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DELAY EQUATIONS OF THE WHEELER–FEYNMAN TYPE

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Abstract. We present an approximate model of Wheeler–Feynman electrodynamics, for which uniqueness of solutions is proved. It is simple enough to be instructive but close enough to Wheeler–Feynman electrodynamics such that we can discuss its natural type of initial data, constants of motion, and stable orbits with regard to Wheeler–Feynman electrodynamics.

1. Introduction

Already in the early stages of classical electrodynamics, it was observed that the coupled equations of motion of Maxwell and Lorentz for point charges are ill-defined. Since then, many attempts have been made to cure this problem, among which the most famous one is Dirac's mass renormalization [5]. However, none of the known cures has yet led to a mathematically well-defined and physically sensible theory of relativistic interaction of point charges. Rather than being a cure of the old theory, Wheeler–Feynman electrodynamics (WF) is a new formulation of classical electrodynamics and is maybe the most promising candidate for a divergence-free theory of electrodynamics of point charges that is capable of describing the phenomenon of radiation damping [10, 11]. For an overview on the mathematical and physical difficulties of classical electrodynamics and the role of WF, we refer the interested reader also to the introductory sections of [3] and [2]. In contrast to textbook electrodynamics, which introduces fields as well as charges, WF is a theory only on charges — fields only occur as mathematical entities and not as dynamical degrees of freedom.

Let N be the number of charges, $\mathbf{q}_{i,t} \in \mathbb{R}^3$ the position of the ith charge at time t, $m_i > 0$ its mass, and

$$\mathbf{p}_{i,t} = \frac{m_i \mathbf{v}_{i,t}}{\sqrt{1-\mathbf{v}_{i,t}^2}}, \qquad \mathbf{v}_{i,t} := \frac{d\mathbf{q}_{i,t}}{dt}$$

its relativistic momentum and velocity. The fundamental equations of WF take the form

$$\frac{d}{dt}\begin{pmatrix}\mathbf{q}_{i,t}\\ \mathbf{p}_{i,t}\end{pmatrix} = \begin{pmatrix}\mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m_i^2+\mathbf{p}_{i,t}^2}}\\ e_i \sum\limits_{j\neq i} \mathbf{F}_t[\mathbf{q}_j](\mathbf{q}_{i,t},\mathbf{p}_{i,t})\end{pmatrix}, \qquad 1 \le i \le N. \tag{1.1}$$

We have chosen units such that the speed of light equals 1. The force term $\mathbf{F}_t[\mathbf{q}_j]$ is a functional of the trajectory $t \mapsto \mathbf{q}_j$ and can be expressed by

$$\mathbf{F}_t[\mathbf{q}_j](\mathbf{x},\mathbf{p}) := e_j \sum_{\pm} \left(\mathbf{E}_t^{\pm}[\mathbf{q}_j](\mathbf{x}) + \mathbf{v}(\mathbf{p}) \wedge \mathbf{B}_t^{\pm}[\mathbf{q}_j](\mathbf{x})\right),$$

where $(\mathbf{E}_t^+[\mathbf{q}_j], \mathbf{B}_t^+[\mathbf{q}_j])$ and $(\mathbf{E}_t^-[\mathbf{q}_j], \mathbf{B}_t^-[\mathbf{q}_j])$ denote the advanced and retarded Liénard–Wiechert fields — in our notation $\mathbf{E}$ represents the electric and $\mathbf{B}$ the magnetic component and $\wedge$ is the outer

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product. The Liénard–Wiechert fields are special solutions to the Maxwell equations corresponding to a prescribed point charge trajectory $t \mapsto \mathbf{q}_{j,t}$; see, e.g., [2]. Their explicit form is

$$\begin{aligned} \mathbf{E}_t^{\pm}[\mathbf{q}_j](\mathbf{x}) &:= \left[\frac{(\mathbf{n}_j \pm \mathbf{v}_j)(1-\mathbf{v}_j^2)}{\|\mathbf{x}-\mathbf{q}_j\|^2 (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)^3} + \frac{\mathbf{n}_j \wedge (\mathbf{n}_j \wedge \mathbf{a}_j)}{\|\mathbf{x}-\mathbf{q}_j\| (1 \pm \mathbf{n}_j \cdot \mathbf{v}_j)^3} \right]_{\pm}, \\ \mathbf{B}_t^{\pm}[\mathbf{q}_j](\mathbf{x}) &:= \mp \mathbf{n}_{j,\pm} \wedge \mathbf{E}_t^{\pm}[\mathbf{q_j}](\mathbf{x}). \end{aligned} \tag{1.2}$$

Here, we have used the abbreviations

$$\mathbf{q}_{j,\pm} = \mathbf{q}_{j,t^{\pm}}, \qquad \mathbf{n}_{j,\pm} = \frac{\mathbf{x}-\mathbf{q}_{j,t_j^{\pm}}}{\left\|\mathbf{x}-\mathbf{q}_{j,t_j^{\pm}}\right\|}, \qquad \mathbf{v}_j = \left.\frac{d\mathbf{q}_{j,t}}{dt}\right|_{t=t_j^{\pm}}, \qquad \mathbf{a}_j = \left.\frac{d^2\mathbf{q}_{j,t}}{dt^2}\right|_{t=t_j^{\pm}}, \tag{1.3}$$

and the delayed times t_j^+ and t_j^- are implicitly defined as solutions to

$$t_j^{\pm}(t,\mathbf{x}) := t \pm \left\|\mathbf{x}-\mathbf{q}_{j,t_j^{\pm}(t,\mathbf{x})}\right\|. \tag{1.4}$$

Given the space-time point $(t,\mathbf{x})$, the delayed times t_j^+ and t_j^- are determined by the intersection points of the trajectory $t \mapsto \mathbf{q}_{j,t}$ with the future and past light cone of $(t,\mathbf{x})$, respectively. As long as $\|\dot{\mathbf{q}}_{j,t}\| < 1$, both times t_j^+ and t_j^- are well defined. As is apparent from (1.4), the WF equations (1.1) involve terms with time-like advanced as well as retarded arguments, which makes them mathematically very hard to handle:

- The question of existence of solutions is completely open with the exception of the following special cases: Schild found explicit solutions formed by two attracting charges that revolve around each other on stable orbits [8], which can be generalized to many particles; see, e.g., [7]. Driver proved existence and uniqueness of solutions for two repelling charges constrained to the straight line whenever initially the relative velocity is zero and the spatial separation is sufficiently large [6]. Furthermore, Bauer proved existence of solutions in the case of two repelling charges constrained to the straight line [1], and a more recent result ensures the existence of solutions on finite but arbitrarily large time intervals for N arbitrary charges in spatial dimension 3 [2].
- It is not even clear what can be considered to be sensible initial data to uniquely identify WF solutions. While Driver's uniqueness result suggests the specification of initial positions and momenta of the charges, Bauer's work points to the use of asymptotic positions and velocities to distinguish scattering solutions, and one sentence below Fig. 3 in [11] hints to a certain configuration of whole trajectory strips of the charges as initial conditions.

Furthermore, one important issue of WF is the justification of the so-called absorber condition and the derivation of the irreversible behavior, which we experience in radiation phenomena [10]. The link between WF and experience must be based on a notion of typical trajectories, i.e., on a measure of typicality for WF dynamics. Such a measure is unknown and at the moment out of reach. A generalization to many particles of the approximate model we consider next provides an excellent case study for introducing a notion of typicality for this kind of non-Markovian dynamics. Note that uniqueness as well as conservation of energy are very likely to be important for defining a measure of typicality in the sense of Boltzmann. This is work in progress.

Our intention behind this work is to provide a pedagogical introduction to the mathematical structures arising from delay differential equations of the WF type by discussing initial data and uniqueness, constants of motion, and stable orbits in a nontrivial approximate model of WF, which was introduced in [4] and for which many mathematical questions can be answered satisfactorily. Compared to (1.1) we make the following simplifications:

- We consider only two charges, i.e., $N = 2$.

- As fundamental equations of this approximate model, we take the form (1.1), where we replace the Liénard–Wiechert fields $(\mathbf{E}_t^{\pm}[\mathbf{q}_j](\mathbf{x}), \mathbf{B}_t^{\pm}[\mathbf{q}_j](\mathbf{x}))$ by

$$\mathbf{E}_t^{\|,\pm}[\mathbf{q}_j](\mathbf{x}) := e_j \frac{\mathbf{x} - \mathbf{q}_{j,t_j^{\pm}(t,\mathbf{x})}}{\left\|\mathbf{x} - \mathbf{q}_{j,t_j^{\pm}(t,\mathbf{x})}\right\|^3}, \qquad \mathbf{B}_t^{\pm}[\mathbf{q}_j] = 0, \tag{1.5}$$

  i.e., the Coulomb fields at the respective delayed times in the future and the past.

The resulting equations are

$$\frac{d}{dt}\begin{pmatrix}\mathbf{q}_{i,t}\\ \mathbf{p}_{i,t}\end{pmatrix} = \begin{pmatrix}\mathbf{v}(\mathbf{p}_{i,t})\\ e_i e_j\left[\mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^{+}(t,\mathbf{q}_{i,t})}\right) + \mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^{-}(t,\mathbf{q}_{i,t})}\right)\right]\end{pmatrix}, \qquad i,j \in \{1,2\}, \quad j \neq i, \tag{1.6}$$

where the force field is given by

$$\mathbf{F} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}, \qquad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) := \frac{\mathbf{x}}{\|\mathbf{x}\|^3}. \tag{1.7}$$

We emphasize that the delay function (1.4) is the same as in WF and that the simplified fields (1.5) are the longitudinal modes of the Liénard–Wiechert fields $(\mathbf{E}_t^{\pm}[\mathbf{q}_j](\mathbf{x}), \mathbf{B}_t^{\pm}[\mathbf{q}_j](\mathbf{x}))$, i.e.,

$$\nabla \cdot \left(\mathbf{E}_t^{\pm}[\mathbf{q}_j](\mathbf{x}) - \mathbf{E}_t^{\|,\pm}[\mathbf{q}_j](\mathbf{x})\right) = 0, \qquad \nabla \wedge \mathbf{E}_t^{\|,\pm}[\mathbf{q}_j](\mathbf{x}) = 0.$$

Furthermore, for small velocities and accelerations of the jth charge one finds $\mathbf{E}_t^{\pm}[\mathbf{q}_j](\mathbf{x}) \approx \mathbf{E}_t^{\|,\pm}[\mathbf{q}_j](\mathbf{x})$. In this sense one can regard this simplified model as a physically interesting approximation to WF. Note also that, in contrast to $\mathbf{E}_t^{\pm}[\mathbf{q}_j]$, the simplified field $\mathbf{E}_t^{\|,\pm}[\mathbf{q}_j]$ does not depend on the acceleration of the jth charge; compare (1.2). This fact makes it easier to control smoothness of solutions but is not the key difference that allows us to prove uniqueness of solutions for the approximate model.

This paper is organized as follows:

(1) In Sec. 2, we discuss natural initial data for the approximate model (Definition 2.1), and show how to construct solutions uniquely depending on given initial data (Theorem 2.1). A byproduct is the observation that in general the specification of initial positions and momenta of the two charges is not sufficient to ensure uniqueness (Corollary 2.1).
(2) In Sec. 3, we discuss the structure of constants of motion by reference to the energy conservation in the approximate model. The energy functional is defined in Definition 3.1, and the energy conservation is shown in Theorem 3.1.
(3) In Sec. 4, we identify stable orbits (Definition 4.1) that solve (1.6) as proved in Theorem 4.1.
(4) We conclude in Sec. 5 by putting these results in perspective to WF.

Notation.

- If not otherwise specified, we use the convention that $i, j \in \{1,2\}$ and $j \neq i$.
- Any derivative at a boundary of an interval is understood as the left-hand or right-hand side derivative.
- Vectors in $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ are denoted by bold letters, the inner product by $\mathbf{x} \cdot \mathbf{y}$, the outer product by $\mathbf{x} \wedge \mathbf{y}$, and the Euclidean norm by $\|\mathbf{x}\|$.
- ∇, $\nabla\cdot$, and $\nabla\wedge$ denote the gradient, divergence, and curl w.r.t. $\mathbf{x}$, respectively.
- Overset dots denote derivatives with respect to time, i.e.,

$$\dot{\mathbf{q}}_{i,t} = \frac{d\mathbf{q}_{i,t}}{dt}, \quad \ddot{\mathbf{q}}_{i,t} = \frac{d^2\mathbf{q}_{i,t}}{dt^2}.$$

- The future and past light cone of the space-time point $(t, \mathbf{x})$ is defined as the set

$$L_{\pm}(t,\mathbf{x}) := \{(s,\mathbf{y}) \mid (t-s)^2 - (\mathbf{x}-\mathbf{y})^2 = 0, t > \pm s\}$$

  for $+$ and $-$, respectively.

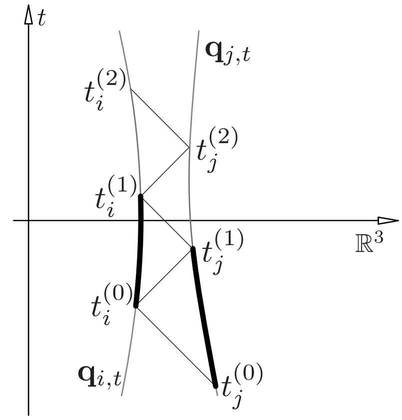

Fig. 1. The thick black stripes represent the initial data.

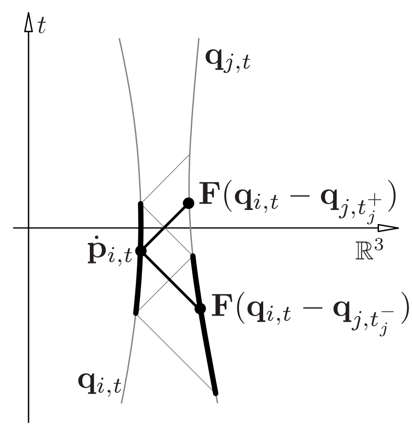

Fig. 2. Nature of the interaction.

2. Construction of Solutions

For this section, we regard the indices (i, j) fixed to either $(1, 2)$ or $(2, 1)$. We consider the following initial data (see Fig. 2):

Definition 2.1. We call a pair $\left(\mathbf{q}_i^{(0)}, \mathbf{p}_i^{(0)}\right)_{i=1,2}$ of smooth position and momentum trajectory stripes that fulfill the conditions

$$\begin{aligned}
\mathbf{q}_i^{(0)} : \quad \mathbb{R} \supset [t_i^{(0)}, t_i^{(1)}] \quad &\to \mathbb{R}^3, \qquad t \mapsto \mathbf{q}_{i,t}^{(0)}, \\
\mathbf{p}_i^{(0)} : \quad \mathbb{R} \supset [t_i^{(0)}, t_i^{(1)}] \quad &\to \mathbb{R}^3, \qquad t \mapsto \mathbf{p}_{i,t}^{(0)}, \qquad i \in \{1, 2\}
\end{aligned}$$

the *initial data* for Eq. (1.6) if

(1) For times $t \in [t_i^{(0)}, t_i^{(1)}] \cap [t_j^{(0)}, t_j^{(1)}]$, we have

$$\mathbf{q}_{i,t}^{(0)} \neq \mathbf{q}_{j,t}^{(0)}.$$

(2) For all $t \in [t_i^{(0)}, t_i^{(1)}]$, we have

$$\frac{d}{dt}\mathbf{q}_{i,t}^{(0)} = \mathbf{v}(\mathbf{p}_{i,t}^{(0)}) \equiv \frac{\mathbf{p}_{i,t}^{(0)}}{\sqrt{m_i^2 + \left\|\mathbf{p}_{i,t}^{(0)}\right\|^2}}, \qquad i \in \{1,2\}. \tag{2.1}$$

(3) The times $t_i^{(0)}, t_i^{(1)}, t_j^{(0)}, t_j^{(1)}$ relate to each other according to

$$t_i^{(0)} = t_i^+\left(t_j^{(0)}, \mathbf{q}_{j,t_j^{(0)}}^{(0)}\right), \qquad t_j^{(1)} = t_j^+\left(t_i^{(0)}, \mathbf{q}_{i,t_i^{(0)}}^{(0)}\right), \qquad t_i^{(1)} = t_i^+\left(t_j^{(1)}, \mathbf{q}_{j,t_j^{(1)}}^{(0)}\right). \tag{2.2}$$

(4) At time $t = t_i^{(0)}$ and for all integers $n \geq 0$ the trajectories obey

$$\frac{d^n}{dt^n}\begin{pmatrix}\dot{\mathbf{q}}_{i,t}\\ \dot{\mathbf{p}}_{i,t}\end{pmatrix} = \frac{d^n}{dt^n}\begin{pmatrix}\mathbf{v}(\mathbf{p}_{i,t})\\ e_ie_j\left[\mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^+(t,\mathbf{q}_{i,t})}\right) + \mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^-(t,\mathbf{q}_{i,t})}\right)\right]\end{pmatrix}. \tag{2.3}$$

(5) At time $t = t_j^{(1)}$ and for all integers $n \geq 0$ the trajectories obey

$$\frac{d^n}{dt^n}\begin{pmatrix}\dot{\mathbf{q}}_{j,t}\\ \dot{\mathbf{p}}_{j,t}\end{pmatrix} = \frac{d^n}{dt^n}\begin{pmatrix}\mathbf{v}(\mathbf{p}_{j,t})\\ e_je_i\left[\mathbf{F}\left(\mathbf{q}_{j,t} - \mathbf{q}_{i,t_i^+(t,\mathbf{q}_{j,t})}\right) + \mathbf{F}\left(\mathbf{q}_{j,t} - \mathbf{q}_{i,t_i^-(t,\mathbf{q}_{j,t})}\right)\right]\end{pmatrix}. \tag{2.4}$$

Remark 2.1. Note that (2.1) requires $\|\dot{\mathbf{q}}_{i,t}\| < 1$ for $i \in \{1,2\}$ and

$$\mathbf{p}_{i,t} = \frac{m_i\dot{\mathbf{q}}_{i,t}}{\sqrt{1-\dot{\mathbf{q}}_{i,t}^2}}.$$

Furthermore, such initial data can be constructed as follows:

(1) Choose times $t_j^{(0)} < t_j^{(1)}$, an arbitrary smooth trajectory strip $\mathbf{q}_j^{(0)} : [t_j^{(0)}, t_j^{(1)}] \to \mathbb{R}^3$ with $\left\|\dot{\mathbf{q}}_{j,t}^{(0)}\right\| < 1$, one space point $(t_i^{(0)}, \mathbf{q}_{i,t_i^{(0)}}^{(0)})$ on the intersection of the forward light cone of $(t_j^{(0)}, \mathbf{q}_{j,t_j^{(0)}}^{(0)})$ with the backward light cone of $(t_j^{(1)}, \mathbf{q}_{j,t_j^{(1)}}^{(0)})$, and one space point $(t_i^{(1)}, \mathbf{q}_{i,t_i^{(1)}}^{(0)})$ somewhere on the forward light cone of $(t_j^{(1)}, \mathbf{q}_{j,t_j^{(1)}}^{(0)})$ and inside the forward light cone of $(t_i^{(0)}, \mathbf{q}_{i,t_i^{(0)}}^{(0)})$.

(2) Define $\mathbf{p}_{j,t}^{(0)} := \frac{m_j\dot{\mathbf{q}}_{j,t}^{(0)}}{\sqrt{1-\dot{\mathbf{q}}_{j,t}^2}}$ for $t \in [t_j^{(0)}, t_j^{(1)}]$ and, using (1.6), compute all derivatives of $\dot{\mathbf{p}}_{i,t}$ at the times $t = t_i^{(0)}$ and $t = t_i^{(1)}$.

(3) Choose a smooth trajectory strip $\mathbf{q}_i^{(0)} : [t_i^{(0)}, t_i^{(1)}] \to \mathbb{R}^3$ through the space-time points $(t_i^{(0)}, \mathbf{q}_{i,t_i^{(0)}}^{(0)})$ and $(t_i^{(1)}, \mathbf{q}_{i,t_i^{(1)}}^{(0)})$ such that $\left\|\dot{\mathbf{q}}_{j,t}^{(0)}\right\| < 1$ and that $t \mapsto \mathbf{p}_{i,t}^{(0)} := \frac{m_i\dot{\mathbf{q}}_{i,t}^{(0)}}{\sqrt{1-\|\dot{\mathbf{q}}_{i,t}\|^2}}$ smoothly connects to the derivatives computed in step 2.

Equipped with such initial data, we will prove our first result:

Theorem 2.1. *Given the initial data* $\left(\mathbf{q}_i^{(0)}, \mathbf{p}_i^{(0)}\right)_{i=1,2}$ *there exist two smooth maps*

$$\mathbb{R} \supseteq D_i \to \mathbb{R}^3 \times \mathbb{R}^3, \qquad t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}), \qquad i \in \{1,2\}, \tag{2.5}$$

such that:

(1) $(\mathbf{q}_{i,t}, \mathbf{p}_{i,t}) = (\mathbf{q}_{i,t}^{(0)}, \mathbf{p}_{i,t}^{(0)})$ *for all* $t \in [t_i^{(0)}, t_i^{(1)}]$ *and* $i \in \{1,2\}$.

(2) $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})$ *solves* (1.6) *on* $D_i := \left(t_i^+(T_j^{\min}, \mathbf{q}_{j,T_j^{\min}}), t_i^-(T_j^{\max}, \mathbf{q}_{j,T_j^{\max}})\right)$ *for* $i \in \{1,2\}$, $j \neq i$.

(3) *For* $i = 1, 2$, *let* $\widetilde{D}_i \subseteq \mathbb{R}$ *be an interval such that* $[t_i^{(0)}, t_i^{(1)}] \subseteq \widetilde{D}_i$ *and let* $\widetilde{D}_i \to \mathbb{R}^3 \times \mathbb{R}^3$, $t \mapsto (\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})$ *be a smooth map such that* $t \mapsto (\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})$ *solves* (1.6) *for* $t \in \widetilde{D}_i$. *Then*

$$(\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t}) = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}) \quad \forall t \in D_i \cap \widetilde{D}_i, i \in \{1,2\}$$

$$\Updownarrow$$

$$(\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t}) = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}) \quad \forall t \in [t_i^{(0)}, t_i^{(1)}], i \in \{1,2\}.$$

Given appropriate constants $d > 0$ *and* $0 \le v < 1$, *the times* $-\infty \le T_i^{\min} \le t_i^{(0)} < t_i^{(1)} \le T_i^{\max} \le \infty$ *are defined such that* $[T_i^{\min}, T_i^{\max}]$ *is the largest interval containing* $[t_i^{(0)}, t_i^{(1)}]$ *with the property*:

$$\|\dot{\mathbf{q}}_{i,t}\| \le v, \qquad \left\| \mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^{\pm}(t,\mathbf{q}_{i,t})} \right\| \ge d \qquad \forall\, t \in [T_i^{\min}, T_i^{\max}]. \tag{2.6}$$

Their value is determined during the construction of (2.5) *in the proof.*

Remark 2.2. Our focus lies on the uniqueness assertion (3) of Theorem 2.1. We do not attempt to give a priori bounds on $T_i^{\min}, T_i^{\max}$, $i \in \{1,2\}$, whose values are determined during the dynamics by condition (2.6). This condition is needed to prevent two types of singularities that can occur: first, the approach of the speed of light, and second, collision or infinitesimal approach of charges. These singularities can also be present in WF, which can be seen directly from the form of the fields (1.2). While the second one is familiar since it is of the same type as seen in the N-body problem of Newtonian gravitation [9], the first one is very specific to WF-type delay problems. Such singularities are due to the nature of the delay times t_j^+ and t_j^- defined in (1.4), which tend to plus or minus infinity if the jth charge approaches the speed of light in the future or the past, respectively; the origin of this singularity can be seen best in (2.11). However, because of angular momentum conservation, it is expected that for $N = 2$ charges one always finds $T_{\min} = -\infty$ and $T_{\max} = \infty$, which is at least true for the solutions given in Sec. 4. A treatment of the N-body problem will require the notion of typicality of solutions.

The key ingredient of the proof, which can be checked by direct computation, is the following:

Lemma 2.1. *The following statements are true*:

(1) *The map* $\mathbf{F}$ *defined in* (1.7) *is bijective and its inverse is given by*

$$\mathbf{I} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}, \qquad \mathbf{y} \mapsto \mathbf{I}(\mathbf{y}) := \frac{\mathbf{y}}{\|\mathbf{y}\|^{3/2}}.$$

(2) *Let* $\left(\mathbf{q}_i^{(0)}, \mathbf{p}_i^{(0)}\right)_{i=1,2}$ *be given initial data. For any integer* $n \ge 0$, *Eqs.* (2.3) *and* (2.4) *are equivalent to*

$$\frac{d^n}{dt^n}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^{\pm}(t,\mathbf{q}_{i,t})}\right) = \frac{d^n}{dt^n}\mathbf{I}\left(\frac{1}{e_i e_j}\dot{\mathbf{p}}_{i,t} - \mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^{\mp}(t,\mathbf{q}_{i,t})}\right)\right) \qquad \text{for } t = t_i^{(0)},$$

and

$$\frac{d^n}{dt^n}\left(\mathbf{q}_{j,t} - \mathbf{q}_{i,t_i^{\pm}(t,\mathbf{q}_{j,t})}\right) = \frac{d^n}{dt^n}\mathbf{I}\left(\frac{1}{e_j e_i}\dot{\mathbf{p}}_{j,t} - \mathbf{F}\left(\mathbf{q}_{j,t} - \mathbf{q}_{i,t_i^{\mp}(t,\mathbf{q}_{j,t})}\right)\right) \qquad \text{for } t = t_j^{(0)},$$

respectively.

Lemma 2.1 ensures that, e.g., in situations as depicted in Fig. 1, we can compute from $\mathbf{F}(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^+})$, which is determined by the initial data and (1.6), the space-time point $(t_j^+, \mathbf{q}_{j,t_j^+})$. This is the key ingredient in our construction.

Proof of Theorem 2.1. As a first step, we construct a smooth extension of $\mathbf{q}_j^{(0)}$ beyond time $t_j^{(1)}$. Let us introduce the short-hand notation

$$t_j^{\pm} = t_j^{\pm}(t, \mathbf{q}_{i,t}).$$

In general, any solution $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{i=1,2}$ to (1.6) has to fulfill

$$\dot{\mathbf{p}}_{i,t} = e_i e_j \left[\mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^+}\right) + \mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^-}\right) \right]. \tag{2.7}$$

With the help of Lemma 2.1(1) we can bring this equation into the form

$$\mathbf{q}_{j,t_j^+} = \mathbf{q}_{i,t} - \mathbf{I}\left(\frac{1}{e_i e_j} \dot{\mathbf{p}}_{i,t} - \mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^{\mp}(t,\mathbf{q}_{i,t})}\right) \right) \tag{2.8}$$

for times $t \in [T_i^{\min}, T_i^{\max}]$, because then

$$\left\| \mathbf{I}\left(\frac{1}{e_i e_j} \dot{\mathbf{p}}_{i,t} - \mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^-}\right) \right) \right\| = \left\| \mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^{\pm}} \right\| \geq d$$

is guaranteed, and hence, the right-hand side of (2.8) is well defined. We now make use of (2.8) to compute

$$\mathbf{q}_j^{(1)} : t \mapsto \mathbf{q}_{j,t}^{(1)}$$

according to

$$\begin{aligned} t_j^+ &= t + \left\| \mathbf{I}\left(\frac{1}{e_i e_j} \dot{\mathbf{p}}_{i,t}^{(0)} - \mathbf{F}\left(\mathbf{q}_{i,t}^{(0)} - \mathbf{q}^{(0)}_{j,t_j^-(t,\mathbf{q}_{i,t})}\right) \right) \right\|, & (2.9) \\ \mathbf{q}^{(1)}_{j,t_j^+} &= \mathbf{q}_{i,t}^{(0)} - \mathbf{I}\left(\frac{1}{e_i e_j} \dot{\mathbf{p}}_{i,t}^{(0)} - \mathbf{F}\left(\mathbf{q}_{i,t}^{(0)} - \mathbf{q}^{(0)}_{j,t_j^-(t,\mathbf{q}_{i,t})}\right) \right) & (2.10) \end{aligned}$$

for all $t \in (t_i^{(0)}, t_i^{(1)}] \cap (T_i^{\min}, T_i^{\max})$. Note that due to (2.2) the right-hand side of (2.9) and (2.10) is well defined. We define

$$t_j^{(2)} = \min\left\{ t_j^+|_{t=t_i^{(1)}}, T_j^{\max} \right\}.$$

Since $\mathbf{q}_i^{(0)}$ is smooth on $[t_i^{(0)}, t_i^{(1)}]$, t_j^+ also depends smoothly on $t \in [t_i^{(0)}, t_i^{(1)}]$, and in consequence, $\mathbf{q}_j^{(1)}$ is smooth on $(t_j^{(1)}, t_j^{(2)}]$. Furthermore, a direct computation gives

$$t \mapsto \frac{dt_j^+}{dt} = \frac{1 + \mathbf{n}_{j,+} \cdot \dot{\mathbf{q}}_{i,t}^{(0)}}{1 + \mathbf{n}_{j,+} \cdot \dot{\mathbf{q}}_{j,t^+}^{(1)}}, \qquad \mathbf{n}_{j,+} := \frac{\mathbf{q}_{i,t}^{(0)} - \mathbf{q}_{j,t^+}^{(1)}}{\left\| \mathbf{q}_{i,t}^{(0)} - \mathbf{q}_{j,t^+}^{(1)} \right\|}, \tag{2.11}$$

so that, using the notation

$$\frac{d}{dt_j^+} = \frac{dt}{dt_j^+} \frac{d}{dt},$$

we may then compute

$$\lim_{s \searrow t_j^{(1)}} \frac{d^n}{ds^n} \mathbf{q}_{j,s}^{(1)} = \lim_{t \searrow t_i^{(0)}} \frac{d^n}{dt_j^{+n}} \mathbf{q}^{(1)}_{j,t_j^+} = \lim_{t \searrow t_i^{(0)}} \frac{d^n}{dt_j^{+n}} \left[\mathbf{q}_{i,t}^{(0)} - \mathbf{I}\left(\frac{1}{e_i e_j} \dot{\mathbf{p}}_{i,t}^{(0)} - \mathbf{F}\left(\mathbf{q}_{i,t}^{(0)} - \mathbf{q}^{(1)}_{j,t_j^-(t,\mathbf{q}_{i,t})}\right) \right) \right] \tag{2.12}$$

for every integer $n \geq 0$. Lemma 2.1(2) ensures that

$$(2.12) = \frac{d^n}{ds^n} \mathbf{q}_{j,s}^{(0)}|_{s=t_j^{(1)}},$$

and hence,

$$t \mapsto \begin{cases} \mathbf{q}_{j,t}^{(0)} & \text{for } t \in [t_j^{(0)}, t_j^{(1)}], \\ \mathbf{q}_{j,t}^{(1)} & \text{for } t \in (t_j^{(1)}, t_j^{(2)}] \end{cases}$$

is a smooth map on $[t_j^{(0)}, t_j^{(2)})$. Furthermore,

$$\mathbf{p}_{j,t}^{(1)} := m\frac{\dot{\mathbf{q}}_{j,t}}{\sqrt{1-\dot{\mathbf{q}}_{j,t}^2}} \qquad \forall\, t \in (t_j^{(1)}, t_j^{(2)}] \tag{2.13}$$

is well defined by (2.6).

In the second step, we use the analogous construction to extend $\mathbf{q}_i^{(0)}$ smoothly beyond time $t_i^{(1)}$. We define

$$\mathbf{q}_i^{(1)} : t \mapsto \mathbf{q}_{i,t}^{(1)}$$

by

$$\begin{aligned} t_i^+ &= t + \left\| \mathbf{q}_{j,t}^{(1)} - \mathbf{I}\left(\frac{1}{e_j e_i}\dot{\mathbf{p}}_{j,t}^{(1)} - \mathbf{F}\left(\mathbf{q}_{j,t}^{(1)} - \mathbf{q}_{i,t_i^-(t,\mathbf{q}_{j,t})}^{(0)}\right)\right)\right\|, & (2.14)\\ \mathbf{q}_{i,t_i^+}^{(1)} &= \mathbf{q}_{j,t}^{(1)} - \mathbf{I}\left(\frac{1}{e_j e_i}\dot{\mathbf{p}}_{j,t}^{(1)} - \mathbf{F}\left(\mathbf{q}_{j,t}^{(1)} - \mathbf{q}_{i,t_i^-(t,\mathbf{q}_{j,t})}^{(0)}\right)\right) & (2.15) \end{aligned}$$

for all $t \in (t_j^{(1)}, t_j^{(2)}] \cap [T_j^{\min}, T_j^{\max}]$ and furthermore

$$t_i^{(2)} = \min\left\{ t_i^+|_{t=t_j^{(2)}}, T_i^{\max} \right\}.$$

As in the first step, one finds that

$$t \mapsto \begin{cases} \mathbf{q}_{i,t}^{(0)} & \text{for } t \in [t_i^{(0)}, t_i^{(1)}], \\ \mathbf{q}_{i,t}^{(1)} & \text{for } t \in (t_i^{(1)}, t_i^{(2)}] \end{cases}$$

is smooth for $t \in [t_i^{(0)}, t_i^{(2)}]$. Finally, due to (2.6), we can define

$$\mathbf{p}_{i,t}^{(1)} := m\frac{\dot{\mathbf{q}}_{i,t}}{\sqrt{1-\dot{\mathbf{q}}_{i,t}^2}} \qquad \forall\, t \in (t_i^{(1)}, t_i^{(2)}]. \tag{2.16}$$

In consequence, the maps

$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}) := \begin{cases} (\mathbf{q}_{i,t}^{(0)}, \mathbf{p}_{i,t}^{(0)}) & \text{for } t \in [t_i^{(0)}, t_i^{(1)}] \\ (\mathbf{q}_{i,t}^{(0)}, \mathbf{p}_{i,t}^{(0)}) & \text{for } t \in (t_i^{(1)}, t_i^{(2)}] \end{cases}$$

for $i = 1, 2$ are smooth, and by virtue of definitions (2.13), (2.16) and (2.10), (2.15), and (2.3), (2.4) of (2.1) they fulfill

$$\begin{pmatrix} \mathbf{q}_{i,t} \\ \mathbf{p}_{i,t} \end{pmatrix} = \begin{pmatrix} \mathbf{v}(\mathbf{p}_{i,t}) \\ e_i e_j \left[\mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^+}\right) + \mathbf{F}\left(\mathbf{q}_{i,t} - \mathbf{q}_{j,t_j^-}\right)\right] \end{pmatrix} \tag{2.17}$$

for $t \in [t_i^{(0)}, t_i^-(t_j^{(2)}, \mathbf{q}_{j,t_j^{(2)}})]$ and

$$\begin{pmatrix} \mathbf{q}_{j,t} \\ \mathbf{p}_{j,t} \end{pmatrix} = \begin{pmatrix} \mathbf{v}(\mathbf{p}_{j,t}) \\ e_j e_i \left[\mathbf{F}\left(\mathbf{q}_{j,t} - \mathbf{q}_{i,t_i^+}\right) + \mathbf{F}\left(\mathbf{q}_{j,t} - \mathbf{q}_{i,t_i^-}\right)\right] \end{pmatrix} \tag{2.18}$$

for $t \in [t_j^{(1)}, t_j^-(t_i^{(2)}, \mathbf{q}_{i,t_i^{(2)}})]$.

This construction can be repeated where in the kth step one constructs the extension

$$(t_i^{(k)}, t_i^{(k+1)}] : t \mapsto (\mathbf{q}_{i,t}^{(k)}, \mathbf{p}_{i,t}^{(k)})_{i=1,2}.$$

For each step, one finds that

$$t_i^{(k+1)} \geq \min\left\{ t^{(k)} + d, T_i^{\max} \right\}, \qquad i \in \{1, 2\}.$$

In consequence, only finite repetitions of this construction are needed to compute

$$[t_i^{(0)}, T_i^{\max}] : t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}) := \begin{cases} (\mathbf{q}_{i,t}^{(0)}, \mathbf{p}_{i,t}^{(0)}) & \text{for } t \in [t_i^{(0)}, t_i^{(1)}], \\ (\mathbf{q}_{i,t}^{(k)}, \mathbf{p}_{i,t}^{(k)}) & \text{for } t \in (t_i^{(k)}, t_i^{(k+1)}], \end{cases} \tag{2.19}$$

which fulfills (2.17) for $t \in [t_i^{(0)}, t_i^-(T_j^{\max}, \mathbf{q}_{j,T_j^{\max}})]$ and (2.18) for $t \in [t_j^{(1)}, t_j^-(T_i^{\max}, \mathbf{q}_{i,T_i^{\max}})]$. The same construction can be carried out into the past, which results in smooth maps

$$[T_i^{\min}, T_i^{\max}] \to \mathbb{R}^3 \times \mathbb{R}^3, \qquad t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}), \qquad i \in \{1, 2\}.$$

From this construction we infer the claims of Theorem 2.1: Claim (1) follows from definition (2.19). Furthermore, due to Lemma 2.1, (2.9)-(2.10), and (2.14)-(2.15), the map $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})$ fulfills (1.6) for times

$$t \in [t_i^+(T_j^{\min}, \mathbf{q}_{j,T_j^{\min}}), t_i^-(T_j^{\max}, \mathbf{q}_{j,T_j^{\max}})]$$

for $i \in \{1, 2\}$ and $j \neq i$ and therefore claim (2). Finally, Lemma 2.1 guarantees that this constructed solution is unique, which proves claim (3) and concludes the proof. $\square$

As a byproduct, we observe that specification of initial positions and momenta of the charges as suggested by the work [6] does not always ensure uniqueness:

Corollary 2.1. *Let* $\left(\mathbf{q}_i^{(0)}, \mathbf{p}_i^{(0)}\right)_{i=1,2}$ *be the initial data such that there is a* $t^* \in (t_i^{(0)}, t_i^{(1)}) \cap (t_j^{(0)}, t_j^{(1)})$, *let* $D_i \to \mathbb{R}^3 \times \mathbb{R}^3, t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})$ *for* $i \in \{1, 2\}$ *be the corresponding solution to* (1.6), *and let* $\mathbf{q}_i^{(0)}, \mathbf{p}_i^{(0)} \in \mathbb{R}^3$ *for* $i = 1, 2$ *be defined as*

$$(\mathbf{q}_i^{(0)}, \mathbf{p}_i^{(0)}) := (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})\big|_{t=t^*}, \qquad i \in \{1, 2\}.$$

There are uncountably many other solutions $\widetilde{D}_i \to \mathbb{R}^3 \times \mathbb{R}^3, t \mapsto (\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})$ *for* $i = 1, 2$ *to* (1.6) *which fulfill*

$$(\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})\big|_{t=t^*} = (\mathbf{q}_i^{(0)}, \mathbf{p}_i^{(0)}), \qquad i \in \{1, 2\}, \tag{2.20}$$

but not

$$(\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t}) = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}) \qquad \forall\, t \in D_i \cap \widetilde{D}_i, \quad i \in \{1, 2\}. \tag{2.21}$$

Proof. Choose $s \in \mathbb{R}$ and $\delta > 0$ such that $(s - \delta, s + \delta) \subset (t_i^{(0)}, t_i^{(1)}) \cap (t_j^{(0)}, t_j^{(1)})$ and $t^* \notin (s - \delta, s + \delta)$. Furthermore, for $\lambda > 0$ and $i = 1, 2$ let $\mathbf{d}_{\lambda,i} : \mathbb{R} \to \mathbb{R}^3$, $t \mapsto \mathbf{d}_{\lambda,i}(t)$ be a smooth function such that

$$\operatorname{supp} \mathbf{d}_{\lambda,i} = [s - \delta, s + \delta], \qquad \sup_{t \in \mathbb{R}} |\dot{\mathbf{d}}_{\lambda,i}(t)| = \lambda.$$

We define

$$t \mapsto \widetilde{\mathbf{q}}_{i,t}^{(0)} := \mathbf{q}_{i,t}^{(0)} + \mathbf{d}_{\lambda,i}(t) \qquad \forall\, t \in [t_i^{(0)}, t_i^{(1)}],$$

choose the parameter $\lambda > 0$ such that $\left\|\dot{\widetilde{\mathbf{q}}}_{i,t}\right\| < 1$, and define

$$t \mapsto \widetilde{\mathbf{p}}_{i,t}^{(0)} := \frac{m_i \dot{\widetilde{\mathbf{q}}}_{i,t}}{\sqrt{1 - \left\|\dot{\widetilde{\mathbf{q}}}_{i,t}\right\|^2}} \qquad \forall\, t \in [t_i^{(0)}, t_i^{(1)}].$$

The maps $(\widetilde{\mathbf{q}}_i^{(0)}, \widetilde{\mathbf{p}}_i^{(0)})_{i=1,2}$ are initial data according to Definition 2.1, which fulfill (2.20). However, according to Theorem 2.1, the solution $\widetilde{D}_i \to \left(\mathbb{R}^3 \times \mathbb{R}^3\right), t \mapsto (\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})$ for $i = 1, 2$ to (1.6) corresponding to $(\widetilde{\mathbf{q}}_i^{(0)}, \widetilde{\mathbf{p}}_i^{(0)})_{i=1,2}$ does not fulfill (2.21). Note that there are uncountably many choices, e.g., in s, δ, λ, and $\mathbf{d}_{\lambda,i}$, to define other $(\widetilde{\mathbf{q}}_i^{(0)}, \widetilde{\mathbf{p}}_i^{(0)})_{i=1,2}$ such that (2.20) holds. None of the corresponding solutions, however, fulfill (2.21). $\square$

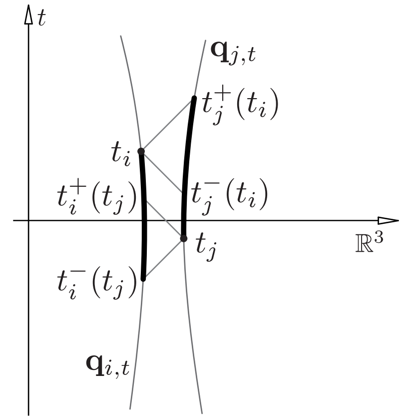

Fig. 3. Given t_i, t_j the thick lines denote the data required to define the energy of the system.

3. Constants of Motion

In the following, we define an energy functional for the approximate model, from which the general structure of constants of motion will become apparent. Throughout this section we consider a solution

$$\mathbb{R} \supseteq D_i \to \mathbb{R}^3 \times \mathbb{R}^3, \qquad t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}), \qquad i \in \{1,2\},$$

to (1.6); see Theorem 2.1.

Definition 3.1. We define a map $H : D_1 \times D_2 \to \mathbb{R}^+$ by

$$\begin{aligned} H(t_1,t_2) :&= \sum_{i=1}^{2} \sqrt{\mathbf{p}_{i,t_i}^2 + m_i^2} + \frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \frac{1}{\left\| \mathbf{q}_{i,t_i} - \mathbf{q}_{j,t_j^{\pm}(t_i)} \right\|} \\ &\quad + \frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_{t_i}^{t_i^{\pm}(t_j)} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{i,s}, \end{aligned}$$

where for the delay functions we have used the short-hand notation

$$t_i^{\pm}(t) \equiv t_i^{\pm}(t, \mathbf{q}_{j,t}), \qquad i \in \{1,2\}, \quad j \neq i.$$

We refer to H as the *energy functional of the system*. See Fig. 3 for an example of which data are needed to define this functional.

Theorem 3.1. *For all $t_1 \in D_1$ and $t_2 \in D_2$, the equality*

$$H(t_1,t_2) = H(0,0)$$

holds true.

Proof. Since we assume a smooth solution $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{i=1,2}$ to (1.6), it is straightforward to verify the claim directly by differentiating with respect to t_1 and t_2. However, here we want to provide a general idea how to find constants of motion for a WF-type delay differential equation, and therefore give a more instructive proof.

We start with the sum of the kinetic energy difference between times 0 and t_i of the two particles $i = 1, 2$, that is,

$$\sum_{i=1}^{2} \int_0^{t_i} ds\, \dot{\mathbf{p}}_{i,s} \cdot \mathbf{v}(\mathbf{p}_{i,s}), \tag{3.1}$$

and make use of Eqs. (1.6) to express this entity in terms of

$$(3.1) = \sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_0^{t_i} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{i,s}. \tag{3.2}$$

It is convenient to split (3.2) into the following summands:

$$\begin{aligned}
(3.2) \quad = \quad & \frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_0^{t_i} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{i,s} && (3.3)\\
& +\frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_0^{t_i} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \left(\dot{\mathbf{q}}_{i,s} - \dot{\mathbf{q}}_{j,t_j^{\pm}(s)} \frac{dt_j^{\pm}(s)}{ds}\right) && (3.4)\\
& +\frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_0^{t_i} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{j,t_j^{\pm}(s)} \frac{dt_j^{\pm}(s)}{ds}. && (3.5)
\end{aligned}$$

The integrand in (3.4) is an exact differential so that

$$(3.4) = -\frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \frac{1}{\left\|\mathbf{q}_{i,t_i} - \mathbf{q}_{j,t_j^{\pm}(t_i)}\right\|} + C, \tag{3.6}$$

where $C \in \mathbb{R}$ is a constant. Next, we exploit the symmetries of the force field and the delay function

$$\mathbf{F}(\mathbf{x}) = -\mathbf{F}(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \mathbb{R}^3 \setminus \{0\}, \tag{3.7}$$

$$t = t_i^{\mp}\left(t_j^{\pm}(t)\right), \qquad i \in \{1,2\}, \quad j \neq i. \tag{3.8}$$

Now (3.8) allows to rewrite (3.5) by substitution of the integration variable according to

$$(3.5) = \frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_{t_j^{\pm}(0)}^{t_j^{\pm}(t_i)} ds\, \mathbf{F}(\mathbf{q}_{i,t_i^{\mp}(s)} - \mathbf{q}_{j,s}) \cdot \dot{\mathbf{q}}_{j,s}. \tag{3.9}$$

Furthermore, we apply (3.7) and after that relabel the indices $i \leftrightarrows j$ and $\pm \leftrightarrows \mp$ to get

$$\begin{aligned}
(3.9) \quad =- \quad & \frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_{t_j^{\pm}(0)}^{t_j^{\pm}(t_i)} ds\, \mathbf{F}(\mathbf{q}_{j,s} - \mathbf{q}_{i,t_i^{\mp}(s)}) \cdot \dot{\mathbf{q}}_{j,s} \\
=- \quad & \frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_{t_i^{\mp}(0)}^{t_i^{\mp}(t_j)} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{i,s} \\
=- \quad & \frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_{t_i^{\pm}(0)}^{0} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{i,s} && (3.10)\\
& -\frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int_0^{t_i} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{i,s} && (3.11)
\end{aligned}$$

$$-\frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int\limits_{t_i}^{t_i^{\mp}(t_j)} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{i,s}. \tag{3.12}$$

Noting that term (3.10) is just another constant and term (3.11) cancels the first term on the right-hand side of (3.3), we write the kinetic energy as

$$(3.1) = -\frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \frac{1}{\left\| \mathbf{q}_{i,t_i} - \mathbf{q}_{j,t_j^{\pm}(t_i)} \right\|} - \frac{1}{2}\sum_{i=1}^{2} e_i \sum_{j\neq i} e_j \sum_{\pm} \int\limits_{t_i}^{t_i^{\mp}(t_j)} ds\, \mathbf{F}(\mathbf{q}_{i,s} - \mathbf{q}_{j,t_j^{\pm}(s)}) \cdot \dot{\mathbf{q}}_{i,s} + C,$$

which proves the claim. □

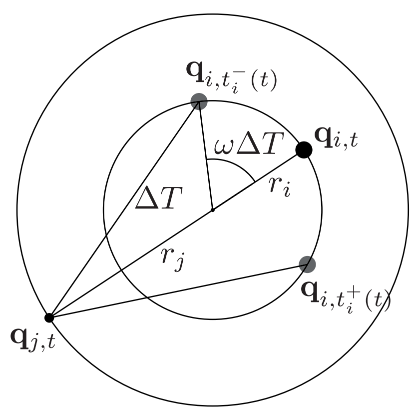

Fig. 4. Stable orbit of two charges revolving around the same center with radii r_i, r_j. ΔT denotes the modulus of the delay.

4. Stable Orbits

By the symmetry of the advanced and retarded delay, it is straightforward to construct stable orbits (compare [8]):

Definition 4.1. For masses $m_1, m_2 > 0$, charges $e_1, e_2 \in \mathbb{R}$, radii $r_1, r_2 > 0$, and angular velocity $\omega \in \mathbb{R} \setminus \{0\}$ such that

$$e_1 e_2 < 0, \qquad \frac{m_1}{m_2} = \frac{\gamma(\omega r_2)}{\gamma(\omega r_1)} \frac{r_2}{r_1}, \qquad 0 < \omega \Delta T < \frac{\pi}{2}, \qquad \Delta T := \sqrt{\frac{r_1^2 + r_2^2}{1 + \frac{r_1 r_2}{e_1 e_2} m_1 \gamma(\omega r_1) r_1 \omega^2}},$$

where

$$\gamma(v) := \frac{1}{\sqrt{1 - v^2}},$$

we define the particle trajectories

$$t \mapsto \mathbf{q}_{1,t} := r_1 \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}, \qquad t \mapsto \mathbf{q}_{2,t} := -r_2 \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}, \tag{4.1}$$

which we call *Schild solutions*; see Fig. 4.

Theorem 4.1. *Given* $m_1, m_2, r_1 > 0$, $e_1 \in \mathbb{R}$, *there are* $r_2 > 0$ *and* $\omega \in \mathbb{R}$ *such that the Schild solutions* (4.1) *obey* (1.6).

Proof. We start by rewriting (1.6) as a second-order equation, i.e.,

$$\frac{d}{dt}\left(m_i\gamma(\dot{\mathbf{q}}_{i,t})\dot{\mathbf{q}}_{i,t}\right) = e_ie_j\left[\mathbf{F}\left(\mathbf{q}_{i,t}-\mathbf{q}_{j,t_j^+(t,\mathbf{q}_{i,t})}\right)+\mathbf{F}\left(\mathbf{q}_{i,t}-\mathbf{q}_{j,t_j^-(t,\mathbf{q}_{i,t})}\right)\right], \qquad i\in\{1,2\}, \quad i\neq j.$$

If the charges stay on the circular orbits (4.1), then velocities and accelerations are constant and fulfill

$$\|\dot{\mathbf{q}}_{i,t}\| = \omega r_i, \qquad \|\ddot{\mathbf{q}}_{i,t}\| = \omega^2 r_i.$$

Furthermore, the kinematics dictate that the net force acting upon the particles must be centripetal w.r.t. the origin and equal, i.e.,

$$\left\|\frac{d}{dt}\left(m_1\gamma(\dot{\mathbf{q}}_{1,t})\dot{\mathbf{q}}_{1,t}\right)\right\| = m_1\gamma(\omega r_1)\omega^2 r_1 = m_2\gamma(\omega r_2)\omega^2 r_2,$$

which implies

$$\frac{m_1}{m_2} = \frac{\gamma(\omega r_2)}{\gamma(\omega r_1)}\frac{r_2}{r_1}. \tag{4.2}$$

Next, we compute the delay function $\Delta T > 0$, which, according to (1.4), was defined by

$$\Delta T^2 := \left(t_j^{\pm}(t)-t\right)^2 = \left\|\mathbf{q}_{i,t}-\mathbf{q}_{j,t_j^{\pm}(t)}\right\|^2 = r_1^2+r_2^2+2r_1r_2\cos\left(\omega\Delta T\right). \tag{4.3}$$

Due to the symmetry in the advanced and retarded delay (see Fig. 4), the net force is centripetal w.r.t. the origin for

$$e_1e_2 < 0$$

and its modulus equals

$$\left\|e_ie_j\left[\mathbf{F}\left(\mathbf{q}_{i,t}-\mathbf{q}_{j,t_j^+(t,\mathbf{q}_{i,t})}\right)+\mathbf{F}\left(\mathbf{q}_{i,t}-\mathbf{q}_{j,t_j^-(t,\mathbf{q}_{i,t})}\right)\right]\right\| = \left|e_1e_2\frac{2\cos\left(\omega\Delta T\right)}{\Delta T^2}\right|.$$

This means in particular that

$$m_1\gamma(\omega r_1)\omega^2 r_1 = \left|e_1e_2\frac{2\cos\left(\omega\Delta T\right)}{\Delta T^2}\right|$$

which by (1.4) allows us to solve for ΔT, according to

$$\Delta T^2 = \frac{r_1^2+r_2^2}{1+\frac{r_1r_2}{e_1e_2}m_1\gamma(\omega r_1)r_1\omega^2} \tag{4.4}$$

for certain values of r_2 and ω.

We can choose $e_2\in\mathbb{R}$ such that $e_1e_2<0$, $r_2>0$ such that (4.2) holds true, and $\omega\in\mathbb{R}\setminus\{0\}$ with $|\omega|$ sufficiently small such that (4.4) is well-defined and $0<\omega\Delta T<\frac{\pi}{2}$. This proves the claim. □

5. What Can We Learn from the Approximate Model?

We emphasize that the presented construction of solutions relies sensitively on the simplicity of the force field $\mathbf{F}$ in (1.7), which was chosen such that it has a global inverse $\mathbf{I}$ as defined in Lemma 2.1. Any generalization of this force field that does not have a global inverse, in particular the one of WF, will require a new technique. However, the approximate model uses the same delay function (1.4) that is also used in WF. Therefore, one can expect that many mathematical structures appearing in the approximate model will also arise in WF. As examples we point out that the data needed to define the energy functional discussed in Sec. 3 is exactly the same as the one needed to define the corresponding energy functional in WF (compare Fig. 3 in [11]), and the stable orbits actually coincide with the ones in WF (compare [8]). Given these similarities, it seems reasonable to expect that also in WF the natural choice of initial data that uniquely identifies solutions will be of the same type as in the approximate model given in Definition 2.1, i.e., Fig. 2. In this respect it is comforting to note that the considered initial data are already sufficient to compute the energy functional. The only additional information in the initial data considered here, i.e., (2.3) and (2.4), is about how smooth the solutions are. Smoothness will become a more delicate issue when considering the full WF interaction as (1.2) also depends on the acceleration.

Concerning the dynamics of many charges, the question of typical behavior becomes relevant and a generalization to many particles of the approximate model is a good candidate for studying measures of typicality for delay dynamics of this kind. Note that while the generalization of the above uniqueness result requires a slightly more sophisticated proof, the results of Sec. 3 and Sec. 4 have straightforward generalizations to many charges.

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