PERGAMON

Nonlinear Analysis 42 (2000) 673–677

Nonlinear Analysis

www.elsevier.nl/locate/na

Existence and uniqueness of H-system's solutions with Dirichlet conditions

P. Amster, M.C. Mariani *, D.F. Rial

Dpto. de Matemática Fac. de Cs, Exactas y Naturales, Universidad de Buenos Aires, Pab. I, Ciudad Universitaria, 1428 Buenos Aires, Argentina

Received 10 June 1998; accepted 22 June 1998

Keywords: Mean curvature equation; Fixed point methods; Dirichlet problem for H-systems

1. Introduction

We consider the Dirichlet problem in the unit disc $B=\{(u,v)\in R^2; u^2+v^2<1\}$ for a vector function $X:\bar{B}\to R^3$ which satisfies the equation of prescribed mean curvature

$$\begin{aligned} &\Delta X = 2H(u,v)X_u \wedge X_v \quad \text{in } B,\\ &X = g \quad \text{in } \partial B, \end{aligned} \tag{1}$$

where $X_u=\partial X/\partial u$, $X_v=\partial X/\partial v$, $\wedge$ denotes the exterior product in R^3 and $H:\bar{B}\to R$ is a given continuous function.

The problem above arises in the Plateau and Dirichlet problems for the prescribed mean curvature equation that has been studied in [1,8,6,4,5,7].

The main results are the following theorems:

Theorem 1. *Let be* $2<p<\infty$, *and* $g\in W^{2,p}(B,R^3)$ *such that* $g-a$ *is small enough for a certain* $a\in R$, *then there exists a solution* $X\in W^{2,p}$ *of* (1).

Theorem 2. *Let* X *be a solution of* (1) *in* $W^{2,p}$ $1<p<\infty$. *Then*:

(i) *If* $p\le 2$ *and* $X\in W^{1,\infty}$ *then* X *is isolated in* $(W^{1,\infty}\cap W^{2,p},\|\cdot\|_{1,\infty})$.

* Corresponding author.

E-mail addresses: pamster@dm.uba.ar (P. Amster), mcmarian@dm.uba.ar (M.C. Mariani), drial@dm.uba.ar (D.F. Rial)

PII: S0362-546X(99)00123-6

(ii) *If* $p > 2$ *then* X *is isolated in* $W^{1,p}$. *Moreover, if* $\overline{\Omega}$ *is a bounded subset of* $W^{1,p}$ *the number of solutions of* (1) *in* $\overline{\Omega}$ *is finite.*

Remark. For H constant Hildebrandt found a solution of (1) in $W^{1,2}$, for the case $|H| \parallel g \parallel_\infty \leq 1$ [3]. Theorem 1 gives a solution for $p > 2$ when g is close to a constant a. Since the equation in B depends only on the derivatives of X, we may suppose $a = 0$

2. Solution by fixed point methods

Systems (2) and (3) are equivalent to (1) with $X = X_0 + Y$:

$$\begin{aligned} &\Delta X_0 = 0 \quad \text{in } B, \\ &X_0 = g \quad \text{in } \partial B, \end{aligned} \tag{2}$$

$$\begin{aligned} &\Delta Y = F(X_0, Y) \quad \text{in } B, \\ &Y = 0 \quad \text{in } \partial B, \end{aligned} \tag{3}$$

where F is given by

$$F(X_0, Y) = 2H(u, v)[(X_{0u} \wedge Y_v + Y_u \wedge X_{0v}) + (Y_u \wedge Y_v + X_{0u} \wedge X_{0v})].$$

For $g \in W^{2,p}(B, R^3)$, (2) admits a unique solution in $W^{2,p}(B, R^3)$. We can rewrite (3) as

$$\begin{aligned} &L(X_0)Y = F_1(X_0, Y) \quad \text{in } B, \\ &Y = 0 \quad \text{in } \partial B, \end{aligned}$$

where $L(X_0)$ is the linear operator

$$L(X_0)Y = \Delta Y - 2H(u, v)(X_{0u} \wedge Y_v + Y_u \wedge X_{0v})$$

and F_1 is defined by

$$F_1(X_0, Y) = 2H(u, v)(Y_u \wedge Y_v + X_{0u} \wedge X_{0v}).$$

Remark. $L(X_0)$ is strictly elliptic and for $p > 2$ its coefficients are bounded since $X_0 \in W^{1,\infty}(B, R^3)$ and $H \in C(\bar{B})$.

We will use the following technical lemmas:

Lemma 4. *Let be* $X_0 \in W^{2,p}(B, R^3)$, *then*:

(i) *If* $2 < p < \infty$,

$$\|F_1(X_0, Y_1)\|_{p/2} \leq 2\|H\|_\infty(\|X_0\|_{1,p}^2 + \|Y_1\|_{1,p}^2), \tag{4}$$

$$\|F_1(X_0, Y_1) - F_1(X_0, Y_2)\|_{p/2} \leq 4\|H\|_\infty R(\|Y_1 - Y_2\|_{1,p}) \tag{5}$$

for $Y_1, Y_2 \in B_R(0) \subset W^{1,p}$.

(ii) *If* $p>1, X_0, Y_1, Y_2 \in W^{1,p}\cap W^{1,\infty}$

$$\|F_1(X_0,Y_1)\|_p \le 2\|H\|_\infty(\|X_0\|_{1,\infty}.\|X_0\|_{1,p}+\|Y_1\|_{1,\infty}.\|Y_1\|_{1,p}), \tag{6}$$

$$\|F_1(X_0,Y_1)-F_1(X_0,Y_2)\|_p \le 4\|H\|_\infty R(\|Y_1-Y_2\|_{1,p}) \tag{7}$$

for $Y_1, Y_2 \in B_R(0)\subset W^{1,\infty}$.

Proof.

$$\|F_1(X_0,Y_1)\|_{p/2} \le 2\|H\|_\infty(\|Y_{1u}\wedge Y_{1v}\|_{p/2}+\|X_{0u}\wedge X_{0v}\|_{p/2})$$

and

$$\|Y_{1u}\wedge Y_{1v}\|_{p/2}^{p/2} \le \int |Y_{1u}|^{p/2}|Y_{1v}|^{p/2} \le \|Y_{1u}\|_p^{p/2}\|Y_{1v}\|_p^{p/2}.$$

The same inequality holds for X_0, and in order to prove (5), we also have that

$$\|Y_{1u}\wedge Y_{1v}-Y_{2u}\wedge Y_{2v}\|_{p/2} \le \|Y_{1u}\wedge(Y_{1v}-Y_{2v})\|_{p/2}+\|Y_{2v}\wedge(Y_{1u}-Y_{2u})\|_{p/2},$$

and the proof follows.

In the same way we obtain (6) and (7), using that $\|a\wedge b\|_p \le \|a\|_\infty.\|b\|_p$. □

Lemma 5. *If* $\|X_0\|_{1,p}\le\delta$, $2<p<\infty$ *with* δ *small enough, then there exists a constant* C *independent of* X_0 *verifying*

$$\|Y\|_{2,p/2} \le C\|L(X_0)Y\|_{p/2},$$

for all $Y\in W^{2,p/2}\cap W_0^{1,p/2}$.

Proof. Writing $L(X_0)Y=\Delta Y-R(X_0)Y$ it follows that

$$\|L(X_0)Y\|_{p/2} \ge \|\Delta Y\|_{p/2}-\|R(X_0)Y\|_{p/2} \ge \frac{1}{C_1}\|Y\|_{2,p/2}-\|R(X_0)Y\|_{p/2}$$

where C_1 is the constant provided by Lemma 9.17 in [2].

As $\|R(X_0)Y\|_{p/2}\le 4\|H\|_\infty\|X_0\|_{1,p}\|Y\|_{1,p}\le 4\bar{c}\|H\|_\infty\|X_0\|_{1,p}\|Y\|_{2,p/2}$ (where $\bar{c}$ is the constant of Sobolev immersion $W^{2,p/2}\hookrightarrow W^{1,p}$), the proof follows. □

Remark. The same Lemma 9.17 in [2] shows that if $\|g\|_{2,p}$ is small, then $\|X_0-g\|_{2,p}$ is small, and then also $\|X_0\|_{1,p}$.

Proposition 6. *Let* $X_0\in W^{2,p}(B,R^3)$ *with* $p>2$; *then the following problem*;

$$\begin{aligned} &L(X_0)Y = F_1(X_0,\overline{Y}) \quad \text{in } B,\\ &Y=0 \quad \text{in } \partial B, \end{aligned} \tag{8}$$

defines a continuous map $T:\overline{Y}\to Y$ *in* $W_0^{1,p}$. *Furthermore, if* $\|X_0\|_{1,p}$ *is small enough, there exists a number* R *such that* T *is a contraction in* $B_R(0)\subset W^{1,p}$.

Proof. From Theorem 9.15 in [2], problem (8) admits a unique solution $Y \in W^{2,p/2}$, so the map T is well defined. Also we have from (5) that if $\overline{Y}, \overline{Z} \in B_R \subset W^{1,p}$ then

$$\begin{aligned}\|Y - Z\|_{2,p/2} &\le C(\|L(X_0)(Y - Z)\|_{p/2} = C\|F_1(X_0, \overline{Y}) - F_1(X_0, \overline{Z})\|_{p/2} \\ &\le 4C\|H\|_\infty R\|\overline{Y} - \overline{Z}\|_{1,p}\end{aligned}$$

and using the Sobolev immersion of $W^{2,p/2}$ in $W^{1,p}$ the continuity of T follows.
Also we have from (4)

$$\|Y\|_{1,p} \le \bar{c}\|Y\|_{2,p/2} \le C\bar{c}\|L(X_0)Y\|_{p/2} \le 2C\bar{c}\|H\|_\infty(\|X_0\|_{1,p}^2 + \|\overline{Y}\|_{1,p}^2).$$

Choosing $\|X_0\|_{1,p}$ and R small enough, we obtain

$$4C\|H\|_\infty R < 1,$$

$$\|Y\|_{1,p} \le R$$

and T is a contraction in B_R. □

Remark. Note that if Y is a fixed point of T, then $Y \in W^{2,p}$, and we obtain Theorem 1 as an immediate consequence of Proposition 6.

3. Local uniqueness of the solutions

Now we will prove Theorem 2.
Let X_0 be a solution of (1). If Y is another solution of (1) then $Z = Y - X_0$ satisfies

$$\begin{aligned}&L(X_0)Z = 2H(u,v)Z_u \wedge Z_v \quad \text{in } B,\\ &Z = 0 \quad \text{in } \partial B.\end{aligned}$$

We consider the associated problem

$$\begin{aligned}&L(X_0)Z = 2H(u,v)\overline{Z}_u \wedge \overline{Z}_v \quad \text{in } B,\\ &Z = 0 \quad \text{in } \partial B.\end{aligned}$$

For (ii), in the same way as before, we obtain

$$\|Z\|_{1,p} \le 2C(X_0)\bar{c}\|H\|_\infty\|\overline{Z}\|_{1,p}^2$$

and if $\overline{Z}_1, \overline{Z}_2 \in B_R(0) \subset W^{1,p}$ then

$$\|Z_1 - Z_2\|_{1,p} \le 4C(X_0)\bar{c}\|H\|_\infty R\|\overline{Z}_1 - \overline{Z}_2\|_{1,p}.$$

If we choose R verifying

$$4C(X_0)\bar{c}\|H\|_\infty R < 1,$$

the map $\overline{T} : \overline{Z} \to Z$ is a contraction in B_R, so X_0 is isolated.

Remark. From Sobolev immersions $\overline{T}$ is compact in $W^{1,p}$ and any solution of (1) being a fixed point of $\overline{T}$, we conclude that the number of solutions in B_R is finite for every R.

In the same way we prove (i) using (6) and (7).

Remark. The number R in the theorems above can be estimated in terms of the Sobolev immersions constants, $\|H\|_\infty$ and $\|g\|_{2,p}$.

References

[1] H. Brezis, J.M. Coron, Multiple solutions of H systems and Rellich's conjecture, Comm. Pure Appl. Math. 37 (1984) 149–187.
[2] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1983.
[3] S. Hildebrandt, On the Plateau problem for surfaces of constant mean curvature, Comm. Pure Appl. Math. 23 (1970) 97–114.
[4] D. Lami, M.C.E. Mariani, A Dirichlet problem for an H system with variable H, Manuscripta Math. 81 (1993) 1–14.
[5] M.C. Mariani, D. Rial, Solutions to the mean cuvature equation by fixed point methods, Bull. Belg. Math. Soc. 4 (1997) 617–620.
[6] M. Struwe, Plateau's problem and the calculus of variations, Lecture Notes, Princeton University Press, Princeton, NJ, 1988.
[7] M. Struwe, Multiple solutions to the Dirichlet problem for the equation of prescribed mean curvature, in: P. Rabinowitz, E. Ehnder (eds.), Research Papers Published in Honor of J. Moser's 60th Birthday, Academic Press, Boston, 1990, pp. 639–666.
[8] G. Wang, The Dirichlet problem for the equation of prescribed mean curvature, Anal. Nonlinéaire 9 (1992) 643–655.