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APPROXIMATION OF PERIODIC ANALYTIC FUNCTIONS BY INTERPOLATION TRIGONOMETRIC POLYNOMIALS

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We obtain asymptotic equalities for the upper bounds of approximations by interpolation trigonometric polynomials on classes of convolutions of periodic functions admitting a regular extension to a fixed strip of the complex plane.

Let C be the space of 2π-periodic continuous functions φ with the norm $\|\varphi\|_C = \max\limits_t |\varphi(t)|$, let L_∞ be the space of 2π-periodic, measurable, essentially bounded functions φ with the norm $\|\varphi\|_\infty = \operatorname*{ess\,sup}\limits_t |\varphi(t)|$, and let $L = L_1$ be the space of 2π-periodic functions summable over the period with the norm

$$\|\varphi\|_L = \|\varphi\|_1 = \int\limits_{-\pi}^{\pi} |\varphi(t)|\, dt.$$

Further, we assume that $f(x)$ is a 2π-periodic function from L,

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

is its Fourier series, $\psi(k)$ is an arbitrary function of natural argument, and β is an arbitrary fixed real number ($\beta \in R$). If the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k \cos\left(kx + \frac{\pi\beta}{2}\right) + b_k \sin\left(kx + \frac{\pi\beta}{2}\right)\right)$$

is the Fourier series of a certain summable function φ, then this function is called (see, e.g., [1, p. 25]) the (ψ, β)-derivative of the function $f(x)$ and is denoted by $f_\beta^\psi(x)$ $(\varphi(x) = f_\beta^\psi(x))$. The set of functions $f(x)$ satisfying this condition is denoted by $f(x)$. If $f \in L_\beta^\psi$ and, simultaneously, $f_\beta^\psi \in \mathfrak{N}$, where $\mathfrak{N}$ is a certain subset of functions from L, then we set $f \in L_\beta^\psi \mathfrak{N}$. If $F_\beta^\psi(x) = f(x)$, then it is natural to call the function $F(\cdot)$ the (ψ, β)-integral of $f(\cdot)$. In this case, we write $F(x) = \mathcal{J}_\beta^\psi(f; x)$.

Further, we set $L_\beta^\psi \cap C = C_\beta^\psi$ and $L_\beta^\psi \mathfrak{N} \cap C = C_\beta^\psi \mathfrak{N}$. In what follows, as the sets $\mathfrak{N}$, we consider the unit balls U_∞^0 in the space L_∞, i.e.,

$$U_\infty^0 = \{\varphi \in L_\infty : \|\varphi\|_\infty \le 1,\ \varphi \perp 1\},$$

or the classes H_ω, namely,

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$$H_\omega = \{\varphi \in C:\ \omega(\varphi, t) \le \omega(t)\},$$

where $\omega(\varphi\,;t)$ is the modulus of continuity of a function φ from C and $\omega(t)$ is a given majorant of the modulus of continuity. In this case, we set $C_\beta^\psi U_\infty^0 = C_{\beta,\infty}^\psi$.

For every fixed $q \in [0, 1)$, we denote by $\mathcal{D}_q$ the set of sequences $\psi(k)$, $k \in N$, for which

$$\lim_{k \to \infty} \frac{\psi(k+1)}{\psi(k)} = q.$$

The principal results of this paper are obtained for the classes $C_\beta^\psi \mathfrak{N}$ defined by sequences $\psi(k)$ from $\mathcal{D}_q$ for certain $q \in [0, 1)$. In this case, as is known, the sets C_β^ψ consist of 2π-periodic functions $f(x)$ admitting a regular extension into the strip $|\operatorname{Im} z| \le \ln \frac{1}{q}$; moreover, at any point x, the following equality holds:

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} f_\beta^\psi(x-t)\,\Psi_\beta(t)\,dt = \frac{a_0}{2} + (f_\beta^\psi * \Psi_\beta)(x), \tag{1}$$

where

$$\Psi_\beta(t) = \sum_{k=1}^{\infty} \psi(k)\cos\left(kt - \frac{\beta\pi}{2}\right)$$

is the kernel of representation (1) (see, e.g., [1, pp. 32, 35]).

Let us introduce additional notation. Let $f(x) \in C$. By $\tilde{S}_n(f\,;x)$, we denote a trigonometric polynomial of degree n that interpolates $f(x)$ at the points $x_k^{(n)} = \frac{2k\pi}{2n+1}$, $k \in Z$, i.e.,

$$\tilde{S}_n(f\,; x_k^{(n)}) = f(x_k^{(n)}), \qquad k \in Z.$$

The space of trigonometric polynomials t_{n-1} whose degree does not exceed $n-1$ is denoted by $\mathcal{T}_{2n-1}$. The quantity

$$E_n(f)_C = \inf_{t_{n-1}} \|f - t_{n-1}\|_C$$

is called the best approximation of f in the metric of the space C by trigonometric polynomials of degree $n-1$.

In the present work, we investigate the quantities $\tilde{\rho}_n(f\,;x) = f(x) - \tilde{S}_{n-1}(f\,;x)$ for $f \in C_\beta^\psi \mathfrak{N}$, where $\mathfrak{N}$ is either U_∞^0 or H_ω, and the quantities

$$\tilde{\mathcal{E}}_n(C_\beta^\psi \mathfrak{N}; x) = \sup_{f \in C_\beta^\psi \mathfrak{N}} |\tilde{\rho}_n(f\,;x)|$$

in order to obtain asymptotic equalities for them in the case where $\psi \in \mathcal{D}_q$, $0 \le q < 1$, and $\beta \in R$.

The principal results of this paper are formulated in the following statements:

Theorem 1. *Let* $q \in \mathfrak{D}_q$, $0 < q < 1$, $\psi(k) > 0$, *and* $\beta \in R$. *If* $f \in C^{\psi}_{\beta} C$, *then*

$$\left|\tilde{\rho}_n(f; x)\right| \leq \psi(n)\left|\sin\frac{2n-1}{2}x\right|\left(\frac{16}{\pi^2}\boldsymbol{K}(q) + O(1)\left(\frac{q}{n(1-q)} + \frac{\varepsilon_n}{(1-q)^2}\right)\right)E_n(f^{\psi}_{\beta})_C \tag{2}$$

for any $n \in N$ *and* $x \in R$. *Moreover, for any* $x \in R$, $n \in N$, *and* $f \in C^{\psi}_{\beta} C$, *there exists a function* $F(t) = F(f; n; x; t)$ *that satisfies the equality*

$$\left|\tilde{\rho}_n(F; x)\right| = \psi(n)\left|\sin\frac{2n-1}{2}x\right|\left(\frac{16}{\pi^2}\boldsymbol{K}(q) + O(1)\frac{\varepsilon_n + 1/n}{(1-q)^2}\right)E_n(F^{\psi}_{\beta})_C \tag{3}$$

and is such that $E_n(F^{\psi}_{\beta})_C = E_n(f^{\psi}_{\beta})_C$. *In relations (2) and (3),*

$$\varepsilon_n = \sup_{k \geq n}\left|\frac{\psi(k+1)}{\psi(k)} - q\right|, \qquad \boldsymbol{K}(q) = \int_0^{\pi/2}\frac{du}{\sqrt{1 - q^2\sin^2 u}},$$

and $O(1)$ *are values uniformly bounded with respect to* x, n, q, β, *and* $f \in C^{\psi}_{\beta} C$.

Theorem 1 shows, in particular, that inequality (2) is asymptotically exact for every $x \in R$ on the entire space $C^{\psi}_{\beta} C$. This inequality is also asymptotically exact on certain important subsets of $C^{\psi}_{\beta} C$. Thus, the following statement is true:

Theorem 2. *Let* $\psi \in \mathfrak{D}_q$, $0 < q < 1$, $\psi(k) > 0$, $\beta \in R$, *and let* $\omega(t)$ *be an arbitrary modulus of continuity. Then the following equalities hold for any* $x \in R$ *as* $n \to \infty$:

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta,\infty}; x) = \left|\sin\frac{2n-1}{2}x\right|\psi(n)\left(\frac{16}{\pi^2}\boldsymbol{K}(q) + O(1)\left(\frac{q}{n(1-q)} + \frac{\varepsilon_n}{(1-q)^2}\right)\right), \tag{4}$$

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta} H_{\omega}; x) = \left|\sin\frac{2n-1}{2}x\right|\psi(n)\left(\frac{8}{\pi^2}e_n(\omega)\boldsymbol{K}(q) + O(1)\frac{\omega(1/n)(\varepsilon_n + 1/n)}{(1-q)^2}\right), \tag{5}$$

where

$$e_n(\omega) = \theta_{\omega}\int_0^{\pi/2}\omega\left(\frac{2t}{n}\right)\sin t\, dt,$$

$\theta_{\omega} \in [1/2, 1]$, $\theta_{\omega} = 1$ *if* $\omega(t)$ *is a convex modulus of continuity,* $\boldsymbol{K}(q)$ *and* ε_n *are the same as in Theorem 1, and* $O(1)$ *are values uniformly bounded with respect to* x, n, q, *and* β.

Note that the asymptotic equalities (4) and (5) for the quantities $\tilde{\mathcal{E}}_n(C^{\psi}_{\beta,\infty}; x)$ and $\tilde{\mathcal{E}}_n(C^{\psi}_{\beta} H_{\omega}; x)$ are interpolation analogs of the asymptotic equalities obtained in Theorems 3 and 5 in [2] (see also [3]) for the values $\mathcal{E}_n(C^{\psi}_{\beta,\infty})_C$ and $\mathcal{E}_n(C^{\psi}_{\beta} H_{\omega})_C$ of the upper bounds of approximation by Fourier sums in the space C on the classes $C^{\psi}_{\beta,\infty}$ and $C^{\psi}_{\beta} H_{\omega}$. Moreover, the following equalities are true:

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta,\infty};x) = 2\left|\sin\frac{2n-1}{2}x\right|\left(\mathcal{E}_n(C^{\psi}_{\beta,\infty})_C + O(1)\,\psi(n)\left(\frac{q}{n(1-q)} + \frac{\varepsilon_n}{(1-q)^2}\right)\right),$$

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta}H_{\omega};x) = 2\left|\sin\frac{2n-1}{2}x\right|\left(\mathcal{E}_n(C^{\psi}_{\beta}H_{\omega})_C + O(1)\frac{\psi(n)\,\omega(1/n)(\varepsilon_n + 1/n)}{(1-q)^2}\right),$$

where the values ε_n and $O(1)$ have the same sense as in Theorem 2.

We also note that, for the known classes W^r_{∞}, the following asymptotic equality was obtained by Nikol'skii [4] in 1941:

$$\tilde{\mathcal{E}}_n(W^r_{\infty};x) = \frac{2K_r}{\pi}\frac{\ln n}{n^r}\left|\sin\frac{2n-1}{2}x\right| + O(1)\,n^{-r},$$

where K_r is the Favard constant,

$$K_r = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(r+1)}}{(2\nu+1)^{r+1}}, \qquad r = 0, 1, \ldots,$$

and $O(1)$ is a value uniformly bounded with respect to x and n.

The Poisson kernels

$$P_{q,\beta}(t) = \sum_{k=1}^{\infty} q^k \cos\left(kt - \frac{\beta\pi}{2}\right), \qquad q \in (0,1), \quad \beta \in R,$$

are an important example of kernels whose coefficients $\psi(k)$ satisfy the condition $\psi \in \mathcal{D}_q$, $0 < q < 1$. In the case where $\psi(k) = q^k$, the classes $C^{\psi}_{\beta}\mathfrak{N}$ are denoted by $C^{q}_{\beta}\mathfrak{N}$ and the corresponding (ψ, β)-derivatives and (ψ, β)-integrals of the function f are denoted by $f^q_{\beta}(\cdot)$ and $\mathcal{J}^q_{\beta}(f;\cdot)$, respectively. Theorem 2 yields the following statement:

Corollary 1. *Let* $0 < q < 1$, $\beta \in R$, *and let* $\omega(t)$ *be an arbitrary modulus of continuity. Then, for any* $x \in R$, *the following asymptotic equalities hold as* $n \to \infty$:

$$\tilde{\mathcal{E}}_n(C^{q}_{\beta,\infty};x) = \left|\sin\frac{2n-1}{2}x\right| q^n\left(\frac{16}{\pi^2}\boldsymbol{K}(q) + O(1)\frac{q}{n(1-q)}\right),$$

$$\tilde{\mathcal{E}}_n(C^{q}_{\beta}H_{\omega};x) = \left|\sin\frac{2n-1}{2}x\right| q^n\left(\frac{8}{\pi^2}e_n(\omega)\boldsymbol{K}(q) + O(1)\frac{\omega(1/n)}{n(1-q)^2}\right),$$

where the quantities $e_n(\omega)$ *and* $\boldsymbol{K}(q)$ *are the same as in Theorem 2 and* $O(1)$ *are values uniformly bounded with respect to* x, n, q, *and* β.

The conditions of Theorem 2 are also satisfied by the coefficients $\psi(k)$ of the biharmonic Poisson kernel

$$B_{q,\beta}(t) = \frac{1}{2} + \sum_{k=1}^{\infty}\left(1 + \frac{1-q^2}{2}k\right)q^k\cos\left(kt - \frac{\beta\pi}{2}\right), \qquad 0 < q < 1, \quad \beta \in R,$$

and the Neumann kernel

$$N_{q,\beta}(t) = \sum_{k=1}^{\infty} \frac{q^k}{k} \cos\left(kt - \frac{\beta\pi}{2}\right), \quad 0 < q < 1, \quad \beta \in R.$$

It is easy to verify that, for the coefficients $\psi(k)$ of the kernels $B_{q,\beta}(t)$ and $N_{q,\beta}(t)$, we have

$$\varepsilon_n = \sup_{k \geq n} \left| \frac{\psi(k+1)}{\psi(k)} - q \right| = \left| \frac{\psi(n+1)}{\psi(n)} - q \right| \leq \frac{q}{n}, \quad n \in N.$$

Therefore, Theorem 2 yields the following statement:

Corollary 2. *Let*

$$\psi(k) = \left(1 + \frac{1-q^2}{2} k\right) q^k, \quad 0 < q < 1, \quad k \in N, \quad \beta \in R,$$

and let $\omega(t)$ *be an arbitrary modulus of continuity. Then*

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta,\infty}; x) = \left|\sin \frac{2n-1}{2} x\right| \left(1 + \frac{1-q^2}{2} n\right) q^n \left(\frac{16}{\pi^2} \boldsymbol{K}(q) + O(1) \frac{q}{n(1-q)^2}\right), \tag{6}$$

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta} H_{\omega}; x) = \left|\sin \frac{2n-1}{2} x\right| \left(1 + \frac{1-q^2}{2} n\right) q^n \left(\frac{8}{\pi^2} e_n(\omega) \boldsymbol{K}(q) + O(1) \frac{\omega(1/n)}{n(1-q)^2}\right) \tag{7}$$

for any $x \in R$ *as* $n \to \infty$; *if* $\psi(k) = \frac{q^k}{k}$, $0 < q < 1$, $k \in N$, *and* $\beta \in R$, *then*

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta,\infty}; x) = \left|\sin \frac{2n-1}{2} x\right| \frac{q^n}{n} \left(\frac{16}{\pi^2} \boldsymbol{K}(q) + O(1) \frac{q}{n(1-q)^2}\right), \tag{8}$$

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta} H_{\omega}; x) = \left|\sin \frac{2n-1}{2} x\right| \frac{q^n}{n} \left(\frac{8}{\pi^2} e_n(\omega) \boldsymbol{K}(q) + O(1) \frac{\omega(1/n)}{n(1-q)^2}\right) \tag{9}$$

for any $x \in R$ *as* $n \to \infty$. *In equalities (6)–(9), the quantities* $e_n(\omega)$ *and* $\boldsymbol{K}(q)$ *have the same sense as in Theorem 2 and* $O(1)$ *are values uniformly bounded with respect to* x, n, q, *and* β.

Theorem 3. *Let* $\psi \in \mathfrak{D}_0$, $\psi(k) > 0$, $\beta \in R$, *and let* $\omega(t)$ *be an arbitrary modulus of continuity. Then, for any* $x \in R$, *the following asymptotic equalities hold as* $n \to \infty$:

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta,\infty}; x) = \left|\sin \frac{2n-1}{2} x\right| \left(\frac{8}{\pi} \psi(n) + O(1) \left(\psi(n+1) \min\left\{\frac{\psi(n+1)}{\psi(n)}, \frac{1}{n}\right\} + \sum_{k=n+2}^{\infty} \psi(k)\right)\right), \tag{10}$$

$$\tilde{\mathcal{E}}_n(C^{\psi}_{\beta} H_{\omega}; x) = \left|\sin \frac{2n-1}{2} x\right| \left(\frac{4}{\pi} \psi(n) e_n(\omega) + O(1) \omega\left(\frac{1}{n}\right) \sum_{k=n+1}^{\infty} \psi(k)\right), \tag{11}$$

where

$$e_n(\omega) = \theta_\omega \int_0^{\pi/2} \omega\left(\frac{2t}{n}\right) \sin t \, dt,$$

$\theta_\omega \in [2/3, 1]$, $\theta_\omega = 1$ *if* $\omega(t)$ *is a convex function, and* $O(1)$ *are values uniformly bounded with respect to all parameters under consideration.*

Note that the conditions $\psi \in \mathcal{D}_0$ and $\psi(k) > 0$ guarantee the validity of the relation

$$\psi(m) = o(1) \sum_{k=m+1}^{\infty} \psi(k).$$

Theorem 3, in fact, completes Theorem 2 in the case $q = 0$.

Prior to the proof of Theorems 1–3, we consider the following lemma, which contains the integral representation of the quantities $\tilde{\rho}_n(f; x)$ on the classes C_β^ψ:

Lemma 1. *Let*

$$\psi(k) > 0, \qquad \sum_{k=1}^{\infty} \psi(k) < \infty,$$

and $\beta \in R$. *Then, for any function* $f \in C_\beta^\psi$, *the following equality holds:*

$$\tilde{\rho}_n(f; x) = \frac{2}{\pi} \sin \frac{2n-1}{2} x \int_{-\pi}^{\pi} \delta_n(t+x) \left(\sum_{\nu=n}^{\infty} \psi(\nu) \cos(\nu t + \gamma_n) + r_n(t) \right) dt \qquad \forall\, x \in R, \tag{12}$$

where $\delta_n(\tau) = f_\beta^\psi(\tau) - t_{n-1}(\tau)$, $t_{n-1}(\cdot)$ *is an arbitrary trigonometric polynomial from* $\mathcal{T}_{2n-1}$, *and the quantities* $r_n(t)$ *and* γ_n *are defined by the equalities*

$$r_n(t) = r_n(\psi; \beta; x; t) = \sum_{k=1}^{\infty} \sum_{\nu=(2k+1)n-k}^{\infty} \psi(\nu) \sin\left(\nu t + \left(k + \frac{1}{2}\right)(2n-1)x + \frac{\pi\beta}{2} \right), \tag{13}$$

$$\gamma_n = \gamma_n(\beta; x) = \frac{(2n-1)x + \pi(\beta-1)}{2}. \tag{14}$$

Proof. The Fourier coefficients a_k and b_k of a function $f \in C_\beta^\psi$ have the form

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \psi(k) \cos\left(kt + \frac{\pi\beta}{2}\right) \varphi(t)\, dt, \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \psi(k) \sin\left(kt + \frac{\pi\beta}{2}\right) \varphi(t)\, dt, \end{aligned} \tag{15}$$

where $\varphi(t) = f_\beta^\psi(t)$. As is known, the coefficients $a_k^{(n)}$ and $b_k^{(n)}$ of the interpolation polynomial

$$\tilde{S}_n(f;x) = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{n} (a_k^{(n)} \cos kx + b_k^{(n)} \sin kx) \tag{16}$$

are expressed in terms of the Fourier coefficients a_k and b_k of the function f according to the following equalities [5, p. 28; 6, p. 213]:

$$a_k^{(n)} = a_k + \sum_{m=1}^{\infty} (a_{m(2n+1)+k} + a_{m(2n+1)-k}), \quad k = 0, 1, \ldots, n, \tag{17}$$

$$b_k^{(n)} = b_k + \sum_{m=1}^{\infty} (b_{m(2n+1)+k} - b_{m(2n+1)-k}), \quad k = 1, 2, \ldots, n. \tag{17$'$}$$

Combining formulas (15)–(17′) and setting

$$\sigma_k = \sigma_k^{(n)} = (2k-1)n + k \quad \text{and} \quad \alpha_k = \alpha_k^{(n)}(\beta; x) = k(2n+1)x + \frac{\beta\pi}{2}, \quad k = 1, 2, \ldots, \quad \forall f \in C_\beta^\psi$$

we obtain

$$f(x) - \tilde{S}_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t+x) \sum_{k=1}^{\infty} \sum_{\nu=\sigma_k}^{\sigma_{k+1}-1} \psi(\nu)\left[\cos\left(\nu t + \frac{\pi\beta}{2}\right) - \cos(\nu t + \alpha_k)\right] dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t+x) \sum_{k=1}^{\infty} \left\{ \sum_{\nu=\sigma_k}^{\infty} \psi(\nu)\left[\cos\left(\nu t + \frac{\pi\beta}{2}\right) - \cos(\nu t + \alpha_k)\right] \right.$$

$$\left. - \sum_{\nu=\sigma_{k+1}}^{\infty} \psi(\nu)\left[\cos\left(\nu t + \frac{\pi\beta}{2}\right) - \cos(\nu t + \alpha_k)\right] \right\} dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t+x) \left(\sum_{\nu=\sigma_1}^{\infty} \psi(\nu)\left[\cos\left(\nu t + \frac{\pi\beta}{2}\right) - \cos(\nu t + \alpha_1)\right] \right.$$

$$+ \sum_{k=1}^{\infty} \left\{ \sum_{\nu=\sigma_{k+1}}^{\infty} \psi(\nu)\left[\cos\left(\nu t + \frac{\pi\beta}{2}\right) - \cos(\nu t + \alpha_{k+1})\right] \right.$$

$$\left.\left. - \sum_{\nu=\sigma_{k+1}}^{\infty} \psi(\nu)\left[\cos\left(\nu t + \frac{\pi\beta}{2}\right) - \cos(\nu t + \alpha_k)\right] \right\} \right) dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t+x) \sum_{k=0}^{\infty} \sum_{\nu=\sigma_{k+1}}^{\infty} \psi(\nu)\left[\cos(\nu t + \alpha_k) - \cos(\nu t + \alpha_{k+1})\right] dt$$

$$= \frac{2}{\pi} \sin\frac{2n+1}{2}x \int_{-\pi}^{\pi} \varphi(t+x) \sum_{k=0}^{\infty} \sum_{\nu=\sigma_{k+1}}^{\infty} \psi(\nu) \sin\left(\nu t + \left(k + \frac{1}{2}\right)(2n+1)x + \frac{\pi\beta}{2}\right) dt. \tag{18}$$

Replacing $n-1$ by n in (18) and taking into account that $\sigma_{k+1}^{(n-1)} = (2k+1)n - k$, we get

$$\tilde{\rho}_n(f;x) = \frac{2}{\pi}\sin\frac{2n-1}{2}x\int_{-\pi}^{\pi} f_\beta^\psi(t+x)\left(\sum_{\nu=n}^{\infty}\psi(\nu)\cos(\nu t+\gamma_n) + r_n(t)\right)dt, \tag{19}$$

where $r_n(t)$ and γ_n are defined by equalities (13) and (14). The functions $\sum_{\nu=n}^{\infty}\psi(\nu)\cos(\nu t+\gamma_n)$ and $r_n(t)$ are orthogonal to any trigonometric polynomial $t_{n-1}\in\mathcal{T}_{2n-1}$ and, therefore, $f_\beta^\psi(u)$ in (19) can be replaced by $\delta_n(u)$. Then relation (19) yields (12) for any function $f\in C_\beta^\psi$.

Proof of Theorem 1. Let $\psi\in\mathcal{D}_q$, $0<q<1$. Then, by virtue of (13), we have

$$r_n(t) \le \sum_{k=1}^{\infty}\sum_{\nu=(2k+1)n-k}^{\infty}\psi(\nu) \tag{20}$$

and

$$\sum_{\nu=(2k+1)n-k}^{\infty}\psi(\nu) = \psi(n)\prod_{i=0}^{(2n-1)k-1}\frac{\psi(n+i+1)}{\psi(n+i)}\left(1+\sum_{m=1}^{\infty}\prod_{j=0}^{m-1}\frac{\psi((2k+1)n-k+j+1)}{\psi((2k+1)n-k+j)}\right)$$

$$\le \psi(n)\prod_{i=0}^{(2n-1)k-1}(q+\varepsilon_{n+i})\left(1+\sum_{m=1}^{\infty}\prod_{j=0}^{m-1}(q+\varepsilon_{(2k+1)n-k+j})\right) \le \psi(n)\frac{(q+\varepsilon_n)^{(2n-1)k}}{1-q-\varepsilon_{3n-1}}.$$

Therefore,

$$|r_n(t)| \le \frac{\psi(n)}{1-q-\varepsilon_{3n-1}}\sum_{k=1}^{\infty}(q+\varepsilon_n)^{(2n-1)k} = \frac{\psi(n)(q+\varepsilon_n)^{2n-1}}{(1-q-\varepsilon_{3n-1})(1-(q+\varepsilon_n)^{2n-1})} = o(1)\frac{q\,\psi(n)}{(1-q)n}. \tag{21}$$

By virtue of Lemma 1 in [3], we have

$$\sum_{\nu=n}^{\infty}\psi(\nu)\cos(\nu t+\gamma_n) = \psi(n)\left(q^{-n}\sum_{k=n}^{\infty}q^k\cos(kt+\gamma_n) + \bar{r}_n(t)\right), \tag{22}$$

where

$$\bar{r}_n(t) = \bar{r}_n(\psi,\gamma_n,t) \overset{\text{df}}{=} \sum_{i=1}^{\infty}\left(\prod_{l=0}^{i-1}\frac{\psi(n+l+1)}{\psi(n+l)} - q^i\right)\cos((n+i)t+\gamma_n);$$

in this case, the following inequality holds for the quantity $\bar{r}_n(t)$ beginning with a certain number n_0:

$$|\bar{r}_n(t)| \le \frac{\varepsilon_n}{(1-q-\varepsilon_n)(1-q)}, \qquad \varepsilon_n = \sup_{k\ge n}\left|\frac{\psi(k+1)}{\psi(k)} - q\right|. \tag{23}$$

Combining relations (12) and (22), as well as (21) and (23), we obtain

$$\tilde{\rho}_n(f;x) = \frac{2}{\pi}\sin\frac{2n-1}{2}x\,\psi(n)\int_{-\pi}^{\pi}\delta(t+x)\left(q^{-n}\sum_{\nu=n}^{\infty}q^{\nu}\cos(\nu t+\gamma_n) + O(1)\left(\frac{\varepsilon_n}{(1-q)^2}+\frac{q}{n(1-q)}\right)\right)dt. \quad (24)$$

Choosing as $t_{n-1}(\cdot)$ in (24) the polynomial of the best approximation of the function $f_\beta^\psi(\cdot)$ in the space C, we get

$$\left|\tilde{\rho}_n(f;x)\right| \le \frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right|\psi(n)\left(q^{-n}\int_{-\pi}^{\pi}\left|\sum_{\nu=n}^{\infty}q^{\nu}\cos(\nu t+\gamma_n)\right|dt\right.$$

$$\left. + O(1)\left(\frac{\varepsilon_n}{(1-q)^2}+\frac{q}{n(1-q)}\right)\right)E_n(f_\beta^\psi)_C. \quad (25)$$

In [7, Sec. 3], Stechkin showed that, for any $0<q<1$ and $\alpha\in R$, the following equality holds:

$$\int_{-\pi}^{\pi}\left|\sum_{\nu=n}^{\infty}q^{\nu}\cos(\nu t+\alpha)\right|dt = q^n\left(\frac{8}{\pi}\boldsymbol{K}(q) + O(1)\frac{q}{n(1-q)}\right), \quad (26)$$

where $O(1)$ is a value uniformly bounded with respect to n, q, and α. Substituting this equality with $\alpha = \gamma_n$ in relation (25), we obtain relation (2).

Let us prove the second part of Theorem 1. By virtue of relations (19) and (21)–(23) and the orthogonality of the function $r_n(t)$ to any polynomial $t_{n-1}\in\mathcal{T}_{2n-1}$, for any $f\in C_\beta^\psi$ we get

$$\tilde{\rho}_n(f;x) = \frac{2}{\pi}\sin\frac{2n-1}{2}x\,\psi(n)\left(q^{-n}\int_{-\pi}^{\pi}f_\beta^\psi(t+x)\sum_{\nu=n}^{\infty}q^{\nu}\cos(\nu t+\gamma_n)\,dt\right.$$

$$\left. + O(1)\left(\frac{\varepsilon_n}{(1-q)^2}+\frac{q}{n(1-q)}\right)E_n(f_\beta^\psi)_C\right). \quad (27)$$

Note that, for every $x\in R$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f_\beta^\psi(t+x)\sum_{\nu=n}^{\infty}q^{\nu}\cos(\nu t+\gamma_n)\,dt = \rho_n(g_x;x),$$

where $\rho_n(f) = \rho_n(f;x) \overset{\text{df}}{=} f(x) - S_{n-1}(f;x)$, $S_n(f) = S_n(f;x)$ are the Fourier partial sums of order n of the function f from L, and

$$g_x(\cdot) \overset{\text{df}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi}f_\beta^\psi(t+\cdot)\sum_{\nu=1}^{\infty}\psi(\nu)\cos(\nu t+\gamma_n)\,dt.$$

Furthermore, according to Theorem 2 in [2], for every $n\in N$, for the function $g_x(\cdot)$ one can find a function $\overline{\varphi}(t) = \overline{\varphi}(n;x;t)$ that satisfies the equalities $E_n(\overline{\varphi})_C = E_n(f_\beta^\psi)_C$ and

$$\|\rho_n(G)\|_C = \left(\frac{8q^n}{\pi^2}\boldsymbol{K}(q) + O(1)\frac{q^n}{(1-q)^2 n}\right)E_n(f_\beta^\psi)_C, \tag{28}$$

where

$$G(\tau) \overset{\text{df}}{=} \mathcal{J}^q_{2\gamma_n/\pi}(\overline{\varphi};\tau) = \frac{1}{\pi}\int_{-\pi}^{\pi}\overline{\varphi}(\tau+t)\sum_{\nu=1}^{\infty}q^\nu\cos(\nu t+\gamma_n)\,dt,$$

and $O(1)$ is a value uniformly bounded with respect to n, q, and γ_n.

Assume that a point x_0 is such that

$$|\rho_n(G;x_0)| = \|\rho_n(G)\|_C. \tag{29}$$

Then $F(t) \overset{\text{df}}{=} \mathcal{J}^\psi_\beta(\overline{\varphi}(t-x+x_0))$ is the function required. Indeed, since $F^\psi_\beta(t) = \overline{\varphi}(t-x+x_0)$, we have $E_n(F^\psi_\beta)_C = E_n(f^\psi_\beta)_C$ and, according to formulas (27)–(29), for given x and n we get

$$\begin{aligned}|\tilde{\rho}_n(F;x)| &= \frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right|\psi(n)\Bigg(q^{-n}\left|\int_{-\pi}^{\pi}\overline{\varphi}(t+x_0)\sum_{\nu=n}^{\infty}q^\nu\cos(\nu t+\gamma_n)\,dt\right| \\ &\qquad + O(1)\left(\frac{\varepsilon_n}{(1-q)^2} + \frac{q}{n(1-q)}\right)E_n(F^\psi_\beta)_C\Bigg) \\ &= 2\left|\sin\frac{2n-1}{2}x\right|\psi(n)\left(q^{-n}\|\rho_n(G)\|_C + O(1)\frac{\varepsilon_n + 1/n}{(1-q)^2}E_n(f^\psi_\beta)_C\right) \\ &= 2\left|\sin\frac{2n-1}{2}x\right|\psi(n)\left(\frac{8}{\pi^2}\boldsymbol{K}(q) + O(1)\frac{\varepsilon_n + 1/n}{(1-q)^2}\right)E_n(f^\psi_\beta)_C.\end{aligned}$$

Theorem 1 is proved.

Proof of Theorem 2. Considering the upper bounds of the absolute values of both sides of equality (24) for given x and $t_{n-1} \equiv 0$ with respect to the class $C^\psi_{\beta,\infty}$ and taking into account the invariance of the set U^0_∞ under translation of its argument, we obtain

$$\begin{aligned}\tilde{\mathcal{E}}_n(C^\psi_{\beta,\infty};x) &= \frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right|\psi(n)\Bigg(q^{-n}\sup_{\varphi\in U^0_\infty}\int_{-\pi}^{\pi}\varphi(t)\sum_{\nu=n}^{\infty}q^\nu\cos(\nu t+\gamma_n)\,dt \\ &\qquad + O(1)\left(\frac{q}{n(1-q)} + \frac{\varepsilon_n}{(1-q)^2}\right)\Bigg).\end{aligned} \tag{30}$$

Taking into account that

$$\sup_{\varphi \in U_\infty^0} \left| \int_{-\pi}^{\pi} \varphi(t) \sum_{\nu=n}^{\infty} q^\nu \cos(\nu t + \gamma_n)\, dt \right| = \int_{-\pi}^{\pi} \left| \sum_{\nu=n}^{\infty} q^\nu \cos(\nu t + \gamma_n) \right| dt + O(1) \frac{q}{n(1-q)}$$

(see, e.g., [7, pp. 137–141]), substituting this equality in formula (30), and using relation (26) (with $\alpha = \gamma_n$), we get

$$\tilde{\mathscr{E}}_n(C^\psi_{\beta,\infty}; x) = \frac{2}{\pi} \left| \sin \frac{2n-1}{2} x \right| \psi(n) \left(q^{-n} \int_{-\pi}^{\pi} \left| \sum_{\nu=n}^{\infty} q^\nu \cos(\nu t + \gamma_n) \right| dt + O(1) \left(\frac{q}{n(1-q)} + \frac{\varepsilon_n}{(1-q)^2} \right) \right)$$

$$= \left| \sin \frac{2n-1}{2} x \right| \psi(n) \left(\frac{16}{\pi^2} \boldsymbol{K}(q) + O(1) \left(\frac{q}{n(1-q)} + \frac{\varepsilon_n}{(1-q)^2} \right) \right).$$

Relation (4) is proved.

Similarly, considering the upper bounds of the absolute values of both sides of equality (12) with respect to the class $C^\psi_\beta H_\omega$ for any fixed x and taking into account the invariance of the set H_ω under translation of its argument, equality (22), and estimate (23), we obtain

$$\tilde{\mathscr{E}}_n(C^\psi_\beta H_\omega; x) = \frac{2}{\pi} \left| \sin \frac{2n-1}{2} x \right| \sup_{\varphi \in H_\omega} \left| \int_{-\pi}^{\pi} \varphi(t)\, \psi(n)\, q^{-n} \sum_{k=n}^{\infty} q^k \cos(kt + \gamma_n)\, dt + R_n(\varphi) \right|, \tag{31}$$

where

$$R_n(\varphi) = R_n(\varphi; x) \overset{\text{df}}{=} \int_{-\pi}^{\pi} \delta^*(t) \left(\psi(n) \bar{r}_n(t) + r_n(t) \right) dt,$$

$$\delta^*_n(\tau) = \varphi(\tau) - t^*_{n-1}(\tau),$$

and $t^*_{n-1}(\cdot)$ is the polynomial of the best approximation of the function φ in the space C. Therefore, by using estimates (21) and (23), we get

$$\left| R_n(\varphi) \right| \le 2\pi \left\| \delta^*_n(\cdot) \right\|_C \left\| \psi(n) \bar{r}_n(\cdot) + r_n(\cdot)) \right\|_C = O(1)\, \psi(n) \left(\frac{q}{(1-q)n} + \frac{\varepsilon_n}{(1-q)^2} \right) E_n(\varphi)_C. \tag{32}$$

The Jackson inequality in the space C

$$E_n(\varphi)_C \le K \omega\left(\varphi; \frac{1}{n} \right) \qquad \forall\, \varphi \in C, \quad \forall\, n \in N, \tag{33}$$

where K is a certain absolute constant (see, e.g., [1, p. 227]), and estimate (32) yield

$$\sup_{\varphi \in H_\omega} \left| R_n(\varphi) \right| = O(1)\, \psi(n)\, \omega\left(\frac{1}{n} \right) \left(\frac{q}{(1-q)n} + \frac{\varepsilon_n}{(1-q)^2} \right). \tag{34}$$

It follows from relations (31) and (34) that

$$\tilde{\mathscr{E}}_n(C^\psi_\beta H_\omega; x) = \frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right|\psi(n)\left(q^{-n}\sup_{\varphi\in H_\omega}\left|\int_{-\pi}^{\pi}\varphi(t)\sum_{k=n}^{\infty}q^k\cos(kt+\gamma_n)\,dt\right|\right.$$

$$\left. + O(1)\,\omega\left(\frac{1}{n}\right)\left(\frac{q}{(1-q)n}+\frac{\varepsilon_n}{(1-q)^2}\right)\right). \tag{35}$$

In Theorem 1 in [9], the following asymptotic equality was established for any $q \in (0, 1)$, $\beta \in R$, and any modulus of continuity $\omega(t)$:

$$\sup_{\varphi\in H_\omega}\left|\int_{-\pi}^{\pi}\varphi(t)\sum_{k=n}^{\infty}q^k\cos\left(kt+\frac{\beta\pi}{2}\right)dt\right| = \frac{4}{\pi}q^n\boldsymbol{K}(q)\,e_n(\omega) + O(1)\frac{q^n\,\omega(1/n)}{(1-q)^2\,n}, \tag{36}$$

where

$$e_n(\omega) = \theta_\omega\int_0^{\pi/2}\omega\left(\frac{2t}{n}\right)\sin t\,dt,$$

$\theta_\omega \in [1/2, 1]$, $\theta_\omega = 1$ if $\omega(t)$ is a convex modulus of continuity, and $O(1)$ is a value uniformly bounded with respect to n, q, and β. In equality (36), we set γ_n instead of $\beta\pi/2$ (the possibility of this substitution follows from the uniform boundedness of the value $O(1)$ in (36) with respect to the parameters n and β). Comparing the equality obtained and representation (35), we get equality (5). Theorem 2 is proved.

Proof of Theorem 3. Let $\psi \in \mathscr{D}_0$, $\psi(k) > 0$, and $\beta \in R$. Considering the upper bounds of the absolute values of both sides of equality (12) with respect to the classes $C^\psi_{\beta,\infty}$ and $C^\psi_\beta H_\omega$ and taking into account estimate (20), we get

$$\tilde{\mathscr{E}}_n(C^\psi_{\beta,\infty}; x) = \frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right|\left(\sup_{\varphi\in U^0_\infty}\left|\int_{-\pi}^{\pi}\varphi(t)\sum_{k=n}^{\infty}\psi(k)\cos(kt+\gamma_n)\,dt\right|\right.$$

$$\left. + O(1)\sum_{k=1}^{\infty}\sum_{\nu=(2k+1)n-k}^{\infty}\psi(\nu)\right), \tag{37}$$

$$\tilde{\mathscr{E}}_n(C^\psi_\beta H_\omega; x) = \frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right|\sup_{\varphi\in H_\omega}\left|\int_{-\pi}^{\pi}\varphi(t)\sum_{k=n}^{\infty}\psi(k)\cos(kt+\gamma_n)\,dt + R^*_n(\varphi)\right|, \tag{38}$$

where

$$R^*_n(\varphi) = R^*_n(\varphi; x) = \int_{-\pi}^{\pi}\delta^*_n(t)\,r_n(t)\,dt, \qquad \delta^*_n(\tau) = \varphi(\tau) - t^*_{n-1}(\tau),$$

$t^*_{n-1}(\cdot)$ is the polynomial of the best approximation of the function φ in the space C, and $O(1)$ is a value uniformly bounded with respect to all parameters under consideration. According to Theorems 2 and 3 in [10, pp. 512–513], we have

$$\sup_{\varphi \in U_\infty^0} \frac{1}{\pi} \left| \int_{-\pi}^{\pi} \varphi(t) \sum_{k=n}^{\infty} \psi(k) \cos(kt + \alpha_k)\, dt \right|$$

$$= \frac{4}{\pi} \psi(n) + O(1) \left(\psi(n+1) \min \left\{ \frac{\psi(n+1)}{\psi(n)}; \frac{1}{n} \right\} + \sum_{k=n+2}^{\infty} \psi(k) \right), \qquad (39)$$

where $\{\alpha_k\}$ is an arbitrary sequence of real numbers and the value $O(1)$ has the same sense as in equality (37). Setting $\alpha_k = \gamma_n$, $k = n, n+1, \ldots,$ in (39), comparing the equality obtained with representation (37), and taking into account that, for any $\psi \in \mathfrak{D}_0$ and sufficiently large n, we have

$$\sum_{k=1}^{\infty} \sum_{\nu=(2k+1)n-k}^{\infty} \psi(\nu) \le \frac{1}{1 - \varepsilon_{3n-1}} \sum_{k=1}^{\infty} \psi\big((2k+1)n - k\big), \qquad (40)$$

where

$$\varepsilon_n = \sup_{k \ge n} \left| \frac{\psi(k+1)}{\psi(k)} \right|,$$

we obtain formula (10).

By using estimates (20) and (33) and inequality (40), we get

$$\left| R_n^*(\varphi) \right| \le 2\pi \left\| \delta_n^* \right\|_C \left\| r_n \right\|_C = O(1) \sum_{k=3n-1}^{\infty} \psi(k) E_n(\varphi)_C = O(1)\, \omega\left(\frac{1}{n}\right) \sum_{k=3n-1}^{\infty} \psi(k). \qquad (41)$$

By virtue of Theorem 7 in [11], we have

$$\sup_{\varphi \in H_\omega} \frac{1}{\pi} \left| \int_{-\pi}^{\pi} \varphi(t) \sum_{k=n}^{\infty} \psi(k) \cos(kt + \beta_k)\, dt \right| = \frac{2\theta_\omega}{\pi} \psi(n) \int_0^{\pi/2} \omega\left(\frac{2t}{n}\right) \sin t\, dt + O(1)\, \omega\left(\frac{1}{n}\right) \sum_{k=n+1}^{\infty} \psi(k), \quad (42)$$

where β_k is an arbitrary sequence of real numbers, $\theta_\omega \in [2/3, 1]$, $\theta_\omega = 1$ if $\omega(t)$ is a convex function, and $O(1)$ is a value uniformly bounded with respect to all parameters under consideration.

Setting $\beta_k = \gamma_k$, $k = n, n+1, \ldots,$ in (42) and comparing the equality obtained with representation (38) and estimate (41), we get equality (11). Theorem 3 is proved.

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