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On the nonstationary sunspot equilibria generated by an unbounded growth model: Comments

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It is a pleasure to be invited to discuss this interesting paper by Professor Shigoka. The paper is concerned with the analysis of long-run growth data. If one looks at such data for several economies, one sees a large variety of growth experience, both in terms of underlying growth rates in GDP, and in terms of cycles around the trend rate.

Suppose one wished to explain these variations. For example, suppose one wished to explain why for over a century the average annual growth rate of GDP in the UK has been approximately 1.7 percent, whereas the same rate for Japan has averaged 2.6 percent. Almost certainly one would begin by looking for fundamental differences in economic factors across the two economies. For example, differences in the marginal rate of substitution of labor for leisure might explain differences in work habits, differences in rates of time preference might explain different savings rates and so on.

Professor Shigoka's paper shows that such 'fundamentalist' explanations of differences in growth behavior may not be necessary. He shows that for some open set of parameter values the dynamics of economic growth are indeterminate, so that for given initial conditions, that is, given tastes, given technology, etc. two economies, say the UK and Japan, can grow at quite different rates. Indeed, for all we know, the growth experience for the whole set of OECD countries may be generated by exactly the same dynamic economic model acting on the same vector of initial conditions. In this case, looking for differences in economic attributes to explain differences in growth would be a waste of time.

This is not to say that in Professor Shigoka's model anything goes. Although the growth rate of the economy is indeterminate, the economy does have several well-determined stochastic features. The model examined is a two-sector model with an externality and exogenous uncertainty (i.e. sunspots). With some delicate mathematical analysis he shows that

1. In this model there exists a continuum of sunspot equilibria with different average growth rates.

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2. The time series of the logarithm of capital and the time series of the logarithm of consumption are co-integrated, both being $I(1)$ processes.

These stochastic properties (2) are testable and are the heart of Professor Shigoka's contribution. I will, however, confine my comments to an earlier part of Professor Shigoka's paper, the part which argues that equilibrium growth is indeterminate. This is such an important claim, changing so fundamentally the way in which we look at growth data, it seems useful to examine it in more detail.

We begin by looking more carefully at the definition of 'indeterminate.' This definition is not from Professor Shigoka's paper, but from an earlier paper to which he refers.

Definition. Indeterminate Equilibrium. Let $\{x_t\}_{t=0}^{\infty}$ *denote an equilibrium for an economy with initial condition* $x_o = k_O$. *We say that it is an* indeterminate equilibrium *if for every e>0 there exists another sequence* $\{y_t\}_{t=0}^{\infty}$ *with* $0 < |y_1 - x_1| < e$ *and,* $y_0 = x_0 = k_0$ *which is also an equilibrium.* (Boldrin and Rustichini (1994 p, 327).

The fundamental dynamic equations (the Euler equations) to which this definition is applied are given in the non-stochastic (Boldrin and Rustichini) model by

$$V_2(x_t, x_{t+1}, x_t) = \delta V_1(x_{t+1}, x_{t+2}, x_{t+1}) = 0 \tag{1}$$

and in Professor Shigoka's sunspot case by

$$a\overline{X}_t^{\eta}(\gamma X_t - aX_{t+1})^{a-1} = \delta\gamma E_t\overline{X}_{t+1}^{\eta}(\gamma X_{t+1} - aX_{t+2})^{a-1} \tag{2}$$

The derivation of Eqs. (1) and (2) is not of importance to us. This, together with the meaning of the various terms, can be found in the original papers. What is of importance is to note that the fundamental dynamics of the state variables (x_i in Eq. (1), X_i in Eq. (2)) are given in each case by a second order difference equation.

Now by the most elementary of arguments, the behavior of second order difference equations is completely determinate once we know the initial conditions. Those initial conditions, however, require that we specify the state of the system not just in period 0, but also in period 1. Looking now at the definition of indeterminacy given above, we see that it amounts to exactly this. If we specify only half of the initial conditions for a dynamic system, the system is indeterminate. On the other hand, the system is fully determinate if we specify the full initial conditions.

For this class of economic models it is thus possible to replace the word 'indeterminate' with the words 'incompletely specified initial conditions.' Of course one may define things any way one pleases, but if the two-sector growth model with externalities is 'indeterminate,' so is all of classical mechanics. When God spoke to Newton, he told him that the laws of mechanics were second order differential equations. Therefore to know the future we must know two initial conditions, position and velocity. Specifying only one, say position, makes classical mechanics, in the sense of the definition above, indeterminate. For example, the behavior of a pendulum around its steady state is indeterminate given that all we know is its initial position.

When God spoke to Ramsey, he also made the laws of growth second order differential equations, or as in Professor Shigoka's discrete case, second order difference equations.

Why? Because at the risk of oversimplification, the dynamic laws consistent with optimal intertemporal savings behavior require that the marginal rate of substitution with respect to the state variables in periods t and $t+1$ equal the marginal rate of substitution with respect to the state variable in periods $t+1$ *and* $t+2$. Hence we require specification of the value of the state variables in two periods to determine the future outcome.

It is very tempting to think of an initial condition as being the state of the system at time $t=0$ arguing that this is what nature chooses, economic agents choosing all future states. However, this need not be the case. What is an initial condition chosen by nature, and what is a variable chosen by the agent depends entirely on how we model choice. For example, if Professor Shigoka replaces his assumption of a Ramsey-type saving model with the Solow–Harrod assumption that saving is proportional to current income, the growth model reduces to a first order difference equation, and the definition given above is appropriate. In a Ramsey-type model, for reasons we discussed above, it is not.

Indeed, it may well be that what is defined here as 'indeterminacy' might better be discussed under the heading of stability. Again the pendulum provides a useful analogy. Around the stable steady state of the pendulum, we may start from any given point x_0 and be sure of arriving at the final rest point for any value of x_1 (velocity) in some open set. On the other hand we may also reverse this argument and say that if we fix x_1 (velocity) we will end up at the same position x_0 for any value of the initial position in an open set. Since we have a stable system, the final steady state is independent of either x_0, x_1, or both.

This interpretation fits the non-stochastic Boldrin–Rustichini model perfectly. It does not quite fit Professor Shigoka's paper because of the assumption of sunspots. But sunspots always produce indeterminacy, independent of the assumptions of two-sector models, externalities and all that. My question is, what new indeterminacies arise if we specify both x_0 and x_1 ?

I have dealt in detail with the problem of indeterminacy because I found it to be the most troubling part of the paper. Professor Shigoka inherited this aspect of his paper from the previous literature, so this should not detract from the fact that he has used some very high-powered mathematics to solve completely a most delicate stochastic growth problem. This solution is likely to be of great interest to specialists in this area.

As to the question of whether or not economists should abandon their standard interpretation of growth data because this class of growth models is indeterminate, I am far from being convinced. These models are only indeterminate if one does not specify the full initial conditions. They are completely determined (up to the sunspot problem) when one does.

References

Boldrin, M. and A. Rustichini, 1994, Growth and indeterminacy in dynamic models with externalities, Econometrica, 62 323–342.