Application of the Methods of Molecular Distribution to Solutions of Large Molecules

Bruno H. Zimm

Additional information on J. Chem. Phys.

with $\nu = 1.773 \times 10^{13}$, $a = 0.36 \times 10^{-8}$ cm. These reasonable values for a are a further check on this theory.

It should be pointed out that the temperature variation of catalytic effects is quite different from that of pure gases, the number of collisions required being nearly independent of tempera-

ture.[15] Various attempts have been made to associate the effectiveness of catalysts with their physical or chemical properties but no generally successful rule seems to have been proposed. Gases that have some chemical affinity, gases with large dipole moments, and gases with small moments of inertia are usually most effective.

Application of the Methods of Molecular Distribution to Solutions of Large Molecules

BRUNO H. ZIMM
Polytechnic Institute of Brooklyn, Brooklyn, New York
(Received December 17, 1945)

The equations for the thermodynamic potentials of the solvent in solutions of ordinary organic molecules are extended to solutions of large molecules by methods using continuous molecular distribution functions. Particular attention is given to the coefficient, A_2, of the second term in the expansion of the osmotic pressure in terms of the concentration, since this coefficient has a simple molecular meaning and is sufficient to describe the deviation of the system from ideality at low concentrations. A_2 is calculated by direct integration for two rigid shapes, the sphere and the long thin rod. A general expression is then developed for A_2 for flexible chain molecules in terms of the interactions of the segments of the chains. In favorable cases it is found possible to relate A_2 for a chain molecule to the solution properties of its small molecule homologues by an equation very similar to those developed by Flory, Huggins, and Miller. In general, however, interactions that depend both on the local structure and also on the over-all shape of the chain molecules seriously modify such a relationship. The nature of these interactions, including the effects of branching and of limited flexibility, is discussed. It is also shown that higher coefficients in the expansion of the osmotic pressure in terms of concentration can be treated in a similar way. Comparison with experimental data confirms the general predictions of the theory both for proteins and chain polymers.

I. INTRODUCTION

IT is a task of statistical mechanics to relate the thermodynamics of solutions to the properties of the molecules that compose them. One problem in this field is the investigation of the extraordinary deviations from Raoult's law that are frequently shown by solutions of high polymers and other large molecules. Several papers in recent years[1-3] have contributed greatly to our understanding of the phenomenon, but they are limited in some respects; they have all been based on a highly idealized lattice model, and they are

restricted to treating linear chain polymers only, leaving the status of the more rigid protein molecules obscure. Moreover the numerical agreement with experiment has not been satisfactory, for reasons that have remained unelucidated.*

The treatment in this paper aims to divest the theory of as many assumptions as possible and to develop it as an extension of the better understood theories of ideal and regular solutions of small organic molecules. Certain inherent limitations should be noted at the outset, however. The theory is not applicable directly to solutions of ionized particles and is therefore not suitable as

---

[1] P. J. Flory, J. Chem. Phys. **9**, 660 (1941); **10**, 51 (1942).
[2] M. L. Huggins, J. Chem. Phys. **9**, 440 (1941); J. Phys. Chem. **46**, 151 (1942); J. Am. Chem. Soc. **64**, 1712 (1942); Ann. N. Y. Acad. Sci. **43**, 1 (1942).
[3] A. R. Miller, Proc. Camb. Phil. Soc. **39**, 54 (1942).

* Note added in proof: A recent article by P. J. Flory (J. Chem. Phys. **13**, 453(1945)) contributes greatly to the clearing up of these discrepancies. Many of the conclusions reached therein are similar to those of this paper.

it stands for aqueous protein solutions, except at their iso-electric points. Moreover, the theory is convenient for the discussion of the partial molar free energy and related quantities of only one component, the solvent, in dilute solutions. (Fortunately these are also the quantities and concentrations most accessible experimentally.)

The essential step in obtaining the partial molar free energy of the solvent is the estimation of the change in the phase integral or partition function of the system, that is, its probability, as the relative amounts of the components are varied.

The statistical method used in this paper for calculating the probabilities of the system is quite different in appearance from the lattice approach since it involves operation with continuous distribution functions. This treatment is based on a recent investigation of the theory of non-electrolyte solutions by McMillan and Mayer[4]; without the simplifications achieved by these authors (fortunately at no loss of accuracy) the continuous method would probably be impracticable.

Because these methods have not yet been widely used, the fundamental equations will be reviewed at some length in the following section.

II. FUNDAMENTAL EQUATIONS[4,5]

II.1. Symbolism

Since we must operate with systems of many particles all influencing each other directly or indirectly, it is necessary to adopt a shorthand to represent sets of many coordinates. The conventions are as follows:

(i) represents all the coordinates of the ith particle

$\{n\}$ represents the coordinates of a set of n particles

$\{n\pm m\}$ represents the coordinates of a set of n plus (or minus) m particles. (The particles are assumed to be distinguishable.)

$d(i)$, $d\{n\}$, $d\{n\pm m\}$ are the respective volume elements in hyperspace.

[4] W. G. McMillan and J. E. Mayer, J. Chem. Phys. **13**, 276 (1945).
[5] J. E. Mayer and E. Montroll, J. Chem. Phys. **9**, 2 (1941).

II.2. Molecular Distribution Functions and Related Quantities[4,5]

In a fluid, or indeed in a real crystal above absolute zero, the molecules are not fixed in a rigid pattern but instead may be found with varying probability at any point in the containing vessel. In line with the fundamental philosophy of quantum-statistical mechanics, one is led to postulate that the most exact description possible of a real system can only give the probabilities of the molecules occurring in certain positions and configurations. For a system containing many molecules in a volume V these probabilities are conveniently expressed by a set of functions, $[F_n\{n\}]$, defined as follows:

$$\frac{1}{V^n}F_n\{n\}d\{n\}, \qquad (1)$$

is the probability that the n specified molecules are each at the positions and have the internal configurations specified by $\{n\}$ in the infinitesimal range of these coordinates represented by $d\{n\}$.

The functions are so normalized that when integrated over the complete range of all the coordinates they are unity:

$$\frac{1}{V^n}\int F_n\{n\}d\{n\} = 1 \qquad (2)$$

(meaning that the probability of finding the n molecules somewhere in V is unity).

If the system contains N molecules altogether, $(1/V^N)F_N\{N\}d\{N\}$ specifies the probability that all these molecules will appear in the configuration symbolized by $\{N\}$. If we are interested in the probability that a certain small group containing n of these N molecules will have the configuration $\{n\}$, irrespective of where the rest of the $N-n$ molecules are, we can obtain it by averaging $(1/V^N)F_N\{N\}d\{N\}$ over all the positions of the $N-n$ undesired molecules. This probability is by definition $(1/V^n)F_n\{n\}d\{n\}$ so that Eq. (3) arises to connect the distribution functions of different sized groups of molecules.

$$F_n\{n\} = \frac{1}{V^{N-n}}\int F_N\{N\}d\{N-n\}. \qquad (3)$$

For non-electrolyte fluid systems there are no long range forces or structures, and a single

molecule is as likely to be found in one place as another, so that its distribution function depends on its internal coordinates only. If it is integrated over these, the integral must then be a constant; this is equal to unity as a result of (2).

$$\int F_1(j)d(j)_i = 1. \qquad (4)$$

(The symbol $(j)_i$ stands for the internal coordinates of molecule j.)

Also, since there is no long range structure, all the higher distribution functions become equal to a simple product of the F_1's of the molecules involved when the distances between all of them are large. It is convenient to make a transformation to a set of functions, the "g-functions," which become zero in like circumstances.

The first two g-functions are explicitly defined here (for the general definition see McMillan and Mayer[4] and Mayer and Montroll:[5]

$$g_2(i, j) = F_2(i, j) - F_1(i)F_1(j), \qquad (5a)$$

$$g_3(i, j, k) = F_3(i, j, k) - F_2(i, j)F_1(k)$$
$$- F_2(j, k)F_1(i) - F_2(k, i)F_1(j)$$
$$+ 2F_1(i)F_1(j)F_1(k). \qquad (5b)$$

Actually the functions so defined do not quite become zero at large distances, but instead approach small values of order $1/N$. The g-functions used later in this paper should therefore be considered to be very slightly modified by the subtraction of a constant so that they go rigorously to zero. The otherwise trivial difference could have been avoided at the cost of simplicity by taking the system as part of an infinite volume of fluid.[4,5]

The function g_2 measures the divergence of the pair distribution function, $F_2(i, j)$, from a simple product of $F_1(i)F_1(j)$; in other words, it is a measure of the effect of the interaction of the two molecules on one another. The higher g-functions have similar meanings for larger groups of molecules.

In this paper it so happens that all the distribution functions are those of the solute molecules in infinitely dilute solution only; the distribution functions of the solvent molecules or those of the solute at finite concentrations are never used.

II.3. Equation for the Osmotic Pressure

By transformation of the grand partition function, appropriately generalized, McMillan and Mayer were able to derive rigorously a series expansion for the osmotic pressure in terms of concentration, which we shall write as

$$\pi = RT\left(\frac{1}{M_2}c + A_2c^2 + A_3c^3 + \cdots\right), \qquad (6)$$

where c is the concentration of solute in mass per unit volume of solution, π the osmotic pressure (pressure increment required to make the activity of the solvent in a solution of concentration c equal to that of the pure solvent at its own vapor pressure), M_2 the molecular weight of the solute, and the A_n are constants at constant temperature. This series converges and is valid with constant coefficients at any concentration between an infinitely dilute and a saturated solution.

The coefficients, A_n, are related to the molecular distribution functions. If the forces between the molecules are of the ordinary van der Waals type, that is, not capable of being saturated, the first coefficients may be obtained from the following equations:

$$A_2 = -\frac{N_0}{2VM_2^2}\int g_2\{2\}d\{2\}, \qquad (7a)$$

$$A_3 = -\frac{N_0^2}{3VM_2^3}\int g_3\{3\}d\{3\} + 4M_2A_2^2; \qquad (7b)$$

N_0 is Avogadro's number and V is the volume of the system. The reader is referred to McMillan and Mayer's paper for the general formula.

In words, the coefficient of the second term in the expansion of the osmotic pressure in terms of concentration is an integral involving the g-function of a pair of solute molecules; the third coefficient is an integral involving the g-functions of a triplet of solvent molecules; all the functions are to be evaluated in the infinitely dilute solution. The higher terms involve larger groups of molecules in a similar way.

It is interesting to note the exact formal correspondence between Eqs. (6) and (7) and the equation of state of an imperfect gas. If π were replaced by the pressure and the g-functions in (7) were defined for the infinitely rarified gas instead of the infinitely dilute solution, the two would be identical. The first term of the series in

each case describes the situation when the molecules do not interact with one another, giving the perfect gas law and van't Hoff's law, respectively, while the higher terms correct successively for the interaction of pairs, triplets, and larger groups of molecules.

III. IDEAL AND REGULAR SOLUTIONS

III.1. Ideal Solutions

If the molecules of the two components are alike in size, shape, and force fields, the distribution functions and hence the integrals over them must be the same for both components. In this case the solution is "ideal" and obeys the well-known equation

$$\pi = -\frac{\Delta \bar{F}_1}{\bar{V}_1} = -\frac{RT}{\bar{V}_1} \ln (1 - N_2). \qquad (8)$$

$\Delta \bar{F}_1$ is the molar free energy dilution, i.e., $\langle \Delta \bar{F} \rangle_1 = \bar{F}_1 - \bar{F}_1{}^0$ where $\bar{F}_1$ is the partial molar free energy of the solvent and $\bar{F}_1{}^0$ is the same quantity at infinite dilution, $\bar{V}_1$ is the solvent's partial molar volume, and N_2 the mole fraction of solute. Equation (8) may be derived from the fundamental Eqs. (6) and (7), but to do so would take us too far afield. It is better here to work backward from Eq. (8) to calculate the values of the coefficients in the general Eq. (6) for this special case.

For the dilute solutions in which we are hereafter interested, it suffices to replace the mole fraction N_2 by its approximate equivalent, $\bar{V}_1 c / M_2$. The logarithm in (8) may then be expanded and the series compared with (6). From this comparison we find the following values for the coefficients A_m for ideal solutions:

$$A_2 = \bar{V}_1 / 2 M_2{}^2, \qquad (9\text{a})$$

$$A_3 = \bar{V}_1{}^2 / 3 M_2{}^3, \qquad (9\text{b})$$

$$\vdots$$

$$A_n = \bar{V}_1{}^{n-1} / n M_2{}^n. \qquad (9\text{c})$$

III.2. Regular Solutions

In an *ideal* solution there is no heat of dilution and the entropy (times T) is equal to the negative of the free energy as given by Eq. (8). In a *regular* solution, after the definition of Hilde-

brand,[6] the heat of dilution is different from zero, but the entropy still has the ideal value. It has been found experimentally that many solutions of organic compounds where the molecules are approximately the same size fit into this classification. The equations of the regular solution therefore form a foundation on which to develop a theory for polymer solutions.

Derivation of the theory of regular solutions themselves from fundamental equations is outside the scope of this paper; here we will content ourselves with setting up formulae to describe the experimental facts.

The partial entropy of dilution equals the negative of the ideal free energy over T and will be given, using (9), by (10)

$$\Delta \bar{S}_1 = R \bar{V}_1 \left(\frac{1}{M_2} c + \sum_{n=2}^{\infty} \frac{\bar{V}_1{}^{n-1}}{n M_2{}^n} c^n \right). \qquad (10)$$

If we follow Hildebrand[6] and Scatchard,[7] we express the corresponding heat in the form

$$\Delta \bar{H}_1 = \sum_{n=2}^{\infty} B_n c^n, \qquad (11)$$

where the coefficients B_n are presumably nearly independent of temperature.

The omission of a term in the first power of c has been discussed theoretically by other authors[6,7] and justified by experiment.[6] It can be simply derived starting with Eq. (6) and calculating $\Delta \bar{H}_1$ thermodynamically (assuming the product, $V_1 c$, to be independent of T):

$$\Delta \bar{F}_1 = -\pi \bar{V}_1 = -RT \bar{V}_1 \left(\frac{1}{M_2} c + \sum_{n=2}^{\infty} A_n c^n \right), \qquad (12)$$

$$\Delta \bar{H}_1 = \frac{\partial (\Delta \bar{F}_1 / T)}{\partial (1/T)} = -R \sum_{n=2}^{\infty} \frac{\partial (A_n / \bar{V}_1{}^{n-1})}{\partial (1/T)} (\bar{V}_1 c)^n. \qquad (13)$$

B_n thus equals

$$-R \bar{V}_1{}^n \frac{\partial (A_n / \bar{V}_1{}^{n-1})}{\partial (1/T)}.$$

The free energy is obtained by adding the heat and entropy contributions:

$$\Delta \bar{F}_1 = \Delta \bar{H}_1 - T \Delta \bar{S}_1 = -RT \bar{V}_1 \left[\frac{1}{M_2} c \right.$$

$$\left. + \sum_{n=2}^{\infty} \left(\frac{\bar{V}_1{}^{n-1}}{n M_2{}^n} - \frac{B_n}{RT \bar{V}_1} \right) c^n \right]. \qquad (14)$$

[6] J. H. Hildebrand, *Solubility* (Reinhold Publishing Corporation, New York, 1936); Chem. Rev. **18**, 315 (1935).
[7] G. Scatchard, Chem. Rev. **8**, 321 (1931).

For the purposes of this paper we will express the osmotic pressure of a regular solution in the form (15).

$$\pi = -\frac{\Delta \bar{F}_1}{\bar{V}_1} = RT\left[\frac{1}{M_2}c + \sum_{n=2}^{\infty}\frac{\bar{V}_1^{n-1}}{nM_2^n}(1-\zeta_n)c^n\right]. \quad (15)$$

The heat coefficients, ζ_n, are given by

$$\zeta_n = \frac{nB_nM_2^n}{RT\bar{V}_1^n}. \quad (16)$$

Comparing (15) with (6), the assumption of a regular solution is seen to be equivalent to expressing the osmotic coefficients A_n in terms of the heat coefficients and molar volume:

$$A_n = \frac{\bar{V}_1^{n-1}}{nM_2^n}(1-\zeta_n). \quad (17)$$

While there is certainly a theoretical basis for this relation (17), it should be understood that its use here is purely empirical, justified by a large amount of experimental data on small-molecule organic compounds.

IV. APPLICATION TO SOLUTIONS OF LARGE MOLECULES

In most of the ensuing discussion only the coefficient A_2 will be considered, since this characterizes the most important deviations from Raoult's law. The succeeding coefficients contribute little in dilute solutions. We will find it convenient to treat first two simple cases, whose results can be used to good advantage in the more complicated treatment of the chain molecule. These two cases are those of large spherical and long rigid rod-like molecules suspended in an effectively continuous and structureless solvent. We will assume that the range of the intermolecular forces is negligible compared to the diameters of the particles.

It is then easy to predict the form of the distribution functions; they will be a constant (determined by Eq. (2)) except for those configurations where the molecules interpenetrate, and since the molecules are assumed to be rigid, the functions, which give the probabilities of these configurations, must be zero there.
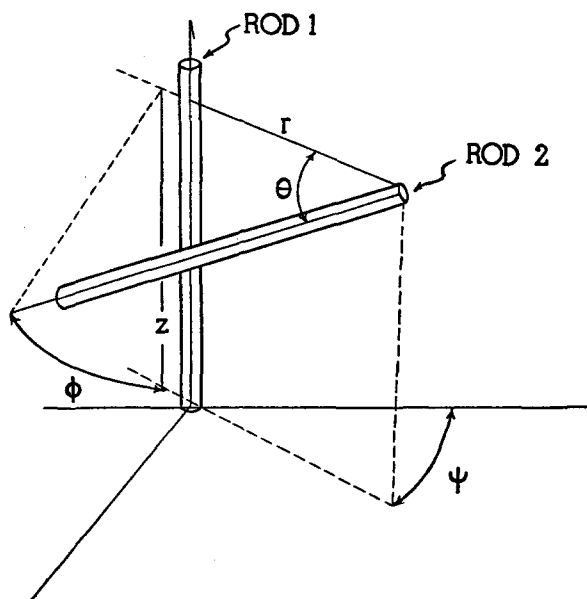

FIG. 1.

IV.1. Rigid Spheres

The spatial distribution function $F_2(1, 2)$, which gives the probability of sphere 1 and sphere 2 being a given distance apart, will be unity because of (2) except when the distance between the centers is less than the diameter a of the spheres.

$$F_2(1, 2) = 1 \quad \text{when} \quad r_{12} > a,$$
$$F_2(1, 2) = 0 \quad \text{when} \quad r_{12} < a, \quad (18)$$

$$g_2(1, 2) = 0 \quad \text{when} \quad r_{12} > a,$$
$$g_2(1, 2) = -1 \quad \text{when} \quad r_{12} < a, \quad (19)$$

$$A_2 = -\frac{N_0}{2VM_2^2}\int g_2(1, 2)d(1)d(2) = \frac{2\pi a^3 N_0}{3M_2^2}. \quad (20)$$

The volume, $\pi a^3/6$, of one sphere will presumably be proportional to the molecular weight, $\pi a^3/6 = M_2 v_0$, where v_0 is the volume per unit molecular weight, so that

$$A_2 = 4N_0 v_0/M_2. \quad (21)$$

IV.2. Long Rigid Rods

Consider two long cylindrical rods, numbered 1 and 2, respectively, of diameter d and length l, with l much greater than d. To describe the positions of this system requires ten coordinates,

three external and two internal for each rod. The distribution function, $F_2(1, 2)$, must be zero when these coordinates have such values that the rods interpenetrate one another.

The ten coordinates may be conveniently chosen as follows (see Fig. 1); the three cartesian coordinates, X, Y, and Z, of one end of rod (1); two polar angles, Θ and Φ, to specify the direction of the axis of rod (1); a system of normal cylindrical coordinates, a height, z, a radius, r, and an azimuth, ψ, to give the position of a selected end of rod (2) with respect to rod (1), with z taken in the direction of the axis of rod (1) and ψ measured from an arbitrary reference plane through the axis; two polar angles, ϑ and φ, to give the orientation of rod (2) with respect to the radius r; ϑ is the angle between the axis of (2) and r, and φ is the angle determined by the two planes through r and the axes of (1) and (2). The volume element in this ten-dimensional space is $dX dY dZ \sin \Theta d\Theta d\Phi dz r dr d\psi \sin \vartheta d\vartheta d\varphi$.

Our problem is to specify those regions of ten-dimensional space in which superposition of the two molecules occurs. For a given value of ϑ less than $\pi/2$, it may be seen from Fig. 1 that the rods only overlap when r is less than $l \cos \vartheta$ and simultaneously φ lies between approximately $d/r \tan \vartheta$ and $-d/r \tan \vartheta$ with z between $-r \tan \vartheta$ and $l-r \tan \vartheta$, or φ between $\pi-d/r \tan \vartheta$ and $\pi+d/r \tan \vartheta$ with z between $r \tan \vartheta$ and $l+r \tan \vartheta$. The distribution function, $F_2(1, 2)$, must be zero in these regions. Outside of these regions it must be constant, since no attractive forces are assumed to act. The value of the constant is determined by the normalization equation, Eq. (2), at $1/16\pi^2$.

Also from Eq. (2) it is found that $F_1(1) = F_1(2) = 1/4\pi$. $F_2(1, 2)$ as well as $g_2(1, 2)$ may now be written down.

$$
\left.\begin{matrix} F_2(1, 2) = 0 \\ g_2(1, 2) = -1/16\pi^2 \end{matrix}\right\}
\begin{cases}
\text{when } 0 < r < l \cos \vartheta \\
\quad 0 < \vartheta < \pi/2, \\
\text{and} \begin{cases} -d/r \tan \vartheta < \varphi < d/r \tan \vartheta \\ r \tan \vartheta < z < l + r \tan \vartheta, \end{cases} \\
\text{or} \begin{cases} \pi - d/r \tan \vartheta < \varphi < \pi + d/r \tan \vartheta \\ -r \tan \vartheta < z < l - r \tan \vartheta; \end{cases}
\end{cases}
$$

$$
\left.\begin{matrix} F_2(1, 2) = 1/16\pi^2 \\ g_2(1, 2) = 0 \end{matrix}\right\} \quad \begin{cases} \text{in all other ranges of the} \\ \text{coordinates.} \end{cases} \quad (22)
$$

The integral, A_2, is the integral of $g_2(1, 2)$ over that region where it is different from zero.

$$
A_2 = \frac{N_0}{32\pi^2 M_2^2 V} \underset{V}{\int \int \int} dX dY dZ \underset{4\pi}{\int \int} \sin \Theta d\Theta d\Phi
$$

$$
\times \int_0^{\pi/2} \sin \vartheta d\vartheta \int_0^{l \cos \vartheta} r dr \int_0^{2\pi} d\psi \left(\int_{r \tan \vartheta}^{l+r \tan \vartheta} dz \right.
$$

$$
\left. \times \int_{-d/r \tan \vartheta}^{d/r \tan \vartheta} d\varphi + \int_{-r \tan \vartheta}^{l-r \tan \vartheta} dz \int_{\pi-d/r \tan \vartheta}^{\pi+d/r \tan \vartheta} d\varphi \right), \quad (23)
$$

$$
A_2 = \pi N_0 d l^2 / 4 M_2^2. \quad (24)
$$

The evaluation is not difficult, but when l and d are of the same order of magnitude, the result must be taken with a grain of salt because of the approximations in the limits on φ. However these limits should be quite adequate when l is greater than d by a factor of ten or more.

It is interesting to note that A_2 is the same constant for a series of rods of the same diameter but of differing lengths, since in this case M_2 is proportional to l. Such constancy is experimentally found for solutions of homologous linear polymers, but does not appear for solutions of the more compact protein molecules. It will be found to arise again in the theory of chain-molecule solutions which follows.

The tremendous difference between the behavior of A_2 for spherical and elongated molecules is also important. With spherical molecules the deviations from Raoult's law are relatively no greater for large molecules than for small ones. With elongated molecules, however, size alone causes *relatively* larger deviations from ideality.

IV.3. Flexible Chain Molecules

A direct extension of the preceding integration to a flexible chain molecule of n links would be very difficult, since the number of coordinates involved would be at least $3n$. However, with the high polymer molecules usually encountered, their length and flexibility are such that the forces of interaction between two such molecules commonly involve only short lengths of the two chains. The parts of the two chains away from the point of collision are out of each other's force fields and have no direct action on one another.

This picture suggests that the pair interaction integral, A_2, depends on two independent things: a function of the size of the molecules only, and a quantity describing the interaction of short segments of the two chains. The following more detailed analysis supports this idea and gives it precise formulation.

IV.4. The Distribution Function F_2 and the Integral A_2 of a Pair of Chain Molecules

For two molecules in solution the form of the distribution function is:

$$
F_2(1, 2) = C^2 \exp \left[- U_2(1, 2)/kT \right], \quad (25)
$$

with a C a normalization constant. The energy quantity $U_2(1, 2)$ is the potential of average force in the infinitely dilute solution.[4,8]

The potential of two isolated molecules, when ordinary van der Waals forces are acting, is known

---

[8] J. G. Kirkwood, J. Chem. Phys. **3**, 300 (1935).

to be almost exactly representable as the sum of the potentials between pairs of atoms or groups of atoms in the two molecules. This assumption of superposition of potential probably does not hold as well for the potential of average force in liquids, but it is still a common and most useful assumption.[8] If we then divide the molecules into small parts, or "segments," the intermolecular potential will be closely equal to a sum of the interaction potentials of all the pairs of segments formed with one member from each molecule. In addition to this, the total potential must include a term which depends on the shape and internal coordinates of each molecule alone, irrespective of the state of the other. In symbols,

$$U_2(1, 2) = U_1(1) + U_2(2) + \sum_{\mu_1=1}^{n} \sum_{\nu_2=1}^{n} u(\mu_1, \nu_2). \quad (26)$$

The functions $U_1(1)$ and $U_1(2)$ are the internal energies of 1 and 2, respectively, while the functions $u(\mu_1, \nu_2)$ are interaction potentials of segment μ_1 of molecule 1 with segment ν_2 of molecule 2.

It is necessary to interject here a word about symbolism. In this paper it will often be necessary to distinguish between the coordinates of whole molecules and those of their constituent segments. In general, numerals will be used for whole molecules and Greek letters for segments. Thus (1) and (2) represent all the coordinates, internal and external, of the molecules 1 and 2, while (μ) and (ν) represent the same thing for the segments μ and ν. Where it is desired to show that a segment belongs to a particular molecule, the symbol for the molecule will follow that for the segment; thus (μ_1) represents the coordinates of segment μ of molecule 1. This double symbol will be avoided where there is no danger of ambiguity.

Classical statistical mechanics also states similar equations for F_1.

$$F_1(1) = C \exp [-U_1(1)/kT];$$
$$F_1(2) = C \exp [-U_1(2)/kT]. \quad (27)$$

If we then define a function $\varphi(\mu, \nu)$, such that

$$\varphi(\mu, \nu) = \exp [-u(\mu, \nu)/kT], \quad (28)$$

we can write $F_2(1, 2)$ as a product, Eq. (29),

$$F_2(1, 2) = F_1(1)F_1(2) \prod_{\mu_1} \prod_{\nu_2} \varphi(\mu_1, \nu_2). \quad (29)$$

It is now desired to transform (29) to a form that separates the interactions of the segments into separate terms.

It was remarked earlier that if 1 and 2 are separated in space by a considerable distance, $F_2(1, 2)$ must be equal to a simple product of $F_1(1)$ and $F_1(2)$, as a result of the lack of long range forces in the liquid.

$$\lim_{r_{12} \to \infty} F_2(1, 2) = F_1(1)F_1(2). \quad (30)$$

Therefore,

$$\lim_{r_{\mu\nu} \to \infty} \varphi(\mu, \nu) = 1. \quad (31)$$

Let us define by Eq. (32) a function, $\chi(\mu, \nu)$, which bears the same relation to $\varphi(\mu, \nu)$ that g_2 bears to F_2.

$$\varphi(\mu, \nu) = 1 + \chi(\mu, \nu). \quad (32)$$

If we substitute (32) in (29), and expand the product, we obtain the desired sum, as follows:

$$F_2(1, 2) = F_1(1)F_2(2)\Big[1 + \sum_{\mu_1} \sum_{\nu_2} \chi(\mu_1, \nu_2)$$
$$+ \sum_{\kappa_1 \leq \mu_1; \lambda_2 \leq \nu_2} \sum \sum \sum \chi(\kappa_1, \lambda_2)\chi(\mu_1, \nu_2) + \cdots\Big]. \quad (33)$$

By Eq. (5a) $g_2(1, 2)$ is found.

$$g_2(1, 2) = F_1(1)F_2(2)\Big[\sum_{\mu_1} \sum_{\nu_2} \chi(\mu_1, \nu_2)$$
$$+ \sum_{\kappa_1 \leq \mu_1; \lambda_2 \leq \nu_2} \sum \sum \sum \chi(\kappa_1, \lambda_2)\chi(\mu_1, \nu_2) + \cdots\Big]. \quad (34)$$

To get A_2, we integrate over $g_2(1, 2)$, eliminating the F_1's from the first term of the sum with the aid of Eq. (4).

$$A_2 = -\frac{N_0}{2M_2{}^2}\Big[\sum_{\mu_1} \sum_{\nu_2} \int \chi(\mu_1, \nu_2)d(\nu_2) + \cdots\Big]. \quad (35)$$

The exact procedure in eliminating the functions F_1 from one of the terms of the first sum is as follows. With μ_1 held fast the function is integrated over all the other coordinates of molecule 1. These occur only in the function $F_1(1)$, so that the rest of the integrand is a constant. $F_1(2)$ is treated in a similar way, holding ν_2 fast. The results of these two integrations are unity by Eq. (4). All that remains, then, is the integration of $\chi(\mu, \nu)$ over $d(\mu_1)d(\nu_2)$. If the coordinates of ν_2 are chosen relative to those of μ_1 as origin, so that the relative positions of μ_1 and ν_2 do not change as that of μ_1 is varied, then $\chi(\mu, \nu)$ is independent of the value of the coordinates (μ_1). Hence the integration over μ_1 leads only to the volume, V, as a multiplicative constant.

Up to this point the discussion has been applicable to any type of large molecule. Henceforth we will restrict ourselves more specifically to chain molecules, *all of whose segments are alike.*

The actual size of the segments is still unchosen, however; they may consist of two, four, ten, or more links of the chain, as long as they are replicas of one another.

When the segments are all alike, the integrals $\int \chi(\mu, \nu) d(\nu)$ are identical for all pairs and are a constant which we shall write $\int \chi d\nu$ for brevity. If there are n segments in each of the molecules, there are n^2 pairs formed with one member from each molecule, and A_2 is given by

$$
\begin{aligned}
A_2 &= -\frac{N_0 n^2}{2 M_2{}^2}\left(\int \chi d\nu + \cdots\right), \\
&= -\frac{N_0}{2 m^2}\left(\int \chi d\nu + \cdots\right),
\end{aligned}
\tag{36}
$$

where $m = M_2/n$ is the molecular weight of a segment.

IV.5. Evaluation of the Constant, $\int \chi d\nu$

As far as $\int \chi d\nu$ itself is concerned, it is possible to evaluate it readily if the segments of the chain can be chosen to be of nearly the same size and shape as the molecules of the solvent.

The functions $\varphi(\mu, \nu)$ are actually defined for liquid solutions only by Eq. (29), but we must nevertheless attempt to evaluate them independently if they are to be of any use. Comparison of Eqs. (28) and (25) suggests that they would be the distribution functions, $F_2(\mu, \nu)$, of the segments μ and ν if μ and ν were cut loose from the rest of their chains and allowed to roam about as free molecules.[9] Therefore,

$$
\chi(\mu_1, \nu_2) = g_2(\mu, \nu), \tag{37a}
$$

$$
-\frac{N_0}{2 m^2}\int \chi d\nu = A_{2, \mu\nu}, \tag{37b}
$$

where $g_2(\mu, \nu)$ is the g-function and $A_{2, \mu\nu}$ the second osmotic coefficient of two free molecules whose constitution is the same as that of the segments μ_1 and ν_2.

We can now use Eq. (17) for $A_{2, \mu\nu}$ and obtain an expression for $\int \chi d\nu$ in this case where the segments are nearly of the same size as the

---

[9] We are assuming that the breaking of their bonds to set them free would have a negligible effect on the van der Waals and repulsive potentials between μ and ν.

solvent molecules.

$$
N_0 \int \chi d\nu = \bar{V}_1 (1 - \zeta_2). \tag{38}
$$

If now the situation is such that the higher integrals in (36) can be neglected, A_2 for the polymer is evaluated:

$$
A_2 = \frac{\bar{V}_1}{2 m^2}(1 - \zeta_2). \tag{39}
$$

Since the segment molecular weight, m, is not precisely defined, it is more convenient to introduce a dimensionless parameter, β, defined as the ratio of the molecular volume of a segment to that of a solvent molecule. If ρ_2 is the density of the polymer segments, A_2 can be written as follows:

$$
A_2 = \frac{(1 - \zeta_2)}{2\beta^2 \bar{V}_1 \rho_2{}^2}. \tag{40}
$$

This equation is very similar to one derived by Flory[1] from a quasi-lattice picture; the only difference is that the segment size, β, is squared in our equation whereas it appears to the first power only in Flory's. In view of the rather indefinite meaning of β, however, this is probably not an important difference.

A really serious defect of Eq. (40), at least when applied to solutions where the solvent molecules are small, is the neglect of the higher terms in (36), which, unfortunately, are not so easily evaluated.

IV.6. Discussion of the Higher Segment Interaction Integrals

The significance of the integrals involving products of several χ-functions will become clearer from an examination of the meaning of the various terms in Eq. (33) for the pair distribution function, $F_2(1, 2)$. The probability of a given configuration of the two molecules is equal to the product of the internal distribution functions of the two plus a series of correction terms. The first group of these, the double sum over the set of $\chi(\mu, \nu)$, corrects the probability of finding a segment of one molecule at a particular place for the chance that a segment of the other molecule will interfere with its being at that

place. The chance that the segment of the other molecule will be there is assumed to be independent of where the remaining segments are. This last assumption obviously requires correction in its turn, since the segments of a molecule are not independent of one another. The correction is made automatically by the succeeding terms in the series; for example, the term $\chi(\kappa_1, \lambda_2)\chi(\mu_1, \nu_2)$ can be considered to describe the effect of the interaction of segments κ_1 and λ_2 on the probability of a meeting between segments μ_1 and ν_2.

Most of these higher integrals must always be small, as can be seen from the following argument: If several pairs of segments are picked at random from two freely coiling chains, in an overwhelming preponderance of the configuration space, at least one pair will be far apart and its χ-function will be zero. As a result the integral over a product of randomly selected χ-functions is chiefly an integral over zero and has a very small value which we will discuss later. Only if the segments are so tied together that when one pair is close the other pairs must be close also is the product of the χ-functions different from zero over an appreciable region. In such a case the integral will not be negligible.

Tying together of this sort can occur if the segments are short and are adjacent to one another on the two chains. It can be seen that when the χ-function between one pair of segments is large as a result of the segments being close, the χ-functions between the neighboring segments will be large also. We might expect then that the integral over this product of χ-functions would be of the same order of magnitude as $\int \chi d\nu$ itself.

Moreover, it may well occur that at least some of the integrals that involve an *even* number of χ-functions will be positive, while $\int \chi d\nu$ and others involving an odd number will be negative (because of the predominately negative character of χ), so that the sum will be smaller than $\int \chi d\nu$ alone. It will be seen that there is both theoretical and experimental evidence for such an effect.

It should be noted that the number of such terms that are tied close together and are not small is proportional to the number, n^2, of pairs of segments that can be formed between the chains, so that even with these terms included A_2 is still practically independent of chain length.

The group of higher terms is thus broken into two subgroups that have similar origins but different effects on A_2. The subgroup we have just considered is proportional to n^2 like the simplest terms, $\sum \int \chi d\nu$; this has the important consequence that A_2 is independent of the molecular weight of the solute. The remaining subgroup, arising from the interactions of distantly connected pairs of segments, will be seen to increase more rapidly with n and to be small for the molecular weights generally investigated. Discussion of these is deferred to Section **VI.2**.

We reemphasize the physical meaning of the important first sub-group of terms. These arise because any pair of segments, e.g., μ_1 and ν_2, are hindered in their motions by interferences between the neighboring segments that are bonded to them. If $\int \chi d\nu$ is negative, meaning that the segments repel one another on the average more than they attract, as is generally the case, the over-all effect of these interferences will be to keep μ_1 and ν_2 apart, thus reducing their contribution to A_2. Therefore it is physically reasonable that many of the terms describing these interferences will be different in sign from $\int \chi d\nu$. The previous neglect of this effect has led to discrepancies between experiment and theory.

It is important to know when this subgroup of the terms after $\sum \int \chi d\nu$ in (36) may be negligible. This will happen if the segments themselves are so long that two adjacent pairs on the two chains are seldom close enough to have their χ-functions simultaneously different from zero. Equation (36) may be restricted to the first term, then, if the segments are many times longer than their width (and the effects discussed in Section **VI.2** are negligible.)

We are thus faced with a dilemma in calculating A_2. If we wish to restrict ourselves to calculating $\int \chi d\nu$, we must divide the chains into long and complicated segments, but if we wish to use small simple segments, we must consider the higher integrals. Two examples will illustrate this point.
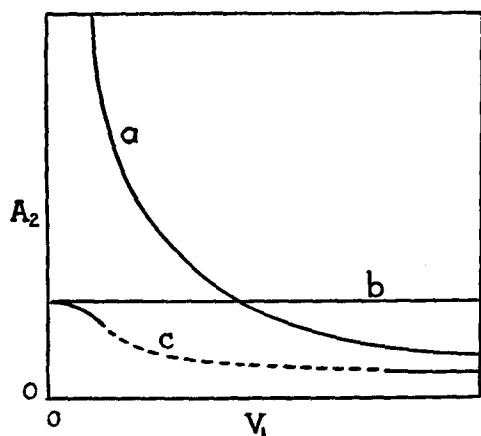
Polystyrene,

$$-CH_2-CH-CH_2-CH-CH_2-CH-$$
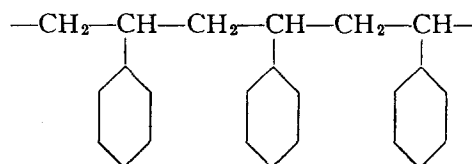
FIG. 2.

dissolves readily in toluene,

$$CH_3$$

to form solutions in which it would be expected that the heat of dilution, $\Delta \bar{H}_1$, and hence ζ_2, would be very small. It would also seem reasonable to choose the styrene units as the segments, so that β is approximately unity. Nevertheless, Eq. (40) with these constants fails to agree with experimental data quoted in Section V by factors varying from eight to ten. A little consideration shows that it is precisely in a case of this kind where the segments are short and bulky that the higher integrals should be largest.

As a second example, consider the following simple case: A pair of model molecules is made by stringing spherical beads on two straight wires. Let the diameter of the beads be d and the length of the wires be l. There are l/d beads on each wire. Suppose the models to be floating in a continuous fluid. If the beads are taken as the segments, then by (20), (36), and (37),

$$\int \chi d\nu = -4\pi d^3/3,$$

$$A_2 = \frac{N_0(l/d)^2}{2M_2} \cdot \frac{4\pi d^3}{3} = \frac{2\pi N_0 dl^2}{3M_2{}^2}. \qquad (41)$$

However, it is also possible to calculate A_2

approximately from Eq. (24) for a pair of stiff rods, using d as the approximate diameter of the rods. In this case A_2 is found to be $\pi N_0 dl^2/4M_2{}^2$.

The discrepancy, a factor of about three, must be due to the neglect of the higher terms.

Before further discussion we will develop an alternative approach, useful when the diameter of the chains is much greater than the solvent molecules, in the following section.

IV.7. Evaluation of A_2 for a Chain Molecule in a Continuous Solvent

Let us first find out what happens when a pair of rod molecules, such as were considered in Section IV.2, are made flexible by introducing universal joints at intervals along the rods. Provided the segments between the joints are still long enough to be treated as thin rods themselves, the constant $\int \chi d\nu$ is obtained by analogy to Eq. (24) as $\pi d\lambda^2/2$ with λ the length of the segment. Since the segments are long in this case, we can neglect the higher terms and obtain A_2 directly:

$$A_2 = \pi N_0 n^2 d\lambda^2/4M_2{}^2. \qquad (42)$$

But since $n\lambda$ is equal to l, the length of the whole chain,

$$A_2 = \pi N_0 dl^2/4M_2{}^2. \qquad (43)$$

This is identical with formula (24) for a pair of rods without joints. Therefore the introduction of a moderate number of joints in a stiff molecule has no appreciable effect on A_2.[10]

It might be worth while to use this formula, (43), for chain molecules when the solvent molecules are small compared with the diameter of the chain, as, for example, the cellulose esters in volatile solvents.

IV.8. General Discussion

Two formulae have now been derived, each of which is a good approximation to the true formula under different special conditions. Equation (40), involving the neglect of interactions between any groups of segments more complicated than pairs, should work best when the segments are very long and extended compared with their diameters. In this case higher inter-

---

[10] M. L. Huggins (see reference 2) has arrived at a similar conclusion on the basis of the quasi-lattice picture.

actions are improbable. Since a segment in this equation is defined as the section of polymer chain most nearly equivalent to the solvent molecules in size, Eq. (40) should then be valid in solvents whose molecules are large compared to the diameter of the chain.

On the other hand, the rod formula, Eq. (42), should be true when the polymer chain is moderately extended and the solvent is effectively continuous, that is, consists of molecules much smaller than the diameter of the chain.

The situation is illustrated in Fig. 2, where A_2 for a given polymer is plotted against V_1 for a series of homologous solvents. (The heat coefficient, ζ_2, is assumed to be constant throughout the series.) The curve a is a plot of Eq. (40) (with β assumed constant); the horizontal line b is Eq. (42); the probable course of the true curve is given by c. At very large values of $\bar{V}_1$ the true curve approaches (40) and at small values (42). In the dotted transition region it probably lies below a because of the negative signs of most of the higher integrals in (36). The exact course of the function here can, however, only be guessed at present.

V. COMPARISON WITH EXPERIMENT

V.1. Three General Parameters

Although there is insufficient data in the literature to test all the conclusions drawn from the theory, there is enough to confirm some of its main features. In order to compare different systems on a common basis, it seems best to write A_2 in terms of three experimentally defined parameters, σ, ζ, and λ, corresponding respectively to the entropy, heat and free energy of dilution, as follows:

$$A_2 = \frac{\sigma(1-\zeta)}{2\bar{V}_1\rho_2{}^2} = \frac{\lambda}{2\bar{V}_1\rho_2{}^2}. \qquad (44)$$

The heat coefficient, ζ, is defined as the ratio of the coefficient of c^2 in the expansion of the molar heat of dilution, $\Delta\bar{H}_1$, to the corresponding coefficient in the expansion for $T\Delta\bar{S}_1$. ζ has the special significance of describing the relative contributions of the heat and entropy to the deviation of the free energy of dilution from van't Hoff's law. Experimentally, ζ may be

determined from either of the following (equivalent) relations:

$$\zeta = \lim_{c \to 0} \frac{1}{T}\left(\frac{\partial^2\Delta\bar{H}_1}{\partial c^2} \Big/ \frac{\partial^2\Delta\bar{S}_1}{\partial c^2}\right), \qquad (45a)$$

$$\zeta = T\frac{\partial(A_2/\bar{V}_1)}{\partial T} \Big/ \frac{\partial(TA_2/\bar{V}_1)}{\partial T}. \qquad (45b)$$

If the segments are chosen so long that higher integrals of the type discussed in Section IV.6 are negligible, and at the same time the other type of higher integrals which lead to the molecular weight dependence of A_2 discussed in Section VI.2 are negligible, ζ can be identified with the heat coefficient of the segments, ζ_2, which was employed in developing the theory of the preceding sections. If the higher order interactions are not negligible, as is usually the case in practice, ζ will depend on these interactions and the heat of dilution characteristics of the polymer may not always be identical with those of its small molecule homologues.[11]

On the other hand the parameter, σ, is proportional to the coefficient of the square term in the expansion of $\Delta\bar{S}_1$ in terms of concentration. If the segments are long and the higher integrals negligible, it should be equal to $1/\beta^2$, where β is the ratio of segment to solvent molecule size as used in Eq. (40). In other cases it will depend not only on the size but also on the shape of the segments and where A_2 varies with the molecular weight it will depend on the shape and size of the molecule as a whole (Eq. (60)). Since σ can be considered to be a function of the way in which the segments of one molecule shield each other from other molecules, the name "internal shielding factor" seems appropriate.

In Fig. 2, σ is β^2 times the ratio of the ordinates of curves c and a.

Experimentally, the internal shielding factor,

---

[11] Cases are well-known where the difference in $\Delta\bar{H}_1$ causes striking differences in the solubility characteristics of the polymer and monomer. This is especially true for highly polar materials, where the intermolecular forces are especially sensitive to small changes in the relative positions of the molecules. For example, polyvinylchloride and polyacrylonitrile are quite insoluble in their respective monomers, although both are amorphous materials and soluble in other solvents. The presence of a double bond, which usually has very little effect on $\Delta\bar{H}_1$, can hardly cause all this difference.

σ, may be defined as follows:

$$\sigma = \lim_{c \to 0} \frac{2\rho_2{}^2}{R}\left(\frac{\partial^2 \Delta \tilde{S}_1}{\partial c^2}\right), \qquad (46a)$$

$$\sigma = 2\rho_2{}^2 \bar{V}_1{}^2 \frac{\partial(TA_2/\bar{V}_1)}{\partial T}. \qquad (46b)$$

It seems reasonable that the molecular structure of the solution will be independent of temperature to a first approximation. To the same degree of approximation $\Delta \tilde{S}_1$ and $\Delta \bar{H}_1$ will not change with the temperature. Therefore, by Eqs. (45a) and (46a), σ will be constant and ζ will vary as $1/T$.[12]

Thus it appears that ζ is of considerable importance for the description of the nature of the forces between the molecules of the solvent and solute, while σ is principally determined by the shape of the segments.*

The free energy parameter, $\lambda = \sigma(1-\zeta)$, is useful mainly as a dimensionless empirical constant to describe the deviation of the free energy of dilution from the classical ideal solution laws.[13]

V.2. Chain Molecules

Sufficient data are available on at least five systems to calculate the three parameters, λ, σ, and ζ. A_2 has been determined for polyisobutylene in cyclohexane,[14] and polystyrene in toluene.[15]

[12] T. Alfrey and P. Doty, J. Chem. Phys. **13**, 77 (1945) and W. C. Orr, Trans. Faraday Soc. **40**, 320 (1944) have considered the effect of temperature-sensitive liquid structure on $\Delta \tilde{S}_1$ and $\Delta \bar{H}_1$, using the lattice model.

* It may be helpful to point out the significance of the various values of ζ. If $\zeta = 0$, the heat of dilution is zero and the solvent is indifferent to the solute. If ζ is positive, heat is absorbed on dilution, meaning that the solvent and solute molecules tend to associate in groups containing others of their own kind. If $\zeta > 1$, this heat effect contributes more to A_2, which determines the deviations of the free energy from Raoult's law, than the entropy of dilution and the system tends to separate into two phases. (A quantitative discussion of the separation of phases requires the investigation of the higher A_n, which is omitted from this paper, except for the few remarks in section VI.4.) On the other hand, if ζ is negative, heat is evolved on dilution and the attraction between unlike molecules is greater than between those of the same kind.

[13] Other authors (references 1 and 2) have introduced a constant, usually designated μ, where $\lambda = 1-2\mu$, for the same purpose. If σ were unity, 2μ would equal ζ. Since σ is usually found to be rather far from unity, the author prefers to use λ, which is directly proportional to the deviation in the free energy, rather than μ.

[14] P. J. Flory, J. Am. Chem. Soc. **65**, 372 (1943).

[15] T. Alfrey, A. Bartovics, and H. Mark, J. Am. Chem. Soc. **65**, 2319 (1943).

TABLE I.

System	Reference	A_2 cm³ g⁻²	λ	ζ	σ
Polyisobutylene-Cyclohexane	14				
Fraction B(0-3), Mol. Wt. 844,000		7.1×10^{-4}	0.13	(0)*	0.13
Fraction B(9-13), Mol. Wt. 79,200		7.1×10^{-4}	0.13	(0)*	0.13
Polystyrene (Polymerized at 120°)—Toluene	15				
Fraction 120-1, Mol. Wt. 340,000		3.82×10^{-4}	0.090	(0)*	0.090
Fraction 120-4, Mol. Wt. 108,000		3.72×10^{-4}	0.088	(0)*	0.088
Polystyrene (Polymerized at 180°)—Toluene	15				
Fraction 180-1, Mol. Wt. 169,000		4.78×10^{-4}	0.124	(0)*	0.124
Fraction 180-4, Mol. Wt. 110,000		4.53×10^{-4}	0.118	(0)*	0.118
Rubber-Benzene	16	9.5×10^{-4}	0.12	0.54	0.26
Rubber-Toluene	17	13.4×10^{-4}	0.21	0.3	0.3
Polyvinyl Chloride-2-Butanone	18	4.9×10^{-4}	0.18	0.0	0.18

* The values in parentheses have been estimated to be zero because both components have similar internal pressures.

Although the heat of dilution has not been measured, it is tentatively assigned the value zero because of the non-polar character and chemical similarity of the solvent and solute. On the systems rubber-benzene,[16] rubber-toluene,[17] and polyvinylchloride-methylethylketone,[18] both the osmotic pressure and its temperature coefficient have been determined. The data and values of the parameters are summarized in Table I.

In the first place, A_2 is actually found to be independent of molecular weight in at least two of these systems. In the second place, σ is uniformly much smaller than unity, as would be expected from the argument of the foregoing sections. The values for polystyrene are of particular interest. Since σ is here affected by the method of polymerization of the material, but not by the molecular weight, it follows that the local structure of the chains must depend on the method of polymerization. One is tempted to guess that the higher values of σ, corresponding to smaller amounts of internal shielding, go with the chains that are more extended because of the arrangement of their carbon skeletons.

V.3. Proteins

Although the experimental data on proteins are even less complete than for chain polymers, the following two examples serve to illustrate the osmotic properties of this interesting class of compounds.

[16] G. Gee and L. R. H. Treloar, Trans. Faraday Soc. **38**, 147 (1942).

[17] K. H. Meyer and C. G. Boissonnas, Helv. Chim. Acta **23**, 430 (1940)

[18] P. Doty, private communication.

Hemoglobin has been studied by Adair[19] and serum albumin by Burk.[20] The osmotic pressure was measured as a function of concentration at the isoelectric point in strong salt solution, where the long range forces between the protein molecules should be at a minimum. For this reason the heat coefficient, ζ, is probably not large. For purposes of illustration it is assumed to be zero.

Ultracentrifugal measurements in Svedberg's laboratory[21] have shown that both these materials have compact molecules; it is stated that they may be approximated by prolate ellipsoids whose axis ratio is no greater than four. Therefore the internal shielding factor, σ, should be very small, and A_2 should approximate the value of $4N_0v_0/M_2^2$ calculated for spherical molecules (Eq. (21)). From the data given in Table II, it can be seen that the materials behave as predicted.

VI. REFINEMENTS AND EXTENSIONS OF THE THEORY FOR CHAIN MOLECULES

VI.1. Polymers of Heterogeneous Molecular Weight[22]

Since the coefficient A_2 is independent of the molecular weight of the solute to the approximation heretofore considered, it is to be expected that it will be unchanged if the solute is a mixture of chain molecules differing only in length. This can, however, be proved directly, as will be shown now.

McMillan and Mayer[4] give an equation for the osmotic pressure of a solution of mixed solutes

which can be written as follows:

$$\pi = RT[\sum_i c_i/M_i + \sum_i \sum_j A_{ij}c_ic_j + \cdots]; \quad (47)$$

c_i is the concentration of the ith species of solute, whose molecules consist here of chains of n_i links, M_i is its molecular weight, and the coefficients A_{ij} are defined by the following equation:

$$A_{ij} = -\frac{N_0}{M_iM_jV} \int g_2(i, j)d(i)d(j). \quad (48)$$

Here i stands for a molecule of the ith species and j for one of the jth species.

By applying exactly the same procedure as previously, these coefficients are immediately found to be

$$A_{ij} = -\frac{N_0}{2m^2} \int \chi d\nu. \quad (49)$$

Inserting these values in (47) the desired relation is obtained.

$$\pi = RT\left[\sum_i c_i/M_i + \frac{N_0}{2m^2}\left(-\int \chi d\nu\right) \times (\sum_i \sum_j c_ic_j) + \cdots \right], \quad (50)$$

$$\pi = RT\left[c/\bar{M} + \frac{N_0}{2m^2}\left(-\int \chi d\nu\right)c^2 + \cdots \right]; \quad (51)$$

$c = \sum c_i$ is the total concentration and

$$\bar{M} = \sum c_i/\sum c_i/M_i = \sum N_iM_i/\sum N_i$$

(where $N_i = c_i/M_i$ is the number of moles of the ith species present) is the so-called "number-average" molecular weight.

VI.2. Dependence of A_2 on Molecular Weight

Previously we have considered only the integral of terms of the type $\chi(\mu, \nu)$ and those of the higher terms whose χ-functions were closely tied together by the chain skeleton. The question now arises as to what effect the interaction of segments far down the chains from the two involved in $\chi(\mu, \nu)$ has on the over-all integral, A_2. Any one of these terms will be small as a result of the argument in Section IV.6, but there are very large numbers of them.

TABLE II.

Substance	Reference	Solvent	A_2 cm^3 g^{-2}	σ	$4N_0v_0/M_2$*
Hemoglobin (Mol. Wt. 67,000)	19	KH$_2$PO$_4$, 0.033M; Na$_2$HPO$_4$, 0.033M; KCl, 0.1M, in Water; pH 6.8	5.6×10^{-5}	0.0037	4.4×10^{-5}
Serum Albumin (Mol. Wt. 75,000)	20	(NH$_4$)$_2$SO$_4$, 0.74M; 0.05M Acetate Buffer in Water; pH 4.8	6.5×10^{-5}	0.0043	4.0×10^{-5}

* The volume per unit molecular weight, v_0, was calculated from a partial specific volume of 0.75.

[19] G. S. Adair, Proc. Roy. Soc. London **120A**, 573 (1928).
[20] N. F. Burk, J. Biol. Chem. **98**, 353 (1932).
[21] T. Svedberg and K. O. Pedersen, *The Ultracentrifuge* (Oxford University Press, Oxford, 1940).
[22] Lattice theory discussions of this topic have been given by P. J. Flory, J. Chem. Phys. **12**, 425 (1944) and by R. L. Scott and M. Magat, *ibid.* **13**, 172 (1945).

Intuitively it can be seen that the more tightly the chains are curled, the more important will be terms of this type, involving the interaction of distant segments. This conclusion is supported by the calculation of the following section, which predicts that A_2 will depend on the molecular weight to a degree determined by the internal distribution function, that is, the average shape, of the molecules. The group of terms we now desire to evaluate in Eq. (35) begins with the following quadruple sum,

$$\sum_{\kappa_1 \leq \mu_1; \ \lambda_2 \leq \nu_2} \sum \sum \sum \frac{1}{V} \int \int \cdots \int F_1(1) F_1(2)$$

$$\times \chi(\kappa_1, \lambda_2) \chi(\mu_1, \nu_2) d(1) d(2), \quad (52)$$

in which μ_1 and ν_2 are not simultaneously equal to κ_1 and λ_2, respectively. Consider a typical term of this sum. The integral may obviously be factored as shown:

$$\frac{1}{V} \int \left[\int F_1(1) d(1 - \kappa, \mu) \right]$$

$$\times \left[\int F_1(2) d(2 - \lambda, \nu) \right]$$

$$\times \chi(\kappa_1, \lambda_2) \chi(\mu_1, \nu_2) d(\kappa_1) d(\mu_1) d(\lambda_2) d(\nu_2). \quad (53)$$

(The symbol $(1 - \kappa, \mu)$ stands for all the coordinates of molecule 1 except those involving segments κ and μ.) By definition (in analogy to Eq. (3)) the partial integral $\int F_1(1) d(1 - \kappa, \mu)$ is the distribution function of segments κ and μ with respect to one another, averaged over all the positions of the other segments. If κ and μ are separated by several links of the chain, this distribution function must be the well-known Gaussian distribution widely used in the theory of rubber elasticity.[23] Therefore,

$$\int F_1(1) d(1 - \kappa, \mu) = (3/2 \pi R_{\kappa\mu}^2)^{\frac{3}{2}}$$

$$\times \exp (-3 r_{\kappa\mu}^2 / 2 R_{\kappa\mu}^2). \quad (54)$$

In this equation $r_{\kappa\mu}$ is the distance between segments κ and μ, while $R_{\kappa\mu}$ is the root-mean-square value of this distance, that is $R_{\kappa\mu}$ satisfies

[23] W. Kuhn, Kolloid Zeits. **68**, 2 (1934).

the identity

$$R_{\kappa\mu}^2 = (3/2 \pi R_{\kappa\mu}^2)^{\frac{3}{2}} \int_0^\infty 4 \pi r_{\kappa\mu}^4$$

$$\times \exp (-3 r_{\kappa\mu}^2 / 2 R_{\kappa\mu}^2) d r_{\kappa\mu}. \quad (55)$$

A similar equation holds for the segments λ and ν, provided there are several links of the chain of molecule 2 between them.

In order to simplify the integral further, use may be made of the special properties of the χ-functions. In (53) we hold the position of segment κ fixed and integrate over the positions of segment λ. In the first place, $\chi(\kappa, \lambda)$ is zero except in the narrow region where λ is very close to κ; in the second place, the only other part of the integrand that depends on the position of λ, $\int F_1(2) d(2 - \lambda, \nu)$, is a slowly varying function of (λ) and therefore may be considered to be a constant for practical purposes whenever $\chi(\kappa, \lambda)$ is different from zero. As a result the integration over the positions of λ may be considered to involve $\chi(\kappa, \lambda)$ alone in the integrand. A similar conclusion is reached concerning segment ν if segment μ is considered fixed. Also, it is noticed that $r_{\lambda\nu}$ must nearly equal $r_{\kappa\mu}$, since either $\chi(\kappa, \lambda)$ or $\chi(\mu, \nu)$ will be zero if this is not so.

Making use of these facts, together with Eq. (54), the integral (53) can now be written in a manageable form, as follows:

$$4\pi \left(\int \chi d\nu \right)^2 \left(\frac{3}{2 \pi R_{\kappa\mu} R_{\lambda\nu}} \right)^3 \int_0^\infty r_{\kappa\mu}^2$$

$$\times \exp \left\{ -\left[\frac{3 (R_{\kappa\mu}^2 + R_{\lambda\nu}^2) r_{\kappa\mu}^2}{2 R_{\kappa\mu}^2 R_{\lambda\nu}^2} \right] \right\} d r_{\kappa\mu}. \quad (56)$$

(Choosing the coordinates of the segment κ as the origin for all the other coordinates of the system, it is obvious that the integrand as a whole will not depend on (κ); therefore the integration over (κ) yields a factor V which cancels that in the denominator of (53). The coordinates of segment μ are now referred to those of κ as origin and are therefore written explicitly in polar form as $4 \pi r_{\kappa\mu}^2 d r_{\kappa\mu}$.)

It may be objected that these two approximations (of using Eq. (54) for the internal distribution functions and assuming that they vary negligibly slowly compared to the χ-functions)

are not valid for those quartets of segments of which any pair are close together on one chain. This is quite true, if the segments are short, as has been remarked previously, but if the segments are long, the size of all such terms is small compared to the aggregate of the terms for which all segments on the same molecule are separated by a considerable length of chain and for which these approximations hold. *Therefore it is to be understood in this section that the χ-functions refer to long segments.* The definition of a segment as the part of chain most nearly equivalent to a solvent molecule, which was used conveniently elsewhere, does *not* apply here.

Equation (56) may be integrated immediately to give:

$$\left[\frac{3}{2\pi(R_{\kappa\mu}{}^2+R_{\lambda\nu}{}^2)}\right]^{\frac{3}{2}}\left(\int\chi d\nu\right)^2. \qquad (57)$$

It is now necessary to subject this expression to the quadruple summation indicated in (52). The variables in this process are the root-mean-square distances, $R_{\kappa\mu}$ and $R_{\lambda\nu}$. It is well known[23] that the squares of these are proportional to the number of segments between κ and μ, or between λ and ν, if the chains are flexible enough to coil at random. In other words, if the segments are numbered consecutively,

$$R_{\kappa\mu}{}^2=b^2(\mu-\kappa). \qquad (58)$$

The proportionality constant, b, is related to the microscopic structure of the chains. We can now replace the sum by an integral and finish the calculation by simple calculus.

$$(3/2\pi b^2)^{\frac{3}{2}}\left(\int\chi d\nu\right)^2\sum_{\kappa\le\mu;\,\lambda\le\nu}\sum\sum\sum 1/(\mu-\kappa+\nu-\lambda)^{\frac{3}{2}}$$

$$=(3/2\pi b^2)^{\frac{3}{2}}\left(\int\chi d\nu\right)^2\int_0^\mu dk\int_0^n d\mu$$

$$\times\int_0^\nu d\lambda\int_0^n 1/(\mu-\kappa+\nu-\lambda)^{\frac{3}{2}}d\nu$$

$$=(4/15)(28-16\sqrt{2})(3/2\pi b^2)^{\frac{3}{2}}n^{5/2}\left(\int\chi d\nu\right)^2$$

$$=0.47n^{5/2}b^{-3}\left(\int\chi d\nu\right)^2. \qquad (59)$$

When this is introduced into Eq. (36) the following amplification of our previous expression for A_2 results:

$$A_2=-\frac{N_0}{2m^2}$$

$$\times\left\{1+0.47\frac{M_2{}^{\frac{1}{2}}\int\chi d\nu}{m^{\frac{1}{2}}b^3}+\cdots\right\}\int\chi d\nu. \qquad (60)$$

Although this equation should not be considered fit for quantitative use until some of the succeeding terms in the series are worked out, it at least shows roughly how A_2 depends on the solute molecular weight. The extent to which this dependence is significant is determined not only by the molecular weight itself but also by the size of the chain-stiffness parameter, b. It is possible, therefore, that A_2 might yield interesting information on the extension of the chains.

However, there seems to be no experimental evidence in the literature to indicate that A_2 depends on M_2 to an important extent. This can only be true if b^3 is larger than $\int\chi d\nu$ by a factor greater than $(M/m)^{\frac{1}{2}}$. It has been calculated,[23] however, that if there were free rotation about the bonds in the backbone of the chain, b should be of the order of magnitude of the length of one of these bonds. We have seen that $\int\chi d\nu$ is generally somewhat smaller than the molecular volume of the solvent, which would be of the order of magnitude of b^3. The fact that a marked change of A_2 with the molecular weight has not been observed is therefore evidence that free rotation about the chain bonds is not of common occurrence.

VI.3. Effect of Branching of the Chains

It is apparent that the presence of a moderate amount of branching in a chain molecule would not affect $\int\chi d\nu$, which depends only on the constitution of the segments. It would, however, increase the dependence of A_2 on the molecular weight, being similar in effect to a tighter coiling of the chain.

The theory therefore predicts a continuous change in the properties of A_2 as the shape of the molecules is varied. When the molecules are rods

or extended chains, the intermolecular interaction is mainly through pairs of segments, so that A_2 is large and independent of the molecular weight in an homologous series. When chain molecules become less extended and more compact, either because of coiling or branching, A_2 is decreased algebraically and becomes dependent on the molecular weight. At the extreme of compactness we have solid spherical molecules (Section **IV.1**), where A_2 is relatively small and varies inversely as the molecular weight.

VI.4. The Coefficient A_3

An equation for A_3 may be obtained from the general form given by McMillan and Mayer (Eq. (7b) of this paper). In the present case where the intermolecular potential is expressed as a sum of pair terms this reduces to

$$A_3 = -\frac{N_0}{3M_2{}^3 V}$$
$$\times \int \frac{g_2(1, 2)g_2(2, 3)g_2(3, 1)}{F_1(1)F_1(2)F_1(3)} d(1, 2, 3). \quad (61)$$

If the expanded form (34) of g_2 is introduced and the same simplifications carried through as before, A_3 can be expanded in series as follows.

$$A_3 = -\frac{N_0{}^2}{3M_2{}^3 V}\left[n^3 \int \chi(\lambda_1, \mu_2) \right.$$
$$\times \chi(\mu_2, \nu_3)\chi(\nu_3, \lambda_1)d(\lambda_1, \mu_2, \nu_3)$$
$$+ \sum_{\kappa_1 \le o_1} \sum_{;} \sum_{\lambda_2 \le \mu_2} \sum_{;} \sum_{\nu_3 \le \xi_3} \sum \int F_1(1)F_1(2)F_1(3)$$
$$\left. \times \chi(\kappa_1, \lambda_2)\chi(\mu_2, \nu_3)\chi(\xi_3, o_1)d(1, 2, 3) + \cdots \right]. \quad (62)$$

By an analogous argument to that in Section **IV.5** the integral in the first term is approximately the coefficient A_3 for a solution of free molecules whose constitution is the same as that of the segments of the polymer. This coefficient is therefore given by Eq. (17) for regular solutions. When we substitute it in Eq. (62) for A_3 for the polymer we get

$$A_3 = \frac{\bar{V}_1{}^2(1 - \zeta_3)}{3m^3} - \frac{N_0{}^2}{3M_2{}^3 V}\left(\sum \cdots \int \cdots \right). \quad (63)$$

The first term in this equation may thus be handled in a way analogous to that used for the expression $\int \chi d\nu$ which was encountered in A_2. This is likewise the only term in A_3 obtained by the quasi-lattice method (as used at present; see especially Huggins[2]).

As with A_2, there are also numerous higher terms whose importance may be considerable. In addition to those whose χ-functions are associated with segments adjoining each other on the chains, and which can be absorbed into the first term if the segments are made long enough, there is another group, the sixfold sum in Eq. (61), associated with segments far removed from each other. Presumably this group can be treated by the same methods used on the similar group in A_2 (Section **VI.2**), and will depend similarly on the lengths and the internal distribution functions of the chains. The integrations have not been carried out, but one fact is immediately evident; this group in A_3 is proportional to $(\int \chi d\nu)^3$, just as the corresponding group in A_2 was proportional to $(\int \chi d\nu)^2$. As the first step in simplifying the integrals the χ-functions are assumed to vary rapidly as compared to the internal distribution functions, F_1. The integral can therefore be factored and the integrations over the χ-functions carried out separately, leading to $(\int \chi d\nu)^3$ as a factor.

It seems probable in view of these results that not only A_2 and A_3, but also all the higher A_n, are closely related to one another in value, and may perhaps all be characterized by a small number of parameters, or even only one. Huggins[2] and Gee[24] have indeed shown that one parameter is sufficient to describe the deviations from ideality of the experimental data of several rubber-solvent systems over a wide range of concentrations.

[24] G. Gee, Trans. Faraday Soc. **41**, 345 (1945).