GRAPH FUNCTIONS WHICH ARE RAMSEY FUNCTIONS

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INTRODUCTION

Ramsey's Theorem, stated informally, says that if $|V(G)|$ is large enough then G contains a large K_n or a large K_n as an induced subgraph. This prompts the pursuit of a very new kind of Ramsey problem which can be stated informally as follows: for a given nonnegative invariant f can we find a finite number of classes of graphs ($f \to \infty$ on each class) so that any graph with a very large f value has a large member of one of the classes as an induced subgraph.

The notation we use is that of [2], where definitions can be found for invariants not specifically defined here. In order to present results for a large number of invariants, only a few proofs will be given here. THEOREM 1 displays the strong relationship of our results to Ramsey's Theorem. The notion of a Ramsey function was first presented in [6].

RAMSEY FUNCTIONS

As it is not unlikely that Ramsey functions will be of interest in other investigations, we define them in terms of partially ordered sets, although we concern ourselves only with the set of all graphs (including the null graph) partially ordered by inclusion as an induced subgraph. If H and G are graphs, we use $H \subset G$ to mean H is an induced subgraph of G.

Let P be a partially ordered set with $0 \in P$, $0 \leq a$ for all $a \in P$. Let f be a nonnegative monotonic function defined on P.

DEFINITION 1. A chain $0 = a_0 < a_1 < a_2 < \cdots$ for which $f(a_i) \to \infty$ is called an *f-chain.*

Let $F = \{(a_{ij}),\ i = 1, 2, \ldots, m;\ j = 0, 1, 2, \ldots\}$ be a finite set of f-chains.

DEFINITION 2. For a given f and $F \neq \phi$, $\underline{t_F(a)} = \max_{a_{ij} \in F}\{j \mid a_{ij} \leq a\}$.

Note: This is defined for all $a \in P$ since the chains contain 0. If we list the chains of F in matrix form, then for a graph G, $t_F(G)$ is the column index j of an induced subgraph of G farthest out in any of the rows of the matrix.

DEFINITION 3. A nonnegative monotonic function f defined on P for which there is a sequence $(a_i) \subset P$, where $f(a_i) \to \infty$, is a *Ramsey function* if there exists a

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finite collection of f-chains F such that for any sequence $S = s_1, s_2, s_3, \ldots$ in P $\sup_S f(s_i) = \infty$ implies $\sup_S t_F(s_i) = \infty$.

EXAMPLE. If $f = |V(G)|$ then f is a Ramsey function and the corresponding $F = \{(0 \subset K_1 \subset K_2 \subset \cdots), (0 \subset \bar{K}_1 \subset \bar{K}_2 \subset \cdots)\}$. If $G = K_2 + K_{1,2}$ then $t_F(G) = 4$.

Requiring the existence of some sequence on which f is unbounded in DEFINITION 3 eliminates trivial cases. We shall need:

LEMMA 1 (Ramsey's Theorem). There exists a smallest positive integer $R(n; m)$ such that any coloring of the edges of the complete graph on $t \geq R(n; m)$ vertices with m colors has a monochromatic complete subgraph of size n.

We write $R(n)$ for $R(n; 2)$.

LEMMA 2 [9, p. 45]. There exists a function $Q(n)$ such that any square (0,1)-matrix of order at least $Q(n)$ with all 1's on the diagonal contains a principal submatrix of order n of form I, J, T, or T^t, where I is the identity matrix, J is the matrix of all 1's, and $T = (t_{ij})$ has $t_{ij} = 1$ iff $i \leq j$.

Clearly $R(n) \geq n$ and $Q(n) \geq n$.

FUNCTIONS WHICH ARE RAMSEY FUNCTIONS

At the end of this section we give a table of functions which we know to be Ramsey functions, along with their f-chains. To give the flavor of the type of proof involved we prove one of these results in THEOREM 1.

We first describe some graphs needed as f-chains below. An *n-legged spider* consists of n distinct P_3's incident at a vertex. nG means n disjoint copies of G. The graph H_n has $2n + 1$ vertices and consists of a K_{2n} and a K_{n+1} with exactly n of their vertices in common. The graph $(K_n \cup \bar{K}_n) + nK_2$ has $2n$ vertices and consists of a K_n and a $\bar{K}_n$ with additional adjacencies: each vertex of K_n is adjacent to exactly one vertex of $\bar{K}_n$. T_n is a bipartite graph with $2n$ vertices which we can label $1, 2, \ldots, n, 1', 2', \ldots, n'$ so that $E(T_n) = \{(i, j') \mid i \leq j\}$. W_n is a bipartite graph with $2n$ vertices which we can label $1, 2, \ldots, n, 1', 2', \ldots, n'$ so that $E(W_n) = \{(i, j') \mid i \neq j\}$.

We describe several of the invariants for which results are given. $\lambda^1(G)$ is the least eigenvalue of $A(G) - xI$. $Match^*(G) = \min\{\beta_1(G), \beta_1(\bar{G})\}$, where $\beta_1(G)$ is the line independence number.

DEFINITION 4. When f is not monotonic we consider $f'(G) = \max_{H \subseteq G} f(H)$.

DEFINITION 5. $v \in V(G)$ is a *cutpoint* of G if the removal of v increases the number of components.

The following are easy consequences of DEFINITION 5:

(a) For any cutpoint v_i, there exist points u_i and l_i adjacent to v_i such that v_i is on every $u_i - l_i$ path.

(b) Given n mutually adjacent cutpoints $v_i \in V(G)$, there exists an x_i adjacent to v_i for each i such that $(x_i, v_j) \notin E(G)$, $i \neq j$, and the x_i's are mutually nonadjacent.

(c) Given n mutually nonadjacent cutpoints $v_i \in V(G)$, all adjacent to the same point w, then there exist points x_i distinct from w such that $(x_i, v_i) \in E(G)$ for each i and (x_i, w), (x_i, x_j), $(x_i, v_j) \notin E(G)$.

Let $\mathrm{Cut}(G)$ = the number of cutpoints in G. When we refer to a *chain* G_n we mean the chain $0 \subset G_1 \subset G_2 \subset \cdots$, where we specify the nth element in the chain. We use (a), (b), and (c) in:

THEOREM 1. $\mathrm{Cut}'(G)$ is a Ramsey function with F consisting of the three chains: nP_3, $(K_n \cup \bar{K}_n) + nK_2$, and an n-legged spider.

Proof: Each chain described is a Cut′-chain since $\mathrm{Cut}'(G_i) \to \infty$ for each chain. We establish that $\mathrm{Cut}'(G)$ is a Ramsey function by showing that $\mathrm{Cut}'(G) \geq R(nQ(R(Q(R(Q(n))))))$ implies $t_F(G) \geq n$. If $\mathrm{Cut}'(G)$ is as stated then there exists $H \subset G$ with $\mathrm{Cut}(H) \geq R(nQ(R(Q(R(Q(n))))))$. Apply LEMMA 1 to the cutpoints of H and find $nQ(R(Q(R(Q(n)))))$ of them are mutually adjacent or that same number are mutually nonadjacent. In the former case (b) implies $(K_n \cup \bar{K}_n) + nK_2 \subset H \subset G$. In the latter case, consider points u_i and l_i adjacent to each of the mutually nonadjacent cutpoints v_i so that v_i is on every $u_i - l_i$ path. We thus have $nQ(R(Q(R(Q(n)))))$ paths of length 2, with a cutpoint in the center of each path. Either n of the paths are mutually incident (via endpoints) or $Q(R(Q(R(Q(n)))))$ of the paths are mutually disjoint. In the former case (c) implies an n-legged spider $\subset H \subset G$. In the latter case, consider the $Q(R(Q(R(Q(n)))))$ paths lined up in a row with u_i vertices as upper vertices, v_i in the middle and l_i as lower vertices (FIGURE 1). We check for addi-

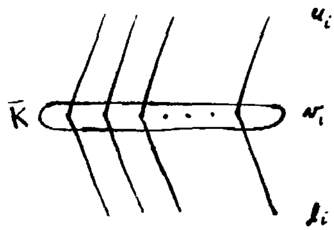

FIGURE 1.

tional adjacencies. Recall that the v_i's are mutually nonadjacent. We may assume that the v_i's have indices $i = 1, 2, \ldots, Q(R(Q(R(Q(n)))))$. Consider the (0, 1)-incidence matrix M, where row i and column i correspond to u_i and v_i, respectively. Matrix M has all 1's on the diagonal since $(u_i, v_i) \in E(G)$. LEMMA 2 applied to M implies M has a principal submatrix M' of order $R(Q(R(Q(n))))$ of form I, J, T, or T^t. If $M' = J$, T, or T^t, there is a vertex u_i adjacent to n of the mutually nonadjacent v_j, thus (c) implies an n-legged spider $\subset H \subset G$. Suppose $M' = I$ and, without loss of generality, assume the u_i and v_i involved have lowest indices. Applying LEMMA 1 to the u_i's we find either $Q(R(Q(n)))$ of them are mutually adjacent or that same number are mutually nonadjacent. In the former case, $(K_n \cup \bar{K}_n) + nK_2 \subset H \subset G$. In the latter case, again assume the u_i's have lowest possible indices. We have an induced matching between u_i and v_i vertices (FIGURE 2). Consider the (0, 1)-incidence matrix N, where row i and column i correspond to v_i and l_i, respectively. Matrix N has all 1's on the diagonal and is of order $Q(R(Q(n)))$. LEMMA 2 implies N has a principal submatrix N' of order $R(Q(n))$ having form I, J, T, or T^t. If $N' = J$, T, or T^t, (c) implies an n-legged spider $\subset H \subset G$. Thus suppose $N' = I$. LEMMA 1 applied to the l_i corresponding to columns of N' implies $Q(n)$ of the l_i are mutually adjacent or $Q(n)$ of the l_i are mutually nonadjacent. In the former case, $(K_n \cup \bar{K}_n) +$

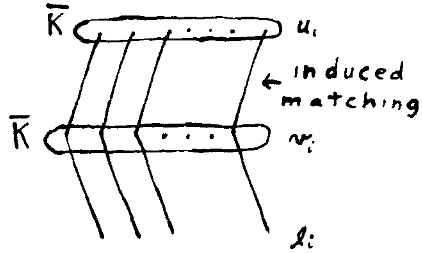

FIGURE 2.

$nK_2 \subset H \subset G$. In the latter case, we may assume the l_i have lowest indices. We are thus left to consider the paths (u_i, v_i, l_i), $i = 1, 2, \ldots, Q(n)$ (FIGURE 3). The only possible additional adjacencies are between vertices u_i and l_j. $(u_i, l_i) \notin E(H)$ for this would contradict the fact that v_i is on every $u_i - l_i$ path. Consider the "antimatching" edges (u_i, l_i) in $\bar{H}$. $(u_i, l_i) \in E(\bar{H})$. Let $\bar{M}$ be the (0, 1)-incidence matrix for $\bar{H}$. Matrix $\bar{H}$ has all 1's on the diagonal and is of order $Q(n)$. LEMMA 2 implies $\bar{M}$ has a principal submatrix, $\bar{M}'$, of order n with form I, J, T, or T^t (for $\bar{H}$). These give a complementary incidence matrix M' (interchange roles of 0 and 1), respectively, of form $J - I$, 0, $T^t - I$, and $T - I$ (for H). If $M' = J - I$, or if $M' = T^t - I$ or $T - I$ and $n = 1$ or 2, n-legged spider $\subset H \subset G$. Otherwise, these are impossible by the $u_i - l_i$ path condition. If $M' = 0$ $nP_3 \subset H \subset G$. Thus for all possible cases, $t_F(G) \geq n$ and Cut'(G) is a Ramsey function. □

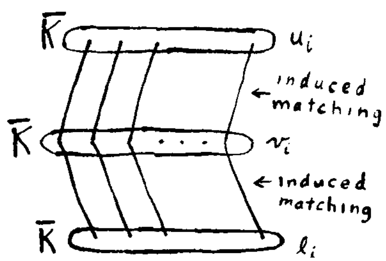

FIGURE 3.

NOTE. We note the strong dependence of the proof on Ramsey's Theorem. This dependence is strengthened by the fact that $Q(n) \leq R(n; 4)$ (see [9]).

Below we list functions we know to be Ramsey functions. LEMMA 1, LEMMA 2, and other Ramsey style theorems (see [4]) are used in proving these results.

Ramsey function f	*f-chains in F*
$\lvert\lambda^1(G)\rvert$, in [5]	$K_{1,n}$, H_n.
diameter' or radius'	P_n.
intersection number	K_n, $\bar{K}_n$.
line chromatic number	K_n, $K_{1,n}$.
line covering number	K_n, $K_{1,n}$, nK_2.
line independence number	K_n, $K_{n,n}$, nK_2.
match*(G), in [7]	nK_2, $\overline{nK_2}$, $K_{n,n}$, $2K_n$, $K_n \cup \bar{K}_n$, $K_n + \bar{K}_n$.
maximum degree of a vertex	K_n, $K_{1,n}$.
number of blocks'	$K_{1,n}$, nK_2, $(K_n \cup \bar{K}_n) \dot{+} nK_2$.
number of cutpoints'	nP_3, $(K_n \cup \bar{K}_n) \dot{+} nK_2$, n-legged spider.

number of edges in the line graph of G	K_n, $K_{1,n}$, nK_3, nP_3.
number of vertices of odd degree'	K_n, $K_{1,n}$, nK_2.
point covering number	K_n, $K_{n,n}$, nK_2.
achromatic number, in [7]	K_n, nK_2, W_n, T_n.

Functions Which Are Not Ramsey Functions

The following gives a method of determining when a function is not a Ramsey function.

THEOREM 2. If for any n a_i's in P with $f(a_i)$ large, there exists an $\bar{a} \subset P$, incomparable to each a_i with $f(\bar{a})$ as large as we wish, then f is not a Ramsey function.

In [1] a theorem much stronger than the following, dealing with Hamiltonian circuits is proved:

LEMMA 3. Given $r \geq 3$, $g \geq 3$, there exists a graph G which is regular of degree r and has girth g.

LEMMA 4. If G and H are graphs, each having finite girth, and f is monotonic (nondecreasing), then $f(H) \geq f(G)$ and $g(H) > g(G)$ together imply that H is incomparable to G.

Proof: If H were comparable to G, f monotonic and $f(H) \geq f(G)$ would imply $G \subset H$. Since G is an induced subgraph of H, $g(H) \leq g(G)$, a contradiction. So H is incomparable to G. □

THEOREM 3. The arboricity, $\gamma(G)$, is not a Ramsey function.

Proof: We show that, for any n graphs G_i with $\gamma(G_i)$ large, we can find a graph B incomparable to each of the G_i's and having $\gamma(B)$ as large as we wish. If B is regular of degree r with girth $g(B)$, then using a formula of Nash-Williams we find

$$\gamma(B) = \max_{C \subset B} \left[\frac{|E(C)|}{|V(C)| - 1}\right] \geq \left[\frac{|E(B)|}{|V(B)| - 1}\right] \geq \left[\frac{nr}{2(n-1)}\right] \geq \left[\frac{r}{2}\right]$$

Now LEMMA 3 implies that we can choose B with $\gamma(B)$ as large as we wish and $g(B) > g(G_i)$, $i = 1, 2, \ldots, n$. Note that $\gamma(G_i)$ large implies $g(G_i)$ is finite. LEMMA 4 shows B is incomparable to each G_i and THEOREM 2 thus implies that $\gamma(G)$ is not a Ramsey function. □

It should be noted that the only method we have at this time for showing that a graph function is not a Ramsey function depends on the properties of girth.

The following are not Ramsey functions: chromatic number, girth, genus, minimum degree of a vertex', number of spanning trees, and the length of the longest trail. The proofs for these rely on THEOREM 2 and the lemmas of this section. The result for chromatic number uses a well-known theorem of Erdös involving girth. The genus result uses an inequality in [8].

The results of this paper are part of the work done in the author's doctoral dissertation, where various other invariants are examined. Several other very interesting invariants are defined and examined in [7] where the concept of *equivalence of Ramsey functions* is explored, as well as whether or not certain Ramsey functions are *spectral functions*. The author would like to thank the referee for helpful comments on the presentation of this material.

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