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On some less familiar properties of anharmonic oscillators

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By considering asymmetric anharmonic oscillators with polynomial potentials of degrees 4, 6, and 8, it is shown that the periods and actions in different potential wells are closely related. The Einstein–Brillouin–Keller (EBK) semiclassical energies are compared against the corresponding quantum energies. It is shown that when all the EBK quantum levels are taken into account, the counting of the levels follow a pattern that is analogous to that for a single minimum oscillator.

I. INTRODUCTION

The continued interests in anharmonic oscillators devolve on their use as physical and mathematical models. In the quantum theory, the literature[1] on anharmonic oscillators is very extensive. The bulk of the quantum studies[2–5] involves calculations of energy levels and resummation of Rayleigh–Schrödinger perturbation series. In recent years, the Birkhoff–Gustavson normal form approach[6–10] has provided ways to compute accurately the semiclassical energies of anharmonic oscillators with polynomial potential of degree 4. In the classical study, the periods have been expressed in terms of single and multiple hypergeometric functions.[11] Although these formal results have their own merits (and limitations), some interesting features of the periods and semiclassical quantum numbers of anharmonic oscillators seem to have escaped proper attention. In this note, we study these features by considering anharmonic oscillators with polynomial potentials of degrees 4, 6, and 8.

II. THE HAMILTONIAN

The Hamiltonian under consideration is defined by the relation

$$H=\tfrac{1}{2}p^2+V(x), \tag{1}$$

where

$$V(x)=\frac{1}{2}x^2+\sum_{n=3}^{M}C_nx^n.$$

The values of the coefficients in $V(x)$ are chosen to obtain the desired number and type of potential wells. For our study, we used the following representative set of values for the $C_n's$:

Case 1: $M=4$; $C_3=0.075$, $C_4=0.0025$. The plot of the potential as a function of x is given in Fig. 1.

Case 2: $M=6$; $C_3=0.027\,353\,189\,8$, $C_4=0.564\,810\,071(-3)$, $C_5=0.508\,132\,747\,9(-5)$, $C_6=0.166\,666\,67(-7)$. The numbers in parentheses indicate powers of ten and a plot of the potential is given in Fig. 2.

Case 3: $M=8$; $C_3=0.007\,986\,118\,4$, $C_4=-0.492\,002\,39(-3)$, $C_5=0.101\,036\,297\,3(-4)$, $C_6=0.696\,618\,14(-7)$, $C_7=0.243\,378\,038(-8)$, $C_8=0.125(-10)$. The plot of the potential is given in Fig. 3.

III. THE PERIOD AND ACTION

The period of oscillation of a particle in a potential well between the turning points $x1$ and $x2$ is given by

$$T=\sqrt{2}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E-V(x)}}, \tag{2}$$

where $E=H$ is the sum of kinetic and potential energy. The action I is defined by the integral

$$I=\frac{\sqrt{2}}{\pi}\int_{x_1}^{x_2}\sqrt{E-V(x)} \tag{3}$$

which gives

$$\frac{dI}{dE}=\frac{T}{2\pi}. \tag{4}$$

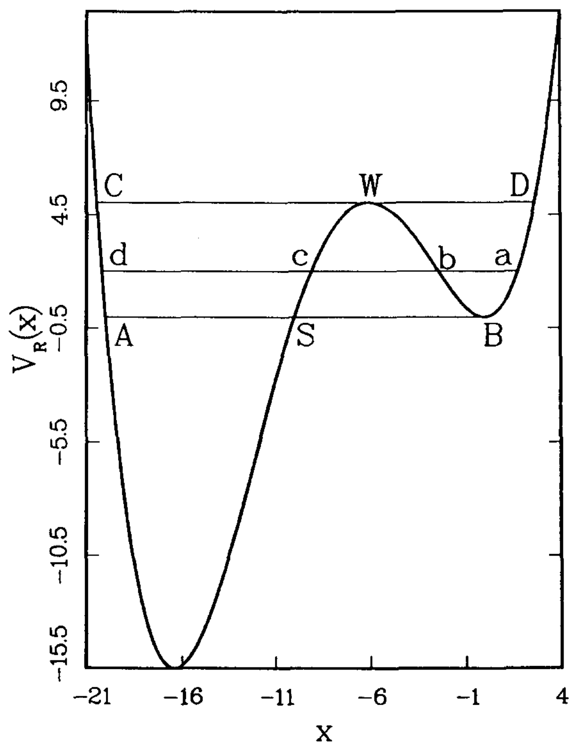

FIG. 1. The potential $V(x)=(1/2)x^2+C_3x^3+C_4x^4$ with $C_3=0.075$ and $C_4=0.0025$.

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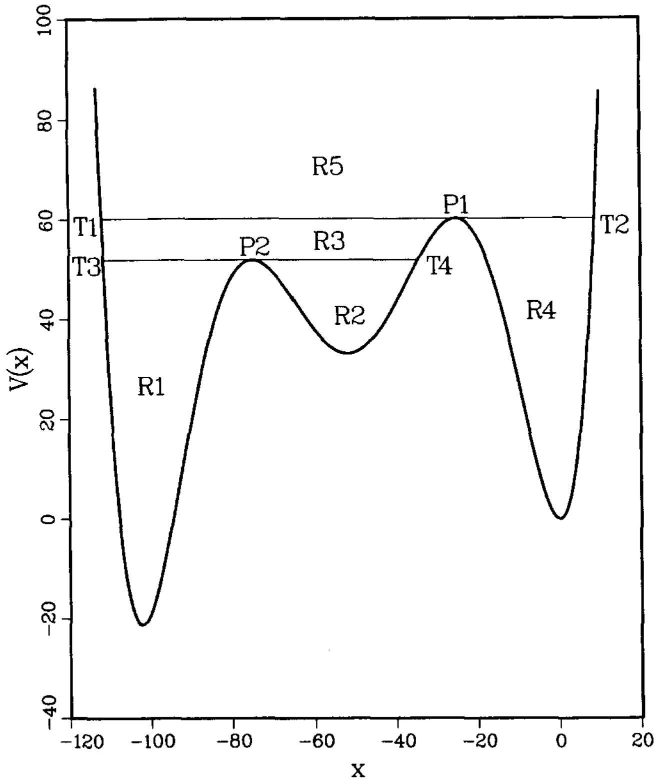

FIG. 2. The potential $V(x)$ for $M = 6$ as a function of x. The coefficients are given in the text and the wells 1, 2, and 3 are counted from the left. With regard to the EBK quantum numbers, the periods and the actions, the regions R_j ($j = 1,2,...,5$) are distinct.

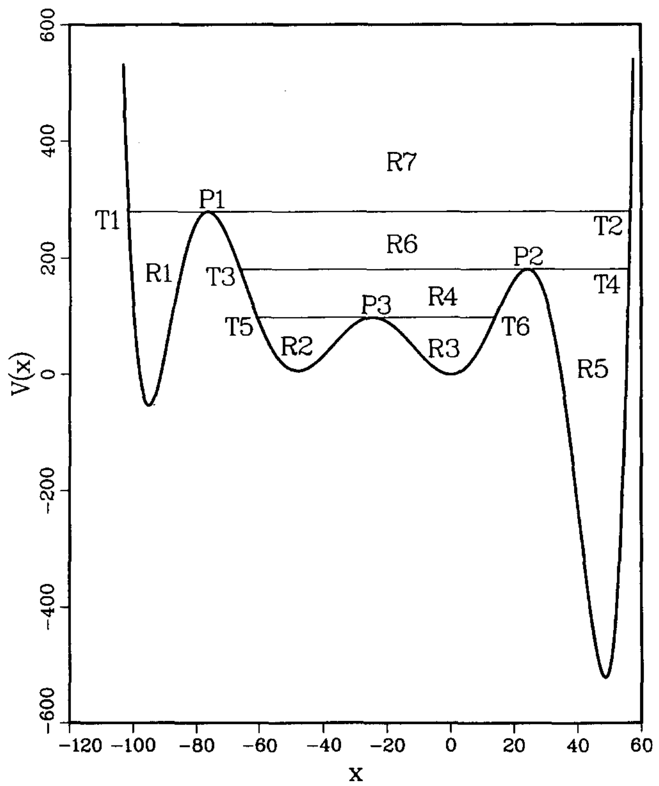

FIG. 3. The potential $V(x)$ for $M = 8$ as a function of x for the coefficients given in the text. Considering the EBK quantum numbers, the periods and the actions, the regions R_j ($j = 1,2,...,7$) are distinct.

It has recently been shown[11] that the period T can be expressed in terms of a sum of multiple hypergeometric series which reduces to a single term when the number of odd monomials in $V(x)$ is less than or equal to one. Our purpose here is to examine if the periods in different wells are related to one another. For this purpose, it is convenient and adequate to compute the periods numerically. The equations of motion for the Hamiltonian in Eq. (1) are given by

$$\frac{dx}{dt} = p, \frac{dp}{dt} = -\frac{dV}{dx}. \tag{5}$$

Equations (5) are solved numerically using the initial conditions $(x,p) = (x_1,0)$. The time required for the system to evolve once to the phase space point $(x_2,0)$ is half the period.

In Figs. 1–3, the potential wells are asymmetric and it is not *a priori* obvious why the period in one well will be related to that in another well. To our surprise, we have observed that in the "overlapping" regions (in an overlapping region the energy E of a classical particle is such that it is energetically capable of oscillating inside any one of the $M/2$ wells) the periods are closely related. In our earlier work[12] we showed for the case $M = 4$ that the periods in the left and right wells in the overlapping region between the lines ASB and CWD of Fig. (1) *are equal* and are given by

$$T = \mu \frac{\pi}{2} {}_2F_1(1/2,1/2,1,m), \mu = \frac{4}{\sqrt{2C_4(a-c)(b-d)}},$$

$$m = \frac{(a-b)(c-d)}{(a-c)(b-d)},$$

where ${}_2F_1(1/2,1/2,1,m)$ is a hypergeometric function.[13] For the case of a general polynomial potential of degree higher than 4, the calculations of periods involve hyperelliptic integrals.[11] The algebra in the analytical proof of the relationships between the periods for the cases $M = 6$ and $M = 8$ is rather complicated. In this work, we present only numerical results in support of the fact that the periods are related. The periods in the overlapping regions for the cases $M = 6$ and $M = 8$ are given in Tables I and II, respectively. It can be seen from Table I that the sum of periods in the first and third well is equal to the period in the second well. From Table II it is seen that the sum of periods in the first and third well is equal to the sum of periods in the second and fourth well. Outside the overlapping regions no such relationships are found to exist. Since the derivative of the action with respect to the energy is proportional to the period Eq. (4), the difference in the action for any two energies behave like the periods. Tables III(A) and III(B) contain actions in the overlapping regions as a function of energy for the $M = 6$ and $M = 8$ potentials, respectively. By taking differences in the actions for any two energies, it is seen that these differences behave like the periods. For example consider the energies 33.172 164 23 and 35 in Table III(A). The differences in the actions in the three wells are, respectively, 2.411 159 702, 4.618 354 77, and 2.207 195 068. Now adding the action differences in wells 1 and 3 gives 4.618 354 77(2.411 159 702 + 2.207 195 068) which is the action difference in the second well. For the $M = 8$ potential, if one adds the action differences in wells 1 and 3, one finds

TABLE I. Periods (in the overlapping region) in the three wells of the $M = 6$ potential as a function of the energy. For any given value of the energy in this region, the sum of the periods in wells 1 and 3 (entries in column 5) equals the period in well 2.

Energy	Period well 1	Period well 2	Period well 3	Sum 1 and 3
$0.331\,721\,642\,3\times10^2$	$0.820\,100\,050\,7\times10^1$	$0.157\,317\,107\,5\times10^2$	$0.753\,071\,024\,5\times10^1$	$0.157\,317\,107\,5\times10^2$
$0.350\,000\,000\,0\times10^2$	$0.837\,905\,406\,6\times10^1$	$0.160\,242\,813\,3\times10^2$	$0.764\,522\,726\,8\times10^1$	$0.160\,242\,813\,3\times10^2$
$0.380\,000\,000\,0\times10^2$	$0.872\,247\,947\,2\times10^1$	$0.165\,770\,328\,4\times10^2$	$0.785\,455\,336\,6\times10^1$	$0.165\,770\,328\,4\times10^2$
$0.420\,000\,000\,0\times10^2$	$0.933\,220\,944\,8\times10^1$	$0.175\,199\,315\,1\times10^2$	$0.818\,772\,206\,5\times10^1$	$0.175\,199\,315\,1\times10^2$
$0.450\,000\,000\,0\times10^2$	$0.999\,758\,268\,0\times10^1$	$0.184\,924\,482\,8\times10^2$	$0.849\,486\,560\,1\times10^1$	$0.184\,924\,482\,8\times10^2$
$0.470\,000\,000\,0\times10^2$	$0.106\,422\,840\,5\times10^2$	$0.193\,808\,897\,6\times10^2$	$0.873\,860\,571\,7\times10^1$	$0.193\,808\,897\,6\times10^2$
$0.490\,000\,000\,0\times10^2$	$0.116\,519\,724\,1\times10^2$	$0.206\,777\,658\,0\times10^2$	$0.902\,579\,339\,2\times10^1$	$0.206\,777\,658\,0\times10^2$
$0.518\,913\,991\,1\times10^2$	$0.232\,074\,363\,6\times10^2$	$0.327\,636\,869\,2\times10^2$	$0.955\,625\,056\,0\times10^1$	$0.327\,636\,869\,2\times10^2$

TABLE II. Periods (in the overlapping region) in the four wells of the $M = 8$ potential as a function of energy. It can be seen that for any given energy the sum of the periods in wells 1 and 3 is equal to the sum of the periods in wells 2 and 4.

Energy	Period well 1	Period well 2	Period well 3	Period well 4
$0.693\,827\,514\,5\times10^1$	$0.206\,694\,581\,9\times10^1$	$0.614\,954\,531\,7\times10^1$	$0.637\,151\,933\,9\times10^1$	$0.228\,891\,984\,2\times10^1$
$0.150\,000\,000\,0\times10^2$	$0.208\,165\,877\,4\times10^1$	$0.625\,101\,106\,7\times10^1$	$0.648\,291\,587\,5\times10^1$	$0.231\,356\,358\,2\times10^1$
$0.250\,000\,000\,0\times10^2$	$0.210\,062\,107\,3\times10^1$	$0.639\,148\,359\,1\times10^1$	$0.663\,690\,117\,5\times10^1$	$0.234\,603\,865\,7\times10^1$
$0.400\,000\,000\,0\times10^2$	$0.213\,068\,062\,0\times10^1$	$0.664\,298\,684\,2\times10^1$	$0.691\,163\,134\,9\times10^1$	$0.239\,932\,512\,7\times10^1$
$0.500\,000\,000\,0\times10^2$	$0.215\,190\,876\,3\times10^1$	$0.684\,973\,573\,0\times10^1$	$0.713\,626\,541\,4\times10^1$	$0.243\,843\,844\,7\times10^1$
$0.750\,000\,000\,0\times10^2$	$0.220\,981\,368\,3\times10^1$	$0.765\,223\,218\,7\times10^1$	$0.799\,498\,800\,1\times10^1$	$0.255\,256\,949\,7\times10^1$
$0.983\,877\,280\,9\times10^2$	$0.227\,155\,631\,3\times10^1$	$0.165\,460\,042\,1\times10^2$	$0.169\,641\,491\,5\times10^2$	$0.268\,970\,125\,0\times10^1$

TABLE III. (A) Actions (in the overlapping region) in the three wells of the $M = 6$ potential as a function of energy. For any two values of the energy, if the differences in actions in the three wells are considered, one finds that the sum of the differences in wells 1 and 3 equals the difference in the second well. (B) Actions (in the overlapping region) in the four wells of the $M = 8$ potential as a function of energy. For any two values of the energy, if one considers the differences in actions in these wells, one finds that the sum of difference in actions in wells 1 and 3 is equal to that in wells 2 and 4.

(A)

Energy	Action well 1	Action well 2	Action well 3
$0.331\,721\,642\,3\times10^2$	$0.591\,947\,318\,3\times10^2$	$0.250\,365\,825\,9\times10^{-1}$	$0.359\,786\,696\,1\times10^2$
$0.350\,000\,000\,0\times10^2$	$0.616\,058\,915\,3\times10^2$	$0.464\,339\,135\,2\times10^1$	$0.381\,858\,646\,8\times10^2$
$0.380\,000\,000\,0\times10^2$	$0.656\,856\,782\,5\times10^2$	$0.124\,223\,197\,9\times10^2$	$0.418\,850\,064\,0\times10^2$
$0.420\,000\,000\,0\times10^2$	$0.714\,208\,462\,4\times10^2$	$0.232\,600\,726\,8\times10^2$	$0.469\,875\,913\,0\times10^2$
$0.450\,000\,000\,0\times10^2$	$0.760\,253\,171\,6\times10^2$	$0.318\,448\,320\,4\times10^2$	$0.509\,678\,797\,4\times10^2$
$0.470\,000\,000\,0\times10^2$	$0.793\,040\,816\,3\times10^2$	$0.378\,654\,049\,2\times10^2$	$0.537\,096\,881\,4\times10^2$
$0.490\,000\,000\,0\times10^2$	$0.828\,377\,086\,3\times10^2$	$0.442\,249\,805\,1\times10^2$	$0.565\,356\,367\,4\times10^2$
$0.510\,000\,000\,0\times10^2$	$0.868\,434\,404\,8\times10^2$	$0.511\,571\,981\,4\times10^2$	$0.594\,621\,225\,1\times10^2$
$0.518\,913\,991\,1\times10^2$	$0.890\,996\,665\,7\times10^2$	$0.547\,560\,167\,3\times10^2$	$0.608\,047\,150\,2\times10^2$

(B)

Energy	Action well 1	Action well 2	Action well 3	Action well 4
$0.693\,827\,514\,5\times10^1$	$0.192\,796\,177\,7\times10^2$	$0.978\,720\,968\,1\times10^{-2}$	$0.698\,645\,628\,8\times10^1$	$0.157\,342\,023\,7\times10^3$
$0.150\,000\,000\,0\times10^2$	$0.219\,410\,262\,5\times10^2$	$0.796\,413\,453\,2\times10^1$	$0.152\,318\,957\,7\times10^2$	$0.160\,294\,524\,3\times10^3$
$0.300\,000\,000\,0\times10^2$	$0.269\,445\,748\,7\times10^2$	$0.231\,390\,249\,9\times10^2$	$0.309\,846\,213\,3\times10^2$	$0.165\,875\,908\,0\times10^3$
$0.450\,000\,000\,0\times10^2$	$0.320\,191\,089\,0\times10^2$	$0.388\,945\,604\,7\times10^2$	$0.473\,720\,138\,2\times10^2$	$0.171\,582\,299\,1\times10^3$
$0.600\,000\,000\,0\times10^2$	$0.371\,697\,071\,5\times10^2$	$0.553\,984\,065\,0\times10^2$	$0.645\,719\,686\,6\times10^2$	$0.177\,429\,006\,1\times10^3$
$0.750\,000\,000\,0\times10^2$	$0.424\,021\,707\,7\times10^2$	$0.729\,590\,173\,6\times10^2$	$0.829\,064\,070\,2\times10^2$	$0.183\,435\,297\,3\times10^3$
$0.900\,000\,000\,0\times10^2$	$0.477\,231\,857\,9\times10^2$	$0.923\,600\,223\,7\times10^2$	$0.103\,177\,232\,5\times10^3$	$0.189\,626\,132\,7\times10^3$
$0.983\,877\,280\,9\times10^2$	$0.507\,400\,458\,5\times10^2$	$0.105\,607\,614\,4\times10^3$	$0.116\,962\,217\,1\times10^3$	$0.193\,180\,385\,3\times10^3$

the sum thus obtained to be equal to the sum of action differences in wells 2 and 4. We now look at a rather interesting property of the semiclassical quantum numbers.

IV. COMPARISON OF SEMICLASSICAL AND QUANTUM ENERGIES

The Einstein–Brillouin–Keller[14] (EBK) semiclassical energies ($E_{\rm SMC}$) are obtained by setting the action $I = m + 1/2$ in Eq. (3) where m is an integer. Some sample numerical values of these energies are presented in Tables IV and V. The quantum energies ($E_{\rm QNTM}$) are obtained by diagonalizing a 1800×1800 Hamiltonian matrix generated by the basis set

$$\phi_n = \left[\frac{\alpha}{\sqrt{\pi}2^n n!}\right]^{1/2} H_n(\alpha x) e^{-(\alpha^2 x^2)/2},$$

where H_n is the Hermite polynomial of order n and α is an adjustable parameter. The vlaue of the parameter α is found by minimizing the trace of the symmetric Hamiltonian matrix having the following elements:

TABLE IV. (A) Comparison of the EBK-semiclassical energy $E_{\rm SMC}$ and the quantum energy $E_{\rm QNTM}$ in regions R_3, R_4, and R_5 of Fig. 2. The lowest EBK-level number in R_5 is $n_5 = 265$. If this number was for a single-well case, it would imply the existence of 265 levels with lower energies. We show here that even for a triple-well system there are 265 levels with lower energies. It can be seen that the total number of levels in R_4 is 76 and the lowest and highest quantum numbers n_3 in region R_3 are, respectively, 144 and 188, indicating a total of 45 levels. (B) Comparison of $E_{\rm SMC}$ and $E_{\rm QNTM}$ in regions R_1 and R_2 of Fig. 2. The total number of levels in regions R_1 and R_2 are, respectively, 89 and 55. Thus, combining the numbers in (A) one obtains 76 + 45 + 89 + 55 = 265!

(A)

R3			R4			R5		
n_3	$E_{\rm SMC}$	$E_{\rm QNTM}$	n_4	$E_{\rm SMC}$	$E_{\rm QNTM}$	n_5	$E_{\rm SMC}$	$E_{\rm QNTM}$
144	51.9673	51.9909	0	0.4995	0.4997	265	60.2978	60.3183
145	52.1057	52.0984	1	1.4956	1.4958	266	60.4184	60.4195
146	52.2543	52.2609	2	2.4877	2.4879	267	60.5445	60.5442
147	52.4096	52.4114	3	3.4758	3.4760	268	60.6747	60.6786
155	53.7842	53.7846	22	21.4482	21.4483			
156	53.9669	53.9673	23	22.3498	22.3494			
157	54.1514	54.1518	24	23.2455	23.2457			
164	55.4825	55.4826	44	40.0707	40.0707			
165	55.6771	55.6773	45	40.8511	40.8511			
166	55.8726	55.8727	46	41.6250	41.6250			
173	57.2588	57.2588	61	52.3462	52.3460			
174	57.4583	57.4583	62	52.9917	52.9915			
175	57.6580	57.6579	63	53.6267	53.6265			
186	59.8162	59.8145	73	59.2251	59.2235			
187	59.9993	59.9941	74	59.6697	59.6669			
188	60.1718	60.1739	75	60.0657	60.0599			

(B)

R1			R2		
n_1	$E_{\rm SMC}$	$E_{\rm QNTM}$	n_2	$E_{\rm SMC}$	$E_{\rm QNTM}$
0	− 20.6583	− 20.6582	0	33.3617	33.3615
1	− 19.6188	− 19.6187	1	33.7596	33.7594
2	− 18.5830	− 18.5828	2	34.1559	34.1557
3	− 17.5508	− 17.5506	3	34.5506	34.5504
23	2.2841	2.2842	14	38.7837	38.7835
24	3.2332	3.2333	15	39.1581	39.1579
25	4.1780	4.1781	16	39.5307	39.5305
44	21.2575	21.2576	28	43.8505	43.8502
45	22.1071	22.1072	29	44.1968	44.1965
46	22.9514	22.9515	30	44.5409	44.5406
65	37.8661	37.8661	38	47.2052	47.2048
66	38.5841	38.5840	39	47.5261	47.5257
67	39.2943	39.2942	40	47.8441	47.8436
86	50.8437	50.8423	52	51.3539	51.3521
87	51.2878	51.2855	53	51.6051	51.6014
88	51.6895	51.6841	54	51.8383	51.8343

TABLE V. (A) Comparison of E_{SMC} and E_{QNTM} in regions R_6 and R_7 of Fig. 3. The entries ⋯ indicate that no reliable quantum energies are available. The lowest level in region R_7 has the number $n_7 = 1037$ while there are 245 levels in region R_6. (B) Same as in (A) in regions R_3, R_4, and R_5 of Fig. 3. The number of levels in these three regions are, respectively, 117, 201, and 236. (C) Same as in (A) in regions R_1 and R_2 of Fig. 3. It is seen that there are 132 and 106 levels in these two regions. Thus, counting the levels in regions R_1 through R_6 we find $132 + 106 + 117 + 201 + 236 + 245 = 1037$ which illustrates the similarity in the level counting for a single- and a multiple-well system.

(A)

R6			R7		
n_6	E_{SMC}	E_{QNTM}	n_7	E_{SMC}	E_{QNTM}
660	180.8025	⋯	1037	279.7854	⋯
661	181.0801	⋯	1038	280.0640	⋯
662	181.3724	⋯	1039	280.3542	⋯
663	181.6733	⋯	1040	280.6517	⋯
723	203.2334	⋯			
724	203.6219	⋯			
725	204.0111	⋯			
785	228.3836	⋯			
786	228.8041	⋯			
787	229.2250	⋯			
840	251.9851	⋯			
841	252.4209	⋯			
842	252.8568	⋯			
902	278.7453	⋯			
903	279.1405	⋯			
904	279.5230	⋯			

(B)

R3			R4			R5		
n_3	E_{SMC}	E_{QNTM}	n_4	E_{SMC}	E_{QNTM}	n_5	E_{SMC}	E_{QNTM}
0	0.4998	0.4995	223	98.5868	98.6102	0	− 519.0164	− 519.0165
1	1.4978	1.4976	224	98.8324	98.8341	1	− 515.1585	− 515.1587
2	2.4939	2.4937	225	99.0924	99.0990	2	− 511.3063	− 511.3064
3	3.4880	3.4878	226	99.3625	99.3651	3	− 507.4597	− 507.4598
35	34.1972	34.1970	280	118.4201	118.4201	59	− 301.4376	− 301.4378
36	35.1203	35.1201	281	118.8166	118.8166	60	− 297.9344	− 297.9345
37	36.0410	36.0408	282	119.2142	119.2142	61	− 294.4376	− 294.4378
59	55.6624	55.6621	328	138.4394	138.4393	133	− 61.1618	− 61.1620
60	56.5235	56.5232	329	138.8738	138.8736	134	− 58.2035	− 58.2037
61	57.3818	57.3815	330	139.3086	139.3085	135	− 55.2538	− 55.2541
89	80.0942	80.0938	390	166.0505	⋯	190	92.0778	⋯
90	80.8507	80.8502	391	166.5000	⋯	191	94.4432	⋯
91	81.6026	81.6021	392	166.9494	⋯	192	96.7952	⋯
114	97.1489	97.1461	421	179.6897	⋯	233	177.6711	⋯
115	97.6872	97.6822	422	180.0975	⋯	234	178.9743	⋯
116	98.1862	98.1720	423	180.4943	⋯	235	180.1594	⋯

(C)

R1			R2		
n_1	E_{SMC}	E_{QNTM}	n_2	E_{SMC}	E_{QNTM}
0	− 51.5257	− 51.5260	0	7.4389	7.4387
1	− 48.3449	− 48.3451	1	8.4586	8.4584
2	− 45.1714	− 45.1716	2	9.4763	9.4761
3	− 42.0055	− 42.0057	3	10.4918	10.4916
25	25.6905	25.6903	36	42.7607	42.7604
26	28.6756	28.6754	37	43.6977	43.6974
27	31.6523	31.6521	38	44.6321	44.6318

TABLE V (*continued*).

R1			R2		
n_1	E_{SMC}	E_{QNTM}	n_2	E_{SMC}	E_{QNTM}
48	92.1680	96.1677	52	57.4236	57.4234
49	94.9504	94.9502	53	58.3155	58.3152
50	97.7235	97.7232	54	59.2042	59.2039
88	195.3573	195.3569	70	72.9685	72.9682
89	197.6972	197.6968	71	73.7976	73.7973
90	200.0236	200.0232	72	74.6227	74.6223
129	276.5061	276.5012	103	97.3041	97.3008
130	277.8259	277.8178	104	97.8518	97.8458
131	279.0303	279.0200	105	98.3384	98.3431

$M = 6;$

$$\langle\phi_n|H|\phi_n\rangle = \frac{\alpha^2(n+1/2)}{2} + \frac{C_2}{2\alpha^2}(2n+1) + \frac{C_4}{2\alpha^4}(6n^2+6n+3) + \frac{C_6}{8\alpha^6}(20n^3+30n^2+40n+15),$$

$$\langle\phi_{n+1}|H|\phi_n\rangle = (n+1)^{1/2}\left\{\frac{C_3}{\alpha^3\sqrt{8}}(3n+3) + \frac{C_5}{\alpha^5\sqrt{32}}(10n^2+20n+15)\right\},$$

$$\langle\phi_{n+2}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}\left\{\frac{-\alpha^2}{4} + \frac{C_2}{2\alpha^2} + \frac{C_4}{4\alpha^4}(4n+6) + \frac{C_6}{8\alpha^6}(15n^2+45n+45)\right\},$$

$$\langle\phi_{n+3}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times\left\{\frac{C_3}{\alpha^3\sqrt{8}} + \frac{C_5}{\alpha^5\sqrt{32}}(5n+10)\right\},$$

$$\langle\phi_{n+4}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2}(n+4)^{1/2} \times\left\{\frac{C_4}{4\alpha^4} + \frac{C_6}{8\alpha^6}(6n+15)\right\},$$

$$\langle\phi_{n+5}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times(n+4)^{1/2}(n+5)^{1/2}\left\{\frac{C_5}{\alpha^5\sqrt{32}}\right\},$$

$$\langle\phi_{n+6}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times(n+4)^{1/2}(n+5)^{1/2}(n+6)^{1/2}\left\{\frac{C_6}{8\alpha^6}\right\}.$$

$M = 8;$

$$\langle\phi_n|H|\phi_n\rangle = \frac{\alpha^2(n+1/2)}{2} + \frac{C_2}{2\alpha^2}(2n+1) + \frac{C_4}{4\alpha^4}(6n^2+6n+3) + \frac{C_6}{8\alpha^6}(20n^3+30n^2+40n+15) + \frac{C_8}{16\alpha^8}(70n^4+140n^3+350n^2+280n+105),$$

$$\langle\phi_{n+1}|H|\phi_n\rangle = (n+1)^{1/2}\left\{\frac{C_3}{\sqrt{8}\alpha^3}(3n+3) + \frac{C_5}{\sqrt{32}\alpha^5}(10n^2+20n+15) + \frac{C_7}{\sqrt{128}\alpha^7}(35n^3+105n^2+175n+105)\right\},$$

$$\langle\phi_{n+2}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2} \times\left\{\frac{-\alpha^2}{4} + \frac{C_2}{2\alpha^2} + \frac{C_4}{4\alpha^4}(4n+6) + \frac{C_6}{8\alpha^6}(15n^2+45n+45) + \frac{C_8}{16\alpha^8}(56n^3+252n^2+532n+420)\right\},$$

$$\langle\phi_{n+3}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times\left\{\frac{C_3}{\sqrt{8}\alpha^3} + \frac{C_5}{\sqrt{32}\alpha^5}(5n+10) + \frac{C_7}{\sqrt{128}\alpha^7}(21n^2+84n+105)\right\},$$

$$\langle\phi_{n+4}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times (n+4)^{1/2}\left\{\frac{C_4}{4\alpha^4}+\frac{C_6}{8\alpha^6}(6n+15) + \frac{C_8}{16\alpha^8}(28n^2+140n+210)\right\},$$

$$\langle\phi_{n+5}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times (n+4)^{1/2}(n+5)^{1/2} \times\left\{\frac{C_5}{\sqrt{32}\alpha^5}+\frac{C_7}{\sqrt{128}\alpha^7}(7n+21)\right\},$$

$$\langle\phi_{n+6}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times (n+4)^{1/2}(n+5)^{1/2}(n+6)^{1/2} \times\left\{\frac{C_6}{8\alpha^6}+\frac{C_8}{16\alpha^8}(8n+28)\right\},$$

$$\langle\phi_{n+7}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times (n+4)^{1/2}(n+5)^{1/2}(n+6)^{1/2} \times (n+7)^{1/2}\left\{\frac{C_7}{\sqrt{128}\alpha^7}\right\},$$

$$\langle\phi_{n+8}|H|\phi_n\rangle = (n+1)^{1/2}(n+2)^{1/2}(n+3)^{1/2} \times (n+4)^{1/2}(n+5)^{1/2}(n+6)^{1/2} \times (n+7)^{1/2}(n+8)^{1/2}\left\{\frac{C_8}{16\alpha^8}\right\}.$$

Tables IV and V show that the $E_{\rm SMC}$ and $E_{\rm QNTM}$ compare quite well. The real merit of semiclassical calculations is appreciated for large values of m for which quantum calculations become prohibitive. We now look at the counting of the EBK quantum numbers in different wells.

In a single minimum case, if there is an energy level with the EBK quantum number m, then it is understood that this is the $(m+1)$th level (levels are counted from $m=0$) with exactly m levels having lower energies. We show here that similar level counting is also valid for a multiminima case. We consider Fig. (2) and Tables IV(A) and IV(B). The lowest level in region R5 above the line joining the turning points T1 and T2 has the quantum number $n_5 = 265$ indicating that it is the 266th level and there are 265 levels in regions below this line. It can be seen from Table IV(A) that there is a total of 76 levels in the third well (region R4) and the highest level in region R3 below the line joining the turning ponts T1 and P1 has the quantum number 188. Thus the total number $76+188+1=265$ matches that mentioned above. Again the lowest level in region R3 has the quantum number $n_3 = 144$ indicating that there should be 144 levels below the line joining the turning points T3 and T4. Indeed the total number of levels in regions R1 and R2 is seen from Table IV(B) to be equal to 144 $(88+54+2)$. It can be seen from Fig. 3 and Tables V that analogous level counting applies for a four minima case. These observations explain why there are apparent discontinuities in the EBK quantum numbers when a potential hump is crossed in a multiminima case.

The properties of the periods, actions and the EBK quantum numbers that we have seen above appear not be widely known.

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