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Calculating and analyzing impulse responses for the vector ARFIMA model

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Abstract

We use the technique of finite-order generating functions to examine impulse responses of the vector fractionally integrated autoregressive moving-average model. We show that impulse responses and their asymptotic variances of such a model evolve at the slow hyperbolic rates. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Mittnik and Zadrozny (1993) and Lütkepohl (1990) recently presented the asymptotic theory for impulse responses in the vector autoregressive moving-average model. In this note we apply the technique of finite-order generating functions to the analysis of impulse responses in the even more general vector fractionally integrated autoregressive moving-average (VARFIMA) model. Mittnik and Zadrozny gave two types of expressions for impulse responses — the recursive forms and the closed forms. We argue that our generating function approach possesses the virtues of both forms. Although being intrinsically recursive, the generating function expressions look like closed forms and are directly available for computer programming. The usefulness of the generating function method is clearly demonstrated in its application to the VARFIMA model.

In the exposition of the VARFIMA model we first derive complete expressions for impulse responses and their asymptotic variances. We then specialize to the univariate ARFIMA model to prove that cumulative impulse responses evolve at slow hyperbolic rates as the lag increases. These

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hyperbolic rates differ from the geometric rates of the ARMA model. The hyperbolic impulse responses of the ARFIMA model imply similar hyperbolic patterns in its autocorrelations, which is usually referred to as long memory as opposed to the short memory of the ARMA model. Because of the sharp distinction in the impulse response patterns between the ARFIMA model and the ARMA model, the impulse response analysis has become an important tool in the study of long memory economic series. See, for example, Diebold and Rudebusch's (1989, 1991) study on aggregate income; Diebold et al. (1991) and Cheung's (1993) work on exchange rates; Baillie et al.'s (1996a,b) paper on inflation rates; and Baillie et al.'s (1996a,b) and Chung's (2000) study on volatility (conditional variances). The results in this note should facilitate the impulse response analysis in this growing literature.

2. Calculating impulse responses

Let us start with the general stationary and invertible vector ARFIMA (VARFIMA) model of the m-variate process $\mathbf{x}_t$:

$$\boldsymbol{\Phi}(B)\,\boldsymbol{\Delta}^d(B)(\boldsymbol{x}_t-\boldsymbol{\mu})=\boldsymbol{\Theta}(B)\,\boldsymbol{\varepsilon}_t$$

where B is the lag operator, $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\varepsilon}_t$ is white noise with $\boldsymbol{E}(\boldsymbol{\varepsilon}_t)=\mathbf{0}$ and $\mathrm{Var}(\boldsymbol{\varepsilon}_t)=\mathbf{RR}'$. The ARMA operators $\boldsymbol{\Phi}(B)=\boldsymbol{\Phi}_0-\Sigma_{j=1}^{p}\boldsymbol{\Phi}_jB^j$ and $\boldsymbol{\Theta}(B)=\boldsymbol{\Theta}_0+\Sigma_{j=1}^{q}\boldsymbol{\Theta}_jB^j$ are $m\times m$ matrix polynomials in B. The coefficients of polynomials $\boldsymbol{\Phi}(B)$and $\boldsymbol{\Theta}(B)$ are assumed to satisfy the standard stationarity and invertibility conditions and are nonlinear functions of some estimable parameters after normalization and identification restrictions are imposed. The operator $\boldsymbol{\Delta}^d(B)$ is an $m\times m$ diagonal matrix characterized by the m parameters $d_1, d_2,\dots,d_m$ as follows:

$$\boldsymbol{\Delta}^d(B)\equiv\begin{bmatrix}(1-B)^{d_1} & 0 & \cdots & 0\\ 0 & (1-B)^{d_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & (1-B)^{d_m}\end{bmatrix}$$

where $d_1, d_2,\dots,d_m$, can be fractional numbers. The term $(1-B)^{d_i}$ is defined by the following power expansion:

$$(1-B)^{-d_i}=\sum_{j=0}^{\infty}\psi_j^{(d_i)}B^j\equiv\psi^{(d_i)}(B)$$

where $\psi_j^{(d_i)}=\Gamma(d_i+j)/[\Gamma(j+1)\,\Gamma(d_i)]$, $\psi_0^{(0)}=1$, and $\psi_j^{(0)}=0$, for $j\neq 1$. Consequently, we have:

$$[\boldsymbol{\Delta}^d(B)]^{-1}=\begin{bmatrix}\psi^{(d_1)}(B) & 0 & \cdots & 0\\ 0 & \psi^{(d_2)}(B) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \psi^{(d_m)}(B)\end{bmatrix}\equiv\boldsymbol{\Psi}^{(d)}(B)=\mathbf{I}_m+\sum_{j=1}^{\infty}\boldsymbol{\Psi}_j^{(d)}B^j \tag{1}$$

where $\mathbf{I}_m$ is the identity matrix of order m and each $\boldsymbol{\Psi}_j^{(d)}$ is an $m\times m$ diagonal matrix with $\psi_j^{(d_i)}$ as the

ith diagonal element. To stress the central role of the fractional differencing parameters $d_1, d_2, \ldots, d_m$ in our analysis, we call $\mathbf{x}_t$ the VARFIMA($d_1, \ldots, d_m$) process.

If $-0.5 < d_i < 0.5$, for all i, then the VARFIMA($d_1, \ldots, d_m$) model is both stationary and invertible (Hosking, 1981, 1996; Chung, 2001) so that it has the infinite moving-average representation: $\mathbf{x}_t = \boldsymbol{\mu} + \mathbf{A}(B)\,\mathbf{u}_t$, where $\mathbf{u}_t = \mathbf{R}^{-1}\boldsymbol{\varepsilon}_t$, and $\mathbf{A}(z) \equiv \mathbf{I}_m + \Sigma_{j=1}^{\infty}\ \mathbf{A}_j z^j$. The moving-average coefficients $\mathbf{A}_j$ are referred to as impulse responses and each of them reflects the change in $\mathbf{x}_t$ caused by a shock in previous $\mathbf{u}_{t-j}$, for $j = 1,2, \ldots$. We note $\mathrm{Var}(\mathbf{u}_t) = \mathbf{I}_m$ for all t. Throughout our analysis, we concentrate on the following truncated version of $\mathbf{A}(z)$: $\mathbf{A}_n(z) = \mathbf{I}_m + \Sigma_{j=1}^{n}\ \mathbf{A}_j z^j$, for a finite n. We will show that the impulse responses analysis can be substantially simplified if we follow a simple rule to express them as products of finite-order power series. Let's first define such a rule of operation.

Given two arbitrary power series (in $m \times m$ matrices) of different orders $\mathbf{P}(z) = \mathbf{I}_m + \Sigma_{j=1}^{\ell}\ \mathbf{P}_j z^j$ and $\mathbf{Q}(z) = \mathbf{I}_m + \Sigma_{j=1}^{h}\ \mathbf{Q}_j z^j$, we use a special notation $\overset{n}{\asymp}$ to denote the operation of truncating the series that result from power series multiplication: $\mathbf{M}_n(z) \overset{n}{\asymp} \mathbf{P}(z)\mathbf{Q}(z)$, where $\mathbf{M}_n(z) = \mathbf{I}_m + \Sigma_{j=1}^{n}\ \mathbf{M}_j z^j$ is power series with the n matrix coefficients being defined by $\mathbf{M}_j = \Sigma_{k=0}^{j}\ \mathbf{P}_k \mathbf{Q}_{j-k}$, $j = 1, \ldots, n$. Here, we set $\mathbf{P}_0 = \mathbf{Q}_0 = \mathbf{I}_m$, $\mathbf{P}_k = \mathbf{0}$ for $k > \ell$, and $\mathbf{Q}_k = \mathbf{0}$ for $k > h$. The basic properties of the operation $\overset{n}{\asymp}$ are included in Lemma 1 in Appendix A. We can also define the truncated series that result from the power series inversion by $\mathbf{N}_n(z) \overset{n}{\asymp} \mathbf{Q}(z)^{-1}\mathbf{P}(z)$, where $\mathbf{N}_n(z) = \mathbf{I}_m + \Sigma_{j=1}^{n}\ \mathbf{N}_j z^j$ is power series with the n matrix coefficients being defined through recursion: $\mathbf{N}_j = \mathbf{P}_j + \Sigma_{k=1}^{s}\ \mathbf{Q}_k \mathbf{N}_{j-k}$, $j = 1, \ldots, n$. Here, we set $\mathbf{P}_k = \mathbf{0}$ for $k > \ell$ and $s \equiv \min\{n,\ j\}$. The basic properties of $\mathbf{N}_n(z)$ are included in Lemma 2 in Appendix A.

The key idea of our formulation is that any n impulse responses in a VARFIMA($d_1, \ldots, d_m$) model can be exactly calculated from the n coefficients of a finite-order power series resulting from truncated power series multiplication and inversion:

$$\mathbf{A}_n(z) = \mathbf{A}_0 + \sum_{j=1}^{n} \mathbf{A}_j z^j \overset{n}{\asymp} \boldsymbol{\Psi}_n^{(d)}(z)\,\boldsymbol{\Phi}(z)^{-1}\,\boldsymbol{\Theta}(z)\,\mathbf{R} \tag{2}$$

where

$$\boldsymbol{\Psi}_n^{(d)}(z) = \mathbf{I}_m + \sum_{j=1}^{n} \boldsymbol{\Psi}_j^{(d)} z^j \overset{n}{\asymp} [\boldsymbol{\Delta}^d(B)]^{-1} \tag{3}$$

is the truncated version of the infinite series $\boldsymbol{\Psi}^{(d)}(z)$ in (1). We call (2) the impulse response generating function for the VARFIMA($d_1, \ldots, d_m$) model. The operator $\overset{n}{\asymp}$ is used simply to represent the truncation rule when only finitely many impulse responses $\mathbf{A}_j$ are desired. We note $\overset{n}{\asymp}$ does *not* mean the n impulse responses $\mathbf{A}_j$ are approximated. They are all computed exactly. Also, if all d_i are equal to zero so that $\boldsymbol{\Psi}_n^{(d)}(z) = \mathbf{I}_m$, then (2) gives the impulse response generating function for the vector ARMA model.

It is also common to consider the (n-horizon) cumulative impulse responses, or (n-horizon) dynamic multipliers, which are defined as $\bar{\mathbf{A}}_n = \mathbf{I}_m + \Sigma_{j=1}^{n}\ \mathbf{A}_j = \mathbf{A}_n(1)$, for all $n > 0$. To construct the generating function for cumulative impulse responses, we need to first define the nth order power expansion of $(\mathbf{I}_m - \mathbf{I}_m z)^{-1}$ which is $\mathbf{L}_n(z) = \mathbf{I}_m + \Sigma_{j=1}^{n}\ \mathbf{I}_m z^j \overset{n}{\asymp} (\mathbf{I}_m - \mathbf{I}_m z)^{-1}$. With $\mathbf{L}_n(z)$, the cumulative impulse response generating function can then be expressed as:

$$\bar{\mathbf{A}}_n(z) = \bar{\mathbf{A}}_0 + \sum_{j=1}^{n} \bar{\mathbf{A}}_j z^j \overset{n}{\asymp} \mathbf{L}_n(z)\, \mathbf{A}_n(z) \overset{n}{\asymp} \mathbf{L}_n(z)\, \boldsymbol{\Psi}_n^{(d)}(z)\, \boldsymbol{\Phi}(z)^{-1}\, \boldsymbol{\Theta}(z)\, \mathbf{R} \tag{4}$$

The series $\mathbf{L}_n(z)$ is a convenient tool as indicated in the following proposition where the relationship between $\mathbf{L}_n(z)$ and the generating functions $\boldsymbol{\Psi}_n^{(d)}(z)$ in (3) is examined. This proposition can be easily proved by induction. Let's first define $\boldsymbol{\Psi}_n^{(d+k)}(z)$ to be the operator which is exactly the same as $\boldsymbol{\Psi}_n^{(d)}(z)$ except that the value of each of the m parameters d_i is increased by k. Also note that $\mathbf{L}_n(z)^{-1} = \mathbf{I}_m - \mathbf{I}_m z$.

Proposition 1. *For any integer* $k = 0, \pm 1, \pm 2, \ldots,$ *we have* $[\mathbf{L}_n(z)]^k\, \boldsymbol{\Psi}_n^{(d)}(z) \overset{n}{\asymp} \boldsymbol{\Psi}_n^{(d+k)}(z)$. *Also,* $\boldsymbol{\Psi}_n^{(d)}(z) = \mathbf{L}_n(z)$ *if* $d_i = 1$ *for all* $i = 1, \ldots, m$.

Corollary 1. *The impulse response generating function of the VARFIMA*$(d_1 + k, \ldots, d_m + k)$ *model with an arbitrary integer k can be computed from* $\boldsymbol{\Psi}_n^{(d+k)}(z)\, \boldsymbol{\Phi}(z)^{-1}\, \boldsymbol{\Theta}(z)\, \mathbf{R} \overset{n}{\asymp} [\mathbf{L}_n(z)]^k\, \boldsymbol{\Psi}_n^{(d)}(z)\, \boldsymbol{\Phi}(z)^{-1}\, \boldsymbol{\Theta}(z)\, \mathbf{R}$

So by repeatedly multiplying the impulse response generating function of the VARFIMA$(d_1, \ldots, d_m)$ model by $\mathbf{L}_n(z)$ or $\mathbf{L}_n(z)^{-1}$, we may obtain the impulse response generating function of the VARFIMA$(d_1 + k, \ldots, d_m + k)$ model. Hence, impulse responses can always be computed even though the VARFIMA$(d_1 + k, \ldots, d_m + k)$ model may not be stationary or invertible. Also note that the cumulative impulse response generating function of the VARFIMA$(d_1, \ldots, d_m)$ model, defined in (4), is simply the impulse response generating function of the VARFIMA$(d_1 + 1, \ldots, d_m + 1)$ model.

Let $\boldsymbol{\lambda}$ be a vector that includes all estimable structural parameters in $\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_p$, $\boldsymbol{\Theta}_0(z)$, $\boldsymbol{\Theta}_1(z), \ldots, \boldsymbol{\Theta}_q(z)$, and $\mathbf{R}$, as well as the m fractional differencing parameters $d_1, d_2, \ldots, d_m$. Given the vectorization operator 'vec(·)', let's also stack vec$(\mathbf{A}_0)$, vec$(\mathbf{A}_1), \ldots,$ and vec$(\mathbf{A}_n)$ into an $nm^2 \times 1$ vector $\boldsymbol{\alpha}$. We note that impulse responses are nonlinear functions of the estimable parameters $\boldsymbol{\lambda}$, which are indirectly defined by (2), and can be written generally as $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\lambda})$. Given an estimator $\hat{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$, impulse responses $\boldsymbol{\alpha}$ can be estimated by $\boldsymbol{\alpha}(\hat{\boldsymbol{\lambda}})$.

Many estimators of $\boldsymbol{\lambda}$ have been proposed in the literature for the univariate ARFIMA model, such as the maximum likelihood estimators, both in the frequency-domain (Fox and Taqqu, 1986) and in the time-domain (Sowell, 1992), the conditional sum of squares estimator (Chung and Baillie, 1993; Baillie et al., 1996a,b), and the minimum distance estimator (Tieslau et al., 1996, Chung and Schmidt, 1999). Moreover, under stationarity and invertibility conditions, all of these estimators have normal limiting distributions. Let $\boldsymbol{\Sigma}(\boldsymbol{\lambda})$ be the corresponding asymptotic variance–covariance matrix, then the asymptotic variance–covariance matrix for $\boldsymbol{\alpha}(\hat{\boldsymbol{\lambda}})$ is $\mathbf{D}(\boldsymbol{\lambda})\, \boldsymbol{\Sigma}(\boldsymbol{\lambda})\, \mathbf{D}(\boldsymbol{\lambda})'$, where $\mathbf{D}(\boldsymbol{\lambda}) = \partial \boldsymbol{\alpha}(\boldsymbol{\lambda}) / \partial \boldsymbol{\lambda}$.

Since the operators of vectorization and Kronecker product are linear, we find $\mathbf{D}(\boldsymbol{\lambda})$ is exactly (not just approximately) equal to the stack of the coefficients of the following matrix generating function:

$$\begin{aligned} \frac{\partial \mathrm{vec}[\mathbf{A}_n(z)]}{\partial \boldsymbol{\lambda}} &\overset{n}{\asymp} [\mathbf{R}'\boldsymbol{\Theta}(z)'\, \boldsymbol{\Phi}(z)'^{-1} \otimes \mathbf{I}_m]\, \frac{\partial \mathrm{vec}[\boldsymbol{\Psi}_n^{(d)}(z)]}{\partial \boldsymbol{\lambda}} \\ &+ [\mathbf{R}' \otimes \boldsymbol{\Psi}_n^{(d)}(z)\, \boldsymbol{\Phi}(z)^{-1}]\, \frac{\partial \mathrm{vec}[\boldsymbol{\Theta}(z)]}{\partial \boldsymbol{\lambda}} + [\mathbf{I}_m \otimes \boldsymbol{\Psi}_n^{(d)}(z)\, \boldsymbol{\Phi}(z)^{-1}\, \boldsymbol{\Theta}(z)]\, \frac{\partial \mathrm{vec}(\mathbf{R})}{\partial \boldsymbol{\lambda}} \\ &- [\mathbf{R}'\boldsymbol{\Theta}(z)'\, \boldsymbol{\Phi}(z)'^{-1} \otimes \boldsymbol{\Psi}_n^{(d)}(z)\, \boldsymbol{\Phi}(z)^{-1}]\, \frac{\partial \mathrm{vec}[\boldsymbol{\Phi}(z)]}{\partial \boldsymbol{\lambda}} \end{aligned} \tag{5}$$

The procedure for computing the derivatives of the cumulative impulse responses is quite similar. All it requires is to multiply (5) by $\mathbf{I}_m \otimes \mathbf{L}_n(z)$.

While the computation involved in (4) and (5) for the vector ARMA case (i.e. $d_1 = \cdots = d_m = 0$) is identical to Mittnik and Zadrozny's (1993) recursive forms [their equations (3) and (5)], it seems easier to write computer programs based on our formulation. The advantage of the expressions (4) and (5) is that, by using the notation of generating functions, we can avoid the messy expressions of sums of products and recursion as well as some unusual large matrices such as the permutation matrix. In the terminology of computer programming, what we do here essentially is using products and inverse of generating functions to symbolize the computer programs that calculate the sums of products and recursion. We also note that inversion is always with respect to the same series $\mathbf{\Phi}(z)$. With such uniformity, the resulting computer programs will be highly structural and efficient. Because (5) includes all parameters, it may in fact be viewed as a kind of closed-form expression that resembles Mittnik and Zadrozny's equation (8).

Note that in the univariate case with $m = 1$, all the $m \times m$ matrices reduce to scalars and the VARFIMA($d_1, \ldots, d_m$) model scales down to the usual univariate ARFIMA(p, d, q) model. In such a case the vector $\boldsymbol{\lambda}$ contains $1 + p + q$ scalar parameters d, $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_p$, $\mathbf{\Theta}_1, \ldots, \mathbf{\Theta}_q$. The $1 + p + q$ columns of $\mathbf{D}(\boldsymbol{\lambda})$ are simply the coefficients of the power series $\partial \mathbf{A}_n(z)/\partial d$, $\partial \mathbf{A}_n(z)/\partial \mathbf{\Phi}_j$, $j = 1, \ldots, p$, and $\partial \mathbf{A}_n(z)/\partial \mathbf{\Theta}_j$, $j = 1, \ldots, q$, respectively, while these power series can be easily computed as follows:

$$\frac{\partial \mathbf{A}_n(z)}{\partial d} \overset{n}{\asymp} \mathbf{\Phi}(z)^{-1}\, \mathbf{\Theta}(z)\, \frac{\partial \mathbf{\Psi}_n^{(d)}(z)}{\partial d} \overset{n}{\asymp} \mathbf{\Phi}(z)^{-1}\, \mathbf{\Theta}(z) \sum_{j=1}^{n} \left\{ \left[\mathbf{\Psi}_j^{(d)} \sum_{k=0}^{j-1} (d+k)^{-1} \right] z^j \right\} \tag{6}$$

$$\frac{\partial \mathbf{A}_n(z)}{\partial \mathbf{\Phi}_j} \overset{n}{\asymp} -\mathbf{\Psi}_n^{(d)}(z)\, \mathbf{\Phi}(z)^{-2}\, \mathbf{\Theta}(z)\, \frac{\partial \mathbf{\Phi}(z)}{\partial \mathbf{\Phi}_j} \overset{n}{\asymp} \mathbf{\Phi}(z)^{-1}\, \mathbf{A}_n(z)\, z^j, \quad j = 1,2, \ldots, p \tag{7}$$

$$\frac{\partial \mathbf{A}_n(z)}{\partial \mathbf{\Theta}_j} \overset{n}{\asymp} \mathbf{\Psi}_n^{(d)}(z)\, \mathbf{\Phi}(z)^{-1}\, \frac{\partial \mathbf{\Theta}(z)}{\partial \mathbf{\Theta}_j} \overset{n}{\asymp} \mathbf{\Psi}_n^{(d)}(z)\, \mathbf{\Phi}(z)^{-1}\, z^j, \quad j = 1,2, \ldots, q \tag{8}$$

The last results in (7) and (8) are based on the facts that $-\partial \mathbf{\Phi}(z)/\partial \mathbf{\Phi}_j = \partial \mathbf{\Theta}(z)/\partial \mathbf{\Theta}_j = z^j$, while the last result in (6) is due to the equalities $\partial \mathbf{\Psi}_j^{(d)}/\partial d = \mathbf{\Psi}_j^{(d)}[\partial \ln \Gamma(d+j)/\partial d - \partial \ln \Gamma(d)/\partial d] = \mathbf{\Psi}_j^{(d)} \Sigma_{k=0}^{j-1} (d+k)^{-1}$. See Gradshteyn and Ryzhik (1980, Eq. (8.365.3)).

3. Analyzing impulse responses

In this section, without loss of generality, we consider the univariate case with $m = 1$; i.e. the usual univariate ARFIMA(p, d, q) model. From a detailed analysis of such a univariate model we will see that the formulation of the impulse response generating function (4) not only facilitates computer programming but is quite useful for the analysis of impulse responses. This point is clearly demonstrated by the results in the following proposition, which have been suggested by various authors (e.g. Diebold and Rudebusch, 1989, 1991) but never formally proven.

Proposition 2. *The cumulative impulse response of the ARMA*(p, q) *model converge to* $[\mathbf{\Theta}(1)/\mathbf{\Phi}(1)]$

at a geometric rate: $\overline{\mathbf{A}}_n \sim [\boldsymbol{\Theta}(1)/\boldsymbol{\Phi}(1)](1 - r^{n+1})$, *as* $n \to \infty$, *for some* $r \in (0, 1)$, *while the cumulative impulse responses of the ARFIMA*(p, d, q) *model with* $d \neq 0$ *evolve at a hyperbolic rate*: $\overline{\mathbf{A}}_n \sim [\boldsymbol{\Theta}(1)/\boldsymbol{\Phi}(1)][\Gamma(d+1)]^{-1} n^d$, *as* $n \to \infty$.

The proof of this proposition is given in Appendix A. From the fact that the cumulative impulse responses of the ARFIMA(p, d, q) model are the same as the impulse responses of the ARFIMA(p, $d+1$, q) model, we immediately get the following result:

Corollary 2. *The impulse response of the ARIMA*$(p, 1, q)$ *model converge to* $[\boldsymbol{\Theta}(1)/\boldsymbol{\Phi}(1)]$ *at a geometric rate*: $\mathbf{A}_n \sim [\boldsymbol{\Theta}(1)/\boldsymbol{\Phi}(1)](1 - r^{n+1})$, *as* $n \to \infty$, *for some* $r \in (0, 1)$, *while the impulse response of the ARFIMA*(p, d, q) *model with* $d \neq 1$ *evolve at a hyperbolic rate*: $\mathbf{A}_n \sim [\boldsymbol{\Theta}(1)/\boldsymbol{\Phi}(1)][\Gamma(d)]^{-1} n^{d-1}$, *as* $n \to \infty$.

These results imply that the impulse responses $\mathbf{A}_n$ of the ARFIMA(p, d, q) model are divergent hyperbolically when $d > 1$, while they converge geometrically to $[\boldsymbol{\Theta}(1)/\boldsymbol{\Phi}(1)]$ when $d = 1$, and hyperbolically to zero when $d < 1$. Some authors (e.g. Diebold et al., 1991) have described a process as *mean reverting* if the corresponding impulse responses are convergent to zero (so that the effect of a shock in the innovation dies out eventually). Here, we note when $0.5 < d < 1$, the ARFIMA(p, d, q) model is mean reverting even though it is not stationary.

The properties of impulse responses of the ARMA(p,q) model can be derived from Corollary 2 as indicated in the following corollary.

Corollary 3. *The impulse responses of the ARMA*(p, q) *model converge to zero at the hyperbolic rates*: $\mathbf{A}_n \sim [\boldsymbol{\Theta}(1)/\boldsymbol{\Phi}(1)]\, n^{-1}$, *as* $n \to \infty$.

By applying the same analysis to the generating functions (6)–(8) for the derivatives of impulse responses, we can also deduce certain patterns in the $n \times n$ asymptotic variance–covariance matrix $\mathbf{D}(\boldsymbol{\lambda})\, \boldsymbol{\Sigma}(\boldsymbol{\lambda})\, \mathbf{D}(\boldsymbol{\lambda})'$, particularly, among its n diagonal elements. It is briefly described in the appendix. Finally, we note the technique suggested here can be extended to some univariate models that are even more general than the ARFIMA model. Gray et al. (1989) and Chung (1996a,b) have considered a class of models that allows persistent cycles in addition to the persistent trend which characterizes the ARFIMA model. The same tool of generating functions is also applicable to that model.

Appendix A

Lemma 1. *Given the definition of power series multiplication in the text, we have the following results.* [1] *The equality* $\mathbf{M}_n(z) = \mathbf{P}(z)\,\mathbf{Q}(z)$ *holds when* $n \geq \ell h$. [2] *If the order* ℓ *of* $\mathbf{P}(z)$ *is infinity, then* $\mathbf{M}_n(z) \stackrel{n}{\asymp} \mathbf{P}_n(z)\ \mathbf{Q}(z)$, *where* $\mathbf{P}_n(z) \equiv \mathbf{I}_m + \Sigma_{j=1}^{n}\ \mathbf{P}_j z^j$ *is the truncated version of* $\mathbf{P}(z)$. *The same truncation procedure can be applied to* $\mathbf{Q}(z)$ *if its order is infinite. That is, truncation can be applied to both sides of* $\stackrel{n}{\asymp}$. [3] *We have the following equalities* $\mathbf{M}_n(1) \equiv \Sigma_{j=0}^{n}\ \mathbf{M}_j = \Sigma_{j=0}^{n}\ \mathbf{P}_j\ \Sigma_{k=0}^{n-j}\ \mathbf{Q}_k$, *where* $\mathbf{P}_k = \mathbf{0}$ *for* $k > \ell$ *and* $\mathbf{Q}_k = \mathbf{0}$ *for* $k > h$. (*The proof of this lemma is straightforward and thus omitted.*)

Lemma 2. *Given the definition of power series division in the text, we have the following results.* [1] $\mathbf{N}_n(z) \stackrel{n}{\asymp} \mathbf{Q}(z)^{-1}\,\mathbf{P}(z)$ *is equivalent to* $\mathbf{Q}(z)\ \mathbf{N}_n(z) \stackrel{n}{\asymp} \mathbf{P}_n(z)$, *where* $\mathbf{P}_n(z) \equiv \mathbf{I}_m + \Sigma_{j=1}^{n}\ \mathbf{P}_j z^j$. [2] $\mathbf{N}_\infty(z) =$

$\mathbf{Q}(z)^{-1}\,\mathbf{P}(z)$. *Hence*, $\mathbf{N}_\infty(1) = \mathbf{Q}(1)^{-1}\,\mathbf{P}(1)$. [3] *Given that* $m = 1$, *if the orders of* $\mathbf{P}(z)$ *and* $\mathbf{Q}(z)$ *are finite and their roots are outside the unit circle, then* $\mathbf{N}_n(1) \equiv \Sigma_{j=0}^{n} \mathbf{N}_j \sim [\mathbf{Q}(1)^{-1}\,\mathbf{P}(1)](1 - r^{n+1})$, *as* $n \to \infty$, *for some* $r \in (0, 1)$. *That is*, $\mathbf{N}_n(1)$ *converges to* $\mathbf{Q}(1)^{-1}\mathbf{P}(1)$ *at a geometric rate.*

Proof. The proofs of [1] and [2] are straightforward and omitted here. To prove [3], we first note that there is another way to compute $\mathbf{N}_j$ as follows. Let ξ_j, $j = 1, \dots, h$ be the reciprocals of the roots of the equation $\mathbf{Q}(z) = 0$. That is, $\mathbf{Q}(z) = \prod_{i=1}^{h}(1 - \xi_i z)$. Since $(1 - \xi_i z)^{-1} = 1 + \Sigma_{j=1}^{\infty}(\xi_i z)^j$, we have $\mathbf{N}_n(z) \overset{n}{\asymp} \mathbf{Q}(z)^{-1}\,\mathbf{P}(z) \overset{n}{\asymp} \prod_{i=1}^{h}\ \xi_{i,n}(z)\ \mathbf{P}(z)$, where $\xi_{i,n}(z) = 1 + \Sigma_{j=1}^{n}(\xi_i z)^j$ is the truncated version of $(1 - \xi_i z)^{-1}$. It suffices to prove the case with $h = 3$. Let's define $\mathbf{Q}_{h,n}(z) \overset{n}{\asymp} \prod_{i=1}^{h}\ \xi_{i,n}(z)$. By item [3] of Lemma 1, we have:

$$\mathbf{Q}_{2,n}(1) = \sum_{j=0}^{n} \xi_1^j \sum_{k=0}^{n-j} \xi_2^k = \sum_{j=0}^{n} \xi_1^j \frac{1 - \xi_2^{n-j+1}}{1 - \xi_2} = \frac{1}{1 - \xi_2}\left[\frac{1 - \xi_1^{n+1}}{1 - \xi_1} - \sum_{j=0}^{n} \xi_1^j \xi_2^{n-j+1}\right]$$
$$\sim \frac{1 - r^{n+1}}{(1 - \xi_1)(1 - \xi_2)}$$

Here, $1 > r > |\xi_*|$ with $\xi_* \equiv \max\{\xi_1, \xi_2\}$. Note that the term $\Sigma_{j=0}^{n}\ \xi_1^j \xi_2^{n-j+1}$ can be omitted from the above approximation since its absolute value is smaller than $(n + 1)|\xi_*|^{n+1}$ which converges to zero at the geometric rate (this argument will be implicitly used throughout this appendix). By repeating the same algebra we have:

$$\mathbf{Q}_{3,n}(1) = \sum_{j=0}^{n} \xi_3^j \mathbf{Q}_{2,n-j}(1) \sim \sum_{j=0}^{n} \xi_3^j \frac{1 - r^{n-j+1}}{(1 - \xi_1)(1 - \xi_2)} \sim \frac{1 - r^{n+1}}{(1 - \xi_1)(1 - \xi_2)(1 - \xi_3)} = \frac{1 - r^{n+1}}{\mathbf{Q}(1)}$$

Finally,

$$\mathbf{N}_n(1) = \sum_{j=0}^{n} \mathbf{Q}_{3,n-j}(1)\,\mathbf{P}_j \sim \mathbf{Q}(1)^{-1} \sum_{j=0}^{n} \mathbf{P}_j(1 - r^{n-j+1}) \sim [\mathbf{Q}(1)^{-1}\,\mathbf{P}(1)](1 - r^{n+1})$$

This completes the proof. Q.E.D.

Proof of Proposition 2. From Lemma 2 we immediately get the result in the first part for the case $d = 0$. To show the result in the second part, let's define $\tau_n(z) \overset{n}{\asymp} \mathbf{\Phi}(z)^{-1}\,\mathbf{\Theta}(z)$, which is the impulse response generating function of the ARMA(p, q) model. From the result in the first part, we have:

$$\tau_n(1) \equiv \sum_{j=0}^{n} \tau_j \sim [\mathbf{\Theta}(1)/\mathbf{\Phi}(1)](1 - r^{n+1})$$

Also, from Proposition 1, we have:

$$\mathbf{\Psi}_n^{(d)}(1) = \sum_{j=0}^{n} \mathbf{\Psi}_j^{(d)} = \mathbf{\Psi}_n^{(d+1)} = \Gamma(d + n + 1)/\Gamma(n + 1)\,\Gamma(d + 1) \sim [\Gamma(d + 1)]^{-1}\, n^d$$

where the approximation is based on Stirling's formula. Now, using these two results as well as Eq.

(2) and item [3] of Lemma 1 we can approximate the cumulative impulse response of the ARFIMA(p, d, q) model with $d \neq 0$ as follows:

$$\begin{aligned}\bar{\mathbf{A}}_n &\overset{n}{\asymp} \mathbf{\Psi}_n^{(d)}(1)\,\tau_n(1) = \sum_{j=0}^{n} \mathbf{\Psi}_j^{(d)} \sum_{k=0}^{n-j} \tau_k \sim \sum_{j=0}^{n} \mathbf{\Psi}_j^{(d)} \frac{\mathbf{\Theta}(1)}{\mathbf{\Phi}(1)} (1 - r^{n-j+1}) \\ &\sim \frac{\mathbf{\Theta}(1)}{\mathbf{\Phi}(1)} \left[\mathbf{\Psi}_n^{(d)}(1) - \sum_{j=0}^{n} \mathbf{\Psi}_j^{(d)} r^{n-j+1} \right] \sim \frac{\mathbf{\Theta}(1)}{\mathbf{\Phi}(1)} \frac{1}{\Gamma(d+1)} n^d\end{aligned}$$

Q.E.D.

In the rest of this appendix we derive a changing pattern in the $n \times n$ asymptotic variance–covariance matrix $\mathbf{D}(\boldsymbol{\lambda})\ \mathbf{\Sigma}(\boldsymbol{\lambda})\ \mathbf{D}(\boldsymbol{\lambda})'$ since it is readily obtainable using the same analysis. Let the diagonal elements of $\mathbf{D}(\boldsymbol{\lambda})\ \mathbf{\Sigma}(\boldsymbol{\lambda})\ \mathbf{D}(\boldsymbol{\lambda})'$ be denoted by $\omega_1^2,\ \omega_2^2, \ldots, \omega_n^2$. The following lemma describes the changing pattern in ω_n as n approaches infinity. (It should be pointed out here that the pattern in ω_n *cannot* be viewed as the pattern in the standard errors of the estimated impulse responses. This is because ω_n can be considered as standard errors only when the sample size T approaches infinity. Moreover, ω_n are not directly computable since they depend on unknown parameters whose estimation brings in additional errors that are not considered here.)

Lemma 3. *For the ARFIMA*(p, d, q) *model we have* $\omega_n \sim [\mathbf{\Theta}(1)/\mathbf{\Phi}(1)][\Gamma(d)]^{-1}\ n^{d-1} \ln(n)$, *as* $n \to \infty$.

Proof. Using the same argument as in the proof of Proposition 2 and (6)–(8), we know $\partial \bar{\mathbf{A}}_n / \partial \boldsymbol{\lambda}$ is a $1 \times (1 + p + q)$ vector whose elements can be approximated, respectively, by $\partial \bar{\mathbf{A}}_n / \partial d \sim [\mathbf{\Theta}(1)/\mathbf{\Phi}(1)] \cdot \{[\partial \mathbf{\Psi}_n^{(d)}(1)/\partial d] - \sum_{j=0}^{n} [\partial \mathbf{\Psi}_j^{(d)}/\partial d]\ \ r^{n-j+1}\}$, $\partial \bar{\mathbf{A}}_n / \partial \mathbf{\Phi}_j \sim [\mathbf{A}_n(1)/\mathbf{\Phi}(1)] \cdot [-\partial \mathbf{\Phi}(1)/\partial \mathbf{\Phi}_j] \sim \mathbf{A}_n(1)/\mathbf{\Phi}(1)$, $j = 1, \ldots, p$, and $\partial \bar{\mathbf{A}}_n / \partial \mathbf{\Theta}_j \sim [\mathbf{\Psi}_n^{(d)}(1)/\mathbf{\Phi}(1)] \cdot [\partial \mathbf{\Theta}(1)/\partial \mathbf{\Theta}_j] \sim \mathbf{\Psi}_n^{(d)}(1)/\mathbf{\Phi}(1)$, $j = 1, \ldots, q$. From Proposition 2 we immediately see that all of $\partial \bar{\mathbf{A}}_n / \partial \mathbf{\Phi}_j$ and $\partial \bar{\mathbf{A}}_n / \partial \mathbf{\Theta}_j$ evolve at the same geometric rates when $d = 0$, or the same hyperbolic n^d rates when $d \neq 0$. In contrast, $\partial \bar{\mathbf{A}}_n / \partial d$ evolves at a slightly faster rate, as will be proven next, so that the behavior of ω_n for cumulative impulse response is determined completely by $\partial \bar{\mathbf{A}}_n / \partial d$ for a sufficiently large n. Since $\partial \mathbf{\Psi}_n^{(d)}(1)/\partial d = \partial \mathbf{\Psi}_n^{(d+1)}/\partial d = \mathbf{\Psi}_n^{(d+1)}[\partial \ln \Gamma(d + 1 + n)/\partial d - \partial \ln \Gamma(d + 1)/\partial d]$ and, from Gradshteyn and Ryzhik's (1980) equation 8.365.5, $\lim_{n\to\infty}[(\partial \ln \Gamma(d + 1 + n)/\partial d) - \ln(n)] = 0$, we have $\partial \mathbf{\Psi}_n^{(d)}(1)/\partial d \sim [\Gamma(d + 1)]^{-1}\ n^d [\ln(n) - 1] \sim [\Gamma(d + 1)]^{-1}\ n^d\ \ln(n)$. This result in turn implies $\partial \bar{\mathbf{A}}_n / \partial d \sim [\mathbf{\Theta}(1)/\mathbf{\Phi}(1)][\Gamma(d + 1)]^{-1}\ n^d\ \ln(n)$. Here, we note the rate $n^d \ln(n)$ is faster than the rate of n^d of other terms. Finally, we note that the cumulative impulse response of the ARFIMA(p, d, q) model are the same as the impulse response of the ARFIMA(p, $d + 1$, q) model. Consequently, for the impulse response of the ARFIMA(p, d, q) model we have $\omega_n \sim [\mathbf{\Theta}(1)/\mathbf{\Phi}(1)][\Gamma(d)]^{-1}\ n^{d-1} \ln(n)$. Q.E.D.

If in the ARIMA model the differencing parameter d is assumed to be 1 and not estimated, then it is straightforward to show that $\omega_n \sim C_1(1 - r^{n+1})$, as $n \to \infty$, where $r \in (0, 1)$ and C_1 is some non-zero constant. The proof follows the same line of arguments as in Lemma 3. So whether the differencing parameter d is estimated or not in the ARIMA(p, 1, q) model affects the pattern of ω_n. If d is unknown and estimated, then ω_n diverges logarithmically, while if d is correctly assumed to be one and not estimated, then ω_n will converge at a geometric rate.

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