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ORIGINAL RESEARCH

Approximate expressions of the bifurcating periodic solutions in a neuron model with delay-dependent parameters by perturbation approach

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Abstract This paper is interested in gaining insights of approximate expressions of the bifurcating periodic solutions in a neuron model. This model shares the property of involving delay-dependent parameters. The presence of such dependence requires the use of suitable criteria which usually makes the analytical work so harder. Most existing methods for studying the nonlinear dynamics fail when applied to such a class of delay models. Although Xu et al. (Phys Lett A 354:126–136, 2006) studied stability switches, Hopf bifurcation and chaos of the neuron model with delay-dependent parameters, the dynamics of this model are still largely undetermined. In this paper, a detailed analysis on approximation to the bifurcating periodic solutions is given by means of the perturbation approach. Moreover, some examples are provided for comparing approximations with numerical solutions of the bifurcating periodic solutions. It shows that the dynamics of the neuron model with delay-dependent parameters is quite different from that of systems with delay-independent parameters only.

Keywords Neuron model · Delay-dependent parameters · Approximation · Periodic solution · Perturbation approach

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Introduction

Since the seminal work for Hopfield neural networks in Hopfield (1982), the nonlinear dynamical behaviors (including stability, instability, periodic oscillatory and chaos) of neural networks without or with delay have received increasing interesting due to their promising potential applications in many fields such as signal processing, pattern recognition, optimization and associative memories. Some important results have been reported. See (Gopalsamy and He 1994; Bélair et al. 1996; Wei and Ruan 1999; Liao et al. 2001a; Guo and Huang 2004; Cao and Song 2006; Cao and Xiao 2007) and the references therein. It is well known that neural networks are complex and large-scale nonlinear dynamical systems, while the dynamics of the delayed neural networks are even rich and more complicated (Wu 2001). In order to obtain a deep and clear understanding of the dynamics of neural networks, many researchers have focused on the studying of simplified systems. One of usual ways is to investigate the delayed neural networks models with two, three or four neurons, see (Kaslik and Balint 2009; Ruan and Fillfil 2004; Zhu and Huang 2007; Yu and Cao 2006). It is expected that we can gain some light for our understanding about the large networks by discussing the dynamics of networks with small number of neurons.

In Gopalsamy and Leung (1997), considered the following single neuron model of integro-differential equation with dynamical threshold effect as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + af\left[x(t) - b\int_{-\infty}^{t} K(t-s)x(s)\mathrm{d}s - c\right], \quad (1)$$

where $x(t)$ denotes the neuron response, a denotes the range of the continuous variable, b can be considered as a measure of the inhibitory influence from the past history, c is a off-set constant, s is time delay which denotes the response time of an action. The term $x(t)$ in the argument of function $f(\cdot)$ in Eq. 1 denotes self excitations. $K : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous delay kernel function. Some necessary and sufficient conditions for the existence of globally asymptotically stable equilibrium of Eq. 1 are derived by Lyapunov's method (Gopalsamy and Leung 1997).

In particular, if the kernel functions is a Dirac delta function of the form

$$K(s) = \delta(s - \tau), \quad \tau > 0, \tag{2}$$

then system (1) is changed into the following model with a discrete delay τ

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + af[x(t) - bx(t - \tau) - c]. \tag{3}$$

Pakdaman and Malta (1998) complemented the study of the asymptotic behavior and delay-induced oscillations of neuron system (3). Moreover, Ruan et al. (2001) studied the stability and Hopf bifurcation of this model by means of the Lyapunov functional approach. Liao et al. (2001b) discussed chaotic behavior of this model with non-monotonously increasing activation function.

The system parameters are constants independent of time delay in all of the aforementioned studies on Eq. 3. However, memory performance of the biological neuron usually depends on time history, and its memory intensity is usually lower and lower as time is gradually far away from the current time. It is easy to conceive that these neural networks may involve some delay-dependent parameters. The presence of such dependence often greatly complicates the task of an analytical study of such model. Most existing methods for studying bifurcation fail when applied to such a class of delay models. Compared with the intensive studies on the neuron models with delay-independent parameters, little progress has been achieved for the systems that have delay-dependent parameters.

In Xu et al. (2006), considered system (3) with $c = 0$ and parameter b depending on time delay τ described by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\mu x(t) + af[x(t) - b(\tau)x(t - \tau)], \tag{4}$$

where $\mu > 0$, $a > 0$, $f \in C^3(R)$, $\tau \geq 0$ is the time delay and $b(\tau) > 0$, which is called memory function, is a strictly decreasing function of τ. A detailed analysis on the stability switches, Hopf bifurcation and chaos of system (4) with delay-dependent parameters is given in Xu et al. (2006). Moreover, the direction and the stability of the bifurcating periodic solutions are obtained by the normal form theory and the center manifold theorem.

However, in almost all studies (Wei and Ruan 1999; Liao et al. 2001a, b; Guo and Huang 2004; Cao and Xiao 2007; Kaslik and Balint 2009; Yu and Cao 2006; Xu et al. 2006), the center manifold theorem and the normal form theory introduced in Hassard et al. (1981) are used to determinate the bifurcating periodic solutions and the stability and directions of the Hopf bifurcation. As the center manifold procedure is much complicated, Campbell (1995) refers to it as "algebraically daunting". Recently, the perturbation method has shown the efficacy in studying the local dynamics near a Hopf bifurcation (Wang and Hu 2003). The idea of using the perturbation method on bifurcation problem in DDEs was first introduced by Casal and Freedman in 1980. Compared with the classical normal form theory, the perturbation method has the following advantages (Yu and Leung 2003): (1) The perturbation technique combines the center manifold theory and the normal form theory in one unified procedure through a perturbation procedure; (2) The technique determines a unique normal form; (3) The perturbation procedure does not increase the number of the equations to be solved in each perturbation order; (4) The approach, unlike other methods which need to solve differential equations, needs to solve only algebraic equations; (5) The method generates explicit solutions for both the normal form and associated nonlinear transformation; (6) The technique uses one constant nonlinear transformation for all equations, which is convenient in applications; (7) The approach is computationally efficient.

Thus, here we attempt to develop the perturbation method to study the dynamic behavior in neuron model (4) which has delay-dependent parameters. It should be noted that although the stability and directions of the Hopf bifurcation have been determined in Xu et al. (2006), explicit solutions for bifurcating oscillations have not been studied. In this paper, a detailed analysis on approximation to the bifurcating periodic solutions is given by means of the perturbation approach. Moreover, some examples are provided for comparing approximations with numerical simulations of the bifurcating periodic solutions.

Hopf bifurcation of system (4)

In this section, we consider the Hopf bifurcation of system (4).

Let $y(t) = x(t) - b(\tau)x(t - \tau)$, then Eq. 4 is recast into Xu et al. (2006):

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\mu y(t) + af[y(t)] - ab(\tau)f[y(t - \tau)]. \tag{5}$$

If y^* denotes an equilibrium of system (5), then it satisfies the following equation (Xu et al. 2006):

$$\mu y^* = a[1 - b(\tau)]f(y^*). \tag{6}$$

Remark 1 The linearized analysis about the equilibrium y^* is highly not trivial due to the fact that y^* explicitly depends on the delay τ and exists just up to a finite value of τ.

The results of Hopf bifurcation for neuron system (4), obtained in Xu et al. (2006), are summarized here for completeness and convenience.

Theorem 1 (Xu et al. 2006) *For system* (4), *assume* $f(0) = 0$. *Then a Hopf bifurcation occurs from trivial equilibrium, when the delay*, τ, *passes through the critical value, the zeros* τ_j *of the functions*

$$S_j(\tau) = \tau - \tau_j(\tau), \quad j = 0, 1, 2, \ldots,$$

where

$$\tau_j(\tau) = \begin{cases} \dfrac{\arctan\left[\dfrac{\sqrt{(ab(\tau)f'(0))^2-(\mu-af'(0))^2}}{af'(0)-\mu}\right]+2j\pi}{\sqrt{(ab(\tau)f'(0))^2-(\mu-af'(0))^2}} & \text{if} \quad af'(0)-\mu > 0, \\ \dfrac{\arctan\left[\dfrac{\sqrt{(ab(\tau)f'(0))^2-(\mu-af'(0))^2}}{af'(0)-\mu}\right]+(2j+1)\pi}{\sqrt{(ab(\tau)f'(0))^2-(\mu-af'(0))^2}} & \text{if} \quad af'(0)-\mu < 0. \end{cases}$$

The direction and the stability of the bifurcating periodic solutions are also obtained in Xu et al. (2006).

Remark 2 For neuron model (4) with delay-dependent parameters, it has been proved in Theorem 1 that Hopf bifurcation may occur as delay τ passes through a critical value, where a family of periodic solutions bifurcate from equilibrium. It should be noted that although the stability and directions of the Hopf bifurcation have also been determined in (Xu et al. 2006), explicit solutions for bifurcating oscillations have not been studied.

Approximating expressions of the bifurcating periodic solutions

In Xu et al. (2006), some conditions which guarantee that the neuron model (4) undergoes a Hopf bifurcation at $\tau = \tau_0$ are obtained. In this section, we will use the perturbation approach (Gopalsamy 1996) to derive the approximating expressions of the periodic solutions bifurcating from equilibrium near the critical value of τ_0.

By the translation $u(t) = y(t) - y^*$, Eq. 5 becomes (Xu et al. 2006):

$$\begin{aligned}\frac{\mathrm{d}u(t)}{\mathrm{d}t} &= -\mu[u(t)+y^*] + af[u(t)+y^*] \\ &\quad - ab(\tau)f[u(t-\tau)+y^*]. \end{aligned} \tag{7}$$

whose linearization is (Xu et al. 2006):

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = (a_1-\mu)u(t) + b_1(\tau)u(t-\tau), \tag{8}$$

where

$$a_1 = af'(y^*), \quad b_1(\tau) = -ab(\tau)f'(y^*). \tag{9}$$

Then the characteristic equation of 8 is (Xu et al. 2006):

$$\lambda - (a_1-\mu) - b_1(\tau)e^{-\lambda\tau} = 0, \tag{10}$$

where a_1 and $b_1(\tau)$ are given in (9).

Suppose that $\lambda = \mathrm{i}\omega_0(\omega_0 > 0)$ is a root of the characteristic Eq. 10 corresponding to the linearization of system (5) when $\tau = \tau_0$, then

$$\mathrm{i}\omega_0 - (a_1-\mu) - b_1(\tau_0)e^{-\mathrm{i}\omega_0\tau} = 0. \tag{11}$$

Separating the real and imaginary parts, we have

$$\cos(\omega_0\tau_0) = -\frac{a_1-\mu}{b_1(\tau_0)}, \qquad \sin(\omega_0\tau_0) = -\frac{\omega_0}{b_1(\tau_0)}. \tag{12}$$

Hence,

$$\omega_0 = \sqrt{b_1^2(\tau_0) - (a_1-\mu)^2}. \tag{13}$$

Suppose that the characteristic Eq. 10 corresponding to the linearization of system (5) has a pair of purely imaginary roots $\lambda = \pm\mathrm{i}\omega_0$ at τ_0, where $\omega_0 = \sqrt{b_1^2(\tau_0) - (a_1-\mu)^2}$, $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)|_{\tau_0} \neq 0$ and the remaining characteristic roots at τ_0 have negative real parts.

We proceed to calculate the bifurcating periodic solution of system (5) by means of perturbation approach in Gopalsamy (1996). We first rescale the variable t by setting $s = \omega(\epsilon)t$, where ϵ is a small positive number so that solutions which are $2\pi/\omega$ periodic in t will correspond to solutions which are 2π periodic in s. Therefore, system (5) can be rewritten as

$$\begin{aligned}\omega\frac{\mathrm{d}u(s)}{\mathrm{d}s} &= -\mu[u(s)+y^*] + af[u(s)+y^*] \\ &\quad - ab(\tau)f[u(s-\omega\tau)+y^*]. \end{aligned} \tag{14}$$

Applying Taylor expansion to the right-hand side of system Eq. 14 at the equilibrium leads to a functional differential equation as

$$\omega\frac{\mathrm{d}u(s)}{\mathrm{d}s} = (a_1-\mu)u(s) + b_1(\tau)u(s-\omega\tau) + a_2u^2(s) + a_3u^3(s) + b_2(\tau)u^2(s-\omega\tau) + b_3(\tau)u^3(s-\omega\tau) + \text{h.o.t.}, \tag{15}$$

where the definition of a_1 and $b_1(\tau)$ are the same as those in (9) and

$$a_2 = \tfrac{1}{2}af''(y^*), \qquad a_3 = \tfrac{1}{6}af'''(y^*),$$
$$b_2(\tau) = -\tfrac{1}{2}ab(\tau)f''(y^*), \qquad b_3(\tau) = -\tfrac{1}{6}ab(\tau)f'''(y^*). \tag{16}$$

The solution of Eq. 15 can be expressed in the form of a perturbation series where

$$U(s,\epsilon) = \epsilon u_0(s) + \epsilon^2u_1(s) + \epsilon^3u_2(s) + \cdots = u(s), \tag{17}$$

with the obvious definition of u_0, u_1, u_2, ... The periodic solutions of the nonlinear system (15) will have their periods depending on the parameter τ. Hence, we perturb both the frequency and delay as follows:

$$\begin{cases} \omega = \omega(\epsilon) = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots, \\ \tau = \tau(\epsilon) = \tau_0 + \epsilon\tau_1 + \epsilon^2\tau_2 + \cdots. \end{cases} \tag{18}$$

From (17) and (18), one can obtain

$$u(s-\omega\tau) = \epsilon u_0(s-\omega\tau) + \epsilon^2u_1(s-\omega\tau) + \epsilon^3u_2(s-\omega\tau) + \cdots, \tag{19}$$

in which

$$u_i(s-\omega\tau) = u_i(s-\omega_0\tau_0) - u_i'(s-\omega_0\tau_0)[\epsilon(\omega_1\tau_0+\omega_0\tau_1) + \epsilon^2(\omega_2\tau_0+\omega_1\tau_1+\omega_0\tau_2)+\cdots] + \frac{1}{2}u_i''(s-\omega_0\tau_0)[\epsilon(\omega_1\tau_0+\omega_0\tau_1)+\cdots]^2 - \cdots.$$

Remark 3 The present perturbation approach starts with an expression Eq. 17 of $u(s)$ in terms of a power series for ϵ that quantifies the deviation from the exactly solvable problem. Also, delay term $u(s-\omega\tau)$ in the right side of (15) can be expanded in the power series for ϵ. Compared with the model without delay, the presence of delay term $u(s-\omega\tau)$ may increase the cost of huge analytical algebra for this perturbation approach.

We note that all $u_i(s)$ are 2π periodic in the variable s. Supplying these expansions into system (15) and using (17–19), we can obtain the following equations by equating the coefficients of the various terms involving powers of ϵ. It is found that $u_0(s)$ is governed by

$$\omega_0\frac{\mathrm{d}u_0(s)}{\mathrm{d}s} = (a_1-\mu)u_0(s) + b_1(\tau_0)u_0(s-\omega_0\tau_0). \tag{20}$$

Remark 4 It should be noted that the delay-dependent parameter $b_1(\tau)$ should be expanded in Taylor's series at $\tau=\tau_0$. The presence of delay-dependent parameter usually makes the analytical work so harder. In what follows, the delay-dependent parameters $b_2(\tau)$ and $b_3(\tau)$ also should be expanded in Taylor's series at $\tau=\tau_0$.

The system (20) is the same as the linearized system (8). In order To find a 2π periodic solution of (20), we let

$$u_0(s) = A_0\sin s + B_0\cos s,$$

where A_0 and B_0 are not necessarily independent constants. We supply this u_0 in (20) and solve for the two unknown quantities. Using Eq. 12, we can find that A_0 and B_0 can be arbitrary. For the sake of easy calculation in the later stages, we impose the initial condition $u_0(0)=0$ and $u'_0(0)=1$ to get that

$$u_0(s) = \sin s. \tag{21}$$

The term $u_1(s)$ in the perturbation is governed by

$$\omega_0\frac{\mathrm{d}u_1(s)}{\mathrm{d}s} + \omega_1\frac{\mathrm{d}u_0(s)}{\mathrm{d}s} = (a_1-\mu)u_1(s) - b_1(\tau_0)u_0'(s-\omega_0\tau_0)(\omega_1\tau_0+\omega_0\tau_1) + b_1(\tau_0)u_1(s-\omega_0\tau_0) + b_1'(\tau_0)\tau_1u_0(s-\omega_0\tau_0) + u_0^2(s) + b_2(\tau_0)u_0^2(s-\omega_0\tau_0). \tag{22}$$

Let

$$u_1(s) = A_1\sin s + B_1\cos s + C_1\sin(2s) + D_1\cos(2s) + G_1. \tag{23}$$

Inserting it into Eq. 22 and using Eq. 12, we can obtain the equation about $\sin s, \cos s, \sin(2s)$ and $\cos(2s)$.Comparing the corresponding coefficients of ϵ^2, we can then get the values of the unknown parameters

$$\omega_1 = \tau_1 = 0,$$
$$C_1 = \frac{\gamma_1\gamma_3+\gamma_2\gamma_4}{\gamma_1^2+\gamma_2^2}, \qquad D_1 = \frac{\gamma_2\gamma_3-\gamma_1\gamma_4}{\gamma_1^2+\gamma_2^2},$$
$$G_1 = \frac{1+b_2(\tau_0)}{2[\mu-a_1-b_1(\tau_0)]}, \tag{24}$$

where

$$\gamma_1 = 2\omega_0 + \tfrac{2(a_1-\mu)\omega_0}{b_1(\tau_0)}, \qquad \gamma_2 = \mu - a_1 + b_1(\tau_0) - \tfrac{2(a_1-\mu)^2}{b_1(\tau_0)},$$
$$\gamma_3 = -\tfrac{1}{2}\left[1 + b_2(\tau_0)\left(\tfrac{2(a_1-\mu)^2}{b_1^2(\tau_0)} - 1\right)\right], \qquad \gamma_4 = b_2(\tau_0)\tfrac{(\mu-a_1)\omega_0}{b_1^2(\tau_0)},$$

with A_1 and B_1 arbitrary.

The equation governing the term $u_2(s)$ can be obtained by comparing the corresponding coefficients of ϵ^3 in Eq. 15 as follows:

$$\begin{aligned}
&\omega_0\frac{\mathrm{d}u_2(s)}{\mathrm{d}s}+\omega_1\frac{\mathrm{d}u_1(s)}{\mathrm{d}s}+\omega_2\frac{\mathrm{d}u_0(s)}{\mathrm{d}s}\\
=&(a_1-\mu)u_2(s)-b_1(\tau_0)u_0'(s-\omega_0\tau_0)(\omega_2\tau_0+\omega_1\tau_1+\omega_0\tau_2)\\
&+\frac{1}{2}b_1(\tau_0)u_0''(s-\omega_0\tau_0)(\omega_1\tau_0+\omega_0\tau_1)^2\\
&-b_1'(\tau_0)\tau_1u_0'(s-\omega_0\tau_0)(\omega_1\tau_0+\omega_0\tau_1)\\
&+b_1(\tau_0)u_2(s-\omega_0\tau_0)+b_1'(\tau_0)\tau_2u_0(s-\omega_0\tau_0)\\
&-b_1(\tau_0)u_1'(s-\omega_0\tau_0)(\omega_1\tau_0+\omega_0\tau_1)\\
&+b_1'(\tau_0)\tau_1u_1(s-\omega_0\tau_0)+\frac{1}{2}b_1''(\tau_0)\tau_1^2u_0(s-\omega_0\tau_0)\\
&+2a_2u_0(s)u_1(s)+a_3u_0^3(s)+b_2'(\tau_0)\tau_1u_0^2(s-\omega_0\tau_0)\\
&+2b_2(\tau_0)u_0(s-\omega_0\tau_0)u_1(s-\omega_0\tau_0)\\
&-2b_2(\tau_0)u_0(s-\omega_0\tau_0)u_0'(s-\omega_0\tau_0)(\omega_1\tau_0+\omega_0\tau_1)\\
&+b_3(\tau_0)u_0^3(s-\omega_0\tau_0).
\end{aligned} \tag{25}$$

Let

$$\begin{aligned}
u_2(s)=&A_2\sin s+B_2\cos s+C_2\sin(2s)+D_2\cos(2s)\\
&+E_2\sin(3s)+F_2\cos(3s)+G_2.
\end{aligned}$$

Substituting $u_2(s)$ into Eq. 15 and using appropriate triangle transformation, we can obtain the following equation about $\sin s, \cos s, \sin(2s), \cos(2s), \sin(3s)$, and $\cos(3s)$:

$$\begin{aligned}
&(\omega_0A_2+\omega_2)\cos s-\omega_0B_2\sin s+2\omega_0C_2\cos(2s)\\
&-2\omega_0D_2\sin(2s)+3\omega_0E_2\cos(3s)\\
&-3\omega_0F_2\sin(3s)\\
=&\left[\omega_0(\omega_2\tau_0+\omega_0\tau_2)-\omega_0B_2-\frac{(a_1-\mu)\tau_2b_1'(\tau_0)}{b_1(\tau_0)}+\rho_1\right]\sin s\\
&+\left[(a_1-\mu)(\omega_2\tau_0+\omega_0\tau_2)+\omega_0A_2+\frac{\omega_0\tau_2b_1'(\tau_0)}{b_1(\tau_0)}+\rho_2\right]\cos s\\
&+\left[(a_1-\mu)C_2+a_2B_1+b_1(\tau_0)(C_2\rho_4+D_2\rho_3)\right.\\
&\left.-b_2(\tau_0)(A_1\rho_3-B_1\rho_4)\right]\sin(2s)\\
&+\left[(a_1-\mu)D_2-a_2A_1+b_1(\tau_0)(-C_2\rho_3+D_2\rho_4)\right.\\
&\left.-b_2(\tau_0)(A_1\rho_4+B_1\rho_3)\right]\cos(2s)\\
&+\left[(a_1-\mu)E_2+b_1(\tau_0)(\rho_5E_2+\rho_6F_2)+b_2(\tau_0)(\rho_5D_1\right.\\
&\left.-\rho_6C_1).-\frac{1}{4}a_3+a_2D_1-\frac{1}{4}b_3(\tau_0)\rho_5\right]\sin(3s)\\
&+\left[(a_1-\mu)F_2+b_1(\tau_0)(\rho_5F_2-\rho_6E_2)-b_2(\tau_0)(\rho_5C_1\right.\\
&\left.+\rho_6D_1).-a_2C_1+\frac{1}{4}b_3(\tau_0)\rho_6\right]\cos(3s)+(a_1-\mu)G_2+a_2A_1\\
&+b_1(\tau_0)G_2+b_2(\tau_0)A_1,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
\rho_1&=-a_2D_1+2a_2G_1+\tfrac{3}{4}a_3-\tfrac{2G_1b_2(\tau_0)(a_1-\mu)}{b_1(\tau_0)}-\tfrac{3(a_1-\mu)b_3(\tau_0)}{4b_1(\tau_0)}-\tfrac{C_1b_2(\tau_0)\omega_0}{b_1(\tau_0)}+\tfrac{D_1b_2(\tau_0)(a_1-\mu)}{b_1(\tau_0)}, & \rho_2&=a_2C_1+\tfrac{3\omega_0b_3(\tau_0)}{4b_1(\tau_0)}+\tfrac{2G_1b_2(\tau_0)\omega_0}{b_1(\tau_0)}-\tfrac{D_1b_2(\tau_0)\omega_0}{b_1(\tau_0)}-\tfrac{C_1b_2(\tau_0)(a_1-\mu)}{b_1(\tau_0)},\\
\rho_3&=\tfrac{2\omega_0(a_1-\mu)}{b_1(\tau_0)^2}, & \rho_4&=\tfrac{2(a_1-\mu)^2}{b_1(\tau_0)^2}-1,\\
\rho_5&=-\tfrac{4(a_1-\mu)^3}{b_1(\tau_0)^3}+\tfrac{3(a_1-\mu)}{b_1(\tau_0)}, & \rho_6&=-\tfrac{3\omega_0}{b_1(\tau_0)}+\tfrac{4\omega_0^3}{b_1(\tau_0)^3}.
\end{aligned}$$

Comparing the coefficients of $\sin s, \cos s, \sin(2s), \cos(2s), \sin(3s)$, and $\cos(3s)$ in Eq. 26, we have the expression of A_2, B_2, C_2, D_2, E_2, F_2, and G_2. Moreover,

$$\begin{aligned}
\omega_2&=\frac{\rho_2\left[\omega_0^2-\frac{(a_1-\mu)b_1'(\tau_0)}{b_1(\tau_0)}\right]-\rho_1\left[(a_1-\mu)\omega_0+\frac{\omega_0b_1'(\tau_0)}{b_1(\tau_0)}\right]}{\omega_0\tau_0\left[(a_1-\mu)\omega_0+\frac{\omega_0b_1'(\tau_0)}{b_1(\tau_0)}\right]-[(a_1-\mu)\tau_0-1]\left[\omega_0^2-\frac{(a_1-\mu)b_1'(\tau_0)}{b_1(\tau_0)}\right]},\\
\tau_2&=\frac{\rho_1[(a_1-\mu)\tau_0-1]-\rho_2\omega_0\tau_0}{\omega_0\tau_0\left[(a_1-\mu)\omega_0+\frac{\omega_0b_1'(\tau_0)}{b_1(\tau_0)}\right]-[(a_1-\mu)\tau_0-1]\left[\omega_0^2-\frac{(a_1-\mu)b_1'(\tau_0)}{b_1(\tau_0)}\right]}.
\end{aligned} \tag{27}$$

After finding the perturbed parameter values, we can write down the approximate solution of Eq. 4 as

$$u(s)=\sqrt{\frac{\tau-\tau_0}{\tau_2}}u_0(s)+\frac{\tau-\tau_0}{\tau_2}u_1(s)+\cdots, \tag{28}$$

where $u_0(s)$ and $u_1(s)$ are given in Eqs. 21 and 23, respectively, and $\tau\approx\tau_0+\epsilon^2\tau_2$ which implies that the parameter τ_2 determines the direction of the Hopf bifurcation and ω_2 determines the period of the bifurcating periodic solutions.

Now, we summarize the above analysis in the following:

Theorem 2 *For system* (4), *the following results hold:*

(i) *If $\tau_2>0(\tau_2<0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau>\tau_0(\tau<\tau_0)$;*
(ii) *If $\omega_2<0(\omega_2>0)$, the period increases (decreases).*

Remark 5 Compared with the classical normal form theory, the perturbation method is more efficient and convenient in determining the bifurcating periodic solutions and the stability and directions of the Hopf bifurcation. Moreover, the approximate expressions of the bifurcating periodic solutions can also be derived by perturbation method.

Numerical simulations

In this section, we will give a critical comparison of the present approximate result (28) to the bifurcating periodic oscillator with those obtained by a more exact numerical solution. The numerical solution is derived by using the fourth-order Runge-Kutta method. For comparison, the same model (5), used in Xu et al. (2006), is discussed, with $\mu = 2,\ a = 1,\ f(\cdot) = \tanh(\cdot)$ and $b(\tau) = 3e^{-0.12\tau}$. The corresponding model assumes the following form

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -2y(t) + \tanh(y(t)) - 3e^{-0.12\tau}\tanh(y(t-\tau)). \tag{29}$$

From Theorem 1, there are two bifurcation points (Xu et al. 2006):

$\tau_{01} = 0.763, \quad \tau_{02} = 8.74.$

For the first bifurcating point $\tau_{01} = 0.763$, using Eqs. 13 and 27, we have $\omega_0 = 2.5483$, $\omega_2 = -5.601$, and $\tau_2 = 1.787 > 0$. Note that since $\tau_2 > 0$, the Hopf bifurcation is supercritical, and the bifurcating periodic solutions exist at least for the value of τ slightly larger than the critical value τ_{01}. Figure 1 shows that a family of periodic solutions bifurcate from the trivial equilibrium when $\tau = 2 > \tau_{01}$. In addition, an approximation to the bifurcating periodic solution of system (14) with the above parameters near τ_{01} is given by

$$\begin{cases} u(s,\epsilon) & \approx & \epsilon\sin s + \epsilon^2[0.6\sin s + 0.59\cos s - 0.0704\sin(2s) + 0.0102\cos(2s) + 0.0501], \\ \omega(\epsilon) & \approx & 2.5483 - 5.601\epsilon^2, \\ \tau(\epsilon) & \approx & 0.763 + 1.787\epsilon^2, \end{cases}$$

and an approximation to the periodic solution of system (29) is given by

$$\begin{aligned} y(t) = {} & (0.5596\tau - 0.427)^{1/2}\sin((4.9353 - 3.063\tau)t) \\ & + (0.5596\tau - 0.427) \times [0.6\sin((4.9353 - 3.063\tau)t) \\ & + 0.59\cos((4.9353 - 3.063\tau)t) \\ & - 0.0704\sin(2(4.9353 - 3.063\tau)t) \\ & + 0.0102\cos(2(4.9353 - 3.063\tau)t) + 0.0501]. \end{aligned} \tag{30}$$

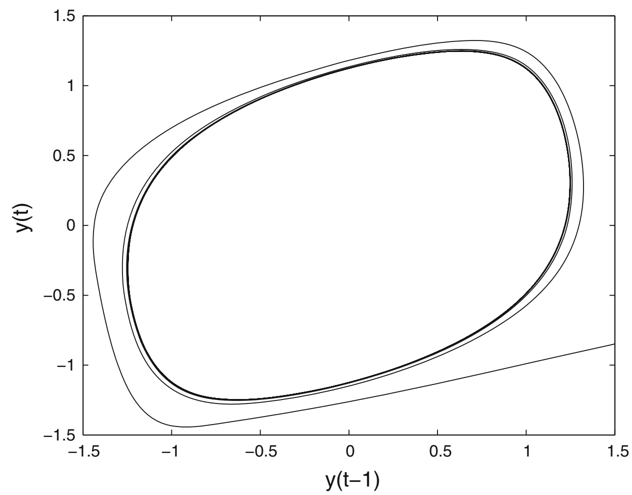

Fig. 1 The periodic oscillations of system (29) with $\tau = 2 > \tau_{01}$

If we choose $\tau = 2$ which is near τ_{01}, we can estimate $\epsilon = 0.832 < 1$. Thus, for approximate solution (30), the contribution of the first order term in ϵ dominates well above the contribution of the second order term in ϵ. We plot approximate analytical solution (30) and the corresponding numerical solution for different time interval in Fig. 2. In Fig. 2a–d, approximate solution and numerical solution of the bifurcating solution are displayed against t for $t \in [0, 50]$, [50, 100], [100, 150] and [150, 200], respectively. From Fig. 2a, b, we find the approximate analytical solutions agree nicely with those of the numerical counterpart for small times notably for $t \in [0, 100]$. In Fig. 2c, d, the phase deviation of the perturbation solution from the numerical one is approximately 10%. The deviation arises with the increase of time due to the violent behavior of the solution near the critical value τ_{01}. This deviation can be reduced if the higher order terms (beyond the second order) in ϵ in Eq. 17 are taken into consideration. Thus, for large time, we have to consider more higher order terms in the power series for ϵ till a convergence is reached.

Similarly, for the second bifurcating point τ_{02}, we obtain $\omega_0 = 0.3237$, $\omega_2 = 0.1534$, and $\tau_2 = -4.3468 < 0$. Then the Hopf bifurcation is subcritical, and the bifurcating periodic solutions exist for $\tau < \tau_{02}$. Figure 3 displays that a family of periodic solutions bifurcate from the trivial equilibrium when $\tau = 8 < \tau_{02}$. The approximation to the bifurcating periodic solution of system (14) near τ_{02} is given by

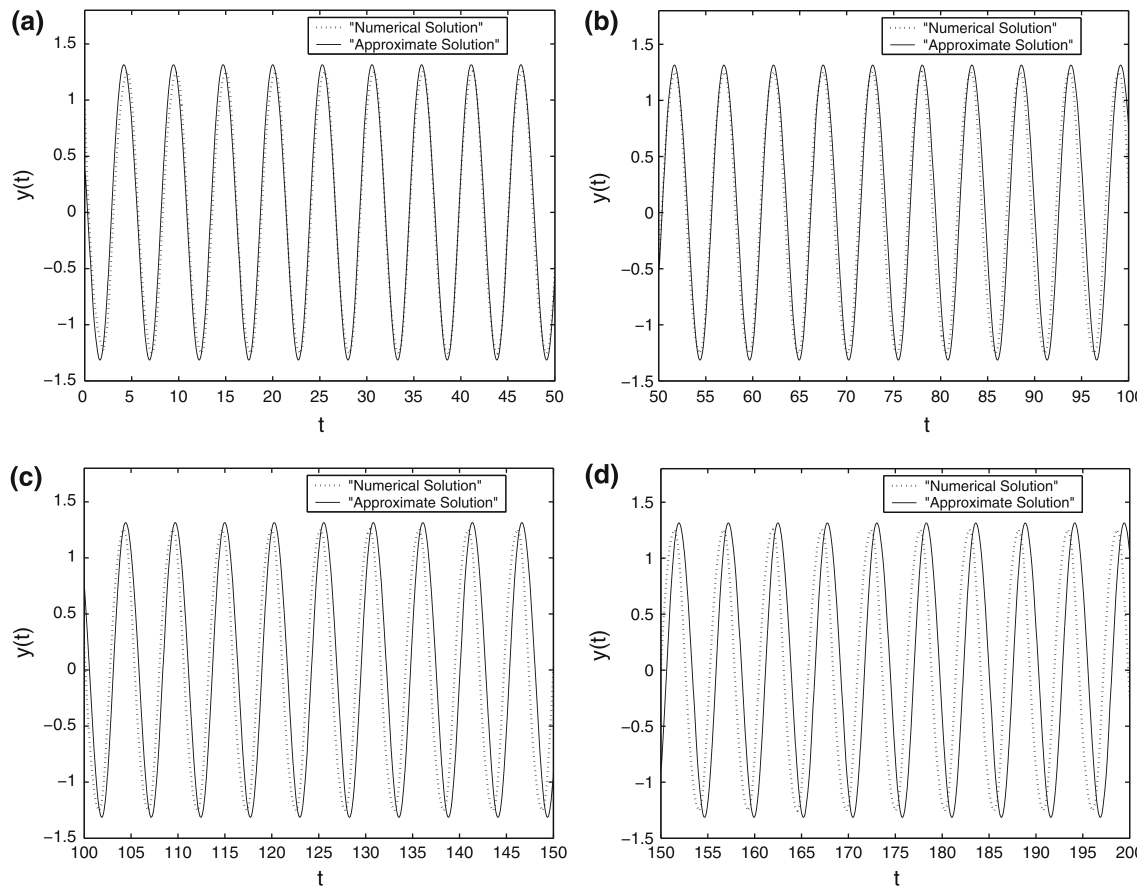

Fig. 2 Approximate solution and numerical solution of the bifurcating periodic solution of system (29) with $\tau = 2 > \tau_{01}$ for **a** $t \in [0, 50]$; **b** [50, 100]; **c** [100, 150]; **d** [150, 200]

$$\begin{cases} u(s,\epsilon) \approx \epsilon \sin s + \epsilon^2[0.3\sin s + 0.3\cos s - 0.1257\sin(2s) - 0.1843\cos(2s) + 0.0501], \\ \omega(\epsilon) \approx 0.3237 + 0.1534\epsilon^2, \\ \tau(\epsilon) \approx 8.74 - 4.3468\epsilon^2, \end{cases}$$

and an approximation to the periodic solution of system (29) is given by

$$\begin{aligned} y(t) = 0 &+ (-0.2301\tau + 2.0107)^{1/2}\sin((0.6321 - 0.0333\tau)t) \\ &+ (-0.2301\tau + 2.0107) \times [0.3\sin((0.6321 - 0.0333\tau)t) \\ &+ 0.3\cos((0.6321 - 0.0333\tau)t) \\ &- 0.1257\sin(2(0.6321 - 0.0333\tau)t) \\ &- 0.1843\cos(2(0.6321 - 0.0333\tau)t) + 0.0501]. \end{aligned} \tag{31}$$

We assume $\tau = 8$ which is near τ_{02}, and plot approximate analytical solution (31) and the corresponding numerical solution for different time interval in Fig. 4. In Fig. 4a, we take $t \in [0, 250]$. It is observed that the approximate solution is not quite consistent with those of the numerical results obtained by the fourth-order Runge-Kutta method. Note that the deviation between the two solutions increases for $t \in [250, 500]$ in Fig. 4b. Maybe the deviation results from the more violent behavior of the solution near the second bifurcation point τ_{02}, which is far away from the first bifurcation point τ_{01}. It is possible to obtain more precise approximation by choosing proper initial condition and considering more higher order terms.

Admittedly, the approximate solution presented in this paper is not always sufficiently accurate. The approximate solution in Fig. 2a, b is quite accurate. On the other hand, the approximate solution in Fig. 2c, d and Fig. 4 is less reliable. The approximate solution in Fig. 2c, d and Fig. 4 may be more reliable if we consider more terms beyond the second order term in Eq. 17 and choose proper initial condition. A comparison of Figs. 2 and 4 exhibits that the approximate solution near the first bifurcation point τ_{01} is

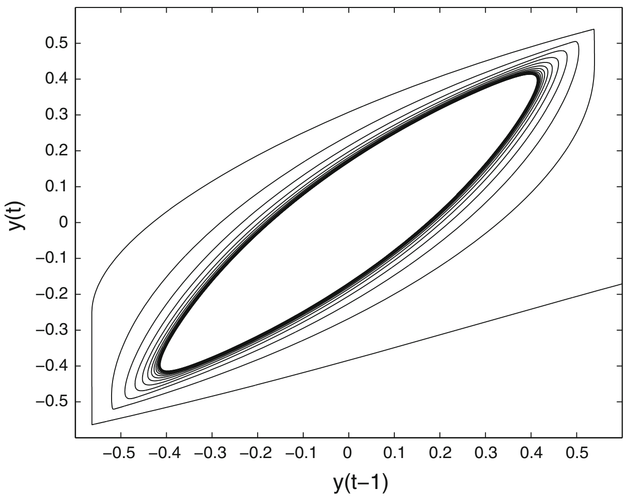

Fig. 3 The periodic oscillations of system (29) with $\tau = 8 < \tau_{02}$

more accurate than the one near the second bifurcation point τ_{02}. It is because, the oscillatory dynamics near the first bifurcation point τ_{01} is more complicated than the one near the second bifurcation point τ_{02}.

Conclusion

The perturbation method introduced by Casal and Freedman in 1980 has been applied to a neuron model with delay-dependent parameters (Xu et al. 2006) which has bifurcating periodic solutions. Although the presence of delay-dependent parameters often greatly complicates the task of analytical study of such delay systems, and most existing methods for studying the nonlinear dynamics fail when used to such a class of delay models, the approximate analytical solution (28) of bifurcating oscillator is obtained by using a complete perturbation analysis. Importantly, the present approximate analytical solution near the first bifurcation point agrees nicely with those numerical solutions for small time. Therefore, the present approach is not merely a supplementary one, rather, it stands on its own.

The present perturbation approach starts with an expression (17) for the desired solution in terms of a power series in some "small" parameter ϵ that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem. The periods of the periodic solutions are depended upon the delay. Hence, we also perturb both the frequency ω and delay τ as Eq. 18. In this paper, $u_0(s)$ is the known solution to the exactly solvable initial problem and $u_1(s)$, $u_2(s)$, … represent the higher-order terms which may be found iteratively by some systematic procedure. For small ϵ these higher order terms in the series become successively smaller. The approximate "perturbation solution" is obtained by truncating the series, by keeping only the first two terms, the initial solution and the "second order" perturbation correction:

$$u(s) \approx \epsilon u_0(s) + \epsilon^2 u_1(s),$$

where $u_0(s)$ and $u_1(s)$ are given in Eqs. 21 and 23, respectively, and $\epsilon \approx \sqrt{\frac{\tau - \tau_0}{\tau_2}}$. For the sake of simplicity, more higher-order terms beyond the second order term in ϵ are completely avoided. The solution, of course, is obtained up to the second order in ϵ. Nevertheless, the present approach is also useful to obtain the solution for higher orders in ϵ.

The approximate perturbation solution presented in this paper has a great deal of academic interests. The second order perturbation solution has been assessed analytically by comparing successive terms in the power series expansions for ϵ, numerically by solving the governing equation by means of an explicit fourth-order accurate Runge-Kutta

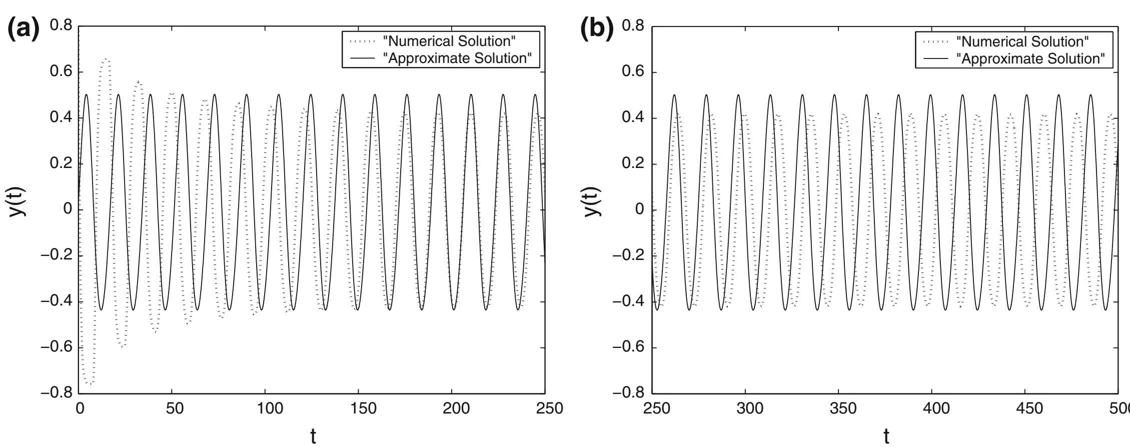

Fig. 4 Approximate solution and numerical solution of the bifurcating solution of system (29) with $\tau = 8 < \tau_{02}$. for **a** $t \in [0, 250]$; **b** [250, 500]

method. The series method for the approximate solution of the present problem depends on the choice of the initial conditions. The accuracy of the perturbation method is found to be very goo d by using appropriate initial conditions. Moreover, the accuracy of the perturbation solution increases with the order of the power series in ϵ. The perturbation solution may be more reliable if we consider more terms beyond the second order term in Eq. 17. Again, the solutions for higher orders may be achieved at the cost of huge analytical algebra. It has been shown that the approximate perturbation solutions near the first bifurcation point agree with those found in the numerical simulations for small times notably for $t < 100$. On the other hand, the approximate perturbation solutions are less reliable when $t > 100$.

Recently, there has been interest in determining analytical and approximate solutions to ordinary differential equations (Ramos 2010; Mandal 2005) and delay-differential equations (Wang and Hu 2003; Gopalsamy 1996). It should be noted that unlike models used in the study of approximate solutions in previous literatures, our neuron model comprises some delay-dependent parameters, which greatly complicates the task of perturbation analytical study of such model.

In the present paper, an approximate expression of the bifurcating periodic solution is given for a neuron system in presence of delay-dependent parameters. For the simplicity of the calculation, the inclusion of second order term in ϵ is incorporated. The inclusion of higher order terms would definitely be a hard one to be achieved. To have some feelings about the present approximate solution, we have displayed few numerical results. A critical comparison of our analytical result with the results obtained by an exact numerical method is exhibited. It is observed that the present approximate solution is quite accurate in some cases.

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Appendix: the procedure of numerical simulations

1. the M. file codes for Fig. 2

```
function neuron
sol=dde23(@ddex1de,[2],@ddex1hist,[0,200]);
figure;
hold on
plot(sol.x,sol.y(1,:),':');
t=0:0.01:200;
y=(0.5596*2-0.427) ^ (1/2)*sin((4.9353-3.063*2)*t)
+(0.5596*2-0.427)*(0.60*sin((4.9353-3.063*2)*t)
+0.59* cos((4.9353-3.063*2)*t)
-0.0704*sin(2*(4.9353-3.063*2)*t)+0.0102*cos(2*
(4.9353-3.063*2)*t)+0.0501);
plot(t,y);
hold off
end
function s=ddex1hist(t)
s=ones(1,1);
function dydt=ddex1de(t,y,Z)
ylag1=Z(:,1);
dydt=[-2*y(1)+tanh(y(1))-3*exp(−0.12*2)*tanh(ylag1(1))];
```

2. the M. file codes for Fig. 4

```
function neuron
sol=dde23(@ddex1de,[8],@ddex1hist,[0,500]);
figure;
hold on
plot(sol.x,sol.y(1,:),':');
t=0:0.01:500;
y=(−0.2301*8+2.0107) ^ (1/2)*sin((0.6321-0.0333*8)*t)
+(−0.2301*8+2.0107)*(0.3*sin((0.6321-0.0333*8)*t)+ 0.3*
cos((0.6321-0.0333*8)*t)
-0.1257*sin(2*(0.6321-0.0333*8)*t)-0.1843* cos(2*(0.6321
-0.0333*8)*t)+0.0501);
plot(t,y);
hold off
end
function s=ddex1hist(t)
s=ones(1,1);
function dydt=ddex1de(t,y,Z)
ylag1=Z(:,1);
dydt=[-2*y(1)+tanh(y(1))-3*exp(−0.12*8)*tanh(ylag1(1))];
```

References

Bélair J, Campbell SA, van den Driessche P (1996) Frustration, stability, and delay-induced oscillations in a neural network model. SIAM J Appl Math 56:245–255

Campbell SA, Belair J, Ohira T, Milton J (1995) Complex dynamics and multistability in a damped harmonic oscillator with delayed negative feedback. Chaos 5:640–645

Cao J, Song Q (2006) Stability in Cohen-Grossberg-type bidirectional associative memory neural networks with time-varying delays. Nonlinearity 19:1601–1617

Cao J, Xiao M (2007) Stability and Hopf bifurcation in a simplified BAM neural network with two time delays. IEEE Trans Neural Networks 18:416–430

Casal A, Freedman M (1980) A Poincare-Lindstedt approach to bifurcation problems for differential-delay equations. IEEE Trans Automat Control 25:967–973

Gopalsamy K (1996) Delay induced periodicity in a neural netlet of excitation and inhibition. Physica D 89:395–426

Gopalsamy K, He X (1994) Stability in asymmetric Hopfield neural networks with transmission delay. Physica D 76:344–358

Gopalsamy K, Leung I (1997) Convergence under dynamical thresholds with delays. IEEE Trans Neural Networks 8:341–348

Guo S, Huang L (2004) Linear stability and Hopf bifurcation in a two-neuron network with three delays. Int J Bifurcation Chaos 8: 2799–2810

Hassard BD, Kazarinoff ND, Wan YH (1981) Theory and application of Hopf bifurcation. Cambridge University Press, Cambridge, MA

Hopfield J (1982) Neural networks and physical systems with emergent collective computations abilities. Proc Natl Acad USA 79:2554–2558

Kaslik E, Balint St (2009) Bifurcation analysis for a discrete-time Hopfield neural network of two neurons with two delays and self-connections, Chaos. Solitons Fractals 39:83–91

Liao XF, Wong KW, Leung CS, Wu ZF (2001a) Hopf bifurcation and chaos in a single delayed neuron equation with non-monotonic activation function, Chaos. Solitons Fractals 12:1535–1547

Liao XF, Wong K-W, Wu Z (2001b) Bifurcation analysis in a two-neuron system with continuously distributed delays. Physica D 149:123–141

Mandal S (2005) The approximate solution of a classical quartic anharmonic oscillator with periodic force: a simple analytical approach. Commun Nonlinear Sci Numer Simul 10:341–352

Pakdaman K, Malta CP (1998) A note on convergence under dynamical thresholds with delays. IEEE Trans Neural Networks 9:231–233

Ramos J (2010) Approximate methods based on order reduction for the periodic solutions of nonlinear third-order ordinary differential equations. Appl Math Comput 215:4304–4319

Ruan S, Fillfil R (2004) Dynamics of a two-neuron system with discrete and distributed delays. Physica D 191:323–342

Ruan J, Li L, Lin W (2001) Dynamics of some neural network models with delay. Phys Rev E 63:051906

Wang HL, Hu HY (2003) Remarks on the perturbations in solving the second-order delay differential equations. Nonlinear Dynam 33:379–398

Wei J, Ruan S (1999) Stability and bifurcation in a neural network model with two delays. Physica D 130:255–272

Wu J (2001) Introduction to neural dynamics and signal transmission delay. Walter de Cruyter, Berlin

Xu X, Hu HY, Wang HL (2006) Stability switches, Hopf bifurcation and chaos of a neuron model with delay-dependent parameters. Phys Lett A 354:126–136

Yu W, Cao J (2006) Stability and Hopf bifurcation analysis on a four-neuron BAM neural network with time delays. Phys Lett A 351: 64–78

Yu P, Leung AYT (2003) A perturbation method for computing the simplest normal forms of dynamical systems. J Sound Vibration 261:123–151

Zhu H, Huang L (2007) Stability and bifurcation in a tri-neuron network model with discrete and distributed delays. Comput Math Appl 188:1742–1756