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Relaxation element method in calculations of stress state of elastic plane with the plastic deformation band

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Abstract

On the basis of relaxation element method an analytical representation of the band of localized plastic deformation, as the defect with its own internal stress field in the plane under tensile loading, is given. The influence of orientation of the band with respect to tensile axis on the stress concentration in the plane is analyzed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The problem of an analytical description of the local fields of plastic deformation and analysis of the influence of gradients of these fields on the stress state of solid is very urgent. However, in the literature, there are practically no papers on the analytical description of any concentrators and stress gradients in the plastically deformed solid. Relaxation element method, elaborated by the author, is very convenient for this purpose. An unambiguous connection between the plastic form changing and the value of stress relaxation in the given region is laid down on the basis of the method [1,2]. The notion of the relaxation element, in the form of fictitious defect in continuous medium, or quantum of plastic deformation, was entered, as a result of relaxation (stress drop) on an infinitely small value $d\sigma$ in the local region of a definite geometric shape. Relaxation element in the form of a circle turns out to be a very convenient object for the construction in continuum of various distributions of plastic deformation and for the definition of non-homogeneous stress fields, connected with them, see [1,2]. Such an approach allowed one to represent the seat of plastic deformation in the form of a defect in continuum with one's own field of internal stresses.

2. Simplest types of relaxation elements of round shape

The type of relaxation element depends on what components of stress tensor in the round region have undergone change under plastic deformation. The change of the components of stress tensor, i.e., a quantitative value of stress relaxation, is characterized by relaxation tensor. The connection between the value of relaxation and plastic deformation is considered in detail for the case of

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relaxation region of the round shape in the infinite plane under tensile loading [1,2]. Let us consider the simplest types of relaxation elements of the round shape, among which we shall select the most fitting for construction of the plastic deformation band.

2.1. Elements of tensile stress relaxation

First-type. Such an element can be obtained when, under tensile stress along an axis y of the Cartesian system of coordinates in circular area, there was a fall of stress σ on size $\mathrm{d}\sigma_y = \mathrm{d}\sigma$. In this case, non-homogeneous field of stress arises and this field of stress is equivalent to a field of stress in a plate with the round orifice under the action of an external stress $\mathrm{d}\sigma$ [1].

The same effect will be caused by homogeneous plastic deformation of material inside the circle, the components of which, according to [2], are equal to

$$\mathrm{d}\varepsilon_y^{\mathrm{p}} = 3\mathrm{d}\sigma/E, \quad \mathrm{d}\varepsilon_x^{\mathrm{p}} = -\mathrm{d}\sigma/E, \quad \mathrm{d}\varepsilon_{xy}^{\mathrm{p}} = 0. \tag{2.1}$$

According to (2.1), a homogeneous field of plastic deformation is characterized by the tensor with two components, which are not equal to 0. A round region in the process of plastic deformation transforms into elliptical one with the half-axis along tensile axis $b = R(1 + 3\mathrm{d}\sigma/E)$ and another half-axis $a = R(1 - \mathrm{d}\sigma/E)$.

Beyond the circle, the internal stress field components of defect, arising at that time are represented in the following form:

$$\begin{aligned}
\frac{\mathrm{d}\sigma_y}{\mathrm{d}\sigma} &= \frac{a^2}{2R^2}\left[1 + \frac{3a^2 + 10y^2}{R^2} - F + G\right],\\
\frac{\mathrm{d}\sigma_x}{\mathrm{d}\sigma} &= \frac{a^2}{2R^2}\left[3 - \frac{3a^2 + 18y^2}{R^2} + F - G\right],\\
\frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}\sigma} &= \frac{a^2yx}{R^4}\left[3 - \frac{2(3a^2 + 4y^2)}{R^2} + \frac{12a^2y^2}{R^4}\right].
\end{aligned} \tag{2.2}$$

Here

$$F = 8y^2(3a^2 + 2y^2)/R^4, \quad G = 24a^2y^4/R^6,$$
$$R^2 = x^2 + y^2.$$

R is the distance from the center of relaxation element to the point with the coordinates (x, y), and a is the radius of the circle.

The distributions of the internal stress field components (2.2) are given in Fig. 1. The relaxing component σ_y is seen to cause the strongest perturbation of stress field and the region of essential perturbation is commensurable with the dimension of the site of plastic deformation. Maximum value of the relation $\mathrm{d}\sigma_y/\mathrm{d}\sigma$ beyond the circle is equal to 2. Minimum value inside the region is equal to -1.

Type II. If the tensile stress σ operates along x-axis, then the decrease of such a stress on the value $\mathrm{d}\sigma_x$ will result in arising of relaxation element with non-zero component $\mathrm{d}\sigma_x = \mathrm{d}\sigma$ of relaxation tensor.

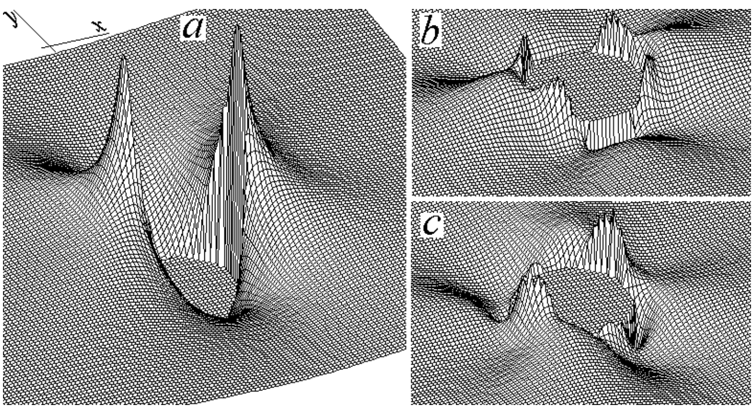

Fig. 1. The distribution of the components of the field of internal stress of defect, according to Eq. (2.2) in the local region, bounded by the circle: (a) $\mathrm{d}\sigma_y/\mathrm{d}\sigma$; (b) $\mathrm{d}\sigma_x/\mathrm{d}\sigma$; (c) $\mathrm{d}\sigma_{xy}/\mathrm{d}\sigma$.

The expressions for the components of internal stresses of a given defect can be derived from Eq. (2.2) by replacing the y-variable on x-variable and x on $-y$. Taking into account that $x^2 = R^2 - y^2$, we shall obtain:

$$\frac{\mathrm{d}\sigma_y}{\mathrm{d}\sigma} = \frac{a^2}{2R^2}\left[1 - \frac{3a^2 + 14y^2}{R^2} + F - G\right],$$
$$\frac{\mathrm{d}\sigma_x}{\mathrm{d}\sigma} = \frac{-a^2}{2R^2}\left[5 - \frac{3a^2 + 22y^2}{R^2} + F - G\right], \tag{2.3}$$
$$\frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}\sigma} = \frac{-a^2yx}{R^4}\left[5 - \frac{2(3a^2 + 4y^2)}{R^2} + \frac{12a^2y^2}{R^4}\right].$$

Homogeneous field of plastic deformation has the components

$$\mathrm{d}\varepsilon_y^{\mathrm{p}} = -\mathrm{d}\sigma/E, \quad \mathrm{d}\varepsilon_x^{\mathrm{p}} = 3\mathrm{d}\sigma/E, \quad \mathrm{d}\varepsilon_{xy}^{\mathrm{p}} = 0. \tag{2.4}$$

The field of internal stresses of the given defect with respect to the direction of x-axis will be the same as for the previous defect with respect to the direction of y-axis (Fig. 1).

2.2. Relaxation element with a quantum of plastic deformation $\mathrm{d}\varepsilon_y^{\mathrm{p}} \neq 0$

Type III. Relaxation element with unique non-zero component $\mathrm{d}\varepsilon_y^{\mathrm{p}}$ will be obtained as a result of superposition of plastic deformation (2.1) and 1/3 of deformation (2.4). In doing so, plastic deformation will not result in the displacement of points of the contour along x-axis. A quantum of plastic deformation of such an element is characterized by the value $\mathrm{d}\varepsilon_y^{\mathrm{p}} = 8\mathrm{d}\sigma/3E$. It is not hard to define that the stress beyond the circle will be characterized by the components:

$$\frac{\mathrm{d}\sigma_y}{\mathrm{d}\sigma} = \frac{a^2}{3R^2}\left[2 + \frac{3a^2 + 8y^2}{R^2} - F + G\right],$$
$$\frac{\mathrm{d}\sigma_x}{\mathrm{d}\sigma} = \frac{a^2}{3R^2}\left[2 - \frac{3a^2 + 16y^2}{R^2} + F - G\right], \tag{2.5}$$
$$\frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}\sigma} = \frac{4a^2yx}{3R^4}\left[1 - \frac{3a^2 + 4y^2}{R^2} + \frac{6a^2y^2}{R^4}\right].$$

The given distributions do not reveal a qualitative difference in distributions, depicted in Fig. 1. Maximum magnitude of relation $\mathrm{d}\sigma_y/\mathrm{d}\sigma$ beyond the circle, in distinction to the first-type, is equal not to 2, but 1.67.

2.3. Relaxation element of pure shear stress

Type IV. The stress field of such a relaxation element is equivalent to the field with a round hole under the operation of external stress of pure shear $\mathrm{d}\sigma_{xy}$. Let us find a solution.

Let us superpose solutions (2.2) and (2.3) with the opposite sign in the system of coordinates, depicted in Fig. 2. Then in the system of coordinates xoy with the axes at an angle of 45° with respect to the direction of applied forces on the distance from the orifice, it will be a state of pure shear. Superposition of fields (2.1) and (2.4) with the opposite sign in the given coordinate system will define a plastic deformation of sufficient pure shear

$$\mathrm{d}\varepsilon_{xy}^{\mathrm{p}} = 4\mathrm{d}\sigma/E. \tag{2.6}$$

It should be noted, that different from the previous cases, a plastic deformation of pure shear does not result in an increase in circle's area. The components of the field of internal stresses of the given relaxation element in the coordinate system xoy have the following form:

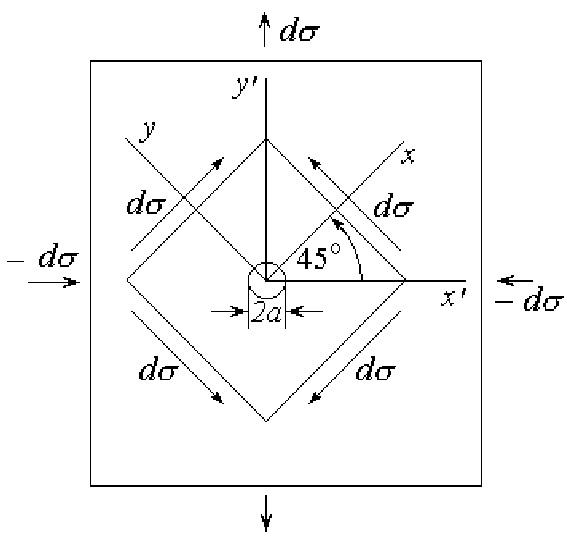

Fig. 2. Boundary conditions for pure shear.

$$\frac{d\sigma_y}{d\sigma} = \frac{4xya^2}{R^4}\left[1 - \frac{3a^2+4y^2}{R^2} + \frac{6a^2y^2}{R^4}\right],$$
$$\frac{d\sigma_x}{d\sigma} = \frac{-4xya^2}{R^4}\left[3 - \frac{3a^2+4y^2}{R^2} + \frac{6a^2y^2}{R^4}\right], \qquad (2.7)$$
$$\frac{d\sigma_{xy}}{d\sigma} = \frac{a^2}{R^2}\left(2 - \frac{3a^2+16y^2}{R^2} + F - G\right).$$

Distributions (2.7) of $d\sigma_y$, $d\sigma_x$ and $d\sigma_{xy}$ are represented in Fig. 3. Pure shear of value (2.6) results in an essential concentration of the components $d\sigma_y$ and $d\sigma_y$ (Fig. 3(a) and (b)):

$$d\sigma_y^{\max}/d\sigma = d\sigma_x^{\max}/d\sigma = 2.3,$$
$$d\sigma_y^{\min}/d\sigma = d\sigma_x^{\min}/d\sigma = -2.3.$$

Stress relaxation of pure shear does not result in such a high concentration of shear stress $d\sigma_{xy} : d\sigma_{xy}^{\max}/d\sigma = 1$ (Fig. 3(c)).

3. The band of localized plastic deformation being perpendicular to tensile axis

Given the relaxation elements we shall use for the construction of bands with various characteristics of the components of plastic deformation.

3.1. The construction of the band of localized plastic deformation

Let us consider the simplest case of the construction of the band of localized plastic deformation, being perpendicular to the tensile axis y by relaxation elements first-type. The construction of bands by relaxation elements of other types is made by a similar way.

Let us set along the straight line within the interval $-l_0 < x < l_0$ a regular distribution of identical relaxation elements of first-type (Fig. 4(a)). This means, along the segment, $2l_0$ in length, relaxation elements are placed at the equal distance dl between each other. Let us define for all elements the same value of stress drop

$$d\sigma_n^r = d\sigma_y = d\sigma = \Delta\sigma dl/2a,$$

where $\Delta\sigma$ is the definite value of stress. In the given case, the quantum of plastic deformation (2.1) and the field of internal stress (2.2) (Fig. 1) are the characteristics of each relaxation element. Let us find the distribution of plastic deformation in a band determined by the given type of relaxation element.

If the point with the coordinates (x, y) is put into n intersected relaxation elements, then the stress relaxation in this point will be equal to $nd\sigma^r$ (Fig. 4(a)). It is easy to follow that n increases as the point moves to the x-axis. The same we can say, when moving along x-axis from the right to the left to the point A.

Apparently, in point A the stress relaxation is maximum and equal to $Nd\sigma^r$, where $N = 2a/dl$ is the total quantity of relaxation elements within the interval $l_0 - a < x < l_0 + a$.

Integrating (2.1) over the variable l from 0 to $2l_0$ gives a maximum value of the components of plastic deformation in point A:

$$\varepsilon^p_{y\max} = 3\Delta\sigma/E, \quad \varepsilon^p_{x\max} = -\Delta\sigma/E.$$

If the point with the coordinates (x, y) gets into the round region at the end of the band, then from

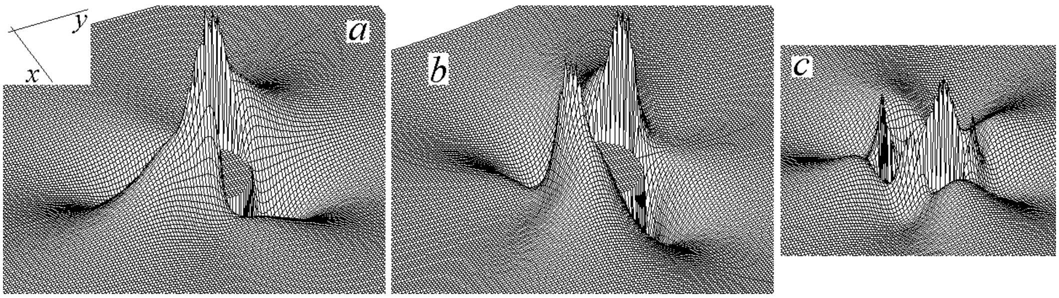

Fig. 3. Distributions of the components of internal stress field (3.18): (a) $d\sigma_y/d\sigma$; (b) $d\sigma_x/d\sigma$; (c) $d\sigma_{xy}/d\sigma$.

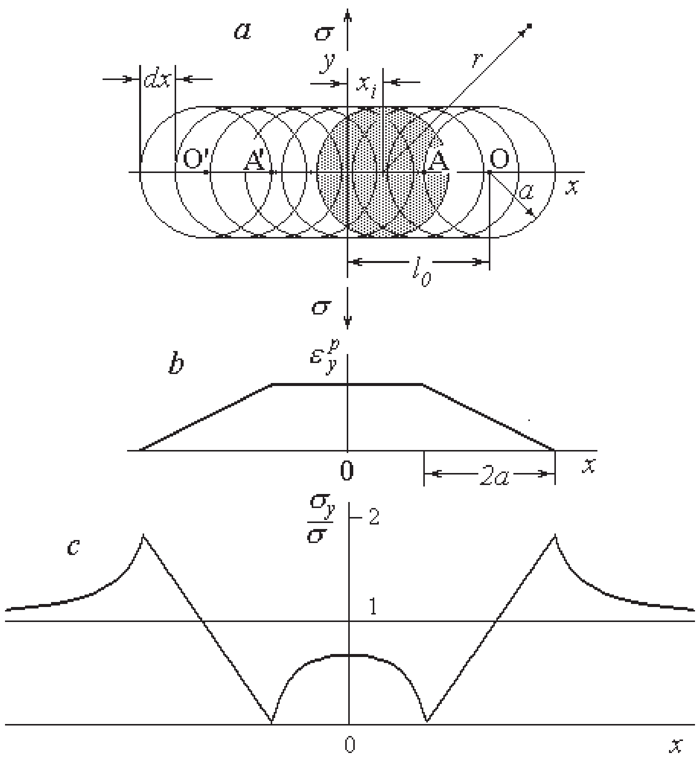

Fig. 4. Even distribution of relaxation elements in the form of circle, placed along the straight line: (a) the profiles of plastic deformation ε_y^p, (b) and stresses σ_y, (c) along x-axis for the band of plastic deformation. l_0 – half-axis of the band.

the geometrical construction it is easy to find, that it is the place, where the stress drop takes place from the contribution of relaxation elements, the quantity of which is

$$N' = N\frac{l_0 - x \pm \sqrt{a^2 - y^2}}{2a},$$

where the sign "+" is true for the circle region on the right end of band and sign "–" for the circle region on the left end of band.

Beyond the round region at the end of the band, a number of relaxation elements, overlapping the point (x, y), are equal to $N'' = N\sqrt{1 - y^2/a^2}$. Hence, the distribution of the component ε_y^p will be described by equations

$$\varepsilon_y^p = \varepsilon_{y\max}^p \begin{cases} 0.5\left[(l_0 - x)/a \pm \sqrt{1 - (y/a)^2}\right] \\ \quad \text{if } (x \mp l_0)^2 + y^2 \leqslant a^2, \\ \sqrt{1 - (y/a)^2} \\ \quad \text{if } (l_0 \mp x)^2 + y^2 \geqslant a^2. \end{cases} \tag{3.1}$$

Analogous equations can be obtained for the component ε_x^p as well. It is seen that at a distance $2a$ from the end of the band, the value of plastic deformation does not depend on the coordinate x (see the low Eq. (3.1)). According to Eq. (2.1), the chosen type of relaxation elements describes plastic deformation, which occurs not only on an increase of width, but also on an essential reduction of length of the band.

The profile of distribution of plastic deformation ε_y^p along x-axis obtained according to Eq. (3.1) is depicted in Fig. 4(b). The spatial distributions of ε_y^p are represented in Fig. 5.

Thus, even distribution of relaxation elements along straight line forms a band of localized plastic deformation with the gradients at the boundary of contact with elastically deformed plane. An integral field of plastic deformation inside the band will be smooth (differentiable). On the boundary of the seat, constructed in such a way, the first derivative ε_y^p with respect to the coordinate

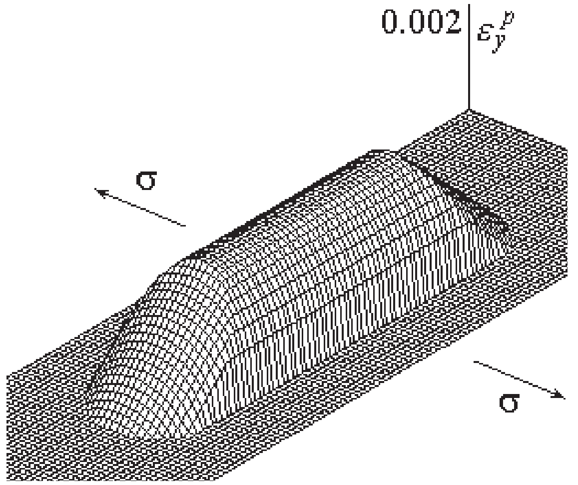

Fig. 5. Spatial distribution of plastic deformation ε_y^p from even distribution of relaxation elements of the round shape along straight line.

undergoes breaks, i.e., the compatibility condition is broken. The breaks of derivative take place in the points of circles on the ends of band too, when $(l_0 \pm x)^2 + y^2 = a^2$. It is obvious that in the qualitative kind of distributions component of a field of plastic deformation inside the band will not depend on the type of relaxation elements.

3.2. Stresses in the plane with the band of localized deformation

The superposition of fields of internal stress from all relaxation elements, obviously, will define the field of internal stress in a plane with a band of plastic deformation. These stresses, obviously, depend on the type of relaxation elements, from which the given strip is constructed. Let us consider some types of strips of the located plastic deformation separately.

3.2.1. The band of localized deformation, which consists of the relaxation elements first-type

In the system of coordinates in Fig. 6 for the arbitrary relaxation element in the band of plastic deformation $R^2 = (x - l_0 + l)^2 + y^2$. Integrating variable l determines the current coordinates of the centers of relaxation element with respect to point $x = l_0$. Inside the circle, according to the chosen

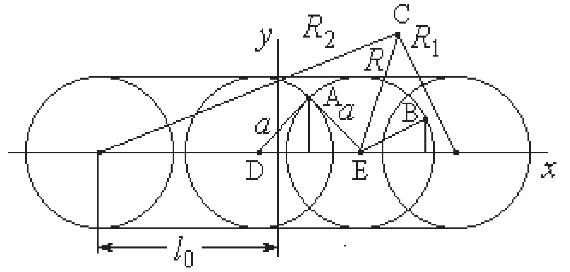

Fig. 6. To the calculation of internal stresses of the plastic deformation band. Explanations are in the text.

relaxation tensor, there exists a homogeneous field of internal stresses of compression:

$$d\sigma_y = -d\sigma = -\sigma dl/2a, \quad d\sigma_x = 0, \quad d\sigma_{xy} = 0. \tag{3.2}$$

Beyond the constructed seat of plastic deformation (point C in Fig. 6) the fields of internal stresses from all the relaxation elements in distribution will be superposed. Thus the resulting stress field will be obtained by an integration of the given equations over the variable l from 0 to $2l_0$. Then, for example, for the component $d\sigma_y$ beyond the band we shall obtain

$$\frac{\sigma_y}{\sigma} = \frac{a}{4}\left\{\frac{l_0 - x}{R_1^2}\left(1 + \frac{a^2 + 4y^2}{R_1^2} - \frac{4a^2y^2}{R_1^4}\right) - \frac{l_0 + x}{R_2^2}\left(1 + \frac{a^2 + 4y^2}{R_2^2} - \frac{4a^2y^2}{R_2^4}\right)\right\}.$$

Here $R_1^2 = (x - l_0)^2 + y^2$ and $R_2^2 = (x + l_0)^2 + y^2$.

In short, it can be written in the form

$$\frac{\sigma_y}{\Delta\sigma} = \frac{-a(l_0 - x + l)}{4R^2}(1 + f - g)\bigg|_0^{2l_0}, \tag{3.3}$$

where $f = (a^2 + 4y^2)/R^2$ and $g = 4a^2y^2/R^4$.

In the band at R_1^2 or $R_2^2 \geqslant a^2$ (see Fig. 6) the point A will not fit into the circle regions at the end of the band. In the interval $l_0 - x - \sqrt{a^2 - y^2} \leqslant l \leqslant l_0 - x + \sqrt{a^2 - y^2}$ this point will fit into the contours of element relaxation. Thus in the given interval it is necessary to integrate Eq. (3.2) and to miss this interval by integration of expression (2.2). One can be convinced that in the given case integration of expression (3.2) over the given interval results in the same result as the integration of expression (2.2) only with the sign reverse. Hence Eq. (3.3) is valid also in the band if R_1^2 or $R_2^2 \geqslant a^2$.

If the point is fit into region R_1^2 or $R_2^2 \leqslant a^2$ (see point B in Fig. 6), then within an interval $0 \leqslant l \leqslant l_0 - x + \sqrt{a^2 - y^2}$ Eq. (3.2) is integrated, and the rest (2.2).

When defining the components σ_x and σ_{xy}, one should bear in mind, which in the corresponding intervals of variable l, according to (3.2) that the relaxation element does not give any contribution to the stress field. It is defined exclusively by integration of the corresponding Eq. (2.2).Thus, under the condition R_1^2 or $R_2^2 \geqslant a^2$ for the given components one can write finally

$$\begin{aligned} \frac{\sigma_x}{\Delta\sigma} &= -\frac{a(l_0 - x + l)}{4R^2}(3 - f + g)\bigg|_A^B, \\ \frac{\sigma_{xy}}{\Delta\sigma} &= -\frac{ay}{4R^2}\left(3 - \frac{3a^2 + 4y^2}{R^2} + g\right)\bigg|_A^B \end{aligned} \tag{3.4}$$

where A and B define the corresponding limits of integration. If $R_1 < a$, then

$$\begin{aligned} \frac{\sigma_y}{\Delta\sigma} &= \frac{a}{4}\left\{2\frac{x - l_0}{a} - \frac{x + l_0}{R_2^2}\right. \\ &\quad \left.\times\left(1 + \frac{a^2 + 4y^2}{R_2^2} - \frac{4y^2a^2}{R_2^4}\right)\right\}, \\ \frac{\sigma_x}{\Delta\sigma} &= \frac{a}{4}\left\{2\sqrt{1 - \frac{y^2}{a^2}} - \frac{x + l_0}{R_2^2}\right. \\ &\quad \left.\times\left(3 - \frac{a^2 + 4y^2}{R_2^2} + \frac{4y^2a^2}{R_2^4}\right)\right\}, \\ \frac{\sigma_{xy}}{\Delta\sigma} &= \frac{-ay}{4R_2^2}\left(3 - \frac{3a^2 + 4y^2}{R_2^2} + \frac{4y^2a^2}{R_2^4}\right). \end{aligned} \tag{3.5}$$

In the case of $R_2 < a$ it is necessary to replace in (3.5) R_2 by R_1, to exchange the places and signs of the values $(x + l_0)$ and $(x - l_0)$.

In Fig. 7 spatial distributions of all components of a stress tensor are given, according to Eqs. (3.2)–(3.5). The distribution of plastic deformation (2.1) is seen to cause a stress concentration of all components at the end of the band. The field disturbance σ_y and σ_{xy} is concentrated mainly at the end of the band, in the region, being commensurable with the width of the band. The concentration of the component σ_y is significantly greater than the component σ_{xy}. Component σ_{xy} contributes a small perturbation to the stress field. It should be noted that any maximums of σ_{xy} do not coincide with maximums of σ_y.

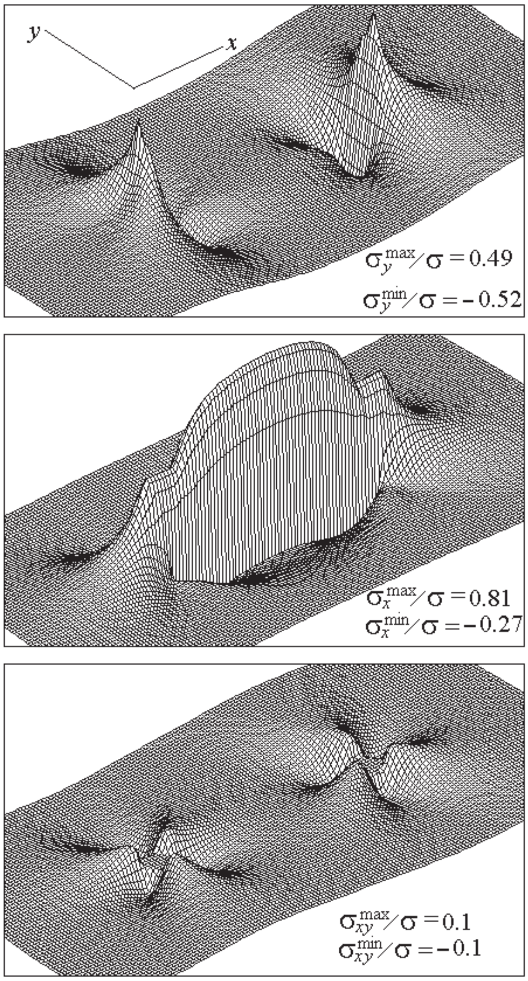

Fig. 7. Distribution of the components of stress fields, according to Eqs. (3.2)–(3.7). $l_0/a = 8$.

The component σ_x makes a strong and specific contribution to stress field. At the end of the band, the value σ_x is of the same order as the value σ_y. This means, that at the end of the band, the material undergoes the state of overall extension. In the band, the material undergoes tensile stresses in a direction perpendicular to tensile axis. Increase in the length of band is accompanied by asymptotic tendency of the maximum value σ_x at the center of the band to the value σ. At the same time, concentration of the component σ_y at the end of the band tends to the value $\sigma/2$. Such a behavior of σ_x is caused by the reaction of a matrix, surrounding the band, on the reduction of length of band owing to the action of plastic deformation components $\varepsilon_y^{\mathrm{p}}$.

3.2.2. The band of plastic deformation from the relaxation elements of type II.

Let us consider the variant of the band of localized deformation, constructed from the relaxation elements of type II. In the given case, each relaxation element will be characterized by the quantum of plastic deformation (2.4) and the components of the field of internal stresses (2.3).

Integrating the components of the field of internal stress of the band of localized deformation, formed as a result of relaxation of only the component σ_x in it, will result in the following equations:

Beyond the band

$$\frac{\sigma_y}{\Delta\sigma} = \frac{-a(x-l_0+l)}{4R^2}(1-f+g)\Big|_A^B,$$
$$\frac{\sigma_x}{\Delta\sigma} = -\frac{a(x-l_0+l)}{4R^2}(5-f+g)\Big|_A^B, \tag{3.6}$$
$$\frac{\sigma_{xy}}{\Delta\sigma} = -\frac{ay}{4R^2}\left(5-\frac{3a^2+4y^2}{R^2}+g\right)\Big|_A^B,$$

where $A=0$ and $B=2l_0$.

Inside the band at R_1^2 or $R_2^2 \geqslant a^2 A = l_0 - x - \sqrt{a^2-y^2}$ and $B = l_0 - x + \sqrt{a^2-y^2}$. In doing so, the expressions for σ_y and σ_{xy} remain like the previous. Integrating the given expression for σ_x within the limits $l_0 - x - \sqrt{a^2-y^2} \leqslant l \leqslant l_0 - x + \sqrt{a^2-y^2}$ by accounting for the contribution from the stresses inside the relaxation elements $\mathrm{d}\sigma_x = -\mathrm{d}\sigma = -\Delta\sigma l/2a$ is written in the form of equation

$$\frac{\sigma_x}{\Delta\sigma} = -\frac{a(l_0-x+l)}{4R^2}(5-f+g)\Big|_A^B - 3\sqrt{1-\frac{y^2}{a^2}}. \tag{3.7}$$

If $R_1^2 \leqslant a^2$, then

$$\frac{\sigma_y}{\Delta\sigma} = \frac{a}{4}\left\{\frac{y}{2a} - \frac{ya}{R_2^2}\left(1-\frac{a^2+4y^2}{R_2^2}+\frac{4y^2a^2}{R_2^4}\right)\right\},$$
$$\frac{\sigma_{xy}}{\Delta\sigma} = \frac{(x+l_0)a}{4R_2^2}\left(1-\frac{a^2+4y^2}{R_2^2}+\frac{4y^2a^2}{R_2^4}\right),$$
$$\frac{\sigma_x}{\Delta\sigma} = \frac{a}{4}\left\{-1.5\sqrt{1-\frac{y^2}{a^2}} - \frac{(x+l_0)a}{R_2^2} \times\left(5-\frac{a^2+4y^2}{R_2^2}+\frac{4y^2a^2}{R_2^4}\right) - \frac{l_0-x}{2a}\right\}. \tag{3.8}$$

In the case of $R_2^2 \leqslant a^2$, in Eq. (3.8) it is necessary to replace R_2 by R_1, interchange the positions and the signs of expressions $(x+l_0)$ and $(x-l_0)$.

In Fig. 8 the distributions of the components of the field of internal stresses, described by Eqs. (3.6)–(3.8) are represented. It is seen, that in such a case, perturbations of the stress fields of the components σ_y and σ_{xy} are negligible. Contrary to the previous case, the σ_x component points to the high compression stress inside the band. These stresses are caused by essential value of plastic deformation $\varepsilon_x^{\mathrm{p}}$ of the material in the band, maximum value of which forms the value $\varepsilon_{x\max}^{\mathrm{p}} = 3\Delta\sigma/E$. As the length of the band increases, the stress σ_x at the center of the band asymptotically tends to the value $\sigma_x^{\max} = -3\Delta\sigma$.

Thus, considered variants of the bands of localized deformation reveal the peculiarity of the component of σ_x inside the band. The rest of the components essentially disturb the field of stresses only at the ends of the band in the volume, commensurable with the width of the band.

3.2.3. The band of plastic deformation from relaxation elements of type III

Let us consider another variant of the band of localized deformation, when plastic deformation in the band is characterized by the unique non-zero component $\varepsilon_y^{\mathrm{p}}$. Plastic deformation there is not accompanied by the changing of the band length. For the construction of such a band, one should use the relaxation element of type III with quantum of plastic deformation $\mathrm{d}\varepsilon_y^{\mathrm{p}} = \mathrm{d}\sigma/E$.

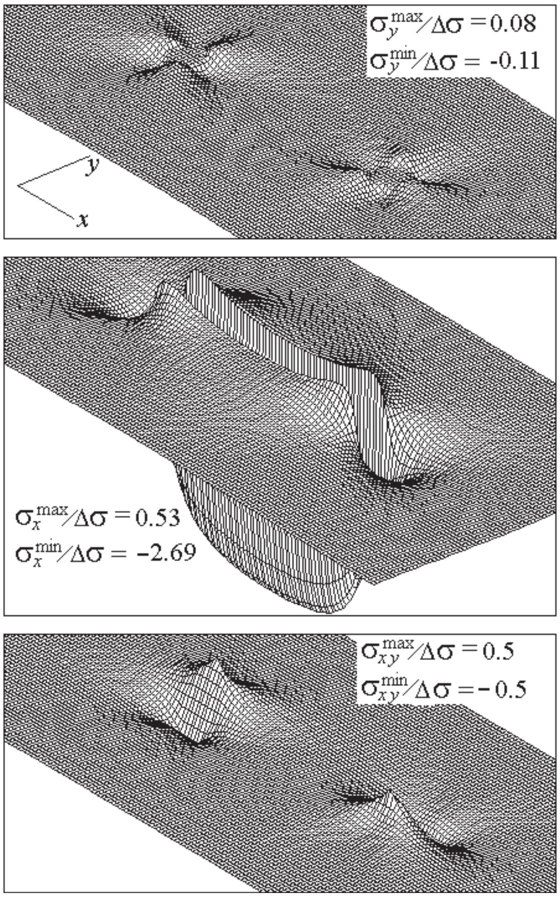

Fig. 8. Distribution of the components of stress fields, according to Eqs. (3.6)–(3.8). $l_0/a = 8$.

We obtain the following equations:

$$\frac{\sigma_y}{\Delta\sigma} = \frac{-a(l_0 - x + l)}{6R^2}(2 + f - g)\bigg|_0^{2l_0},$$
$$\frac{\sigma_x}{\Delta\sigma} = -\frac{a(l_0 - x + l)}{6R^2}(2 - f + g)\bigg|_0^{2l_0},$$
$$\frac{\sigma_{xy}}{\Delta\sigma} = -\frac{ay}{6R^2}\left(2 - \frac{3a^2 + 4y^2}{R^2} + g\right)\bigg|_0^{2l_0} \tag{3.9}$$

if R_1^2 or $R_2^2 \geqslant a^2$.

$$\frac{\sigma_y}{\Delta\sigma} = \frac{a}{6}\left\{-\frac{(x + l_0)a}{R_2^2}\left(2 + \frac{a^2 + 4y^2}{R_2^2} - \frac{4y^2a^2}{R_2^4}\right) - \frac{l_0 - x}{2a}\right\},$$

$$\frac{\sigma_{xy}}{\Delta\sigma} = \frac{-ay}{6R_2^2}\left(2 - \frac{3a^2 + 4y^2}{R_2^2} + \frac{4y^2a^2}{R_2^4}\right),$$
$$\frac{\sigma_x}{\Delta\sigma} = \frac{a}{4}\left\{-\frac{(x + l_0)a}{R_2^2}\left(2 - \frac{a^2 + 4y^2}{R_2^2} + \frac{4y^2a^2}{R_2^4}\right) - \frac{l_0 - x}{2a}\right\} \quad \text{if } R_1^2 \leqslant a^2. \tag{3.10}$$

In the case of $R_2^2 \leqslant a^2$ it is necessary in Eq. (3.10) to replace R_2 by R_1, interchange the positions and signs $(x + l_0)$ and $(x - l_0)$.

Shown in Fig. 9 are the distributions of the components of the field of internal stresses, described by Eqs. (3.9) and (3.10). Comparison with the case, represented in Fig. 7 shows, that quantitative and qualitative distributions of the components σ_y practically have not changed. At the same time, the component σ_x represents a completely different distribution. Perturbation of stress field σ_x is negligible now and is concentrated only at the ends of the band. The perturbation of the same order at the end of the band is observed for the component σ_{xy} too. Observed distinctions testify that the energy of the field of internal stresses of the band of localized plastic deformation of type III is essentially lower than that for the band of I or II type.

Really, according to [3] the work of external forces on the plastic displacements increases the elastic energy of solid by a value.

$$U^* = 0.5\int_{V_0} C_{ijmn}\varepsilon_{ij}^{\mathrm{p}}\varepsilon_{mn}^{\mathrm{p}}\,\mathrm{d}V, \tag{3.11}$$

which determines the energy of the field of internal stresses for the field of plastic deformation in the region of stress relaxation. Here, C_{ijmn} are the elastic constants of a material. The energy (3.11) remains in the volume of solid with defect in the form of sites of plastic deformation and after relieving of an external load. Integral is taken over by the volume V_0 of the site of plastic deformation. Elastic energy of the band of localized deformation, apparently, will be proportional to the elastic energy of relaxation elements of the type, from which the given band of localized plastic deformation is constructed. The characteristics of the

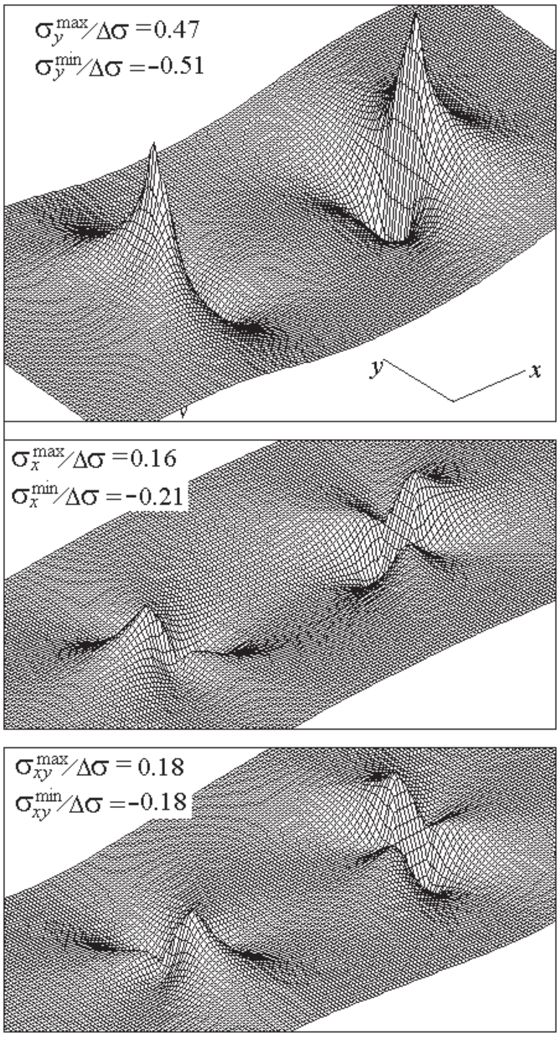

Fig. 9. Distribution of the components of stress fields, according to Eqs. (3.9), (3.10). $l_0/a = 8$.

quantum of plastic deformation for relaxation elements of first-types and type II are represented by Eqs. (2.1) and (2.4). In the isotropic medium, elastic energies of the given relaxation elements per unit of thickness h, according to (3.11) are identical and equal to

$$\mathrm{d}U_{\mathrm{I}} = \mathrm{d}U_{\mathrm{II}} = \pi a^2 E\left[\left(\mathrm{d}\varepsilon_y^{\mathrm{p}}\right)^2 + \left(\mathrm{d}\varepsilon_x^{\mathrm{p}}\right)^2\right]/h = 10\pi a^2(\mathrm{d}\sigma)^2/Eh.$$

Quantum of plastic deformation of relaxation element III type is equal to $\mathrm{d}\varepsilon_y^{\mathrm{p}} = 8\mathrm{d}\sigma/3E$. Hence, an elastic energy of the given relaxation element is equal to

$$\mathrm{d}U_{\mathrm{III}} = \pi a^2(\mathrm{d}\varepsilon_y^{\mathrm{p}})^2/h = 7.1\pi a^2(\mathrm{d}\sigma)^2/Eh.$$

Comparison shows that it forms, practically only 2/3 part of elastic energy of relaxation elements first- or second-types. Hence, the formation of the band of localized deformation of type III under the same other conditions is preferable, i.e., it requires less work of external applied forces.

3.2.4. The band of plastic deformation from relaxation elements of IV type

Plastic deformation in such a band apparently results in change of shape without changing the band area.

Integration of the field of stresses of relaxation element of pure shear results in the following equations:

$$\begin{aligned}
\frac{\sigma_y}{\Delta\sigma} &= -\frac{ay}{2R^2}\left(2 - \frac{3a^2+4y^2}{R^2} + g\right)\Bigg|_0^{2l_0}, \\
\frac{\sigma_x}{\Delta\sigma} &= -\frac{ay}{2R^2}\left(6 - \frac{3a^2+4y^2}{R^2} + g\right)\Bigg|_0^{2l_0}, \\
\frac{\sigma_{xy}}{\Delta\sigma} &= \frac{-a(l_0 - x + l)}{2R^2}(2 - f + g)\Bigg|_0^{2l_0}
\end{aligned} \tag{3.12}$$

if R_1^2 or $R_2^2 \geqslant a^2$.

$$\begin{aligned}
\frac{\sigma_y}{\Delta\sigma} &= \frac{-ay}{2R_2^2}\left(2 - \frac{3a^2+4y^2}{R_2^2} - \frac{4y^2a^2}{R_2^4}\right) - \frac{y}{2a}, \\
\frac{\sigma_x}{\Delta\sigma} &= \frac{-ay}{2R_2^2}\left(6 - \frac{3a^2+4y^2}{R_2^2} - \frac{4y^2a^2}{R_2^4}\right) + \frac{3y}{2a}, \\
\frac{\sigma_{xy}}{\Delta\sigma} &= \frac{-a(x+l_0)}{2R_2^2}\left(2 - \frac{a^2+4y^2}{R_2^2} + \frac{4y^2a^2}{R_2^4}\right) + \frac{x-l_0}{2a} \quad \text{if } R_1^2 \leqslant a^2.
\end{aligned} \tag{3.13}$$

In the case of $R_2^2 \leqslant a^2$ it is necessary in Eq. (3.13) to replace R_2 by R_1, interchange the positions and signs of the equations $(x + l_0)$ and $(x - l_0)$.

In Fig. 10 the distributions of the components of the field of internal stresses, described by

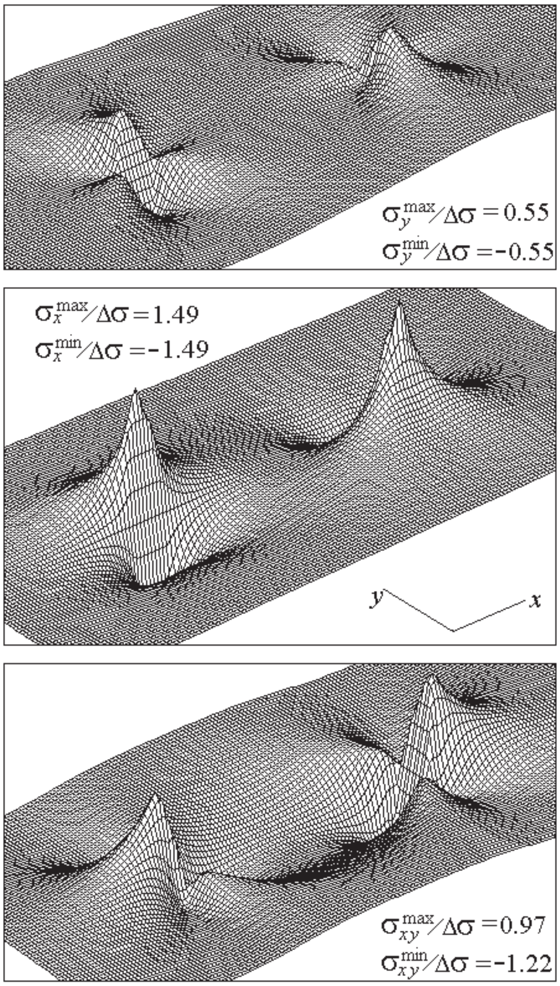

Fig. 10. Distribution of the stress components, according to Eqs. (3.12) and (3.13). $l_0/a = 8$.

Eqs. (3.12) and (3.13) are represented. It is seen that in the given case at R_1^2 or $R_2^2 \geqslant a^2$ we obtain a smooth (differentiated) stress field. At the end of the band there exists an essential concentration of all components of stress tensor. Non-shear component of stress field σ_x along the band of localized plastic deformation undergoes the highest concentration. It exceeds the maximum value σ_y practically three times. Maximum σ_x and σ_{xy} are not coincided. σ_y component undergoes minimum concentration, although its maximum value is not less than that observed in the previous case.

Elastic energy of the relaxation element for the band of plastic deformation of pure shear per unit of the plane thickness, according to Eq. (3.13) accounting for (2.6) is equal to

$$\mathrm{d}U_{\mathrm{IV}} = G\pi a^2 \left[\mathrm{d}\varepsilon_{xy}^{\mathrm{p}}\right]^2 = 6.2\pi a^2 (\mathrm{d}\sigma)^2/E, \qquad (3.14)$$

where $G = E/2(1+\nu)$ is the shear modulus, and ν is Poisson's ratio. In the calculation, we put $\nu = 0.3$. The value $\mathrm{d}U_{\mathrm{IV}}$ turned out to be somewhat less than $\mathrm{d}U_{\mathrm{III}}$. Consequently, energy of the field of internal stresses of the band of plastic deformation in case of pure shear has the same order as the energy of the band from relaxation elements of type III.

Performed analysis shows that formation of the bands of plastic deformation from relaxation elements of III or IV types under another equal condition is preferable, as it requires less work of external applied forces, than under formation of the bands from relaxation elements of first or second types.

4. The band of plastic deformation under arbitrary angle to tensile axis.

Assume that tensile stress σ is directed along y-axis. Then plastic deformation in the band will

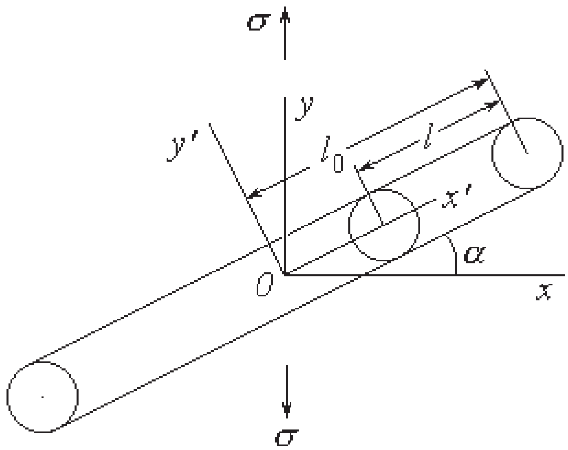

Fig. 11. The scheme of the band of localized deformation under angle α to tensile axis.

proceed preferably owing to relaxation of the component σ_y in the plane. Let us consider the band of plastic deformation at an arbitrary angle α to tensile axis.

Formation of the band requires minimum work of external applied forces in two cases. Firstly, when relaxation is realized through the shear components σ_{xy} of stress tensor. Then plastic deformation does not result in a change of the band's length. First case we can realize that when plastic deformation is fulfilled only by one or two slip systems. Second case is possible only at the multiple slip, ensuring the component ε_z^p of the tensor of plastic deformation.

In the coordinate system $x'oy'$ in Fig. 11 one can ensure the stress drop $\Delta\sigma_y = \Delta\sigma = 2\Delta\sigma_{y'x'} \sin\alpha\cos\alpha$ through the shear stress along the band $\Delta\sigma_{y'x'}$. Hence, maximum value of stress drop σ_y due to plastic deformation of shear will be observed at the orientation of the band at an angle $\alpha = 45°$ to tensile axis. At $\alpha = 0$ plastic deformation of pure shear does not result in stress relaxation σ_y. The expressions for the components of the field of internal stresses of the pure shear band can be obtained by substitution of Eqs. (3.12) and (3.13) in the coordinate system $x'oy'$, and having transformed them by rotation of the system through an angle $-\alpha$.

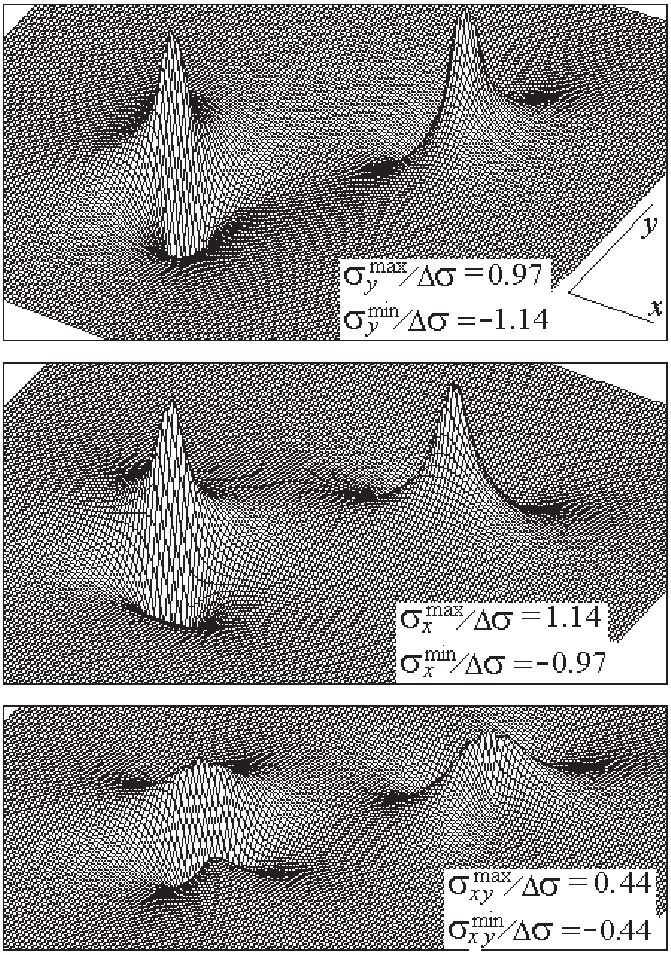

Fig. 12. Distributions of the stress components of the band of plastic deformation of pure shear at an angle 45° degrees to tensile axis. $l_0/a = 8$.

Shown in Fig. 12 are the distributions of the components of the field of internal stresses of the band of plastic deformation of pure shear, being oriented at an angle 45° to tensile axis. It is seen that essential stress concentration of the components σ_y and σ_x is observed. The peculiarity is the fact that the region of elevated stress concentration is not in front of the band end, but to the side of it. It is well demonstrated on the pattern of stress isolines (Fig. 13). Maximum stress gradient takes place at the end of the band, in a direction, being perpendicular to the band axis.

Qualitatively another distribution is observed for shear stress τ at an angle 45° to the tensile axis (Fig. 14). In the given direction it is maximum. The region of high shear stress concentration is situated ahead of the band. Maximum shear stress gradient has the direction along the band at their ends. In such a manner, in the case of operation of unique slip system, the propagation of the band of localized deformation at an angle 45° degrees requires minimum external applied tensile stress and formation of the band in the given direction is monitored by critical shear stress.

At the multiple slipping, allowing for the component ε_z^p of tensor of plastic deformation, there exists a possibility of σ stress relaxation due to all components of stress tensor in the coordinate system $x'oy'$ (Fig. 11).

Stress drop $\Delta\sigma_y = \Delta\sigma$ in the coordinate system $x'oy'$ is represented in the form of the components:

$$\Delta\sigma_{y'} = \Delta\sigma\cos^2\alpha, \quad \Delta\sigma_{x'} = \Delta\sigma\sin^2\alpha, \quad \Delta\sigma_{y'x'} = \Delta\sigma\cos\alpha\sin\alpha. \tag{4.1}$$

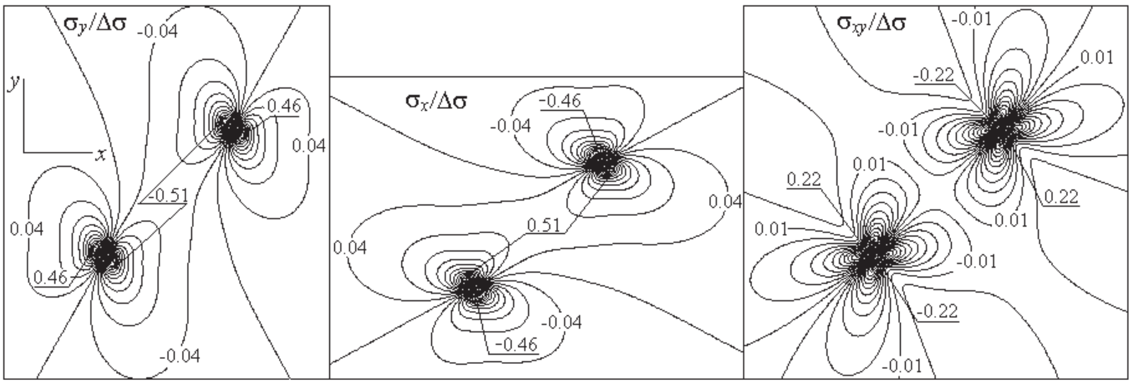

Fig. 13. The stress isolines of the shear band at an angle 45° degrees to tensile axis. $l_0/a = 8$.

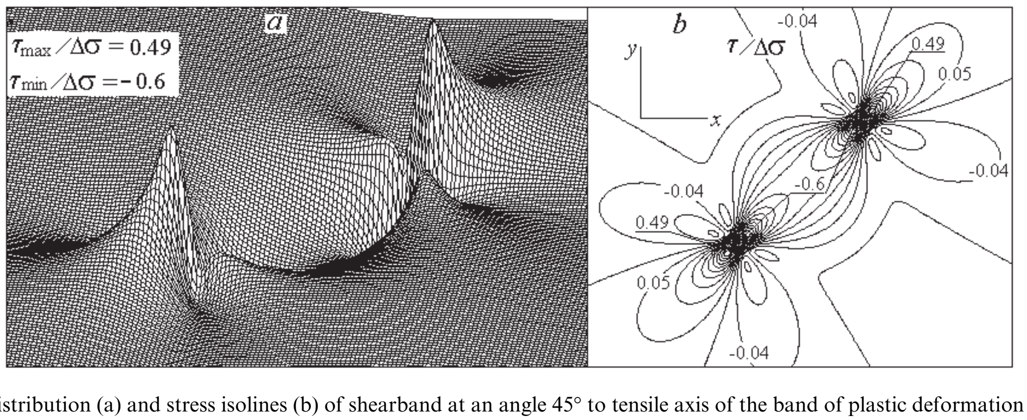

Fig. 14. Distribution (a) and stress isolines (b) of shearband at an angle 45° to tensile axis of the band of plastic deformation $l_0/a = 8$.

Thus, stress field in a plane can be represented in the form of superposition of three solutions for relaxation the stresses $\sigma_{y'}$, being normal to the band's axis, $\Delta\sigma_{x'}$ along the band axis and $\Delta\sigma_{y'x}$ along coordinate axes x' and y' Eq. (4.1) defines the values of relaxation $\Delta\sigma_{'}$, $\Delta\sigma_{x'}$, $\Delta\sigma_{y'x'}$.

Corresponding solutions we considered in the previous section. In the given case, it is necessary to substitute the value $\Delta\sigma_{y'}$, instead of the value $\Delta\sigma$ in Eqs. (3.3)–(3.5), the value $\Delta\sigma_x$ in Eqs. (3.6)–(3.8) and the value $\Delta\sigma_{y'x'}$ in Eqs. (3.12) and (3.13), then to rotate the coordinate system through the angle $-\alpha$.

Eqs. (3.3)–(3.5) and (3.6)–(3.8) reveal the peculiarities in the band in the form of high tensile (Fig. 7) or compression stresses (Fig. 8). Thus, in the common case these peculiarities, in one way or another, will manifest themselves in the band also, being oriented at an arbitrary angle to tensile axis. The exception is the band orientation at an angle of 60° to tensile axis (at an angle $\alpha = 30°$ to x-axis in Fig. 11). In such a situation, the component $\Delta\sigma_{x'}$ is 1/3 of the component $\Delta\sigma_{y'}$ and we have a case of the band of localized plastic deformation of III type, being characterized by smooth field of

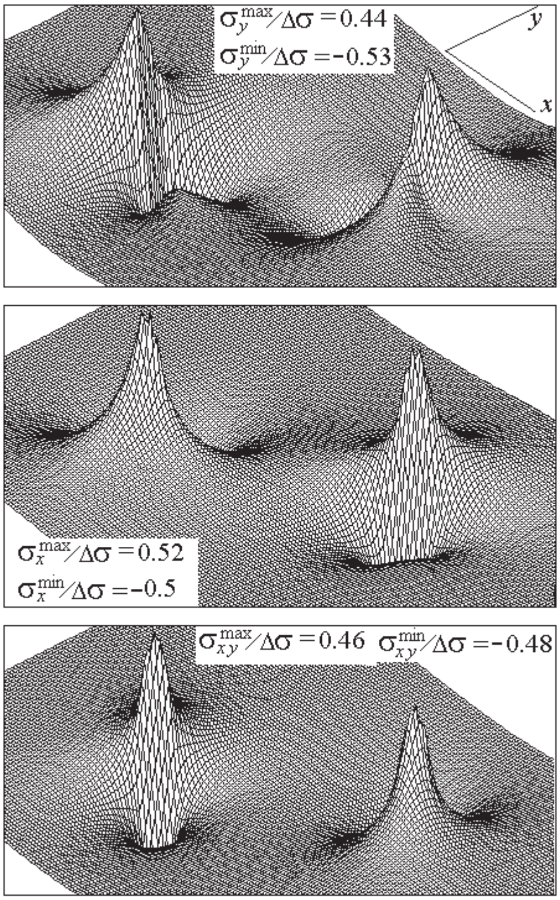

Fig. 15. Distributions of the components of stress fields of the band of plastic deformation at an angle 60° to tensile axis at a multiply slip. $l_0/a = 8$.

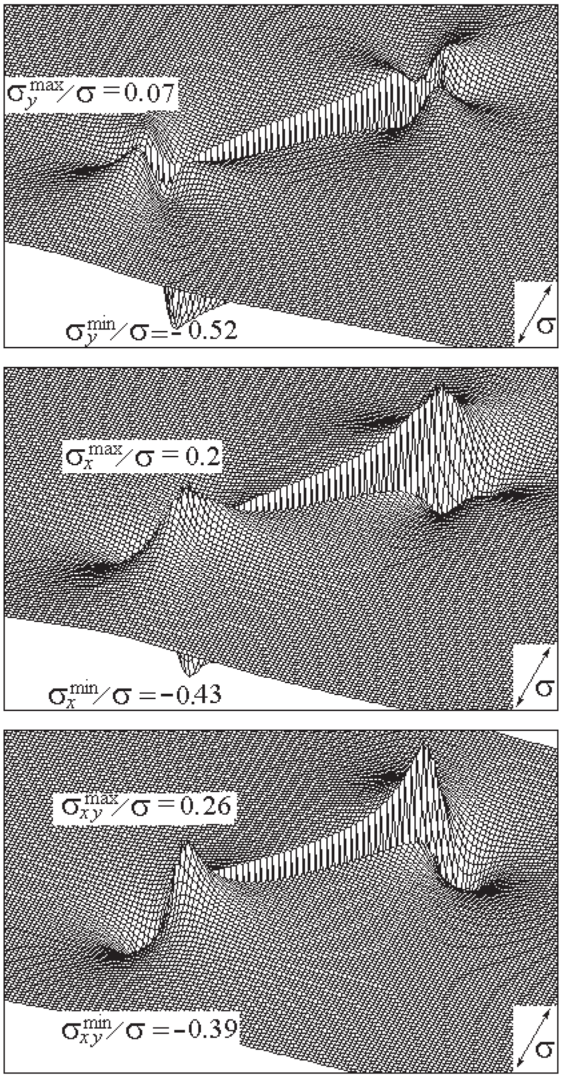

Fig. 16. Distributions of the components of stress fields of the band of plastic deformation at an angle 45° to tensile axis at a multiply slip. $l_0/a = 8$.

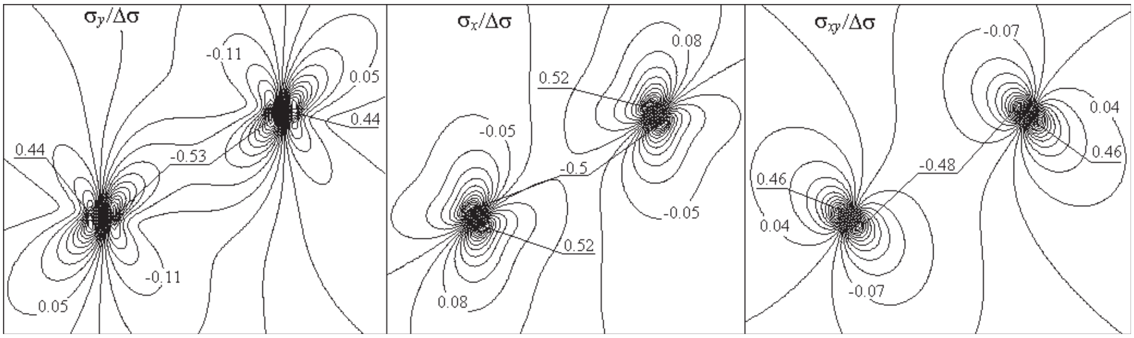

Fig. 17. Stress isolines of the plastic deformation band at an angle 60° to tensile axis at the multiply slipping. $l_0/a = 8$.

stresses (see Eqs. (3.9), (3.10) and Fig. 9). We have already noted that the energy of the field of internal stresses of the band is minimum. At that time, changing of the length of the band due to plastic deformation does not occur.

Distributions of all components of stress fields for the case of band orientation at an angle 30° to x-axis are represented in Fig. 15. It is seen that stress field is disturbed significantly only at the ends of the band, where a great concentration of all tensor components is observed. Apparently, as the length of the band increases, the region of stress concentration will displace together with the end of the band. At a sufficient distance from the end, the stress field in the band will be undisturbed, i.e. it practically became homogeneous.

When violating this rule, differentiability of the stress gradients on the band boundary is broken. An example is given in Fig. 16 for all components of stress tensor at $\alpha = 45°$, which have a topological peculiarity on the boundary in the form of jump, accompanied by stress drop. For this reason, the band of localized plastic deformation clearly develops in the patterns of distributions. As the inclination angle of the band increases, the stress relaxation increases. As this takes place, the stress concentration σ_y at the end of the band successively decreases.

The picture of stress isolines for the case of band orientation at an angle 30° to x-axis presented in Fig. 17. It points to the high probability of changing the band direction on conjugate one during the time of formation. It is often observed in experiments.

5. Conclusion

Analysis of the pattern of distributions of the components σ_y, σ_x and σ_{xy} reveals, that minimum changing of the field of internal stresses depends not only on the band orientation, but also on the possibility of relaxation of non-shear component of stress tensor along tensile axis. Relaxation of the non-shear component of stress tensor in the band of localized deformation without breaking the material continuity can be provided only by a combination of shear component of stress tensor along different directions, i.e., under condition of the operation of the mechanism of multiple slipping. These cases are realized very often [4,5]. In the band, being oriented at an angle of 60° to tensile axis, the relaxation of the components $\sigma_{y'}$ and $\sigma_{x'}$ of external stress does not result in plastic deformation along the band. In such a case one should obtain a minimum elastic energy of the field of internal stresses of the localized deformation and maximum stress concentration at the end of the band or the values of plastic deformation in the band itself (see Fig. 17). This means, that under multiple slip, the most favorable variant for the accumulation of plastic deformation is the angle 60° to tensile axis. As a consequence, the formation of the band of localized deformation along the given direction will occur at the minimum external applied stress.

Variants of the accumulation of plastic deformation by relaxation of the only shear component does not result in arising of high gradients and topological peculiarities of stress field far from the ends of the band (see Fig. 17) for all components of stress tensor. If the accumulation of plastic deformation in the band of localized deformation is defined only by shear mechanism on the only one slip system, then the most favorable way for accumulation of plastic deformation is the direction of shear along the band and the angle of inclination of the band 45° to tensile axis. Given angle of orientation of the band of localized deformation then the minimum elastic energy of the field of internal stresses of localized deformation as a defect in continuous medium and minimum external stress of the formation of the band of localized plastic deformation are ensured.

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