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The cohomology ring of subspaces of universal S^1-space with finite orbit types

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ABSTRACT

We calculate the ring of the equivariant cohomologies of an $\mathcal{F}$-classifying G-space for finite open family $\mathfrak{F} \subset \mathrm{Conj}_G$ of orbit types and the acting group $G = \mathrm{SO}(2)$.

1. Introduction

R. Palais was among the first to suggest the study of the compact transformation groups in the context of generalized principal G-bundles [1, 2.6]. He proved the covering homotopy theorem for G-spaces and introduced the concept of universal G-spaces that was a far-reaching generalization of universal principal G-bundles. Refining the analogy with the theory of principal G-bundles, R. Palais sets the problem on defining and calculating the characteristic classes of G-spaces which is of great importance for equivariant algebraic topology.

R. Palais established the existence of universal G-spaces – in our terms *universal isovariant extensors* or *injective objects of the isovariant category* ISOV-TOP, only in the finite-dimensional case with some restrictions on the orbit types [1, 2.6]. The general problem on existence of such spaces was solved in [2,3]. Recall that these spaces contain an important portion of information about the whole category, and their orbit spaces perform the classification of all G-spaces. As a result, the category ISOV-TOP turns out to be homotopical [4].

A future solution of Palais's problem on calculating of characteristic classes of G-spaces can be obtained on the base of the theory of isovariant absolute extensors. For this aim it is useful to extend the functor of equivariant Borel cohomology to the case of arbitrary family $\mathcal{F}$ of orbit types (which may be understood as a subset of the set Conj_G of conjugate classes of closed subgroups of G) and to construct the functor $H^*_{\mathcal{F}}$ of equivariant cohomology with respect to $\mathcal{F}$. (Borel's construction corresponds to the single-element family $\mathcal{F}$.) It turns out that the equivariant cohomologies $H^*_{\mathcal{F}}(\mathbb{X})$ of a G-space $\mathbb{X}$ is an algebra over the ring $R_{\mathcal{F}}$ of the equivariant cohomologies of an $\mathcal{F}$-classifying G-space (which is defined as a final object in the equivariant homotopy category $\mathrm{EQUIV}_{\mathcal{F}}$-HOMOT of G-$\mathcal{F}$-spaces).

Letting $\mathbb{W}$ be an isovariant extensor, we note that the $\mathcal{F}$-orbit bundle $\mathbb{W}_{\mathcal{F}} \subset \mathbb{W}$ is an $\mathcal{F}$-classifying G-space[1] and the ring $R_{\mathcal{F}}$ equals the equivariant cohomologies of $\mathbb{W}_{\mathcal{F}}$ (which in turn coincides with the ordinary cohomologies $H^*(W_{\mathcal{F}})$ of

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[1] The possibility to locate all $\mathcal{F}$-classifying G-spaces within one space $\mathbb{W}$ is the key ingredient of the proofs.

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its orbit space $W_{\mathcal{F}}$). These rings $R_{\mathcal{F}}$ are important invariants of the group G itself. In particular, the locking cohomology of the group corresponds to the case of $\mathcal{F} = \mathrm{Conj}_G \setminus \{(G)\}$.

The present paper is the continuation of [2], where the locking cohomologies of groups SO(2) and O(2) were calculated. Here we continue these investigations and prove the following result:

Theorem 1. *Let $\mathbb{W}$ be an isovariant extensor for $G = \mathrm{SO}(2)$, $\mathfrak{F}$ a finite open family of orbit types and $W_{\mathfrak{F}}$ the orbit space of the $\mathcal{F}$-orbit bundle $\mathbb{W}_{\mathfrak{F}}$. Then the (Čech) cohomology ring $\check{H}^*(W_{\mathfrak{F}}; \mathbb{Q})$ with coefficients in the rationals equals the polynomial algebra $\mathbb{Q}[x]$ on a generator x of dimension 2.*

2. Preliminary facts and results

In what follows we shall assume all spaces (all maps) to be metric (continuous, respectively), if they do not arise as a result of some constructions or if the opposite is not claimed; all acting groups are assumed to be compact Lie groups.

We present the basic notions of the theory of G-spaces [5]. An action of a compact group G on a space $\mathbb{X}$ is a continuous map μ from the product $G \times \mathbb{X}$ into $\mathbb{X}$ satisfying the following properties: [5]

(1) $\mu(g, \mu(h, x)) = \mu(g \cdot h, x)$; and
(2) $\mu(e, x) = x$ for all $x \in \mathbb{X}$, $g, h \in G$ (here e is the unit of the group G).

As a rule, $\mu(g, x)$ will be written as $g \cdot x$ or just gx. A space $\mathbb{X}$ with an action of the group G is called a *G-space*. The map $f : \mathbb{X} \to \mathbb{Y}$ of G-spaces is called a *G-map* or an *equivariant map* if $f(g \cdot x) = g \cdot f(x)$ for all $x \in \mathbb{X}$, $g \in G$.

The subset $\{g \cdot x \mid g \in G\} = G \cdot x$ is called the *orbit* $G(x)$ of the point $x \in \mathbb{X}$ which turns out to be closed. The natural map $\pi = \pi_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}/G$, $x \mapsto G(x)$, of the space $\mathbb{X}$ into the space $\mathbb{X}/G$ of quotient partition said to be the *orbit projection*. We call the space of quotient partition, equipped with the quotient topology induced by π, the *orbit space*. We will denote it by $X \rightleftharpoons \mathbb{X}/G$, provided that no confusion occurs. The subset A is called *invariant* or a *G-subset* if $\pi^{-1}\pi(A) = G \cdot A$.

For each point $x \in \mathbb{X}$ the subset $G_x = \{g \in G \mid g \cdot x = x\}$ is a closed subgroup of G and is called a *stabilizer of* x. For each closed subgroup $H < G$ let us consider the following subsets $\mathbb{X}$:

$$\mathbb{X}^H = \{x \in \mathbb{X} \mid H \cdot x = x\} = \{x \in \mathbb{X} \mid H \subset G_x\}$$

(*the set of H-fixed points*),

$$\mathbb{X}_H = \{x \in \mathbb{X} \mid H = G_x\}, \qquad \mathbb{X}_{(H)} = \{x \in \mathbb{X} \mid H \text{ conjugates with } G_x\}.$$

Let $\mathcal{F}$ be a family of orbit types (that is, a subset of the set Conj_G of conjugacy classes of closed subgroups of G). Then the set $\mathbb{X}_{\mathcal{F}} \rightleftharpoons \{x \mid (G_x) \in \mathcal{F}\} \subset \mathbb{X}$ is called an *$\mathcal{F}$-orbit bundle* $\mathbb{X}$. We say that the G-space $\mathbb{X}$ is of *orbit type* $\mathcal{F}$, or briefly the *G-$\mathcal{F}$-space*, provided that $\mathbb{X} = \mathbb{X}_{\mathcal{F}}$.

The family $\mathcal{F}$ of orbit types is called *open* if $(K) < (H)$ and $(H) \in \mathcal{F}$ implies that $(K) \in \mathcal{F}$ (here $(K) < (H)$ means the existence of a subgroup K' such that K' is conjugated to K and $K' \subset H$). The family $\mathcal{F}_H = \{(K) \leqslant (H)\}$ is called a *simple open family*, and it serves as an example of open family. It is easy to prove the following fact.

Lemma 1. *Each finite open family of the orbit types coincides with the union $\bigcup\{\mathcal{F}_{H_i} \mid H_1, H_2, \ldots, H_m < G\}$ of simple open families.*

We denote the space $\mathbb{X}_{\mathfrak{F}_H}$ by $\mathbb{X}_{\leqslant H}$ and its orbit space by $X_{\leqslant H}$. It follows that $\mathbb{X}_{\mathfrak{F}} = \bigcup\{\mathbb{X}_{\leqslant H_i} \mid i \leqslant m\}$.

Note that metric G-spaces of the orbit type $\mathcal{F}$ and G-maps between them generate the category, which is denoted by $G_{\mathcal{F}}$-TOP or EQUIV$_{\mathcal{F}}$-TOP if all are clear about the group G in question. We will freely use the symbol “G-” or “Equiv-” meaning *equivariant*. If “$***$” is any notion from nonequivariant topology, then “G-$***$” or “Equiv-$***$” means the corresponding equivariant analogue.

The equivariant map $f : \mathbb{X} \to \mathbb{Y}$ is called *isovariant* if f preserves stabilizers, that is, $G_x = G_{f(x)}$ for every $x \in \mathbb{X}$. The category generated by metric G-spaces of orbit type $\mathcal{F}$ and isovariant maps is denoted by ISOV$_{\mathcal{F}}$-TOP (it is always clear about the group G in question).

We introduce several concepts related to extension of G-maps in the category $\mathcal{C}$ coinciding with ISOV$_{\mathcal{F}}$-TOP or with EQUIV$_{\mathcal{F}}$-TOP. A space $\mathbb{X}$ with an action of compact group G which is an object of the category $\mathcal{C}$ is called an *absolute neighbourhood $\mathcal{C}$-extensor* (is denoted by $\mathbb{X} \in \mathcal{C}$-ANE) if for each morphism $\varphi : \mathbb{A} \to \mathbb{X}$ from $\mathcal{C}$, defined on a closed G-subset $\mathbb{A} \subset \mathbb{Z}$ of a G-space $\mathbb{Z}$ and called a *partial $\mathcal{C}$-morphism*, can be extended into some G-neighbourhood $\mathbb{U} \subset \mathbb{Z}$ of $\mathbb{A}$ to a morphism $\widehat{\varphi} : \mathbb{U} \to \mathbb{X} \in \mathcal{C}$. If it is always possible to make $\mathbb{U}$ equal to $\mathbb{Z}$, then $\mathbb{X}$ is called an *absolute $\mathcal{C}$-extensor*, $\mathbb{X} \in \mathcal{C}$-AE. If the acting group G is trivial (that is, all spaces are considered without actions), then this notion is transformed into the notion of absolute [neighbourhood] extensors for metric spaces – A[N]E (see [6,7]).

If $\mathcal{C}$ coincides with the category EQUIV-TOP (ISOV-TOP), then absolute [neighbourhood] $\mathcal{C}$-extensors will be called equivariant [neighbourhood] extensors (isovariant [neighbourhood] extensors) or briefly – Equiv-A[N]E- or Equiv$_G$-A[N]E-spaces (briefly as Isov-A[N]E- or Isov$_G$-A[N]E-spaces). In what follows we will denote injective objects of the category EQUIV$_{\mathcal{F}}$-TOP (ISOV$_{\mathcal{F}}$-TOP) by Equiv$_{\mathcal{F}}$-A[N]E (Isov$_{\mathcal{F}}$-A[N]E).

We fix a G-space $\mathbb{W} \in$ Isov-AE, a family $\mathcal{F} \subset \mathrm{Conj}_G$ and consider the Borel functor

$$E_{\mathcal{F}} : \text{EQUIV-TOP} \to \text{ISOV}_{\mathcal{F}}\text{-TOP}, \quad E_{\mathcal{F}}(\mathbb{X}) \rightleftharpoons \bigl\{(w,x) \bigm| G_w \subset G_x,\ (G_w) \in \mathcal{F}\bigr\},$$

which assigns to a G-map $f : \mathbb{X} \to \mathbb{Y}$ the isovariant map $E_f : E_{\mathcal{F}}(\mathbb{X}) \to E_{\mathcal{F}}(\mathbb{Y})$ defined by the formula $E_f(w,x) = (w, f(x))$. It is known (see [2,3]) that

(1) $E_{\mathcal{F}}(\mathbb{X})$ is an $\mathrm{Isov}_{\mathcal{F}}$-ANE-space, provided that $\mathbb{X} \in \mathrm{Equiv}_{\mathcal{F}}$-ANE;
(2) there exists an isovariant map from $E_{\mathcal{F}}(\mathbb{X})$ into $\mathbb{W}_{\mathcal{F}}$ unique to isovariant homotopy, which coincides with the natural projection $p : E_{\mathcal{F}}(\mathbb{X}) \to \mathbb{W}_{\mathcal{F}}$, $p(w,x) = w$.

We note also that for each Equiv-ANE-space $\mathbb{X}$, p is a locally $\mathrm{Isov}_{\mathcal{F}}$-soft map, the induced map $\tilde{p} : E_{\mathcal{F}}(X) \to W_{\mathcal{F}}$ of orbit spaces is locally soft, and the orbit projection $\pi : E_{\mathcal{F}}(\mathbb{X}) \to E_{\mathcal{F}}(X)$ is the pullback of the orbit projection $\pi : \mathbb{W}_{\mathcal{F}} \to W_{\mathcal{F}}$ via $\tilde{p}$.

Let us consider the commutative diagrams

$$\begin{tikzcd}
\mathbb{W}_{\mathcal{F}} \arrow[d, "\pi"] & E_{\mathcal{F}}(\mathbb{X}) \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\pi"] & \mathbb{X} \arrow[d, "\pi"] \\
W_{\mathcal{F}} & E_{\mathcal{F}}(X) \arrow[l, "\tilde{p}"'] \arrow[r, "\tilde{q}"] & X,
\end{tikzcd}$$

in which the natural projection $q : E_{\mathcal{F}}(\mathbb{X}) \to \mathbb{X}$, $q(w,x) = x$, is an equivariant and $\mathrm{Equiv}_{\mathcal{F}}$-soft map for each G-space $\mathbb{X}$. Therefore the G-space $\mathbb{X}$ and the G-$\mathcal{F}$-space $E_{\mathcal{F}}(\mathbb{X})$ have the same equivariant homotopy properties concerning with the family $\mathcal{F} \subset \mathrm{Conj}_G$.

Now define the contravariant functor $H^*_{\mathcal{F}}$ of (rational) equivariant cohomology with respect to the family $\mathcal{F}$ of orbit types, which assigns to a G-space $\mathbb{X}$ the algebra $H^*_{\mathcal{F}}(\mathbb{X}) = H^*(E_{\mathcal{F}}(X);\mathbb{Q})$ and assigns to a G-map $f : \mathbb{X} \to \mathbb{Y}$ the homomorphism $H^*_{\mathcal{F}}(f) : H^*_{\mathcal{F}}(\mathbb{Y}) \to H^*_{\mathcal{F}}(\mathbb{X})$ of the algebras coinciding with the induced homomorphism $(\tilde{E}_f)^* : H^*(E_{\mathcal{F}}(Y);\mathbb{Q}) \to H^*(E_{\mathcal{F}}(X);\mathbb{Q})$.

Since $E_{\mathcal{F}}(*) = \{(w,*) \mid G_w < G,\ (G_w) \in \mathcal{F}\} = \mathbb{W}_{\mathcal{F}}$, we have $H^*_{\mathcal{F}}(*) = H^*(W_{\mathcal{F}};\mathbb{Q})$. Therefore $H^*_{\mathcal{F}}(\mathbb{X})$ is an algebra over the ring $H^*_{\mathcal{F}}(*)$: if $r \in H^*_{\mathcal{F}}(*)$, $x \in H^*_{\mathcal{F}}(\mathbb{X})$, then $r \cdot x$ equals the cup product $(\tilde{p})^*(r) \smile x$. As it can be easily checked, $(E_{\mathcal{F}}(G/H))/G = (\mathbb{W}_{\mathcal{F}} \cap W_{\leqslant H})/H$. Hence the coefficients $H^*_{\mathcal{F}}(G/H)$ of the $H^*_{\mathcal{F}}$-theory equals $H^*_{\mathcal{F}}(G/H) = H^*((\mathbb{W}_{\mathcal{F}} \cap W_{\leqslant H})/H;\mathbb{Q})$.

By the Eilenberg–MacLane complex $K(\pi,n)$ we mean an ANE-space with the following homotopy groups:

$$\pi_q\bigl(K(\pi,n)\bigr) = \begin{cases} \pi, & q = n, \\ 0, & q \neq n. \end{cases}$$

For example $S^1 = \mathrm{SO}(2)$ is $K(\mathbb{Z},1)$. It is known that any two Eilenberg–MacLane complexes of the same type are homotopy equivalent [8, p. 93]. Hence, the Čech and Alexander–Spanier cohomologies of any two Eilenberg–MacLane complexes equal. It is known [8, p. 224] that:

Proposition 1. *The cohomology ring $\check{H}^*(K(\mathbb{Z},2);\mathbb{Q})$ of the Eilenberg–MacLane space $K(\mathbb{Z},2)$ is the polynomial algebra $\mathbb{Q}[x]$ on a generator x of dimension* 2:

$$\check{H}^k\bigl(K(\mathbb{Z},2),\mathbb{Q}\bigr) = \begin{cases} \mathbb{Q}, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}. \end{cases}$$

3. The proof of Theorem 1

In what follows we shall consider the group SO(2) as the subset $S^1 = \{e^{i\varphi} \mid \varphi \in \mathbb{R}\}$ of the complex plane $\mathbb{C}$. Let H denote a proper subgroup of S^1. Since any proper closed subgroup of S^1 is $\mathbb{Z}_n = \{e^{\frac{2\pi k}{n} i} \mid k \in \mathbb{N}\}$ for appropriate $n \in \mathbb{N}$, the quotient group S^1/H is isomorphic to S^1.

Lemma 2. *Let $\mathbb{W}$ be an Isov_{S^1}-AE-space. Then the cohomology ring $\check{H}^*(W_{\leqslant H};\mathbb{Q})$ is equal to the polynomial algebra $\mathbb{Q}[x]$ on a generator x of dimension* 2.

Proof. In view of Proposition 1, Lemma 2 is reduced to the fact that $W_{\leqslant H}$ is the Eilenberg–MacLane space $K(\mathbb{Z},2)$.

Let us consider $\mathbb{W}_{\leqslant H}$ as an H-space. Since $\mathbb{W}_{\leqslant H}$ is an Isov_H-AE-space (see [2, p. 9]), it is H-contractible and the orbit space $\mathbb{W}_{\leqslant H}/H$ is contractible. Moreover, $\mathbb{W}_{\leqslant H}/H$ is a free S^1/H-space and therefore the S^1/H-orbit projection $p : \mathbb{W}_{\leqslant H}/H \to W_{\leqslant H}$ is the principal bundle with the fibre S^1/H. From the corresponding exact homotopy sequence

$$\cdots \to \pi_{n+1}(W_{\leqslant H}) \to \pi_n\bigl(S^1/H\bigr) \to \pi_n(\mathbb{W}_{\leqslant H}/H) \to \pi_n(W_{\leqslant H}) \to \pi_{n-1}\bigl(S^1/H\bigr) \to \pi_{n-1}(\mathbb{W}_{\leqslant H}/H) \to \cdots,$$

it follows by contractibility of the space $\mathbb{W}_{\leqslant H}/H$ that $\pi_n(W_{\leqslant H})$ is isomorphic to $\pi_{n-1}(S^1/H)$ for all $n>0$. Since $\mathbb{W}_{\leqslant H}$ is connected, $W_{\leqslant H}$ is also connected, and hence $\mathbb{W}_{\leqslant H}$ is $K(\mathbb{Z},2)$. $\square$

Proof of Theorem 1. The proof will be performed by induction on the number of subgroups contained in $\mathfrak{F}$. The inductive hypothesis consists in coincidence $\check{H}^*(W_{\mathfrak{F}};\mathbb{Q})$ with the polynomial algebra $\mathbb{Q}[x]$, $\deg x=2$.

If $|\mathfrak{F}|=1$, then $\mathfrak{F}$ is a trivial simple open family, and Theorem 1 is proved by Lemma 2. Let the inductive hypothesis have been proved for all $\mathfrak{F}$ with $|\mathfrak{F}|<n$. We consider an open family $\mathfrak{F}$ with $|\mathfrak{F}|=n$ and assume for definiteness that $\mathfrak{F}=\bigcup_{i=1}^m \mathfrak{F}_{H_i}$, where each $\mathfrak{F}_{H_i}$ is a simple open family, and H_i is a proper subgroup of S^1. It is clear that $\mathfrak{S}=\bigcup_{i=1}^{m-1}\mathfrak{F}_{H_m}$ and $\mathfrak{X}=\mathfrak{S}\cap\mathfrak{F}_{H_m}$ are open finite families with $|\mathfrak{X}|<n$, $|\mathfrak{S}|<n$ and $|\mathfrak{F}_m|<n$.

We get as an easy corollary of the Slice Theorem (see [5, p. 92]) that $W_{\mathfrak{X}}$, $W_{\mathfrak{S}}$ and $W_m \rightleftharpoons W_{\mathfrak{F}_{H_m}}$ are open in $W_{\mathfrak{F}}$. It follows that the Mayer–Vietoris sequence for cohomology can be applied (see [8, p. 125]):

$$0\to \check{H}^0(W_{\mathfrak{F}},\mathbb{Q})\xrightarrow{f^0}\check{H}^0(W_{\mathfrak{S}},\mathbb{Q})\oplus\check{H}^0(W_m,\mathbb{Q})\xrightarrow{i^0}\check{H}^0(W_{\mathfrak{X}},\mathbb{Q})$$
$$\xrightarrow{\delta^0}\check{H}^1(W_{\mathfrak{F}},\mathbb{Q})\xrightarrow{f^1}\check{H}^1(W_{\mathfrak{S}},\mathbb{Q})\oplus\check{H}^1(W_m,\mathbb{Q})\xrightarrow{i^1}\check{H}^1(W_{\mathfrak{X}},\mathbb{Q})\xrightarrow{\delta^1}\check{H}^2(W_{\mathfrak{F}},\mathbb{Q})\xrightarrow{f^2}\cdots,$$

where $i^*(a,b)=i_1^*(a)-i_2^*(b)$, and homomorphisms i_1^* and i_2^* are induced by the embeddings of $W_{\mathfrak{X}}$ into $W_{\mathfrak{S}}$ and W_m, respectively.

Since $\mathbb{Q}$ is a field, all cohomologies $\check{H}^k(X,\mathbb{Q})$ are vector spaces over the field $\mathbb{Q}$ and the corresponding induced homomorphisms i^*, f^* and δ^* are linear operators in view of the known properties of multiplication in cohomologies.

In view of Lemma 2 and the inductive hypothesis (that is, $W_{\mathfrak{X}}$, $W_{\mathfrak{S}}$ and W_m satisfy Theorem 1), we can write down the Mayer–Vietoris sequence $(*)$ as follows:

$$0\to\check{H}^0(W_{\mathfrak{F}},\mathbb{Q})\xrightarrow{f^0}\mathbb{Q}\oplus\mathbb{Q}\xrightarrow{i^0}\mathbb{Q}\xrightarrow{\delta^0}\check{H}^1(W_{\mathfrak{F}},\mathbb{Q})\xrightarrow{f^1}0\xrightarrow{i^1}0$$
$$\xrightarrow{\delta^1}\check{H}^2(W_{\mathfrak{F}},\mathbb{Q})\xrightarrow{f^2}\mathbb{Q}\oplus\mathbb{Q}\xrightarrow{i^2}\mathbb{Q}\xrightarrow{\delta^2}\check{H}^3(W_{\mathfrak{F}},\mathbb{Q})\xrightarrow{f^3}0\xrightarrow{i^3}0\xrightarrow{\delta^3}\cdots.$$

Since $\mathbb{Q}$ is a one-dimensional vector space, the linear operator $i^{2q}:\mathbb{Q}\oplus\mathbb{Q}\to\mathbb{Q}$ is surjective if and only if it is nontrivial. Below we shall prove that $i^{2q}\neq 0$ (and therefore i^{2q} is an epimorphism), but now let us just assume this is the case.

Since i^{2q} is an epimorphism and δ^{2q} is epimorphic by $(*)$, the odd cohomologies $\check{H}^{2n+1}(W_{\mathfrak{F}},\mathbb{Q})$ are equal to zero, and a long exact sequence $(*)$ is split into the short exact sequences:

$$0\to\check{H}^{2n}(W_{\mathfrak{F}},\mathbb{Q})\xrightarrow{f^{2q}}\mathbb{Q}\oplus\mathbb{Q}\xrightarrow{i^{2q}}\mathbb{Q}\xrightarrow{\delta^{2q}}0,$$

with help of which one can calculate the even cohomologies. Since f^{2q} is monomorphic and $\dim(\mathrm{Ker}(i^{2q}))+\dim(\mathrm{Im}(i^{2q}))=\dim(\mathbb{Q}\oplus\mathbb{Q})=2$, it follows that $\mathrm{Ker}(i^{2q})=\mathbb{Q}$ and $\mathrm{Im}(f^{2q})=\mathrm{Ker}(i^{2q})=\mathbb{Q}$. Since $\check{H}^{2q}(W_{\mathfrak{F}},\mathbb{Q})=\mathrm{Im}(f^{2q})$, we have $\check{H}^{2q}(W_{\mathfrak{F}},\mathbb{Q})=\mathbb{Q}$.

Now we show that the rings $\check{H}^*(W_{\mathfrak{F}},\mathbb{Q})$ and $\mathbb{Q}[x]$, $\deg(x)=2$, are isomorphic. Recall that $\check{H}^*(X,\mathbb{Q})=\bigoplus_q\check{H}^q(X,\mathbb{Q})$ is an associative skew-commutative ring with the cohomology multiplication – the cup product $a\smile b$. We define for the integer $n>1$ the map $\alpha:\check{H}^2(W_{\mathfrak{F}},\mathbb{Q})\to\check{H}^{2n}(W_{\mathfrak{F}},\mathbb{Q})$ by $\alpha(a)=\underbrace{a\smile a\smile\cdots\smile a}_{n}\rightleftharpoons a^n$ and consider the following commutative diagram:

$$\begin{tikzcd}
\check{H}^2(W_{\mathfrak{F}},\mathbb{Q}) \arrow[r, "f^2"] \arrow[d, "\alpha"] & \check{H}^2(W_{\mathfrak{S}},\mathbb{Q})\oplus\check{H}^2(W_m,\mathbb{Q}) \arrow[d, "\alpha\oplus\alpha"] \\
\check{H}^{2n}(W_{\mathfrak{F}},\mathbb{Q}) \arrow[r, "f^{2n}"] & \check{H}^{2n}(W_{\mathfrak{S}},\mathbb{Q})\oplus\check{H}^{2n}(W_m,\mathbb{Q}).
\end{tikzcd}$$

If $0\neq a\in\check{H}^2(W_{\mathfrak{F}},\mathbb{Q})$, then by the injectivity of the linear operator f^2, we get that $f^2(a)\neq 0$. It follows by the inductive hypothesis (that is, $\check{H}^2(W_{\mathfrak{S}},\mathbb{Q})$ and $\check{H}^2(W_m,\mathbb{Q})$ are isomorphic to the ring $\mathbb{Q}[x]$, $\deg x=2$) that $(\alpha\oplus\alpha)(f^2(a))\neq 0$. In view of commutativity of diagram it follows that $f^{2n}(\alpha(a))\neq 0$. Since f^{2n} is a monomorphism, we have $\alpha(a)\neq 0$. It means that the cohomology multiplication in the ring $\check{H}^2(W_{\mathfrak{F}},\mathbb{Q})$ is nondegenerate, and it easily leads to the establishing of the desired isomorphism of the rings, that is, that $\check{H}^*(W_{\mathfrak{F}},\mathbb{Q})$ is isomorphic to $\mathbb{Q}[x]$, $\deg(x)=2$.

If we show that i^{2q} is a nontrivial linear operator, then Theorem 1 will be proved. Since $i^{2q}(a,b)=(i_1)^{2q}(a)-(i_2)^{2q}(b)$, it suffices to show that $(i_2)^{2q}\neq 0$.

We denote $W_{\leqslant e}$, where e is the unit subgroup, by W_e and consider the embeddings $h:W_e\hookrightarrow W_m$, $g:W_e\hookrightarrow W_{\mathfrak{X}}$ and $i_2:W_{\mathfrak{X}}\hookrightarrow W_m$, which induce the linear operators of the corresponding cohomology rings (which we denote for brevity by the same letters). Let us show that the linear operator $h^{2q}=g^{2q}\circ(i_2)^{2q}:\check{H}^{2q}(W_m,\mathbb{Q})\to\check{H}^{2q}(W_e,\mathbb{Q})$ is an isomorphism and therefore, $(i_2)^{2q}\neq 0$.

Let us consider the linear operator $\kappa:\check{H}^2(W_{\mathfrak{F}_H},\mathbb{Q})\to\check{H}^{2s}(W_{\mathfrak{F}_H},\mathbb{Q})$ defined by the formula $\kappa(a)=\underbrace{1\smile 1\smile\cdots\smile 1}_{s-1}\smile a$.

As the structure of the cohomology ring $\check{H}^*(W_{\mathfrak{F}_H},\mathbb{Q})$ for any simple open family $\mathfrak{F}_H$ is known, the linear operator κ is nonzero and thus, it is an isomorphism. Since the cup product $\smile$ commutes with all homomorphisms h^{2q}, the following diagram is commutative:

$$\begin{tikzcd} \check{H}^2(W_m,\mathbb{Q}) \arrow[r, "h^2"] \arrow[d, "\kappa"] & \check{H}^2(W_e,\mathbb{Q}) \arrow[d, "\kappa"] \\ \check{H}^{2s}(W_m,\mathbb{Q}) \arrow[r, "h^{2s}"] & \check{H}^{2s}(W_e,\mathbb{Q}). \end{tikzcd}$$

Hence, h^2 is a nonzero linear operator if and only if $h^{2s} \neq 0$. Further we investigate the homomorphisms of homotopy groups induced by the embedding of W_e into W_m. For this aim we consider the orbit projections $\pi' : \mathbb{W}_e \to W_e$ and $\pi'' : \mathbb{W}_m/H_m \to \mathbb{W}_m/S^1 = W_m$, which are principal bundles, and define the morphism of these principal bundles

$$\begin{tikzcd} \mathbb{W}_e \arrow[r, "f"] \arrow[d, "\pi'"] & \mathbb{W}_m/H_m \arrow[d, "\pi''"] \\ W_e \arrow[r, hook, "h"] & W_m, \end{tikzcd}$$

defined by the formula $f(a) = H_m \cdot a$. This morphism of the bundles naturally generates the morphism of exact homotopy sequences:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(\mathbb{W}_e) \arrow[r] \arrow[d, "f_{n+1}"] & \pi_{n+1}(W_e) \arrow[r, "\delta"] \arrow[d, "h_{n+1}"] & \pi_n(S^1) \arrow[r] \arrow[d, "p_n"] & \pi_n(\mathbb{W}_e) \arrow[r] \arrow[d, "f_n"] & \cdots \\ \cdots \arrow[r] & \pi_{n+1}(\mathbb{W}_m/H_m) \arrow[r] & \pi_{n+1}(W_m) \arrow[r, "\delta"] & \pi_n(S^1/H_m) \arrow[r] & \pi_n(\mathbb{W}_m/H_m) \arrow[r] & \cdots. \end{tikzcd}$$

It follows by contractibility of the spaces $\mathbb{W}_m/H_m$ and $\mathbb{W}_e$ that only four central groups are nontrivial, so the homomorphism h_{n+1} is isomorphic to the homomorphism p_n, which is generated by the quotient map S^1 into S^1/H_m (for more details see [2]). Thus, $h_2 \neq 0$ and $h^{2s} \neq 0$.

As the homotopy and homology groups of W_m and W_e are known, we make use of the Hurewicz theorem [8, 120] and get the following commutative diagram:

$$\begin{tikzcd} \pi_2(W_e,\mathbb{Z}) \arrow[r, "h_2"] \arrow[d, "\eta"'] & \pi_2(W_m,\mathbb{Z}) \arrow[d, "\eta"] \\ H_2(W_e,\mathbb{Z}) \arrow[r, "h_2"] & H_2(W_m,\mathbb{Z}), \end{tikzcd}$$

where η is the Hurewicz isomorphism. It follows that $h_2 : H_2(W_1,\mathbb{Z}) \to H_2(W_m,\mathbb{Z})$ is a nonzero homomorphism of two-dimensional homology groups.

The well-known universal coefficient theorem for cohomology [8, 128] asserts that the following sequences

$$0 \to \operatorname{Ext}\big(H_{n-1}(C), G\big) \to H^n(\mathcal{C}; G) \xrightarrow{\tau} \operatorname{Hom}\big(H_n(C; G)\big) \to 0$$

are exact. The homomorphism τ is defined as $(\tau(\gamma))(\alpha) = \langle\gamma, \alpha\rangle$, where $\gamma \in H^n(C; G)$, $\alpha \in H_n(C; G)$ and $\langle\cdot,\cdot\rangle$ denotes the pairing of cohomologies and homologies. Since $\mathbb{Q}$ is a ring, the homomorphism $\tau : \check{H}^2(W_m,\mathbb{Q}) \to \operatorname{Hom}(\check{H}_2(W_m,\mathbb{Z}),\mathbb{Q})$ is an isomorphism.

Let $s : \operatorname{Hom}(H_2(W_m,\mathbb{Z}),\mathbb{Q}) \to \operatorname{Hom}(H_2(W_e,\mathbb{Z}),\mathbb{Q})$ be a map dual to h_2. It means that $s(\varphi) = \varphi \circ h_2$ and therefore $s \neq 0$. Taking into account that $(\langle h^2(\gamma), \alpha\rangle = \langle\gamma, h_2(\alpha)\rangle)$ for each $\gamma \in H^n(C; G)$, $\alpha \in H_n(C; G)$, we infer that the following diagram

$$\begin{tikzcd} \check{H}^2(W_m,\mathbb{Q}) \arrow[r, "\tau"] \arrow[d, "h^2"'] & \operatorname{Hom}(\check{H}_2(W_m,\mathbb{Z}),\mathbb{Q}) \arrow[d, "s"] \\ \check{H}^2(W_e,\mathbb{Q}) \arrow[r, "\tau"] & \operatorname{Hom}(\check{H}_2(W_e,\mathbb{Z}),\mathbb{Q}) \end{tikzcd}$$

commutes. It follows by $s \neq 0$ that $h^2 \neq 0$ – the proof of Theorem 1 is completed. □

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