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On a class of damped vibration problems with obstacles☆

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ABSTRACT

The main purpose of this paper is to study the following damped vibration problem

$$-\ddot{x}=g(t)\dot{x}+f(t,x) \tag{1.1}$$

satisfying

$$x(0)-x(2\pi)=\dot{x}(0)-\dot{x}(2\pi)=0, \qquad x(t)\geq 0, \quad \forall t\in R \tag{1.2}$$

$$\dot{x}(t_0^-)=-\dot{x}(t_0^+), \quad \text{if } x(t_0)=0. \tag{1.3}$$

The variational principles are given and some existence and multiplicity results of nonzero periodic solutions satisfying (1.1)–(1.3) are obtained.

1. Introduction

Throughout the paper, we shall consider, as usual, $u=v$ means $u(t)=v(t)$ for a.e. $t\in R$. Let $f: R\times R^+\to R$ be a continuous function and $f(t,x)$ 2π-periodic in t. We look for the solutions of

$$-\ddot{x}=g(t)\dot{x}+f(t,x) \tag{1.1}$$

satisfying the following conditions

$$x(0)-x(2\pi)=\dot{x}(0)-\dot{x}(2\pi)=0, \qquad x(t)\geq 0, \quad \forall\, t\in R \tag{1.2}$$

$$\dot{x}(t_0^-)=-\dot{x}(t_0^+), \quad \text{if } x(t_0)=0 \tag{1.3}$$

where $g: R\to R$ is continuous with $G(2\pi)=0$, $G(t)=\int_0^t g(s)\mathrm{d}s$, and

$$\dot{x}(t_0^-)=\lim_{t\to t_0-0}\dot{x}(t), \qquad \dot{x}(t_0^+)=\lim_{t\to t_0+0}\dot{x}(t).$$

Such a solution is called a bouncing periodic solution of (1.1). Physically, it means that the particle bounces in a perfectly elastic way when it hits the obstacle $x=0$.

As $g\equiv 0$, the existence of the bouncing periodic solutions and quasiperiodic solutions of (1.1) has been considered by several authors in the last decade (see [1–7]). But, their methods are not variational. In 2005, Mei-Yue Jiang [8] took the lead in using the variational methods to study the existence of a sequence of periodic bouncing solutions for

$$-\ddot{x}=f(t,x),$$

and the following results were given.

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Theorem A. *If*

$$f(t,x) = p(t) + a(t)x + o(x), \quad x \to 0^+ \tag{f_1}$$

and $p(t) > 0$ *for* $t \in [0, 2\pi]$, *then problem* (1.1) *with* $g \equiv 0$ *has a sequence of* 2π*-periodic solutions* $\{x_j\}$ *satisfying* $\|x_j\|_\infty \to 0$ *as* $j \to \infty$ *and*

(1) $x_j(t) \geq 0$ *for* $t \in R$;
(2) *For each j, the set* $\sigma_j = \{t \in [0, 2\pi] : x_j(t) = 0\}$ *is finite;*
(3) $\dot{x}_j(t^-) = -\dot{x}_j(t^+)$ *for* $t \in \sigma_j$.

Theorem B. *Let* $F(t,x) = \int_0^x f(t,s)\mathrm{d}s : R \times R^+ \to R$ *be a* C^1 *function and satisfy: there are constants* $\theta > 2, r > 0$ *such that*

$$xf(t,x) \geq \theta F(t,x) > 0, \quad \text{for } x \geq r, \tag{f_2}$$

and

$$\left|\frac{\partial F}{\partial t}\right| \leq C(1 + F(t,x)), \quad \text{for } x \in R^+ \tag{f_3}$$

for some constant $C > 0$. *Then problem* (1.1) *with* $g \equiv 0$ *has an unbounded sequence of* 2π*-periodic solutions* $\{x_j\}$ *satisfying*

(1) $x_j(t) \geq 0$ *for* $t \in R$;
(2) *For each j, the set* $\sigma_j = \{t \in [0, 2\pi] : x_j(t) = 0\}$ *is finite;*
(3) $\dot{x}_j(t^-) = -\dot{x}_j(t^+)$ *for* $t \in \sigma_j$.

In the present paper, our purposes are to research the variational principles and the existence and multiplicity of 2π-periodic bouncing solutions for problem (1.1) with g is a nonzero continuous function. Our results not only contain Theorems A and B, but also shows that the condition (f_3) is unnecessary in Theorem B. Moreover, some new results are given as f is super-linear or asymptotically linear at infinity.

2. Preliminaries

Let X be a real Banach space and X^* the dual space of X. A functional $J : X \to R$ is called locally Lipschitz if for each $u \in X$ there exist a neighborhood U of u and a constant $L \geq 0$ such that

$$|J(v) - J(w)| \leq L\|v - w\|, \quad \forall v, w \in U.$$

For any $u, v \in X$, we define the generalized directional derivative $J^0(u; v)$ of J at point u along the direction v as

$$J^0(u; v) = \overline{\lim_{h \to 0, \lambda \downarrow 0}} \frac{1}{\lambda}[J(u + h + \lambda v) - J(u + h)].$$

The generalized gradient of the function J at u, denoted by $\partial J(u)$, is the set

$$\partial J(u) = \{w \in X^* : \langle w, v\rangle \leq J^0(u; v), \forall v \in X\}.$$

Set

$$\lambda(u) = \min_{w \in \partial J(u)} \|w\|.$$

A point $u \in X$ is said to be a critical point of J if $\lambda(u) = 0$. Let X be a normed linear space and $f : X \to R$ a locally Lipschitz function. We say that f satisfies the (C) (or $(C)_c$) condition, if any sequence $\{x_n\} \subset X$ along which $f(x_n)$ is bounded (or $f(x_n) \to c$) and $(1 + \|x_n\|)\lambda(x_n) \to 0$ possesses a convergent subsequence. We say that f satisfies the P.S condition, if any sequence $\{x_n\} \subset X$ along which $f(x_n)$ is bounded and $\lambda(x_n) \to 0$ possesses a convergent subsequence.

In [9], the following deformation theorem was obtained.

Theorem 2.1. *Let X be a reflexive Banach space and* $f : X \to R$ *a locally Lipschitz function with the condition (C) in* $f^{-1}((a, b))$. *Then for any* $c \in (a, b)$, *any* $\varepsilon_0 > 0$ *and any neighborhood N of* $K_c := \{x \in X : f'(x) = 0, f(x) = c\}$, *there exist* $\varepsilon \in (0, \varepsilon_0)$ *and a continuous mapping* $\eta : [0, 1] \times X \to X$ *such that for all* $(t, x) \in [0, 1] \times X$ *we have*

(a) $\|\eta(t,x) - x\| \leq e(1 + \|x\|)t$, *where e is a constant;*
(b) $|f(x) - c| \geq \varepsilon_0 \Rightarrow \eta(t,x) = x$;
(c) $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon} \bigcup N$;
(d) $f(\eta(t,x))$ *is non-increasing in t;*
(e) $\eta(t, \cdot) : X \to X$ *is a homeomorphism;*
(f) $\eta(t,x) \neq x \Rightarrow f(\eta(t,x)) < f(x)$.

Using the deformation theorem, we can prove the following the non-smooth version of symmetric mountain pass theorem as Theorem 9.12 in [10].

Theorem 2.2. *Let E be a reflexive infinite dimensional Banach space and let $I : E \to R$ be a even locally Lipschitz function with the condition (C) and $I(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and I satisfies*

(i) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \bigcap X} \geq \alpha$, and*
(ii) *for each finite dimensional subspace $\tilde{E} \subset E$, there is an $r = r(\tilde{E})$ such that $I \leq 0$ on $\tilde{E} \setminus B_r$,*

then I possesses an unbounded sequence of critical values, where $B_r = \{x \in E : \|x\| < r\}$.

Moreover, the following result appeared in [11].

Theorem 2.3. *Let X be a reflexive Banach space with direct decomposition $X = X_1 \oplus X_2$ and X_1 is finite dimensional. Suppose $f : X \to R$ be a locally Lipschitz function with the P.S. condition. If*

(i) *there exist a constant $r > 0$ such that*

$$f(x) \leq 0, \quad \forall\, x \in X_1 \text{with } \|x\| \leq r$$

and

$$f(x) \geq 0, \quad \forall\, x \in X_2 \text{ with } \|x\| \leq r,$$

(ii) *f is bounded below and $\alpha = \inf_{x \in X} f(x) < 0$, then f has at least two nonzero critical points.*

Using Theorem 2.1, by standard arguments (for example, see [10]), we can prove the following two results.

Theorem 2.4. *Let X be a reflexive Banach space with direct decomposition $X = X_1 \oplus X_2$, $\dim(X_1) < +\infty$ and $f : X \to R$ a locally Lipschitz function with the condition (C) in $f^{-1}(0, +\infty)$. If there exist three constants $r > \rho > 0$, $\beta > 0$ and a point $e \in X_2$ with $\|e\| = 1$ such that*

(i) *$f|_{s_\rho \cap X_2} \geq \beta$, where $s_\rho = \partial B_\rho$, $B_\rho = \{x \in X : \|x\| < \rho\}$,*
(ii) *$f|_{(B_r \cap X_1) \cup (\partial B_r \cap (X_1 \times R^+ e))} \leq 0$,*

then $c = \inf_{\varphi \in \Gamma} \sup_{x \in Q} f(\varphi(x)) \geq \beta$ is a critical value of f, where $Q = \{x_1 + te : x_1 \in X_1, t \geq 0, \|x_1\|^2 + t^2 \leq r^2\}$, $\Gamma = \{\varphi \in C(X, X) : \varphi|_{\partial Q} = id|_{\partial Q}\}$.

Theorem 2.5. *Let X be a reflexive Banach space and $f : X \to R$ a locally Lipschitz function. Assume that there exist a neighborhood U of 0, a point $x_0 \notin U$ and a constant β such that*

$$f(0), f(x_0) < \beta, \qquad f|_{\partial U} \geq \beta.$$

Let $\Gamma = \{\varphi \in C([0, 1], X) : \varphi(0) = 0, \varphi(1) = x_0\}$ and $c = \inf_{\varphi \in \Gamma} \max_{t \in [0,1]} f(\varphi(t))$. Then $c \geq \beta$ and there exists a sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and

$$(1 + \|x_n\|)\lambda(x_n) \to 0.$$

Furthermore, if f satisfies the $(C)_c$ condition, then c is a critical value of f.

3. The variational principle

Obviously, we have the following proposition.

Proposition 3.1. *Suppose that x is a 2π-periodic bouncing solution of (1.1) with isolated zeros $0 < t_1 < t_2 < \cdots < t_r < 2\pi = t_{r+1}$. Set*

$$\tilde{x}(t) = -x(t), \quad t \in [t_{2i-1}, t_{2i}];$$
$$\tilde{x}(t) = x(t), \quad t \in [0, t_1] \cup [t_{2i}, t_{2i+1}],$$

where $i = 1, 2, \ldots, [\frac{r+1}{2}]$, $[x]$ denotes the largest integer which does not exceed x. Then $\tilde{x}$ is a solution of

$$-\ddot{x} = g(t)\dot{x} + f(t, |x|)sgn(x) \tag{1.4}$$

satisfying the periodic boundary condition:

$$\tilde{x}(0) - \tilde{x}(2\pi) = \dot{\tilde{x}}(0) - \dot{\tilde{x}}(2\pi) = 0. \tag{1.5}$$

if r is even; and the anti-periodic boundary condition:

$$\tilde{x}(0) = -\tilde{x}(2\pi), \qquad \dot{\tilde{x}}(0) = -\dot{\tilde{x}}(2\pi). \tag{1.6}$$

if r is odd. Conversely, if $\tilde{x}$ is a 2π-periodic solution of (1.4) satisfying the boundary condition (1.5) or (1.6), and all zeros of $\tilde{x}$ are isolated, then $x = |\tilde{x}|$ is a bouncing 2π-periodic solution of (1.1).

We will state the variational principle for periodic solutions of (1.4).

Let $E = H^1_{2\pi} = \{x : [0, 2\pi] \to R : x \textit{ is absolute continuous}, x(0) = x(2\pi), \dot{x} \in L^2(0, 2\pi)\}$. Then E is a Hilbert space with the norm

$$\|x\|_0 = \left[\int_0^{2\pi} |x(t)|^2 \mathrm{d}t + \int_0^{2\pi} |\dot{x}(t)|^2 \mathrm{d}t\right]^{\frac{1}{2}}.$$

Obviously, the norm $\|\cdot\|_0$ equivalent to the norm defined by

$$\|x\| = \left(\int_0^T \mathrm{e}^{G(t)} |x(t)|^2 \mathrm{d}t + \int_0^T \mathrm{e}^{G(t)} |\dot{x}(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}},$$

where $G(t) = \int_0^t g(s)\mathrm{d}s$. Define the functional I on E, given by

$$I(x) = \frac{1}{2}\int_0^{2\pi} \mathrm{e}^{G(t)} \dot{x}^2 \mathrm{d}t - \int_0^{2\pi} \mathrm{e}^{G(t)} F(t, |x|)\mathrm{d}t$$

where $F(t, |x|) = \int_0^{|x|} f(t, s)\mathrm{d}s$. It is easy to see that the functional I is locally Lipschitz on E. But it may be not continuously differentiable on E.

Theorem 3.2. *Let $f : R \times R^+ \to R$ be a continuous function, $f(t, x)$ be 2π-periodic in t and u be a critical point of the functional I on E.*

(i) *If all zero points of u are isolated, then u is a 2π-periodic solutions of* (1.4) *with* (1.5).

(ii) *If the following conditions hold:*

(a)

$$\liminf_{|x| \to +\infty} F(t, |x|) \geq 0, \quad \textit{uniformly, for a.e. } t \in [0, 2\pi],$$

(b) *there exists $t_0 \in [0, 2\pi]$ such that $u(t_0) = \dot{u}(t_0) = 0$,*

then $u = 0$ on $[0, 2\pi]$. Particularly, if $u \neq 0$, then the zeros of u in $[0, 2\pi]$ are isolated.

(iii) *If the following conditions hold:*

(c)

$$f(t, x) = p(t) + a(t)x + o(x) \quad \textit{as } x \to 0^+,$$

where a and p are 2π-periodic and continuous,

(d) *there exists $t_0 \in [0, 2\pi]$ such that $p(t_0) > 0$ and $u(t_0) = \dot{u}(t_0) = 0$,*

then $u = 0$ on σ_0, where σ_0 is the connect component of the set $\{t \in [0, 2\pi] : p(t) > 0\}$ containing t_0. Particularly, if $u \neq 0$ on σ_0, then the zeros of u in σ_0 are isolated.

Proof. Since $u \in H^1_{2\pi}$ is a critical point of I, $0 \in \partial I(u)$. Set $J(u) = \int_0^{2\pi} \mathrm{e}^{G(t)} F(t, |u(t)|)\mathrm{d}t$. Then

$$\partial J(u) \subset \mathrm{e}^{G(t)}[\underline{f}(t, |u(t)|), \overline{f}(t, |u(t)|)], \quad \text{a.e. } t \in [0, 2\pi],$$

where

$$\underline{f}(t, s) = \min\{\lim_{\tau \to s-0} f(t, |\tau|)\mathrm{sgn}(\tau), \lim_{\tau \to s+0} f(t, |\tau|)\mathrm{sgn}(\tau)\}$$

and

$$\overline{f}(t, s) = \max\{\lim_{\tau \to s-0} f(t, |\tau|)\mathrm{sgn}(\tau), \lim_{\tau \to s+0} f(t, |\tau|)\mathrm{sgn}(\tau)\}.$$

Hence there exists a function $\xi(t) \in [\underline{f}(t, |u(t)|), \overline{f}(t, |u(t)|)]$ such that

$$\int_0^{2\pi} \mathrm{e}^{G(t)}[\dot{u}(t)\dot{v}(t) - \xi(t)v(t)]\mathrm{d}t = 0$$

for every $v \in H^1_T$. By Fundamental Lemma and Remarks. 1 in [12, p. 6–7] we know that $\mathrm{e}^{G(t)}\dot{u}(t)$ has a weak derivative, and

$$[\mathrm{e}^{G(t)}\dot{u}(t)]' = -\mathrm{e}^{G(t)}\xi(t), \quad \text{a.e. on } \in [0, 2\pi], \tag{3.1}$$

$$\mathrm{e}^{G(t)}\dot{u}(t) = -\int_0^t \mathrm{e}^{G(s)}\xi(s)\mathrm{d}s + c, \quad \text{a.e. on } \in [0, 2\pi], \tag{3.2}$$

$$-\int_0^T \mathrm{e}^{G(t)}\xi(t)\mathrm{d}t = 0, \tag{3.3}$$

where c is a constant. We identify the equivalence class $\mathrm{e}^{G(t)}\dot{u}(t)$ and its continuous representant $\int_0^t -\mathrm{e}^{G(s)}\xi(s)\mathrm{d}s + c$. Then $\dot{u}$ is absolutely continuous, and by (3.2), (3.3), one has

$$\dot{u}(0) - \dot{u}(2\pi) = u(0) - u(2\pi) = 0.$$

(i) Notice that all zero points of u are isolated. By (3.1) we know

$$-\ddot{u}(t) = g(t)\dot{u}(t) + f(t, |u(t)|)\mathrm{sgn}(u(t)), \quad \text{a.e. } t \in [0, 2\pi].$$

Hence u is a 2π-periodic solution of (1.4) with (1.5).

(ii) Let $\sigma = \{t \in [0, 2\pi] : u(t) = \dot{u}(t) = 0\}$. Then $\sigma \neq \emptyset$ by (b). For each $t^* \in \sigma$ and for each t near t^*. We can assume $t > t^*$. By (a), adding a positive constant we may assume that F is nonnegative, too. Then

$$H(t) = \frac{1}{2}\mathrm{e}^{G(t)}|\dot{u}(t)|^2 + \mathrm{e}^{G(t)}F(t, |u(t)|) \geq 0.$$

On the other hand, by (3.1) one has

$$\begin{aligned} H(t) - \mathrm{e}^{G(t)}F(t, |u(t)|) &= \frac{1}{2}\mathrm{e}^{G(t)}|\dot{u}(t)|^2 \\ &= \int_{t^*}^{t} \left[\frac{1}{2}\mathrm{e}^{G(s)}|\dot{u}(s)|^2\right]' \mathrm{d}s \\ &= \int_{t^*}^{t} \left[\frac{1}{2}g(s)\mathrm{e}^{G(s)}|\dot{u}(s)|^2 + \mathrm{e}^{G(s)}\ddot{u}(s)\dot{u}(s)\right] \mathrm{d}s \\ &= -\frac{1}{2}\int_{t^*}^{t} g(s)\mathrm{e}^{G(s)}|\dot{u}(s)|^2\mathrm{d}s - \int_{t^*}^{t} \mathrm{e}^{G(s)}\dot{u}(s)\xi(s)\mathrm{d}s \\ &\leq C_1\int_{t^*}^{t} \frac{1}{2}\mathrm{e}^{G(s)}|\dot{u}(s)|^2\mathrm{d}s - \int_{t^*}^{t} \mathrm{e}^{G(s)}\dot{u}(s)\xi(s)\mathrm{d}s. \end{aligned}$$

By the continuity of f and g, there is constant $C_2 > 0$ such that

$$H(t) \leq C_1\int_{t^*}^{t} H(s)\mathrm{d}s + C_2.$$

Whenever $\int_{t^*}^{t} \mathrm{e}^{G(s)}|\dot{u}(s)|^2\mathrm{d}s > 0$, there is constant $C > 0$ such that

$$H(t) \leq C\int_{t^*}^{t} H(s)\mathrm{d}s.$$

By Gronwall Inequality, one has $H(t) \leq 0$ for t near t^*. Hence $H(t) = 0$ for t near t^*, so that $u(t) = u(t^*) = 0$ for t near t^*, a contradiction. Hence $\int_{t^*}^{t} \mathrm{e}^{G(s)}|\dot{u}(s)|^2\mathrm{d}s = 0$. This implies $\dot{u}(t) = 0$ and so $u(t) = u(t^*) = 0$ for t near t^*. This shows σ is a nonempty open set of $[0, 2\pi]$. Moreover, obviously, σ is closed set of $[0, 2\pi]$. Hence $\sigma = [0, 2\pi]$, and hence $u = 0$.

If $u \neq 0$ and the zeros of u in $[0, 2\pi]$ are not isolated, we can assume that t_0 is not a isolated zero, then there exists $\{t_n\}$ with $u(t_n) = 0$ and $t_n \to t_0$. By the definition of derivative we have $\dot{u}(t_0) = 0$. So $u = 0$, a contradiction. Hence the zeros of u in $[0, 2\pi]$ are isolated.

(iii) Let $\sigma_1 = \{t \in \sigma_0 : u(t) = \dot{u}(t) = 0\}$. Then $\sigma_1 \neq \emptyset$. For each $t^* \in \sigma_1$ and for each t near t^*. We can assume $t > t^*$. Set

$$H(t) = \frac{1}{2}\mathrm{e}^{G(t)}|\dot{u}(t)|^2 + \mathrm{e}^{G(t)}p(t^*)|u(t)|.$$

Then $H(t) \geq 0$.

On the other hand, one has

$$H(t) - \mathrm{e}^{G(t)}p(t^*)|u(t)| = \frac{1}{2}\mathrm{e}^{G(t)}|\dot{u}(t)|^2.$$

The rests of the proof are similar to (ii) in corresponding parts. □

4. Existence of a sequence of periodic bouncing solutions

Theorem 4.1. *If there are constants $\theta > 2$, $r > 0$ such that*

$$xf(t, x) \geq \theta F(t, x) > 0, \quad \forall x \geq r, \tag{f_2}$$

then (1.1) has an unbounded sequence of 2π-periodic solutions $\{u_j\}$ satisfying (1.2), (1.3) and for each j, the set $\sigma_j = \{t \in [0, 2\pi] : u_j(t) = 0\}$ is finite.

Proof. Let X_2 be a finite dimensional subspace of E given by

$$X_2 = \left\{ \sum_{j=0}^{k} (a_j \cos jt + b_j \sin jt) | a_j, b_j \in R, j = 0, \ldots, k \right\}$$

and let $X_1 = X_2^{\perp}$. Then $E = X_1 \oplus X_2$. It is obvious that we have

$$\|\dot{x}\|_2^2 \le k^2 \|x\|_2^2 \quad \forall\, x \in X_2.$$
$$\|\dot{x}\|_2^2 \ge (k+1)^2 \|x\|_2^2 \quad \forall\, x \in X_1.$$

By the continuity of f and the boundedness of $e^{G(t)}$ we know that for $|x| \le 1$, there is a constant $M > 0$ such that $e^{G(t)} |F(t, |x|)| \le M|x|$. Set $d_1 = \min_{t\in[0,2\pi]} e^{G(t)}$, $d_2 = \max_{t\in[0,2\pi]} e^{G(t)}$. Then $0 < d_1 \le d_2 < +\infty$. Consequently, for $u \in X_1$ with small $\|u\|$, one has

$$\begin{aligned} I(u) &\ge \frac{1}{2}\int_0^{2\pi} e^{G(t)} |\dot{u}(t)|^2 \mathrm{d}t - M\int_0^{2\pi} |u(t)| \mathrm{d}t \\ &\ge \frac{1}{2} d_1 \int_0^{2\pi} |\dot{u}(t)|^2 \mathrm{d}t - M\int_0^{2\pi} |u(t)| \mathrm{d}t \\ &\ge \frac{1}{2} d_1 \int_0^{2\pi} |\dot{u}(t)|^2 \mathrm{d}t - \frac{\sqrt{2\pi}}{k+1} M \|\dot{u}\|_2 \\ &\ge \frac{1}{4} d_1 \|u\|_0^2 - \frac{\sqrt{2\pi}}{k+1} M \|u\|_0. \end{aligned}$$

Take a small $\rho > 0$ and a integer $k > \frac{4\sqrt{2M\pi}}{d_1\rho} + 1$. Then for $\|u\|_0 = \rho$, one has $I(u) \ge \frac{1}{4} d_1 \rho^2 - \frac{\sqrt{2\pi}}{k+1} M\rho = \alpha_k > 0$.

By (f_2), one has

$$F(t, |x|) \ge c_1 |x|^{\theta} - c_2, \quad \forall (t, x) \in R \times R,$$

where $c_1 > 0$, $c_2 > 0$ are constants. Since all norms are equivalent in a finite dimensional space, for any finite dimensional subspace $X \subset E$ and any $u \in X$, one has

$$\begin{aligned} I(u) &= \frac{1}{2}\int_0^{2\pi} e^{G(t)} |\dot{u}(t)|^2 \mathrm{d}t - \int_0^{2\pi} e^{G(t)} F(t, |u(t)|) \\ &\le \frac{1}{2}\int_0^{2\pi} e^{G(t)} |\dot{u}(t)|^2 \mathrm{d}t - \int_0^{2\pi} e^{G(t)} [c_1 |u(t)|^{\theta} - c_2] \mathrm{d}t \\ &\le c_3 \|u\|_0^2 - c_4 \|u\|_0^{\theta} + c_5, \end{aligned}$$

where c_3, c_4 and c_5 are positive constants. Notice that $\theta > 2$. There is a large $L > 0$ such that $I(u) \le 0$ for all $u \in X$ with $\|u\|_0 \ge L$.

Now, we prove $I(x)$ satisfies the (C) condition.

Assume $\{x_n\} \subset E$ with $\{I(x_n)\}$ is bounded and $(1 + \|x_n\|)\lambda(x_n) \to 0$ as $n \to +\infty$, where $\lambda(x_n) = \min_{\omega\in\partial I(x_n)} \|\omega\|$. Choose $\omega_n \in \partial I(x_n)$ be such that $\|\omega_n\| = \lambda(x_n)$. Then there exists $y_n \in [\underline{f}(t, |x_n(t)|), \overline{f}(t, |x_n(t)|)]$ such that

$$\langle \omega_n, y \rangle = \int_0^{2\pi} e^{G(t)} \dot{x}_n(t)\dot{y}(t) \mathrm{d}t - \int_0^{2\pi} e^{G(t)} y_n(t) y(t) \mathrm{d}t, \quad \forall\, y \in E.$$

Hence

$$\begin{aligned} \theta I(x_n) - \langle \omega_n, x_n \rangle &= \left(\frac{\theta}{2} - 1\right) \int_0^{2\pi} e^{G(t)} |\dot{x}_n(t)|^2 \mathrm{d}t + \int_0^{2\pi} e^{G(t)} [y_n(t) x_n(t) - \theta F(t, |x_n(t)|)] \mathrm{d}t \\ &= \left(\frac{\theta}{2} - 1\right) \int_0^{2\pi} e^{G(t)} |\dot{x}_n(t)|^2 \mathrm{d}t + \int_0^{2\pi} e^{G(t)} [f(t, |x_n(t)|)|x_n(t)| - \theta F(t, |x_n(t)|)] \mathrm{d}t \\ &\ge \left(\frac{\theta}{2} - 1\right) \int_0^{2\pi} e^{G(t)} |\dot{x}_n(t)|^2 \mathrm{d}t - \int_0^{2\pi} e^{G(t)} \max_{|u|\le r} |f(t, |u|)|u| - \theta F(t, |u|)| \mathrm{d}t \\ &= \left(\frac{\theta}{2} - 1\right) \int_0^{2\pi} e^{G(t)} |\dot{x}_n(t)|^2 \mathrm{d}t - c_6 \\ &\ge d_1 \left(\frac{\theta}{2} - 1\right) \int_0^{2\pi} |\dot{x}_n(t)|^2 \mathrm{d}t - c_6. \end{aligned}$$

This implies that $\{\int_0^{2\pi} |\dot{x}_n(t)|^2 \mathrm{d}t\}$ is bounded. Moreover, by

$$I(x_n) \leq \frac{1}{2}\int_0^{2\pi} \mathrm{e}^{G(t)}|\dot{x}_n(t)|^2\mathrm{d}t - \int_0^{2\pi} \mathrm{e}^{G(t)}[c_1|x_n(t)|^\theta - c_2]\mathrm{d}t$$

one has

$$c_1\int_0^{2\pi} \mathrm{e}^{G(t)}|x_n(t)|^\theta \mathrm{d}t \leq d_2\frac{1}{2}\int_0^{2\pi} |\dot{x}_n(t)|^2\mathrm{d}t - I(x_n) + c_2\int_0^{2\pi} \mathrm{e}^{G(t)}\mathrm{d}t,$$

and hence $\{\|x_n\|_{L^\theta}\}$ is bounded. Notice that $\theta > 2$. Hence $\{\|x_n\|_{L^2}\}$ is bounded. Therefore, $\{x_n\}$ is bounded. As the proof 2° of Theorem 4.3 in [13] we may prove that $\{x_n\}$ possess a convergent subsequence. So I satisfies condition (C). $\square$

Obviously, I is even and $I(0) = 0$. By virtue of Theorem 2.2, I has an unbounded sequence of critical points $\{x_j\}$ on E. Consequently, Theorem 4.1 follows from Theorem 3.2 and Proposition 3.1.

Remark 4.1. Theorem 4.1 not only contains Theorem B, but also shows that the condition (f_3) is unnecessary in Theorem B.

Remark 4.2. Using our Theorem 3.2 and Proposition 3.1, as the proof of Theorem 1 in [8] we can prove Theorem A holds for our problem (1.1), too.

Theorem 4.2. *Suppose that the following conditions hold.*

(i) *There exists $r > 2$ such that*

$$\limsup_{s\to+\infty} \frac{|f(t,s)|}{s^{r-1}} < +\infty, \quad \textit{uniformly, for a.e. } t \in [0, 2\pi].$$

(ii) *There exists $\mu > 2$ such that*

$$\liminf_{|x|\to+\infty} \frac{F(t,|x|)}{|x|^\mu} > 0, \quad \textit{uniformly, for a.e. } t \in [0, 2\pi].$$

(iii) *There exists $\nu > r - 2$ such that*

$$\liminf_{|\xi|\to+\infty} \frac{|\xi|f(t,|\xi|) - 2F(t,|\xi|)}{|\xi|^\nu} > 0, \quad \textit{uniformly, for a.e. } t \in [0, 2\pi].$$

Then problem (1.1) has an unbounded sequence of 2π-periodic solutions $\{u_j\}$ satisfying (1.2), (1.3) and for each j, the set $\sigma_j = \{t \in [0, 2\pi] : u_j(t) = 0\}$ is finite.

Proof. By (ii), one has

$$F(t,|x|) \geq c_1|x|^\mu - c_2, \quad \forall x \in R \text{ and a.e. } t \in R,$$

where $c_1 > 0, c_2 > 0$ are constants. Hence all conditions of Theorem 2.2 hold, except the condition (C), by the proof of Theorem 4.1. Therefore, it is enough to prove that I satisfies the condition (C).

Assume $\{x_n\} \subset E$ with $\{I(x_n)\}$ is bounded and $(1+\|x_n\|)\lambda(x_n) \to 0$ as $n \to +\infty$, where $\lambda(x_n) = \min_{\omega\in\partial I(x_n)} \|\omega\|$. Choose $\omega_n \in \partial I(x_n)$ be such that $\|\omega_n\| = \lambda(x_n)$. Then there exists $y_n \in [\underline{f}(t,|x_n(t)|), \overline{f}(t,|x_n(t)|)]$ such that

$$\langle\omega_n, y\rangle = \int_0^{2\pi} \mathrm{e}^{G(t)}\dot{x}_n(t)\dot{y}(t)\mathrm{d}t - \int_0^{2\pi} \mathrm{e}^{G(t)}y_n(t)y(t)\mathrm{d}t, \quad \forall y \in E.$$

Hence, by (iii),

$$\begin{aligned}
2I(x_n) - \langle\omega_n, x_n\rangle &= \int_0^{2\pi} \mathrm{e}^{G(t)}[y_n(t)x_n(t) - 2F(t,|x_n(t)|)]\mathrm{d}t \\
&= \int_0^{2\pi} \mathrm{e}^{G(t)}[f(t,|x_n(t)|)|x_n(t)| - 2F(t,|x_n(t)|)]\mathrm{d}t \\
&\geq b_1\int_0^{2\pi} |x_n(t)|^\nu \mathrm{d}t - b_2,
\end{aligned}$$

where $b_1 > 0, b_2 > 0$ are constants. This implies that $\{\int_0^{2\pi} |x_n(t)|^\nu dt\}$ is bounded. Moreover, by (i)

$$\begin{aligned}\frac{1}{2}\|x_n\|^2 &= I(x_n) + \int_0^{2\pi} e^{G(t)} F(t, |x_n|) dt + \frac{1}{2}\int_0^{2\pi} e^{G(t)} x_n^2 dt \\ &\le M + d_2 \int_0^{2\pi} (c_1 |x_n|^r + M_1 |x_n|) dt + d_2 \frac{1}{2}\int_0^{2\pi} x_n^2 dt \\ &\le \alpha \int_0^{2\pi} |x_n|^r dt + \beta.\end{aligned}$$

If $\nu > r$, by Hölder inequality

$$\int_0^{2\pi} |x_n|^r dt \le (2\pi)^{\frac{\nu-r}{\nu}} \left(\int_0^{2\pi} |x_n|^\nu dt\right)^{\frac{r}{\nu}},$$

and hence $\|x_n\|$ is bounded. If $\nu \le r$, one has

$$\begin{aligned}\int_0^{2\pi} |x_n|^r dt &= \int_0^{2\pi} |x_n|^{r-\nu} \cdot |x_n|^\nu dt \\ &\le \|x_n\|_\infty^{r-\nu} \int_0^{2\pi} |x_n|^\nu dt \\ &\le c_0^{r-\nu} \|x_n\|^{r-\nu} \int_0^{2\pi} |x_n|^\nu dt.\end{aligned}$$

By $r - \nu < 2$, we know that $\|x_n\|$ is bounded, too. Consequently, as the proof 2° of Theorem 4.3 in [13] we may prove that $\{x_n\}$ possess a convergent subsequence. This shows that I satisfies the (C) condition. □

Theorem 4.3. *Suppose that the following conditions hold:*

(i) $\lim_{\xi\to+\infty} \frac{f(t,\xi)}{\xi} = +\infty$, *uniformly, for a.e.* $t \in [0, 2\pi]$;
(ii) $\frac{f(t,\xi)}{\xi}$ *is non-decreased in* ξ *for all* $t \in [0, 2\pi]$;
(iii) *there exists a constant* $r \in (2, +\infty)$ *such that*

$$\lim_{\xi\to+\infty} \frac{f(t,\xi)}{\xi^{r-1}} = 0, \quad \text{uniformly, for all } t \in [0, 2\pi];$$

(iv) $f(t, \xi) \ge (\not\equiv) 0$ *for all* $t \in [0, 2\pi]$.

Then (1.1) *has an unbounded sequence of* 2π*-periodic solutions* $\{u_j\}$ *satisfying* (1.2), (1.3) *and for each* j, *the set* $\sigma_j = \{t \in [0, 2\pi] : u_j(t) = 0\}$ *is finite.*

Proof. Let X_2 be a finite dimensional subspace of E given by

$$X_2 = \left\{\sum_{j=0}^{k_0} (a_j \cos jt + b_j \sin jt) | a_j, b_j \in R, j = 0, \dots, k_0\right\}$$

and let $X_1 = X_2^\perp$. Then $E = X_1 \oplus X_2$. By first section of the proof of Theorem 4.1 we know that for large k_0, there is small $\rho > 0$ such that $I(u) \ge \alpha_{k_0} > 0$ for $u \in X_1$ with $\|u\|_0 = \rho$.

For any finite dimensional subspace $X \subset E$, there is a large integer k such that $X \subset H_k := \{\sum_{j=0}^k (a_j \cos jt + b_j \sin jt) | a_j, b_j \in R, j = 0, \dots, k\}$. Take large $M > 0$ be such that $Md_1 > d_2 k^2$, where $d_1 = \min_{t\in[0,2\pi]} e^{G(t)}$, $d_2 = \max_{t\in[0,2\pi]} e^{G(t)}$. By (i) there is a $r_1 > 0$ such that $f(t, \xi) \ge M\xi$ for all $\xi \ge r_1$ and a.e. $t \in [0, 2\pi]$. Hence, by (iv),

$$f(t, \xi) \ge M\xi - Mr_1, \quad \forall \xi \in R \text{ and a.e. } t \in [0, 2\pi],$$

and hence

$$F(t, |x|) = \int_0^{|x|} f(t, \xi) d\xi \ge \frac{1}{2} M |x|^2 - Mr_1 |x|.$$

Consequently, for any $u \in X$, one has

$$\begin{aligned} I(u) &= \frac{1}{2}\int_0^{2\pi} e^{G(t)}|\dot{u}(t)|^2 dt - \int_0^{2\pi} e^{G(t)} F(t, |u(t)|) \\ &\le \frac{d_2}{2}\int_0^{2\pi} |\dot{u}(t)|^2 dt - d_1 \int_0^{2\pi}\left[\frac{1}{2}M|u(t)|^2 - Mr_1|u|\right] dt \\ &\le \frac{d_2}{2}k^2\|u\|_2^2 - \frac{1}{2}Md_1\|u\|_2^2 + Mr_1 d_1\|u\|_1 \\ &= \frac{1}{2}(d_2k^2 - Md_1)\|u\|_2^2 + Mr_1d_1\|u\|_1. \end{aligned}$$

Since all norms are equivalent in a finite dimensional space, there is an $L = L(X) > 0$ such that $I(u) < 0$ for all $u \in X$ with $\|u\| \ge L$.

Now we prove I satisfies the (C) condition. Assume $\{x_n\} \subset E$ with $\{I(x_n)\}$ is bounded and $(1 + \|x_n\|)\lambda(x_n) \to 0$ as $n \to +\infty$, where $\lambda(x_n) = \min_{\omega \in \partial I(x_n)} \|\omega\|$. Passing a subsequence if necessary, we can assume that $I(x_n) \to c$. Choose $x_n^* \in \partial I(x_n)$ be such that $\|x_n^*\| = \lambda(x_n)$. Then there exists $z_n \in [\underline{f}(t, |x_n(t)|), \bar{f}(t, |x_n(t)|)]$ such that

$$\langle x_n^*, y\rangle = \int_0^{2\pi} e^{G(t)}\dot{x}_n(t)\dot{y}(t)dt - \int_0^{2\pi} e^{G(t)} z_n(t)y(t)dt, \quad \forall\, y \in E.$$

Hence

$$\langle x_n^*, x_n\rangle = \int_0^{2\pi} e^{G(t)}|\dot{x}_n(t)|^2 dt - \int_0^{2\pi} e^{G(t)} z_n(t)x_n(t)dt.$$

We claim that the sequence $\{x_n\}$ is bounded. Otherwise, we can assume $\|x_n\| \to \infty$. Set $\omega_n = \frac{x_n}{\|x_n\|}$. Then $\{\omega_n\}$ is bounded in E. Since $H^1_{2\pi}$ is a Hilbert space, we can assume that there exists $\omega \in H^1_{2\pi}$ such that

$$\omega_n \rightharpoonup \omega \text{ in } H^1_{2\pi},$$

and hence $\{\omega_n\}$ converges uniformly to ω on $[0, 2\pi]$ by Proposition 1.2 in [12]. Set $\Omega = \{t \in [0, 2\pi] : \omega(t) \ne 0\}$. If the measure $|\Omega| \ne 0$, then $|x_n(t)| \to \infty$ for a.e. $t \in \Omega$, and hence

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{1}{\|x_n\|^2}\int_0^{2\pi} e^{G(t)}[z_n(t)x_n(t) + |x_n(t)|^2]dt \\ &\ge \lim_{n\to\infty}\int_\Omega e^{G(t)}\left[\frac{f(t, |x_n|)}{|x_n|} + 1\right]\omega_n^2 dt \\ &= +\infty. \end{aligned}$$

This is a contradiction. Hence $|\Omega| = 0$, namely $\omega(t) = 0$, $a.e.\ t \in [0, 2\pi]$.

By (iii), for any $\varepsilon > 0$, there exists $k > 0$ such that

$$|f(t, \xi)| \le \varepsilon\xi^{r-1}, \quad \forall \xi > k.$$

Since f is continuous, $M = \sup_{(t,\xi)\in[0,2\pi]\times[0,k]} |f(t, \xi)|$ is finite, and hence

$$|f(t, \xi)| \le \varepsilon\xi^{r-1} + M.$$

Therefore,

$$\begin{aligned} F(t, |x|) &\le \int_0^{|x|} |f(t, \xi)|d\xi \\ &\le \int_0^{|x|} (\varepsilon\xi^{r-1} + M)d\xi \\ &= \frac{\varepsilon}{r}|x|^r + M|x|. \end{aligned}$$

Since

$$\begin{aligned} 0 &\le \lim_{n\to+\infty}\int_0^{2\pi} e^{G(t)} F(t, 2\sqrt{c}|\omega_n(t)|)dt \\ &\le \lim_{n\to+\infty}\int_0^{2\pi} d_2\left[\frac{\varepsilon}{r}(2\sqrt{c})^r|\omega_n(t)|^r + M2\sqrt{c}|\omega_n(t)|\right]dt \\ &= 0, \end{aligned}$$

$$\lim_{n\to+\infty}\int_0^{2\pi} e^{G(t)}F(t, 2\sqrt{c}|\omega_n(t)|)dt = 0.$$

Hence,

$$\begin{aligned}\lim_{n\to+\infty} I(2\sqrt{c}\omega_n) &= \lim_{n\to+\infty}\left[2c\int_0^{2\pi} e^{G(t)}|\dot{\omega}_n|^2dt - \int_0^{2\pi} e^{G(t)}F(t, 2\sqrt{c}|\omega_n(t)|)dt\right]\\ &= 2c\lim_{n\to+\infty}\left(\|\omega_n\|^2 - \int_0^{2\pi} e^{G(t)}|\omega_n|^2dt\right)\\ &= 2c.\end{aligned}$$

Since

$$|\langle x_n^*, x_n\rangle| \le \lambda(x_n)(1+\|x_n\|) \to 0,$$
$$\int_0^{2\pi} e^{G(t)}|\dot{x}_n(t)|^2dt - \int_0^{2\pi} e^{G(t)}z_n(t)x_n(t)dt \to 0.$$

Hence we may assume that

$$-\frac{1}{n} < \int_0^{2\pi} e^{G(t)}|\dot{x}_n(t)|^2dt - \int_0^{2\pi} e^{G(t)}z_n(t)x_n(t)dt < \frac{1}{n}, \quad \forall\, n \ge 1.$$

For any $s \ge 0$,

$$\begin{aligned} I(sx_n) &= \frac{1}{2}s^2\int_0^{2\pi} e^{G(t)}|\dot{x}_n(t)|^2dt - \int_0^{2\pi} e^{G(t)}F(t, |sx_n(t)|)dt\\ &< \frac{1}{2}s^2\left[\frac{1}{n} + \int_0^{2\pi} e^{G(t)}z_n(t)x_n(t)dt\right] - \int_0^{2\pi} e^{G(t)}F(t, |sx_n(t)|)dt\\ &= \frac{s^2}{2n} + \frac{1}{2}s^2\int_0^{2\pi} e^{G(t)}z_n(t)x_n(t)dt - \int_0^{2\pi} e^{G(t)}F(t, |sx_n(t)|)dt\\ &= \frac{s^2}{2n} + \int_{\{t:\, t\in[0,2\pi],\ x_n(t)\neq 0\}} e^{G(t)}\left[\frac{1}{2}s^2|x_n(t)|f(t, |x_n(t)|) - F(t, |sx_n(t)|)\right]dt.\end{aligned}$$

For any fixed $t \in [0, 2\pi]$ and positive integer n, set

$$h(s) = \frac{1}{2}s^2|x_n(t)|f(t, |x_n(t)|) - F(t, |sx_n(t)|).$$

Then the function $h(s)$ is absolutely continuous in any closed interval $[a, b] \subset [0, +\infty)$, and differentiable almost everywhere in $(0, +\infty)$ and one has

$$\frac{d}{ds}h(s) = s|x_n|f(t, |x_n|) - f(t, |sx_n|)|x_n|.$$

Hence, whenever $0 \le s_1 \le 1$ and $x_n(t) \neq 0$, by (ii) we have

$$\begin{aligned} h(1) - h(s_1) &= \int_{s_1}^1 \frac{d}{ds}h(s)ds\\ &= \frac{1}{2}(1-s_1^2)|x_n|f(t, |x_n|) - \int_{s_1}^1 f(t, s|x_n|)|x_n|ds\\ &= \frac{1}{2}(1-s_1^2)|x_n|f(t, |x_n|) - \int_{s_1}^1 \frac{f(t, s|x_n|)}{s|x_n|}s|x_n|^2ds\\ &\ge \frac{1}{2}(1-s_1^2)|x_n|f(t, |x_n|) - \frac{1}{2}(1-s_1^2)|x_n|f(t, |x_n|)\\ &= 0,\end{aligned}$$

i.e. $h(s_1) \le h(1)$ for all $s_1 \in [0, 1]$. Consequently, for all $s \in [0, 1]$, one has

$$I(sx_n) \le \frac{s^2}{2n} + \int_{\{t:\, t\in[0,2\pi],\ x_n(t)\neq 0\}} e^{G(t)}\left[\frac{1}{2}|x_n(t)|f(t, |x_n(t)|) - F(t, |x_n(t)|)\right]dt.$$

On the other hand,

$$\begin{aligned} I(x_n) &= \frac{1}{2}\int_0^{2\pi} e^{G(t)}|\dot{x}_n(t)|^2 dt - \int_0^{2\pi} e^{G(t)}F(t,|x_n(t)|)dt \\ &> \frac{1}{2}\left[\int_0^{2\pi} e^{G(t)}z_n(t)x_n(t)dt - \frac{1}{n}\right] - \int_0^{2\pi} e^{G(t)}F(t,|x_n(t)|)dt \\ &= -\frac{1}{2n} + \frac{1}{2}\int_0^{2\pi} e^{G(t)}z_n(t)x_n(t)dt - \int_0^{2\pi} e^{G(t)}F(t,|x_n(t)|)dt \\ &= -\frac{1}{2n} + \int_{\{t:\, t\in[0,2\pi],\ x_n(t)\neq 0\}} e^{G(t)}\left[\frac{1}{2}|x_n(t)|f(t,|x_n|) - F(t,|x_n(t)|)\right]dt. \end{aligned}$$

Hence, for all $s \in [0, 1]$, one has

$$I(sx_n) \le \frac{1+s^2}{2n} + I(x_n).$$

Consequently, for large n, one has

$$I(2\sqrt{c}\omega_n) = I\left(\frac{2\sqrt{c}}{\|x_n\|}x_n\right) \le \frac{1}{2n}\left(1 + \frac{4c}{\|x_n\|^2}\right) + I(x_n)$$

and hence $2c \le c$. This contradicts that $c > 0$. Hence the sequence $\{x_n\}$ is bounded in E. Consequently, as the proof 2^0 of Theorem 4.3 in [13] we may prove that $\{x_n\}$ possess a convergent subsequence, and hence I satisfies the (C) condition. □

By virtue of Theorem 2.2, I has an unbounded sequence of critical points $\{x_j\}$ on E. Consequently, Theorem 4.3 follows from Theorem 3.2 and Proposition 3.1.

Remark. In fact, there are many functions satisfying all conditions of Theorems 4.2 and 4.3, respectively. For example:

$$F(t,|x|) = |x| + e^{\sin t}\cdot|x|^3 \quad \text{or} \quad f(t,|x|) = 1 + 3e^{\sin t}\cdot|x|^2.$$

5. Existence and multiplicity of nonzero periodic bouncing solutions

Theorem 5.1. *Let $f > 0$ and there exist $p, q \in L^1([0, 2\pi], R^+)$, $\alpha \in [0, 1)$ such that*

$$\lim_{|x|\to\infty} |x|^{-2\alpha}\int_0^{2\pi} F(t,|x|)dt \to +\infty$$

and

$$f(t,x) \le p(t)|x|^\alpha + q(t)$$

for all $x \in R$ and a.e. $t \in [0, 2\pi]$. Assume that there exist $r > 0$ and an integer $k \ge 1$, such that

$$e^{G(t)}F(t,|x|) \le \frac{1}{2}(k+1)^2 d_1|x|^2 \tag{f_4}$$

for all $|x| \le r$ and a.e. $t \in [0, 2\pi]$, where $d_1 = \min_{t\in[0,2\pi]} e^{G(t)}$. Then problem (1.1) has at least two distinct nonzero solutions x_i $(i = 1, 2)$ satisfying

(1) *$x_i(t) \ge 0$, for $t \in R$;*
(2) *for each i, the set $\sigma_i = \{t \in [0, 2\pi] : x_i(t) = 0\}$ is finite;*
(3) *$\dot{x}_i(t^-) = -\dot{x}_i(t^+)$ for all $t \in \sigma_i$.*

Proof. Let X_2 be a finite dimensional subspace of E given by

$$X_2 = \left\{\sum_{j=0}^{k}(a_j\cos jt + b_j\sin jt) | a_j, b_j \in R, j = 0,\ldots,k\right\}$$

and let $X_1 = X_2^{\perp}$. Then $E = X_1 \oplus X_2$. It is obvious that we have

$$\|\dot{x}\|_2^2 \le k^2\|x\|_2^2 \quad \forall\, x \in X_2.$$
$$\|\dot{x}\|_2^2 \ge (k+1)^2\|x\|_2^2 \quad \forall\, x \in X_1.$$

By $f > 0$, the continuity of f and the boundedness of $e^{G(t)}$ in $[0, 2\pi]$ we know that for $|x| \leq 1$, there is a constant $m > 0$ such that $e^{G(t)}F(t, |x|) \geq m|x|$ for all $t \in [0, 2\pi]$. Set $d_2 = \max_{t\in[0,2\pi]} e^{G(t)}$. Then $0 < d_1 \leq d_2 < +\infty$. Consequently, by the equivalence of norms in a finite dimensional space, for $u \in X_2$ with small $\|u\|_0$, one has

$$\begin{aligned} I(u) &\leq \frac{1}{2}\int_0^{2\pi} e^{G(t)}|\dot{u}(t)|^2 dt - m\int_0^{2\pi} |u(t)|dt \\ &\leq \frac{1}{2}d_2\int_0^{2\pi} |\dot{u}(t)|^2 dt - m\int_0^{2\pi} |u(t)|dt \\ &\leq \frac{1}{2}d_2\|u\|_0^2 - c\|u\|_0 \\ &\leq 0. \end{aligned}$$

Moreover, by (f_4) one has

$$I(u) \geq \frac{1}{2}d_1\int_0^{2\pi} \dot{u}^2 dt - \frac{1}{2}d_1(k+1)^2\int_0^{2\pi} u^2 dt \geq 0$$

for all $u \in X_1$ with $\|u\|_0 \leq c_0^{-1}r$, where c_0 is the optimum positive constant satisfying $\|x\|_\infty \leq c_0\|x\|_0 (\forall x \in H^1_{2\pi})$. For each $u \in H^1_{2\pi}$, set $u(t) = \bar{u} + \tilde{u}(t)$, where $\bar{u} = \frac{1}{2\pi}\int_0^{2\pi} u(t)dt$. Since

$$f(t, x) \leq p(t)|x|^\alpha + q(t),$$

by the mean value theorem for locally Lipschitz function, one has

$$\begin{aligned} \int_0^{2\pi} e^{G(t)}|F(t, |u(t)|) - F(t, |\bar{u}|)|dt &\leq d_2\int_0^{2\pi} [p(t)(|\bar{u}| + |\tilde{u}(t)|)^\alpha + q(t)]|\tilde{u}(t)|dt \\ &\leq d_2\|\tilde{u}\|_\infty(|\bar{u}| + \|\tilde{u}\|_\infty)^\alpha\int_0^{2\pi} p(t)dt + d_2\left\|\int_0^{2\pi} q(t)dt\right\| \\ &\leq 2c_1d_2\|\tilde{u}\|_\infty(|\bar{u}|^\alpha + \|\tilde{u}\|_\infty^\alpha) + c_2d_2\|\tilde{u}\|_\infty \\ &\leq \frac{2\pi}{3d_1}c_1^2d_2^2|\bar{u}|^{2\alpha} + \frac{3}{2\pi}d_1\|\tilde{u}\|_\infty^2 + 2c_1d_2\|\tilde{u}\|_\infty^{\alpha+1} + c_2d_2\|\tilde{u}\|_\infty \\ &\leq \frac{1}{4}d_1\|\dot{u}\|_2^2 + c_3|\bar{u}|^{2\alpha} + c_4\|\dot{u}\|_2^{\alpha+1} + c_5\|\dot{u}\|_2. \end{aligned}$$

Hence

$$I(u) \geq \frac{d_1}{4}\|\dot{u}\|_2^2 - c_4\|\dot{u}\|_2^{\alpha+1} + d_1|\bar{u}|^{2\alpha}\left(|\bar{u}|^{-2\alpha}\int_0^{2\pi} F(t, |\bar{u}|)dt - c_6\right) - c_5\|\dot{u}\|_2.$$

Notice that $\|u\|_0 \to \infty$ if and only if $(|\bar{u}|^2 + \|\dot{u}\|_2^2)^{\frac{1}{2}} \to \infty$. Therefore,

$$\lim_{\|u\|_0\to\infty} I(u) = +\infty.$$

Hence $I(x)$ is bounded from below and every P.S sequence $\{x_n\} \subset H^1_{2\pi}$ is bounded. Consequently, as the proof 2° of Theorem 4.3 in [13] we may prove that $\{x_n\}$ possess a convergent subsequence, and hence I satisfies the P.S condition and is bounded from below. If $\inf_{x\in H^1_{2\pi}} I(x) < 0$, then, by Theorem 2.3, I possesses tow nonzero critical points in E. If $\inf_{x\in H^1_{2\pi}} I(x) \geq 0$ then $I(x) = 0$ for all $x \in X_2$ with small $\|x\|_0$, which implies that all $x \in X_2$ with small $\|x\|_0$ are critical points of I. Consequently, Theorem 5.1 follows from Theorem 3.2 and Proposition 3.1. □

Theorem 5.2. *Suppose that the following conditions hold:*

(i)

$$\liminf_{|x|\to+\infty} \frac{F(t, |x|)}{|x|^2} > \frac{d_2}{2}$$

for a.e. $t \in [0, 2\pi]$, *where* $d_2 = \max_{t\in[0,2\pi]}$;

(ii) *there exist* $r > 2$ *and* $\mu > r - 2$ *such that*

(a) $\limsup_{|x|\to+\infty} \frac{f(t,|x|)}{|x|^{r-1}} < +\infty,$ *for a.e.* $t \in [0, 2\pi]$;

(b) $\liminf_{|\xi|\to+\infty} \frac{|\xi|f(t,|\xi|)-2F(t,|\xi|)}{|\xi|^\mu} > 0,$ *for a.e.* $t \in [0, 2\pi]$.

Then problem (1.1) has at least one nonzero periodic solution x satisfying (1.2) and (1.3).

Proof. By (ii)(a) there exist $c_1 > 0$ and $k_1 > 0$ such that

$$F(t, |x|) \le c_1|x|^r, \quad \forall |x| \ge k_1 \text{ and a.e. } t \in [0, 2\pi].$$

Set $M = \max_{(t,s)\in[0,2\pi]\times[0,k_1]} |f(t, s)|$. Then

$$F(t, |x|) \le \int_0^{|x|} |f(t, s)|\mathrm{d}s \le M|x|.$$

Therefore

$$F(t, |x|) \le c_1|x|^r + M|x|.$$

Let $X_2 = \{x \in E | \int_0^{2\pi} x(t)\mathrm{d}t = 0\}$ and $X_1 = X_2^{\perp} = R$. Then $E = X_1 \oplus X_2$. For all $x \in X_2$, by Wirtinger inequality, we have

$$\|x\|_2^2 \le \|\dot{x}\|_2^2, \qquad \|x\|_0^2 \le 2\|\dot{x}\|_2^2 \quad \text{and} \quad \int_0^{2\pi} |x|\mathrm{d}t \le \alpha_2 \int_0^{2\pi} |x|^r\mathrm{d}t.$$

Hence

$$\begin{aligned} I(x) &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^{G(t)}\dot{x}^2\mathrm{d}t - \int_0^{2\pi} \mathrm{e}^{G(t)}F(t, |x|)\mathrm{d}t \\ &\ge \frac{d_1}{4}\|x\|_0^2 - c_1d_2\int_0^{2\pi} |x|^r\mathrm{d}t - Md_2\int_0^{2\pi} |x|\mathrm{d}t \\ &\ge \frac{d_1}{4}\|x\|_0^2 - d_2(c_1 + \alpha_2M)\int_0^{2\pi} |x|^r\mathrm{d}t \\ &\ge \frac{d_1}{4}\|x\|_0^2 - d_3\|x\|^r. \end{aligned}$$

Hence there exist $b > 0$ and $\rho \in (0, 1)$ such that $I(x) \ge b$ for all $x \in X_2$ with $\|x\| = \rho$, and hence,

$$I|_{X_2\cap\partial B_\rho} \ge b.$$

By (i) there exist ε_0 and $k_2 > 0$ such that

$$F(t, |x|) \ge \left(\frac{d_2}{2} + \varepsilon_0\right)|x|^2, \quad \forall |x| \ge k_2.$$

Since f is continuous, by the above inequality, adding a positive constant we may assume that F is nonnegative, too. Hence

$$F(t, |x|) \ge \left(\frac{d_2}{2} + \varepsilon_0\right)|x|^2 - \left(\frac{d_2}{2} + \varepsilon_0\right)k_2^2.$$

Choose $e = \frac{1}{\sqrt{2\pi}}\sin t \in X_2$ and $\|e\|_0 = 1$. Let $E_1 = R \oplus \mathrm{span}\{e\}$. Then $\dim(E_1) < \infty$. Therefore there exists $\delta > 0$ such that

$$\int_0^{2\pi} |x|^2\mathrm{d}t \ge \delta\|x\|_0^2, \quad \forall\, x \in E_1.$$

Let $Q = \{x + se : x \in R, s \ge 0, \|x\|_0^2 + s^2 \le r_1^2\}$, where $r_1 = \max\{2, (\delta\varepsilon_0)^{-\frac{1}{2}}c_3^{\frac{1}{2}}\}$ and $c_3 = d_2(\frac{d_2}{2} + \varepsilon_0)k_2^2 \cdot 2\pi$. Then for every $x + se \in Q$, one has

$$\begin{aligned} I(x + se) &= \frac{1}{2}\int_0^{2\pi} \mathrm{e}^{G(t)}|s\dot{e}(t)|^2\mathrm{d}t - \int_0^{2\pi} \mathrm{e}^{G(t)}F(t, |x + se|)\mathrm{d}t \\ &\le \frac{d_2}{2}\int_0^{2\pi} |x + se|^2\mathrm{d}t - \left(\frac{d_2}{2} + \varepsilon_0\right)\int_0^{2\pi} |x + se|^2\mathrm{d}t + c_3 \\ &= -\varepsilon_0\int_0^{2\pi} |x + se|^2\mathrm{d}t + c_3 \\ &\le -\varepsilon_0\delta\|x + se\|_0^2 + c_3. \end{aligned}$$

Hence $\sup_{u\in\partial Q} I(x) \le 0$.

Moreover, from the proof of Theorem 4.2, we know that I satisfies the condition (C). So $c = \inf_{\varphi\in\Gamma}\sup_{x\in Q} I(\varphi(x)) \ge b > 0$ is a critical value of I by Theorem 2.4, where $\Gamma = \{\varphi \in C(E, E) : \varphi|_{\partial Q} = id|_{\partial Q}\}$. Consequently, Theorem 5.2 follows from Theorem 3.2 and Proposition 3.1. □

Theorem 5.3. *Assume the following conditions hold:*

(i) $\lim_{\xi\to 0^+}\frac{f(t,\xi)}{\xi}=m(t)$,

(ii) $\lim_{\xi\to +\infty}\frac{f(t,\xi)}{\xi}=q(t)$,

where, $q(t)>0$, $m(t),\ q(t)\in L^\infty(0,2\pi)$. *If*

$$\Lambda=\inf\left\{\int_0^{2\pi}\mathrm{e}^{G(t)}\dot{x}^2\mathrm{d}t : x\in H^1_{2\pi},\int_0^{2\pi}q(t)\mathrm{e}^{G(t)}x^2\mathrm{d}t=1\right\}<1,$$

then problem (1.1) has at least one nonzero solution x satisfying

(1) $x(t)\geq 0$, *for* $t\in R$;

(2) *The set* $\sigma=\{t\in[0,2\pi]:x(t)=0\}$ *is finite;*

(3) $\dot{x}(t^-)=-\dot{x}(t^+)$ *for each* $t\in\sigma$.

Proof. Since $\Lambda<1$, there is $\varepsilon>0$ such that $\Lambda+\varepsilon<1$. By the definition of Λ there exists a $x_0\in H^1_{2\pi}$ with $\int_0^{2\pi}q(t)\mathrm{e}^{G(t)}x_0^2\mathrm{d}t=1$ such that $\int_0^{2\pi}\mathrm{e}^{G(t)}\dot{x_0}^2\mathrm{d}t<\Lambda+\varepsilon$.

Hence

$$\begin{aligned}\lim_{s\to+\infty}\frac{I(sx_0)}{s^2}&=\frac{1}{2}\int_0^{2\pi}\mathrm{e}^{G(t)}|\dot{x_0}|^2\mathrm{d}t-\lim_{s\to+\infty}\int_0^{2\pi}\mathrm{e}^{G(t)}\frac{F(t,s|x_0|)}{s^2}\mathrm{d}t\\&\leq\frac{1}{2}\int_0^{2\pi}\mathrm{e}^{G(t)}|\dot{x_0}|^2\mathrm{d}t-\int_0^{2\pi}\lim_{s\to+\infty}\mathrm{e}^{G(t)}\frac{F(t,s|x_0|)}{s^2}\mathrm{d}t\\&=\frac{1}{2}(\Lambda+\varepsilon)-\frac{1}{2}\int_0^{2\pi}\mathrm{e}^{G(t)}q(t)|x_0|^2\mathrm{d}t\\&=\frac{1}{2}(\Lambda+\varepsilon-1)\\&<0.\end{aligned}$$

Hence there exists large $s_0>0$ such that $\|s_0x_0\|>\rho$ and $I(s_0x_0)<0$.

By (ii), for any $\varepsilon>0$, there exists $k>1$ such that

$$|f(t,\xi)|\leq(\|q\|_\infty+\varepsilon)\xi\leq(\|q\|_\infty+\varepsilon)\xi^{r-1},\quad\forall\,\xi>k(r>2).$$

Since f is continuous, $M_1=\sup_{(t,\xi)\in[0,2\pi]\times[0,k]}|f_1(t,\xi)|$ is finite, and hence

$$|f(t,\xi)|\leq(\|q\|_\infty+\varepsilon)\xi^{r-1}+M_1.$$

Hence,

$$\begin{aligned}F(t,|x|)&=\int_0^{|x|}f(t,\xi)\mathrm{d}\xi\\&\leq\int_0^{|x|}[(\|q\|_\infty+\varepsilon)\xi^{r-1}+M_1]\mathrm{d}\xi\\&=\frac{1}{r}(\|q\|_\infty+\varepsilon)|x|^r+M_1|x|.\end{aligned}$$

Therefore,

$$\begin{aligned}I(x)&=\frac{1}{2}\int_0^{2\pi}\mathrm{e}^{G(t)}\dot{x}^2\mathrm{d}t-\int_0^{2\pi}\mathrm{e}^{G(t)}F(t,|x|)\mathrm{d}t\\&\geq\frac{d_1}{2}\left[\int_0^{2\pi}\dot{x}^2\mathrm{d}t+\int_0^{2\pi}|x|^2\mathrm{d}t\right]-d_2\left[\frac{1}{r}(\|q\|_\infty+\varepsilon)\int_0^{2\pi}|x|^r\mathrm{d}t-M_1\int_0^{2\pi}|x|\mathrm{d}t-\frac{1}{2}\int_0^{2\pi}|x|^2\mathrm{d}t\right]\\&\geq\frac{d_1}{2}\|x\|_0^2-c\int_0^{2\pi}|x|^r\mathrm{d}t\\&\geq\frac{1}{2}\|x\|_0^2-2\pi c\|x\|_\infty^r\\&\geq\frac{1}{2}\|x\|_0^2-c_1\|x\|_0^r,\end{aligned}$$

where constants $c>0$, $c_1>0$. Since $r>2$, so there exists small $\rho>0$ such that $I(x)\geq\frac{1}{4}:=\beta$ for all $x\in E$ with $\|x\|=\rho$.

Set $\Gamma = \{\varphi \in C([0,1], H^1_{2\pi}) : \varphi(0) = 0, \varphi(1) = s_0 x_0\}$, $c = \inf_{\varphi\in\Gamma} \max_{t\in[0,1]} I(\varphi(t))$. Then $c \geq \beta > 0$.

Now, we prove I satisfies the $(C)_c$ condition. Indeed, for any sequence $\{x_n\} \subset E$ with $I(x_n) \to c$ and $(1+\|x_n\|)\lambda(x_n) \to 0$, we shall prove that $\{x_n\}$ has a convergent subsequence.

Choose $x^*{}_n \in \partial I(x_n)$ be such that $\|x^*{}_n\| = \lambda(x_n)$. Then there exists $z_n \in [\underline{f}(t, |x_n(t)|), \overline{f}(t, |x_n(t)|)]$ such that

$$\langle x^*{}_n, y\rangle = \int_0^{2\pi} e^{G(t)} \dot{x}_n(t)\dot{y}(t)dt - \int_0^{2\pi} e^{G(t)} z_n(t) y(t)dt, \quad \forall\, y \in E.$$

Hence

$$\langle x^*{}_n, x_n\rangle = \int_0^{2\pi} e^{G(t)} |\dot{x}_n(t)|^2 dt - \int_0^{2\pi} e^{G(t)} z_n(t) x_n(t)dt, \quad \forall\, y \in E.$$

We claim that the sequence $\{x_n\}$ is bounded in E. Otherwise, we can assume $\|x_n\| \to \infty$. Set $\omega_n = \frac{x_n}{\|x_n\|}$. Then $\{\omega_n\}$ is bounded. Since $H^1_{2\pi}$ is a Hilbert space, we can assume that there exists $\omega \in H^1_{2\pi}$ such that

$$\omega_n \rightharpoonup \omega \text{ in } H^1_{2\pi}, \qquad \omega_n \to \omega \text{ in } L^2([0, 2\pi]).$$

By (i), for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(t,\xi)| \leq (\|m\|_\infty + \varepsilon)\xi, \quad \forall\, 0 \leq \xi < \delta.$$

For the above ε, there exists $k_0 > \delta$ such that

$$|f(t,\xi)| < (\|q\|_\infty + \varepsilon)\xi, \quad \forall\, \xi > k_0.$$

Since f is continuous, there exists $c_1 > 0$ such that

$$|f(t,\xi)| \leq c_1\xi, \quad \forall\, (t,\xi) \in [0, 2\pi] \times [0, +\infty).$$

If $\omega = 0$, then

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{1}{\|x_n\|^2} \int_0^{2\pi} e^{G(t)} [z_n(t)x_n(t) + |x_n(t)|^2]dt \\ &\leq d_2(c_1+1) \lim_{n\to\infty} \int_0^{2\pi} \omega_n^2 dt \\ &= 0. \end{aligned}$$

This is a contradiction. Hence $\omega \neq 0$. Consequently, for a.e. $t \in [0, 2\pi]$, $x_n(t) \to +\infty$ (as $n \to +\infty$). Hence, for a.e. $t \in [0, 2\pi]$, $\lim_{n\to\infty} \frac{f(t,x_n(t))}{x_n(t)} = q(t)$ and $\lim_{n\to\infty} \underline{f}(t, |x_n(t)|) = \lim_{n\to\infty} \overline{f}(t, |x_n(t)|) = \lim_{n\to\infty} f(t, |x_n(t)|)$. Set

$$p_n(t) = \begin{cases} \dfrac{z_n(t)}{x_n(t)}, & x_n(t) \neq 0, \\ 0, & x_n(t) = 0. \end{cases}$$

Since $|p_n(t)| \leq c_1$, one has, for each $\varphi(t) \in H^1_{2\pi}$,

$$\begin{aligned} \int_0^{2\pi} e^{G(t)} p_n(t)\omega_n(t)\varphi(t)dt &= \int_0^{2\pi} e^{G(t)} p_n(t)[\omega_n(t) - \omega(t)]\varphi(t)dt + \int_0^{2\pi} e^{G(t)} p_n(t)\omega(t)\varphi(t)dt \\ &\to \int_0^{2\pi} e^{G(t)} q(t)\omega(t)\varphi(t)dt. \end{aligned}$$

Moreover,

$$\begin{aligned} \left| \int_0^{2\pi} e^{G(t)} \dot{\omega}_n\dot{\varphi}dt - \int_0^{2\pi} e^{G(t)} p_n(t)\omega_n(t)\varphi(t)dt \right| &= \frac{1}{\|x_n\|} |\langle x_n^*, \varphi\rangle| \\ &\leq \frac{1}{\|x_n\|} \lambda(x_n)\|\varphi\| \to 0. \end{aligned}$$

Notice that

$$\omega_n \rightharpoonup \omega \text{ in } H^1_{2\pi}, \qquad \omega_n \to \omega \text{ in } L^2([0, 2\pi]).$$

We have

$$\lim_{n\to\infty} \int_0^{2\pi} e^{G(t)} [\dot{\omega}_n(t) - \dot{\omega}(t)]\dot{\varphi}(t)dt = 0.$$

Hence

$$\int_0^{2\pi} \mathrm{e}^{G(t)} \dot{\omega}\dot{\varphi}\mathrm{d}t - \int_0^{2\pi} \mathrm{e}^{G(t)} q(t)\omega(t)\varphi(t)\mathrm{d}t = 0.$$

This contradicts that $\Lambda < 1$. Therefore, $\{x_n\}$ is bounded. Consequently, as the proof 2^0 of Theorem 4.3 in [13] we may prove that $\{x_n\}$ possesses a convergent subsequence and hence I satisfies the $(C)_c$ condition. By Theorem 2.5 we know that c is a critical value of I. Consequently, Theorem 5.3 follows from Theorem 3.2 and Proposition 3.1. □

References

[1] D. Bonheure, C. Fabry, Periodic motions in impact oscillators with perfectly elastic bounces, Nonlinearity 15 (2002) 1281–1297.
[2] C.J. Budd, The global dynamics of impact oscillators, in: Real and Complex Dynamical Systems, Nato Adv. Sci. Inst, Series C, Math. Phys. Sci. 644 (1995) 27–46.
[3] C.J. Budd, F. Dux, Intermittency in impact oscillators close to resonance, Nonlinearity 7 (1994) 1191–1224.
[4] H. Lamba, Chaotic regular and unbounded behavior in the elastic impact oscillator, Physica D 82 (1995) 117–135.
[5] A.C. Lazer, P.J. Mckenna, Periodic bouncing for a forced linear spring with obstacle, Differential Integral Equations 5 (1992) 165–172.
[6] R. Ortega, Dynamics of a forced oscillator with obstacle, in: Variantional and Topological Methods in the study of Nonlinear Phenomena, Birkhäuser, Boston-Based-Berlin, 2002, pp. 75–87.
[7] V. Zharnitsky, Invariant tori in Hamiltonian systems with impacts, Comm. Math. phys. 211 (2000) 289–302.
[8] M.Y. Jiang, Periodic solutions of second order differential equations with an obstacle, Nonlinearity 19 (2006) 1165–1183.
[9] N.C. Kourogenis, N.S. Papageorgiou, Nonsmooth critical point theory and nonlinear elliptic equations at resonance, J. Aust. Math. Soc. 69 (2000) 245–271.
[10] P.H. Rabinowitz, Minimax Methods in critical point theory with applications to differential equations, in: CBMS, Regional Conf. Ser. in Math. vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[11] X. Wu, A new critical point theorem for locally Lipschitz functionals with applications to differential equations, Nonlinear Anal. 66 (2007) 624–638.
[12] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York Berlin Heidelberg, London Paris Tokyo, 1989.
[13] K.-C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. 80 (1981) 103–129.