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Propagation of Sound in Gases

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The propagation of sound and supersonic waves in gases has been attacked by Kneser, Richards and Reid, Rose, and others, without taking viscosity and heat conduction into account. Bourgin neglects viscosity. Herzfeld and Rice have taken heat conduction and viscosity into account along with the effect of internal temperature lag of one state. In the present paper, these effects are considered together with the effect of many internal energy states. The general equation is derived for wave propagation in a pure gas of any number of energy states. The probability constants involved are functions of temperature and gas composition only. This equation is first solved for the case of a single two state gas. The solution is then extended to mixtures of gases, including air.

I. VELOCITY AND ABSORPTION IN A PURE GAS WITH ANY NUMBER OF ENERGY STATES

A

WHEN a sound or supersonic wave is propagated through a gas, a compression is accompanied by a rise in temperature and an increase in pressure. The respective changes, δT , δp , and $\delta \rho$ in temperature, pressure and density, as well as the mass velocity, u , have phase relations which depend upon the undisturbed conditions of the gas and the frequency of the wave. If the gas contains molecules in various states (e.g., those having energy of translation only, others having vibrational or rotational energy as well), the number of molecules per unit volume in the i th state, N_i , will depend upon the total number of molecules N , upon the temperature, T , and also upon the phase relations between δN_i , δN , and δT . This last dependency is most easily described from the physical conditions which effect the change in N_i , namely, the number of collisions between particles in different energy states and the probability that a particular class of collisions will result in a particular type of transition from one energy state to another.

$$\frac{\partial N_i}{\partial t} = \frac{\partial N_i}{\partial N} \frac{\partial N}{\partial t} + \sum_j N_j \sum_k [N_k (f_{ki})_i - N_i (f_{ik})_j]. \quad (1.01)$$

The first term on the right of (1.01) is the rate of change of N_i because of the compression of the gas; the second is the additional rate of change of N_i because of collisions and is dependent on

temperature changes. The number of collisions per second of a certain type (as between molecules in the j th and k th state) is proportional to the product $N_j N_k$. The number of transitions per second of a given type is proportional to the number of these collisions per second and thus proportional to $N_j N_k$. The proportionality constant $(f_{ki})_i$ ¹ relates the product $N_j N_k$ to the number of transitions from the k th state to the i th state per second, and so on. In each case the internal subscripts designate the transition under consideration, and the external subscript designates the state of the molecule with which the transition molecule collides. Now

$$N_i = N_{i0} + \delta N_i \quad (1.02)$$

$$\text{and} \quad (f_{ki})_i = (f_{ki})_i^0 + \delta (f_{ki})_i. \quad (1.03)$$

Also the principle of detailed balancing may be stated:

$$[N_{k0} (f_{ki})_i^0 - N_{i0} (f_{ik})_j^0] = 0. \quad (1.04)$$

We have, upon substituting (1.02), (1.03), and (1.04) in (1.01):

$$\begin{aligned} \frac{\partial(\delta N_i)}{\partial t} &= \frac{\partial(\delta N_i)}{\partial(\delta N)} \frac{\partial(\delta N)}{\partial t} \\ &+ \sum_j N_{j0} \sum_k [\delta N_k (f_{ki})_i^0 - \delta N_i (f_{ik})_j^0] \\ &+ \sum_j N_{j0} \sum_k [N_{k0} \delta (f_{ki})_i - N_{i0} \delta (f_{ik})_j]. \end{aligned} \quad (1.05)$$

¹ With this treatment $(f_{ki})_i$ and $(S_{ki})_i$ to be introduced later, are dependent only upon gas composition and temperature. This simple dependence is not possible for parameters relating transition rates to a fixed number of molecules under consideration. In this connection as well as in some others, our method of attack is very similar to that of D. G. Bourgin, *Phys. Rev.* **42**, 721 (1932).

Assuming that for given colliding states, k and j , $(f_{ki})_j$ depends only on the kinetic energy of the colliding molecules, i.e., on the temperature, and is always in equilibrium with the temperature, $\delta(f_{ki})_j$ may be calculated from equilibrium conditions. We have for the last expression in brackets in (1.05)

$$N_{k0}\delta(f_{ki})_j - N_{i0}\delta(f_{ik})_j = \left[N_{k0} \frac{\partial(f_{ki})_j}{\partial T} - N_{i0} \frac{\partial(f_{ik})_j}{\partial T} \right] \delta T. \quad (1.06)$$

Furthermore, since the principle of detailed balancing is fulfilled at all temperatures, (1.04) gives

$$\begin{aligned} 0 &= \frac{\partial}{\partial T} [N_{k0}(f_{ki})_j^0 - N_{i0}(f_{ik})_j^0] \\ &= \frac{\partial N_{k0}}{\partial T} (f_{ki})_j^0 - \frac{\partial N_{i0}}{\partial T} (f_{ik})_j^0 \\ &\quad + N_{k0} \frac{\partial(f_{ki})_j^0}{\partial T} - N_{i0} \frac{\partial(f_{ik})_j^0}{\partial T}. \end{aligned}$$

Using (1.04) again and the equation

$$\frac{\partial N_{k0}}{\partial T} = \frac{N_{k0}\epsilon_k}{kT^2},$$

in which k is Boltzmann's constant, and ϵ_k is the energy associated with the k th state of one molecule, we find

$$N_{k0} \frac{\partial(f_{ki})_j^0}{\partial T} - N_{i0} \frac{\partial(f_{ik})_j^0}{\partial T} = \frac{(f_{ki})_j^0 N_{k0}(\epsilon_i - \epsilon_k)}{kT^2}. \quad (1.07)$$

Substituting (1.07) in (1.06) gives us the evaluation we desire.

It will be observed that N_r and $(f_{rs})_t$ occur in (1.05) only with zero subscripts or superscripts, indicating equilibrium values, constants for given undisturbed conditions. Accordingly, the zero subscripts and superscripts will henceforth be omitted though understood. (1.05) now becomes

$$\begin{aligned} \frac{\partial(\delta N_i)}{\partial t} &= \frac{\partial(\delta N_i)}{\partial(\delta N)} \frac{\partial(\delta N)}{\partial t} + \sum_j N_j \sum_k \left[(f_{ki})_j \delta N_k \right. \\ &\quad \left. - (f_{ik})_j \delta N_i + \frac{(f_{ki})_j N_k (\epsilon_i - \epsilon_k) \delta T}{kT^2} \right]. \quad (1.08) \end{aligned}$$

When we observe that

$$[\partial(\delta N_i)]/[\partial(\delta N)] = N_i/N$$

and assume that all changes are of the form $Ae^{j\omega t}$, (1.08) goes over into

$$\begin{aligned} j\omega \delta N_i &= \frac{j\omega N_i \delta N}{N} + \sum_j N_j \sum_k \left[(f_{ki})_j \delta N_k \right. \\ &\quad \left. - (f_{ik})_j \delta N_i + \frac{(f_{ki})_j N_k (\epsilon_i - \epsilon_k) \delta T}{kT^2} \right]. \quad (1.09) \end{aligned}$$

We now introduce the following simplifying notation:

$$\begin{aligned} \rho_0 S_{ki} &= \sum_j N_j (f_{ki})_j, \\ \rho_0 Q_i &= \sum_j \sum_k N_j (f_{ik})_j = \rho_0 \sum_k S_{ik}, \end{aligned} \quad (1.10)$$

and obtain on substituting in (1.09) and again using (1.04)

$$\begin{aligned} (\rho_0 Q_i + j\omega) \delta N_i &= \frac{j\omega N_i}{N} \delta N + \rho_0 \sum_k S_{ki} \delta N_k \\ &\quad + \rho_0 \sum_k N_k (\epsilon_i - \epsilon_k) S_{ki} \delta T / kT^2. \quad (1.11) \end{aligned}$$

From Eqs. (1.11), for the various values of i , δN_i may be solved for in terms of δN and δT .

B

The equation of continuity, and the equation of motion (written to include the effect of viscosity) yield for the complex velocity of sound, v' , (defined by $u = u_0 e^{j\omega(t - x/v')}$)

$$v'^2 = \delta p / \delta \rho + j4\eta\omega / 3\rho. \quad (1.12)$$

It should be noted that this complex velocity includes absorption. (See Eq. (2.09).) Using the general gas law to eliminate δp we have

$$v'^2 = \frac{RT}{M} \left(1 + \frac{\rho}{T} \frac{\delta T}{\delta \rho} \right) + j4\eta\omega / 3\rho. \quad (1.13)$$

The equation of energy may be written

$$ML \frac{\partial^2 T}{\partial x^2} = -RT \frac{\partial \rho}{\partial t} + C_\infty \rho \frac{\partial T}{\partial t} + \sum_i \epsilon_i M \left(\frac{\partial N_i}{\partial t} - \frac{N_i}{N} \frac{\partial N}{\partial t} \right), \quad (1.14)^2$$

in which C_∞ is the molar heat at constant volume associated with the external degrees of freedom, ϵ_i is the internal energy of a single molecule in the i th state, and L is the coefficient of heat conduction. The last term in (1.14) is necessary because (since the equation is written for a fixed number of molecules) the quantity $\partial N_i / \partial t$ must have subtracted from it that part of itself which is caused by compression.

When we assume $\delta \rho \propto \delta T \propto \delta N_i \propto e^{j\omega(t-x/v')}$ Eq. (1.14) takes the form

$$(\rho C_\infty - j\omega ML/v'^2) \delta T - RT \delta \rho = - \sum_i \epsilon_i M (\delta N_i - N_i \delta N / N). \quad (1.15)$$

The right-hand member may be evaluated from equations (1.11), the ϵ_i generally being known from spectroscopic data. Then $\delta T / \delta \rho$ in (1.13) can be determined and v'^2 becomes completely specified.

II. TWO-STATE GAS

A

For the two-state gas, the subscript k in (1.11) takes on the values 0, and 1. Then $Q_0 = S_{01}$ and

$Q_1 = S_{10}$. Eq. (1.11) therefore becomes:

$$(\rho_0 S_{10} + j\omega) \delta N_1 = \frac{j\omega N_1}{N} \delta N + \rho_0 S_{01} \delta N_0 + \frac{\rho_0 N_0}{kT^2} \epsilon_1 S_{01} \delta T, \quad (2.01)$$

and taking into account

$$\delta N = \delta N_1 + \delta N_0, \quad (2.02)$$

we get

$$[\rho_0(S_{10} + S_{01}) + j\omega] \delta N_1 = \left(\rho_0 S_{01} + \frac{j\omega N_1}{N} \right) \delta N + \frac{\rho_0 N_0}{kT^2} \epsilon_1 S_{01} \delta T. \quad (2.03)$$

Rose³ has demonstrated that certain types of collisions produce a higher order effect and may be neglected. If this is done, S_{10} and S_{01} have a different interpretation but otherwise equation (2.03) is unchanged.

An alternative way of writing (2.03) which will be more useful in subsequent development is

$$[\rho_0(S_{01} + S_{10}) + j\omega] \delta N_1 = \left(\rho_0 S_{01} + \frac{j\omega \rho_1}{\rho} \right) \frac{R \delta \rho}{Mk} + \frac{R \rho_0^2}{Mk^2 T^2} \epsilon_1 S_{01} \delta T, \quad (2.04)$$

in which k is Boltzmann's constant.

If we now use the value of δN_1 , given in (2.04), in Eq. (1.15) and solve for $\delta T / \delta \rho$, there results:

$$\frac{\delta T}{\delta \rho} = \frac{RT(S_{10} + j\omega/\rho)}{(\rho C_\infty - j\omega ML/v'^2)(S_{10} + j\omega/\rho) + R \rho_1 \rho_0 \epsilon_1^2 S_{10} / \rho k^2 T^2}. \quad (2.05)$$

In obtaining the last equation we made use of the fact that $S_{01} = N_1 S_{10} / N_0$ for a two-state gas, and consequently $[(\rho_1/\rho)(S_{01} + S_{10}) - S_{01}] = 0$.

² In the Landau-Teller theory (Physik. Zeits. Sowjetunion, July 1936), $\sum_i \left(\frac{\partial N_i}{\partial t} - \frac{N_i}{N} \frac{\partial N}{\partial t} \right)$ is replaced by $\epsilon \frac{dZ}{dt}$ where Z is the total number of vibrational quanta associated with the fixed number of molecules, N , under consideration. They proceed to show that for the case of a single vibrational mode, which is harmonic with evenly spaced energy levels, $dZ/dt = kN[k_{01}(Z+N) - k_{10}Z]$, in which K is the proportionality constant relating NN_i to number of collisions experienced by a molecule in the i th vibrational state per second, and k_{ij} is a second proportionality constant relating number of collisions per second experienced by molecules in the i th vibrational state to

number of transitions per second from the i th to the j th state. In what follows in their paper, they treat k , k_{01} , and k_{10} as constants. This procedure, though frequently resorted to in papers on sound propagation, is unjustified. k depends upon both pressure and temperature, and k_{ij} on temperature. The importance of this paper should not be overlooked, however, since it appears that the authors have introduced a great simplification for cases in which higher vibrational levels are of importance. But in the many cases where the ratio N_{i+1}/N_i is small (about 0.0005 for O_2 at room temperature), the higher levels may be neglected with sufficient precision.

³ M. E. Rose, J. Chem. Phys. 2, 260 (1934).

Eliminating $\delta T/\delta\rho$ from (2.05) and (1.13) gives

$$v'^2 = \frac{RT}{M} \left[1 + \frac{j4\eta\omega M}{3\rho RT} + \frac{R}{C_\infty} \frac{S_{10} + j\omega/\rho}{[S_{10}(1 + R\rho_1\rho_0\epsilon_1^2/C_\infty\rho^2k^2T^2) + \omega^2 ML/\rho^2 v'^2 C_\infty + j\omega(1 - MLS_{10}/v'^2 C_\infty)/\rho]} \right]. \quad (2.06)$$

At this point we can make the single simplifying assumption that $MLS/v'^2 C_\infty$ is negligible in comparison to 1. We also make use of the relation

$$\left(1 + \frac{R\rho_1\rho_0\epsilon_1^2}{C_\infty\rho^2k^2T^2} \right) = C_0/C_\infty. \quad (2.07)$$

Then, separating v'^2 into its real and imaginary parts we obtain

$$v'^2 = \frac{RT}{M} \left[1 + \frac{R}{C_\infty} \frac{S_{10}^2 C_0/C_\infty + \omega^2/\rho^2}{S_{10}^2 C_0^2/C_\infty^2 + \omega^2/\rho^2 + M^2 L^2 \omega^4/v'^4 C_\infty^2 \rho^4} \right] + \frac{j\omega}{\rho} \left[4\eta/3 + \frac{R^2 T}{M C_\infty} \frac{S_{10}(C_0 - C_\infty)/C_\infty + (\omega^2 ML/\rho^2 v'^2 C_\infty)}{S_{10}^2 C_0^2/C_\infty^2 + \omega^2/\rho^2 + M^2 L^2 \omega^4/v'^4 C_\infty^2 \rho^4} \right]. \quad (2.08)$$

The complex velocity of sound, v' , is related to the real velocity, v , and the absorption constant per wave-length, μ , as follows:

$$1/v' = (1 - j\mu/4\pi)/v. \quad (2.09)$$

Inverting and squaring, and neglecting $\mu^2/16\pi^2$ in comparison with unity, we find

$$v'^2 = v^2(1 + j\mu/2\pi). \quad (2.10)$$

Comparing (2.10) and (2.08) there are obtained the equations for v^2 and μ :

$$v^2 = \frac{RT}{M} \left[1 + \frac{R}{C_\infty} \frac{S_{10}^2 C_0/C_\infty + \omega^2/\rho^2}{S_{10}^2 C_0^2/C_\infty^2 + \omega^2/\rho^2 + M^2 L^2 \omega^4/v^4 C_\infty^2 \rho^4} \right], \quad (2.11)$$

$$\mu = \frac{2\pi\omega}{\rho} \frac{\frac{4\eta M}{3RT} \left(\frac{S_{10}^2 C_0^2}{C_\infty^2} + \frac{\omega^2}{\rho^2} + \frac{M^2 L^2 \omega^4}{v^4 C_\infty^2 \rho^4} \right) + \frac{R}{C_\infty} \left(\frac{C_0 - C_\infty}{C_\infty} S_{10} + \frac{\omega^2 ML}{\rho^2 v^2 C_\infty} \right)}{C_0(R + C_0)S_{10}^2/C_\infty^2 + (R + C_\infty)\omega^2/C_\infty\rho^2 + M^2 L^2 \omega^4/v^4 C_\infty^2 \rho^4} \quad (2.12)$$

One important special case occurs when, in (2.12), the terms involving η and L do not become appreciable until such high frequency is reached that the constant terms in this equation involving S_{10} are negligible in comparison with the terms in ω^2/ρ^2 . In this case the fraction on the right side of (2.12) can be separated into two parts, and the equation written in the form

$$\mu = \frac{2\pi\omega}{\rho} \left[\frac{R(C_0 - C_\infty)S_{10}}{(R + C_0)C_0S_{10}^2 + (R + C_\infty)C_\infty\omega^2/\rho^2} + \frac{C_\infty}{R + C_\infty} \left(\frac{4\eta M}{3RT} + \frac{MLR}{v^2 C_\infty^2} \right) \right], \quad (2.13)$$

in which the first term is important at relatively lower and the second at relatively higher fre-

quencies. In the lower frequency region μ goes through a maximum at some critical frequency, and, neglecting the second term, the value of μ_m is found to be

$$\mu_m = \frac{\pi R(C_0 - C_\infty)}{(C_0 C_\infty (R + C_0)(R + C_\infty))^{1/2}}. \quad (2.14)$$

Such cases are discussed, for instance, by Knudsen.⁴ We believe our (2.13) and (2.14) to be more precise than the corresponding Knudsen-Kneser⁵ formulas because of the simplifying assumptions they have made, but doubt if the difference can be detected experimentally.

⁴ V. O. Knudsen, J. Acous. Soc. Am. **5**, 112 (1933).

⁵ H. O. Kneser, Ann. d. Physik **11**, 761 (1931).

In this case, we may consider also the effect at very high frequencies when $\mu^2/16\pi^2$ is not negligible in comparison to unity. Then

$$v^2 = (\text{real part } v'^2)(1 + \mu^2/16\pi^2) \quad (2.15)$$

to a sufficient approximation. Under these circumstances, using (2.13) and retaining the first-order terms only, we obtain

$$v^2 = \frac{RT}{M} \left\{ 1 + \frac{R}{C_\infty} \frac{S_{10}^2 C_0 / C_\infty + \omega^2 / \rho^2}{S_{10}^2 C_0^2 / C_\infty^2 + \omega^2 / \rho^2} + \frac{\omega^2 C_\infty}{\rho^2 (R + C_\infty)} \right. \\ \left. \times \left(\frac{4\eta^2 M^2}{9R^2 T^2} + \frac{2L\eta M^2}{3v^2 C_\infty^2 T} - \frac{M^2 L^2 R (3R + 4C_\infty)}{4v^4 C_\infty^4} \right) \right\}. \quad (2.16)$$

(2.13) and (2.16) should be especially useful over a wide range of values of ω/ρ . However, the approximations made should be critically scrutinized in connection with any particular application.

A second special case, that of the one-state gas, is also important. The value of μ may be obtained directly from (2.13) by dropping out the first term

$$\mu = \frac{2\pi\omega C_\infty}{\rho(R + C_\infty)} \left(\frac{4\eta M}{3RT} + \frac{MLR}{v^2 C_\infty^2} \right). \quad (2.17)$$

Similarly, (2.16) becomes

$$v^2 = \frac{RT}{M} \left\{ 1 + \frac{R}{C_\infty} + \frac{\omega^2 C_\infty}{\rho^2 (R + C_\infty)} \left[\frac{4\eta^2 M^2}{9R^2 T^2} + \frac{2L\eta M^2}{3v^2 C_\infty^2 T} - \frac{M^2 L^2 R (3R + 4C_\infty)}{4v^4 C_\infty^4} \right] \right\}. \quad (2.18)$$

In these last two equations C_∞ is identical with C_0 . These equations must be equivalent to those obtained by Herzfeld and Rice⁶ for a one-state gas since our fundamental equations become identical with theirs for this case.

Herzfeld⁷ has carried out a more detailed analysis of the case of the one-state gas. He finds other effects on velocity of comparable magnitude to those of heat conduction and viscosity. The additional terms obtained for absorption are too small to be detected experimentally.

III. MIXTURE OF GASES: PROPAGATION IN AIR

This case is of particular importance because it includes air. In strict treatment for air, one would consider the carbon dioxide component as a three-state gas, since two vibrational modes in pure carbon dioxide are sufficiently excited to be of importance at room temperature. However, since one of these modes, the transverse vibration, is so much more important than the other and since the carbon dioxide content of air is small, this transverse vibration alone may be treated. Then, air is included under the case of a mixture of one-state and two-state gases.

It is necessary to include a new symbolism as follows:

$h_j = N_j/N$ the numerical fraction of the mixture composed of molecules of the j th component,

$M = \sum_{j=1}^n h_j M_j$, the effective molecular weight of the mixture,

$C_\infty = \sum_{j=1}^n h_j C_{\infty j}$, effective C_∞ for the mixture.

Since a numerical subscript on an N , ρ or h denotes a particular component, a superscript (0 or 1) is used to denote molecules in the ground or first vibrational state. Thus ρ_3' is the density of the excited molecules of the third component. ρ_3 , with no superscript, is used to denote total density of the third component. No subscript or superscript is used when reference is had to properties of the whole mixture.

We shall define $(S_i^k)_r$ by the relation

$$\frac{\rho \rho_i^0}{\rho_i} (S_i^k)_r = \sum_i N_i [(f_{ki})_i]_r.$$

Where the r denotes the gas component under consideration and N_i^0 and N_i' should be taken separately in the summation. For the two-state gas mixture only $(S_0')_r$ and $(S_1^0)_r$ will occur, and since these are related; $(S_1^0)_r = (N_r' / N_r^0) (S_0')_r$;

⁶ K. F. Herzfeld and F. O. Rice, Phys. Rev. **31**, 691 (1928).

⁷ K. F. Herzfeld, Ann. d. Physik **23**, 465 (1935).

we need use only the former, writing it S_r . S_r is dependent upon the composition of the mixture as well as the temperature.

L and η are the effective values for the mixture.

Equation (1.12) for the complex velocity of sound remains unchanged for this case. In order to keep equation (1.13) as it was, the above definition of M is necessary. The equation is renumbered

$$v'^2 = \frac{RT}{M} \left(1 + \frac{\rho}{T} \frac{\delta T}{\delta \rho} \right) + j \frac{4\eta\omega}{3\rho}. \quad (3.01)$$

(1.15) becomes

$$(\rho C_\infty - j\omega ML/v'^2) \delta T - \left(RT + \sum_i \epsilon_i \frac{\rho_i' M}{\rho M_i} \frac{R}{K} \right) \delta \rho = - \sum_i \epsilon_i M N_i'. \quad (3.02)$$

Eqs. (1.11) become

$$\delta N_i' = \frac{\left[\frac{\rho_i' M}{M_i} S_i + j \frac{\omega \rho_i' M}{\rho M_i} \right] \delta N + \frac{\rho \rho_i' \rho_i^0 R}{\rho_i M_i} \frac{\epsilon_i}{K^2 T^2} S_i \delta T}{\rho S_i + j\omega}. \quad (3.03)$$

Substituting (3.03) in (3.02) and solving for $\delta T/\delta \rho$, we obtain, using $C_i' = C_{0i} - C_{\infty i}$ as the vibrational molar heat of the i th component, and $a_i = (h_i' C_i S_i)/C_\infty$:

$$\frac{\delta T}{\delta \rho} = \frac{RT \left[\left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right) + \frac{j\omega}{\rho} \left(\frac{ML}{v'^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2} \right) \right]}{\rho C_\infty \left[\left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right)^2 + \frac{\omega^2}{\rho^2} \left(\frac{ML}{v'^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2} \right)^2 \right]}. \quad (3.04)$$

When (3.04) is substituted in (3.01), there is obtained for v'^2 :

$$v'^2 = v^2 (1 + j\mu/2\pi) = \frac{RT}{M} \left\{ 1 + \frac{R}{C_\infty} \frac{1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2}}{\left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right)^2 + \frac{\omega^2}{\rho^2} \left(\frac{ML}{v'^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2} \right)^2} + \frac{j\omega}{\rho} \left[\frac{4\eta M}{3RT} + \frac{\frac{ML}{v'^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2}}{\left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right)^2 + \frac{\omega^2}{\rho^2} \left(\frac{ML}{v'^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2} \right)^2} \right] \right\}. \quad (3.05)$$

It follows that

$$v^2 = \frac{RT}{M} \left[1 + \frac{R}{C_\infty} \frac{1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2}}{\left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right)^2 + \frac{\omega^2}{\rho^2} \left(\frac{ML}{v^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2} \right)^2} \right], \quad (3.06)$$

$$\mu = \frac{2\pi\omega}{\rho} \left[\frac{4\eta M}{3RT} + \frac{\frac{ML}{v^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2} - \frac{4\eta M}{3C_\infty T} \left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right)}{\left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right)^2 + \frac{\omega^2}{\rho^2} \left(\frac{ML}{v^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2} \right)^2 + \frac{R}{C_\infty} \left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right)} \right]. \quad (3.07)$$

The square of a summation in the denominator is rather cumbersome. The operation may be carried out for the general case, however, yielding these forms:

$$v^2 = \frac{RT}{M} \left[1 + \frac{R}{C_\infty} \frac{1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2}}{1 + \frac{\omega^2 M^2 L^2}{\rho^2 v^4 C_\infty^2} + \sum_i \frac{a_i(a_i + 2S_i + 2\omega^2 ML/\rho^2 v^2 C_\infty)}{S_i^2 + \omega^2/\rho^2} + \sum_{i,j} \frac{2a_i a_j (S_i S_j + \omega^2/\rho^2)}{(S_i^2 + \omega^2/\rho^2)(S_j^2 + \omega^2/\rho^2)}} \right], \quad (3.06')$$

$$\mu = \frac{2\pi\omega}{\rho} \left[\frac{4\eta M}{3RT} + \frac{\frac{ML}{v^2 C_\infty} + \sum_i \frac{a_i}{S_i^2 + \omega^2/\rho^2} - \frac{4\eta M}{3C_\infty T} \left(1 + \sum_i \frac{a_i S_i}{S_i^2 + \omega^2/\rho^2} \right)}{1 + \frac{R}{C_\infty} + \frac{\omega^2 M^2 L^2}{\rho^2 v^4 C_\infty^2} + \sum_i \frac{a_i[a_i + (2 + R/C_\infty)S_i + 2\omega^2 ML/\rho^2 v^2 C_\infty]}{S_i^2 + \omega^2/\rho^2} + \sum_{i,j} \frac{2a_i a_j (S_i S_j + \omega^2/\rho^2)}{(S_i^2 + \omega^2/\rho^2)(S_j^2 + \omega^2/\rho^2)}} \right]. \quad (3.07')$$

in which \sum_{ij} means that each combination of i, j ($i \neq j$) is to be taken once.

Various approximations may be made for special cases. If, for instance, there is but one two-state gas present in the mixture, or if all components but one may be treated as one-state gases in the range of ω/ρ under consideration, the last summation in the denominator is zero and the remaining summation signs may be dropped along with the subscript i . One then obtains, multiplying numerator and denominator by $S^2 + \omega^2/\rho^2$, and neglecting $M^2 L^2 S^2/v^4 C_\infty^2$ and $2MLa/v^2 C_\infty$ in comparison to unity:

$$v^2 = \frac{RT}{M} \left[1 + \frac{R}{C_\infty} \frac{S(S+a) + \omega^2/\rho^2}{(S+a)^2 + \omega^2/\rho^2 + M^2 L^2 \omega^4/v^4 C_\infty^2 \rho^4} \right], \quad (3.08)$$

$$\mu = \frac{2\pi\omega}{\rho} \left[\frac{4\eta M}{3RT} + \frac{a + \frac{ML}{v^2 C_\infty} (S^2 + \omega^2/\rho^2) - \frac{4\eta M}{3C_\infty T} [S(S+a) + \omega^2/\rho^2]}{(S+a) \left(S + a + \frac{R}{C_\infty} S \right) + \frac{\omega^2}{\rho^2} (1 + R/C_\infty) + \frac{\omega^4 M^2 L^2}{\rho^4 v^4 C_\infty^2}} \right]. \quad (3.09)$$

For the further specialization of making this single two-state gas, the only component present in the mixture, (3.08) and (3.09) reduce respectively to (2.11) and (2.12) affording a check on the work of this part of the paper.

When the effects of heat conduction and viscosity are negligible, a very considerable simplification of (3.06') and (3.07') results. The simplification is so obvious that we do not record it here.