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Isotropic Hypersurfaces and Minimal Extensions of Lipschitz Functions

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Abstract. The existence and uniqueness theorem for isotropic hypersurfaces with prescribed boundary in Lorentzian warped products is proved. The proof is based on minimal Lipschitz extensions of functions.

Key words: Lorentzian space, isotropic surface, Lipschitz function, minimal extension of a Lipschitz function.

1. Let M be a C^2 Riemannian manifold with metric g, and let $\delta(m) > 0$ be a C^2 function on M. Let H be a Lorentzian space with metric h. Following [1, Sec. 2.6], we define the *Lorentzian warped product* $\mathscr{W} = M \times_\delta H$ to be the manifold $M \times H$ equipped with the Lorentz metric $\overline{g}$ defined by the rule

$$\overline{g}(u,v) = g(\pi u, \pi v) + \delta(\pi(p))h(\eta u, \eta v),$$

where $p \in \mathscr{W}$, π and η are the natural projections onto M and H, respectively, and $u, v \in T_p(\mathscr{W})$. It is clear from the form of the metric that the tangent spaces $T_{\pi(p)}M$ and $T_{\eta(p)}H$ are orthogonal to each other.

A vector $u \in T_p(\mathscr{W})$ is said to be *spacelike* if $\overline{g}(u,u) > 0$. In the degenerate case $\overline{g}(u,u) = 0$, the vector u is said to be *isotropic* (or *lightlike*). We say that a surface $F \subset \mathscr{W}$ is *spacelike* if so is each tangent vector to F. A surface F whose tangent space at each point contains both spacelike and isotropic vectors is said to be *isotropic* (or *lightlike*).

Consider the Lorentzian warped product $\mathscr{W} = M \times_\delta \mathbb{R}$, where $\mathbb{R}$ is the real line with the metric $-dt^2$. In what follows, we consider a hypersurface $F \subset \mathscr{W}$ that is the graph of a C^1 function $f(m)$ defined in a domain $\Omega \subset M$.

Let us write out conditions under which F is spacelike. Take a point $m \in M$ and an orthonormal basis $\{E_i\}_{i=1}^n$ in T_mM. Let $E_0 \in \mathbb{R}$ be a vector with $\overline{g}(E_0, E_0) = -1$ such that the orientation of the ordered $(n+1)$-tuple of vectors $E_0, E_1, \dots, E_n$ coincides with the orientation of $\mathscr{W}$.

A surface F is spacelike if and only if $\overline{g}(X,X) > 0$ for each tangent vector field $X \in TF$. Let us represent X in the form $X = \sum_{i=1}^n \xi_i X_i$, where $X_i = E_i + \nabla_{E_i}(f)E_0$ and ∇ is a connection on M. Then the condition that F is spacelike is equivalent to the inequality

$$\sum_{i,j=1}^n \overline{g}(X_i, X_j)\,\xi_i\xi_j > 0. \tag{1}$$

Clearly, $X_i \in TF$. Indeed, take a point $p = (m,t) \in F$. Let $\gamma(s) \subset M$ be an arbitrary smooth curve such that $\gamma(0) = m$ and $\dot\gamma(0) = E_i$. We have $\gamma^*(s) = (\gamma(s), f(\gamma(s))) \subset F$ and $\gamma^*(0) = p$. Thus

$$\frac{d}{ds}h(\gamma(s), f(\gamma(s)))|_{s=0} = \nabla_{E_i}h + \nabla_{E_0}h\nabla_{E_i}f = X_ih$$

for each C^1 function $h\colon F \to \mathbb{R}$, and consequently, X_i is tangent to F.

Relation (1) is equivalent to the system

$$\det(\overline{g}(X_i, X_j))_{i,j=1}^k > 0, \qquad k = 1, \dots, n.$$

Let us evaluate the determinants. We have

$$\det(\overline{g}(X_i, X_j))_{i,j=1}^k = \det(\delta_{ij} - \delta(m)\nabla_{E_i} f \nabla_{E_j} f) = 1 - \delta(m) \sum_{i=1}^k (\nabla_{E_i} f)^2.$$

Thus the graph F of the function $t = f(m)$ is spacelike if and only if

$$\delta^{1/2}(m)|\nabla f(m)| < 1.$$

Accordingly, F is isotropic if and only if

$$\delta^{1/2}(m)|\nabla f(m)| = 1 \quad \text{for all } \ m \in \Omega.$$

Now suppose that M is connected and Ω is a domain in M. Consider the intrinsic metric

$$r_\Omega(m_1, m_2) = \inf \int_\gamma \delta^{-1/2}(m)\, ds, \tag{2}$$

where ds is the length element on M and the infimum is taken over all arcs $\gamma \subset \Omega$ connecting the points $m_1, m_2 \in \Omega$. Let Ω_r be the completion of Ω with respect to the metric r_Ω, and let $\partial\Omega_r = \Omega_r \setminus \Omega$. By $D_R(m)$ we denote the ball of radius $R > 0$ (with respect to the metric r_Ω) centered at m:

$$D_R(m) = \{m' \in \Omega_r : r_\Omega(m, m') < R\}.$$

Consider the set

$$\Gamma(m_1, m_2) = \{m \in \Omega_r : r_\Omega(m_1, m_2) = r_\Omega(m_1, m) + r_\Omega(m, m_2)\}. \tag{3}$$

Clearly, $\Gamma(m_1, m_2)$ is nonempty, since it contains at least m_1 and m_2. Note also that $\Gamma(m, m) = \{m\}$.

2. Let a function $\varphi\colon \partial\Omega_r \to \mathbb{R}$ be given. In [2, 3], the problem on a spacelike extension of φ from the boundary $\partial\Omega_r$ to the entire Ω was investigated. For example, one faces this problem when describing the set of admissible functions in the variational problem

$$\int_\Omega \sqrt{1 - \delta\,|\nabla f|^2}\, dv_M \to \max$$

or analyzing the solvability [4, 5] of the Dirichlet problem for the equation

$$\operatorname{div}\left(\frac{\delta\,\nabla f}{\sqrt{1 - \delta\,|\nabla f|^2}}\right) = 0.$$

In particular, φ has a spacelike extension if and only if

$$|\varphi(m_1) - \varphi(m_2)| \leqslant r_\Omega(m_1, m_2) \quad \text{for all } \ m_1, m_2 \in \partial\Omega_r, \tag{4}$$

and moreover,

$$|\varphi(m_1) - \varphi(m_2)| < r_\Omega(m_1, m_2) \tag{5}$$

whenever $\Gamma(m_1, m_2) \setminus \partial\Omega_r \neq \varnothing$ [3, Theorem 1].

The existence of an isotropic extension of φ to the entire Ω has not been studied yet. In what follows, we fill the gap. Using the technique of minimal Lipschitz extensions developed in [6–9] for other aims, we prove a criterion for isotropic extendability.

3. Let X be a metric space with metric d, and let E be a compact subset of X. Suppose that $\varphi\colon E \to \mathbb{R}$ is a function satisfying the Lipschitz condition

$$|\varphi(x') - \varphi(x'')| \leqslant L\, d(x', x'') \quad \text{for every } \ x', x'' \in E.$$

By $\operatorname{Lip}(\varphi, E)$ we denote the least constant L in this inequality.

Every Lipschitz function $\varphi\colon E \to \mathbb{R}$ can be extended to X as a Lipschitz function $f\colon X \to \mathbb{R}$ such that $f|_E = \varphi$ and $\operatorname{Lip}(\varphi, E) = \operatorname{Lip}(f, X)$ (see [10, 2.10.44]). The function f is called a *minimal*

Lipschitz extension of φ. Apparently, such extensions were suggested for the first time in [13, 14] in the form

$$\overline{f}(x) = \inf_{y\in E}\{\varphi(y) + \mathrm{Lip}(\varphi, E)\, d(x, y)\},$$
$$\underline{f}(x) = \sup_{y\in E}\{\varphi(y) - \mathrm{Lip}(\varphi, E)\, d(x, y)\}. \tag{6}$$

One can readily prove that if f is an arbitrary minimal Lipschitz extension of φ, then

$$\underline{f}(x) \leqslant f(x) \leqslant \overline{f}(x) \quad \text{everywhere in } X. \tag{7}$$

Indeed,

$$|f(x) - \varphi(y)| \leqslant \mathrm{Lip}(\varphi, E)\, d(x, y),$$

for arbitrary $x \in X$ and $y \in E$, whence it follows, for example, that

$$f(x) \leqslant \varphi(y) + \mathrm{Lip}(\varphi, E)\, d(x, y).$$

Since $y \in E$ is arbitrary, we obtain

$$f(x) \leqslant \inf_{y\in E}\{\varphi(y) + \mathrm{Lip}(\varphi, E)\, d(x, y)\} = \overline{f}(x).$$

In a similar way, one proves the inequality $f(x) \geqslant \underline{f}(x)$. For the case $X = \mathbb{R}^n$, see [6, Theorem 1].

The functions $\overline{f}$ and $\underline{f}$ are called the *upper* and the *lower* extension of φ, respectively. Note, however, that the minimal Lipschitz extensions take into account only the global Lipschitz constants and largely ignore the local structure of functions.

The set

$$U_\varphi = \{x : x \notin E,\, \overline{f}(x) = \underline{f}(x)\}$$

is called the *uniqueness set* of φ.

We set

$$\Gamma(x_1, x_2) = \{x \in X : d(x_1, x_2) = d(x_1, x) + d(x, x_2)\},$$
$$I_\varphi = \bigcup\{\Gamma(x_1, x_2) : x_1, x_2 \in E \text{ and } |\varphi(x_1) - \varphi(x_2)| = \mathrm{Lip}(\varphi, E)\, d(x_1, x_2)\}.$$

In fact, the uniqueness set coincides with I_φ on $X \setminus E$. Namely, the following assertion holds.

Theorem 1. *Let $E \subset X$ be a compact set, and let φ be a Lipschitz function on E. Then*

$$U_\varphi = I_\varphi \setminus E. \tag{8}$$

Proof. To be definite, we shall assume that $\mathrm{Lip}(\varphi, E) = 1$. Consider the upper and lower extensions $\overline{f}(x)$ and $\underline{f}(x)$ of φ. Clearly, $\overline{f}(x) \geqslant \underline{f}(x)$ for $x \in X$ and $\overline{f}(x) = \underline{f}(x) = \varphi(x)$ for $x \in E$.

Let $x_0 \in I_\varphi \setminus E$. Let us show that $\overline{f}(x_0) = \underline{f}(x_0)$. Since $x_0 \in I_\varphi$, it follows that there exist points $y_1, y_2 \in E$ such that $|\varphi(y_1) - \varphi(y_2)| = d(y_1, y_2)$ and $x_0 \in \Gamma(y_1, y_2)$. Without loss of generality, we shall assume that

$$\varphi(y_1) - \varphi(y_2) = d(y_1, y_2). \tag{9}$$

By the definition of $\overline{f}$,

$$\overline{f}(x_0) \leqslant \varphi(y_2) + d(y_2, x_0). \tag{10}$$

Then, using the definition of $\underline{f}$, Eq. (9), the fact that $x_0 \in \Gamma(y_1, y_2)$, and inequality (10), we obtain

$$\underline{f}(x_0) \geqslant \varphi(y_1) - d(y_1, x_0) = \varphi(y_2) + d(y_1, y_2) - d(y_1, x_0) = \varphi(y_2) + d(y_2, x_0) \geqslant \overline{f}(x_0).$$

Hence $\overline{f}(x_0) = \underline{f}(x_0)$.

Now suppose that $x_0 \in U_\varphi$. Since E is compact, it follows that there exist points $y_1, y_2 \in E$ such that

$$\overline{f}(x_0) = \varphi(y_1) + d(y_1, x_0), \qquad \underline{f}(x_0) = \varphi(y_2) - d(y_2, x_0).$$

Since $x_0 \in U_\varphi$, we see that

$$d(y_1, y_2) \geqslant \varphi(y_2) - \varphi(y_1) = d(y_1, x_0) + d(y_2, x_0).$$

Consequently, $x_0 \in \Gamma(y_1, y_2)$ and $\varphi(y_2) - \varphi(y_1) = d(y_1, y_2)$; that is, $x_0 \in I_\varphi$. □

4. We use minimal Lipschitz extensions to describe conditions for the existence of an isotropic surface spanning a given contour. First, it is easily seen that (5) is equivalent to

$$I_\varphi \cap \Omega = \varnothing,$$

where the set I_φ is defined for the metric r_Ω.

We need the following two auxiliary assertions.

Lemma 1. *Let M be a Riemannian C^2 manifold, and let $m_0 \in M$. Then the function*

$$r(m) = r_\Omega(m_0, m)$$

is continuously differentiable in a sufficiently small punctured neighborhood $D_\varepsilon(m_0) \setminus \{m_0\}$.

The proof is the same as for C^∞ manifolds (e.g., see [12, Sec. 8.1]).

Before proceeding to the following auxiliary assertion, note that

$$\overline{f}(m) = \inf_{m' \in \partial B} \{\overline{f}(m') + r_\Omega(m, m')\}, \tag{11}$$

$$\underline{f}(m) = \sup_{m' \in \partial B} \{\underline{f}(m') - r_\Omega(m, m')\} \tag{12}$$

for an arbitrary subdomain $B \Subset \Omega$.

For example, let us prove the first relation. We set

$$\overline{f}^*(m) = \inf_{m' \in \partial B} \{\overline{f}(m') + r_\Omega(m, m')\}.$$

It follows from (7) that $\overline{f}^*(m) \geqslant \overline{f}(m)$. Suppose that $m \in \Omega$ and $m_k \in \partial\Omega_r$ is a sequence of points satisfying

$$\overline{f}(m) = \varphi(m_k) + r_\Omega(m, m_k) - \varepsilon_k,$$

where $\varepsilon_k \geqslant 0$ and $\varepsilon_k \to 0$ as $k \to \infty$. Then there exist points $m_k' \in \partial B \cap \Gamma(m, m_k)$. Indeed, for an arbitrary $\varepsilon > 0$ consider a curve $\gamma_\varepsilon \colon [0,1] \to \Omega_r$ such that

$$\gamma_\varepsilon(0) = m, \quad \gamma_\varepsilon(1) = m_k,$$

and

$$\int_0^1 \delta^{-1/2} g(\dot\gamma_\varepsilon(t), \dot\gamma_\varepsilon(t))\, dt \leqslant r_\Omega(m, m_k) + \varepsilon.$$

Take some points $m_k^\varepsilon \in \partial B \cap \gamma_\varepsilon$. Since ∂B is compact, we can assume without loss of generality that there exist points $m_k' \in \partial B$ such that $m_k^\varepsilon \to m_k'$ as $\varepsilon \to 0$. It is easily seen that

$$\begin{aligned} r_\Omega(m, m_k^\varepsilon) + r_\Omega(m_k^\varepsilon, m_k) &\leqslant \int_0^{t_k^\varepsilon} \delta^{-1/2} g(\dot\gamma_\varepsilon(t), \dot\gamma_\varepsilon(t))\, dt + \int_{t_k^\varepsilon}^1 \delta^{-1/2} g(\dot\gamma_\varepsilon(t), \dot\gamma_\varepsilon(t))\, dt \\ &= \int_0^1 \delta^{-1/2} g(\dot\gamma_\varepsilon(t), \dot\gamma_\varepsilon(t))\, dt \leqslant r_\Omega(m, m_k) + \varepsilon. \end{aligned}$$

By letting $\varepsilon \to 0$ in this relation, we obtain

$$r_\Omega(m, m_k') + r_\Omega(m_k', m_k) \leqslant r_\Omega(m, m_k),$$

whence $m_k' \in \Gamma(m, m_k)$ by the triangle inequality. Therefore,

$$\begin{aligned} \overline{f}^*(m) \leqslant \overline{f}(m_k') + r_\Omega(m_k', m) &\leqslant \varphi(m_k) + r_\Omega(m_k, m_k') + r_\Omega(m_k', m) \\ &= \varphi(m_k) + r_\Omega(m_k, m) = f(m) + \varepsilon_k. \end{aligned}$$

By passing to limit, we obtain the inequality $\overline{f}^*(m) \leqslant \overline{f}(m)$, whence (11) follows.

Lemma 2. *Let $E \subset \Omega_r$ be a closed set. Let $\varphi \colon E \to \mathbb{R}$ and*

$$f(m) = \inf_{m' \in E} \{\varphi(m') + r_\Omega(m, m')\}, \qquad m \in \Omega.$$

Then $|\nabla f| = \delta^{-1/2}(m)$ at each point $m \in \Omega \setminus E$ where $f(m)$ is differentiable.

Proof. Let $m_0 \in \Omega \setminus E$ be a point such that $\nabla f(m_0)$ exists. One can readily see from (11) that there exists an $m_0' \in \partial D_\rho(m_0)$ such that

$$f(m_0) = f(m_0') + r_\Omega(m_0, m_0'),$$

where ρ is so small that $D_\rho(m_0) \subset \Omega \setminus E$.

Let $\gamma(s)$ be the geodesic joining m_0 and m_0', where s is the natural parameter. We claim that $f(m) = f(m_0) - r_\Omega(m, m_0)$ for $m \in \gamma$. Indeed, suppose the opposite. Since $\mathrm{Lip}(E, f) = 1$, it follows that there exists a point $m' \in \gamma$ such that $f(m') > f(m_0) - r_\Omega(m', m_0)$. Now

$$\begin{aligned} f(m_0) - f(m_0') &= f(m_0) - f(m') + f(m') - f(m_0') \leqslant f(m_0) - f(m') + r_\Omega(m', m_0') \\ &< r_\Omega(m_0, m') + r_\Omega(m', m_0') = r_\Omega(m_0, m_0'), \end{aligned}$$

and we arrive at a contradiction. Consequently,

$$\langle \nabla f(m_0), \dot\gamma(0) \rangle = \lim_{s\to 0} \frac{f(\gamma(s)) - f(x_0)}{s} = -\lim_{s\to 0} \frac{r_\Omega(\gamma(s), x_0)}{s} = -\delta^{-1/2}(m_0)$$

and since $|\nabla f(m_0)| \leqslant \delta^{-1/2}(m_0)$, we obtain the desired statement. □

The following theorem is the main result of this paper.

Theorem 2. *Let $M\times_\delta\mathbb{R}$ be a Lorentzian warped product, and let $\Omega \subset M\times_\delta\mathbb{R}$ be a domain. Suppose that $\partial\Omega_r$ is compact with respect to the metric r_Ω, and let a function $\varphi\colon \partial\Omega_r \to \mathbb{R}$ be given. A necessary and sufficient condition for the existence of a function $f(m) \in C^1(\Omega)$ having an isotropic graph $F \subset M\times_\delta\mathbb{R}$ and satisfying $f|_{\partial\Omega_r} = \varphi$ is that $\Omega \subset I_\varphi$ and (4) holds.*

Proof. First, let us prove the sufficiency. Consider the upper and lower extensions $\overline{f}(m)$ and $\underline{f}(m)$ of φ. Clearly, $\overline{f}(m) \geqslant \underline{f}(m)$ for $m \in \Omega_r$ and $\overline{f}(m) = \underline{f}(m) = \varphi(m)$ for $m \in \partial\Omega_r$. It follows from Theorem 1 and the inclusion $\Omega \subset I_\varphi$ that $\overline{f} \equiv \underline{f}$.

Let us prove that $\overline{f} \in C^1(\Omega)$. First, note that the relation $\overline{f} \equiv \underline{f}$ implies that the relations

$$\overline{f}(m) = \inf_{m'\in\partial D_R} \{\overline{f}(m') + r_\Omega(m', m)\}, \qquad \underline{f}(m) = \inf_{m'\in\partial D_R} \{\underline{f}(m') + r_\Omega(m', m)\}$$

hold for each geodesic ball $D_R(m_0) \Subset \Omega$ by virtue of (11) and (12). Let $m_0 \in \Omega$. Using Lemma 1, we choose $R > 0$ small enough that $r_\Omega(m_1, m) \in C^1(D_R(m_0))$ with respect to the variable m for each $m_1 \in \partial D_R(m_0)$.

Since $\partial D_R(m_0)$ is compact, it follows that there exist points $m_1, m_2 \in \partial D_R(m_0)$ for which

$$\overline{f}(m_0) = \overline{f}(m_1) + r_\Omega(m_1, m_0), \qquad \underline{f}(m_0) = \underline{f}(m_2) - r_\Omega(m_2, m_0).$$

We set $\overline{v}(m) = \overline{f}(m_1) + r_\Omega(m_1, m)$ and $\underline{v}(m) = \underline{f}(m_2) - r_\Omega(m_2, m)$. By the definition of $\overline{f}$ and $\underline{f}$,

$$\underline{v}(m) \leqslant \underline{f}(m) = \overline{f}(m) \leqslant \overline{v}(m)$$

for each $m \in D_R(m_0)$. Since $\underline{v}(m_0) = \overline{v}(m_0)$ and $\overline{v}, \underline{v} \in C^1(D_R(m_0))$, we conclude that $\overline{f}$ is differentiable at m_0. Moreover, the relation $|\nabla\overline{f}(m_0)| = |\nabla r_\Omega(m_1, m)||_{m=m_0} = \delta^{-1/2}(m_0)$ follows from Lemma 2.

Now let us prove the continuity of $\nabla\overline{f}$ at m_0. Suppose the contrary. Then there exists a sequence $m^k \to m_0$ of points in the ball $D_R(m_0)$ such that $\nabla\overline{f}(m^k)$ does not converge to $\nabla\overline{f}(m_0)$. By m_1^k and m_2^k we denote some points of $\partial D_R(m_0)$ such that

$$\overline{f}(m^k) = \overline{f}(m_1^k) + r_\Omega(m_1^k, m^k), \qquad \overline{f}(m^k) = \underline{f}(m^k) = \underline{f}(m_2^k) - r_\Omega(m_2^k, m^k).$$

Note that it follows from the preceding that $\nabla\overline{f}(m^k) = \nabla r_\Omega(m_1^k, m)|_{m=m^k}$.

Using the compactness of $\partial D_R(m_0)$, one can find converging subsequences $m_1^{k_l} \to m_1'$ and $m_2^{k_l} \to m_2'$ for some points $m_1', m_2' \in \partial D_R(m_0)$. Since $\overline{f}$ is continuous, we obtain

$$\overline{f}(m_0) = \overline{f}(m_1') + r_\Omega(m_1', m_0), \qquad \overline{f}(m_0) = \overline{f}(m_2') - r_\Omega(m_2', m_0).$$

Thus $\nabla\overline{f}(m_0) = \nabla r_\Omega(m_1', m)|_{m=m_0}$, and consequently,

$$\nabla\overline{f}(m^k) = \nabla r_\Omega(m_1^k, m^k) \to \nabla r_\Omega(m_1', m)|_{m=m_0} = \nabla\overline{f}(m_0).$$

We arrive at a contradiction with our assumption, and therefore, the function $\overline{f}(m)$ is the desired extension.

To prove the necessity, it suffices to note that first, (4) holds, and second, the identity $\delta^{1/2}|\nabla f| \equiv 1$ implies that for each point $m_0 \in \Omega$ there exist points $m_1, m_2 \in \partial\Omega_r$ such that

$$f(m_0) = \varphi(m_1) + r_\Omega(m_1, m_0), \qquad f(m_0) = \varphi(m_2) - r_\Omega(m_2, m_0).$$

To find these points, one should consider the solution $\gamma(s) \in \Omega$ of the equation $\dot\gamma = \nabla f(\gamma)$ with the initial condition $\gamma(0) = m$. Let (s_1, s_2) be the maximal interval on which the solution is defined. Then $\gamma(s_1)$ and $\gamma(s_2)$ are the desired points. Thus

$$|\varphi(m_1) - \varphi(m_2)| = r_\Omega(m_1, m_0) + r_\Omega(m_2, m_0) \geqslant r_\Omega(m_1, m_2);$$

that is, $m_0 \in I_\varphi$. The proof of the theorem is complete. □

5. Theorems 1 and 2 imply the following assertion.

Theorem 3. *Let $M \times_\delta \mathbb{R}$ be a Lorentzian warped product, where the manifold M and the function δ belong to C^2. Let $\Omega \subset M$ be a subdomain with compact boundary $\partial\Omega_r$ with respect to the metric* (2)*, and let $\varphi\colon \partial\Omega_r \to \mathbb{R}$ be a Lipschitz function in the metric r_Ω. Suppose that the uniqueness set of φ satisfies $U_\varphi = \Omega$, and let $\tilde f = \underline{f}|_\Omega = \overline{f}|_\Omega$. Then $\tilde f \in C^1(\Omega)$ and $\delta^{1/2}(m)|\nabla\tilde f(m)|$ is constant on Ω; moreover*

$$\delta^{1/2}(m)|\nabla\tilde f(m)| \equiv \sup_{\substack{m',m''\in\partial\Omega_r\\ m'\neq m''}} \frac{|\varphi(m') - \varphi(m'')|}{r_\Omega(m', m'')}\,.$$

Proof. Just as in the proof of Theorem 2, we can assume that

$$\sup_{\substack{m',m''\in\partial\Omega_r\\ m'\neq m''}} \frac{|\varphi(m') - \varphi(m'')|}{r_\Omega(m', m'')} = 1.$$

Then $I_\varphi = U_\varphi = \Omega$ by Theorem 1 and $\tilde f = \overline{f} \in C^1(\Omega)$ by Theorem 2; moreover, $\delta^{1/2}(m)|\nabla\tilde f(m)| \equiv 1$. □

The corresponding assertion for graphs in $\mathbb{R}^n$ is due to Aronsson [6].

6. Note that the isotropy of a surface implies a special structure of F. Namely, F consists of lightlike segments, i.e. straight lines with lightlike tangent vector at each point. However, there may be no isotropic extension if we replace the condition $\Omega \subset I_\varphi$ by the following weaker condition. Each point $(m_1, \varphi(m_1))$ of the contour can be joined with some other point $(m_2, \varphi(m_2))$ of the contour by lightlike line. Indeed, consider the following example.

Let $M = \mathbb{R}^2$ and $\delta \equiv 1$; thus $M \times_\delta \mathbb{R}$ is the Minkowski space-time $\mathbb{R}^3_1$. Let Ω be the ball $B = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$. Consider the circle ∂B divided by the points $(1/\sqrt2, 1/\sqrt2)$, $(1/\sqrt2, -1/\sqrt2)$, $(-1/\sqrt2, -1/\sqrt2)$, and $(-1/\sqrt2, 1/\sqrt2)$ into four arcs S_1, S_2, S_3, and S_4, numbered counterclockwise starting from the upper arc. Then the function

$$\varphi(x) = \begin{cases} x_1 & \text{for } x \in S_1,\\ x_2 & \text{for } x \in S_2,\\ -x_1 & \text{for } x \in S_3,\\ -x_2 & \text{for } x \in S_4 \end{cases}$$

has all above-mentioned properties. Moreover, $I_\varphi = \overline{B} \setminus K$, where K is the open square

$$(-1/\sqrt2, 1/\sqrt2) \times (-1/\sqrt2, 1/\sqrt2)\,;$$

i.e., B does not lie in I_φ. It follows from our theorem that there is no smooth isotropic extension of φ into the entire domain B.

Condition (4) also cannot be omitted. It suffices to consider the following example. Let $\Omega = \{x = (x_1, x_2) : 1 < x_1^2 + x_2^2 < R^2\}$, $\varphi = 0$ for $|x| = 1$, and $\varphi = \sqrt{R^2 - 1}$ for $|x| = R$. Clearly, there is no isotropic extension in spite of the fact that I_φ contains Ω. $\square$

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