Cross-sections of chaotic attractors

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We present an efficient algorithm for constructing cross-sections of chaotic attractors. The technique is particularly useful for studying the structure and fractal dimension of higher dimensional attractors.

One of the central topics in nonlinear dynamical systems theory is the study of the structure and organization of invariant sets under the dynamics. In particular, the geometry of strange attractors [1] is of particular interest. For such studies, the visualization of the strange attractor is important for revealing structure as well as characterizing the attractor. This presents problems when higher dimensional attractors are encountered. For example, the projection of an attractor whose fractal dimension is greater than two to a plane yields a fuzzy blob. Questions such as whether the local structure of a typical higher dimensional strange attractor is the product of a continuum with a Cantor set [2] or is more complex than this cannot be answered by simply taking a projection of the attractor. In addition, numerical determination of the dimension of higher dimensional fractal sets by box-counting algorithms can require enormous memory storage and CPU time. If feasible, taking *cross-sections* of the attractor (i.e., intersections of the attractor with a surface) might offer a way of both elucidating the geometry of the attractor and of estimating its dimension.

In this regard, two procedures for taking a cross-section of a chaotic attractor were proposed by Lorenz [2], and the first of them was extended and further developed by Kostelich and Yorke [3]. This latter procedure is basically as follows. An orbit on the chaotic attractor is followed until it comes near the desired cross-section plane. Through a subsidiary calculation, a local approximation to the unstable manifold through that point is found. Then the intersection of the approximate unstable manifold and the desired cross-section plane is determined, thus projecting the orbit point onto the cross-section plane. Assuming the attractor is smooth in the unstable direction (or directions), this intersection approximates to a point in the cross-section of the attractor. Repeating this procedure many times as an orbit is followed, a cross-section picture of the attractor is built up.

In this note, we consider Lorenz's second procedure for taking numerical cross-section. Compared to the first procedure, this procedure can be easier to implement and yield faster computer computation. On the other hand, the method has certain limitations which will be discussed. Consider an N-dimensional invertible map, $x_{n+1}=F(x_n)$. Choose a compact volume V which contains the chaotic attractor. We shall find the cross-section of an m-dimensional hyperplane with the unstable manifolds of the invariant sets contained in V. This will typically include the attractor. By inverting the map, the attractor becomes a repellor. Consider a point x in V and examine its preimages $F^{-1}(x)$, $F^{-2}(x)$, ..., $F^{-n}(x)$. Let $T(x)$ denote the smallest value of n such that $F^{-n}(x)$ is not in V. We call $T(x)$ the inverse escape time from V. Under the inverse map, all points in the region V will finally escape except for those on the unstable manifolds of the invariant sets con-

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tained in V. This set, of course, includes the repellor of the inverse map originating from the chaotic attractor of the forward map. Points on the unstable manifolds of the invariant sets in V correspond to singular points of the inverse escape time function ($T(\boldsymbol{x})=\infty$). (We assume that the inverse map has no attractors in V. For example, the inverse of a strictly contractive map (e.g., the Hénon map) can have no attractors.) Thus, if we start initial conditions from a hyperplane and collect all the singular points of the inverse escape time function of the map, we find the intersection of the hyperplane with the unstable manifolds of the invariant sets in V, and this typically includes the cross-section of the attractor with the hyperplane. In practice, we not not determine the singular points but rather we determine a succession of nested sets containing the singular points. We do this by computing $\boldsymbol{x}$ values for which $T(\boldsymbol{x})\geqslant N$ for successively larger values of N. To obtain the intersection with the attractor, one should reject points which satisfy $T(\boldsymbol{x})\geqslant N$ but do not lie approximately on the attractor. In principle, this can be done by calculating the Lyapunov exponents (or other ergodic quantities) of $\boldsymbol{F}^{-1}$ for each $\boldsymbol{x}$ satisfying $T(\boldsymbol{x})\geqslant N$ along the orbit $\boldsymbol{x}$, $\boldsymbol{F}^{-1}(\boldsymbol{x})$, ..., $\boldsymbol{F}^{-(N-1)}(\boldsymbol{x})$. For large N, these exponents will approximate the negatives of the Lyapunov exponents of the forward map on the attractor, provided that $\boldsymbol{x}$ lies approximately on the attractor. If $\boldsymbol{x}$ does not lie approximately on the attractor, then the Lyapunov exponents for the inverse map starting from $\boldsymbol{x}$ will approximate those for another invariant set in V and will differ substantially from the exponents of the attractor. In this case the point $\boldsymbol{x}$ is rejected. It will not always be possible to apply this Lyapunov exponent test, because N must be sufficiently large to obtain reliable estimates of the Lyapunov exponents of the inverse map. Alternatively, one can omit the Lyapunov exponent test altogether. In this case, the set obtained may be larger than that for the attractor. Thus a calculation of the fractal dimension of this set yields an upper bound for the fractal dimension of the attractor. In our numerical examples, we have not applied the Lyapunov exponent test. Nonetheless, as shown below, for these examples, the method appears to yield very good approximations to the actual attractor, and the calculated dimensions agree with previous calculations which resolved only the attractor set.

The dimension of the intersection set in the cross-section plane is related to the dimension of the unstable manifold set by a result of Mattila [6]. If the Hausdorff dimension D of a bounded fractal set lying in an N-dimensional space is greater than $N-m$, then a random cut by an m-dimensional hyperplane intersects the set with positive probability; if it does intersect the fractal set, the dimension d of the intersection set is related to D by

$$D=d+(N-m) \tag{1}$$

with probability one. Hence, by generating the cross-section of the attractor and measuring the dimension of the cross-section set, we determine the dimension of the strange attractor.

To illustrate out algorithm, we first calculate one-dimensional cross-sections of the Hénon attractor. The Hénon attractor is generated by the following map,

$$x_{n+1}=a-x_n^2+by_n\,,\quad y_{n+1}=x_n\,. \tag{2}$$

At parameter values $a=1.4$, $b=0.3$, Hénon observed that there exists a chaotic attractor. Numerical box counting techniques for the calculation of the dimension of a strange attractor were first applied by Russell et al. [4], who obtained a result for the dimension of the Hénon attractor. A more accurate result was obtained by Grassberger who found that the capacity dimension is approximately 1.28 ± 0.01 [5]. However, from different least squares fits of the slope, the dimension takes values between 1.22 and 1.30.

Fig. 1 shows the Hénon attractor. It can be shown that the attractor is included in the square $[-2.0, 2.0]\times[-2.0, 2.0]$. This is the region V which we use for calculating the inverse escape time function. We take a horizontal one-dimensional cross-section through the point $x=0$, $y=0$ and calculate $T(\boldsymbol{x})$ at regularly spaced intervals along this line. This is shown in fig. 2a. We see there is a natural Cantor set level structure in the inverse escape time function. At level 0, there is one interval from which it requires at least one backward iterate to escape the square; at level 1, there are two intervals from which it requires at least two backward iterates to escape the square; etc. The intersection of all these intervals is the cross-section of the Hénon attractor. Fig. 2b

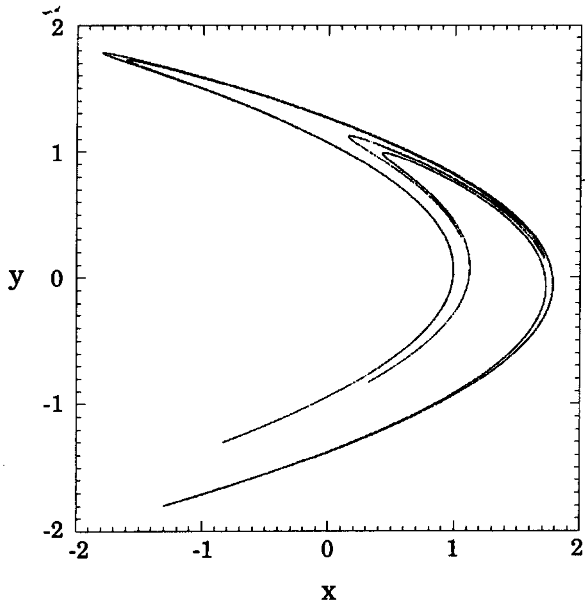

Fig. 1. The Hénon attractor.

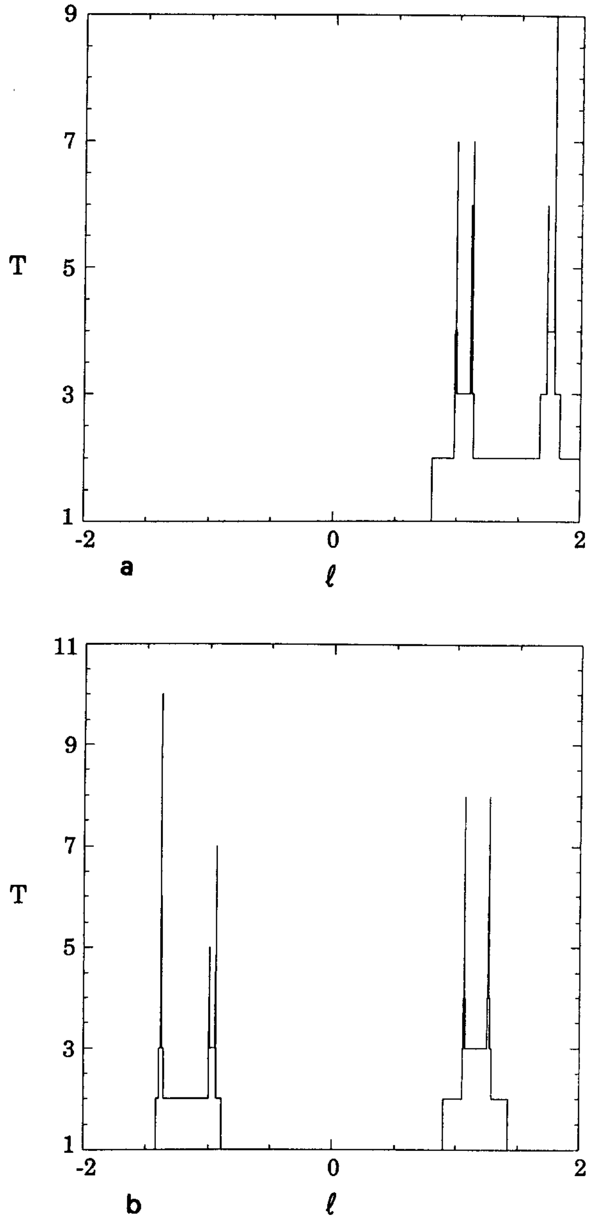

Fig. 2. Inverse escape time function for the Hénon map. (a) Horizontal cut through $x=0$, $y=0$. (b) Vertical cut through $x=0$, $y=0$.

shows the same function for the vertical cross-section through the same point $x=0$, $y=0$.

To get the fractal dimension of these cross-section sets, we use the following procedure. We denote the lengths of the intervals at level i by l_j^i. Then we form the Hausdorff sum

$$K^i(s)=\sum_j (l_j^i)^s\,, \tag{3}$$

where the sum is taken over all intervals at level i. When i tends to infinity, this sum is the Hausdorff s-dimensional measure [7]. Therefore, it is infinite when s is less than the Hausdorff dimension d of the fractal set, and is zero when s is greater than d. Hence, we expect that for large i, the sums $K^i(s)$ versus s for different levels will intersect with each other at approximately the same point $s=d$ given by the Hausdorff dimension of the one-dimensional fractal set [#1]. In fig. 3, we show results for the Hausdorff sums for different levels for a typical one-dimensional cut. The lines for this case have intersections in the range $d\approx 0.24$ to 0.30. Examining many different one-dimensional horizontal and vertical cuts, we estimate d to lie in the range 0.20 to 0.34. From formula (1), the dimension of the Hénon attractor is approximately $D\approx 1.20$–1.34. The whole calculation for a

[#1] The numerical application of the Hausdorff sum (3) to find the fractal dimension has been previously used to study chaotic scattering [8]. Results of Nusse and Yorke [9] guarantee that for hyperbolic horseshoes, an interval with successive nested increasing $T(x)$ contains a point where $T(x)=\infty$.

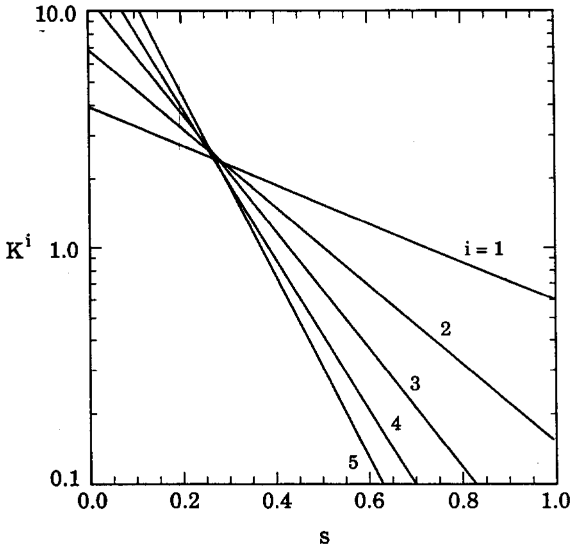

Fig. 3. The Hausdorff sum $K^i(s)$ as a function of s for different levels i for the one-dimensional vertical cut through $x=0.8$, $y=0.0$.

cut involved very little computer memory and took less than 5 seconds on the Cray XMP computer.

Our second example is the double rotor attractor generated by the following four-dimensional, volume-contracting map [10],

$$\begin{pmatrix} x_1^{n+1} \\ x_2^{n+1} \end{pmatrix} = M_1 \begin{pmatrix} y_1^n \\ y_2^2 \end{pmatrix} + \begin{pmatrix} x_1^n \\ x_2^n \end{pmatrix} \bmod 1 ,$$

$$\begin{pmatrix} y_1^{n+1} \\ y_2^{n+1} \end{pmatrix} = M_2 \begin{pmatrix} y_1^n \\ y_2^n \end{pmatrix} + \begin{pmatrix} (c_1/2\pi)\sin(2\pi x_1^{n+1}) \\ (c_2/2\pi)\sin(2\pi x_2^{n+1}) \end{pmatrix} . \qquad (4)$$

Here x_1, x_2 take values from the unit interval $[0, 1)$, and y_1 and y_2 take values from the real line. At parameter values given by

$$M_1 = \begin{pmatrix} -5.8 & -6.602 \\ -6.602 & -12.40 \end{pmatrix} ,$$

$$M_2 = \begin{pmatrix} 0.7496 & 0.1203 \\ 0.1203 & 0.8699 \end{pmatrix} ,$$

$$c_1 = 0.3536, \quad c_2 = 0.5 ,$$

Kostelich and Yorke [3] find that there is a chaotic attractor. Since the two x-directions of the double rotor map are compact, we choose for V the hypercube box given by $\max(|y_1|, |y_2|) \leqslant y_{\max}$. Starting from a uniform distribution of initial points in the cross-section plane, we collect those points from which after some chosen maximum number of iterates of the inverse map $n_{\max}$, the point remains in the hypercube region. Fig. 4 shows two two-dimensional cross-sections of the attractor using our algorithm ($y_{\max}=0.5$ and $n_{\max}=15$). The pictures in fig. 4 appear to be identical to those in ref. [3].

To find the fractal dimension of the chaotic at-

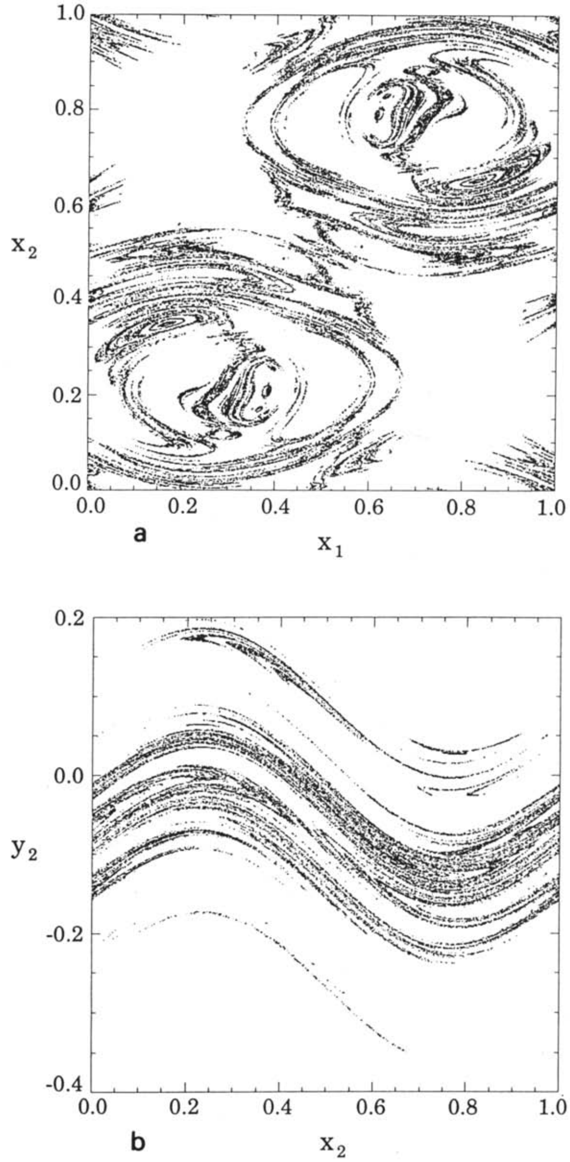

Fig. 4. Cross-sections of the double rotor attractor. (a) Cross-section at $y_1=0$, $y_2=0$. (b) Cross-section at $y_1=0$, $x_1=2.2/2\pi$.

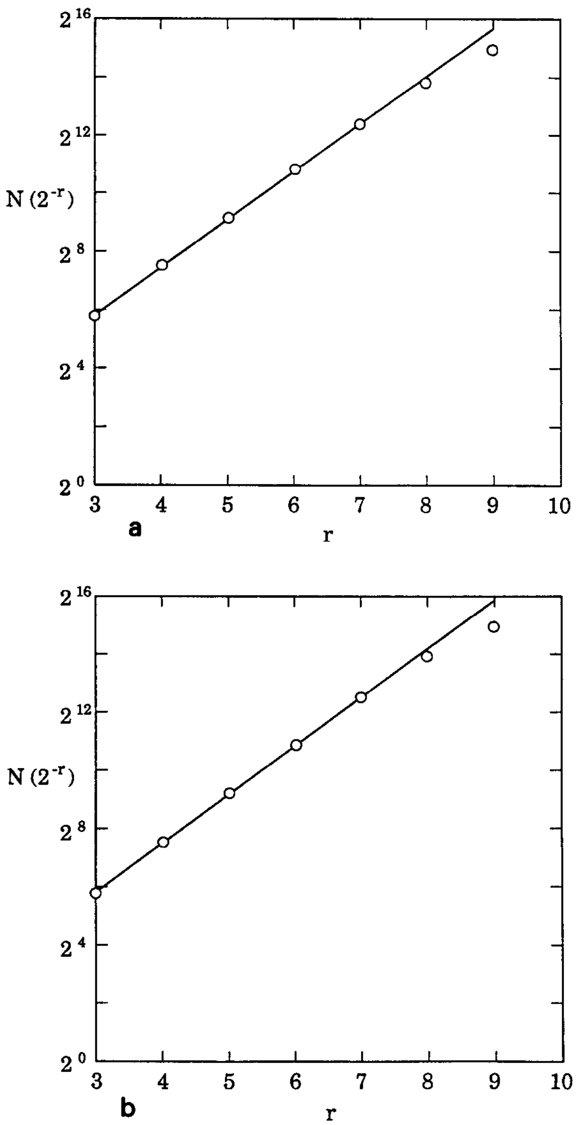

Fig. 5. (a) $N(\epsilon)$ as a function of ϵ for the cross-section set in fig. 4a. The least squares fit gives a capacity dimension $d=1.67\pm0.05$. (b) The same plot for fig. 4b. The least squares fit gives $d=1.63\pm0.05$.

tractor, we used a box counting algorithm. We cover the resulting cross-section set with squares from a grid of edge length ϵ. In the limit $\epsilon\to0$, the number of squares $N(\epsilon)$ needed for the covering scales as

$$N(\epsilon)\sim\epsilon^{-d}. \tag{5}$$

The exponent d is determined by a least squares fit of a straight line to a log–log plot of $N(\epsilon)$. In fig. 5, we calculate the capacity dimension d for the cross-section sets of figs. 4a and 4b. The two values of $D=d+2$ determined from least squares fitting are 3.67 and 3.63. According to the estimates of ref. [3], the information dimension lies in the range 3.61 to 3.68. Thus we find that the values of the capacity and information dimensions (the latter must be smaller) are apparently quite close to each other.

In conclusion, we have presented an efficient algorithm for calculating cross-sections of strange attractors. This method may be useful for the estimation of the fractal dimension of higher dimensional chaotic attractors.

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