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Global attraction to the origin in a parametrically driven nonlinear oscillator

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Abstract

We consider a parametrically driven nonlinear ODE, which encompasses a simple model of an electronic circuit known as a parametric amplifier, whose linearisation has a zero eigenvalue. By adopting two different approaches, we obtain conditions for the origin to be a global attractor which is approached (a) non-monotonically and (b) monotonically. In case (b), we obtain an asymptotic expression for the convergence to the origin. Some further numerical results are reported.
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1. Introduction

In this paper, we consider the behaviour of the following nonlinear, parametrically driven ODE

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + f(t)x^{2m+1} = 0, \tag{1.1}$$

where $\gamma > 0$, $m \in \mathbb{Z}^+$ and $f(t) > 0$ is bounded above for all $t \geqslant 0$. This differential equation arises as a simple model of a nonlinear electronic circuit known

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as a parametric amplifier [1], in which $f(t)$ is the signal to be amplified, and $x(t)$ is the output of the circuit. We denote the upper and lower bounds of $f(t)$ as f^+ and f^-, respectively. Eq. (1.1) can be re-written as a pair of coupled, first-order differential equations as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\gamma y - f(t)x^{2m+1}, \end{aligned} \tag{1.2}$$

where $(x,y) \in \mathbb{R}^2$ and a dot placed above a variable indicates differentiation with respect to time. We refer to the differential equation in this form as the 'original system'.

It is well known that the phase space structure of time-dependent dynamical systems is in general very intricate: there are systems possessing bounded periodic and non-periodic solutions, quasi-periodic solutions and unbounded non-periodic solutions of oscillatory type with any prescribed number of zeros, see for example the papers by Alekseev [2].

There exist some results on a large class of time-dependent potentials in the undamped case. For example, it has been proved that for superquadratic potentials which are parametrically driven by a positive time-periodic function, as in (1.1) when $\gamma = 0$ and $f(t + T_0) = f(t)$, all the solutions are bounded for all time and in fact most of the motion is quasiperiodic [3,4].

Moreover the search for periodic solutions in time-dependent dynamical systems has proven very fruitful by using a number of techniques such as the calculus of variations [5] and the Poincaré–Birkhoff theorem [6–8].

Some work has also been done on the classification of the asymptotic behaviour of solutions of nonlinear differential equations; see [9] and references therein.

Another natural question one may ask is: do there exist time-dependent potentials such that all the solutions of the corresponding differential equations are bounded in phase space? The answer is naturally of fundamental importance for the stability properties of dynamical systems and there currently exists a substantial body of work on this problem in the zero-damping case [10–15]. The essential idea used to show boundedness of solutions for these potentials is to transform the system, by a sequence of canonical transformations, into a near-integrable one, and then to apply KAM theory and the Moser twist theorem.

It is not difficult to prove that a certain amount of damping guarantees boundedness of solutions of Eq. (1.1), but we go further in this paper, and derive conditions under which the origin is a global attractor. We also derive a further bound on the damping which forces all solutions to decay to the origin in a non-oscillatory way.

The system (1.2) has just one fixed point, namely the origin, $(x,y) = (0,0)$. The eigenvalues of the linearised system at the origin are 0 and $-\gamma$, so that the origin is marginally stable as a solution of the linearised system. Our aim is to establish sufficient conditions for the origin to be a global attractor, i.e., for all initial values $(x(0), y(0)) \in \mathbb{R}^2$, $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$. With zero as an eigenvalue of the linearised problem, even the *local* stability of the origin is not determined by the linearised equations and thus the problem of the global attractivity is a particularly delicate one. This paper will establish the global attractivity of the origin for sufficiently large $\gamma > 0$, and also a condition on γ which guarantees that the convergence is monotone.

2. Global attractivity of the origin

2.1. Oscillatory decay

We use the following transformation, which is due to Liouville:

$$\tau = G(t) = \int_0^t \sqrt{f(s)}\,\mathrm{d}s.$$

Since $f(t) > 0$ and bounded above for all t, $G(t)$ is a monotonically increasing function of t and G^{-1} exists. Introducing $F(\tau) := f(t) = f(G^{-1}(\tau))$, Eq. (1.1) is transformed into the system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{y}{\sqrt{F}}\left(\frac{F'}{2\sqrt{F}} + \gamma\right) - x^{2m+1}, \end{aligned} \tag{2.1}$$

where prime denotes differentiation with respect to τ. Defining the Hamiltonian, $H(x, y)$, as $H = x^{2m+2}/(2m+2) + y^2/2$ gives

$$H' = -\frac{y^2}{\sqrt{F}}\left(\frac{F'}{2\sqrt{F}} + \gamma\right). \tag{2.2}$$

Hence, provided

$$\gamma > -\min_{\tau \geqslant 0}\left(\frac{F'(\tau)}{2\sqrt{F(\tau)}}\right) = -\min_{t \geqslant 0}\left(\frac{\dot{f}(t)}{2f(t)}\right) \tag{2.3}$$

we have $H'(\tau) \leqslant 0$ so that x and y are both bounded. Also, for all $\tau \geqslant 0$,

$$H(\tau) + \int_0^\tau \frac{y^2(s)}{\sqrt{F(s)}}\left(\frac{F'(s)}{2\sqrt{F(s)}} + \gamma\right)\mathrm{d}s = H(0).$$

Letting $\tau \to \infty$ and using (2.3), and also the boundedness above and below of $F(\tau)$, we conclude that

$$\min_{s \geqslant 0} \left\{ \frac{1}{\sqrt{F(s)}} \left(\frac{F'(s)}{2\sqrt{F(s)}} + \gamma \right) \right\} \int_0^\infty y^2(s)\,\mathrm{d}s < \infty.$$

Therefore, $y \to 0$ as $\tau \to \infty$ (and hence also as $t \to \infty$). We shall now revert to the system in the form (1.2) to show that $x \to 0$ as $t \to \infty$. Let $k > 0$ be arbitrary and $U = \{(x, y) \in \mathbf{R}^2 : H(x, y) < k\}$. Let $S = \{(x, y) \in \overline{U} : \dot{H} = 0\}$ and let M denote the largest invariant set in S. Then, by the La Salle's invariance principle [16, Theorem 9.22], every solution of (1.2) that starts in U has its ω-limit set in M. In this case, S consists of points with $y = 0$ and, from (1.2), it follows immediately that M consists solely of the origin. Since $k > 0$ was arbitrary, we may now state:

Proposition 1. *Let γ satisfy* (2.3). *Then all solutions of* (1.2) *satisfy* $(x(t), y(t)) \to (0, 0)$ *as* $t \to \infty$.

2.2. Non-oscillatory decay

This section aims to discover further constraints on γ and on the initial conditions which will guarantee that convergence of solutions of (1.2) to the origin is monotone.

It will be advantageous to introduce the new state variables $u(t)$ and $v(t)$ defined by

$$u = x^{-2m}, \quad v = \dot{u}. \tag{2.4}$$

The main result of this section follows by a careful analysis of the various regions of the (u, v) phase plane, and this analysis is carried out below.

With $u = x^{-2m}$, we have on differentiating

$$\dot{x} = -\frac{x^{2m+1}}{2m}\dot{u} \quad \text{and} \quad \ddot{x} = \frac{x^{2m+1}}{2m}\left[\frac{(2m+1)}{2m}\frac{\dot{u}^2}{u} - \ddot{u}\right].$$

Substituting these into (1.1) and defining $\dot{u} = v$ gives

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= 2mf(t) - \gamma v + \frac{(2m+1)}{2m}\frac{v^2}{u}, \end{aligned} \tag{2.5}$$

which we refer to as the transformed system. In the transformed system, $u(t) > 0$, since $u = x^{-2m}$, but $v(t)$ may be of either sign. We therefore need to conduct a thorough investigation of the right half $u > 0$ of the (u, v) phase plane. Note that the transformed system (2.5) cannot have any fixed points. First, we have

Proposition 2. *Let the hypotheses of Proposition* 1 *hold. Then, trajectories of* (2.5) *must enter the open first quadrant in finite time, and then remain there.*

Proof. It is trivial to see that the open first quadrant is positively invariant and so we simply need to address the fate of a trajectory that starts in the fourth quadrant, i.e., has $v(0) < 0$. Since, by Proposition 1, x and y are bounded, any crossing of the v-axis is impossible so $u(t) > 0$ is bounded below for all t. Also

$$\dot{v} = 2mf(t) - \gamma v + \frac{(2m+1)}{2m}\frac{v^2}{u} \geqslant 2mf^- - \gamma v. \tag{2.6}$$

By a standard comparison argument, we conclude that $v(t)$ becomes positive in finite time. The proof is complete. □

In the light of the above proposition, it is sufficient to consider trajectories that start in the open first quadrant and thus we may assume from now on that $u(t), v(t) > 0$ for all $t \geqslant 0$.

The next proposition is important. Of course, our transformation $u = x^{-2m}$ clearly requires $x \neq 0$. It is implicitly assumed at the outset that the convergence of $x(t)$ and $y(t)$ to zero (the main result we are aiming for) is a non-oscillatory convergence, i.e., either $x(t) > 0$ for all $t \geqslant 0$, or $x(t) < 0$ for all $t \geqslant 0$. As we might expect physically, this will be the case only if the damping parameter γ is sufficiently large (see Proposition 3). We address this point below by giving conditions on γ, and on the initial conditions, which ensure that neither $u(t)$ nor $v(t)$ reaches infinity in finite time.

Proposition 3. *Suppose* (*WLOG*) *that* $u(0), v(0) > 0$, *that the hypotheses of Proposition* 1 *hold, that*

$$\gamma^2 > \frac{4f^+}{u(0)} \tag{2.7}$$

and that

$$\frac{v(0)}{u(0)} < m\left(\gamma + \sqrt{\gamma^2 - 4f^+/u(0)}\right).$$

Then $u(t)$ *and* $v(t)$ *remain finite for all times* t.

Proof. It is sufficient to prove that $v(t)/u(t)$ is bounded for all t, for then we have

$$\dot{u}(t) \leqslant \text{const. } u(t)$$

so that $u(t)$ (and hence also $v(t)$) does not blow up. Let

$$\phi(t) = \frac{v(t)}{u(t)}.$$

Straightforward calculations yield that

$$\dot{\phi} = \frac{2mf(t)}{u(t)} - \gamma\phi + \frac{1}{2m}\phi^2.$$

Now, the open first quadrant is positively invariant and so $\dot{u} = v > 0$. Hence $u(t)$ is strictly increasing and so $u(t) > u(0)$ for all $t > 0$. Hence

$$\dot{\phi} \leqslant \frac{2mf^+}{u(0)} - \gamma\phi + \frac{1}{2m}\phi^2.$$

Solutions of the above differential inequality are bounded by the solution of the corresponding differential equation. Simple arguments for one-dimensional ODEs, together with the hypotheses on γ and $\phi(0)$, immediately yield that $\phi(t)$ is bounded for all t. The proof is complete. □

We may now state the main theorem of this section.

Theorem 1. *Let the hypotheses of Proposition* 3 *hold. Then solutions of* (1.2) *satisfy*

$$(x(t), y(t)) \to (0, 0)$$

monotonically, *as* $t \to \infty$.

Proof. To show that $x(t) \to 0$ we must show $u(t) \to \infty$ as $t \to \infty$. Also, since

$$2my = -\frac{v}{u^{(2m+1)/(2m)}}$$

then, to show that $y(t) \to 0$ we shall show that $\psi(t) \to 0$ where

$$\psi = \frac{v}{u^{(2m+1)/(2m)}} > 0. \tag{2.8}$$

From inequality (2.6) we have that

$$\liminf_{t\to\infty} v(t) \geqslant 2mf^-/\gamma.$$

Hence, there exists $t_1 > 0$ such that, for all $t \geqslant t_1$,

$$v(t) \geqslant mf^-/\gamma.$$

Therefore, for $t \geqslant t_1$,

$$u(t) = u(t_1) + \int_{t_1}^{t} v(s)\,\mathrm{d}s \geqslant u(t_1) + \frac{mf^-}{\gamma}(t - t_1)$$

so that $u(t) \to \infty$.

Now, with ψ defined by (2.8) it is straightforward to see that

$$\dot{\psi} = \frac{2mf(t)}{(u(t))^{(2m+1)/(2m)}} - \gamma\psi.$$

Let $\varepsilon > 0$ be arbitrary. Since $u(t) \to \infty$ as $t \to \infty$, and since $f(t)$ is bounded, there exists $t_2 > 0$ such that, for all $t \geqslant t_2$,

$$\frac{2mf(t)}{(u(t))^{(2m+1)/(2m)}} < \varepsilon.$$

Then, for $t \geqslant t_2$, we have

$$\dot{\psi} \leqslant \varepsilon - \gamma\psi$$

from which it follows that

$$\limsup_{t\to\infty} \psi(t) \leqslant \frac{\varepsilon}{\gamma}.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\limsup_{t\to\infty} \psi(t) \leqslant 0$. But $\psi(t) > 0$ for all t. Hence $\lim_{t\to\infty} \psi(t) = 0$ and so $\lim_{t\to\infty} y(t) = 0$ also. The proof of the theorem is complete. □

3. Asymptotics and numerical simulations

In this section, we shall conduct some further examination of the (u, v) phase space, together with some numerical simulations, to gain further insight into the dynamics of the system and, in particular, the manner of the convergence of $(x(t), y(t))$ to $(0, 0)$ as $t \to \infty$ for suitably large $\gamma > 0$.

Our numerical simulations indicate that, as $t \to \infty$, trajectories in the (u, v) phase plane typically become trapped between the horizontal asymptotes of the two curves Γ^+ and Γ^- defined below, and shown in Fig. 1. Note that

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\dot{v}}{\dot{u}} = \frac{2mf(t)}{v} - \gamma + \frac{(2m+1)}{2m}\frac{v}{u}$$

and so, in particular, $\mathrm{d}v/\mathrm{d}u < 0$ in regions where

$$\frac{2mf^+}{v} - \gamma + \frac{(2m+1)}{2m}\frac{v}{u} < 0 \tag{3.1}$$

and $\mathrm{d}v/\mathrm{d}u > 0$ in regions where

$$\frac{2mf^-}{v} - \gamma + \frac{(2m+1)}{2m}\frac{v}{u} > 0. \tag{3.2}$$

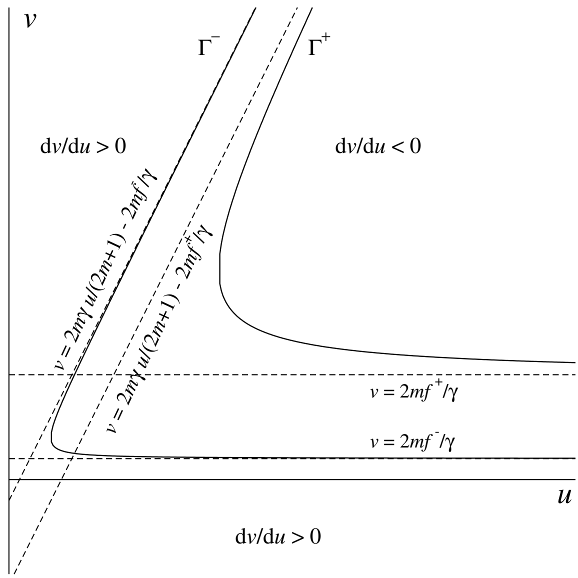

Fig. 1. The curves Γ^+ and Γ^- and their asymptotes. To the left of Γ^-, dv/du is positive and to the right of Γ^+ it is negative. Between Γ^+ and Γ^-, dv/du can be positive, negative or zero.

Recalling that $u(t)$ increases in the open first quadrant, any trajectory can be characterised as the graph of a function, $v = v(u)$. A trajectory has a turning point at any time when $dv/du = 0$. This will happen at any time with

$$v = \frac{m\gamma u}{2m+1}\left[1 \pm \sqrt{1 - \frac{4(2m+1)f(t)}{\gamma^2 u}}\right]. \tag{3.3}$$

Replacing $f(t)$ in (3.3) by its upper and lower bounds defines two non-intersecting curves (provided $f^+ \neq f^-$) in the (u, v) plane; we refer to these curves as Γ^+ and Γ^- (they are also defined by replacing the inequalities in (3.1) and (3.2) by equalities). Only in the region between Γ^+ and Γ^- can a trajectory $v(u)$ have a turning point. The curves Γ^+ and Γ^- each possess two asymptotes, obtained by Taylor expanding (3.3). These are

$$\text{Asymptotes to } \Gamma^{\pm} = \frac{2m\gamma}{2m+1}u - \frac{2m}{\gamma}f^{\pm}, \frac{2m}{\gamma}f^{\pm} \tag{3.4}$$

Since the horizontal asymptotes both have $v > 0$, and the other two asymptotes have negative intercept with the v-axis, Γ^+ and Γ^- must be confined to the first quadrant $u,\ v > 0$. Furthermore, in this quadrant $\dot{v}$, and hence dv/du, are positive everywhere to the left of Γ^-; similarly, $dv/du < 0$ everywhere to the

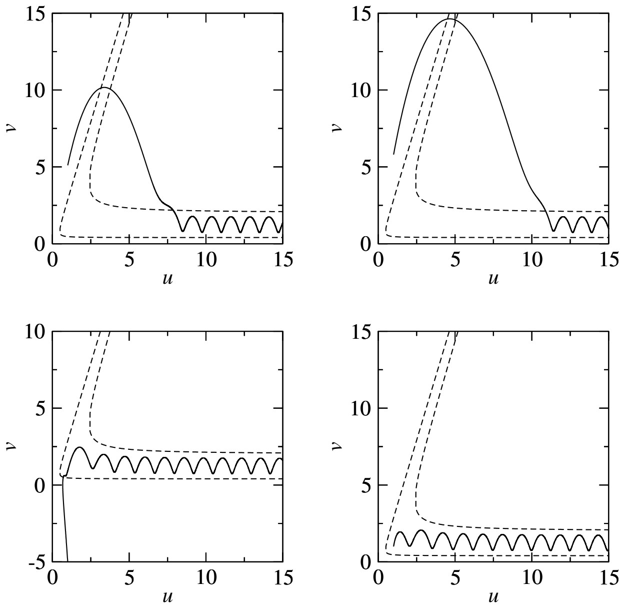

Fig. 2. Numerically obtained periodic solutions starting from various points in the u, v-plane, with $f(t) = 3 + 2\cos 2\pi t$ so $f^+ = 5$, and parameters that satisfy the conditions of Theorem 1: $m = 1$, $u(0) = 1$, $\gamma = 5$. The curves Γ^- and Γ^+ are shown dotted.

right of Γ^+. For $v \leqslant 0$, $\mathrm{d}v/\mathrm{d}u$ is everywhere positive. The situation is sketched in Fig. 1.

Typical solutions for various different initial data satisfying the conditions of Theorem 1 are illustrated in Fig. 2. The observations we have made so far indicate that, typically, $u(t) \to \infty$ with $v(t)$ remaining bounded, and these are borne out by the numerical solutions in Fig. 2. For any value of γ and any initial data giving such an outcome, we can formally approximate the differential equations (2.5) in the limit as $t \to \infty$ to yield the approximated system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= 2mf(t) - \gamma v. \end{aligned} \tag{3.5}$$

Solving the second of these equations and ignoring the transient term gives

$$v(t) = 2m \int_0^t \mathrm{e}^{-\gamma(t-s)} f(s)\,\mathrm{d}s$$

from which it follows that $u(t)$ is given asymptotically by

$$u(t) \sim 2m \int_0^t \int_0^\xi \mathrm{e}^{-\gamma(\xi-s)} f(s)\,\mathrm{d}s\,\mathrm{d}\xi$$

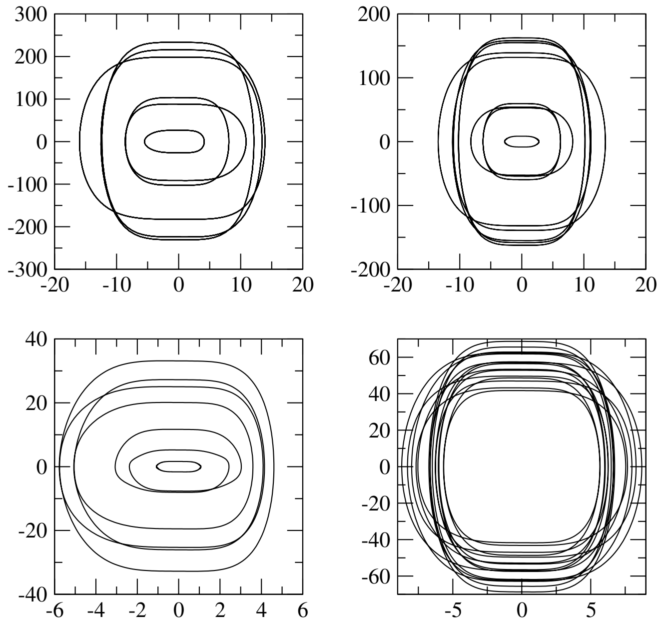

Fig. 3. Numerical solutions to (1.1) with $f(t) = 3 + 2\cos 2\pi t$ and $\gamma = 0.04$. In all four graphs, x is plotted horizontally and y vertically. The top left graph shows three period-one (same period as $f(t)$) solutions; the top right graph shows three period-two solutions (repeating every two periods of $f(t)$). Bottom left: three period-four solutions; bottom right, one period-ten solution.

or, on reversing the order of integration,

$$u(t) \sim \frac{2m}{\gamma}\int_0^t f(s)(1 - \mathrm{e}^{-\gamma(t-s)})\,\mathrm{d}s.$$

Thus, for suitably large γ and suitable initial data, the convergence of $x(t)$ to zero is given asymptotically by

$$x(t) \sim \frac{1}{\left(\frac{2m}{\gamma}\int_0^t f(s)(1 - \mathrm{e}^{-\gamma(t-s)})\,\mathrm{d}s\right)^{1/2m}}.$$

Attraction to the origin is not the only dynamics displayed by Eq. (1.1); a variety of periodic solutions exist as well, as shown in Fig. 3.

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