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Non-dissipative boundary feedback for Rayleigh and Timoshenko beams

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ABSTRACT

We show that a non-dissipative feedback that has been shown in the literature to exponentially stabilize an Euler–Bernoulli beam makes a Rayleigh beam and a Timoshenko beam unstable.

1. Introduction

Feedback control of beams is a much studied topic, in part due to its applications to the control of robot arms. The feedback control strategies used are often of the static output feedback kind and the input and output are usually chosen to make the closed loop system dissipative. An intriguing non-dissipative control strategy was however chosen in [1]. We refer to that article for the physical interpretation of their choice of feedback. As an open-loop model they consider an undamped Euler–Bernoulli beam. Dissipative static output feedback strategies give rise to a closed loop system that has eigenvalues asymptotic to a line $\operatorname{Re}\lambda = -c$ for some constant $c > 0$ (see e.g. [2]). The eigenvalues of the non-dissipative closed-loop system were shown in [1] to be asymptotic to the parts of the parabolas $\operatorname{Im}\lambda = \pm c\,(\operatorname{Re}\lambda)^2$ in the left half-plane (see Fig. 1(a)). This indicates that high frequencies are much better damped by the non-dissipative feedback than by dissipative feedbacks, a very attractive property.

Besides the above asymptotics, [1] also showed that – as in the dissipative case – the eigenvalues of the closed loop system are all in the open left half-plane. However, for partial differential equations certain pathologies may occur that prevent the stability of a system to be determined from the location of its eigenvalues. Due to this, [1] only managed to show the exponential stability of the closed-loop system for smooth initial conditions in spite of the fact that all its eigenvalues are in the open left half-plane and are bounded away from the imaginary axis. Using estimates of the Green function [3] showed that the closed-loop system is a Riesz spectral system and since for Riesz spectral systems the location of the eigenvalues does determine the stability, exponential stability followed (also for non-smooth initial data). Subsequently, [4] gave a more direct proof that the closed-loop system is a Riesz spectral system and [5] gave a proof of exponential stability based on microlocal analysis instead of on the Riesz basis property.

As mentioned, [1] chose an Euler–Bernoulli beam model (and the subsequent articles mentioned followed suit). This neglects the fact that the beam has a moment of inertia (and probably less importantly it neglects shear effects and non-linear effects). The Rayleigh beam model does incorporate the fact that a beam has a positive moment of inertia. The eigenvalues based on a finite element approximation of the Rayleigh beam with a non-dissipative feedback analogous to the one in [1] are given in Fig. 1(b). Surprisingly, the eigenvalues are very different from those in the Euler–Bernoulli case. In particular, there are many unstable eigenvalues. In this article we prove that indeed the Rayleigh beam with non-dissipative feedback has infinitely many unstable eigenvalues. We also prove that the addition of shear effects on top of a nonzero moment of inertia (i.e. replacing the Rayleigh model by the Timoshenko model) gives no qualitative difference: also in that case there are infinitely many eigenvalues with a positive real part. We conclude that a static non-dissipative feedback as considered [1] is a *worse* choice for stability than dissipative feedback for Rayleigh and Timoshenko beam models.

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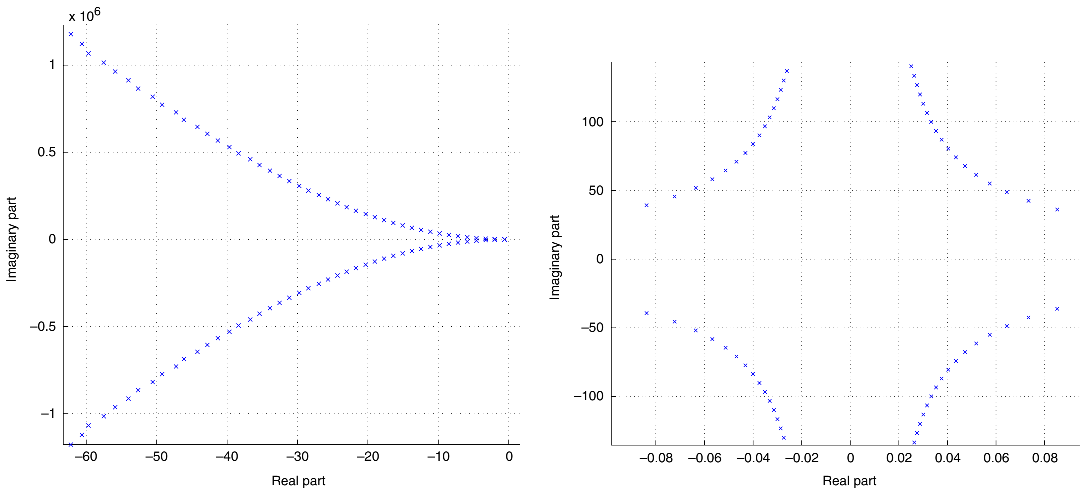

(a) Euler–Bernoulli beam.

(b) Rayleigh beam.

Fig. 1. Numerical approximations for eigenvalues of the Euler–Bernoulli and Rayleigh beam models.

2. Main results

2.1. Rayleigh beam case

We consider first the following Rayleigh beam problem:

$$\begin{aligned} &EIw_{\xi\xi\xi\xi} + \rho w_{tt} - I_\rho w_{\xi\xi tt} = 0, \\ &w = w(\xi, t), \quad t \in \mathbb{R}_+,\ \xi \in [a, b] \subset \mathbb{R}, \end{aligned} \tag{1a}$$

where $w(\xi, t)$ is the transverse displacement of the beam at position ξ and time t. We use the notation $w_t = \frac{\partial w}{\partial t}$ and $w_\xi = \frac{\partial w}{\partial \xi}$. The constants EI, ρ and I_ρ are physical parameters associated with the beam, for details see [6], or most elementary vibration textbooks. The choice of boundary feedbacks is analogous to the choice in [1,3–5] and are for $t \geq 0$:

$$\begin{aligned} &w(a, t) = 0, \\ &w_\xi(a, t) = 0, \\ &-k_1 w_t(b, t) = w_{\xi\xi}(b, t), \\ &-k_2 w_{\xi_t}(b, t) = (I_\rho w_{\xi tt} - EIw_{\xi\xi\xi})(b, t), \end{aligned} \tag{1b}$$

where $k_1, k_2 \geq 0$ are the feedback constants. The beam is clamped at the left endpoint which is described by the first two equations in (1b). To help understand the motivation for the third and fourth equations in (1b), recall that the energy of the Rayleigh beam is given by:

$$E(t) = \frac{1}{2} \int_a^b EI|w_{\xi\xi}|^2 + \rho|w_t|^2 + I_\rho |w_{t\xi}|^2 \,\mathrm{d}\xi.$$

Differentiating with respect to t, substituting using (1a), integrating by parts and then applying the boundary conditions at $\xi = a$ gives:

$$\begin{aligned} E_t(t) &= \left\langle \begin{pmatrix} w_t(b, t) \\ w_{\xi t}(b, t) \end{pmatrix}, \begin{pmatrix} I_\rho w_{\xi t_t}(b, t) - EIw_{\xi\xi\xi}(b, t) \\ EIw_{\xi\xi}(b, t) \end{pmatrix} \right\rangle \\ &=: \langle y(t), u(t) \rangle, \end{aligned} \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^2$ and $u(t)$ is the input. From Lyapunov theory, it is sensible to choose u such that $E_t(t) < 0$ along solutions w. Therefore, an obvious choice of u is

$$u(t) = Ky(t), \tag{3}$$

with K negative definite, which is the so-called dissipative boundary feedback. Inserting (3) into (2) gives:

$$E_t(t) = \langle y(t), Ky(t) \rangle < 0.$$

The canonical negative definite matrix is

$$K = \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}, \quad k_1, k_2 > 0.$$

The choice of boundary conditions in [3] for the Euler–Bernoulli case (i.e. (1a) and (1b) with $I_\rho = 0$) is to instead take

$$K = \begin{pmatrix} 0 & -k_2 \\ -k_1 & 0 \end{pmatrix}, \tag{4}$$

which is an indefinite matrix (and leads to non-dissipative boundary feedback). Exponential stability is proven when $k_1 = 0$ and $k_2 > 0$. The same result also holds in the alternate case with $k_1 > 0$, $k_2 = 0$ which follows by a duality argument.

The choice of feedback matrix (4) in the Rayleigh case gives the third and fourth equations in (1b).

Denote by (1) the partial differential equation (1a) and the boundary conditions (1b). In this article we prove that not only is the Rayleigh system (1) *not* exponentially stable, but further that the system is in fact *unstable*.

To that end, we make the ansatz that a non-trivial solution to (1) has the form:

$$w(\xi, t) = \mathrm{e}^{st}\mathrm{e}^{\lambda(\xi - a)}, \quad s, \lambda \in \mathbb{C}. \tag{5}$$

Throughout this paper we will assume that $s \neq 0$. In order for such an ansatz (5) to be a solution λ, s must satisfy an algebraic condition given by the PDE (1a) and a characteristic equation given by the boundary conditions (1b). The algebraic condition is:

$$\lambda^4 - \frac{s^2 I_\rho}{EI}\lambda^2 + \frac{s^2\rho}{EI} = 0, \tag{6}$$

giving

$$\begin{aligned} \lambda_1 &= \sqrt{\frac{\frac{s^2 I_\rho}{EI} + \sqrt{\frac{s^4 I_\rho^2}{EI^2} - 4\frac{s^2\rho}{EI}}}{2}}, \\ \lambda_2 &= \sqrt{\frac{\frac{s^2 I_\rho}{EI} - \sqrt{\frac{s^4 I_\rho^2}{EI^2} - 4\frac{s^2\rho}{EI}}}{2}}, \qquad \lambda_3 = -\lambda_1, \qquad \lambda_4 = -\lambda_2. \end{aligned} \tag{7}$$

It follows that a non-trivial solution to (1a) is given by

$$w(\xi, t) = \mathrm{e}^{st} \sum_{i=1}^{4} c_i \mathrm{e}^{\lambda_i(s)(\xi - a)}, \quad s \in \mathbb{C},\ c_i \in \mathbb{R} \text{ not all zero}. \tag{8}$$

The boundary conditions (1b) applied to (8) yields the second condition for λ, s in the form of a linear system for the c_i, given below:

$$Pc := \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & -\lambda_1 & -\lambda_2 \\ \varepsilon_1 e^{\lambda_1 \Delta} & \varepsilon_2 e^{\lambda_2 \Delta} & \varepsilon_1 e^{-\lambda_1 \Delta} & \varepsilon_2 e^{-\lambda_2 \Delta} \\ \lambda_1 \eta_1 e^{\lambda_1 \Delta} & \lambda_2 \eta_2 e^{\lambda_2 \Delta} & -\lambda_1 \eta_1 e^{-\lambda_1 \Delta} & -\lambda_2 \eta_2 e^{-\lambda_2 \Delta} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0, \tag{9}$$

where $\Delta := b - a$, $\varepsilon_i = \lambda_i^2 + k_1 s$ and $\eta_i = (-k_2 s - s^2 I_\rho + EI\lambda_i^2)$. Eq. (9) has a non-trivial solution c if and only if $\det P = 0$. Computing $\det P = 0$ and dividing through by s^5 results in the following characteristic equation:

$$\begin{aligned} 0 &= \lambda_1\lambda_2 \left[\frac{I_\rho^2}{EIs} + \frac{k_2 I_\rho}{EIs^2} + \frac{k_1 I_\rho}{s^2} + 2\frac{k_1 k_2 - \rho}{s^3} \right] \\ &\times \cosh(\lambda_1\Delta)\cosh(\lambda_2\Delta) - \left[\frac{\rho I_\rho}{EIs} + \frac{k_1 k_2 I_\rho}{EIs} + \frac{2k_2\rho}{EIs^2} + \frac{2k_1\rho}{s^2} \right] \\ &\times \sinh(\lambda_1\Delta)\sinh(\lambda_2\Delta) \\ &- \lambda_1\lambda_2 \left[\frac{k_2 I_\rho}{EIs^2} + \frac{k_1 I_\rho}{s^2} + 2\frac{\rho + k_1 k_2}{s^2} \right]. \end{aligned} \tag{10}$$

We prove the instability of the system (1) by investigating the sign of $\operatorname{Re} s$, for s a zero of (10) and ultimately proving (10) has zeros with positive real part. In this case we have a solution of (1) in the form (8) with $\operatorname{Re} s > 0$, and instability follows. We mention again that in [3] only one of the feedback parameters is required to be non-zero in order to achieve exponential stability. To give full generality we consider all three possible cases. These are where exactly one of k_1 and k_2 is zero, and also where both k_1, k_2 are positive. Our main results are now stated beneath:

Theorem 2.1. *For all $k_1, k_2 \geq 0$ with $k_1 + k_2 > 0$ the Eq. (10) has zeros $s_n \in \mathbb{C}$, $n \in \mathbb{N}$ which satisfy*

$$\left| s_n - \frac{(\pi n + \frac{\pi}{2})\,\mathrm{i}}{b-a}\sqrt{\frac{EI}{I_\rho}} \right| \to 0 \quad \text{as } n \to \infty.$$

Further, $\operatorname{Re} s_n > 0$ for infinitely many $n \in \mathbb{N}$.

We then deduce the following corollary.

Corollary 2.2. *For all $k_1, k_2 \geq 0$ with $k_1 + k_2 > 0$ the system (1) is unstable.*

2.2. Timoshenko beam case

We consider next the Timoshenko beam equation:

$$\begin{aligned} &w = w(\xi, t), \quad t \in \mathbb{R}_+,\ \xi \in [a, b] \subset \mathbb{R}, \\ &EI w_{\xi\xi\xi\xi} + \rho w_{tt} - \left(I_\rho + \frac{EI\rho}{K} \right) w_{\xi\xi tt} + \frac{I_\rho \rho}{K} w_{tttt} = 0, \end{aligned} \tag{11}$$

where K is an additional physical parameter, the shear modulus. It is also convenient to write (11) as the coupled wave equations

$$\begin{aligned} \rho w_{tt} &= K w_{\xi\xi} - K\phi_\xi, \\ I_\rho \phi_{tt} &= EI\phi_{\xi\xi} - K\phi + K w_\xi, \end{aligned} \tag{12a}$$

where ϕ is the angular displacement. Note that as the parameter K tends to infinity the Eq. (11) collapses to (1a), the PDE for the Rayleigh beam, which represents the beam becoming rigid to shear. The non-dissipative boundary feedbacks for the Timoshenko beam are:

$$\begin{aligned} &w_t(a, t) = \phi_t(a, t) = 0, \\ &w_\xi(b, t) - \phi(b, t) = -k_1 I_\rho \phi_t(b, t), \\ &\phi_\xi(b, t) = -k_2 \rho w_t(b, t), \end{aligned} \tag{12b}$$

where $k_1, k_2 \geq 0$ are the feedback constants.

There is an elegant formulation of the Timoshenko beam problem using state variables x_1, x_2, x_3, x_4 where

$$\begin{aligned} x_1 &= w_\xi - \phi, \\ x_2 &= \rho w_t, \\ x_3 &= \phi_\xi, \\ x_4 &= I_\rho \phi_t. \end{aligned}$$

In these variables the energy of the Timoshenko beam is

$$E(t) = \frac{1}{2}\int_a^b K|x_1|^2 + \frac{1}{\rho}|x_2|^2 + EI|x_3|^2 + \frac{1}{I_\rho}|x_4|^2 \, d\xi.$$

Arguing as in the Rayleigh case it is not difficult to see that (12b) are indeed the analogous choice of non-dissipative boundary conditions for this problem. For more information on the state variable approach to the Timoshenko beam we refer the reader to Villegas' thesis [7].

Let (12) denote the PDE (12a) and boundary conditions (12b). We proceed as in the Rayleigh case and make the ansatz for a solution of (12)

$$\begin{aligned} w(\xi, t) &= e^{st} \sum_{i=1}^{4} c_i e^{\lambda_i(s)(\xi - a)}, \\ \phi(\xi, t) &= e^{st} \sum_{i=1}^{4} c_i e^{\lambda_i(s)(\xi - a)} \left(\lambda_i - \frac{\rho s^2}{K\lambda_i} \right), \end{aligned} \tag{13}$$

for $c_i \in \mathbb{R}$ not all zero. The λ, s satisfy algebraic conditions from the PDE (12a) and the boundary conditions (12b). For each $s \in \mathbb{C}$, the λ_i are the four roots of

$$EI\lambda^4 - \left(I_\rho + \frac{EI\rho}{K} \right) s^2\lambda^2 + \left(\rho s^2 + s^4 \frac{\rho I_\rho}{K} \right) = 0. \tag{14}$$

The second condition, the corresponding linear system for the c_i, is given by:

$$Q(s)c := \begin{bmatrix} 1 & 1 & 1 & 1 \\ \varepsilon_1 & \varepsilon_2 & -\varepsilon_1 & -\varepsilon_2 \\ \eta_1 e^{\lambda_1\Delta} & \eta_2 e^{\lambda_2\Delta} & -\eta_1 e^{-\lambda_1\Delta} & -\eta_2 e^{-\lambda_2\Delta} \\ \chi_1 e^{\lambda_1\Delta} & \chi_2 e^{\lambda_2\Delta} & \chi_1 e^{-\lambda_1\Delta} & \chi_2 e^{-\lambda_2\Delta} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0, \tag{15}$$

where $\Delta := b - a$ and for $i \in \{1, 2\}$

$$\begin{aligned} &\varepsilon_i = \lambda_i - \frac{\rho s^2}{K\lambda_i}, \qquad \eta_i = k_1 I_\rho \lambda_i + \frac{\rho s}{K\lambda_i} - \frac{k_1 I_\rho \rho s^2}{K\lambda_i}, \\ &\chi_i = \lambda_i^2 - \frac{\rho s^2}{K} + k_2\rho s. \end{aligned} \tag{16}$$

Again, we seek s such that $\det Q = 0$. The resulting characteristic equation is:

$$\begin{aligned} 0 = {} & R(s, \lambda_1, \lambda_2)\cosh(\lambda_1\Delta)\cosh(\lambda_2\Delta) \\ & + P(s, \lambda_1, \lambda_2)\sinh(\lambda_1\Delta)\sinh(\lambda_2\Delta) + T(s, \lambda_1, \lambda_2) \end{aligned} \tag{17}$$

where P, R and T are polynomials in several variables and are given in more detail in Section 4.

As before, zeros of the characteristic equation (17) will give a solution to the Timoshenko beam system (12) in the form of our ansatz (13). We prove (12) is *not* exponentially stable by proving (17) has zeros with positive real part.

Theorem 2.3. *For all positive ρ, EI, I_ρ and K with $\frac{I_\rho}{EI} \neq \frac{\rho}{K}$ and all non-negative k_1, k_2 with $k_1 + k_2 > 0$ and $k_1 k_2 \neq \frac{1}{KI_\rho}$, the Eq. (17) has infinitely many zeros, $s_n \in \mathbb{C}$, with $\operatorname{Re} s_n > 0$.*

If $\frac{I_\rho}{EI} = \frac{\rho}{K}$ and $k_1k_2 > 0$, $k_1k_2 \neq \frac{1}{KI_\rho}$ then the above result holds. If $\frac{I_\rho}{EI} = \frac{\rho}{K}$ and $k_1k_2 = 0$ then the above result holds provided that additionally $\cos\left(\frac{(b-a)}{2}\sqrt{\frac{\rho}{I_\rho}}\right) \neq 0$.

We deduce the following corollary.

Corollary 2.4. *Assuming the hypotheses of Theorem 2.3, the system (12) is unstable.*

3. Proofs for the Rayleigh beam

The work that follows is an analysis of the characteristic equation (10) which eventually allows us to deduce Theorem 2.1. The main ingredient in the proof is Rouché's theorem, which we first apply to the Eq. (10) on circles centred on the imaginary axis. We work with the identity:

$$0 = \cosh(\lambda_1\Delta)\cosh(\lambda_2\Delta) + \sum_{i=1}^{2}\frac{a_i}{s^i}\cosh(\lambda_1\Delta)\cosh(\lambda_2\Delta) + \sum_{i=1}^{2}\frac{b_i}{s^i}\sinh(\lambda_1\Delta)\sinh(\lambda_2\Delta) + \sum_{i=1}^{2}\frac{c_i}{s^i}, \tag{18}$$

where the numbers a_i, b_i, c_i are constants. We observe that since $\lambda_1\lambda_2 = \sqrt{s^2\frac{\rho}{EI}}$ the characteristic equation (10) is an example of (18) with a particular choice of constants. We seek to eliminate the λ_i terms from (18) and to do this we will make use of their Taylor expansions, however first we make a remark to ease the following notation.

Remark 3.1. For complex numbers z and indices n we use the notation $O(z^n)$ in place of $O(|z|^n)$.

The Taylor expansions of $\mathbb{C} \ni z \mapsto \lambda_1(z), \lambda_2(z)$ at infinity are given respectively by:

$$\lambda_1(z) = z\sqrt{\frac{I_\rho}{EI}} + O(z^{-1}), \tag{19}$$

$$\text{and} \quad \lambda_2(z) = \sqrt{\frac{\rho}{I_\rho}} + O(z^{-2}). \tag{20}$$

Remark 3.2. In what follows we will only be considering complex s with bounded real part and large modulus. For such s it follows that

$$\cosh(\mu s), \sinh(\mu s) = O(1), \quad \forall \mu \in \mathbb{R}.$$

Let $d_1 := \Delta\sqrt{\frac{I_\rho}{EI}}$, $d_2 := -\Delta\frac{\rho\sqrt{EI}}{2I_\rho\sqrt{I_\rho}}$ and $d_3 := \Delta\sqrt{\frac{\rho}{I_\rho}}$. Using the Maclaurin series

$$\cosh x = 1 + \frac{x^2}{2} + O(x^4), \tag{21}$$

$$\sinh x = x + \frac{x^3}{6} + O(x^5), \tag{22}$$

the Taylor expansions (19) and (20), the hyperbolic addition formulae and Remark 3.2 we obtain

$$\begin{aligned}\cosh(\Delta\lambda_1) &= \cosh(d_1 s) + \left(\frac{d_2}{s} + O(s^{-2})\right)\sinh(d_1 s) + O(s^{-2})\cosh(d_1 s)\\ &= \cosh(d_1 s) + \frac{d_2}{s}\sinh(d_1 s) + O(s^{-2}),\end{aligned} \tag{23}$$

$$\cosh(\Delta\lambda_2) = \cosh d_3 + O(s^{-2}). \tag{24}$$

Similarly

$$\sinh(\Delta\lambda_1) = \sinh(d_1 s) + \frac{d_2}{s}\cosh(d_1 s) + O(s^{-2}), \tag{25}$$

$$\sinh(\Delta\lambda_2) = \sinh(d_3) + O(s^{-2}). \tag{26}$$

Substituting (23)–(26) into Eq. (18) gives

$$\begin{aligned}0 = \cosh(d_1 s)\cosh d_3 + \frac{1}{s}\Big[&d_2\cosh d_3\sinh(d_1 s) + a_1\cosh(d_1 s)\cosh d_3\\ &+ b_1\sinh(d_1 s)\sinh d_3 + c_1\Big] + O(s^{-2}).\end{aligned} \tag{27}$$

Define:

$$f(s) := \cosh d_1 s, \tag{28}$$

$$\begin{aligned}g(s) := \frac{1}{s\cosh d_3}&[d_2\cosh d_3\sinh(d_1 s) + a_1\cosh(d_1 s)\cosh d_3\\ &+ b_1\sinh(d_1 s)\sinh d_3 + c_1] + O(s^{-2})\end{aligned} \tag{29}$$

so that Eqs. (10) and (27) are equivalent to $f + g = 0$. In order to apply Rouché's theorem to $f + g$ we will need an upper bound for g and a lower bound for f on appropriately chosen contours in the complex plane.

Definition 3.3. The arguments that follow will make use of the points $t_n\mathrm{i} \in \mathbb{C}$ which are given by:

$$t_n\mathrm{i} := \frac{(\pi n + \frac{\pi}{2})\,\mathrm{i}}{\Delta}\sqrt{\frac{EI}{I_\rho}} = \frac{(\pi n + \frac{\pi}{2})\,\mathrm{i}}{d_1}, \quad n \in \mathbb{Z}. \tag{30}$$

By construction the points $t_n\mathrm{i}$ are the zeros of f.

Our next task is to bound g from above.

Lemma 3.4. *There is a positive constant C_1 such that for complex z with $\operatorname{Re} z \leq 1$ and $|z|$ sufficiently large the following bound holds:*

$$|g(z)| \leq \frac{C_1}{|z|}.$$

Moreover, there is another positive constant C_2 such that for all complex δ with $|\delta| \leq 1$ and sufficiently large positive integers, n, we have

$$|g(t_n\mathrm{i} + \delta)| \leq \frac{C_2}{n}. \tag{31}$$

Proof. The first bound follows easily from the definition of g, see Eq. (29), the triangle inequality and Remark 3.2. The second inequality follows quickly from the first. □

Lemma 3.5. *For sufficiently large positive integers, n, and for all $\delta_n \in \mathbb{C}$ with $|\delta_n| = \frac{2C_2}{d_1\sqrt{n}}$ the following bound holds*

$$|f(t_n\mathrm{i} + \delta_n)| \geq \frac{C_2}{\sqrt{n}}. \tag{32}$$

Proof. For $\delta \in \mathbb{C}$ the Taylor expansion of f about $t_n\mathrm{i}$ is

$$\begin{aligned}f(t_n\mathrm{i} + \delta) &= \underbrace{f(t_n\mathrm{i})}_{=0} + \delta f'(t_n\mathrm{i}) + \underbrace{\delta^2 f''(t_n\mathrm{i})}_{=0} + \sum_{j=3}^{\infty}\frac{\delta^j f^{(j)}(t_n\mathrm{i})}{j!}\\ &= \delta d_1\mathrm{i}(-1)^n + \delta^3\sum_{j=0}^{\infty}\frac{\delta^j f^{(j+3)}(t_n\mathrm{i})}{(j+3)!},\end{aligned}$$

$$\begin{aligned}\text{so that } |f(t_n\mathrm{i} + \delta)| &\geq d_1|\delta| - |\delta|^3.\underbrace{\left|\sum_{j=0}^{\infty}\frac{\delta^j f^{(j+3)}(t_n\mathrm{i})}{(j+3)!}\right|}_{\leq D,\text{constant}}\\ &\geq |\delta|(d_1 - D|\delta|^2).\end{aligned}$$

If $\delta_n \in \mathbb{C}$ with $|\delta_n| = \frac{2C_2}{d_1\sqrt{n}}$ for n sufficiently large so that $1 - \frac{4C_2^2 D}{d_1^3 n} \geq \frac{1}{2}$ then

$$|f(t_n\mathrm{i} + \delta_n)| \geq \frac{C_2}{\sqrt{n}},$$

as required. □

Corollary 3.6. *For sufficiently large positive integers, n, the functions f and $f+g$ have precisely one zero, $t_n\mathrm{i}$ and s_n respectively, in the discs centred at $t_n\mathrm{i}$ with radius $\frac{2C_2}{d_1\sqrt{n}}$. By construction $\operatorname{Re} s_n \to 0$ as $n \to \infty$.*

Proof. Take $\delta_n \in \mathbb{C}$ with $|\delta_n| = \frac{2C_2}{d_1\sqrt{n}}$. Choosing n sufficiently large so that Lemmas 3.4 and 3.5 hold we compare Eqs. (31) and (32) which gives

$$|f(t_n\mathrm{i} + \delta_n)| \geq \frac{C_2}{\sqrt{n}} > \frac{C_2}{n} \geq |g(t_n\mathrm{i} + \delta_n)|. \tag{33}$$

Invoking Rouché's theorem we deduce the corollary. □

The statement of Corollary 3.6 is the first part of Theorem 2.1. We now prove there are infinitely many zeros s_n with positive real part.

For $n \in \mathbb{N}$, write $s_n = t_n\mathrm{i} + \varepsilon_n = \frac{(n\pi + \frac{\pi}{2})\mathrm{i}}{d_1} + \varepsilon_n$. By Corollary 3.6, we know $\varepsilon_n \to 0$ as $n \to \infty$.

To make the following arguments slightly clearer, we consider the identity $s_n(f+g)(s_n) \equiv 0$. Taking the Taylor expansion of f at s_n about $t_n\mathrm{i}$ gives

$$\begin{aligned}
0 &= (t_n\mathrm{i} + \varepsilon_n) \\
&\quad \times \left[\underbrace{f(t_n\mathrm{i})}_{=0} + \varepsilon_n f'(t_n\mathrm{i}) + \frac{\varepsilon_n^2}{2}\underbrace{f''(t_n\mathrm{i})}_{=0} + O(\varepsilon_n^3)\right] + s_n g(s_n) \\
&= d_1\mathrm{i}^2(-1)^n t_n\varepsilon_n \cosh d_3 + t_n\mathrm{i}O(\varepsilon_n^3) + O(\varepsilon_n^2) \\
&\quad + d_2\cosh d_3 \sinh(d_1 s_n) + a_1\cosh(d_1 s_n)\cosh d_3 \\
&\quad + b_1\sinh(d_1 s_n)\sinh d_3 + c_1 + O(s_n^{-2}).
\end{aligned} \tag{34}$$

We note that using the Maclaurin series (21) and (22) yields

$$\cosh(d_1 s_n) = \mathrm{i}(-1)^n \sinh(d_1\varepsilon_n) = \mathrm{i}(-1)^n d_1\varepsilon_n + O(\varepsilon_n^3),$$
$$\sinh(d_1 s_n) = \mathrm{i}(-1)^n \cosh(d_1\varepsilon_n) = \mathrm{i}(-1)^n + O(\varepsilon_n^2).$$

Thus Eq. (34) becomes

$$\begin{aligned}
0 &= -d_1(-1)^n t_n\varepsilon_n\cosh d_3 + d_2\cosh d_3\mathrm{i}(-1)^n \\
&\quad + a_1 d_1\cosh d_3\mathrm{i}(-1)^n\varepsilon_n + b_1\sinh d_3\mathrm{i}(-1)^n + c_1 \\
&\quad + O(s_n^{-2}) + t_n\mathrm{i}O(\varepsilon_n^3) + O(\varepsilon_n^2).
\end{aligned} \tag{35}$$

We would like to split equation (35) into two parts so that we can find an expression for $\operatorname{Re}\varepsilon_n$ and ultimately apply Rouché's theorem again. As such write (35) as

$$0 = \psi_{1,n}(\varepsilon_n) + \psi_{2,n}(\varepsilon_n)$$

where

$$\begin{aligned}
\psi_{1,n}(z) &= -d_1(-1)^n t_n z\cosh d_3 + d_2\mathrm{i}(-1)^n\cosh d_3 \\
&\quad + b_1\sinh d_3\mathrm{i}(-1)^n + c_1,
\end{aligned} \tag{36}$$

$$\psi_{2,n}(z) = (t_n\mathrm{i} + z)(f+g)(t_n\mathrm{i} + z) - \psi_{1,n}(z). \tag{37}$$

It follows immediately that $\psi_{1,n}$ has zeros $\tilde{\varepsilon}_n$ with

$$\operatorname{Re}\tilde{\varepsilon}_n = \frac{c_1(-1)^n}{(n\pi + \frac{\pi}{2})\cosh d_3}, \quad n \in \mathbb{N}.$$

Moreover, by (36) the following bound for $|\tilde{\varepsilon}_n|$ holds

$$\begin{aligned}
|\tilde{\varepsilon}_n| &\leq \frac{|c_1| + |d_2|\cosh d_3 + |b_1|\sinh d_3}{d_1 t_n\cosh d_3} \\
&\leq \frac{D}{n}, \quad D \text{ constant.}
\end{aligned} \tag{38}$$

We deduce that $\operatorname{Re}\tilde{\varepsilon}_n$ takes both positive and negative sign for infinitely many n, so long as c_1 is not zero. By considering the original characteristic equation (10), we see that provided $k_1+k_2 > 0$, c_1 is always non-zero.

Take n sufficiently large (so that Corollary 3.6 holds) and such that $\operatorname{Re}\tilde{\varepsilon}_n$ is positive. Let $\nu_n := \frac{\operatorname{Re}\tilde{\varepsilon}_n}{2}e^{\mathrm{i}\theta}$ for $\theta \in [0, 2\pi)$. Then

$$\begin{aligned}
|\psi_{1,n}(\tilde{\varepsilon}_n + \nu_n)| &= d_1|\nu_n t_n|\cosh d_3 \\
&= \frac{c_1}{2} > 0, \quad \text{independently of } n \text{ and } \theta.
\end{aligned} \tag{39}$$

Eqs. (37) and (38) yield another constant D' such that

$$|\psi_{2,n}(\tilde{\varepsilon}_n + \nu_n)| \leq \frac{D'}{n}, \tag{40}$$

whence $|\psi_{1,n}(\tilde{\varepsilon}_n + \nu_n)| > |\psi_{2,n}(\tilde{\varepsilon}_n + \nu_n)|$,
for n sufficiently large.

Since θ was arbitrary we can invoke Rouché's theorem to conclude that the functions $\psi_{1,n}$ and $\psi_{1,n} + \psi_{2,n}$ both have one zero in the discs $\{z \in \mathbb{C} : |z - \tilde{\varepsilon}_n| \leq \frac{\operatorname{Re}\tilde{\varepsilon}_n}{2}\}$, $\tilde{\varepsilon}_n$ and ε_n respectively. Further, by construction $\operatorname{Re} s_n = \operatorname{Re}\varepsilon_n$ and thus $\operatorname{Re} s_n \geq \frac{\operatorname{Re}\tilde{\varepsilon}_n}{2} > 0$, which concludes the proof of Theorem 2.1.

4. Proofs for the Timoshenko beam

We now turn our attention to the Timoshenko beam. Our starting point is the equation $\det Q = 0$, where Q is given in (15). A short calculation gives

$$\begin{aligned}
0 = -\det Q &= -\begin{vmatrix} 1 & 1 & 1 & 1 \\ \varepsilon_1 & \varepsilon_2 & -\varepsilon_1 & -\varepsilon_2 \\ \eta_1 e^{\lambda_1\Delta} & \eta_2 e^{\lambda_2\Delta} & -\eta_1 e^{-\lambda_1\Delta} & -\eta_2 e^{-\lambda_2\Delta} \\ \chi_1 e^{\lambda_1\Delta} & \chi_2 e^{\lambda_2\Delta} & \chi_1 e^{-\lambda_1\Delta} & \chi_2 e^{-\lambda_2\Delta} \end{vmatrix} \\
&= (\varepsilon_1\chi_1\eta_2 + \varepsilon_2\chi_2\eta_1)\cosh(\Delta\lambda_1)\cosh(\Delta\lambda_2) \\
&\quad - (\varepsilon_2\chi_1\eta_2 + \varepsilon_1\chi_2\eta_1)\sinh(\Delta\lambda_1)\sinh(\Delta\lambda_2) \\
&\quad - (\varepsilon_2\chi_1\eta_1 + \varepsilon_1\chi_2\eta_2),
\end{aligned}$$

where ε_i, η_i and χ_i are stated in (16). Expanding these terms is a laborious but elementary process which uses the relations

$$\lambda_1^2 + \lambda_2^2 = \left(\frac{I_\rho}{EI} + \frac{\rho}{K}\right)s^2 \quad \text{and} \quad \lambda_1^2\lambda_2^2 = \frac{\rho}{EI}s^2 + \frac{\rho I_\rho}{EIK}s^4.$$

After multiplying through by $\frac{(\lambda_1\lambda_2)^2}{\rho s^3}$ we eventually infer (17) with

$$R(s,\lambda_1,\lambda_2) = \lambda_1\lambda_2\left\{\frac{\theta^2}{K}s^2 + \theta\varphi s + \frac{2\rho I_\rho}{EI}\left(k_1k_2 - \frac{1}{KI_\rho}\right)\right\},$$

$$\begin{aligned}
P(s,\lambda_1,\lambda_2) &= -\left\{\frac{\rho I_\rho\theta^2}{K}k_1k_2s^4 + \frac{\rho\theta\varphi}{K}s^3 + \left[\frac{\rho}{EIK}\left(\frac{I_\rho}{EI} + \frac{\rho}{K}\right)\right.\right. \\
&\quad \left.\left. + \frac{I_\rho\rho}{EI}\left(\frac{I_\rho}{EI} - \frac{3\rho}{K}\right)k_1k_2\right]s^2 + 2\frac{\rho\varphi}{EI}s\right\}
\end{aligned}$$

$$T(s,\lambda_1,\lambda_2) = -\lambda_1\lambda_2\left\{\theta\varphi s + 2\frac{\rho I_\rho}{EI}\left(k_1k_2 + \frac{1}{KI_\rho}\right)\right\},$$

where

$$\theta := \frac{I_\rho}{EI} - \frac{\rho}{K}, \quad \text{and} \quad \varphi := \frac{I_\rho}{EI}k_1 + \frac{\rho}{K}k_2. \tag{41}$$

We consider the following equation

$$\begin{aligned}0 &= \lambda_1\lambda_2[a_1s^2+a_2s+a_3]\cosh(\Delta\lambda_1)\cosh(\Delta\lambda_2)\\&\quad+[b_1s^4+b_2s^3+b_3s^2+b_4s]\sinh(\Delta\lambda_1)\sinh(\Delta\lambda_2)\\&\quad+\lambda_1\lambda_2[c_1s+c_2],\end{aligned}\tag{42}$$

where a_i, b_i and c_i are constants. We comment that by choosing the constants a_i, b_i, c_i appropriately, we recover from (42) the characteristic equation (17).

For the time being we assume $\frac{I_\rho}{EI} > \frac{\rho}{K}$ and so $\theta > 0$, though the arguments that follow can be altered if $\theta < 0$. The arguments change if $\theta = 0$, which will be considered at the very end. We need the Taylor expansions

$$\begin{aligned}\lambda_1(s) &= \sqrt{\frac{I_\rho}{EI}}s-\frac{\rho}{2\left(I_\rho-\frac{EI\rho}{K}\right)s}\sqrt{\frac{EI}{I_\rho}}+O(s^{-3})\\&=:\ d_1s+\frac{d_2}{s}+O(s^{-3})\end{aligned}$$

and
$$\begin{aligned}\lambda_2(s) &= \sqrt{\frac{\rho}{K}}s+\frac{\rho}{2\left(I_\rho-\frac{EI\rho}{K}\right)s}\sqrt{\frac{K}{\rho}}+O(s^{-3})\\&=:\ e_1s+\frac{e_2}{s}+O(s^{-3}).\end{aligned}$$

Note that by assumption $d_1 > e_1$. It follows that

$$\begin{aligned}\lambda_1\lambda_2 &= \sqrt{\frac{\rho I\rho}{EIK}}s^2+\sqrt{\frac{\rho K}{4I_\rho EI}}+O(s^{-2})\\&=:\ \alpha_1s^2+\alpha_2+O(s^{-2}).\end{aligned}\tag{43}$$

Substituting (43) into (42), expanding the hyperbolic terms, and dividing through by $\alpha_1a_1s^4$ yields

$$\begin{aligned}0 &= \cosh(\Delta d_1s)\cosh(\Delta e_1s)-L\sinh(\Delta d_1s)\sinh(\Delta e_1s)\\&\quad+\frac{\tilde{a}_2}{s}\cosh(\Delta d_1s)\cosh(\Delta e_1s)+\frac{\tilde{b}_2}{s}\sinh(\Delta d_1s)\sinh(\Delta e_1s)\\&\quad+\frac{\tilde{c}_1}{s}+\frac{\Delta(d_2-Le_2)}{s}\cosh(\Delta e_1s)\sinh(\Delta d_1s)\\&\quad+\frac{\Delta(e_2-Ld_2)}{s}\cosh(\Delta d_1s)\sinh(\Delta e_1s)+O(s^{-2}).\end{aligned}\tag{44}$$

The constants $\tilde{a}_2, \tilde{b}_2, \tilde{c}_2$ and L are important. They are $\frac{a_2}{a_1}, \frac{b_2}{\alpha_1a_1}, \frac{c_1}{a_1}$ respectively and

$$L := \sqrt{EIK\rho I_\rho}k_1k_2 \geq 0.\tag{45}$$

We set

$$f(s) := \cosh(\Delta d_1s)\cosh(\Delta e_1s)-L\sinh(\Delta d_1s)\sinh(\Delta e_1s),\tag{46}$$

$$\begin{aligned}g(s) &:= \frac{\tilde{a}_2}{s}\cosh(\Delta d_1s)\cosh(\Delta e_1s)+\frac{\tilde{b}_2}{s}\sinh(\Delta d_1s)\sinh(\Delta e_1s)\\&\quad+\frac{\tilde{c}_1}{s}+\frac{\Delta(d_2-Le_2)}{s}\cosh(\Delta e_1s)\sinh(\Delta d_1s)\\&\quad+\frac{\Delta(e_2-Ld_2)}{s}\cosh(\Delta d_1s)\sinh(\Delta e_1s)+O(s^{-2}),\end{aligned}\tag{47}$$

so that (44) can be written $f(s)+g(s)=0$.

We first prove that the zeros of $f+g$ converge to the imaginary axis. For this we will need the following bound on g.

Lemma 4.1. *There is a positive constant C_1 such that for complex s with sufficiently large modulus and bounded real part*

$$|g(s)| \leq \frac{C_1}{|s|}.$$

In particular there is another positive constant C_2 such that for all complex δ with small modulus we have

$$|g(t_n\mathrm{i}+\delta)| \leq \frac{C_2}{n}.\tag{48}$$

Proof. The arguments are identical to that for the Rayleigh beam, see Lemma 3.5. □

We now describe the zeros of the function f, defined by (46). The constant L defined by (45) plays a crucial role.

Lemma 4.2. *If $L = 0$ then f has zeros*

$$t_{n,0}\mathrm{i} := \frac{\left(n\pi+\frac{\pi}{2}\right)}{\Delta d_1}\mathrm{i},\quad n\in\mathbb{Z}.$$

If $L = 1$ then f has zeros

$$t_{n,1}\mathrm{i} := \frac{\left(n\pi+\frac{\pi}{2}\right)}{\Delta(d_1-e_1)}\mathrm{i},\quad n\in\mathbb{Z}.$$

Otherwise for every integer n the function f has at least one zero, denoted $t_{n,L}\mathrm{i}$, on the imaginary axis with modulus in the interval $\left[\frac{n\pi}{\Delta(d_1-e_1)}, \frac{(n+1)\pi}{\Delta(d_1-e_1)}\right]$. Further if $f'(t\mathrm{i})\in\mathbb{R}$, if $f'(t_{2n,L}\mathrm{i})\leq 0$ and if $f'(t_{2n+1,L}\mathrm{i})\geq 0$.

Proof. The first two parts are trivial, noting for example that when $L = 0$

$$f(s) = \cosh(\Delta d_1s)\cosh(\Delta e_1s).$$

For the last part let $s = \mathrm{i}t$ for real t. Then

$$\begin{aligned}f(s) &= \cos(\Delta d_1t)\cos(\Delta e_1t)+L\sin(\Delta d_1t)\sin(\Delta e_1t)\\&= \frac{(1+L)}{2}\cos\Delta(d_1-e_1)t+\frac{(1-L)}{2}\cos\Delta(d_1+e_1)t\\&=:\ f_{\mathbb{R}}(t).\end{aligned}$$

The function $f_{\mathbb{R}}$ is a real valued, smooth function. Since $L > 0$ we have $\frac{1+L}{2} > \frac{1-L}{2}$ and so by the intermediate value theorem $f_{\mathbb{R}}$ has a zero in every interval $\left[\frac{n\pi}{\Delta(d_1-e_1)}, \frac{(n+1)\pi}{\Delta(d_1-e_1)}\right]$, for $n \in \mathbb{N}$. Because the function $f_{\mathbb{R}}$ decreases (increases) between $\frac{2n\pi}{\Delta(d_1-e_1)}$ and $\frac{(2n+1)\pi}{\Delta(d_1-e_1)}$ $\left(\frac{(2n+1)\pi}{\Delta(d_1-e_1)}\text{ and }\frac{2(n+1)\pi}{\Delta(d_1-e_1)}\right)$, though not necessarily monotonically, for every integer n we conclude there must be a zero with $f'_{\mathbb{R}}(t_{2n,L}) \leq 0$ ($f'_{\mathbb{R}}(t_{2n+1,L}) \geq 0$). Finally by the chain rule $f'_{\mathbb{R}}(t) = \mathrm{i}f'(t\mathrm{i})$. □

We now seek a lower bound for f which will require a subsequence when $L = 0$.

Lemma 4.3. *There is an infinite subsequence of zeros $\left(t_{n_j,0}\mathrm{i}\right)_{j\in\mathbb{N}}$ with the following two properties:*

$$\forall j\in\mathbb{N} : |\cos\left(e_1\Delta t_{n_j,0}\right)| \geq B_0 > 0,\quad \textit{independently of } j.\tag{49}$$

There are infinitely many j such that $n_{j+1}-n_j = 1$ and for these j

$$\begin{aligned}\exists k\in\mathbb{N} : \left(k+\frac{1}{2}\right)\pi &< e_1\Delta t_{n_j,0} < e_1\Delta t_{n_{j+1},0}\\&< \left(k+1+\frac{1}{2}\right)\pi.\end{aligned}\tag{50}$$

Proof. Recall first that $t_{n,0} = \frac{\left(n\pi+\frac{\pi}{2}\right)}{d_1\Delta}$ and so successive terms $e_1\Delta t_{n+1,0}$ and $e_1\Delta t_{n,0}$ are separated by $\frac{e_1\pi}{d_1} < \pi$. The lower bound holds because $\cos x$ is zero if and only if $x = m\pi+\frac{\pi}{2}$ for integer m. As the iterates $e_1\Delta t_{n,0} = \frac{e_1}{d_1}\left(n\pi+\frac{\pi}{2}\right)$ are either periodic mod π

or dense in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$ mod π we can choose a subsequence avoiding $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ (both mod π) by some finite distance, hence the bound. To prove the second property we assume first that the iterates $\left(e_1\Delta t_{n,0}\right)$ are dense in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$ mod π. Then there is some integer n with $-\frac{\pi}{2} < e_1\Delta t_{n,0} < \left(\frac{\pi}{2} - \frac{e_1\pi}{d_1}\right)$. Given such an n, $e_1\Delta t_{n+1,0}$ satisfies $\frac{-\pi}{2} < e_1\Delta t_{n,0} < e_1\Delta t_{n+1,0} < \frac{\pi}{2}$ mod π. The case when the iterates are periodic is similar. $\square$

Remark 4.4. We use the notation $t_n\mathrm{i}$ to denote a zero of f when the value of L is unimportant, otherwise we use the double subscript $t_{n,L}$. For reasons apparent below, if $L = 0$ we will need to restrict ourselves to the subsequence of zeros $t_{n_j,0}\mathrm{i}$ defined in Lemma 4.3. For ease of presentation we drop the subsequence notation from now on.

Lemma 4.5. *For integers n with sufficiently large modulus and all complex δ_n with $|\delta_n| = \frac{2C_2}{B_1\sqrt{n}}$ the bound*

$$|f(t_n\mathrm{i} + \delta_n)| \geq \frac{C_2}{\sqrt{n}} \tag{51}$$

holds, where B_1 is a positive constant given below.

Proof. As in the proof of Lemma 3.5, the Taylor expansion of f about $t_n\mathrm{i}$ yields

$$|f(t_n\mathrm{i} + \delta)| \geq |f'(t_n\mathrm{i})|\,|\delta| - |\delta|^2 \left|\sum_{k=0}^{\infty} \frac{f^{(k+2)}(t_n\mathrm{i})\delta^k}{(k+2)!}\right|. \tag{52}$$

We consider $L = 0$ first. We have

$$f'(t_{n,0}\mathrm{i}) = \Delta d_1\mathrm{i}(-1)^n \cosh\left(e_1\Delta t_{n,0}\mathrm{i}\right) \Rightarrow$$
$$|f'(t_{n,0}\mathrm{i})| = \Delta d_1|\cos(e_1\Delta t_{n,0})|.$$

The subsequence $(t_{n,0})$ has been chosen in such a way that these terms are bounded from below, see Lemma 4.3. Thus there is a positive constant B_1 such that

$$|f'(t_{n,0})| \geq B_1. \tag{53}$$

Second, when $L = 1$:

$$f'(t_{n,1}\mathrm{i}) = \Delta(d_1 - e_1)\sinh(\Delta d_1 t_{n,1}\mathrm{i}) = \Delta(d_1 - e_1)(-1)^n\mathrm{i}$$
$$\therefore |f'(t_{n,1}\mathrm{i})| = \Delta(d_1 - e_1) =: B_1 > 0. \tag{54}$$

Finally, when $L \notin \{0, 1\}$, set $r_1 := d_1 - e_1$, $r_2 := d_1 + e_1$, $\theta_1 = \frac{1+L}{2}$, $\theta_2 = \frac{1-L}{2}$. Then $f_{\mathbb{R}}$ as defined in the proof of Lemma 4.2 can be written

$$f_{\mathbb{R}}(t) = \theta_1\cos(\Delta r_1 t) + \theta_2\cos(\Delta r_2 t),$$

which when we differentiate yields

$$f_{\mathbb{R}}'(t_{n,L}) = -\Delta(\theta_1 r_1\sin r_1 t_{n,L} + \theta_2 r_2\sin r_2 t_{n,L}).$$

Now observe that since $f_{\mathbb{R}}(t_{n,L}) = 0$

$$|f'(t_{n,L}\mathrm{i})|^2 = |f_{\mathbb{R}}'(t_{n,L})|^2 = \Delta^2(r_1^2 + r_2^2)|f_{\mathbb{R}}(t_{n,L})|^2 + |f_{\mathbb{R}}'(t_{n,L})|^2.$$

Expanding and collecting gives

$$\begin{aligned} &= \Delta^2\Bigg[r_1^2\theta_1^2 + r_2^2\theta_2^2 + r_2^2\theta_1^2\cos^2 r_1 t_{n,L} + r_1^2\theta_2^2\cos^2 r_2 t_{n,L} \\ &\quad + 2\theta_1\theta_2\underbrace{(r_1^2 + r_2^2)}_{\geq r_1 r_2}\cos r_1 t_{n,L}\cos r_2 t_{n,L} \\ &\quad + 2\theta_1\theta_2 r_1 r_2\underbrace{\sin r_1 t_{n,L}\sin r_2 t_{n,L}}_{\geq -1}\Bigg] \\ &\geq \Delta^2(r_1\theta_1 - r_2\theta_2)^2 + \Delta^2(r_2\theta_1\cos r_1 t_{n,L} + r_1\theta_2\cos r_2 t_{n,L})^2 \\ &\geq \Delta^2(r_1\theta_1 - r_2\theta_2)^2 = \frac{\Delta^2\rho I_\rho}{2}\left(\frac{1}{KI_\rho} - k_1k_2\right)^2 \\ &:= B_1^2 > 0, \end{aligned} \tag{55}$$

where we have used the assumption $k_1k_2 \neq \frac{1}{KI_\rho}$. Note that (55) is the same bound as (53) and (54).

Moving on, it is easy to see that there is a positive constant B_2 such that

$$\left|\sum_{k=0}^{\infty}\frac{f^{(k+2)}(t_n\mathrm{i})\delta^k}{(k+2)!}\right| \leq B_2. \tag{56}$$

Inserting the bound (56) and the applicable bound from (53)–(55) (which depends on $L \geq 0$) into inequality (52) yields

$$|f(t_n\mathrm{i} + \delta)| \geq B_1|\delta| - B_2|\delta|^2. \tag{57}$$

Take complex δ_n with $|\delta_n| = \frac{2C_2}{B_1\sqrt{n}}$ and n large enough so that $1 - \frac{2B_2C_2}{B_1^2\sqrt{n}} \geq \frac{1}{2}$. By (57) it now follows that

$$\begin{aligned} |f(t_n\mathrm{i} + \delta_n)| &\geq |\delta_n|B_1\left(1 - \frac{B_2|\delta_n|}{B_1}\right) = \frac{2C_2}{\sqrt{n}}\left(1 - \frac{2B_2C_2}{B_1^2\sqrt{n}}\right) \\ &\geq \frac{C_2}{\sqrt{n}}. \end{aligned}$$ $\square$

Corollary 4.6. *The zeros $t_n\mathrm{i}$ of f are simple. Moreover, for $n \in \mathbb{N}$ and $L > 0$, $\mathrm{i}f'(t_{2n,L}\mathrm{i}) < 0$ and $\mathrm{i}f'(t_{2n+1,L}\mathrm{i}) > 0$.*

Proof. The bounds (53)–(55) show that $f'(t_n\mathrm{i}) \neq 0$. When $L = 1$:

$$\mathrm{i}f'(t_{n,1}\mathrm{i}) = -\Delta d_1(-1)^n\begin{cases} < 0 & n \text{ even} \\ > 0 & n \text{ odd}. \end{cases}$$

For $L > 0, L \neq 1$ combining $f'(t_n\mathrm{i}) \neq 0$ with the statement of Lemma 4.2 we obtain the desired strict inequalities. $\square$

Corollary 4.7. *For integers n with sufficiently large modulus, the functions f and $f + g$ have the same number of zeros, i.e. at least one, in the discs centred at $t_n\mathrm{i}$ with radius $\delta_n = \frac{2C_2}{B_1\sqrt{n}}$. We call one of these zeros s_n. By construction* $\operatorname{Re} s_n \to 0$ *as* $n \to \infty$.

Proof. This is an application of Rouché's theorem on inequalities (48) and (51). $\square$

We next prove there are infinitely many s_n with a positive real part. Writing $s_n =: t_n\mathrm{i} + \varepsilon_n$ a zero of $f + g$, we note that

$$\begin{aligned} 0 &= (t_n\mathrm{i} + \varepsilon_n)(f + g)(s_n) = (t_n\mathrm{i} + \varepsilon_n)(f + g)(t_n\mathrm{i} + \varepsilon_n) \\ &= (t_n\mathrm{i} + \varepsilon_n)\Bigg[\underbrace{f(t_n\mathrm{i})}_{=0} + f'(t_n\mathrm{i})\varepsilon_n + O(\varepsilon_n^2)\Bigg] \\ &\quad + (t_n\mathrm{i} + \varepsilon_n)g(t_n\mathrm{i} + \varepsilon_n). \end{aligned} \tag{58}$$

First consider the related problem, namely finding $\operatorname{Re}\tilde{\varepsilon}_{n,L}$ for the equation

$$\begin{aligned} -f'(t_n\mathrm{i})t_n\mathrm{i}\tilde{\varepsilon}_{n,L} &= \tilde{a}_2\cosh(\Delta d_1 t_n\mathrm{i})\cosh(\Delta e_1 t_n\mathrm{i}) \\ &\quad + \tilde{b}_2\sinh(\Delta d_1 t_n\mathrm{i})\sinh(\Delta e_1 t_n\mathrm{i}) + \tilde{c}_1 \\ &\quad + \Delta(e_2 - Ld_2)\cosh(\Delta d_1 t_n\mathrm{i})\sinh(\Delta e_1 t_n\mathrm{i}) \\ &\quad + \Delta(d_2 - Le_2)\cosh(\Delta e_1 t_n\mathrm{i})\sinh(\Delta d_1 t_n\mathrm{i}). \end{aligned} \tag{59}$$

Note the Eq. (59) contains all the asymptotically largest terms from Eq. (58). The L dependence of $\tilde{\varepsilon}_n$ has also been highlighted with a

$$\begin{aligned}\operatorname{Re}\tilde{\varepsilon}_{n,L} &= \frac{\tilde{a}_2\cos(\Delta d_1 t_{n,L})\cos(\Delta e_1 t_{n,L}) - \tilde{b}_2\sin(\Delta d_1 t_{n,L})\sin(\Delta e_1 t_{n,L}) + \tilde{c}_1}{-t_{n,L}\mathrm{i}f'(t_{n,L}\mathrm{i})},\\ &= \frac{-\varphi K}{\theta t_{n,L}\mathrm{i}f'(t_{n,L}\mathrm{i})}\left[\cos(\Delta d_1 t_{n,L})\cos(\Delta e_1 t_{n,L}) + \frac{e_1}{d_1}\sin(\Delta d_1 t_{n,L})\sin(\Delta e_1 t_{n,L}) - 1\right]\\ &=: \frac{-\varphi K}{\theta t_{n,L}\mathrm{i}f'(t_{n,L}\mathrm{i})}h(t_{n,L}) \end{aligned} \tag{60}$$

Box I.

subscript. Using the relations

$$\left.\begin{aligned}\cosh(x\mathrm{i}) &= \cos x\\ \sinh(x\mathrm{i}) &= \mathrm{i}\sin x\end{aligned}\right\} \quad \forall x\in\mathbb{R},$$

we obtain Eq. (60) given in Box I, which is well defined by Corollary 4.6 (and the term $\mathrm{i}f'(t_{n,L}\mathrm{i})$ is real).

Lemma 4.8. *For all $L \geq 0$ the terms of the sequences $(h(t_{n,L}))_{n\in\mathbb{N}}$ do not change sign and are not zero.*

Proof. Let $n \in \mathbb{N}$. We look at the three cases $L = 0$, $L = 1$ and $L \in (0,\infty)\setminus\{0,1\}$ separately. First

$$h(t_{n,0}) = \underbrace{\frac{e_1}{d_1}}_{<1}(-1)^n\sin\left(\frac{e_1}{d_1}\left(n\pi + \frac{\pi}{2}\right)\right) - 1 < 0.$$

Second, when $L = 1$ it follows from $f(t_{n,1}) = 0$ that

$$\sin(\Delta d_1 t_{n,1})\sin(\Delta e_1 t_{n,1}) = -\cos(\Delta d_1 t_{n,1})\cos(\Delta e_1 t_{n,1})$$

and so $h(t_{n,1}) = \left(1 - \dfrac{e_1}{d_1}\right)\cos(\Delta d_1 t_{n,1})\cos(\Delta e_1 t_{n,1}) - 1 < 0.$

Third, for general $L \notin \{0,1\}$ and $r_1 := d_1 - e_1$, $r_2 := d_1 + e_1$

$$h(t_{n,L}) = \frac{1+\frac{e_1}{d_1}}{2}\cos(\Delta r_1 t_{n,L}) + \frac{1-\frac{e_1}{d_1}}{2}\cos(\Delta r_2 t_{n,L}) - 1.$$

Observe that both of the coefficients of the cosines in the above expression are positive and less than one and recall $t_{n,L}$ satisfies

$$0 = (1+L)\cos(\Delta r_1 t_{n,L}) + (1-L)\cos(\Delta r_2 t_{n,L}).$$

Suppose first that $1-L$ is positive. If $\cos(\Delta r_1 t_{n,L})$ is positive then $\cos(\Delta r_2 t_{n,L})$ must be negative and so $h(t_{n,L})$ is negative. Conversely, if $\cos(\Delta r_1 t_{n,L})$ is negative then $\cos(\Delta r_2 t_{n,L})$ is positive and again $h(t_{n,L})$ is negative. A similar argument in the case when $1-L$ is negative proves $h(t_{n,L})$ is negative, and so we infer the result. □

Lemma 4.9. *The ratio $\operatorname{Re}\tilde{\varepsilon}_{n,L} = \frac{-\varphi K}{\theta t_{n,L}\mathrm{i}f'(t_{n,L}\mathrm{i})}h(t_{n,L})$ takes both signs infinitely often for all non-negative L.*

Proof. By Lemma 4.8 we know that the numerator $-\varphi K h(t_{n,L})$ is not zero and does not change sign. By assumption $\theta > 0$ and additionally, t_n is real and positive for positive integers n. So we need to consider the denominator $\mathrm{i}f'(t_{n,L}\mathrm{i})$. When $L = 0$:

$$\mathrm{i}f'(t_{n,0}\mathrm{i}) = -\Delta d_1(-1)^n\cos\left(e_1\Delta t_{n,0}\right). \tag{61}$$

By our choice of original subsequence, namely property (50), we know there are pairs of consecutive integers, n, where cos does not change sign. Examining (61) we see that $\mathrm{i}f'(t_{n+1,0}\mathrm{i})$ and $\mathrm{i}f'(t_{n,0}\mathrm{i})$ are different signs, hence $\operatorname{Re}\tilde{\varepsilon}_{n+1,0}$ and $\operatorname{Re}\tilde{\varepsilon}_{n,0}$ are different signs. Again, by construction of our subsequence, this process repeats infinitely often. For $L > 0$ we invoke the result of Corollary 4.6, which completes the proof. □

We are now ready to prove that $\operatorname{Re}\tilde{\varepsilon}_{n,L}$ is the asymptotically largest part of $\operatorname{Re}s_n$. Define:

$$\begin{aligned}\psi_{1,n,L}(z) &= f'(t_n\mathrm{i})t_n\mathrm{i}\tilde{\varepsilon}_{n,L} + \Delta(e_2 - Ld_2)\\ &\quad\times\cosh(\Delta d_1 t_n\mathrm{i})\sinh(\Delta e_1 t_n\mathrm{i})\\ &\quad+\tilde{b}_2\sinh(\Delta d_1 t_n\mathrm{i})\sinh(\Delta e_1 t_n\mathrm{i})\\ &\quad+\Delta(d_2 - Le_2)\cosh(\Delta e_1 t_n\mathrm{i})\sinh(\Delta d_1 t_n\mathrm{i})\\ &\quad+\tilde{a}_2\cosh(\Delta d_1 t_n\mathrm{i})\cosh(\Delta e_1 t_n\mathrm{i}) + \tilde{c}_1\end{aligned} \tag{62}$$

which has zero $\tilde{\varepsilon}_n$, (see Eq. (59)). Similarly, define

$$\psi_{2,n,L}(z) = (t_n\mathrm{i} + z)(f+g)(t_n\mathrm{i} + z) - \psi_{1,n,L}(z). \tag{63}$$

For $n \in \mathbb{N}$ with sufficiently large modulus and such that $\operatorname{Re}\tilde{\varepsilon}_n > 0$, let $\nu_n := \frac{\operatorname{Re}\tilde{\varepsilon}_n}{2}\mathrm{e}^{\mathrm{i}\beta}$ for $\beta \in [0, 2\pi)$. Then as in the Rayleigh case there are constants D and D' such that

$$\left.\begin{aligned}|\psi_{1,n,L}(\tilde{\varepsilon}_n + \nu_n)| &= |f'(t_n\mathrm{i})t_n\nu_n| \geq D > 0,\\ |\psi_{2,n,L}(\tilde{\varepsilon}_n + \nu_n)| &\leq \frac{D'}{n},\end{aligned}\right\}$$

independently of n and β.

Hence

$$|\psi_{1,n,L}(\tilde{\varepsilon}_n + \nu_n)| > |\psi_{2,n,L}(\tilde{\varepsilon}_n + \nu_n)|, \quad \text{for } n \text{ sufficiently large.}$$

See inequalities (39) and (40) for the details (the arguments here are virtually identical). Since β was arbitrary we can invoke Rouché's theorem to conclude that the functions $\psi_{1,n,L}$ and $\psi_{1,n,L} + \psi_{2,n,L}$ both have at least one zero in the discs $\{z \in \mathbb{C} : |z - \tilde{\varepsilon}_n| \leq \frac{\operatorname{Re}\tilde{\varepsilon}_n}{2}\}$, $\tilde{\varepsilon}_n$ and ε_n respectively. Further, by construction $\operatorname{Re}s_n = \operatorname{Re}\varepsilon_n$ and thus $\operatorname{Re}s_n \geq \frac{\operatorname{Re}\tilde{\varepsilon}_n}{2} > 0$, which concludes the proof of Theorem 2.3.

Finally we consider the situation where $\frac{\rho}{K} = \frac{I_\rho}{EI}$, i.e. $\theta = 0$. From (14) it follows that

$$\begin{aligned}\lambda_1(s) &= \sqrt{\frac{I_\rho}{EI}}s + \frac{1}{2}\sqrt{\frac{\rho}{I_\rho}}\mathrm{i} + \frac{\rho}{8I_\rho s}\sqrt{\frac{EI}{I_\rho}} + O(s^{-2})\\ &=: d_1 s + d_2 + \frac{d_3}{s} + O(s^{-2}),\\ \lambda_2(s) &= \sqrt{\frac{I_\rho}{EI}}s - \frac{1}{2}\sqrt{\frac{\rho}{I_\rho}}\mathrm{i} + \frac{\rho}{8I_\rho s}\sqrt{\frac{EI}{I_\rho}} + O(s^{-2})\\ &=: d_1 s - d_2 + \frac{d_3}{s} + O(s^{-2}).\end{aligned}$$

Further the coefficients R, S and T from the beginning of Section 4 simplify considerably so that the characteristic equation (17) reduces to

$$0 = \cosh^2(\Delta d_1 s) + \sinh^2(\Delta d_1 s) + c_0 + O(s^{-1}), \tag{64}$$

where c_0 is given by

$$c_0 := -\frac{\left(k_1k_2 + \frac{1}{KI_\rho}\right)}{\left(k_1k_2 - \frac{1}{KI_\rho}\right)}. \tag{65}$$

Note that c_0 is well defined by the assumptions of Theorem 2.3. We define f and g respectively as

$$f(s) := \cosh^2(\Delta d_1 s) + \sinh^2(\Delta d_1 s) + c_0,$$
$$g(s) := O(s^{-1}).$$

We consider the cases $k_1k_2 > 0$ and $k_1k_2 = 0$ separately.

Case 1: $k_1k_2 > 0$.

From (65), c_0 has modulus greater than 1. As such, f has zeros $t_n := t_0 + \frac{2n\pi \mathrm{i}}{\Delta d_1}$ for integers n. Moreover $\operatorname{Re} t_0$ is non-zero, and since f is an even function, we can assume without loss of generality that $\operatorname{Re} t_0 > 0$.

Arguing as in the proofs of the earlier Lemmas 3.4 and 3.5 it follows that there is a positive constant C such that for sufficiently large positive integers, n, and complex δ_n with $|\delta_n| = \frac{C}{\sqrt{n}}$ the inequality

$$|f(t_n + \delta_n)| > |g(t_n + \delta_n)| \tag{66}$$

holds.

The immediate consequence of inequality (66) is that for integers n with sufficiently large modulus the functions f and $f+g$ have precisely one zero, denoted by t_n and s_n respectively, in the discs centred at t_n with radius $\delta_n = \frac{C}{\sqrt{n}}$. By construction $\operatorname{Re} s_n \to \operatorname{Re} t_0 > 0$ as $n \to \infty$.

Case 2: $k_1k_2 = 0$.

From (65), $c_0 = 1$ and so f becomes

$$f(s) = 2\cosh^2(\Delta d_1 s), \tag{67}$$

which has zeros $t_n\mathrm{i} = \frac{(n\pi + \frac{\pi}{2})\mathrm{i}}{\Delta d_1}$, $n \in \mathbb{Z}$. Now $f'(t_n\mathrm{i}) = 0$ and

$$f''(t_n\mathrm{i}) = 4d_1^2\Delta^2(\mathrm{i}(-1)^n)^2, \quad \therefore |f''(t_n\mathrm{i})| = 4d_1^2\Delta^2 > 0.$$

The Taylor expansion of f then is

$$f(t_n\mathrm{i} + \delta) = \underbrace{f(t_n\mathrm{i})}_{=0} + \underbrace{\delta f'(t_n\mathrm{i})}_{=0} + \delta^2 \frac{f''(t_n\mathrm{i})}{2} + O(\delta^3).$$

When we take complex δ_n with $|\delta_n| = \frac{C}{\sqrt[4]{n}}$ for some constant C, which may alter from line to line, we obtain

$$\begin{aligned} |f(t_n\mathrm{i} + \delta_n)| &\geq \frac{C}{\sqrt{n}} \\ &> \frac{C}{n} \geq |g(t_n\mathrm{i} + \delta_n)|, \end{aligned}$$

for sufficiently large positive integers n. We deduce that $f+g$ has at least one zero, s_n, in the circles centred at $t_n\mathrm{i}$ with radius $\frac{C}{\sqrt[4]{n}}$. Hence $s_n \to t_n\mathrm{i}$ as $n \to \infty$.

Arguing as before, we write $s_n = t_n\mathrm{i} + \varepsilon_n$ and by splitting $s_n(f+g)(s_n) = 0$ according to the order of its terms it follows that $\operatorname{Re} s_n = \operatorname{Re} \varepsilon_n \geq \frac{\operatorname{Re} \tilde{\varepsilon}_n}{2}$ where

$$\tilde{\varepsilon}_n^2 = \frac{\rho E I^2 (k_1 + k_2)\cos^2\left(\frac{\Delta}{2}\sqrt{\frac{\rho}{I_\rho}}\right)\mathrm{i}}{2I_\rho^2 \Delta^2 t_n}.$$

The above argument is very similar to that outlined in Eqs. (58), (59), (62) and (63). Hence there are zeros s_n of $f+g$ with a positive real part for every $n \in \mathbb{N}$. Observe that we have required our assumption from Theorem 2.3, namely that $\cos\left(\frac{\Delta}{2}\sqrt{\frac{\rho}{I_\rho}}\right) \neq 0$.

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