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# Integral transform solution of boundary layer equations in stream function-only formulation

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**Abstract**

The boundary layer equations for steady incompressible laminar channel flow are solved through the integral transform method, by adopting the stream function-only formulation of the governing equations, instead of the more commonly used primitive variables formulation. This hybrid numerical–analytical approach provides benchmark results under user-prescribed accuracy targets and is recognized in the validation of purely numerical schemes. The relative merits of the stream function formulation are illustrated through numerical results for the convergence behavior in the case of a plane Poiseuille flow. 

*Keywords:* Integral transforms; Boundary layer equations; Hybrid methods; Channel flow

## 1. Introduction

The boundary layer equations remain an extremely important model in the analysis of various flow problems of practical relevance in the engineering sciences. Several numerical approaches have been proposed along the last few decades, for a stable and consistent solution of this set of equations, which even in the present days compete for the researchers' and engineers' preference in applications [1,2]. In the vast majority of related developments, the primitive variables formulation was adopted as the most suitable one for discrete approaches associated with the approximation of the velocity components' partial derivatives.

More recently, the ideas on classical analytical methods have experienced a revival, in connection with the proposition of novel hybrid numerical–analytical schemes for partial differential systems. One such trend is related to the so-called generalized integral transform technique, as reviewed in Refs. [3–5], which is an eigenfunction expansion approach that yields a computational procedure with automatic global error control and mild cost increase in multidimensional situations.

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This approach, over the last few years, proved itself as a powerful benchmarking tool and a potential alternative to numerical methods in various applications, including linear and non-linear diffusion, convection–diffusion and eigenvalue problems, as well as in the solution of the boundary layer and Navier–Stokes equations. Its use in the solution of the boundary layer equations under the primitive variables formulation is well established in different contributions [6–8]. In parallel, and even more successfully, the integral transform approach was employed to solve the full Navier–Stokes equations, for both cavity and channel flows [9–13]. But, in this case, the vorticity transport equation in the stream function-only formulation was preferred. The evident question was then raised as to whether the equivalent stream function-only form of the boundary layer equations, when handled through the integral transform method, would offer advantages over the primitive variables formulation, in terms of improved convergence behavior. Essentially, the change in formulation of the governing equations alters the differential operators involved and, therefore, the choice of auxiliary eigenvalue problem associated with the expansion basis. The present study addresses this question and explores the relative merits of the stream function-only formulation, which results in a more natural choice for eigenfunction expansion-type approaches. This note is first of all aimed at illustrating the solution of the boundary layer equations, through a spectral-type approach, in the stream function-only formulation. These conclusions are supported by a numerical example for developing laminar flow in a plane Poiseuille flow, as follows.

## 2. Analysis

In the present study, steady incompressible laminar flow of a Newtonian fluid, developing within a parallel plates channel is studied. The channel is assumed infinite in the $z$ direction and semi-infinite in the direction of positive $x$. The fluid enters the channel with a uniform velocity profile. Within the range of validity for the boundary layer hypothesis, the vorticity transport equation in stream function-only formulation [14] for this problem is rewritten, in dimensionless form, as [15,16]

$$\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x\,\partial y^2}-\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}=\frac{1}{\mathrm{Re}}\frac{\partial^4\psi}{\partial y^4},$$

$$0<y<1,\quad x>0 \tag{1a}$$

where the stream function is defined in terms of the velocity components in the transversal and longitudinal directions, $y$ and $x$, respectively, as

$$\frac{\partial\psi}{\partial y}=u, \tag{1b}$$

$$\frac{\partial\psi}{\partial x}=-v. \tag{1c}$$

The inlet and boundary conditions for this problem, in terms of the stream function, are given, respectively, by

$$\psi(0,y)=y, \tag{2a}$$

$$\psi(x,0)=0, \tag{2b}$$

$$\frac{\partial^2\psi(x,0)}{\partial y^2}=0, \tag{2c}$$

$$\psi(x,1)=1, \tag{2d}$$

$$\frac{\partial\psi(x,1)}{\partial y}=0. \tag{2e}$$

The various dimensionless groups are defined as

$$x=\frac{\bar{x}}{\bar{y}_{\mathrm{w}}} \tag{3a}$$

$$y=\frac{\bar{y}}{\bar{y}_{\mathrm{w}}}, \tag{3b}$$

$$u=\frac{\bar{u}}{\bar{u}_0}, \tag{3c}$$

$$v=\frac{\bar{v}}{\bar{u}_0}, \tag{3d}$$

$$p=\frac{\bar{p}}{\rho\bar{u}_0^2}, \tag{3e}$$

$$\mathrm{Re}=\frac{\bar{u}_0\,\bar{y}_{\mathrm{w}}}{\nu}. \tag{3f}$$

The favorable aspect of the stream function-only formulation is the reduction of the primitive variables two-equation system (continuity and $x$-momentum) with three unknowns ($u$, $v$ and $p$), to a single partial differential Eq. (1a) with only one dependent variable, $\psi$. Although of fourth order, Eq. (1a) also brings the advantage of not requiring previous evaluation of the pressure gradient, as needed for the primitive variables counterpart. The fourth-order derivative, which is certainly a less desirable feature for discrete approaches, is readily handled by the integral transformation process. Moreover, the stream function definition (1b) and (1c) automatically satisfies the local (as well as the global) continuity, with no need for numerical compensation.

Following the ideas in the generalized integral transform technique (GITT) [3, 9–13], in order to select the appropriate auxiliary eigenvalue problem, which shall provide the basis for the eigenfunction expansion, the original system is made homogeneous in the boundary conditions for the coordinate selected to be eliminated through integral transformation. Therefore, the stream function is rewritten as

$$\psi(x, y) = \psi_\infty(y) + \psi^*(x, y), \tag{4}$$

where $\psi_\infty(y)$ represents the fully developed flow stream function of the plane Poiseuille flow, or

$$\psi_\infty(y) = \frac{3}{2}\left(y - \frac{y^3}{3}\right). \tag{5}$$

Substituting Eq. (4) back into Eqs. (1a) and (2a)–(2e), the problem formulation for $\psi^*(x, y)$ becomes

$$\left[\frac{\mathrm{d}\psi_\infty}{\mathrm{d}y}\frac{\partial^3\psi^*}{\partial x\,\partial y^2} - \frac{\mathrm{d}^3\psi_\infty}{\mathrm{d}y^3}\frac{\partial\psi^*}{\partial x}\right] + \left[\frac{\partial\psi^*}{\partial y}\frac{\partial^3\psi^*}{\partial x\,\partial y^2} - \frac{\partial\psi^*}{\partial x}\frac{\partial^3\psi^*}{\partial y^3}\right] = \frac{1}{\mathrm{Re}}\frac{\partial^4\psi^*}{\partial y^4}, \quad 0 < y < 1, \quad x > 0 \tag{6}$$

with inlet and boundary conditions

$$\psi^*(0, y) = y - \psi_\infty(y), \tag{7a}$$

$$\psi^*(x, 0) = 0, \tag{7b}$$

$$\frac{\partial^2\psi^*(x, 0)}{\partial y^2} = 0, \tag{7c}$$

$$\psi^*(x, 1) = 0, \tag{7d}$$

$$\frac{\partial\psi^*(x, 1)}{\partial y} = 0. \tag{7e}$$

For the integral transform solution of Eq. (6), the appropriate eigenvalue problem is taken as [3, 9–13]

$$\frac{\mathrm{d}^4 Y_i(y)}{\mathrm{d}y^4} = \varphi_i^4 Y_i(y), \quad 0 < y < 1 \tag{8a}$$

with the following boundary conditions:

$$Y_i(0) = 0, \tag{8b}$$

$$\frac{\mathrm{d}^2 Y_i(0)}{\mathrm{d}y^2} = 0, \tag{8c}$$

$$Y_i(1) = 0, \tag{8d}$$

$$\frac{\mathrm{d}Y_i(1)}{\mathrm{d}y} = 0. \tag{8e}$$

Problem (8) can be solved analytically to yield

$$Y_i(y) = \frac{\sin(\varphi_i y)}{\sin(\varphi_i)} - \frac{\sinh(\varphi_i y)}{\sinh(\varphi_i)}, \quad i = 1, 2, \ldots . \tag{9}$$

The eigenvalues, $\varphi_i$'s, are obtained from the transcendental equation

$$\tanh(\varphi_i) = \tan(\varphi_i), \quad i = 1, 2, \ldots , \tag{10}$$

while the eigenfunctions, $Y_i(y)$'s, can be shown to have the orthogonality property below

$$\int_0^1 Y_i(y)\,Y_i(y)\,\mathrm{d}y = \delta_{ij}, \tag{11a}$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{11b}$$

The property above allows definition of the integral transform pair:

$$\bar{\Psi}_i(x) = \int_0^1 Y_i(y)\psi^*(x, y)\,\mathrm{d}y \quad \text{transform,} \tag{12a}$$

$$\psi^*(x, y) = \sum_{i=1}^{\infty} Y_i(y)\bar{\Psi}_i(x) \quad \text{inverse.} \tag{12b}$$

From the inversion formula (12b), it may be verified that the evaluation of the stream function field reduces to the evaluation of the transformed potentials, $\bar{\Psi}_i$'s. To obtain the resulting equations for the transformed potentials, the partial differential Eq. (6) is integral transformed through the operator, $\int_0^1 Y_i(y)\,\mathrm{d}y$, yielding, after recalling the orthogonality property (11), the following system of first-order ordinary differential equations:

$$\sum_{k=1}^{\infty} E^*_{ik} \frac{\mathrm{d}\bar{\Psi}_k}{\mathrm{d}x} = \sum_{k=1}^{\infty} D^*_{ik} \bar{\Psi}_k, \qquad i = 1, 2, 3, \ldots, \tag{13a}$$

where

$$E^*_{ik} = P_{ik} + \sum_{j=1}^{\infty} C_{ijk} \bar{\Psi}_j, \tag{13b}$$

$$D^*_{ik} = \frac{\varphi_k^4}{\mathrm{Re}} \delta_{ik}, \tag{13c}$$

$$P_{ik} = B_{\infty ik} - \psi'''_{\infty} \delta_{ik}, \tag{14a}$$

$$B_{\infty ik} = \int_0^1 \psi'_{\infty} Y_i(y) Y''_k(y)\,\mathrm{d}y, \tag{14b}$$

$$C_{ijk} = \int_0^1 Y_i(y) Y'_j(y) Y''_k(y)\,\mathrm{d}y - \int_0^1 Y_i(y) Y'''_j(y) Y_k(y)\,\mathrm{d}y. \tag{14c}$$

Integral transformation of the inlet condition (7a) yields the following initial condition for system (13a)–(13c):

$$\bar{\Psi}_i(0) = \frac{1}{2} \int_0^1 (y^3 - y) Y_i(y)\,\mathrm{d}y. \tag{15}$$

Once Eqs. (13a)–(13c) are solved for the transformed potentials $\bar{\Psi}_i(x)$'s, as discussed below, the inversion formula (12b) is recalled to provide the stream function field. Moreover, through differentiation of Eq. (12b), expressions for the longitudinal and transversal velocity fields may be obtained as

$$u(x, y) = \psi'_{\infty}(y) + \sum_{i=1}^{\infty} Y'_i(y) \bar{\Psi}_i(x), \tag{16a}$$

$$v(x, y) = \sum_{i=1}^{\infty} U_i(y) \frac{\mathrm{d}\bar{Y}_i(x)}{\mathrm{d}x}. \tag{16b}$$

The infinite system (13a)–(13c) is numerically solved, after truncation to an order $N$, large enough to guarantee convergence of the stream function field to the user-prescribed accuracy. The truncated version of system (13a)–(13c) can be numerically solved for $\bar{\Psi}_i(x)$, with the feature of adaptively controlling the local error in the final solution of the transformed potentials, by well-established subroutines, from both commercial and public domain libraries, such as DIVPAG (IMSL library) or HSSODE (BOEING library), which are dedicated to the solution of stiff initial value problems.

Since the transformed potentials are calculated within a prescribed accuracy, the global error in the stream function field is controlled by the truncation order of system (13a)–(13c) as the eigenfunction expansion (12b) converges within the requested accuracy at selected domain positions. The required truncation order may be automatically determined along the integration path by the so-called adaptive procedure. In order to be successful, the first $x$ position must be fully converged, with a sufficiently large truncation order ($N$). An estimate of the error in truncating the series (12b) in a lower order ($N^*$) may then be computed. A suggestion for this estimate, used in the present study, is

$$\varepsilon_{\psi} = \max_{y} \left| \frac{\sum_{i=N^*}^{N} Y_i(y) \bar{\Psi}_i(x)}{\psi(x, y)} \right|, \quad 1 < N^* < (N-3) \tag{17}$$

which is a conservative estimate of the global relative error (residual) incurred if the stream function were calculated with a lower truncation order, $N^*$, instead of $N$. Since upstream channel positions need less equations for the results to be fully converged [3–5], the smaller $N^*$, that keeps $\varepsilon_{\psi}$ satisfying the tolerance test, may be used as the truncation order for the numerical integration to the next $x$ position desired.

Numerical results that illustrate the computational procedure and the relative merits of this approach are now presented and discussed.

## 3. Results and discussion

The numerical results shown in this section were obtained from a FORTRAN 77 code implemented in a PC 486 DX2 (66 MHz) microcomputer.

The results were obtained for Re = 2000. However, it should be noticed that, when the longitudinal coordinate is written in dimensionless form as

$$X^{+} = \frac{10^{3}\,x}{D_{\mathrm{h}}\,\mathrm{Re}_{\mathrm{h}}}, \tag{18}$$

the results expressed as a function of $X^{+}$ become independent of the Reynolds number.

First of all, a convergence analysis was performed, in order to inspect the comparative behavior of the expansion for the longitudinal velocity field, with respect to the primitive variables case. A relative error target of $10^{-4}$ was selected, and convergence was considered to be attained within $\pm 1$ in the fourth digit of the stream function field. Table 1 shows the convergence behavior of the duct centerline longitudinal velocity at selected $x$ positions, for increasing truncation orders in the stream function eigenfunction expansion. This table also shows a comparison with the finite differences work of Bodoia and Osterle [17], as published in Ref. [18].

With $N = 40$, taking the fully converged results as benchmarks, the maximum error is already less than 0.14%, thus below the 0.2% error obtained in Refs. [17, 18]. With higher truncation orders, progressively smaller relative errors are obtained, as the results for $N = 60$, that are fully converged to four digits.

The adaptive procedure was successful in providing accurate results at minimum cost (Table 1 and Fig. 1), i.e., recovering the same fully converged results with lower truncation orders, thus with a reduced computational cost. The implementation of the adaptive procedure is so straightforward (essentially an additional loop for estimating the relative error), it makes the cost reduction an even more attractive feature of this approach.

Figs. 2a and b illustrate the comparative convergence behavior of the longitudinal velocity component for, respectively, the primitive variables and the stream function-only formulations

An interesting result, as expected in this formulation, is the automatic satisfaction of the global conservation of mass, even for the lowest truncation orders. The mean flow velocity is already equal to unit for $N = 5$, even though the velocity profile still oscillates due to the low frequency of the first five eigenfunctions, as illustrated in Fig. 2b. It should be noticed that for $N = 10$ the solution does not oscillate anymore and with $N = 30$ it is almost converged in graphical scale. Such convergence behavior is notoriously improved over the primitive variables formulation choice [7], as observable from direct comparison of Figs. 2a and b.

The feature of automatically preserving continuity is not present in the primitive variables formulation, neither in the finite differences, nor in the GITT solution ([7], as seen in Fig. 2a). The GITT solution of the boundary layer equations for internal flow in the stream function-only formulation offers both the inherent global conservation of mass, and a faster rate of convergence. The main

Table 1
Convergence behavior of longitudinal velocity component at duct centerline, $u(x, 0)$, and comparison with finite differences results [17, 18]

| $X^{+}$ | $N = 5$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 40$ | $N = 50$ | $N = 60$ | ADAPT. | FIN.DIF. [17,18] |
|---|---|---|---|---|---|---|---|---|---|
| 0.0625 | 0.97325 | 1.0252 | 1.0436 | 1.0475 | 1.0494 | 1.0504 | 1.0509 | 1.0509 | – |
| 0.125 | 1.0119 | 1.0512 | 1.0661 | 1.0697 | 1.0712 | 1.0719 | 1.0724 | 1.0724 | – |
| 0.375 | 1.0948 | 1.1111 | 1.1201 | 1.1225 | 1.1235 | 1.1240 | 1.1243 | 1.1243 | 1.1244 |
| 0.75 | 1.1491 | 1.1636 | 1.1707 | 1.1725 | 1.1733 | 1.1736 | 1.1738 | 1.1738 | 1.1767 |
| 1.5 | 1.2222 | 1.2351 | 1.2404 | 1.2417 | 1.2423 | 1.2425 | 1.2427 | 1.2427 | 1.2463 |
| 2.5 | 1.2940 | 1.3038 | 1.3078 | 1.3088 | 1.3092 | 1.3094 | 1.3095 | 1.3095 | 1.3121 |
| 5.0 | 1.4043 | 1.4089 | 1.4108 | 1.4113 | 1.4115 | 1.4116 | 1.4116 | 1.4116 | 1.4111 |
| 12.5 | 1.4903 | 1.4907 | 1.4909 | 1.4910 | 1.4910 | 1.4910 | 1.4910 | 1.4910 | 1.4903 |
| 62.5 | 1.5000 | 1.5000 | 1.5000 | 1.5000 | 1.5000 | 1.5000 | 1.5000 | 1.5000 | 1.4999 |

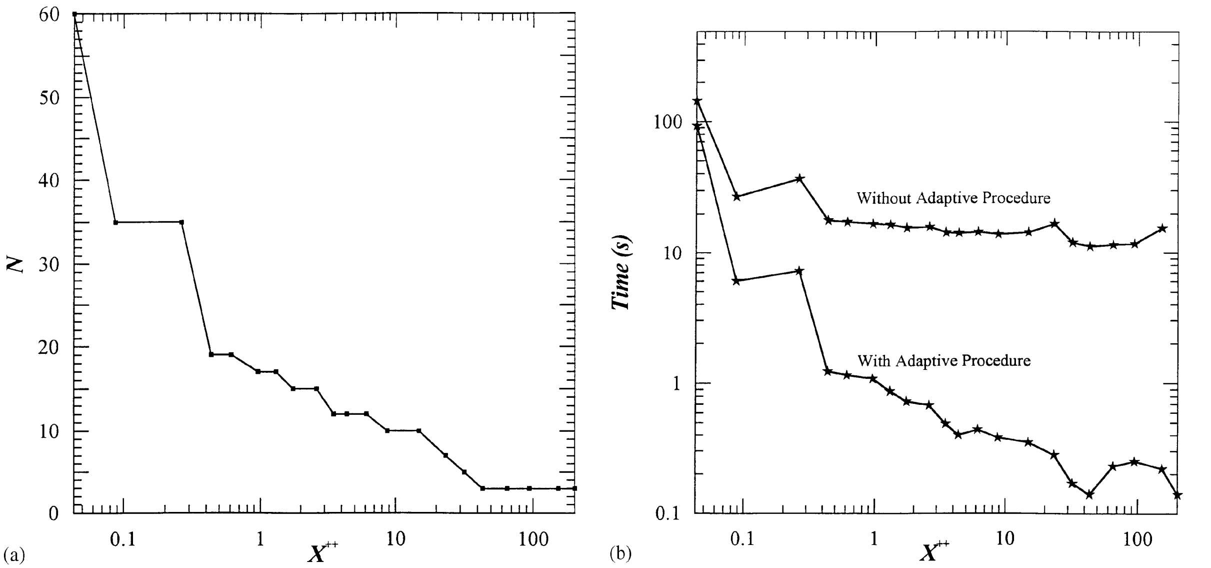



Fig. 1. Adaptive procedure: (a) reduction of truncation order along integration path, (b) comparison of total computational cost with and without adaptive procedure.

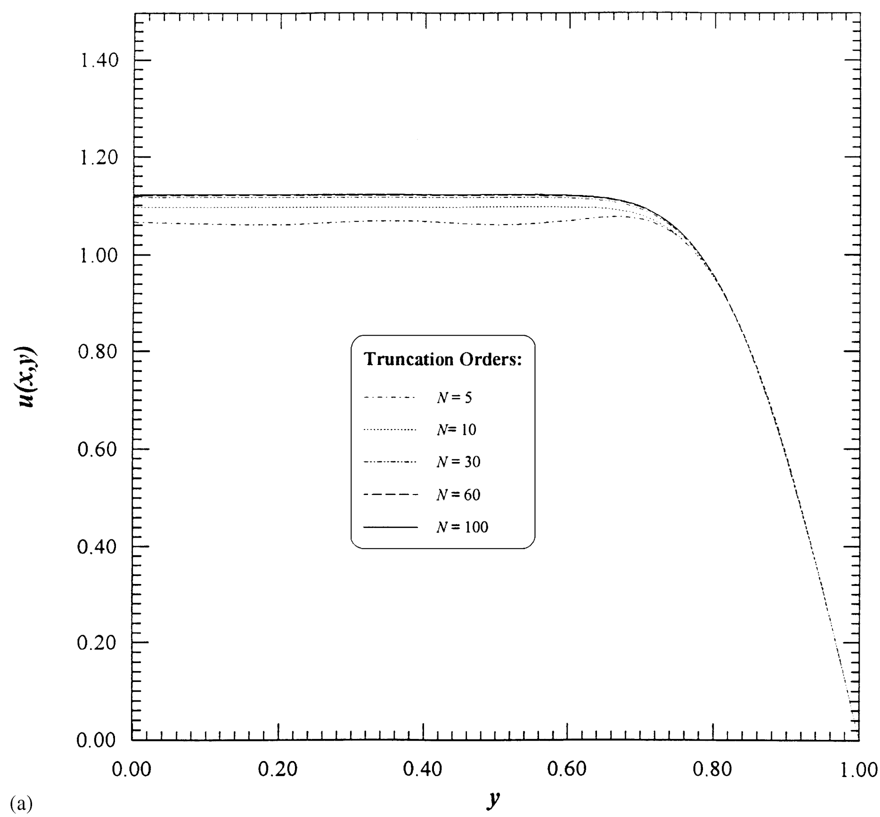
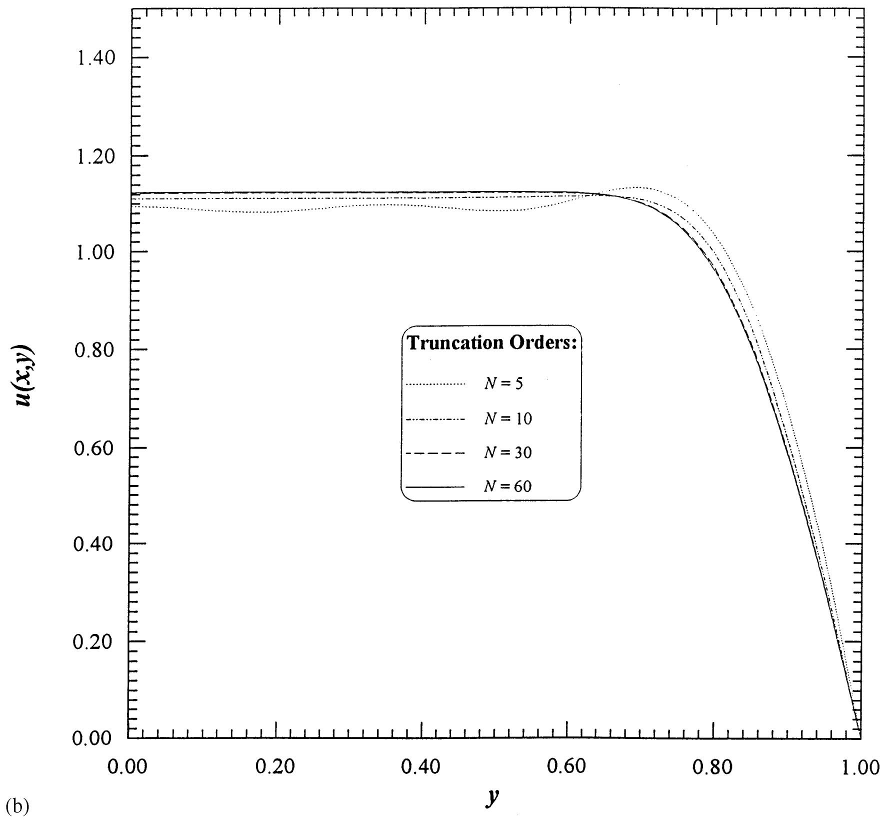



Fig. 2. *Continued on facing page.*




(b)

Fig. 2. Convergence behavior of the longitudinal velocity component, $u(x, y)$, for $X^+ = 0.375$: (a) primitive variables formulation, (b) stream function-only formulation.

disadvantage of the GITT solution in the primitive variables formulation at the present stage is the slow rate of convergence, that makes the computational cost less attractive. Some attempts of improving this behavior were made [19], including the proposition of filtering solutions [8] and alternative schemes for satisfaction of the global mass conservation. However, none of these procedures offered advantages over the change of basis in fact promoted by the use of the stream function-only formulation. The finite differences solution presented in Refs. [17, 18] is a classical benchmark set, though not so accurate as the present one, as illustrated in Fig. 3, specially for positions closer to the channel inlet. Thus, it is suggested that the present results replace those in refs [17,18] as benchmarks for the internal flow boundary layer equations. The benchmark results are presented in Table 2, for the longitudinal velocity component profiles along the channel.

Comparisons are also performed relating both distinct mathematical models for incompressible internal flow, the Navier–Stokes and boundary layer equations. Fig. 4a shows the longitudinal velocity component distributions, along the channel length, allowing for two different inlet conditions in the Navier–Stokes model, i.e., irrotational ($u = 1$, $w = 0$) or uniform parallel flow ($u = 1$, $v = 0$). It is then noticeable that the boundary layer solution is closer to the Navier–Stokes solutions under the irrotational inlet flow condition, improving further away from the channel entrance. Fig. 4b presents the transversal velocity component profiles at different positions along the channel, again comparing the two models. Quite marked differences are found between the two models results, specially at

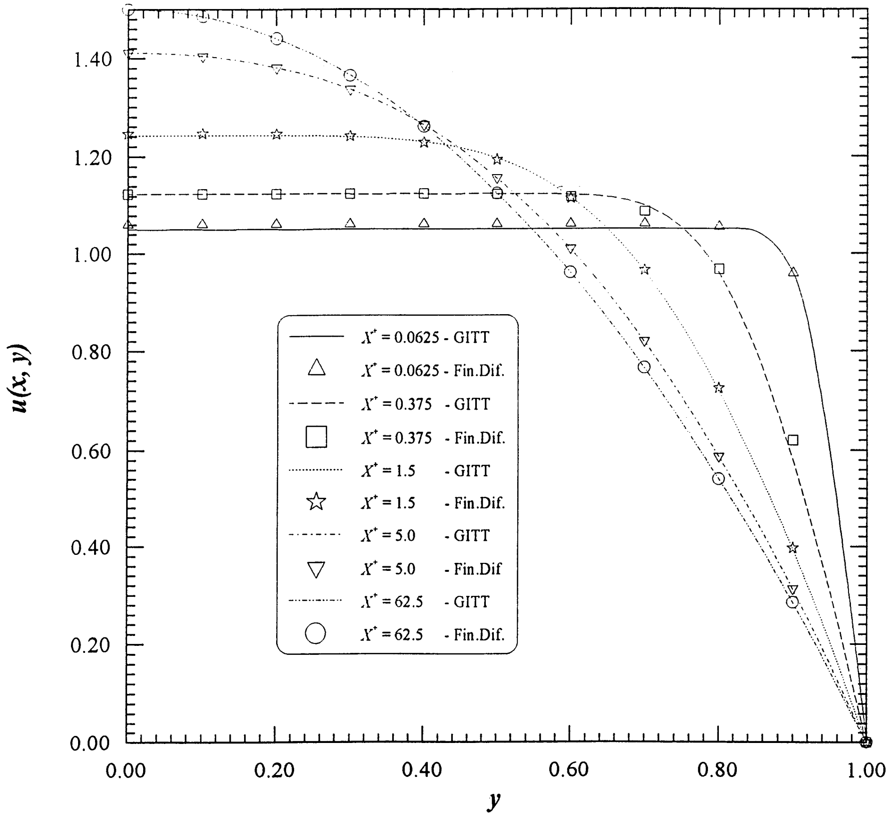



Fig. 3. Comparison of integral transform and finite differences [17, 18] results for the longitudinal velocity profiles.

Table 2
Reference results for the longitudinal velocity component profiles, $u(x, y)$ at different positions

| | $X^+$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0.0625 | 0.125 | 0.25 | 0.375 | 0.5 | 0.625 | 0.75 | 1.0 | 1.25 | 1.5 |
| 0.0 | 1.0516 | 1.0729 | 1.1024 | 1.1246 | 1.1432 | 1.1595 | 1.1741 | 1.1999 | 1.2225 | 1.2429 |
| 0.1 | 1.0516 | 1.0729 | 1.1024 | 1.1246 | 1.1432 | 1.1595 | 1.1741 | 1.1999 | 1.2225 | 1.2428 |
| 0.2 | 1.0516 | 1.0729 | 1.1024 | 1.1246 | 1.1432 | 1.1595 | 1.1741 | 1.1999 | 1.2224 | 1.2423 |
| 0.3 | 1.0516 | 1.0729 | 1.1024 | 1.1246 | 1.1432 | 1.1595 | 1.1741 | 1.1996 | 1.2213 | 1.2397 |
| 0.4 | 1.0516 | 1.0729 | 1.1024 | 1.1246 | 1.1432 | 1.1594 | 1.1736 | 1.1974 | 1.2156 | 1.2292 |
| 0.5 | 1.0516 | 1.0729 | 1.1024 | 1.1246 | 1.1428 | 1.1576 | 1.1695 | 1.1853 | 1.1933 | 1.1962 |
| 0.6 | 1.0516 | 1.0729 | 1.1023 | 1.1230 | 1.1362 | 1.1428 | 1.1446 | 1.1394 | 1.1286 | 1.1162 |
| 0.7 | 1.0517 | 1.0730 | 1.0972 | 1.0994 | 1.0881 | 1.0712 | 1.0529 | 1.0179 | 0.98762 | 0.96213 |
| 0.8 | 1.0518 | 1.0621 | 1.0179 | 0.95870 | 0.90840 | 0.86772 | 0.83464 | 0.78432 | 0.74778 | 0.71991 |
| 0.9 | 0.95489 | 0.81039 | 0.65746 | 0.57974 | 0.53135 | 0.49764 | 0.47249 | 0.43690 | 0.41251 | 0.39453 |
| 1.0 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |

Table 2. (*continued*).

| | $X^+$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 1.75 | 2.0 | 2.5 | 3.125 | 3.75 | 5.0 | 6.25 | 9.375 | 12.5 | 62.5 |
| 0.0 | 1.2615 | 1.2788 | 1.3097 | 1.3426 | 1.3701 | 1.4117 | 1.4399 | 1.4769 | 1.4910 | 1.5000 |
| 0.1 | 1.2614 | 1.2784 | 1.3085 | 1.3401 | 1.3659 | 1.4044 | 1.4302 | 1.4639 | 1.4768 | 1.4850 |
| 0.2 | 1.2602 | 1.2762 | 1.3035 | 1.3304 | 1.3512 | 1.3808 | 1.4000 | 1.4246 | 1.4340 | 1.4400 |
| 0.3 | 1.2554 | 1.2686 | 1.2892 | 1.3073 | 1.3199 | 1.3360 | 1.3456 | 1.3576 | 1.3621 | 1.3650 |
| 0.4 | 1.2392 | 1.2465 | 1.2556 | 1.2609 | 1.2629 | 1.2633 | 1.2625 | 1.2609 | 1.2604 | 1.2600 |
| 0.5 | 1.1962 | 1.1943 | 1.1880 | 1.1787 | 1.1700 | 1.1559 | 1.1460 | 1.1330 | 1.1281 | 1.1250 |
| 0.6 | 1.1037 | 1.0919 | 1.0708 | 1.0495 | 1.0326 | 1.0085 | 0.99278 | 0.97258 | 0.96488 | 0.96000 |
| 0.7 | 0.94065 | 0.92241 | 0.89321 | 0.86622 | 0.84616 | 0.81865 | 0.80117 | 0.77892 | 0.77042 | 0.76500 |
| 0.8 | 0.69785 | 0.67991 | 0.65235 | 0.62786 | 0.61011 | 0.58620 | 0.57117 | 0.55204 | 0.54470 | 0.54000 |
| 0.9 | 0.38060 | 0.36943 | 0.35253 | 0.33769 | 0.32702 | 0.31272 | 0.30374 | 0.29227 | 0.28784 | 0.28500 |
| 1.0 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |

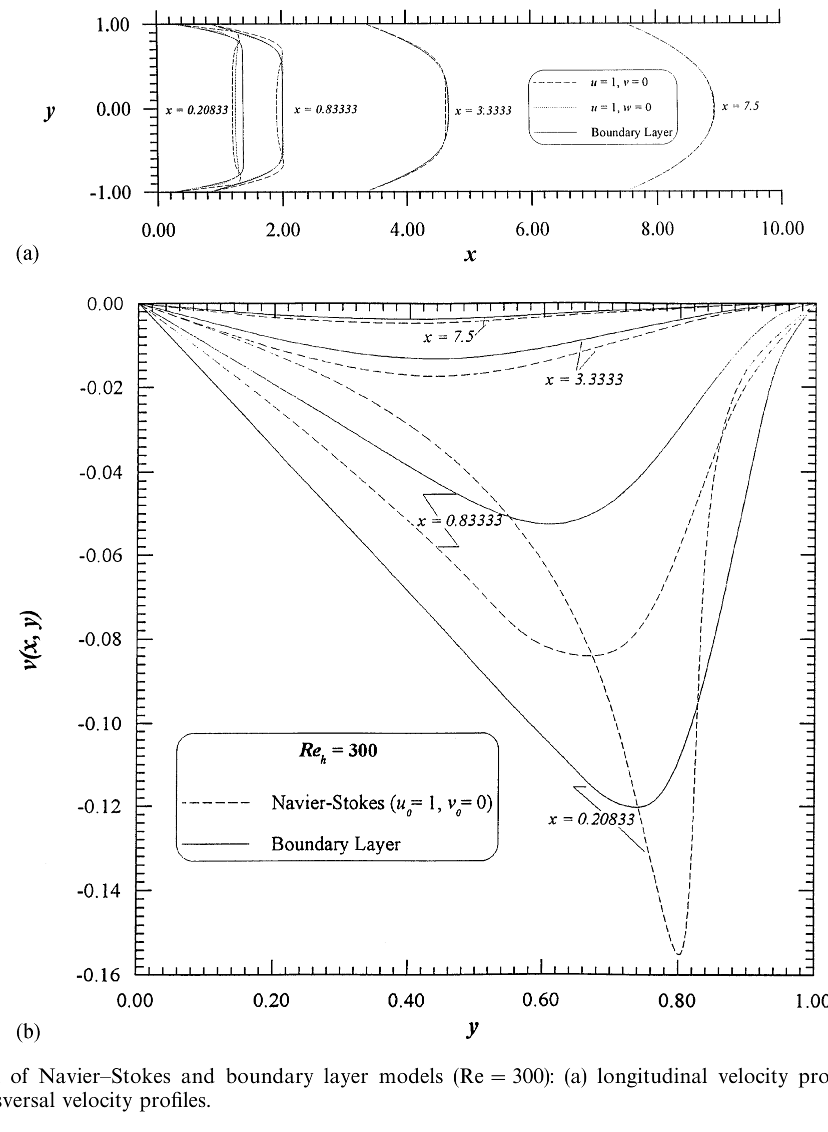



Fig. 4. Comparison of Navier–Stokes and boundary layer models (Re = 300): (a) longitudinal velocity profiles for distinct inlet conditions, (b) transversal velocity profiles.

regions close to the channel entrance, where longitudinal momentum diffusion and transversal pressure gradients can be of some relevance, and are not present in the boundary layer formulation.

## Nomenclature

| | |
|---|---|
| $D_h$ | hydraulic diameter |
| $N$ | truncation order |
| $p$, $\bar{p}$ | pressure, dimensionless and dimensional |
| Re | Reynolds number |
| $\mathrm{Re_h}$ | Reynolds number |
| $u$, $\bar{u}$ | longitudinal velocity component, dimensionless and dimensional |
| $\bar{u}_0$ | inlet flow velocity |
| $v$, $\bar{v}$ | transversal velocity component, dimensionless and dimensional |
| $x$, $\bar{x}$ | longitudinal coordinate, dimensionless and dimensional |
| $X^+$ | dimensionless longitudinal coordinate ($=10^3\,\bar{x}/D_h\,\mathrm{Re_h}$) |
| $y$, $\bar{y}$ | transversal coordinate, dimensionless and dimensional |
| $Y_i$ | normalized eigenfunctions |

*Greek symbols*

| | |
|---|---|
| $\varepsilon_\psi$ | residue estimate in adaptive procedure |
| $\varphi_i$ | eigenvalue of order $i$ |
| $\nu$ | kinematic viscosity |
| $\psi$ | dimensionless stream function |
| $\psi^*$ | filtered solution |
| $\psi_\infty$ | fully developed stream function profile |
| $\bar{\Psi}_i$ | transformed filtered stream function |

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