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Discontinuous Control of the Brockett Integrator

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The problem of asymptotic stabilisation of the Brockett integrator has been addressed and solved in recent years with a variety of methods and approaches. In particular, several discontinuous control laws guaranteeing exponential convergence in an open and dense set have been proposed. In this work we show that all such discontinuous controllers can be obtained as special cases of a more general class of controllers. Furthermore, the problem of stabilisation with bounded control is also discussed and solved. Finally, we address the problem of controlling the kinematic model of an under-actuated satellite. Simulation results complete the work.

Keywords: Discontinuous stabilisation; Non-holonomic systems; Non-linear control

1. Introduction

Since the publication of a paper in which some necessary conditions for smooth stabilisation have been established [6], much attention has been devoted to the class of systems described by equations of the form

$$\dot{x} = G(x)u \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, with $m < n$, and $G(x)$ is a matrix of proper dimensions with smooth entries defined on $\mathbb{R}^n$ and with constant rank equal to m. The reason for such an interest lies in the fact that systems described by equations (1) cannot be asymptotically stabilised by any smooth, static or dynamic, time-invariant feedback law.

Among all systems described by equations of the form (1) a special place is occupied by the so-called Brockett integrator (or non-holonomic integrator), i.e. the system

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2 \end{aligned}\right\} \tag{2}$$

System (2) has both an historical relevance, as it was the first example of a locally controllable non-linear system which is not smoothly stabilisable,[1] and a theoretical importance, as it can be considered a benchmark for the design of discontinuous control laws yielding exponential convergence. Moreover, system (2) can be regarded as the simplest, in the class of systems described by equation (1), which displays the main features of such a class.[2] Finally, system (2) also has practical relevance as it describes, after a feedback transformation, the motion of a simple wheeled mobile robot (see [7] for details) and also describes a part of the dynamics of an induction motor with high-gain current loops [12]. Therefore several researchers have dealt with the system (2) as a prototype to design discontinuous, static, time-invariant feedback laws. In particular, discontinuous *stabilisers* have been designed via sliding modes [3–5], via the invariant manifold technique [9] and via discontinuous transformations [1].

The present work is organised as follows. In Section 2 we introduce the definition of *almost exponential stability*. In Sections 3, 4 and 5 we derive a

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[1]To be precise the first such system was proposed by Sussmann [14]; however, the Sussmann example requires bounds on the control signals and global considerations, whereas system 2 does not.
[2]It is even possible to show that any smooth system described by equations of the form (1) with $n = 3$ and $m = 2$ is (locally) feedback equivalent to system (2).

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family of almost stabilising controllers for the system (2), we discuss some features of the members of such a family and we show that, despite their different origins, all the control laws derived in the aforementioned works are particular members of this family. In Section 6 we address the problem of almost exponential stabilisation with bounded control. Finally, Section 7 applies the proposed results to the stabilisation of a (kinematic) under-actuated satellite.

2. Background Material

Throughout this paper we restrict our attention to system (2) and we consider discontinuous, time-invariant, state feedback control laws, i.e. control laws modelled by equations of the form

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \alpha_1(x) \\ \alpha_2(x) \end{bmatrix} \tag{3}$$

where the $\alpha_i : \mathbb{R}^3 \to \mathbb{R}$, for $i = 1, 2$, are discontinuous functions of their arguments. It must be noted that control laws which are not defined at $x = 0$, i.e. are unbounded at $x = 0$, are allowed. In particular, the term discontinuous will be used throughout this work in the sense introduced in [1], i.e. to denote functions which are unbounded, hence undefined, in a certain set; e.g. the function $1/x$ is discontinuous at $x = 0$. Moreover, in order to state our results in a compact and simple form we introduce the following definition.

Definition 1. A state feedback control law described by equations of the form (3) *almost exponentially stabilises*[3] the system (2) in the region Ω_0 if the following holds:

(i) for all initial states $x_0 \in \Omega_0$ the closed loop system admits a unique (forward) solution which remains in Ω_0 for all finite $t \geq 0$;
(ii) for all initial states $x_0 \in \Omega_0$ there exists a function $c_0 = c_0(x_0) > 0$, which is smooth in Ω_0, and a constant $\lambda_0 > 0$ such that for all $t \geq 0$ one has, along the trajectories of the closed loop system, $\|x(t)\| \leq c_0 \exp^{-\lambda_0 t}$.

It must be noted that the property of almost exponential stability is weaker than exponential stability (in the sense of Lyapunov) in various respects. Firstly, an almost exponentially stable system is not (in general) Lyapunov stable. Secondly, the set Ω_0 does not contain any neighbourhood of the origin. Finally, there exists $x^\star \in \partial\Omega_0$ such that $\lim_{x\to x^\star} c_0(x) = \infty$, or what is the same, the function $c_0(x)$ is not bounded in $\mathbb{R}^n$.

Nevertheless, this notion has proved to be adequate in the study of systems, like the Brockett integrator, which are not continuously stabilisable via static state feedback (see e.g. [1,2]).

[3] This terminology differs from that introduced in [13].

3. Preliminary Results

We now describe how a family of almost exponentially stabilising control laws can be obtained on the basis of the following simple and natural heuristic. The results of this section will be generalised in Sections 4 and 5.

Heuristic 1. To have exponential convergence of the state x_3 to zero, regardless of the states x_1 and x_2, we need to have

$$\dot{x}_3 = x_2 u_1 - x_1 u_2 = -g x_3 \tag{4}$$

for all $x_3 \in \mathbb{R}$, for all (x_1, x_2) in an open and dense set and for some positive constant g.

Lemma 1. Condition (4) cannot be fulfilled by any pair of continuous functions $u_1 = \alpha_1(x)$ and $u_2 = \alpha_2(x)$.

Proof. Proceed by contradiction. Suppose there exist continuous $\alpha_i(x)$, for $i = 1, 2$, fulfilling condition (4). It follows also that the function

$$x_2\alpha_1(x_1, x_2, x_3) - x_1\alpha_2(x_1, x_2, x_3)$$

is continuous. Hence, the limit

$$l_0 = \lim_{(x_1,x_2)\to(0,0)} \Big\{ \Big(x_2\alpha_1(x_1, x_2, x_3) - x_1\alpha_2(x_1, x_2, x_3) \Big) \Big\}$$

is well defined. We conclude

$$0 = l_0 = -g x_3$$

for all $x_3 \in \mathbb{R}$. Hence, a contradiction. □

Therefore, to satisfy condition (4), we need to make use of discontinuous $\alpha_i(x)$. However, in order to have an idea of the kind of discontinuity involved in the problem we prove the following stronger result.

Lemma 2. Condition (4) cannot be fulfilled by any pair of functions $u_1 = \alpha_1(x)$ and $u_2 = \alpha_2(x)$ which are continuous in $\mathbb{R}^3 - \{(0, 0, 0)\}$.

Proof. The proof is similar to that of Lemma 1, hence it is omitted. □

As a consequence of the previous discussion, we infer that the functions $\alpha_i(x)$ which must be used to fulfil condition (4) cannot be just discontinuous at the origin, i.e. on a set of dimension zero, but must be discontinuous on a larger set. We now proceed in the following way:

Step 1: We assume that the discontinuity in the $\alpha_i(x)$ has a special parameterised form, i.e. we set a structure for the $\alpha_i(x)$.

Step 2: We show that for such a form of discontinuity it is always possible to find (discontinuous) $u_1 = \alpha_1(x)$ and $u_2 = \alpha_2(x)$ such that condition (4) holds.

Step 3: We show that the region where the control is discontinuous is never crossed and can be rendered exponentially attractive. Moreover, we show that all trajectories starting outside the region where the control is discontinuous are defined for all $t \geq 0$.

Step 4: We derive a general almost exponential stabilisation result for the system (2), which recovers and generalises the results in [1,9].

Remark 1. The control law in [3–5] can be derived using a modified version of condition (4) (see Section 4).

3.1. Step 1

We consider control laws described by equations of the form

$$\left.\begin{aligned} u_1 &= -kx_1 + \frac{A_1}{ax_1^n + bx_2^n} \\ u_2 &= -kx_2 + \frac{A_2}{ax_1^n + bx_2^n} \end{aligned}\right\} \tag{5}$$

with k constant, $A_i = A_i(x_1, x_2, x_3)$ smooth functions to be determined, n integer bigger than or equal to one and a and b constants such that $|a| + |b| > 0$.

The region where the control law (5) is discontinuous is the set

$$D = \{x \in \mathbb{R}^3 \mid ax_1^n + bx_2^n = 0\}, \tag{6}$$

whereas the region where it is defined is the set

$$\Omega = \{x \in \mathbb{R}^3 \mid ax_1^n + bx_2^n \neq 0\} \tag{7}$$

Note that $D \cup \Omega = \mathbb{R}^3$. Observe, moreover, that the set D can be either a one dimensional surface of $\mathbb{R}^3$, e.g. when n is even and $ab > 0$, or a two-dimensional surface of $\mathbb{R}^3$, e.g. when n is odd and $ab > 0$, or, finally, a union of two-dimensional surfaces of $\mathbb{R}^3$, e.g. when n is even and $ab < 0$.

3.2. Step 2

Proposition 1. Let u_1 and u_2 be as in (5). Then there exist two smooth functions $A_1 = A_1(x_1, x_2, x_3)$ and $A_2 = A_2(x_1, x_2, x_3)$ such that condition (4) holds for any x_3 and for any (x_1, x_2) in an open and dense set.

Moreover, two such functions are

$$\left.\begin{aligned} A_1(x_1, x_2, x_3) &= -gbx_2^{n-1}x_3 \\ A_2(x_1, x_2, x_3) &= gax_1^{n-1}x_3 \end{aligned}\right\} \tag{8}$$

Proof. It suffices to substitute Eqs (8) in (5) and then in (4). □

In the remainder of this section we restrict our attention to the family of discontinuous control laws

$$\left.\begin{aligned} u_1 &= -kx_1 - g\frac{bx_2^{n-1}}{ax_1^n + bx_2^n}x_3 \\ u_2 &= -kx_2 + g\frac{ax_1^{n-1}}{ax_1^n + bx_2^n}x_3 \end{aligned}\right\} \tag{9}$$

and to the resulting (discontinuous) closed loop system

$$\left.\begin{aligned} \dot{x}_1 &= -kx_1 - g\frac{bx_2^{n-1}}{ax_1^n + bx_2^n}x_3 \\ \dot{x}_2 &= -kx_2 + g\frac{ax_1^{n-1}}{ax_1^n + bx_2^n}x_3 \\ \dot{x}_3 &= -gx_3 \end{aligned}\right\} \tag{10}$$

3.3. Step 3

Proposition 2. Consider the system (10). Any trajectory starting outside the set D, defined in (6), never crosses the discontinuity surface D. Moreover, if $k > 0$, D is exponentially attractive.

Proof. Let $\phi = ax_1^n + bx_2^n$. Simple calculations yield $\dot{\phi} = -kn\phi$, hence the claim. □

The above result can be used to show that any trajectory of the system (10) starting in

$$\Omega = \{x \in \mathbb{R}^3 \mid ax_1^n + bx_2^n \neq 0\}$$

exists for all $t \geq 0$, i.e. escape to infinity in finite time cannot occur, as illustrated in the following statement.

Proposition 3. Consider the system (10). Let $x(t) = (x_1(t), x_2(t), x_3(t)) : [0, T) \to \mathbb{R}^3$ denote a trajectory starting in Ω, i.e. $ax_1^n(0) + bx_2^n(0) \neq 0$.

Then $T = \infty$, i.e. the trajectory $x(t)$ is defined for all $t \geq 0$.

Proof. The proof is contained in the Appendix A. □

3.4. Step 4

Theorem 1. Consider the system (2) and the control law (9). Assume $g > nk > 0$. Then:

(i) The state feedback control law (9) almost exponentially stabilises system (2) in the open and dense set

$$\Omega = \{x \in \mathbb{R}^3 \mid ax_1^n + bx_2^n \neq 0\};$$

(ii) The control signals (9) are bounded along any trajectory of the closed loop system (2)–(9) starting in Ω.

Remark 2. The assumption $g > nk$, in Theorem 1, is necessary to ensure that

$$\frac{x_3}{ax_1^n + bx_2^n}$$

is bounded and converges (exponentially) to zero for any trajectory of the closed loop system (2)–(9) starting in Ω.

Proof. The proof is a trivial consequence of Propositions 1, 2 and 3. □

3.5. Connections with Previous Works

The control laws proposed in [9] can be obtained setting $n = 2$, $a = 1$ and $b = 1$ in the control law (9), whereas the control law in [1] can be obtained setting $b = 0$ in (9).

4. A New Heuristic

A natural extension of Heuristic 1 is the following:

Heuristic 2. To have asymptotic (exponential, finite time) convergence of the state x_3 to zero, regardless of the states x_1 and x_2, we need to have

$$\dot{x}_3 = x_2 u_1 - x_1 u_2 = \theta(x_3) \tag{11}$$

for all $x_3 \in \mathbb{R}$ and for all (x_1, x_2) in an open and dense set, where $\theta(x_3)$ is any function such that the trajectories of the system

$$\dot{x}_3 = \theta(x_3)$$

converge to zero asymptotically (exponentially, in finite time).

Heuristic 1 is a special case of Heuristic 2, hence a wider set of discontinuous control laws can be obtained making using of Heuristic 2. For example, setting $n = 2$, $a = b = 1$ and

$$\theta(x_3) = -\text{sign}(x_3)$$

in (9) (with $\theta(x_3)$ in place of $-gx_3$), we obtain the control law proposed in [3–5], i.e. the control law

$$u_1 = -x_1 - x_2 \frac{1}{x_1^2 + x_2^2} \text{sign}(x_3)$$

$$u_2 = -x_2 + x_1 \frac{1}{x_1^2 + x_2^2} \text{sign}(x_3)$$

which not only almost exponentially stabilises the system (2), also drives the state x_3 to zero in finite time.

5. A Family of Controllers

In the present section we show how more general discontinuous control laws can be built. Following the discussion in Section 3 it seems natural to consider control laws described by equations of the form

$$\left.\begin{aligned} u_1 &= -kx_1 + \frac{B_1(x_1, x_2)}{\phi(x_1, x_2)} \theta(x_3) \\ u_2 &= -kx_2 - \frac{B_2(x_1, x_2)}{\phi(x_1, x_2)} \theta(x_3) \end{aligned}\right\} \tag{12}$$

where $B_1(x_1, x_2)$, $B_2(x_1, x_2)$ and $\phi(x_1, x_2)$ are smooth functions to be determined. Notice that, on the basis of Lemmas 1 and 2, $\phi(0, 0) = 0$. The region where the control law (12) is discontinuous is the set

$$D_\phi = \{x \in \mathbb{R}^3 \mid \phi(x_1, x_2) = 0\} \tag{13}$$

whereas the region where it is defined is the set

$$\Omega_\phi = \{x \in \mathbb{R}^3 \mid \phi(x_1, x_2) \neq 0\} \tag{14}$$

Note that $D_\phi \cup \Omega_\phi = \mathbb{R}^3$.

The functions $B_1(x_1, x_2)$, $B_2(x_1, x_2)$ and $\phi(x_1, x_2)$ are to be determined in such a way that the following two requirements are fulfilled

(R1) $\dot{\phi} = \lambda(\phi)$, for some smooth function $\lambda(\cdot)$, such that $\lambda(0) = 0$ and $\lambda(\cdot) \neq 0$ elsewhere.

(R2) $\dot{x}_3 = x_2 u_1 - x_1 u_2 = \theta(x_3)$ for all $x_3 \in \mathbb{R}$ and for all (x_1, x_2) in an open and dense set.

The first requirement implies that no trajectory of the closed loop system can cross (or leave) the set D_ϕ and that the set D_ϕ is asymptotically attractive either in forward or in backward time, whereas the second implies that the state x_3 can be controlled independently, as discussed already in Sections 3 and 4.

Remark 3. We admit that the above requirements are rather arbitrary, i.e. they do not have any theoretical justification; however, on the basis of the previous discussion, we believe that they are quite natural and reasonable. Our aim here is not to discuss their necessity but to exploit their implications and to show

how they can be used to build discontinuous stabilisers.

Remark 4. At this stage we do not make any assumption on the shape of the set D_ϕ, as we will show that any control law described by equations of the form (12) and fulfilling the requirements R1 and R2 induces a very special shape on the set D_ϕ.

A first preliminary result is contained in the following statement.

Lemma 3. There exist smooth $B_1(x_1, x_2)$, $B_2(x_1, x_2)$ and $\phi(x_1, x_2)$ fulfilling R1 and R2 iff there exist smooth $B_1(x_1, x_2)$, $B_2(x_1, x_2)$ and $\phi(x_1, x_2)$ fulfilling the following set of conditions:

$$\left.\begin{aligned} \phi_{x_1}x_1 + \phi_{x_2}x_2 &= \tilde{\lambda}(\phi(x_1, x_2)) \\ B_1(x_1, x_2)\phi_{x_1} - B_2(x_1, x_2)\phi_{x_2} &= 0 \\ B_1(x_1, x_2)x_2 + B_2(x_1, x_2)x_1 &= c\phi(x_1, x_2) \end{aligned}\right\} \tag{15}$$

for some smooth function $\tilde{\lambda}(\cdot)$.

Proof. (If). Suppose there exist smooth $B_1(x_1, x_2)$, $B_2(x_1, x_2)$ and $\phi(x_1, x_2)$ fulfilling the set of equations (15). Then, simple calculations show that requirements R1 and R2 are fulfilled with $\lambda(\cdot) = -k\tilde{\lambda}(\cdot)$.

(Only if). Suppose there exist $B_1(x_1, x_2)$, $B_2(x_1, x_2)$ and $\phi(x_1, x_2)$ fulfilling requirements R1, i.e.

$$\begin{aligned} \dot{\phi} &= \phi_{x_1}u_1 + \phi_{x_2}u_2 \\ &= -k\left(\phi_{x_1}x_1 + \phi_{x_2}x_2\right) \\ &\quad - \frac{\theta(x_3)}{\phi(x_1, x_2)}\left(B_1(x_1, x_2)\phi_{x_1} - B_2(x_1, x_2)\phi_{x_2}\right) \\ &= \lambda(\phi(x_1, x_2)) \end{aligned}$$

for all $(x_1, x_2, x_3) \in \Omega_\phi$. Setting $x_3 = 0$ in the last two terms we get

$$\phi_{x_1}x_1 + \phi_{x_2}x_2 = -\frac{\lambda(\phi(x_1, x_2))}{k}$$

which also implies

$$B_1(x_1, x_2)\phi_{x_1} - B_2(x_1, x_2)\phi_{x_2} = 0$$

i.e. the first and the second of Eqs (15), with $\tilde{\lambda}(\cdot) = -\lambda(\cdot)/k$. Finally, simple computations show that R2 is equivalent to the third of Eqs (15). □

Equations (15) have a nice structure, which allows to find a solution in two steps, as expressed in the following simple result.

Lemma 4. Let $\phi(x_1, x_2)$ be a solution of the p.d.e.

$$\phi_{x_1}x_1 + \phi_{x_2}x_2 = \tilde{\lambda}(\phi(x_1, x_2)) \tag{16}$$

for a given function $\tilde{\lambda}(\cdot)$, such that $\tilde{\lambda}(0) = 0$, $\tilde{\lambda}(\cdot) \neq 0$ elsewhere and $\mathrm{d}\tilde{\lambda}(0) \neq 0$.

Then the two smooth functions

$$\left.\begin{aligned} B_1(x_1, x_2) &= \frac{\phi(x_1, x_2)}{\tilde{\lambda}(\phi(x_1, x_2))}\phi_{x_2} \\ B_2(x_1, x_2) &= \frac{\phi(x_1, x_2)}{\tilde{\lambda}(\phi(x_1, x_2))}\phi_{x_1} \end{aligned}\right\} \tag{17}$$

solve the second and the third of Eqs (15).

Proof. Observe that the second and third of equations (15) can be regarded as a system of linear equations in the unknowns $B_1(x_1, x_2)$ and $B_2(x_1, x_2)$ and with coefficients matrix not singular for all $x \in \Omega_\phi$. Such a system can be solved for all non-zero (x_1, x_2) and the corresponding solution is given by Eqs (17). It remains to show that the solution is smooth for all (x_1, x_2). To this end note that, by the hypotheses $\tilde{\lambda}(0) = 0$ and $d\tilde{\lambda}(0) \neq 0$, the function

$$\frac{y}{\tilde{\lambda}(y)}$$

is smooth for all y and that composition of smooth functions is again a smooth function. □

We conclude that the solvability of the set of equations (15) depends on the solvability of the p.d.e. (16). This last issue will be now addressed.

5.1. On the Solvability of the p.d.e. (16)

We distinguish between two cases. In the former we assume that the function $\tilde{\lambda}(\cdot)$ is linear, whereas in the latter we consider a complete general $\tilde{\lambda}(\cdot)$.

5.1.1. Case 1: Linear $\tilde{\lambda}(\cdot)$

Proposition 4. Consider the p.d.e. (16). Assume $\tilde{\lambda}(\cdot)$ is a linear function, i.e. $\tilde{\lambda}(\phi) = n\phi$, *with* $n > 0$.

Then any homogeneous function of degree $n > 0$ solves the p.d.e. (16).

Proof. The claim is a consequence of the Euler theorem on homogeneous functions. □

5.1.2. Case 2: General $\tilde{\lambda}(\cdot)$

Proposition 5. Consider the p.d.e. (16). Let $\psi(x_1, x_2)$ be a homogeneous function of degree $m > 0$. Let $H(x)$ be a solution of the o.d.e.

$$m\frac{\mathrm{d}H(x)}{\mathrm{d}x}x = \tilde{\lambda}(H(x)) \tag{18}$$

Then the function $\phi(x_1, x_2) = H(\psi(x_1, x_2))$ solves the p.d.e. (16).

Proof. To begin with, observe that, by the Euler Theorem, we have

$$\psi_{x_1}x_1 + \psi_{x_2}x_2 = m\psi(x_1, x_2)$$

and that

$$\phi_{x_1} = \frac{\mathrm{d}H}{\mathrm{d}\psi}\psi_{x_1} \qquad \phi_{x_2} = \frac{\mathrm{d}H}{\mathrm{d}\psi}\psi_{x_2}$$

Finally,

$$\begin{aligned}\phi_{x_1}x_1 + \phi_{x_2}x_2 &= \frac{\mathrm{d}H}{\mathrm{d}\psi}(\psi_{x_1}x_1 + \psi_{x_2}x_2)\\ &= \frac{dH}{d\psi}m\psi(x_1, x_2)\\ &= \tilde{\lambda}(H(\psi(x_1, x_2))\\ &= \tilde{\lambda}(\phi(x_1, x_2)\end{aligned}$$

and the proof is complete. □

The conclusion that can be drawn from the results summarised in Propositions 4 and 5 is that the p.d.e. (16) admits a solution of the form

$$\phi(x_1, x_2) = G(\psi(x_1, x_2)) \tag{19}$$

where $\psi(x_1, x_2)$ is a homogeneous function of degree $\alpha > 0$ and $G(\cdot)$ is a smooth function. This particular structure induces a very special shape on the set D_ϕ.

5.2. On the Shape of the Set D_ϕ

Proposition 6. Let

$$\phi(x_1, x_2) = G(\psi(x_1, x_2))$$

with $G(\cdot)$ smooth function. Assume

R3 $G(0) = 0$ and $G(\cdot) \neq 0$ elsewhere.

Then the set D_ϕ defined in equation (13) is composed of the x_3-axis and a collection of hyperplanes of the form

$$H = \{x \in \mathbb{R}^3 \, |ax_1 + bx_2 = 0\}$$

with $a \in \mathbb{R}$ and $b \in \mathbb{R}$. See Fig. 1 for further detail.

Proof. To begin with, observe that, by R3, $D_\phi = D_\psi$, where

$$D_\psi = \{x \in \mathbb{R}^3 \, |\psi(x_1, x_2) = 0\}$$

Then, as the function $\psi(x_1, x_2)$ is homogeneous of degree $\alpha > 0$, we have $\psi(0, 0) = 0$, implying that the x_3 axis is indeed contained in D_ψ. Moreover, by the homogeneity property of the function we conclude that the set D_ψ can be described considering the zeros of the functions

$$\psi_1(\xi) = \psi(\xi, 1) \qquad \psi_2(\xi) = \psi(1, \xi)$$

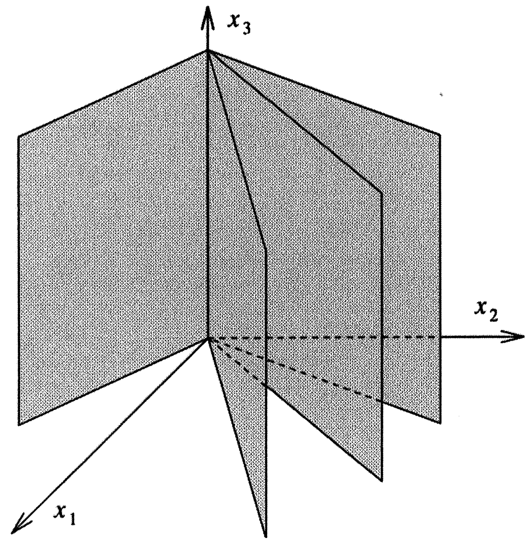

Fig. 1. Typical shape of the set D_ϕ.

If $\xi^\star$ is any solution of the equation $\psi_1(\xi) = 0$, then $\psi(x_1, x_2)|_{(x_1 - \xi^\star x_2 = 0)} = 0$, whereas, if $\xi^\star$ is any solution of the equation $\psi_2(\xi) = 0$, then $\psi(x_1, x_2)|_{(x_2 - \xi^\star x_1 = 0)} = 0$. Hence, the claim. □

In what follows we say that the function $\phi(x_1, x_2) = G(\psi(x_1, x_2))$ resulting from the solution of the p.d.e. (16) is good if $G(\cdot)$ fulfils R3.

5.3. A New Family of Almost Stabilising Controllers

The results established so far can be used to describe in a simple way a large family of control laws almost stabilising the Brockett integrator: for any given pair of functions $\tilde{\lambda}(\cdot)$ and $\theta(\cdot)$, we obtain a member of the family, namely

$$\left.\begin{aligned} u_1 &= -kx_1 + \frac{\phi_{x_2}}{\tilde{\lambda}(\phi(x_1, x_2)}\,\theta(x_3)\\ u_2 &= -kx_2 - \frac{\phi_{x_1}}{\tilde{\lambda}(\phi(x_1, x_2)}\,\theta(x_3)\end{aligned}\right\} \tag{20}$$

where $\phi(x_1, x_2)$ is a solution of the p.d.e. (16).

We now briefly state two properties of the closed loop system (2)–(20), which are the counterpart of Propositions 2 and 3.

Proposition 7. Consider the system (2) and the control (20) with $\phi(x_1, x_2)$ good. Any trajectory of the closed loop system starting outside the set D_ϕ, defined in (13), never crosses the discontinuity set D_ϕ. Moreover, if the function $\tilde{\lambda}(\cdot)$ is such that the system $\dot{\xi} = -\tilde{\lambda}(\xi)$ has $\xi = 0$ as a globally asymptotically (exponentially) stable equilibrium, then D_ϕ is asymptotically (exponentially) attractive.

Proposition 8. Consider the system (2) and the control (20) with $\phi(x_1, x_2)$ good. Let $x(t) = (x_1(t), x_2(t), x_3(t)) : [0, T) \to \mathbb{R}^3$ denote a trajectory of the closed loop system starting in Ω_ϕ, i.e. $\phi(x_1(0), x_2(0)) \neq 0$.

Then $T = \infty$, i.e. the trajectory $x(t)$ is defined for all $t \geq 0$.

We now conclude with the following interesting statement, the proof of which is a trivial consequence of the above discussion.

Theorem 2. Let $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}$. Suppose the following.

(H1) $\lambda(\xi)$ is a smooth real valued function such that
  (i) $\lambda(0) = 0$, $\lambda(\cdot) \neq 0$ elsewhere, $d\lambda(0) \neq 0$;
  (ii) the system $\dot{\xi} = \lambda(\xi)$ is globally asymptotically stable.

(H2) $\theta(\eta)$ is a smooth real valued function such that
  (i) $\theta(0) = 0$;
  (ii) the system $\dot{\eta} = \theta(\eta)$ is globally asymptotically (exponentially) stable.

(H3) For any non-zero $\xi(0)$ and for any $\eta(0)$ the corresponding trajectories $\xi(t)$ and $\eta(t)$ of the above defined systems are such that the quantity

$$\frac{\eta(t)}{\xi(t)}$$

is defined and bounded for all $t \geq 0$.

(H4) The solution $\phi(x_1, x_2)$ of the p.d.e. (16) with $\tilde{\lambda}(\cdot) = -\lambda(\cdot)/k$ is good.

Then,

(i) the discontinuous control law (20), with $k > 0$, almost asymptotically (exponentially) stabilises the system (2) in the open and dense set Ω_ϕ, defined in equation (14);
(ii) the control signals (20) are bounded along any trajectory of the closed loop system (2)–(20) starting in Ω_ϕ.

The result summarised in the previous statement lends itself to the following interpretation. To design an almost asymptotic (exponential) stabiliser for the system (2), it suffices to choose two real valued functions $\lambda(\cdot)$ and $\theta(\cdot)$ fulfilling the hypotheses H1, H2, H3 and H4. This is a very trivial task! Finally, once such a pair of functions has been fixed, the almost asymptotic (exponential) stabiliser can be designed automatically.

We illustrate this procedure with a simple example.

Example. Let $\lambda(\xi) = -\alpha\xi$ for $\alpha > 0$ and $\theta(\eta) = -\beta\eta - \gamma\eta^3$ for $\beta > 0$ and $\gamma > 0$. Note that such functions fulfil the hypotheses H1 and H2. Moreover, simple calculations show that hypothesis H3 holds if $\beta > \alpha$.

One solution $H(x)$ of the o.d.e. (18) is

$$H(x) = x^{\left(\frac{\alpha}{km}\right)}$$

where $k > 0$ and $m > 0$. Let $\psi(x_1, x_2)$ be any homogeneous function of degree m, e.g.

$$\psi(x_1, x_2) = \left(\frac{x_1^4 + x_2^4}{x_1^2 + 3x_2^2}\right)^{\frac{m}{2}}$$

Then, the function

$$\phi(x_1, x_2) = \sqrt{(x_1^4 + x_2^4)^{\frac{\alpha}{k}}((x_1^2 + 3x_2^2)^{\frac{\alpha}{k}})^{-1}}$$

is a solution of the p.d.e. (16) and it is good. Hence, using equations (17) and (12), an almost exponential stabiliser can be computed. □

6. Almost Exponential Stabilisation with Bounded Control

In this section we discuss the problem of designing an almost exponential stabiliser for the system (2), assuming that the control signals u_1 and u_2 are bounded, i.e. $u_i \in [-1, 1]$, for $i = 1, 2$. Rather than proposing a family of stabilisers, we restrict our attention to a particular solution. It must be noted that, to the best of our knowledge, no stabilising control law for the system (2) with bounded control has been proposed.

Proposition 9. Consider the system (2). Let

$$\left.\begin{aligned} u_1 &= -k\frac{x_1}{1+V} + \psi\left(\frac{x_2}{V}\right)\theta(x_3) \\ u_2 &= -k\frac{x_2}{1+V} - \psi\left(\frac{x_1}{V}\right)\theta(x_3) \end{aligned}\right\} \tag{21}$$

with $k \in (0, 2)$, $V = x_1^2 + x_2^2$

$$\psi(\cdot) = \min(\max(\cdot, -1), 1)$$

and

$$\theta = -\left(1 - \frac{k}{2}\right)\operatorname{sign}(x_3)$$

Then,

(i) The state feedback control law (21) almost exponentially stabilises system (2) in the open and dense set

$$\Omega_b = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \neq 0\}$$

(ii) The control signals (21) are such that $|u_i| \leq 1$, for all $x \in \mathbb{R}^3$ and $i = 1, 2$ and, along the trajectories of the closed loop system

$$\lim_{t\to\infty} u_i = \lim_{t\to\infty} u_i(x(t)) = 0$$

for $i = 1, 2$.

Proof. We break up the proof in several steps.
(i) Let

$$B([a,b],\rho) = \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1-a)^2 + (x_2-b)^2 < \rho^2\}$$

i.e. $B([a,b],\rho)$ is an open ball of centre $(x_1,x_2) = (a,b)$ and radius equal to ρ. Note that

$$\psi\left(\frac{x_1}{V}\right) = \frac{x_1}{V}$$

in the set $S_1 = \mathbb{R}^2 - \{B([1/2,0],1/2) \cup B([-1/2,0],1/2)\}$;

$$\psi\left(\frac{x_2}{V}\right) = \frac{x_2}{V}$$

in the set $S_2 = \mathbb{R}^2 - \{B([0,1/2],1/2) \cup B([0,-1/2],1/2)\}$ and, finally,

$$\psi\left(\frac{x_1}{V}\right) = \frac{x_1}{V} \quad \text{and} \quad \psi\left(\frac{x_2}{V}\right) = \frac{x_2}{V}$$

in the set $S = \mathbb{R}^2 - B([0,0],1)$. See Fig. 2 for further detail.

Hence, in S we have

$$\dot V = -2k\frac{V}{1+V}$$
$$\dot x_3 = \theta(x_3)$$

implying that any trajectory of the closed loop system starting in

$$\mathbb{R}^3 - \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid V < 1\}$$

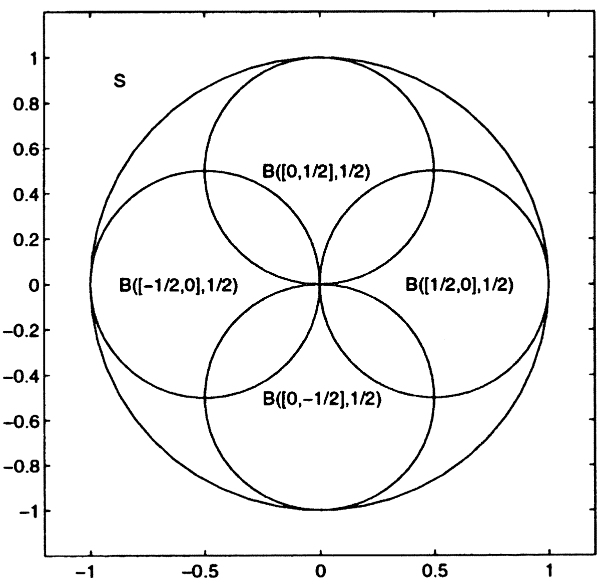

Fig. 2. Sketch of the regions S, $B([1/2,0],1/2)$, $B([-1/2,0],1/2)$, $B([0,1/2],1/2)$ and $B([0,-1/2],1/2)$.

converges to the (unbounded) set

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid V < 1\}$$

(ii) We now show that $|x_3(t)|$ is a non-increasing function of time. Observe that, in $\mathbb{R}^3$

$$\begin{aligned}\dot x_3 &= \left[x_2\psi\left(\frac{x_2}{V}\right) + x_1\psi\left(\frac{x_1}{V}\right)\right]\theta(x_3)\\ &= \Sigma(x_1,x_2)\theta(x_3)\end{aligned} \tag{22}$$

and

$$\Sigma(x_1,x_2) = \left[x_2\psi\left(\frac{x_2}{V}\right) + x_1\psi\left(\frac{x_1}{V}\right)\right] > 0$$

for all non-zero (x_1,x_2). Hence, $\dot x_3$ is non-negative for negative x_3 and non-positive for positive x_3. We conclude that for any trajectory of the closed loop system

$$\lim_{t\to\infty} |x_3(t)| = c < \infty$$

and that $cx_3(0) \geq 0$.
(iii) (Existence of a positively invariant set) From the discussion in (i) and (ii) we conclude that any trajectory of the closed loop system converges (in finite time) to the bounded set

$$\Omega_p = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid V < 1 \quad \text{and} \quad |x_3| \leq |x_3(0)|\}$$

and remains inside Ω_p thereafter.
(iv) (Asymptotic convergence)

Lemma 5. Every trajectory of the closed loop system (2)–(21) starting in $\Omega_p \cap \Omega_b$ converges asymptotically to the origin.

Proof. The proof is contained in Appendix A. □

(v) (Exponential convergence) From the discussion in the proof of Lemma 5 we infer that $V = x_1^2 + x_2^2$ converges to zero only if $x_3 = 0$. This fact, together with equation (22) implies that x_3 converges to zero in finite time. Afterwards, one has

$$\dot V = -2k\frac{V}{1+V} \tag{23}$$

From this we infer exponential convergence of V to zero, implying exponential convergence of x_1 and x_2 to zero, as expressed in the following statement.

Lemma 6. The solution $V(t)$ of the o.d.e. (23) from the initial condition V_0 converges exponentially to zero.

Proof. The proof is contained in Appendix A. □

(vi) (Boundedness of the control signals) Simple calculations yield

$$|u_i| = k\left|\frac{x_i}{1+V}\right| + \left|\psi\left(\frac{x_j}{V}\right)\right|\left(1-\frac{k}{2}\right)$$

where $j = 1$, if $i = 2$, and $j = 2$, if $i = 1$. Hence, by the definition of the function $\psi(\cdot)$ and by

$$\frac{x_i}{1+V} \leq \frac{1}{2}$$

we conclude

$$|u_i| \leq \frac{k}{2} + \left(1-\frac{k}{2}\right) \leq 1$$

Finally, as $\lim_{t\to\infty} x(t) = 0$, we obtain $\lim_{t\to\infty} u_i(t) = 0$, for $i = 1, 2$. □

7. A Simple Application

In this section we show how the proposed control laws can be used to *stabilise* the kinematic equations describing the attitude of a rigid body (satellite) using only two velocity controls. This is a control system with state space SO(3) and two controls; see [11] for further details. Traditionally, the control problem is addressed selecting a local parametrisation of SO(3) around the identity. In what follows we use the so-called Rodrigues parametrisation, yielding a system described by equations of the form (see [11])

$$\left.\begin{aligned} \dot{x}_1 &= v_1 - v_2 x_3 + (v_1 x_1 + v_2 x_2) x_1 \\ \dot{x}_2 &= v_2 + v_1 x_3 + (v_1 x_1 + v_2 x_2) x_2 \\ \dot{x}_3 &= v_2 x_1 - v_1 x_2 + (v_1 x_1 + v_2 x_2) x_3 \end{aligned}\right\} \qquad (24)$$

where (x_1, x_2, x_3) are the Rodrigues parameters and v_1 and v_2 denote the angular velocity vectors along the principal axes.

The main result of this section is summarised in the following statement.

Proposition 10. Consider the system (24). Then there exists a (regular) input transformation

$$v = B(x)u$$

such that the transformed system, with input u and state x, is described by equations of the form (2). Moreover, all the entries of the matrix $B(x)$ are bounded by one.

Proof. Let

$$B(x) = \begin{bmatrix} \dfrac{1+x_2^2}{1+x_1^2+x_2^2+x_3^2} & -\dfrac{x_2x_1 - x_3}{1+x_1^2+x_2^2+x_3^2} \\ -\dfrac{x_2x_1 + x_3}{1+x_1^2+x_2^2+x_3^2} & \dfrac{1+x_1^2}{1+x_1^2+x_2^2+x_3^2} \end{bmatrix}$$

Tedious but straightforward calculations show that, in the transformed input coordinates, the system (24) is precisely described by equations of the form (2). Finally, inspection of the entries of the matrix $B(x)$ proves the last claim. □

As a simple consequence of Proposition 10 we have the following statement.

Corollary 1. For any (given) $\epsilon > 0$ the system (24) can be almost exponentially stabilised by a state feedback control law $u = \text{col}(u_1(x), u_2(x))$ such that $u_i(x) \in [-\epsilon, \epsilon]$ for $i = 1, 2$.

8. Simulation Results

In this section we present some simulation results, showing the features of the control laws discussed so far.

8.1. The Control Law (9) with $n = 1$

Simulations have been carried out with the control law (9) setting $n = 1$. The constants k, g, a and b have been set as follows: $k = 0.5$, $g = 1$, $a = x_{10}$ and $b = x_{20}$, where $x_{10} = x_1(0)$ and $x_{20} = x_2(0)$.

Remark 5. The constants a and b have been set in such a way that (on the (x_1, x_2) plane) the straight line connecting the point (x_{10}, x_{20}) to the origin is orthogonal to the discontinuity surface. See Fig. 3 for details.

Figure 4 displays the simulation results for a prototype initial condition: $(x_{10}, x_{20}, x_{30}) = (1, 1, 1)$. Note the exponential convergence of the states and the boundedness of the control signals along the trajectories of the closed loop system.

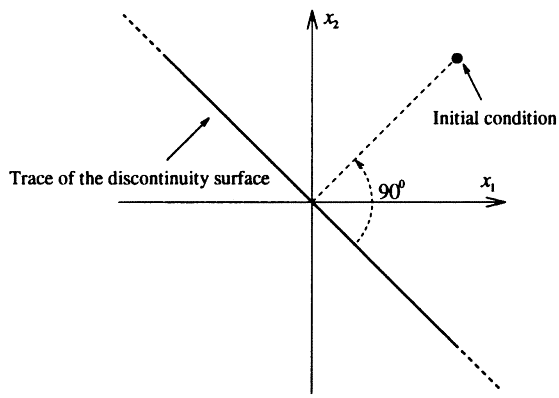

Fig. 3. Relative position of the initial condition (x_{10}, x_{20}) and the discontinuity surface on the (x_1, x_2)-plane.

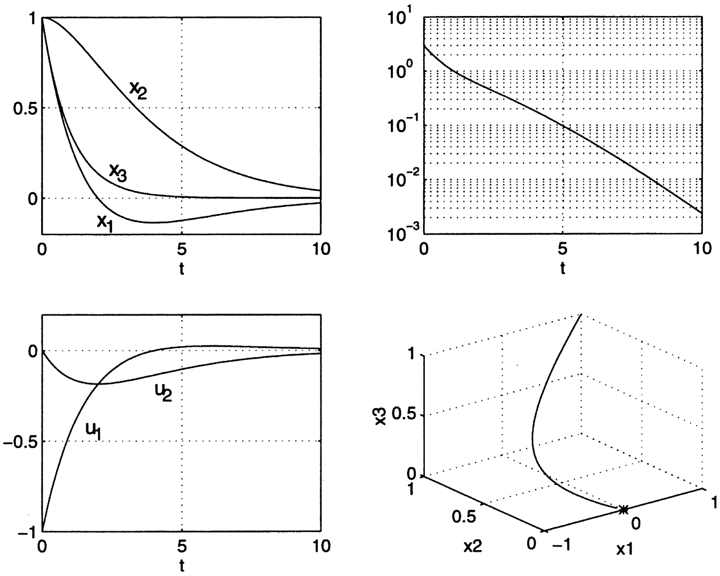

Fig. 4. Simulations result 1 ($n = 1$). State histories (top left), $\log(\|x(t)\|^2)$ (top right), control signals (bottom left) and phase portrait (bottom right).

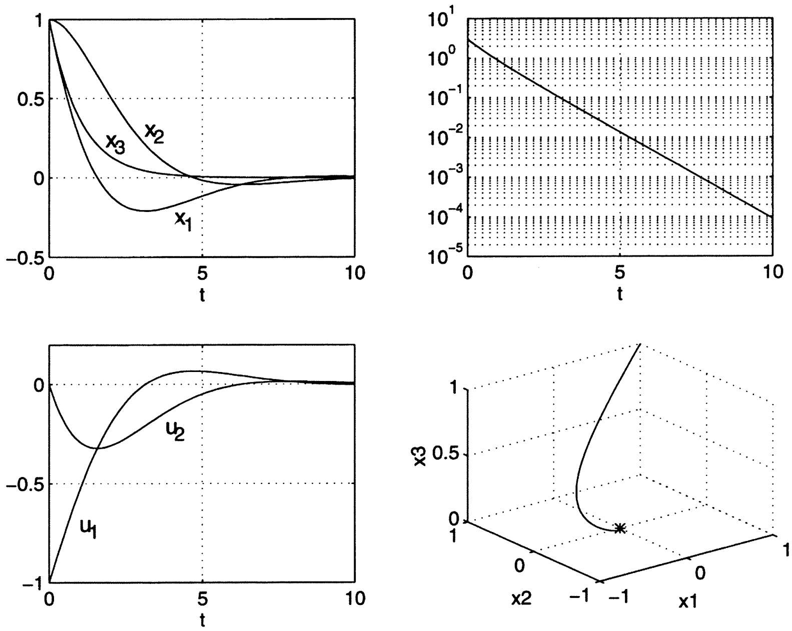

Fig. 5. Simulations result 2 ($n = 2$). State histories (top left), $\log(\|x(t)\|^2)$ (top right), control signals (bottom left) and phase portrait (bottom right).

8.2. The Control Law (9) with $n = 2$

Simulations have been carried out with the control law (9) setting $n = 2$. The constants k, g, a and b have been set as follows: $k = 0.5$, $g = 1$, $a = 1$ and $b = 1$. Figure 5 displays the simulation results for the initial condition $(x_{10}, x_{20}, x_{30}) = (1, 1, 1)$. Note the exponential convergence of the states and the boundedness of the control signals along the trajectories of the closed loop system.

8.3. The Control Law (21)

Simulations have been also carried out with the control law (21) with the constant $k = 0.2$. Figure 6 dis-

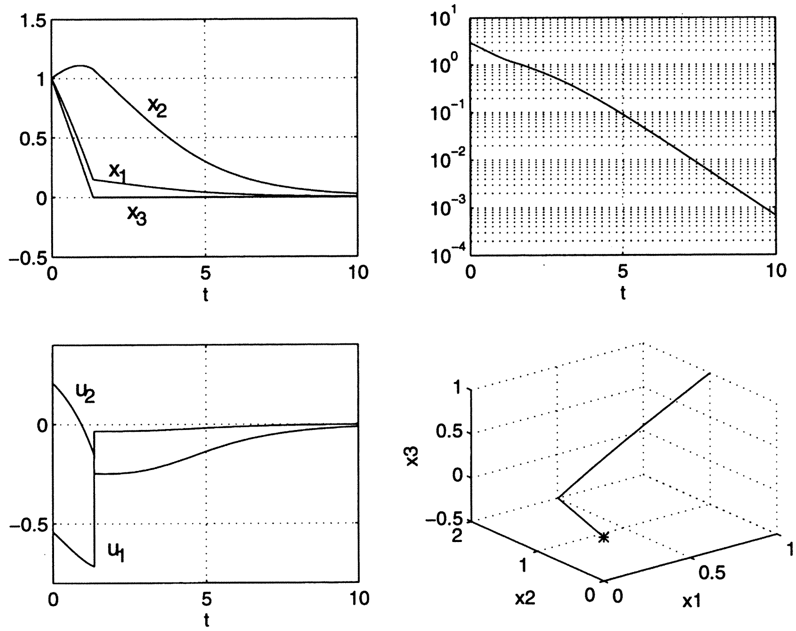

Fig. 6. Simulations result 3 (bounded control). State histories (top left), $\log(\|x(t)\|^2)$ (top right), control signals (bottom left) and phase portrait (bottom right).

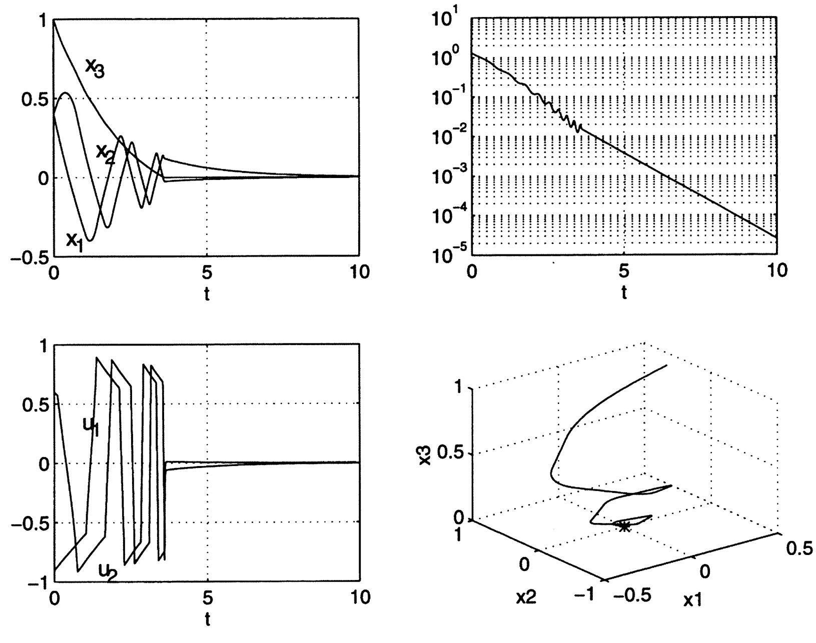

Fig. 7. Simulations result 4 (bounded control). State histories (top left), $\log(\|x(t)\|^2)$ (top right), control signals (bottom left) and phase portrait (bottom right).

plays the simulation results for the initial condition $(x_{10}, x_{20}, x_{30}) = (1, 1, 1)$, whereas Fig. 7 displays the simulation results for the initial condition $(x_{10}, x_{20}, x_{30}) = (0.4, 0.4, 1)$.

8.4. Discussion of the Simulation Results

The performances of the control law (9) for $n = 1$ and $n = 2$ are comparable. However, this is the result of the particular good choice of the constants a and b in the case $n = 1$. In general, if such constants are fixed and cannot be adjusted using the knowledge of the initial conditions, the control law with $n = 2$ will probably perform better. Note that, if in the control law (9) with $n = 1$ we let a and b be function of the current state, i.e. $a = a(x)$ and $b = b(x)$ and we select them in such a way that the singularity surface is always orthogonal to the shortest straight line connecting the actual state to the x_3 axis, we have $a = x_1$ and $b = x_2$, i.e. the control law (9) with $n = 2$!

In all the presented simulation results the control variables are bounded, in modulo, by one. However, using only the control law (21) we have a guarantee that the control signals are bounded, in modulo, by one, for any initial conditions.

The phase portraits, from the initial conditions $(1, 1, 1)$ and $(2, -2, 2)$, resulting from the three control laws discussed, are displayed in Figs 8 and 9, respectively. Note the qualitatively different shape of the trajectory produced by the bounded control (21).

The performance of the control law (21) depends strongly on the initial conditions. More precisely, if $x_3(0)$ is of the same order of magnitude as $x_1(0)^2 + x_2(0)^2$, then the states converge exponentially to the origin without oscillation (see Fig. 6, where $x_3(0) = 1$ and $x_1(0)^2 + x_2(0)^2 = 2$), whereas, if $x_3(0) \gg x_1(0)^2 + x_2(0)^2$, the control signals as well as the states have an oscillatory behaviour (see Fig. 7, where $x_3(0) = 1$ and $x_1(0)^2 + x_2(0)^2 = 0.32$). In the second case the trajectories of the closed loop system resemble the trajectories obtained using time-varying (periodic) feedback laws (see e.g. [10,15]). However, it must be noted that, in the present case, the oscillations are the result of the input saturation and the particular choice of initial conditions.

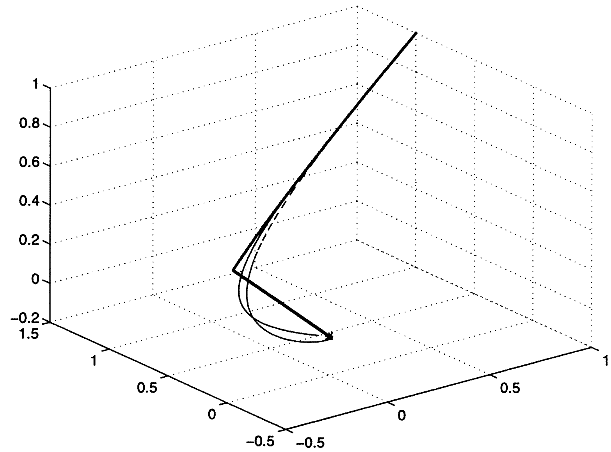

Fig. 8. Phase portraits for $n = 1$ (solid), $n = 2$ (dashed) and bounded control (strong) from the initial condition $(1, 1, 1)$.

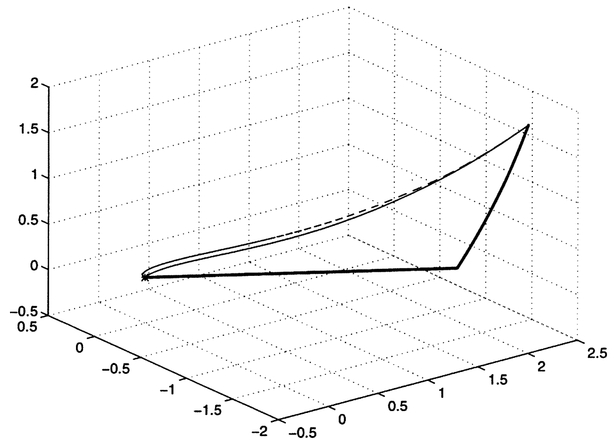

Fig. 9. Phase portraits for $n = 1$ (solid), $n = 2$ (dashed) and bounded control (strong) from the initial condition $(2, -2, 2)$.

9. Conclusions

The paper has discussed the problem of almost exponential stabilisation for the Brockett integrator. A family of discontinuous control laws has been derived and a simple procedure to select one member of this family has been proposed (Proposition 2). Moreover, the problem of stabilisation with bounded control has also been addressed. We have shown that, despite the input saturation, almost exponential stability can be achieved by a simple control law and for any bound (even arbitrarily small) on the input signals (Proposition 9). Application of the proposed method to the control of the under-actuated kinematic satellite has also been briefly discussed. The main result (Proposition 10) shows that the system can be almost exponentially stabilised with bounded control. Simulation results displaying some of the features of the proposed control laws complete the work.

Several problems are left open in this work. Firstly, how to select a member in the proposed family of control laws. This selection must be done on the basis of an optimality criterion. Then, whether it is possible to extend the proposed method to a larger class of systems, e.g. systems in chained form. Finally, we have proposed only one saturated control, but we believe that it is a special element of a larger family. Further work needs to be done to characterise such a family of saturating control laws.

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Appendix A

Proof of Proposition 3. Obviously $x_3(t)$ is defined for all $t \geq 0$. Hence, we need to show that the same holds for $x_1(t)$ and $x_2(t)$. We consider three cases.

Case 1: $ab = 0$.

Assume, without lack of generality, $b = 0$. Then the closed loop system (10) is described, for any n, by the equations

$$\begin{aligned}\dot{x}_1 &= -kx_1\\ \dot{x}_2 &= -kx_2 + g\frac{1}{x_1}x_3\\ \dot{x}_3 &= -gx_3.\end{aligned}$$

Hence, we also conclude that x_1 is defined for all $t \geq 0$. Finally, observe that, for all trajectories starting in Ω, one has

$$\dot{x}_2 = -kx_2 + g\exp^{(k-g)t}\frac{x_3(0)}{x_1(0)}$$

with

$$\frac{x_3(0)}{x_1(0)}$$

bounded, which implies the existence of $x_2(t)$ for all $t \geq 0$.

Case 2: $ab \neq 0$ and $n = 1$.
In this case the closed loop system (10) is described by the equations

$$\begin{aligned}\dot{x}_1 &= -kx_1 - g\frac{b}{ax_1 + bx_2}x_3\\ \dot{x}_2 &= -kx_2 + g\frac{a}{ax_1 + bx_2}x_3\\ \dot{x}_3 &= -gx_3.\end{aligned}$$

Hence, for any trajectory starting in Ω one has

$$\begin{aligned}\dot{x}_1 &= -kx_1 - gb\exp^{(k-g)t}\frac{x_3(0)}{ax_1(0) + bx_2(0)}\\ \dot{x}_2 &= -kx_2 + ga\exp^{(k-g)t}\frac{x_3(0)}{ax_1(0) + bx_2(0)}\end{aligned}$$

with

$$\frac{x_3(0)}{ax_1(0) + bx_2(0)}$$

bounded, which implies the existence of $x_1(t)$ and $x_2(t)$ for all $t \geq 0$.

Case 3: $ab \neq 0$ and $n \geq 2$.
We proceed by contradiction, i.e. we assume that either $x_1(t)$ or $x_2(t)$ have finite escape time and we show that this leads to a contradiction. We break up the proof into several steps.

Fact 0: By Proposition 2 we have

$$ax_1^n(t) + bx_2^n(t) = (ax_1^n(0) + bx_2^n(0))\exp^{-knt}$$

for all trajectories starting in Ω. Hence, $ax_1^n(t) + bx_2^n(t)$ is bounded for all $t \geq 0$ and is never zero.

Fact 1: $x_1(t)$ and $x_2(t)$ are defined over the same maximum time interval.

Proof. Suppose $x_1 : [0, T_1) \to \mathbb{R}$ and $x_2 : [0, T_2) \to \mathbb{R}$ with $T_1 < T_2$. Then

$$|\lim_{t\to T_1}(ax_1^n(t) + bx_2^n(t)| = |\lim_{t\to T_1}(ax_1^n(t)) + bx_2^n(T_1)| = \infty$$

which contradicts Fact 0. The same conclusion can be drawn if we assume $T_1 > T_2$. Hence, $T_1 = T_2 = T^\star$. □

Fact 2: $T^\star = \infty$.

Proof. We distinguish between two subcases.

Subcase 1: The equation $as^n + b = 0$ has no solution.

In this case, the function $ax_1^n + bx_2^n$ is positive (or negative) definite for all $(x_1, x_2) \neq (0, 0)$. Hence, by Fact 0, we conclude that $x_1(t)$ and $x_2(t)$ are bounded for any finite $t \geq 0$.

Subcase 2: The equation $as^n + b = 0$ has at least one solution $s^\star$.

First of all note that the equation $as^n + b = 0$ cannot have multiple solutions. Hence, we can write

$$ax_1^n + bx_2^n = (x_1 - s^\star x_2)F(x_1, x_2) \tag{25}$$

for some smooth function $F(x_1, x_2)$ such that

$$F(s^\star x_2, x_2) \neq 0 \tag{26}$$

Let

$$x_1(t) = \pi_1(t) + \psi_1(t)$$

and

$$x_2(t) = \pi_2(t) + \psi_2(t)$$

with $\pi_i(t)$ defined for all $t \geq 0$ and $\psi_i(t)$ defined for all $t \in [0, T^\star)$, i.e. $x_1(t)$ and $x_2(t)$ escape to infinity in finite time. Substituting these expressions into equations (25) yields

$$\begin{aligned} ax_1^n(t) + bx_2^n(t) &= ((\pi_1(t) + \psi_1(t)) \\ &\quad - s^\star(\pi_2(t) + \psi_2(t))) \\ &\quad \times F(\pi_1(t) + \psi_1(t), \pi_2(t) + \psi_2(t)). \end{aligned}$$

Observe that both factors must be bounded and that none of them can be zero, by Proposition 2 and the assumption $x(0) \in \Omega$. Hence, necessarily

$$\lim_{t\to T^\star} |x_1(t) - s^\star x_2(t)| < \infty$$

which implies

$$\lim_{t\to T^\star} (\psi_1(t) - s^\star \psi_2(t)) = 0$$

Let

$$\begin{aligned} F^\star &= \lim_{t\to T^\star} F(\pi_1(t) + \psi_1(t), \pi_2(t) + \psi_2(t)) \\ &= \lim_{t\to T^\star} F(\pi_1(t) + s^\star\psi_2(t), \pi_2(t) + \psi_2(t)) \end{aligned}$$

and observe that, by smoothness of $F(\cdot, \cdot)$ and by equation (26), we conclude $|F^\star| = \infty$ for any $\pi_1(t)$ and $\pi_2(t)$, i.e. a contradiction. □

The proof of Proposition 3 is now complete. □

Proof of Lemma 5. We divide the proof into several facts.

Fact 1: By (iii) in the proof of Proposition 9 we know that, in closed loop

$$\lim_{t\to\infty} x_3(t) = c$$

for some finite constant c. We now show that $c = 0$ if the initial condition $(x_1(0), x_2(0), x_3(0)) \in \Omega_b$. Proceed by contradiction and suppose $c \neq 0$. Then, as $\lim_{t\to\infty} \dot{x}_3(t) = 0$, necessarily

$$\lim_{t\to\infty} \Sigma(x_1(t), x_2(t)) = 0$$

which, in turn, implies

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty}(x_1^2(t) + x_2^2(t)) = 0$$

We now show that this last implication is false.

Fact 2: If $\eta \neq 0$, any trajectory of the system

$$\left.\begin{aligned} \dot{x}_1 &= -k\frac{x_1}{1+V} + \psi\left(\frac{x_2}{V}\right)\eta \\ \dot{x}_2 &= -k\frac{x_2}{1+V} - \psi\left(\frac{x_1}{V}\right)\eta \end{aligned}\right\} \tag{27}$$

starting from non-zero initial conditions does not converge to the origin.

Proof. Consider the following list of implications. (Note $V(t) = x_1^2(t) + x_2^2(t)$.)

$$\lim_{t\to\infty}(x_1(t), x_2(t)) = (0, 0)$$

$$\Downarrow$$

$$\lim_{t\to\infty}\left(\psi\left(\frac{x_2(t)}{V(t)}\right), \psi\left(\frac{x_1(t)}{V(t)}\right)\right) = (0, 0)$$

$$\Updownarrow$$

$$\lim_{t\to\infty}\left(\frac{x_2(t)}{V(t)}, \frac{x_1(t)}{V(t)}\right) = (0, 0)$$

$$\Updownarrow$$

$$\lim_{t\to\infty}\left(\frac{x_2(t)}{x_1^2(t)}, \frac{x_1(t)}{x_2^2(t)}\right) = (0, 0)$$

Observe now that the last limits cannot be simultaneously zero, hence a contradiction. We conclude that, if $\eta \neq 0$, the trajectories of the system (27), start ing from non-zero initial conditions, do not converge to the origin. □

Fact 3: Facts 1 and 2 imply that $x_3(t)$ converges to zero as t goes to infinity. We now also show that x_1 and x_2 converge to zero. For, observe that

$$\dot{V} = -2k\frac{V}{1+V} + \Delta(x_1, x_2)\theta(x_3)$$

where

$$\Delta(x_1, x_2) = x_1\psi\left(\frac{x_2}{V}\right) - x_2\psi\left(\frac{x_1}{V}\right)$$

is a bounded function. By (iii) in the proof of Proposition 9 we conclude that $V(t)$ exists for all t. Finally, as x_3 converges to zero as t goes to infinity and $\theta(0) = 0$, we conclude that $V(t)$ converges to zero as well.

The proof of Lemma 5 is now complete. □

Proof of Lemma 6. Simple but tedious calculations show that the differential Eq. (23) admits the closed integral

$$V(t) = W(V_0 \exp^{(-kt+V_0)})$$

where $W(\cdot)$ is Lambert's W function (see Appendix B and [8] for further details). Exponential convergence of $V(t)$ to zero is then a consequence of Eq. (28). □

Appendix B

The Lambert's W function, see [8], satisfies the equation

$$W(x)\exp(W(x)) = x$$

However, as the equation $y\exp(y) = x$ has an infinite number of solutions y for each non-zero value of x, the function W has an infinite number of branches. Only one of these branches, referred to as the principal branch, is analytic at $x = 0$. In this work we denote the principal branch with $W(\cdot)$.

The function $W(\cdot)$ has several properties. Among them we recall that

$$\lim_{x\to 0}\frac{W(x)}{x} = 1 \qquad (28)$$