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Note on Becker's Theory of the Shock Front

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Becker, in a classical paper, concludes that weak shocks are many free paths thick and may be treated by ordinary hydrodynamics including the effects of viscosity and thermal conductivity of the medium; that moderately strong shocks have a thickness of the order of a free path and must be treated by a direct attack on the relevant Boltzmann equation; while violent shocks have a thickness small compared with a free path and even the Boltzmann equation is no longer applicable. It is shown here that this last conclusion rests on an oversight; the thickness of a shock front is always at least of the order of magnitude of a free path, and it is to be expected that the Boltzmann equation can be applied even for the most violent shocks.

INTRODUCTION

VISCOSITY and heat conduction are often neglected in the equations of motion of a compressible fluid. The researches of Earnshaw[1] and Riemann[2] showed that a continuous solution of the resulting equations may be impossible for longer than a finite time if an initial disturbance moves from high to low pressure. The velocity of sound being greater at greater temperature produced by adiabatic compression, a disturbance on the high pressure side moves faster than on the low, and the disturbance becomes more and more concentrated, until finally no continuous solution is possible. Physically, a shock wave has been built up. The subsequent motion of the shock front may be obtained by regarding it as a surface of discontinuity across which there is continuous flow of mass and momentum, leading to equations given by Stokes[3] and later by Rankine.[4] In order that energy also should be conserved the entropy must increase across the front leading to a further condition owing to Huguoniot.[5] (A rarefaction shock, across which entropy would have to diminish, is therefore not possible.)

It was explained by Rankine and later by Rayleigh[6] and by Taylor[7] that if viscosity and heat conduction are taken into account, the shock wave would never become an infinitely sharp discontinuity, but would have a finite thickness (3×10^{-5} cm for moderately strong shock waves in air). Becker,[8] in a classical paper, obtained the exact solution of these equations for a one-dimensional stationary shock wave in a simple gas. He found that for a violent shock ($p_2/p_1=8$, where p_2 and p_1 are the pressures on opposite sides of the shock), the thickness for a gas was of the order of a mean free path (10^{-5} cm). The equations of hydrodynamics may be derived from kinetic theory only provided that large macroscopic changes do not occur over a distance of the order of a free path. Becker therefore concludes that these equations would not apply within a moderately violent shock, and that recourse must then be had to a direct solution of the Boltzmann equation of kinetic theory.

Becker finds further that for a very violent shock ($p_2/p_1=2000$) the thickness of the shock wave on the basis of these (inapplicable) equations is so much smaller yet as to approach the average intermolecular distance (3×10^{-7} cm) and he inferred that the same order of magnitude holds for liquids. He therefore doubts even the applicability of the Boltzmann equation to very violent shocks.

Unfortunately Becker overlooks the increase of the coefficients of thermal conductivity and of viscosity with increasing temperature and pressure, using the values for normal temperature and pressure as constants. If this oversight is corrected the hydrodynamical equations never lead to a thickness small compared to the length

* On leave of absence from Ohio State University.
[1] S. Earnshaw, Phil. Trans. **150**, 133 (1858).
[2] B. Riemann, Gott. Abh. **8**, 43 (1858).
[3] E. E. Stokes, Phil. Mag. **33**, 349 (1848).
[4] W. V. M. Rankine, Phil. Trans. **160**, 277 (1870).
[5] A. Hugoniot, J. École Poly. 58, (1889).
[6] Lord Rayleigh, Proc. Roy. Soc. **A84**, 371 (1910).
[7] G. I. Taylor, Proc. Roy. Soc. **A84**, 371 (1910).
[8] R. Becker, Zeits. f. Physik **8**, 321 (1922).

of a free path. Thus the ordinary approximation of the kinetic theory of non-uniform gases is valid for the less violent shocks and leads to the equations of hydrodynamics without viscosity and heat conduction as a first approximation, including these in the second approximation, in which approximation Becker's solution is obtained. The coefficients of the terms required for the third approximation have been given by Burnett.[9] The inclusion of these terms leads to an even greater thickness, but the numerical values indicate that the series of approximations may not converge for very violent shocks, for which the departure from a Maxwellian distribution of velocities at points within the shock front may be large. However, methods similar to those of Milne[10] and Hopf[11] for the transfer of radiation through the solar photosphere are still available for solving the Boltzmann equation.

THE EQUATIONS OF HYDRODYNAMICS AS OBTAINED FROM KINETIC THEORY[12]

For steady motion in one dimension (x) the equations of continuity, momentum, and energy, are obtained from kinetic theory in the form

$$u\frac{d\rho}{dx}=-\rho\frac{du}{dx}, \tag{1}$$

$$u\frac{d}{dx}\rho u=-\rho u\frac{du}{dx}-\frac{d}{dx}p, \tag{2}$$

$$u\frac{d}{dx}E=-E\frac{du}{dx}-p\frac{d}{dx}u-\frac{d}{dx}q, \tag{3}$$

where u is the mean molecular velocity in the x direction, ρ is the density, E is the energy per unit volume referred to axes moving with velocity u, p is the mean rate of flow of momentum in the x direction per unit area perpendicular to the x direction referred to these axes, and q is the mean rate of flow of energy defined similarly.

These equations have the first integrals[8]

$$\rho u=a, \tag{4}$$

$$au+p=b, \tag{5}$$

$$uE+ub-\tfrac{1}{2}au^2+q=c, \tag{6}$$

expressing that the rates of flow of mass, momentum, and energy, in the x direction, are the same for all values of x.

These integrated equations must hold generally and show that, provided there exists a solution of the Boltzmann equation corresponding to steady flow from uniform steady motion for large negative x to uniform steady conditions for large positive x, the relation between these uniform conditions is that given by the Rankine-Hugoniot equations to which these equations lead.

The usual first approximation of kinetic theory gives, for a gas with energy only of molecular translation,

$$\rho=nm,\quad E=\tfrac{3}{2}nkT,\quad p=p^{(0)}=nkT,\quad q=0, \tag{7}$$

where n is the number of molecules per unit volume, T is the absolute temperature, k is Boltzmann's constant, and m is the mass of a molecule.

The methods of Chapman and Enskog, using $\rho=nm$, $E=\frac{3}{2}nkT$ (the latter essentially as a definition of T), lead to a solution of Boltzmann's equation by successive approximation in series giving

$$\begin{aligned}p&=p^{(0)}+p^{(1)}+p^{(2)}+\cdots,\\ q&=q^{(1)}+q^{(2)}+\cdots,\end{aligned} \tag{8}$$

where, in the one-dimensional case,[12]

$$p^{(0)}=nkT,\quad p^{(1)}=-\frac{4}{3}\mu\frac{du}{dx},\quad q^{(1)}=-\lambda\frac{dT}{dx}, \tag{9}$$

$$\begin{aligned}p^{(2)}=\tilde{\omega}_1\frac{\mu^2}{p^{(0)}}\frac{du}{dx}\frac{2}{3}\frac{du}{dx}+\tilde{\omega}_2\frac{\mu^2}{p^{(0)}}\left\{-\frac{2}{3}\frac{d}{dx}\left(\frac{1}{\rho}\frac{dp^{(0)}}{dx}\right)\right.\\ \left.-\frac{2}{3}\left(\frac{du}{dx}\right)^2-\frac{8}{9}\left(\frac{du}{dx}\right)^2\right\}+\tilde{\omega}_3\frac{\mu^2}{\rho T}\frac{2}{3}\frac{d^2T}{dx^2}\\ +\tilde{\omega}_4\frac{\mu^2}{\rho p^{(0)}T^3}\frac{2}{3}\frac{dp^{(0)}}{dx}\frac{dT}{dx}\\ +\tilde{\omega}_5\frac{\mu^2}{\rho T^2}\frac{2}{3}\left(\frac{dT}{dx}\right)^2+\tilde{\omega}_6\frac{\mu^2}{p^{(0)}}\left(\frac{du}{dx}\right)^2,\end{aligned}$$

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[9] E. S. Burnett, Proc. Lond. Math. Soc. **39**, 385 (1935).
[10] E. A. Milne, M. N. R. A. S. **88**, 493 (1928).
[11] E. Hopf, "Mathematical problems of radiative equilibrium," *Cambridge Tracts in Mathematics and Mathematical Physics* (Cambridge University Press, Cambridge, England, 1934).
[12] S. Chapman and T. E. Cowling, *Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, England, 1939), Chapters 3, 7, 15.

$$q^{(2)}=\vartheta_1\frac{\mu^2}{\rho T}\frac{du}{dx}\frac{dT}{dx}+\vartheta_2\frac{\mu^2}{\rho T}\left\{\frac{2}{3}\frac{d}{dx}\left(T\frac{du}{dx}\right)+2\frac{du}{dx}\frac{dT}{dx}\right\}+\vartheta_3\frac{\mu^2}{\rho p^{(0)}}\frac{dp^{(0)}}{dx}\frac{2}{3}\frac{du}{dx}+\vartheta_4\frac{\mu^2}{\rho}\frac{d}{dx}\left(\frac{2}{3}\frac{du}{dx}\right)+\vartheta_5\frac{\mu^2}{\rho T}\frac{dT}{dx}\frac{2}{3}\frac{du}{dx}, \quad (10)$$

where μ is the coefficient of viscosity, λ the coefficient of thermal conductivity, the forms of which have been computed for various laws of interaction between the molecules.

The coefficients $\vartheta_1, \cdots, \vartheta_5$, and $\tilde{\omega}_1, \cdots, \tilde{\omega}_6$, have been evaluated for molecules attracting according to an inverse fifth power law:[9]

$$\vartheta_1=75/8, \quad \vartheta_2=45/8, \quad \vartheta_3=-3, \quad \vartheta_4=3, \quad \vartheta_5=6, \quad (11)$$

$$\tilde{\omega}_1=10/3, \quad \tilde{\omega}_2=2, \quad \tilde{\omega}_3=3, \quad \tilde{\omega}_4=0, \quad \tilde{\omega}_6=3, \quad \tilde{\omega}_6=8. \quad (12)$$

The coefficients $\tilde{\omega}_1, \cdots, \tilde{\omega}_6$, have also been evaluated for rigid elastic spheres.

Successive approximations to the solution of the Boltzmann equation may now be obtained from Eqs. (4)–(6), by using only the terms $p^{(0)}+p^{(1)}$ and $q^{(1)}$; or only the terms $p^{(0)}+p^{(1)}+p^{(2)}$ and $q^{(1)}+q^{(2)}$; and so on; and subject to the convergence of this process, a solution of the Boltzmann equation of the desired form would be obtained.

For a gas made up of molecules with internal degrees of freedom this is not entirely correct; energy does not partition itself immediately into the internal degrees of freedom. Bethe and Teller[13] have given arguments which show that while energy passes rapidly to rotation of the molecules, it passes only slowly to internal vibration. Through the shock front itself, when of only a few mean-free-paths thickness, we need take account only of the molecular rotation. The shock front is then followed by a region in which the energy is gradually distributed until the equilibrium state is reached.

Through the shock front it would still be necessary, strictly, to take account of the delay in exchanging with molecular rotation. In the only such case for which the theory of non-uniform gases has been worked out, that of rough spherical molecules, the first approximation departs little from what would be obtained by taking the equations (4)–(6), with

$$E=\tfrac{1}{2}NnkT, \quad \lambda=f\mu\frac{kN}{2m}, \quad (13)$$

where N is the number of effective degrees of freedom (3 for monatomic, 5 for diatomic, and 6 for polyatomic molecules) and f is a constant ($\approx\frac{1}{4}(9\gamma-5)$ where $\gamma=(N+2)/N$ is the ratio of specific heats.

Finally, when we require the variation of μ with temperature, we have, for elastic spheres of diameter σ,

$$\mu=\frac{v}{\pi^{\frac{3}{2}}}\frac{(kmT)^{\frac{1}{2}}}{\sigma^2}, \quad (14)$$

where $v=0.998$, while the mean free path l is given by

$$l=1/\sqrt{2}\pi n\sigma^2. \quad (15)$$

The variation of μ with temperature for actual gases is given more nearly by a formula of Sutherland's type, but this gives variation of the above form at high temperatures.

BECKER'S THEORY OF THE THICKNESS OF SHOCK FRONTS[8]

If in Eqs. (4)–(6), we replace p by $p^{(0)}+p^{(1)}$, i.e., $nkT-(4/3)\mu(du/dx)$, and q by $q^{(1)}$, i.e., $-\lambda(dT/dx)$, they may be regarded as equations to determine du/dx and dT/dx in terms of u and T. Making the substitutions

$$u=\frac{b}{a}\omega, \quad T=\frac{m}{k}\frac{b^2}{a^2}\theta \quad (16)$$

and writing

$$\lambda=f\mu\frac{kN}{2m},$$

these become Becker's equations

$$\begin{aligned}\frac{1}{a}\frac{4}{3}\mu\frac{d\omega}{dx}&=\omega+\frac{\theta}{\omega}-1,\\ \frac{1}{a}f\mu\frac{d\theta}{dx}&=\theta-\delta[(1-\omega)^2+\alpha],\end{aligned} \quad (17)$$

[13] H. A. Bethe and E. Teller, "Deviations from thermal equilibrium in shock waves" (unpublished).

where

$$\delta=\frac{1}{N}, \quad \alpha=\frac{2ac}{b^2}-1,$$

which give

$$\frac{d\theta}{d\omega}=\frac{4}{3f}\omega\frac{\theta-\delta[(1-\omega)^2+\alpha]}{\omega^2-\omega+\theta} \tag{18}$$

to determine the variation of θ with ω across the shock.

Following Becker the solution of this equation for which x extends from $-\infty$ to ∞ extends from $\theta_1=\omega_1(1-\omega_1)$ to $\theta_2=\omega_2(1-\omega_2)$, where

$$\omega_1+\omega_2=\frac{N+2}{N+1}, \quad \omega_1\omega_2=\frac{2ac/b^2}{N+1}.$$

In case

$$\frac{3}{4}f=1+\frac{2}{N},$$

this special solution has the simple form

$$\theta=\frac{1}{N+2}\left(\frac{2ac}{b^2}-\omega^2\right)$$
$$=\frac{\theta_1\omega_2{}^2-\theta_2\omega_1{}^2}{\omega_1{}^2-\omega_2{}^2}-\frac{\theta_2-\theta_1}{\omega_1{}^2-\omega_2{}^2}\omega. \tag{19}$$

For air $N=5$, $f=1.95$ (observed); if we take $f=1.86$, following Becker, we have this simple case, and we may expect this change of f to make only small differences in the result.

Thus far, according to Becker: To find the thickness of the shock front, however, for violent shocks, for which there is considerable change of pressure and temperature, it is essential to take account of the increase of viscosity and thermal conductivity with temperature.

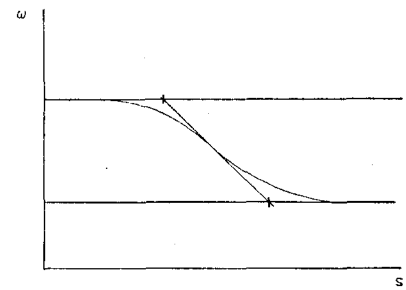

FIG. 1. Definition of "thickness" of the shock front.

Using (14) and (15), if we measure "distance" in terms of "free paths," i.e., in terms of s where

$$\frac{ds}{dx}=l, \tag{20}$$

we replace Becker's equations (17) by

$$\begin{aligned}&\frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}v\frac{\sqrt{\theta}}{\omega}\frac{d\omega}{ds}=\omega+\frac{\theta}{\omega}-1,\\ &f\left(\frac{2}{\pi}\right)^{\frac{1}{2}}v\frac{\sqrt{\theta}}{\omega}\frac{d\theta}{ds}=\theta\\ &\qquad-\frac{1}{N}\left[(1-\omega)^2+\left(\frac{2ac}{b^2}-1\right)\right].\end{aligned} \tag{21}$$

We may measure the thickness of the shock front by the whole change in ω divided by the greatest slope $d\omega/ds$ (Fig. 1).

$$(\omega_1-\omega_2)\left(\frac{ds}{d\omega}\right)_{\min}=\frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\nu\left\{\frac{(\omega_1-\omega_2)[(\omega_1+\omega_2)(\omega_1\omega_2-(\omega_1+\omega_2-1)\omega^2)]^{\frac{1}{2}}}{(\omega_1-\omega)(\omega-\omega_2)}\right\}_{\min}. \tag{22}$$

This thickness, as well as the maximum value of $p^{(1)}/p^{(0)}$ in the front,

$$\left(\frac{p^{(1)}}{p^{(0)}}=-\frac{4}{3}\frac{\mu}{p}\frac{du}{dx}\right)_{\max}$$
$$=-\left\{\frac{(\omega^2-\omega)(\omega_1+\omega_2)}{\omega^2(1-\omega_1-\omega_2)+\omega_1\omega_2}+1\right\}_{\max} \tag{23}$$

is given for shocks of various intensity in Table I.

If now the expressions (17) for $d\omega/dx$ and $d\theta/dx$ are modified by including $p^{(2)}$ and $q^{(2)}$ evaluated from the above solution on the right-hand sides, they give

$$\begin{aligned}&\frac{1}{a}\frac{4}{3}\mu\frac{d\omega}{dx}=\left(\omega+\frac{\omega}{\theta}-1\right)\Big/\left(1+\frac{p^{(2)}}{p^{(1)}}\right),\\ &\frac{1}{a}f\mu\frac{d\theta}{dx}=[\theta-\delta\{(1-\omega)^2+\alpha\}]\Big/\left(1+\frac{q^{(2)}}{q^{(1)}}\right),\end{aligned} \tag{24}$$

TABLE I.

$\frac{\rho_2}{\rho_1}$	$\frac{p_2}{p_1}$	$\frac{T_2}{T_1}$	Thickness (free paths)	$\frac{p^{(1)}}{p^{(0)}}$
1.0000	1.0000	1.0000	∞	0.0000
2.6661	4.4981	1.6871	3.98	0.3460
3.7899	9.8365	2.5954	3.08	0.7198
4.6382	19.700_3	4.2474	2.25	1.0967
5.2931	43.387_6	8.1998	1.98	1.4854
6.0000	∞	∞	1.74	2.5000

where, from Burnett's formulae, (9)–(12),

$$p^{(2)}/p^{(1)} \approx p^{(1)}/p^{(0)},$$

and

$$q^{(2)}/q^{(1)} \approx 10p^{(1)}/p^{(0)}.$$

We see that, while there is no guarantee of convergence, the general effect of the third-order terms is to make the front thicker.

Lastly, in the notation of kinetic theory, the Boltzmann equation, the solutions of which we are approximating, is

$$u\frac{\partial f}{\partial x} = \int\int\int\int (f'f_1' - ff_1) gbdbd\epsilon dv_1 du_1$$

to determine $f(x, u, v)$ and, if the method of Chapman and Enskog fails to converge, this may be rewritten

$$u\frac{\partial f}{\partial x} + f\int\int\int\int f_1 gbdbd\epsilon dv_1 du_1 = \int\int\int\int f'f_1' gbdbd\epsilon dv_1 du_1$$

and may be approximated by iteration using an already determined approximation both on the right-hand side and in the integral on the left, in the manner used in astrophysics for the transport of radiation in a stellar photosphere.[10 11]

CONCLUSIONS

All shock waves in air but the weakest are a few free paths thick. If the shock is strong the approximation process of the theory of non-uniform gases probably does not converge, because the distribution function of molecular velocities within the front departs too far from the Maxwellian. There is, however, no reason to doubt that the Boltzmann equation possesses a suitable solution.

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The Charge Effect in Photographic Development: Role of the Gelatin*

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The approach of negatively charged developer ions to the surface of a silver bromide grain is opposed by charge barriers originating both with the gelatin and with the adsorbed bromide ions. It is shown that the effect of neutral salt upon the rate of development by negative ions results primarily from a depression of the gelatin charge barrier. The effect of salt upon the bromide barrier is relatively unimportant. On the other hand, a change in the bromide barrier is mainly responsible for certain other kinetic effects, such as the induction period in development and the accelerating action of phenosafranin.

A COLLOIDAL particle of silver bromide, suspended in a solution containing excess soluble bromide, is negatively charged by the adsorption of excess bromide ions to the surface. This charge acts to concentrate positive ions in the solution near the surface and to oppose the approach of other negative ions to the surface. The resulting electric double layer consists on the one hand of the adsorbed bromide-ion layer, together with a more or less firmly held water shell, and on the other hand a diffuse ionic atmosphere in which positive ions predominate near the solid surface but decrease in predominance as the distance from the surface increases.

If the system is placed in an electric field, the

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