta)(x)dx$$
$$= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} f_x(\eta(x))dx.$$

This and (f) imply that, for some sequence $x_n \to +\infty$,

$$\lim_{n\to+\infty} f_{x_n}(\eta(x_n)) = 0.$$

This contradiction with the assumption that $\mathrm{Spec}(Y) \cap (-\varepsilon, \varepsilon) = \emptyset$ proves the theorem. □

3 Proof of Theorems 1.2 and 1.3

To prove Theorem 1.2 we shall need the following.

Lemma 3.1 (Lemma 6.2.11 of [12]) *Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded ring (A need not be commutative). Let $a \in A_d$ for some $d \geq 1$. Then $1 + a$ is invertible in A if and only if a is nilpotent.*

Proof of Theorem 1.2 We start as in the proof of Proposition 8.1.8 of [12]. By the Reduction Theorem (See [2,11,31]) it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F = (-X_1 + H_1, \ldots, -X_n + H_n)$$

where $X_i : \mathbb{C}^n \to \mathbb{C}$ denotes the canonical i-th coordinate function, each H_i is either zero or homogeneous of degree 3 and JH (with $H = (H_1, H_2, \ldots, H_n)$) is nilpotent. Consider the polynomial map $\tilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$\tilde{F} = (ReF_1, ImF_1, \ldots, ReF_n, ImF_n).$$

So we have $\tilde{F} = -\tilde{X} + \tilde{H}$, where $\tilde{H}$ is homogeneous of degree 3. Since JH is nilpotent, $JF = -I + JH$ is invertible by Lemma 3.1 and $\det J\tilde{F} = |\det JF|^2 = 1$, whence $J\tilde{F}$ is invertible. So by Lemma 3.1, $J\tilde{H}$ is nilpotent and consequently $\mathrm{Spec}(\tilde{F}) = \{-1\}$.

Now we proceed as in the proof of Proposition 8.3 of [22]. By [23,24], we get that the set S_F has complex codimension 1, hence $S_{\tilde{F}}$ has real codimension 2. Now F is bijective if, and only if, $\tilde{F}$ is bijective. Therefore if the assumption of this theorem is satisfied, it follows from Theorem Białynicki-Rosenlicht [3], that F is bijective. □

To prove Theorem 1.3 we shall need the following results of Jelonek [22]:

Theorem 3.2 *If $Y : \mathbb{R}^n \to \mathbb{R}^n$ is a real polynomial mapping with nonzero Jacobian everywhere and* $\mathrm{codim}(S_Y) \geq 3$, *then Y is a bijection (and consequently $S_Y = \emptyset$).*

Theorem 3.3 *Let $Y : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial mapping. Then the set S_Y is closed, semi-algebraic and for every non-empty connected component $S \subset S_Y$ we have* $1 \leq \dim(S) \leq n - 1$. *Moreover, for every point $q \in S_Y$ there exists a polynomial mapping $\phi : \mathbb{R} \to S_Y$ such that $\phi(\mathbb{R})$ is a semi-algebraic curve containing $\{q\}$.*

The proof of the following lemma is easy and will be omitted.

Lemma 3.4 *Let $Y : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map such that* $\mathrm{Spec}(Y) \cap \{0\} = \emptyset$. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. If $Z = A \circ Y \circ A^{-1}$ then* $\mathrm{Spec}(Y) = \mathrm{Spec}(Z)$ *and $S_Z = A(S_Y)$.*

Proposition 3.5 *Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map such that* $\mathrm{Spec}(Y) \cap (-\varepsilon, \varepsilon) = \emptyset$, *for some $\varepsilon > 0$. If* $\mathrm{codim}(S_Y) \geq 2$ *then Y is a bijection.*

Proof Suppose that Y is not bijective. By Theorem 3.2, we must have $\dim(S_Y) = 1$. Then by Theorem 3.3, we obtain that

(a) $Y(\mathbb{R}^3) \supset \mathbb{R}^3 \backslash S_Y$.

Therefore, using again Theorem 3.3, and Lemma 3.4, we may suppose that S_Y contains an analytic regular curve $\gamma : (a-\varepsilon, a+\varepsilon) \mapsto \mathbb{R}^3$ meeting the plane $\{x = a\}$ transversally at the point $p = \gamma(a) = (a, b, c)$. In this way

(b) the plane of $\{x = a\}$ contains a smooth embedded disc $D(a)$ such that $\{\gamma(a)\} = D(a) \cap S_Y$ and $C(a) \cap S_Y = \emptyset$, where $C(a)$ is the boundary of $D(a)$.

It is well known that there exists a positive integer K such that

(c) for all $q \in \mathbb{R}^3$, $\#Y^{-1}(q) \leq K$.

This implies that $Y^{-1}(C(a))$ is the union of finitely many embedded circles $C_1, C_2, \ldots, C_k$ contained in $f^{-1}(a)$. Each $Y_{|C_i} : C_i \to C(a)$ is a finite covering. As, by Theorem 1.1, each connected component of $f^{-1}(a)$ is a plane, we have that, for all $i = 1, 2, \ldots, k$, there exists a compact disc $D_i \subset f^{-1}(a)$ bounded by C_i. It follows that, for all $i = 1, 2, \ldots, k$, $Y(D_i) = D(a)$. As $D(a)$ is simply connected $Y_{|D_i} : D_i \to D(a)$ is a diffeomorphism for all $i \in \{1, 2, \ldots, k\}$. Hence, if $q \in C(a)$, $\#Y^{-1}(q) = k$. As $D(a) \cap S_Y = \{\gamma(a)\}$ and $\#Y^{-1}$ is locally constant, $\#Y^{-1}$ must be identically equal to k in $D(a)\backslash\{p\}$ and therefore $Y^{-1}(D(a) - \{p\}) \subset \cup_{i=1}^k D_i$. As Y is a local diffeomorphism, by using a limiting procedure,

(d) for all $q \in D(a)$we have $\#Y^{-1}(q) = k$, and so $Y^{-1}(D(a)) = \cup_{i=1}^k D_i$.

Notice that $D(a)$ can be taken to be of the form $D(a) = \{a\} \times D$, where D is a 2-disc of $\mathbb{R}^2$ centered at (b, c); in this way $C(a) = \{a\} \times \partial D$. We have that for $\varepsilon > 0$ small enough,

(e) if $s \in [a-\varepsilon, a+\varepsilon]$, $D(s) = \{s\} \times D$ and $C(s) = \{s\} \times \partial D$, then $\{\gamma(s)\} = D(s) \cap S_Y$ and $C(s) \cap S_Y = \emptyset$.

Proceeding as above, we find that for all $s \in [a-\varepsilon, a+\varepsilon]$ there are k embedded circles $C_1(s), C_2(s), \ldots, C_k(s)$, with $C_1(a) = C_1, C_2(a) = C_2, \ldots, C_k(a) = C_k$, contained in $f^{-1}(s)$ and such that $Y^{-1}(C(s)) = \cup_{i=1}^k C(k(s))$. Moreover each $C_i(s)$ depends continuously on s. Therefore,

(f) for all $s \in [a-\varepsilon, a+\varepsilon]$ and for all $i = 1, 2, \ldots, k$ there exists a compact disc $D_i(s) \subset f^{-1}(s)$ bounded by $C_i(s)$ such that $Y(D_i(s)) = D(s)$, and $D_i(s)$ depends continuously on s.

Proceeding as in the proof of (d) we obtain that

(g) $\#Y^{-1}(q) = k$ for all $s \in [a-\varepsilon, a+\varepsilon]$ and for all $q \in D(s)$, and $Y^{-1}(D(s)) = \cup_{i=1}^k D_i(s)$.

As $[a-\varepsilon, a+\varepsilon] \times D$ is a compact neighborhood of $\gamma(a)$ and $Y^{-1}([a-\varepsilon, a+\varepsilon] \times D)$ is compact we obtain a contradiction with the assumption $p \in S_Y$. $\square$

The proof of the following lemma can be found in [13,14]. We include it here for the sake of completeness.

Lemma 3.6 *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a differentiable map such that* $\det(F'(x)) \neq 0$ *for all* x *in* $\mathbb{R}^n$. *Given* $t \in \mathbb{R}$, *let* $F_t : \mathbb{R}^n \to \mathbb{R}^n$ *denote the map* $F_t(x) = F(x) - tx$. *If there exists a sequence* $\{t_m\}$ *of real numbers converging to* 0 *such that every map* $F_{t_m} : \mathbb{R}^n \to \mathbb{R}^n$ *is injective, then* F *is injective.*

Proof Choose $x_1, x_2 \in \mathbb{R}^n$ such that $F(x_1) = y = F(x_2)$. We will prove $x_1 = x_2$. By the Inverse Mapping Theorem for differentiable maps (A. V. Černavskii's Theorem [6,7]), we may find neighborhoods U_1, U_2, V of x_1, x_2, y, respectively, such that, for $i = 1, 2$, $F|_{U_i} : U_i \to V$ is a diffeomorphism and $U_1 \cap U_2 = \emptyset$. If m is large enough, then $F_{t_m}(U_1) \cap F_{t_m}(U_2)$ will contain a neighborhood W of y. In this way, for all $w \in W$, $\#(F_{t_m}^{-1}(w)) \geq 2$. This contradiction with the assumptions proves the lemma. □

Remark 3.7 Even if $n = 1$ and the maps F_{t_m} in Lemma 3.6 are smooth diffeomorphisms, we cannot conclude that F is a diffeomorphism. For instance, if $F : \mathbb{R} \to (0, 1)$ is an orientation reversing diffeomorphism, then for every $t > 0$, the map $F_t : \mathbb{R} \to \mathbb{R}$ (defined by $F_t(x) = F(x) - tx$) will be an orientation reversing global diffeomorphism.

Theorem 1.3 *Let $Y = (f, g, h) : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map such that* $\mathrm{Spec}(Y) \cap [0, \varepsilon) = \emptyset$ *for some $\varepsilon > 0$. If* $\mathrm{codim}(S_Y) \geq 2$ *then Y is a bijection.*

Proof We claim that for each $0 < t < \epsilon$, the map $Y_t : \mathbb{R}^3 \to \mathbb{R}^3$, given by $Y_t(x) = Y(x) - tx$, is injective.

In fact, as $D(Y_t)(x) = DY(x) - tI$, (where I is the Identity map), we obtain that if $0 < a < \min\{t, \epsilon - t\}$, then $\mathrm{Spec}(Y_t) \cap (-a, a) = \emptyset$. It follows immediately from Lemma 3.6 and Proposition 3.5 that Y is injective. The conclusion of this theorem is obtained by using the Theorem Białynicki-Rosenlicht [3]. □

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