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ORIGINAL PAPER

Solving Partition Problems with Colour-Bipartitions

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Abstract Polar, monopolar, and unipolar graphs are defined in terms of the existence of certain vertex partitions. Although it is polynomial to determine whether a graph is unipolar and to find whenever possible a unipolar partition, the problems of recognizing polar and monopolar graphs are both NP-complete in general. These problems have recently been studied for chordal, claw-free, and permutation graphs. Polynomial time algorithms have been found for solving the problems for these classes of graphs, with one exception: polarity recognition remains NP-complete in claw-free graphs. In this paper, we connect these problems to edge-coloured homomorphism problems. We show that finding unipolar partitions in general and finding monopolar partitions for certain classes of graphs can be efficiently reduced to a polynomial-time solvable 2-edge-coloured homomorphism problem, which we call the *colour-bipartition problem*. This approach unifies the currently known results on monopolarity and extends them to new classes of graphs.

Keywords Monopolar graphs · Unipolar graphs · Edge-coloured graphs · Partition problems · Polynomial algorithm

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1 Introduction

We mostly follow the standard terminology and notation from [22]; in particular, we use P_k and C_k to denote the path and cycle on k vertices, respectively. A *2-edge-coloured graph* G is a vertex set $V(G)$ along with symmetric binary relations $E_b(G)$, $E_r(G)$ on $V(G)$ which are viewed as blue and red edge sets, respectively. The unqualified term *graph* is reserved for the corresponding structure with only one (uncoloured) vertex set. We permit graphs and 2-edge-coloured graphs to have loops unless otherwise stated. A 2-edge-coloured graph may have parallel edges provided they are of different colours; that is, a pair of vertices may be joined by both a blue edge and a red edge.

A *colour-bipartition* of a 2-edge-coloured graph G is a partition (S, T) of $V(G)$ such that the subgraph induced by S contains no blue edge and the subgraph induced by T contains no red edge. In other words, S and T induce blue and red independent sets, respectively. If G admits such a partition, it is called *colour-bipartite*. It is easily shown that a colour-bipartition of G is equivalent to a homomorphism from G to the 2-edge-coloured graph with vertex set $\{s, t\}$, blue edges $\{st, tt\}$, and red edges $\{st, ss\}$. Brewster points out in [2] that this homomorphism problem reduces to 2-SAT and is hence solvable in polynomial time. In this paper, we present a simple $O(nm)$ solution which gives additional information about the possible colour-bipartitions of the input. Although this algorithm is not radically different from existing techniques (cf. [16]), we use it to find novel solutions to several graph partition problems.

A loopless graph G is called *polar* if its vertex set can be partitioned into A and B which induce a complete multipartite graph and a disjoint union of cliques, respectively. When A is restricted to be an independent set, G is called *monopolar*; when A is restricted to be a clique, G is called *unipolar*. The corresponding partitions are called *polar*, *monopolar*, and *unipolar partitions*. The classes of polar, monopolar, and unipolar graphs enjoy various interesting properties; in particular, all three are closed under taking induced subgraphs.

Polar graphs were introduced in [21] as a generalization of bipartite and split graphs. (*Split graphs* are the graphs which can be partitioned into an independent set and a clique.) Despite this, the recognition of polar [3] and monopolar [14] graphs is NP-complete. The recognition of unipolar graphs, on the other hand, is polynomial: an $O(n^3)$ recognition algorithm for unipolar graphs is found in [20].

Recent research has focused on the polar and monopolar partition problems in restricted classes of graphs. Ekim et al. [12] showed that there are only four minimal non-polar cographs and hence that polar cographs are recognizable in polynomial time. Efficient algorithms for recognizing polar and monopolar chordal graphs are given in [10] despite the fact that there are infinitely many minimal non-monopolar chordal graphs (catalogued in [19]). Both recognition problems have also been solved in polynomial time for permutation graphs [9].

Many of the ideas in this paper stem from the recent work on polar line graphs and claw-free graphs. In [17], Huang and Xu characterized (in terms of forbidden subgraphs) the bipartite graphs whose line graphs are polar. A linear recognition algorithm for such graphs was found by Ekim and Huang in [11]. These led the current authors to the study of polarity and monopolarity of general line graphs, claw-free graphs, and triangle-free graphs. We showed that both the polarity and monopolarity of line graphs can be recognized in linear time [6]. In contrast, for claw-free graphs, while the monopolarity can be recognized in polynomial time, recognizing the polarity remains an NP-complete problem [5]. For triangle-free graphs, recognizing either polarity or monopolarity is NP-complete [5].

Where no confusion arises, we sometimes refer interchangeably to a vertex set S and the subgraph it induces. If G, H are graphs, we say that G is *H-free* if it does not contain H as an induced subgraph. Complete multipartite graphs are precisely the $\overline{P_3}$-free graphs, and disjoint unions of cliques are the P_3-free graphs. This remark is helpful when arguing that a partition (A, B) is a polar, monopolar, or unipolar partition: to show that B induces a disjoint union of cliques it suffices to show that B is P_3-free.

In Sect. 2, we present the essential terminology and results on colour-bipartitions. This sets the general framework from which the rest of the paper follows. In Sect. 3 we apply this framework to the unipolar partition problem for general graphs and in Sect. 4 to the monopolar partition problem restricted to a large graph class containing all claw-free graphs and all graphs with no induced cycles of length ≥ 5. Finally, in Sect. 5 we show how to use these techniques to determine the monopolarity of a graph with a known polar partition.

2 General Framework: Finding Colour-Bipartitions

Let G be a 2-edge-coloured graph. A blue-first alternating walk is a sequence of vertices $Q : v_0v_1 \dots v_k$ such that v_i, v_{i+1} are connected by a blue edge for even i and a red edge for odd i. Red-first alternating walks are defined similarly with the edge colours swapped. As the following proposition shows, alternating walks give us useful information about possible colour-bipartitions of G.

Proposition 1 *Suppose $Q : v_0v_1v_2 \dots v_k$ is a blue-first (respectively, red-first) alternating walk in G and (S, T) is a colour-bipartition of G with $v_0 \in S$ (respectively, $v_0 \in T$). Then $v_i \in S$ for each even (respectively, odd) index i and $v_i \in T$ otherwise.*

Proof Suppose $v_i \in S$ and i is even. Since Q is a blue-first alternating walk, v_iv_{i+1} is a blue edge and S cannot contain both v_i, v_{i+1}. Hence $v_{i+1} \in T$. A similar argument shows that $v_{i+1} \in S$ if $v_i \in T$ and i is odd. The result holds by induction on i. □

Proposition 1 suggests the following definitions. If $v_0v_1 \dots v_k$ is a blue-first alternating walk and k is odd, then v_0 and v_k are said to be blue-incompatible; such vertices cannot both be contained in S. In particular, a vertex which is blue-incompatible with itself must be contained in T. A vertex set which contains no blue-incompatible pair is blue-compatible. Corresponding definitions can be made for red-incompatibility using red-first alternating walks.

The following theorem characterizes colour-bipartite graphs using blue- and red-incompatibility. The first statement can be extracted from the homomorphism duality theorem in [1], which uses different terminology. We prove it as a consequence of the second statement, which is more important for our purposes.

Theorem 2 *A 2-edge-coloured graph G is colour-bipartite if and only if no vertex is both blue-incompatible and red-incompatible with itself. Moreover, if S and T are (respectively) maximal blue- and red-compatible sets in a colour-bipartite G, then $(S, V(G) \setminus S)$ and $(V(G) \setminus T, T)$ are colour-bipartitions of G.*

Proof The necessity has already been shown by Proposition 1. So suppose G has no vertex which is both blue-incompatible and red-incompatible with itself. Let S be a maximal blue-compatible set in G; we show that $(S, V(G) \setminus S)$ is a colour-bipartition of G. Observe that S is a blue independent set. Its maximality implies that each vertex $u \in V(G) \setminus S$ is blue-incompatible with itself or some vertex in S. Let $u, v \in V(G) \setminus S$ and let $uu_1u_2 \ldots u_j$ and $vv_1v_2 \ldots v_k$ be blue-first alternating walks with j, k odd, $u_j \in S \cup \{u\}$, and $v_k \in S \cup \{v\}$.

Suppose uv is a red edge. Then $u_j \ldots u_2u_1uvv_1v_2 \ldots v_k$ is a blue-first alternating walk of odd length. Since S is blue-compatible, it contains at most one of u_j, v_k. Assume without loss of generality that $v_k \notin S$ and hence $v_k = v$. Then the blue-first alternating walk $u_j \ldots u_1uvv_1v_2 \ldots v_{k-1}vuu_1 \ldots u_j$ of odd length certifies that u_j is blue-incompatible with itself. Thus S does not contain u_j, and $u_j = u$. But then the red-first alternating walk $uvv_1v_2 \ldots v_{k-1}vu$ of odd length shows that u is also red-incompatible with itself. This contradicts the assumption that no vertex is both blue- and red-incompatible with itself. Hence $V(G) \setminus S$ is a red independent set and $(S, V(G) \setminus S)$ is a colour-bipartition as desired. A similar argument shows that $(V(G) \setminus T, T)$ is a colour-bipartition if T is a maximal red-compatible set. □

If G is a 2-edge-coloured graph, denote by G^{inc} the incompatibility graph of G, obtained by adding a blue (respectively, red) edge uv wherever u and v are blue-incompatible (respectively, red-incompatible). Using Theorem 2, we can learn a lot about the colour-bipartitions of G by looking at G^{inc}. For example:

Proposition 3 *G is colour-bipartite if and only if G^{inc} has no vertex with a blue and red loop. Every colour-bipartition of G is a colour-bipartition of G^{inc}, and vice versa. If S and T are (respectively) maximal blue and red independent sets in a colour-bipartite G^{inc}, then $(S, V(G) \setminus S)$ and $(V(G) \setminus T, T)$ are colour-bipartitions of G.* □

In the following sections, we reduce various graph partition problems to colour-bipartition problems. In many cases, knowing G^{inc} can simplify the reduction or provide additional information about the possible desired graph partitions. As it turns out, G^{inc} can be constructed from an input G in $O(nm)$ time. For each vertex u, a straightforward modification of breadth-first search (Algorithm 1) traverses the blue-first alternating walks beginning with u to discover the vertices which are blue-incompatible with u. Applying the same algorithm with the roles of red and blue reversed yields the vertices which are red-incompatible with u. Once all the incompatible pairs in G are found, they are added as edges of the appropriate colours.

Proposition 4 *Algorithm 1 is correct and runs in $O(m)$ time.*

Proof Algorithm 1 considers each blue edge and each red edge at most once, so it runs in $O(m)$ time. Suppose that it misses a vertex; let $uu_1u_2 \ldots u_k$ be a shortest

Algorithm 1 Blue-First Breadth-First Search

Require: A 2-edge-coloured graph G; a root vertex u.
Ensure: The set B_u of vertices which are blue-incompatible with u
initialize $B_u = \emptyset$ and the queue $\mathcal{Q} := \{(u, \texttt{blue})\}$
while $\mathcal{Q}$ is nonempty **do**
  remove the head $(x, \texttt{col})$ from $\mathcal{Q}$
  for all unmarked edges xy of colour $\texttt{col}$ **do**
    if $\texttt{col} = \texttt{blue}$ **then**
      mark xy as visited, add y to B_u, and append $(y, \texttt{red})$ to $\mathcal{Q}$
    else if $\texttt{col} = \texttt{red}$ **then**
      mark xy as visited and append $(y, \texttt{blue})$ to $\mathcal{Q}$
return B_u

blue-first alternating walk of odd length such that u_k is not added to B_u. In particular, the algorithm never enqueued $(u_{k-1}, \texttt{blue})$ nor any $(x, \texttt{red})$ such that xu_{k-1} is a red edge. But since the walk was chosen to be shortest, either $u_{k-2} \in B_u$ or $u_{k-2} = u$; in either case, the algorithm should have enqueued $(u_{k-2}, \texttt{red})$. Hence Algorithm 1 does not miss any vertices. On the other hand, it clearly adds a vertex v to B_u only if it has traversed a blue-first alternating walk of odd length from u to v (i.e. if u and v are blue-incompatible). Therefore, Algorithm 1 is correct. □

Constructing G^{inc} is simply a matter of running Algorithm 1 on each vertex u and modifying G according to the results. Since Algorithm 1 takes $O(m)$ time per application, this process takes $O(nm)$ time in total.

Theorem 5 *The incompatibility graph G^{inc} of a given 2-edge-coloured G can be found in $O(nm)$ time.*

3 Finding Unipolar Partitions

We now set out to apply colour-bipartitions to several graph partition problems. Our first target is the so-called *unipolar partition problem*, which asks whether the vertices of a given graph G can be partitioned into A and B inducing a clique and a disjoint union of cliques, respectively. Unipolar graphs have already been shown to be recognizable in $O(n^3)$ time [20]. In this section, we present an alternate $O(n^2m)$ algorithm using colour-bipartitions.

Given an input graph G, construct a 2-edge-coloured graph H from a blue copy of the complement $\overline{G}$ as follows. Add a blue loop on every vertex with three non-neighbours inducing a P_3 in G, and add a red edge xy whenever at least one of the following is true: G contains an induced $P_4 : wxyz$ or an induced $C_4 : xwyz$. (These cases are illustrated in Fig. 1.) Finally, add a red edge xy whenever G contains an induced $P_3 : xwy$ such that w has a blue loop in H.

Theorem 6 *G is unipolar if and only if H is colour-bipartite. If S is a maximal blue-compatible set in H, then the colour-bipartition $(S, V(G) \setminus S)$ is a unipolar partition of G.*

Proof Suppose (A, B) is a unipolar partition of G. Then A is a clique and B is P_3-free (as it is a disjoint union of cliques). Clearly, then, A is a blue independent set in H.

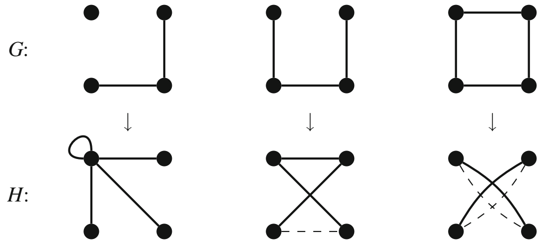

Fig. 1 Examples of the construction of H. *Solid* and *dashed lines* in H are *blue* and *red*, respectively

Suppose xy is a red edge of H. Then either G contains an induced $P_3 : xwy$ such that w has a blue loop in H or G contains two nonadjacent vertices w, z such that each of the sets $\{w, x, y\}$ and $\{x, y, z\}$ induce a P_3. Since A is a blue independent set, B must contain w in the first case and at least one of w, z in the second case. Either way, one of x, y is in A since B is P_3-free. It follows that B is a red independent set and (A, B) is a colour-bipartition of H.

Conversely, suppose H is colour-bipartite and let S be a maximal blue-compatible set in H. By definition S induces a clique in G. Suppose that $w, x, y \in V(G) \setminus S$ induce the $P_3 : wxy$ in G. According to Theorem 2, $V(G) \setminus S$ is a red independent set; hence x does not have a blue loop because wy is not a red edge in H. This fact, along with the maximality of S, implies that there exists some $z \in S$ which is nonadjacent to x in G. If z is nonadjacent to both w, y in G, then it has a blue loop in H, contradicting the blue-compatibility of S. If z is adjacent to both w, y in G, then $wxyz$ is an induced C_4 in G, contradicting the assumption that $V(G) \setminus S$ is a red independent set in H. Finally, if z is adjacent to exactly one of w, y, then either $wxyz$ or $zwxy$ is an induced P_4, again contradicting the assumption that $V(G) \setminus S$ is a red independent set in H. All cases lead to a contradiction, so $V(G) \setminus S$ contains no such induced P_3. Consequently, $(S, V(G) \setminus S)$ is a unipolar partition. □

Corollary 7 *There exists an $O(n^2m)$ algorithm to determine whether a given graph is unipolar, and to find a unipolar partition if one exists.*

Proof Given a graph G with n vertices and m edges, we can find every C_4, P_4, P_3, and $P_3 \cup K_1$ in G in time $O(n^2m)$. This can be done by examining every edge and every pair of vertices of G. Hence the 2-edge-coloured graph H (as defined above) can be constructed in time $O(n^2m)$. Since H has $O(n^2)$ edges, its colour-bipartiteness can be decided in $O(n^3)$ time by computing H^{inc} as in Theorem 5. The overall complexity is thus $O(n^2m)$. □

4 Finding Monopolar Partitions in Certain Classes of Graphs

In contrast to the unipolar partition problem, it is difficult to find monopolar partitions in general [14]. Recall that a graph is *monopolar* if its vertices can be partitioned into an independent set A and a disjoint union of cliques B. Much recent work has studied monopolarity in restricted graph classes, including cographs [12], chordal graphs [10,19], line graphs [6,11,17], claw-free graphs [5,7,8], and permutation graphs [9]. Polynomial time algorithms have been found to solve the monopolar partition problem in each of the aforementioned classes. In this section, we unify these results using colour-bipartitions.

The inspiration for this section comes from the characterization of monopolar claw-free graphs given in [5]. It is equivalent to the following. Given an input claw-free graph G, let H be the 2-edge-coloured graph obtained from a blue copy of G by adding red edges xy whenever G contains an induced paw consisting of a triangle ywz and a pendant edge xy, or G contains an induced kite consisting of the two triangles xwz and wzy. The names "paw" and "kite" are taken from [22].

Lemma 8 ([5]) *A claw-free graph G is monopolar if and only if no induced P_3 in G consists of vertices each blue-incompatible with itself in H. If S is a maximal blue-compatible set in H, then $(S, V(G) \setminus S)$ is a monopolar partition of G if one exists.*

An $O(n^3)$ recognition algorithm follows, using a breadth-first search algorithm similar to the one discussed in Sect. 2 (see also [5]). We can obtain a characterization which is more in the spirit of this paper by adding a few red edges to H. Let H' be obtained from H by adding a red edge xy whenever G contains an induced $P_3 : wxy$ or xwy such that w is blue-incompatible with itself in H. Then the above characterization becomes

Theorem 9 *A claw-free graph G is monopolar if and only if H' is colour-bipartite. If S is a maximal blue-compatible set in H', then the colour-bipartition $(S, V(G) \setminus S)$ is a monopolar partition of G.* □

This suggests a method to characterize monopolarity in a larger class of graphs. Given an input graph G, let H_0 be the 2-edge-coloured graph obtained from a blue copy of G by adding red edges xy whenever at least one of the following is true: G contains an induced paw consisting of a triangle ywz and a pendant edge xy, or G contains an induced kite consisting of the two triangles xwz and wzy, or G contains an induced $C_4 : wxyz$. For $i > 0$, inductively construct H_i from H_{i-1} by adding a red edge xy wherever G contains an induced $P_3 : wxy$ or xwy such that w is blue-incompatible with itself in H_{i-1}. Finally, let $H^* = (H_n)^{inc}$. Note that if $\{w, x, y\}$ induce a P_3 in G and w is blue-incompatible with itself in H, then xy is a red edge and w has a blue loop in H^*.

It turns out that this construction is sufficient to characterize monopolarity for every graph G having the following property: if $d_G(x) \geq 3$ and wxy is an induced P_3 wholly contained in some cycle of G, then the $P_3 : wxy$ shares a vertex with a triangle of G or an edge with an induced C_4. We use $\mathcal{G}$ to denote the class of graphs having this property.

Although $\mathcal{G}$ seems like a strange class of graphs to consider, it contains many well-studied graph classes—including all classes for which monopolarity recognition algorithms are known. As alluded to above, $\mathcal{G}$ contains the claw-free graphs: any vertex of degree ≥ 3 in a claw-free graph is contained in some triangle. It also contains all graphs which contain no induced C_k with $k \geq 5$, as any P_3 wholly contained in a cycle of such a graph is contained in a C_4. Therefore, the following characterization applies to cographs, chordal graphs, and (via the characterization by Gallai [15]) permutation graphs and co-comparability graphs, all of which have no induced cycles of length ≥ 5. The last result answers a question asked in [9]; in contrast, we have been able to show that recognizing monopolar comparability graphs is NP-complete [4].

Theorem 10 *A graph $G \in \mathscr{G}$ is monopolar if and only if H^* is colour-bipartite.*

Proof Let $P(G)$ be the set of vertices of degree ≥ 3 in G each of which

- is contained in no triangle of G,
- is incident with an edge in G which is not in any induced C_4,
- has no loop in H^*, and
- is not adjacent in G to any vertex with a blue loop in H^*.

By definition of $\mathscr{G}$, no vertex in $P(G)$ is contained in any induced cycle of length ≥ 5 in G. We prove, by induction on $|P(G)|$, that G has a monopolar partition (S, T) with a given vertex $s \in S$ provided H^* is colour-bipartite and s has no blue loop in H^*.

Suppose first that $P(G) = \emptyset$. Choose S to be a maximal blue-compatible set containing s and let $T = V(G) \setminus S$. By Theorem 2 and the fact that every edge of G is a blue edge in H^*, S is an independent set in G and T is a red independent set in H. Suppose that T contains some induced $P_3 : xyz$ of G. Since xy or yz are not red edges, neither is contained in an induced C_4 of G. Similarly, neither edge is the pendant edge of an induced paw in G, which implies y is not contained in any triangle. Because none of xy, yz, xz are red edges, y has no loop in H^* and is not adjacent in G to any vertex with a blue loop in H^*. It follows—since $P(G) = \emptyset$—that $d(y) = 2$. This contradicts the assumption that S is maximal. Thus T contains no such induced P_3 in G and (S, T) is the desired monopolar partition.

Now suppose $|P(G)| > 0$ and the inductive hypothesis is true for every G' having $|P(G')| < |P(G)|$. Let $v \in P(G)$. Now, G can be written as $G = \bigcup G_i$ where each G_i is induced by v and a component of $G - v$. By the definitions of $\mathscr{G}$ and $P(G)$, v is a cut vertex of G. Moreover, for each i, either v has degree one in G_i or every edge of G_i incident with v is contained in an induced C_4 of G_i. In either case, $|P(G_i)| < |P(G)|$ for every i and the inductive hypothesis applies to G_i. (Note that H^* contains as subgraphs each 2-edge-coloured graph H_i^* which would be constructed from G_i by the above process.)

Assume without loss of generality that the specified vertex s is contained in V_1. By the inductive hypothesis, G_1 has a monopolar partition (S_1, T_1) with $s \in S_1$. There are two cases, based on which part contains v. If $v \in S_1$, then apply the inductive hypothesis to the remaining G_i to find monopolar partitions (S_i, T_i) with $v \in S_i$. Otherwise, if $v \in T_1$, choose neighbours n_i of v in each G_i and apply the inductive hypothesis to find monopolar partitions (S_i, T_i) with $n_i \in S_i$. In either case, $\left(\bigcup S_i, \bigcup T_i\right)$ is a desired monopolar partition of G. This is obvious in the first case. In the second, it turns out that every neighbour of v is contained in $\bigcup S_i$: this follows from the fact that v either has degree one in G_i or every edge incident with v in G_i is a blue and red edge in H_i.

Conversely, suppose (A, B) is a monopolar partition of G. Since every blue edge of H_0 is an edge of G, A is a blue independent set in H_0. If xy is a red edge in H_0, then G contains an induced paw, kite, or cycle containing x, y and adjacent w, z such that $\{w, x, y\}$ and $\{x, y, z\}$ are induced P_3s. As A induces an independent set and B contains no induced P_3 of G, at most one of x, y is in B. Hence (A, B) is a colour-bipartition of H_0. Now, if (A, B) is colour-bipartition for some H_{i-1}, $i > 0$, then B contains every vertex which is blue-incompatible with itself in H_i. If w is such a vertex and $\{w, x, y\}$ is a induced P_3 in G, then at least one of x, y is contained in A. So (A, B) is colour-bipartition of H_i, and, by Proposition 3, of H_i^{inc}. In particular, it is a colour-bipartition of $H^* = H_n^{inc}$. □

The Proof of Theorem 10 suggests an algorithm to find a monopolar partition when one exists, which we offer without proof. Construct a blue-compatible set S_P by recursively adding vertices as follows: if there exists any $p \in P(G)$ which is blue-incompatible with a vertex in S_P, add $N_G(p)$ to S_P; otherwise, choose a $p \in P(G)$ which is not blue-incompatible with any vertex in S_P and add it to S_P. Then if S is a maximal blue-compatible set containing S_P and G is monopolar, then $(S, V(G) \setminus S)$ is a monopolar partition of G.

Corollary 11 *There exists an $O(n^4)$ algorithm to determine whether a given $G \in \mathscr{G}$ is monopolar, and to find a monopolar partition if one exists.*

Proof The 2-edge-coloured graph H_0 can clearly be constructed in time $O(n^4)$: it amounts to finding all induced C_4s, paws, and kites of G and adding the appropriate red edges to G. To construct each successive H_i, we construct H_{i-1}^{inc} as in Theorem 5 to find the vertices which are blue-incompatible with themselves, and then find the induced P_3s in G which contain such vertices. This can be done in $O(n^3)$ time for each i. In total, $H^* = H_n^{inc}$, which has n vertices and $O(n^2)$ edges, is constructed in $O(n^4)$ time. Checking whether $H^* = H_n^{inc}$ is colour-bipartite takes $O(n)$ time: it needs only be checked whether any vertex has a blue and red loop (see Proposition 3). We leave it to the reader to implement the algorithm mentioned above to construct a monopolar partition; with H^* already constructed, it can be done without affecting the overall complexity of $O(n^4)$. □

We should mention that the above techniques can be easily modified to solve the list-monopolar partition problem for graphs in $\mathscr{G}$, where some vertices are specified to be in S or T: when constructing H_0, just add red and blue loops to the appropriate specified vertices. In fact, this strategy works even for general graphs, provided loops are added in such a way that no cycle of length ≥ 5 contains a vertex in the set $P(G)$ in the above proof.

5 Finding Monopolar Partitions of Polar Graphs

As mentioned in Sect. 1, both the polar and monopolar partition problems are hard for general graphs. Moreover, every monopolar partition is also a polar partition. Would the monopolar partition problem become easier if a polar partition of the input graph were already known? We show in Theorem 14 that this is indeed the case.

We first consider the following related problem. Given an input graph G and a vertex partition (X, Y) such that Y induces a disjoint union of cliques, does G have a monopolar partition (S, T) with $X \subseteq T$? This question is easily answered using colour-bipartitions: as the next proposition shows, such a partition is equivalent to a colour-bipartition of the 2-edge-coloured graph H obtained from a blue copy of G by adding blue loops on the vertices of X and red edges yz whenever G contains a P_3 induced by $\{x, y, z\}$ with $x \in X$.

Proposition 12 *A colour-bipartition (S, T) of H is a monopolar partition of G with $X \subseteq T$, and vice versa.*

Proof The condition that S is a blue independent set is clearly equivalent to requiring S to be an independent set in G disjoint from X. If T is a disjoint union of cliques, then it contains no induced P_3 of G and in particular no red edge of H. Conversely, every P_3 in G has a vertex in X because Y is a disjoint union of cliques. Thus if T is not a disjoint union of cliques, it contains a P_3 with a vertex in X; i.e., T contains a red edge of H. □

Corollary 13 *Suppose X_1, X_2, Y partition the vertices of G such that Y is a disjoint union of cliques. It can be decided in $O(nm+n^2)$ time whether G admits a monopolar partition (S, T) with $X_1 \subseteq S$ and $X_2 \subseteq T$.*

Proof Apply Proposition 12 to the graph $G - X_1$ with $X = X_2 \cup N(X_1)$ and $Y = V(G - X_1) \setminus X$. Clearly, any desired monopolar partition (S, T) restricts to a monopolar partition of $G - X_1$ with $X_2 \cup N(X_1) \subseteq T$. Conversely, if (S, T) is a monopolar partition of $G - X_1$ with $X_2 \cup N(X_1) \subseteq T$, then $(S \cup X_1, T)$ is a desired monopolar partition of G.

The 2-edge-coloured graph H can be constructed as above in $O(nm)$ time. We could use the result of Theorem 5 to construct H^{inc} and test whether it is colour-bipartite; however, since H may have $\Omega(n^2)$ edges, this may take $\Omega(n^3)$ time. A more efficient method of deciding whether H is colour-bipartite is to use the linear reduction to 2-SAT suggested by [2] followed by a linear 2-SAT solver (e.g. [13]): as H has n vertices and $O(n^2)$ edges, this reduction takes $O(n^2)$ time. In total, the construction of H and test of colour-bipartiteness takes $O(nm + n^2)$ time. □

Suppose G is known to have a polar partition (A, B). The above proposition allows us to determine in $O(nm)$ time whether a monopolar partition of A extends to a monopolar partition of G. As it turns out, there are not too many possible monopolar partitions of A and we can afford to check them all.

Theorem 14 *If G is known to admit the polar partition (A, B), it can be decided in $O(n^2m)$ time whether G is monopolar.*

Proof If A is an independent set, then we are already done as (A, B) is a monopolar partition. If A contains a $K_{2,2,1}$, then it is not monopolar. Otherwise, A induces either a complete bipartite graph containing a C_4 or the join of a (possibly empty) independent set and a clique. If A is a complete bipartite graph containing a C_4, then any monopolar partition of A is a bipartition. We need only find the (unique) bipartition (A_1, A_2) of A and apply Proposition 13 twice: once with $X_1 = A_1$, $X_2 = A_2 \cup N_G(A_1)$, and $Y = B - N_G(A_1)$, and once with $X_1 = A_2$, $X_2 = A_1 \cup N_G(A_2)$, and $Y = B - N_G(A_2)$.

Finally, if A is the join of a (possibly empty) independent set A_1 and a clique A_2, then any monopolar partition (S, T) of A takes one of three forms: either $S = \{a\}$ for some $a \in A_2$, or $S = A_1$, or $S = A_1 - \{a\}$ for some $a \in A_1$. Between these there are at most $2n + 1$ possible monopolar partitions for A; apply Proposition 13 to each of these.

It can be determined in $O(n + m)$ time which case A falls into: it amounts to checking whether A induces an independent set, a complete bipartite graph, a split graph, or none of the above. If A is an independent set, we answer "yes" and report the monopolar partition (A, B). If A is a complete bipartite graph containing a C_4, a bipartition can be constructed in $O(m)$ time; applying Proposition 13 takes $O(nm)$ time to determine whether G is monopolar and to find a monopolar partition. Finally, if A is the join of an independent set and a clique, we can find each of the (at most) $2n + 1$ possible partitions in linear time; applying Corollary 13 to each of these gives a total complexity of $O(n^2m)$. □

6 Concluding Remarks

As mentioned at the end of Sect. 4, our techniques can easily be adapted to solve the "list" or "precoloured" versions of the above problems. Each of our main theorems constructs a colour-bipartition (S, T) of a 2-edge-coloured graph H which is also a desired partition of the input graph. By adding red and blue loops during the construction of H, we can force vertices of G to be in S or T. If the new H is still colour-bipartite, a desired partition of G is produced by the given proofs of Theorems 6, 10, and 14.

The general techniques of this paper applies to more partition problems than the polar-type problems considered here. For instance, the authors have adapted the argument of Sect. 3 to solve the problem of partitioning the vertices of a graph into a clique and a triangle-free graph. It would be interesting to see which other partition problems can be solved with similar techniques. The polynomial time solvable edge-coloured homomorphism problems given in [2] generalize colour-bipartitions and may be useful for solving problems involving partitions into more than two vertex sets.

Van Bang Le and Ragnar Nevries recently took a 2-SAT approach to monopolarity in a number of different classes [18]. Notably, they found a reduction for a class containing all maximal planar graphs such that the monopolar partitions of the input graph coincide exactly with the solutions to the output 2-SAT instance. This is not always the case in our reductions: Theorems 6 and 10 reduce the unipolarity and monopolarity of an input G to the colour-bipartiteness of an output H, but not every colour-bipartition of H is a desired partition of G.

The forbidden structures for colour-bipartite graphs are given by the homomorphism duality theorem originally from [1] and reproduced in Theorem 2. By interpreting blue and red edges as subgraphs according to our reductions, we can get a general idea of what the forbidden induced subgraphs for unipolar and monopolar graphs look like. It should be possible to catalogue these forbidden structures precisely—either explicitly (as in [17] for line-monopolar bipartite graphs) or using graph grammars (as in [19] for monopolar chordal graphs)—using Theorems 6 and 10.

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